Factorization in generalized Calogero-Moser spaces

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Outline

1. The rational Cherednik algebra
2. The generalized Calogero-Moser space and Etingof-Ginzburg sheaf
3. Factorization

Consequences

1. A reduction theorem
2. Example G_2

The rational Cherednik algebra

- W a complex reflection group with representation $\mathfrak{h}$ over $\mathbb{C}$
- $\mathcal{S} \subset W$ the set of complex reflections and $\mathbf{c} : \mathcal{S}/W \longrightarrow \mathbb{C}$ a function (our "parameter")
- Fix $\omega : \bigwedge^2(\mathfrak{h} \oplus \mathfrak{h}^*) \longrightarrow \mathbb{C}$,

$$\omega((f_1, f_2),(g_1, g_2)) = g_2(f_1) - f_2(g_1)$$

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- For $s \in \mathcal{S}$, ω_s is ω on $\mathrm{im}(1 - s)$ and zero on $\ker(1 - s)$

The rational Cherednik algebra

We can form the rational Cherednik algebra

$$H_{t,c}(W) = \frac{T(\mathfrak{h} \oplus \mathfrak{h}^*)\# W}{\langle [x,y] = t\omega(x,y) - \sum_{s \in S} \mathbf{c}(s)\omega_s(x,y)s \rangle}$$

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PBW Theorem (Etingof - Ginzburg)

As a vector space $H_{t,\mathbf{c}} \cong \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}W$

The generalized Calogero-Moser space

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$\mathbb{C}[\mathfrak{h}]^W$ and $\mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow Z_{0,\mathbf{c}}(W)$ so we have a map

$$\pi_W : X_{\mathbf{c}}(W) \twoheadrightarrow \mathfrak{h}/W$$

Example: $W = S_2$

The Etingof-Ginzburg sheaf

Let e be the idempotent in $\mathbb{C}W \subset H_{\mathbf{c}}$ corresponding to the trivial W-module

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Definition

The Etingof-Ginzburg sheaf, $\mathcal{R}[W]$, on $X_{\mathbf{c}}$ is the sheaf defined by $\Gamma(X_{\mathbf{c}}, \mathcal{R}[W]) = H_{\mathbf{c}}e$

The sheaf $\mathcal{R}$ "contains all the information about $H_{\mathbf{c}}$"

Theorem (Etingof - Ginzburg)

$$\mathrm{End}_{Z_{\mathbf{c}}}(He) \cong H_{\mathbf{c}}$$

Relation to simple modules

Let U be a Zariski-open affine subset of $X_{\mathbf{c}}$

Theorem (Etingof-Ginzburg)

If $U \subseteq \mathrm{Smooth}(X_{\mathbf{c}})$ then

1. The sheaf $\mathcal{R}_U$ is locally free and $End_U(\mathcal{R}_U) \cong \mathcal{H}_{\mathbf{c},U}$
2. Any simple $\mathcal{H}_{\mathbf{c},U}$-module is isomorphic to $\mathcal{R}(x)$ for some $x \in U$
3. Any simple $\mathcal{H}_{\mathbf{c},U}$-module has dimension $|W|$ and is isomorphic to the regular representation as a W-module

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Thus, $\mathcal{R}[S_n]$ is a vector bundle on $X_{\mathbf{c}}$ and, $\forall\, x \in X_{\mathbf{c}}$,

$$\mathcal{R}[S_n](x) \cong \mathbb{C}S_n \quad \text{as a } S_n\text{-module}$$

In particular, $\dim \mathcal{R}[S_n](x) = n!$

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It was hoped that $\mathcal{R}[S_n]$ would be related to the Procesi bundle on the Hilbert scheme

Factorization

To $b \in \mathbb{C}^n/S_n$ we associate (up to conjugation) a stabilizer subgroup

$$W_b = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k} \qquad n_1 + \cdots + n_k = n$$

Then Wilson showed:

Factorization

$$\pi_{S_n}^{-1}(b) \cong \pi_{S_{n_1}}^{-1}(0) \times \cdots \times \pi_{S_{n_k}}^{-1}(0)$$

Conjecture

Fix $Y = \pi_{S_n}^{-1}(b)$. Based on the analogy with the Procesi bundle, Etingof and Ginzburg made

Conjecture

There is a factorization of the Etingof-Ginzburg bundle

$$\mathcal{R}[S_n]_{|Y} \cong \mathtt{Ind}_{S_{n_1} \times \cdots \times S_{n_k}}^{S_n} \mathcal{R}[S_{n_1}] \boxtimes \cdots \boxtimes \mathcal{R}[S_{n_k}]_{|Y}$$

as S_n-equivariant bundles

The main results

Theorem - Factorization of the gen Calogero-Moser space (B)

Let W be a complex relection group, $b \in \mathfrak{h}/W$ with stabilizer W_b, then there is a scheme theoretic isomorphism

$$\pi_W^{-1}(b) \cong \pi_{W_b}^{-1}(0)$$

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Theorem - Factorization of the Etingof-Ginzburg sheaf (B)

For W, b and W_b as above,

$$\mathcal{R}[W]|_{\pi_W^{-1}(b)} \cong \mathtt{Ind}_{W_b}^{W} \mathcal{R}[W_b]|_{\pi_{W_b}^{-1}(0)}$$

as W-equivariant sheaves

Proof is based on a recent result of Bezrukavnikov and Etingof

Poisson structure on $X_{\mathbf{c}}$

Since $Z_{0,\mathbf{c}} \cong eH_{0,\mathbf{c}}e$ has a flat noncommutative deformation, $eH_{t,\mathbf{c}}e$, it is a Poisson algebra with bracket

$$\{-,-\} : Z_{0,\mathbf{c}} \times Z_{0,\mathbf{c}} \longrightarrow Z_{0,\mathbf{c}}$$

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In this situation, $X_{\mathbf{c}}$ is stratified by symplectic leaves

A result of Brown and Gordon says that there are only finitely many leaves and they are algebraic

i.e. each leaf is Zariski locally closed

Finite dimensional quotients

A point $x \in X_{\mathbf{c}}$ corresponds to a maximal ideal $\mathfrak{m}_x \subset Z_{\mathbf{c}}$
Fix

$$H_{\mathbf{c},x} := H_{\mathbf{c}}/\mathfrak{m}_x H_{\mathbf{c}} \qquad (= \mathcal{H}_{\mathbf{c}}(x))$$

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Then the following holds

Theorem (Brown-Gordon)

Let $\mathcal{L} \subset X_{\mathbf{c}}$ be a symplectic leaf and $x, y \in \mathcal{L}$, then there is an algebra isomorphism

$$H_{\mathbf{c},x} \cong H_{\mathbf{c},y}$$

Reduction to zero dimsional leaves

In fact, when describing $H_{\mathbf{c},x}$ we need only consider the "worst" case:

Theorem (B)

Let $\mathcal{L} \subset X_{\mathbf{c}}(W)$ be a symplectic leaf of dimension $2l$ and $x \in \mathcal{L}$. Then there exists a parabolic subgroup W' of W, a point $y \in X_{\mathbf{c}'}(W')$ such that $\{y\}$ is a symplectic leaf and an algebra isomorphism

$$H_{\mathbf{c},x} \cong \mathtt{Mat}_{|W/W'|}(H_{\mathbf{c}',y})$$

If $\dim X_{\mathbf{c}}(W) = 2n$ then rank $W' = n - l$

Example G2