Canonical and semicanonical bases

(Journées Jacques Alev, Reims)

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Overview

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- Lusztig, Kashiwara (1990):

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- Lusztig, Kashiwara (1990): canonical basis of $U_q(\mathfrak{n})$.

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Problem

Compare these bases.

Algebras

The algebra $\mathcal{A}$

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- Relations :

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u_i u_{i+1} &= q^{-2} u_{i+1} u_i, && (0 \leq i \leq 2),\\
u_i u_{i+2} &= q^{-2} u_{i+2} u_i + (q^{-2} - 1) u_{i+1}^2, && (0 \leq i \leq 1),\\
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Standard monomials

$$u[\mathbf{a}] := u_3^{a_3} u_2^{a_2} u_1^{a_1} u_0^{a_0}, \qquad (\mathbf{a} = (a_3, a_2, a_1, a_0) \in \mathbb{N}^4).$$

form a $\mathbb{Q}(q)$-basis of $\mathcal{A}$.

The algebra $\mathcal{A}$

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Definition

$\{E[\mathbf{a}] \mid \mathbf{a} \in \mathbb{N}^4\}$ is the dual PBW basis of $\mathcal{A}$.

The algebra $\mathcal{A}$

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$$p_0 \quad := \quad u_2 u_0 - q^2 u_1^2 \quad = \quad E[0,1,0,1] - qE[0,0,2,0],$$

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$$\begin{aligned} p_0 &:= u_2 u_0 - q^2 u_1^2 &=& E[0,1,0,1] - qE[0,0,2,0], \\ p_1 &:= u_3 u_1 - q^2 u_2^2 &=& E[1,0,1,0] - qE[0,2,0,0]. \end{aligned}$$

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p_0 and p_1 are q-central

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$$p_0 u_0 = q^2 u_0 p_0, \;\; p_0 u_1 = u_1 p_0, \;\; p_0 u_2 = q^{-2} u_2 p_0, \;\; p_0 u_3 = q^{-4} u_3 p_0,$$

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$$p_1 u_0 = q^4 u_0 p_1,\ \ p_1 u_1 = q^2 u_1 p_1,\ \ p_1 u_2 = u_2 p_1,\ \ p_1 u_3 = q^{-2} u_3 p_1.$$

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$$\underline{\mathcal{A}} := \mathbb{Q} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{A}_{\mathbb{Z}} = \mathbb{Q}[x_0, x_1, x_2, x_3],$$

where $x_i = 1 \otimes u_i$.

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$$f_0 := 1 \otimes p_0 = x_2x_0 - x_1^2,$$

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The algebra $\underline{\mathcal{A}}$

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- cluster variables x_n $(n \in \mathbb{Z})$,
- clusters $\{x_n, x_{n+1}\}$ $(n \in \mathbb{Z})$.

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Proposition (Geiss-L-Schröer)

The assignment

$$x_0 \mapsto \Delta_{s_0(\varpi_0)}, x_1 \mapsto \Delta_{s_0 s_1(\varpi_1)}, f_0 \mapsto \Delta_{s_0 s_1 s_0(\varpi_0)}, f_1 \mapsto \Delta_{s_0 s_1 s_0 s_1(\varpi_1)},$$

extends to an isomorphism $\underline{\mathcal{A}} \cong \mathbb{C}[N(w)]$.

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Proposition

$\mathcal{A}$ is the quantum cluster algebra with initial seed $((u_1, u_2, p_0, p_1), B, L)$, where

$$L := \begin{bmatrix} 0 & -2 & 0 & -2 \\ 2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}.$$

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- $\mathbb{C}[N(w)] \cong \mathbb{C}[N]^{N'(w)}$ (where $N'(w) = N \cap (w^{-1}Nw)$), the polynomial subalgebra of $\mathbb{C}[N]$ with generators

$$\Delta_{s_0(\varpi_0)},\ \Delta_{s_0 s_1(\varpi_1)},\ \Delta_{s_0(\varpi_0),\, s_1 s_0 s_0(\varpi_0)},\ \Delta_{s_0 s_1(\varpi_1),\, s_0 s_1 s_0 s_1(\varpi_1)}.$$

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Proposition

$\mathcal{A} \cong \mathbb{C}_q[N(w)]$, the subalgebra of $\mathbb{C}_q[N]$ generated by

$$u_0 = \Delta^q_{s_0(\varpi_0)}, \quad u_1 = \Delta^q_{s_0 s_1(\varpi_1)},$$

$$u_2 = \Delta^q_{s_0(\varpi_0),\, s_1 s_0 s_0(\varpi_0)}, \ u_3 = \Delta^q_{s_0 s_1(\varpi_1),\, s_0 s_1 s_0 s_1(\varpi_1)}.$$

Bases

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- Cluster monomials: $f_0^{k_0} f_1^{k_1} x_n^{a_n} x_{n+1}^{a_{n+1}}, \quad n, k_0, k_1, a_n, a_{n+1} \in \mathbb{N}.$

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Theorem (Sherman-Zelevinsky)

$\underline{\mathcal{B}}$ is a $\mathbb{Q}$-basis of $\underline{\mathcal{A}}$, characterized by positivity properties.

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$\underline{\mathcal{B}}$ is a $\mathbb{Q}$-basis of $\underline{\mathcal{A}}$, characterized by positivity properties.

Example: $C_2(t) = t^2 - 2$, hence $z^2 - 2f_0 f_1 \in \underline{\mathcal{B}}$.

Semicanonical basis of $\underline{\mathcal{A}}$

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- Can show that $\underline{\Sigma}^* = \{\text{cluster monomials}\} \cup \{z^k \mid k \geq 1\}$.
- **Remark** : This is the same as Dupont's basis coming from generic representations of the Kronecker quiver.

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- $S(\mathbf{a}) := \{\mathbf{b} \in \mathbb{N}^4 \mid \mathbf{a} \lhd \mathbf{b} \text{ and } \mathbf{b} \neq \mathbf{a}\}$ is finite.

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There is a unique $\mathbb{Q}(q)$*-basis* $\mathcal{B} = \{B[\mathbf{a}] \mid \mathbf{a} \in \mathbb{N}^4\}$ *of* $\mathcal{A}$ *satisfying*

(i) $B[\mathbf{a}] - E[\mathbf{a}] \in \bigoplus_{\mathbf{b} \in S(\mathbf{a})} q\mathbb{Z}[q]E[\mathbf{b}]$,

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$B[1,0,0,0] = u_3$, $B[0,1,0,1] = p_0$, $B[1,0,1,0] = p_1$.

$B[2,0,0,1] = E[2,0,0,1] - (q+q^3)E[1,1,1,0] + q^2E[0,3,0,0]$.

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Remark: Via $\mathcal{A} \hookrightarrow \mathbb{C}_q[N]$, $\mathcal{B}$ is a subset of the dual of Lusztig's canonical basis of $U_q(\mathfrak{n})$.

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- *For* $a_0, a_1, a_2, a_3 \in \mathbb{N}$,

$$\begin{aligned} B[0,0,a_1,a_0] &= E[0,0,a_1,a_0], \\ B[0,a_2,a_1,0] &= E[0,a_2,a_1,0], \\ B[a_3,a_2,0,0] &= E[a_3,a_2,0,0]. \end{aligned}$$

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- *For* $\mathbf{a} = [a_3, a_2, a_1, a_0] \in \mathbb{N}^4$,

$$\begin{aligned} B[\mathbf{a}]p_0 &= q^{-(a_2+2a_1+3a_0)} B[a_3, a_2+1, a_1, a_0+1] \\ &= q^{2(2a_3+a_2-a_0)} p_0 B[\mathbf{a}], \end{aligned}$$

$$\begin{aligned} p_1 B[\mathbf{a}] &= q^{-(3a_3+2a_2+a_1)} B[a_3+1, a_2, a_1+1, a_0] \\ &= q^{2(-a_3+a_1+2a_0)} B[\mathbf{a}]p_1. \end{aligned}$$

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Proposition implies:

- every element of $\mathcal{B}$ is product of a monomial in q, p_0, p_1 times an element of the form:

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- We are left with type $B[a_3, 0, 0, a_0]$.

Imaginary elements of $\mathcal{B}$

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$$Z := B[1,0,0,1] = E[1,0,0,1] - q^2 E[0,1,1,0] = u_3 u_0 - q^2 u_2 u_1 .$$

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Specializing $q \mapsto 1$, we get $z^2 - f_0 f_1 \in \underline{\mathcal{A}}$.

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The $q \mapsto 1$ specialization of Luzstig-Kashiwara's dual canonical basis $\mathcal{B}$ is not equal to the Sherman-Zelevinsky canonical basis $\underline{\mathcal{B}}$.

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Conjecture

For $k \in \mathbb{N}$:

$$q^{4k} B[1,0,0,1]\, B[k,0,0,k] = B[k+1,0,0,k+1] + B[k,1,1,k].$$

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For $k \geq 2$ we have

$$B[k+1,0,0,k]B[k-1,0,0,k-2] = q^{-4k(k-1)}(qB[2k,0,0,2k-2] + B[k+1,k-2,k+1,k-2]),$$

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If $a_3 \neq a_0$ then $B[a_3,0,0,a_0]$ is a quantum cluster monomial.

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When $q \mapsto 1$, Lusztig-Kashiwara's dual canonical basis $\mathcal{B}$ of $\mathcal{A}$ specializes to Caldero-Zelevinsky's semicanonical basis $\underline{\mathcal{S}}$ of $\underline{A}$.

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- Can we describe the $q \mapsto 1$ specialization of $\mathcal{B}$ in terms of representation theory of quivers, or preprojective algebras ?
- What is the positive cone spanned by $\mathcal{B}_{|q=1}$?