Factorization in generalized Calogero-Moser spaces

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Introduction

The Etingof-Ginzburg sheaf

Consequences

Outline

1. The rational Cherednik algebra
2. The generalized Calogero-Moser space and Etingof-Ginzburg sheaf
3. Factorization

Consequences

1. A reduction theorem
2. Example $G_2$
The rational Cherednik algebra

- $\mathcal{W}$ a complex reflection group with representation $\mathfrak{h}$ over $\mathbb{C}$
- $S \subset \mathcal{W}$ the set of complex reflections and $c : S/\mathcal{W} \longrightarrow \mathbb{C}$ a function (our “parameter”)
- Fix $\omega : \bigwedge^2(\mathfrak{h} \oplus \mathfrak{h}^*) \longrightarrow \mathbb{C}$,

$$\omega((f_1, f_2), (g_1, g_2)) = g_2(f_1) - f_2(g_1)$$
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- For $s \in S$, $\omega_s$ is $\omega$ on $\text{im}(1 - s)$ and zero on $\ker(1 - s)$
The rational Cherednik algebra

We can form the rational Cherednik algebra

\[ H_{t,c}(W) = \frac{T(\mathfrak{h} \oplus \mathfrak{h}^*) \# W}{\langle [x, y] = t\omega(x, y) - \sum_{s \in S} c(s)\omega_s(x, y)s \rangle} \]

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**PBW Theorem (Etingof - Ginzburg)**

As a vector space \( H_{t,c} \cong \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}W \)
The generalized Calogero-Moser space

When $t = 0$, $H_{0,c}$ is a finite module over its center $Z_{0,c}$, but the center of $H$ is $\mathbb{C}$ when $t \neq 0$
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From now on we assume \( t = 0 \) and write \( X_c(W) \) for the reduced affine variety \( \text{Spec}(Z(H_{0,c})) \)

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$\mathbb{C}[\mathfrak{h}]^W$ and $\mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow Z_{0,c}(W)$ so we have a map

$\pi_W : X_c(W) \rightarrow \mathfrak{h}/W$

Example: $W = S_2$
The Etingof-Ginzburg sheaf

Let $e$ be the idempotent in $\mathbb{C}W \subset H_c$ corresponding to the trivial $W$-module.
Then $H_ce$ is a left $H_c$-module and a (right) $Z_c$-module.
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**Definition**

The Etingof-Ginzburg sheaf, $\mathcal{R}[W]$, on $X_c$ is the sheaf defined by
$\Gamma(X_c, \mathcal{R}[W]) = H_c e$.

The sheaf $\mathcal{R}$ “contains all the information about $H_c$”

**Theorem (Etingof - Ginzburg)**

$$\text{End}_{Z_c}(He) \cong H_c$$
Relation to simple modules

Let $U$ be a Zariski-open affine subset of $X_c$

**Theorem (Etingof-Ginzburg)**

If $U \subseteq \text{Smooth}(X_c)$ then

1. The sheaf $\mathcal{R}_U$ is locally free and $\text{End}_U(\mathcal{R}_U) \cong \mathcal{H}_{c,U}$
2. Any simple $\mathcal{H}_{c,U}$-module is isomorphic to $\mathcal{R}(x)$ for some $x \in U$
3. Any simple $\mathcal{H}_{c,U}$-module has dimension $|\mathcal{W}|$ and is isomorphic to the regular representation as a $\mathcal{W}$-module
The case $W = S_n$ and $c \neq 0$

In this situation $X_c$ is smooth and isomorphic to the “classical” Calogero-Moser space studied by Wilson.
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\[
\mathcal{R}[S_n](x) \cong \mathbb{C}S_n \quad \text{as a } S_n\text{-module}
\]

In particular, \( \dim \mathcal{R}[S_n](x) = n! \)
The case $\mathcal{W} = S_n$ and $c \neq 0$

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It was hoped that $\mathcal{R}[S_n]$ would be related to the Procesi bundle on the Hilbert scheme.
Factorization

To $b \in \mathbb{C}^n / S_n$ we associate (up to conjugation) a stabilizer subgroup

$$W_b = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k} \quad n_1 + \cdots + n_k = n$$

Then Wilson showed:

$$\pi_{S_n}^{-1}(b) \cong \pi_{S_{n_1}}^{-1}(0) \times \cdots \times \pi_{S_{n_k}}^{-1}(0)$$
Conjecture

Fix $Y = \pi_{\mathcal{S}_n}^{-1}(b)$. Based on the analogy with the Procesi bundle, Etingof and Ginzburg made

There is a factorization of the Etingof-Ginzburg bundle

$$\mathcal{R}[S_n]|_Y \cong \text{Ind}_{\mathcal{S}_{n_1} \times \ldots \times \mathcal{S}_{n_k}} \mathcal{R}[S_{n_1}] \boxtimes \cdots \boxtimes \mathcal{R}[S_{n_k}]|_Y$$

as $S_n$-equivariant bundles.
The main results

Theorem - Factorization of the gen Calogero-Moser space (B)

Let \( W \) be a complex reflection group, \( b \in \mathfrak{h}/W \) with stabilizer \( W_b \), then there is a scheme theoretic isomorphism

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\pi_{W}^{-1}(b) \cong \pi_{W_b}^{-1}(0)
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The main results

**Theorem - Factorization of the gen Calogero-Moser space (B)**

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**Theorem - Factorization of the Etingof-Ginzburg sheaf (B)**

For $W$, $b$ and $W_b$ as above,

$$
\mathcal{R}[W]|_{\pi_W^{-1}(b)} \cong \text{Ind}_{W_b}^W \mathcal{R}[W_b]|_{\pi_{W_b}^{-1}(0)}
$$

as $W$-equivariant sheaves

Proof is based on a recent result of Bezrukavnikov and Etingof
Poisson structure on $X_c$

Since $Z_{0,c} \cong eH_{0,c}e$ has a flat noncommutative deformation, $eH_{t,c}e$, it is a Poisson algebra with bracket

$$\{ - , - \} : Z_{0,c} \times Z_{0,c} \longrightarrow Z_{0,c}$$

i.e. $(Z_{0,c}, \{ - , - \})$ is a Lie algebra and $\{ z , - \}$ a derivation

$\forall a \in Z_{0,c}$
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In this situation, $X_c$ is stratified by symplectic leaves

A result of Brown and Gordon says that there are only finitely many leaves and they are algebraic

i.e. each leaf is Zariski locally closed
Finite dimensional quotients

A point $x \in X_c$ corresponds to a maximal ideal $m_x \subset Z_c$

Fix

$$H_{c,x} := H_c/m_x H_c \quad (= H_c(x))$$
Finite dimensional quotients

A point \( x \in X_c \) corresponds to a maximal ideal \( m_x \subset \mathbb{Z}_c \)

Fix

\[ H_{c,x} := H_c/m_x H_c \quad (= \mathcal{H}_c(x)) \]

Then the following holds

**Theorem (Brown-Gordon)**

Let \( \mathcal{L} \subset X_c \) be a symplectic leaf and \( x, y \in \mathcal{L} \), then there is an algebra isomorphism

\[ H_{c,x} \cong H_{c,y} \]
Reduction to zero dimensional leaves

In fact, when describing $H_{c,x}$ we need only consider the “worst” case:

**Theorem (B)**

Let $\mathcal{L} \subset X_{c}(W)$ be a symplectic leaf of dimension $2l$ and $x \in \mathcal{L}$. Then there exists a parabolic subgroup $W'$ of $W$, a point $y \in X_{c'}(W')$ such that $\{y\}$ is a symplectic leaf and an algebra isomorphism

$$H_{c,x} \cong \text{Mat}_{|W/W'|}(H_{c',y})$$

If $\dim X_{c}(W) = 2n$ then $\text{rank } W' = n - l$
Example G2