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ALGEBRES DE HOPF COMBINATOIRES

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Introduction

Mes travaux de recherche s'inscrivent dans le cadre des algèbres de Hopf combinatoires ; plus spécifiquement, ils portent sur l'étude des algèbres de Hopf d'arbres de Connes-Kreimer et de ses liens avec d'autres objets, comme par exemple l'algèbre de Hopf de Malvenuto-Reutenauer.

Les principaux objets étudiés dans ce mémoire sont les suivants :

- L'algèbre de Hopf des arbres enracinés $\mathcal{H}_R$ et plus généralement les algèbres de Hopf d'arbres enracinés décorés $\mathcal{H}_R^{\mathcal{D}}$, introduites dans [CK99] et étudiées dans [Kre98, Kre99, Kre02] et [2]. Ces algèbres de Hopf sont commutatives, non cocommutatives. Elles furent introduites pour traiter de manière algébrique la Renormalisation, procédure itérative d'extraction de pôles en Théorie Quantique des Champs. L'algèbre de Hopf $\mathcal{H}_R$, graduée et connexe, est le dual gradué d'une algèbre enveloppante, connue sous le nom d'algèbre de Grossman-Larson [GL89, GL90]. L'algèbre de Lie sous-jacente, de base indexée par les arbres enracinés, est l'algèbre prélie libre sur un générateur [CL01].
- L'algèbre de Hopf des arbres enracinés plans $\mathcal{H}$ et ses versions décorées $\mathcal{H}^{\mathcal{D}}$, introduites simultanément dans [3, 4] et [Hol03]. Ces algèbres de Hopf ne sont ni commutatives, ni cocommutatives et on peut montrer qu'elles sont auto-duales. Il est immédiat que $\mathcal{H}_R$ est un quotient de $\mathcal{H}$. Ainsi, le dual de $\mathcal{H}_R$ s'identifie alors à une sous-algèbre de $\mathcal{H}$.

Ces objets ont été étudiés dans différents domaines. En théorie Quantique des Champs, l'utilité de $\mathcal{H}_R$ pour la Renormalisation est explicitée dans [BK05, BK06, BK00a, BK00b, CQRV02, CK00, CK01a, CK01b, EFGK04, EFGK05, FGB01, FGB04, KW99, Kre98, BK06, Kre99, Kre02]. L'algèbre $\mathcal{H}_R$ apparaît aussi comme algèbre des coordonnées du groupe de Butcher des méthodes de Runge-Kutta [Bro04] et dans le cadre du calcul moulien [Men07]. D'un point de vue plus algébrique, $\mathcal{H}_R$ et $\mathcal{H}$ munies de structures supplémentaires sont utilisés opéradiquement dans [Cha02, CL01, Moe01, Mur06, OG05, vdLM06].

Ce mémoire parcourt mes travaux de recherche dans ce cadre pour la période 2002-2009. Il se découpe en quatre chapitres. Dans le premier chapitre sont exposées les différentes notions dont j'aurai besoin pour la suite.

Le deuxième chapitre est dédié aux résultats des articles [6, 12]. L'algèbre de Hopf $\mathcal{H}$ (et plus généralement $\mathcal{H}^{\mathcal{D}}$) est munie d'une structure supplémentaire par un scindement d'associativité qui en fait une algèbre de Hopf dendriforme au sens de [Lod01]. De plus, $\mathcal{H}$ munie de sa structure dendriforme est un objet libre et fournit ainsi une alternative à la description de l'algèbre dendriforme libre en termes d'arbres binaires planaires. D'autres algèbres de Hopf dendriformes sont connues, comme par exemple l'algèbre de Malvenuto-Reutenauer ou l'algèbre des fonctions de parking.

Les travaux ici présentés donnent un outil permettant de démontrer qu'une algèbre de Hopf dendriforme est libre en utilisant un scindement de coassociativité. Citons quelques corollaires : auto-dualité de l'algèbre de Hopf étudiée, liberté de son algèbre de Lie, liberté en tant qu'algèbre et coliberté en tant que cogèbre... Plus précisément, la notion de bigèbre bidendriforme est introduite dans la section 2.2. Un théorème de rigidité (théorème de Milnor-Moore bidendriforme) montre que toute bigèbre bidendriforme connexe est libre en tant qu'algèbre dendriforme. Ce résultat est appliqué par exemple à l'algèbre de Malvenuto-Reutenauer **FQSym**.

Il reste alors à décrire un système de générateurs de **FQSym**, ce qui est fait dans la section 2.3. Par le théorème de Milnor-Moore dendriforme [Cha02, Ron01], $Prim(\mathbf{FQSym})$ est une algèbre brace libre et il est alors équivalent de trouver un système de générateurs de $Prim(\mathbf{FQSym})$. Une base de cet espace est donné inductivement à l'aide de la structure brace et de l'insertion de la lettre $(n+1)$ dans les éléments du groupe symétrique S_n représentés par des mots de n lettres. Cette base est indexée par un ensemble d'arbres plans partiellement décorés. Les éléments primitifs au sens bidendriforme correspondent aux éléments indexés par des arbres réduits à leur racine et fournissent un système de générateurs de **FQSym** au sens dendriforme.

D'autres résultats (non décrits ici) sur les algèbres braces peuvent être trouvés dans [10], où il est montré qu'une algèbre brace libre et aussi libre en tant qu'algèbre prélie.

Le chapitre suivant est dédié à mes travaux sur les équations de Dyson-Schwinger combinatoires [9, 11]. Ces équations [BK06, Kre07, KY06] sont de la forme $X = B^+(f(X))$, où X est un élément de la complétion de $\mathcal{H}_R$ pour la topologie induite par la graduation et $f(h)$ une série formelle. Chaque équation de Dyson-Schwinger possède une unique solution et les composantes homogènes de cette solution définissent une sous-algèbre $\mathcal{H}_f$ de $\mathcal{H}_R$.

La question est de déterminer les séries formelles f telles que $\mathcal{H}_f$ soit une sous-algèbre de Hopf. Une réponse complète est apportée dans la section 3.2, avec quelques indications sur la preuve de ce résultat et une description des générateurs des sous-algèbres obtenues. Une version non commutative de ces résultats est exposée dans [9].

On obtient ainsi une famille de sous-algèbres $\mathcal{H}_f$ de Hopf de $\mathcal{H}_R$, indexées par deux paramètres α et β. Nous montrons dans la section 3.3 qu'hormis deux cas dégénérés pour lesquels la sous-algèbre devient cocommutative, $\mathcal{H}_f$ est isomorphe à l'algèbre de Hopf de Faà di Bruno, algèbre des coordonnées du groupe des difféomorphismes formels de la droite tangents à l'identité en 0. Une explication possible de ce fait est la suivante : toutes ces algèbres, à l'exception du cas $\alpha = 0$, sont duales d'une algèbre enveloppante dont l'algèbre de Lie a pour série formelle $\frac{t}{1-t}$. En ajoutant une hypothèse de non commutativité, il est montré qu'à isomorphisme près, il n'existe que trois telles algèbres de Lie. Avec une condition plus forte, il n'en existe qu'une, l'algèbre de Lie de Faà di Bruno. Les duales des algèbres enveloppantes des deux autres algèbres de Lie obtenues sont également décrites comme sous-algèbres de $\mathcal{H}_R$.

La dernière section de ce chapitre donne quelques résultats sur les systèmes d'équations de Dyson-Schwinger issus de mes travaux en préparation [17]. Ces systèmes sont des généralisations aux cas décorés des objets précédents. Leur étude est néanmoins plus complexe. Différents exemples sont données dans la section 3.3, ainsi qu'un procédé (la dilatation) permettant d'augmenter le nombre de décorations. Enfin, un théorème montre que tout système dont les séries formelles vérifient une certaine condition de dépendance mutuelle, est une dilatation d'un système formé d'une seule équation.

Le dernier chapitre expose quelques liens entre ces algèbres de Hopf et la théorie des algèbres enveloppantes quantiques, objet des travaux exposés dans [5, 13]. Les algèbres $\mathcal{H}^{\mathcal{D}}$ sont traitées à la manière des algèbres enveloppantes de la partie positive $\mathfrak{g}^+$ d'une certaine algèbre de Lie simple $\mathfrak{g}$. Dans les deux cas, il s'agit d'une algèbre de Hopf graduée et connexe, admettant une quantification à un paramètre q donnant une famille d'algèbres de Hopf tressées, ces objets étant auto-duaux lorsque $q \neq 1$. La construction classique de $\mathcal{U}_q(\mathfrak{g})$ à partir de la quantification $\mathcal{U}_q(\mathfrak{g}^+)$ à l'aide d'une bosonisation et d'un double quantique de Drinfeld est étendue au cas de $\mathcal{H}^{\mathcal{D}}$ et on obtient ainsi une algèbre de Hopf $D(\mathcal{H}_q^{\mathcal{D}})$, comprenant un tore et deux copies de $\mathcal{H}_q^{\mathcal{D}}$. Cette construction est décrite dans la section 4.1. Cette algèbre $D(\mathcal{H}_q^{\mathcal{D}})$, pour un bon choix de la quantification, possède un sous-quotient isomorphe à $\mathcal{U}_q(\mathfrak{g})$. Une version "classique" de cet objet est l'algèbre de Lie double des arbres, étudiée ainsi que certains de ses modules dans [7].

Poursuivant le parallèle avec la théorie des groupes quantiques, une notion de modules de plus haut poids sur $D(\mathcal{H}_q^{\mathcal{D}})$ est introduite dans la section 4.2. Les modules de Verma et les modules simples sont décrits et il est montré qu'un produit tensoriel de modules simples est semi-simple. D'autre part, lorsque $\mathcal{H}_q^{\mathcal{D}}$ est primitivement engendrée (hypothèse se traduisant sur le choix des paramètres de la quantification de $\mathcal{H}^{\mathcal{D}}$), $D(\mathcal{H}_q^{\mathcal{D}})$ est engendrée en tant qu'algèbre par une famille de sous-algèbres isomorphes à $\mathcal{U}_q(\mathfrak{sl}(2))$, ce qui permet d'introduire la notion de base cristalline d'un module simple de plus haut poids.

Dans la dernière section, l'existence et l'unicité d'une base cristalline est démontrée pour chaque module simple de plus haut poids dominant. Ce résultat est utilisé pour décrire la décomposition d'un produit tensoriel de deux modules de plus haut poids de manière combinatoire, en utilisant le graphe cristallin associée aux bases cristallines de chacun des deux modules.

Pour terminer cette introduction, signalons qu'une partie de mes travaux, non détaillée ici, porte sur des calculs d'homologie de Poisson de certaines algèbres d'invariants, en collaboration avec Jacques Alev [1, 8]. Je n'ai pas non plus évoqué mes travaux sur l'algèbre infinitésimale des arbres plans, qu'on peut obtenir en envoyant le paramètre q à 0 dans les quantifications du quatrième chapitre [14, 15, 16]. Dans ces travaux apparaissent des liens avec le poset de Tamari et certaines opérades quadratiques de Koszul.

La bibliographie de ce mémoire est séparée en deux sections différentes : mes publications et preprints sont séparées des autres références.

Chapter 1

Hopf algebras of trees

We introduce in this chapter the objects we shall study in the sequel: the Connes-Kreimer Hopf algebra of (decorated) rooted trees and its non commutative version the Hopf algebra of (decorated) planar rooted trees.

1.1 The Connes-Kreimer Hopf algebra

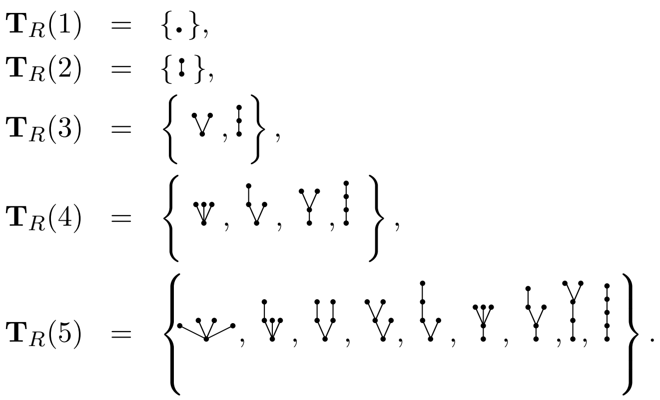
1.1.1 Rooted trees

Let us first recall the definition of a rooted tree.

Definition 1 [Sta86, Sta99]

1. A rooted tree is a finite graph, connected and without loops, with a special vertex called the root.
2. The weight of a rooted tree is the number of its vertices.
3. The set of rooted trees will be denoted by $\mathbf{T}_R$. For all $n \in \mathbb{N}^*$, the set of rooted trees of weight n will be denoted by $\mathbf{T}_R(n)$.

Examples.



1.1.2 Bialgebra of rooted trees

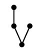
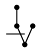
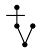
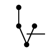
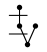
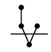
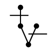
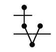
The Hopf algebra $\mathcal{H}_R$ of rooted trees is introduced in [CK99]. As an algebra, $\mathcal{H}_R$ is the free associative, commutative, unitary K-algebra generated by $\mathbf{T}_R$. In other terms, a K-basis of $\mathcal{H}_R$ is given by rooted forests, that is to say non necessarily connected graphs F such that each connected component of F is a rooted tree. The set of rooted forests will be denoted by $\mathbf{F}_R$. The product of $\mathcal{H}_R$ is given by the concatenation of rooted forests, and the unit is the empty forest, denoted by 1.

Examples. Here are the rooted forests of weight ≤ 4:

$$1, \tun, \tun\tun, \tdeux, \tun\tun\tun, \tdeux\tun, \ttroisun, \ttroisdeux, \tun\tun\tun\tun, \tdeux\tun\tun, \tdeux\tdeux, \ttroisun\tun, \ttroisdeux\tun, \tquatreun, \tquatredeux, \tquatrequatre, \tquatrecinq .$$

In order to make $\mathcal{H}_R$ a bialgebra, we now introduce the notion of cut of a tree t. A *cut* c of a tree t is a choice of edges of t. Deleting the chosen edges, the cut sends t to a forest denoted by $W^c(t)$. A non-empty cut c is *admissible* if any path in the tree meets at most one cut edge. For such a cut, the tree of $W^c(t)$ which contains the root of t is denoted by $R^c(t)$ and the product of the other trees of $W^c(t)$ is denoted by $P^c(t)$. We also add the total cut, which is by convention an admissible cut such that $R^c(t) = 1$ and $P^c(t) = W^c(t) = t$. The set of admissible cuts of t is denoted by $Adm(t)$.

Example. Let us consider the rooted tree $t = \tquatredeux$. As it as 3 edges, it has 2^3 non total cuts.

cut c									total
Admissible?	yes	yes	yes	yes	no	yes	yes	no	yes
$W^c(t)$	$\tquatredeux$	$\tdeux\tdeux$	$\tun\ttroisun$	$\ttroisdeux\tun$	$\tun\tun\tdeux$	$\tdeux\tun\tun$	$\tdeux\tun\tun$	$\tun\tun\tun\tun$	$\tquatredeux$
$R^c(t)$	$\tquatredeux$	$\tdeux$	$\ttroisun$	$\ttroisdeux$	$\times$	$\tun$	$\tdeux$	$\times$	1
$P^c(t)$	1	$\tdeux$	$\tun$	$\tun$	$\times$	$\tdeux\tun$	$\tun\tun$	$\times$	$\tquatredeux$

The coproduct of $\mathcal{H}_R$ is defined as the unique algebra morphism from $\mathcal{H}_R$ to $\mathcal{H}_R \otimes \mathcal{H}_R$ such that, for all rooted tree $t \in \mathbf{T}_R$:

$$\Delta(t) = \sum_{c \in Adm(t)} P^c(t) \otimes R^c(t).$$

Example. We obtain:

$$\Delta(\tquatredeux) = \tquatredeux \otimes 1 + 1 \otimes \tquatredeux + \tdeux \otimes \tdeux + \tun \otimes \ttroisun + \tun \otimes \ttroisdeux + \tdeux\tun \otimes \tun + \tun\tun \otimes \tdeux .$$

A study of admissible cuts of a tree proves the following lemma:

Lemma 2 *We define $B^+ : \mathcal{H}_R \longrightarrow \mathcal{H}_R$ as the operator which associates to any rooted forest $t_1 \ldots t_n$, the rooted tree obtained by grafting the roots of $t_1, \ldots, t_n$ on a common new root. Then, for all $x \in \mathcal{H}_R$:*

$$\Delta \circ B^+(x) = B^+(x) \otimes 1 + (Id \otimes B^+) \circ \Delta(x).$$

For example, $B^+(\tun\tdeux) = \tquatredeux$. This operator B^+ is used to inductively prove the coassociativity of Δ, so:

Theorem 3 *With this coproduct, $\mathcal{H}_R$ is a bialgebra. The counit of $\mathcal{H}_R$ is given by:*

$$\varepsilon : \left\{ \begin{array}{rcl} \mathcal{H}_R & \longrightarrow & K \\ F \in \mathbf{F}_R & \longrightarrow & \delta_{1,F}. \end{array} \right.$$

The bialgebra $\mathcal{H}_R$ is clearly graded by the weight. It is connected, that is to say the homogeneous component $\mathcal{H}_R(0)$ of degree 0 is K. So, $\mathcal{H}_R$ has an antipode S. It is given by the following theorem:

Theorem 4 *Let $t \in \mathbf{T}_R$. Then:*

$$S(t) = \sum_{c \text{ non total cut of } t} (-1)^{n_c+1} W^c(t),$$

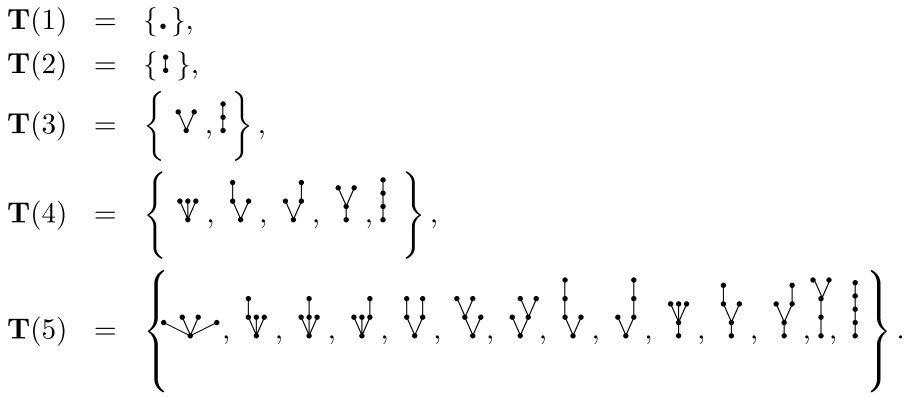
where n_c is the number of cut edges in c.

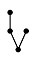
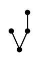
1.2 A non commutative version of $\mathcal{H}_R$

1.2.1 Planar rooted trees

Definition 5 [Sta86, Sta99] A *planar* (or *plane*) *rooted tree* is a rooted tree t such that for each vertex s of t, the children of s are totally ordered. The set of planar rooted trees will be denoted by $\mathbf{T}$. For every $n \in \mathbb{N}^*$, the set of planar rooted trees of weight n will be denoted by $\mathbf{T}(n)$.

Example. Planar rooted are drawn such that the total order on the children of each vertex is given from left to right.



In particular,  and  are equal as rooted trees, but not as planar rooted trees.

1.2.2 The Hopf algebra of planar rooted trees

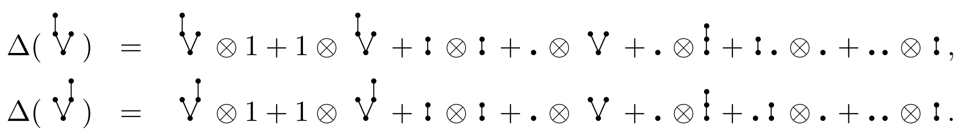
The Hopf algebra of planar rooted tree $\mathcal{H}$ was introduced simultaneously in [3] and [Hol03]. As an algebra, $\mathcal{H}$ is the free associative unitary algebra generated by $\mathbf{T}$. In other terms, a K-basis of $\mathcal{H}$ is given by planar rooted forests, that is to say non necessarily connected graphs F such that each connected component of F is a planar rooted tree, and the roots of these rooted trees are totally ordered. The set of planar rooted forests will be denoted by $\mathbf{F}$. For all $n \in \mathbb{N}$, the set of rooted forests of weight n will be denoted by $\mathbf{F}(n)$. The product of $\mathcal{H}$ is given by the concatenation of planar rooted forests, and the unit is the empty forest, denoted by 1.

If t is a planar tree and c is an admissible cut of c, then the rooted tree $R^c(t)$ is naturally a planar tree. Moreover, as c is admissible, the different rooted trees of the forest $P^c(t)$ are planar and totally ordered from left to right, so $P^c(t)$ is a planar forest. We then define a coproduct on $\mathcal{H}$ as the unique algebra morphism from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{H}$ such that, for all planar rooted tree $t \in \mathbf{T}$:

$$\Delta(t) = \sum_{c \in Adm(t)} P^c(t) \otimes R^c(t).$$

As $\mathcal{H}$ is the free algebra generated by $\mathbf{T}$, this makes sense.

Examples.



An operator B^+ is also defined on $\mathcal{H}$, with a non commutative version of lemma 2. So:

Theorem 6 *With this coproduct, $\mathcal{H}$ is a bialgebra. The counit of $\mathcal{H}$ is given by:*

$$\varepsilon : \left\{ \begin{array}{rcl} \mathcal{H} & \longrightarrow & K \\ F \in \mathbf{F}_R & \longrightarrow & \delta_{1,F}. \end{array} \right.$$

The bialgebra $\mathcal{H}$ is graded by the weight and is connected, so it has an antipode.

The Hopf algebra $\mathcal{H}$ satisfies a universal property:

Theorem 7 (Universal property of $\mathcal{H}$) *Let A be an algebra and let $L : A \longrightarrow A$ be a linear map.*

1. *There exists a unique algebra morphism $\phi : \mathcal{H} \longrightarrow A$, such that $\phi \circ B^+ = L \circ \phi$.*
2. *If A is a Hopf algebra and if L satisfies: for all $x \in A$,*

$$\Delta \circ L(x) = L(x) \otimes 1 + (Id \otimes L) \circ \Delta(x),$$

then ϕ is a Hopf algebra morphism.

The linear maps satisfying the condition of this theorem will be called 1-cocycle of A [CK99]. The Hopf algebra $\mathcal{H}_R$ satsfies a similar universal property, when A is commutative.

1.2.3 Dual Hopf algebra and self-duality

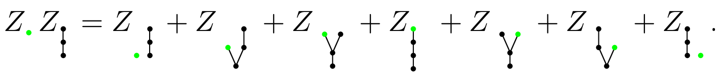
For any $F \in \mathbf{F}$, we define the following element of the graded dual $\mathcal{H}^*$:

$$Z_F : \left\{ \begin{array}{rcl} \mathcal{H} & \longrightarrow & K \\ G \in \mathbf{F} & \longrightarrow & \delta_{F,G}. \end{array} \right.$$

Then $(Z_F)_{F \in \mathbf{F}}$ is a basis of $\mathcal{H}^*$. The coproduct of $\mathcal{H}^*$ is given by:

$$\Delta(Z_{t_1 \ldots t_n}) = \sum_{i=0}^{n} Z_{t_1 \ldots t_i} \otimes Z_{t_{i+1} \ldots t_n}.$$

The product of Z_F and Z_G is given by planar graftings. Note that there are several ways to graft a planar tree on a vertex of a planar forest, and this implies the use of *angles* of a planar forest [CL01].

Example. 

In order to prove the self-duality of $\mathcal{H}$, we introduce the application γ:

$$\gamma : \left\{ \begin{array}{rcl} \mathcal{H} & \longrightarrow & \mathcal{H} \\ t_1 \ldots t_n \in \mathbf{F} & \longrightarrow & t_1 \ldots t_{n-1} \delta_{t_n, \bullet}. \end{array} \right.$$

It is clearly homogeneous of degree -1, so its transpose $\gamma^* : \mathcal{H}^* \longrightarrow \mathcal{H}^*$ exists and is homogeneous of degree $+1$. Moreover, γ^* is a 1-cocycle of $\mathcal{H}^*$, so by the universal property of $\mathcal{H}$ there exists a unique Hopf algebra morphism $\phi : \mathcal{H} \longrightarrow \mathcal{H}^*$, such that $\phi \circ B^+ = \gamma^* \circ \phi$. It is possible to prove:

Theorem 8 *ϕ is an isomorphism, homogeneous of degree 0.*

There are two alternative ways to see this isomorphism ϕ. The first one is in terms of a Hopf pairing. We put, for all $x, y \in \mathcal{H}$, $\langle x, y \rangle = \phi(x)(y)$. As ϕ is a Hopf algebra morphism, this pairing satisfies the following properties:

- For all $x \in \mathcal{H}$, $\langle 1, x \rangle = \langle x, 1 \rangle = \varepsilon(x)$.

- For all $x,y,z\in\mathcal{H}$, $\langle xy,z\rangle=\langle x\otimes y,\Delta(z)\rangle$, and $\langle x,yz\rangle=\langle \Delta(x),y\otimes z\rangle$.
- For all $x,y\in\mathcal{H}$, $\langle S(x),y\rangle=\langle x,S(y)\rangle$.

In other terms, $\langle -,-\rangle$ is a Hopf pairing. As ϕ is homogeneous of degree 0:

- For all $x,y\in\mathcal{H}$, homogeneous of different degrees, $\langle x,y\rangle=0$.

As $\phi\circ B^+=\gamma^*\circ\phi$:

- For all $x,y\in\mathcal{H}$, $\langle B^+(x),y\rangle=\langle x,\gamma(y)\rangle$.

As ϕ is an isomorphism, $\langle -,-\rangle$ is non degenerate. It is shown that this pairing is also symmetric. It admits combinatorial interpretations in term of partial orders and it can be inductively computed, using the preceding properties.

Examples. The following arrays give the values of $\langle -,-\rangle$ taken on forests of weight ≤ 3:

	$\tun$
$\tun$	1

	$\tun\tun$	$\tdeux$
$\tun\tun$	2	1
$\tdeux$	1	0

	$\tun\tun\tun$	$\tdeux\tun$	$\tun\tdeux$	$\ttroisun$	$\ttroisdeux$
$\tun\tun\tun$	6	3	3	2	1
$\tdeux\tun$	3	1	1	1	0
$\tun\tdeux$	3	1	1	0	0
$\ttroisun$	2	1	0	0	0
$\ttroisdeux$	1	0	0	0	0

The third way to see the isomorphism ϕ is in terms of a new basis. For all $F\in\mathbf{F}$, we put $e_F=\phi^{-1}(Z_F)$. Alternatively, e_F is the unique element of $\mathcal{H}$ such that, for all $G\in\mathbf{F}$, $\langle e_F,G\rangle=\delta_{F,G}$. This basis satisfies the following property:

- For all $F\in\mathbf{F}$, $\Delta(e_F)=\displaystyle\sum_{F_1F_2=F} e_{F_1}\otimes e_{F_2}$.

In particular, $(e_t)_{t\in\mathbf{T}_R}$ is a basis of $Prim(\mathcal{H})$.

Examples.

$$\begin{array}{rcl}
e_{\tun}&=&\tun,\\
e_{\tdeux}&=&\tun\tun-2\tdeux,\\
e_{\tun\tun\tun}&=&\ttroisdeux,\\
e_{\tun\tdeux}&=&\tun\tdeux-\ttroisun-\ttroisdeux,\\
e_{\tun\tun}&=&\tdeux,\\
e_{\tdeux\tun}&=&\ttroisun-2\ttroisdeux,\\
e_{\ttroisun}&=&\tdeux\tun-\tun\tdeux,\\
e_{\ttroisdeux}&=&\tun\tun\tun-2\tdeux\tun-\tun\tdeux+3\ttroisdeux.
\end{array}$$

The product of two elements of the dual basis $(e_F)_{F\in\mathbf{F}}$ is described in terms of graftings, as for $(Z_F)_{F\in\mathbf{F}}$.

1.3 Decorated versions

It is also possible to define decorated versions of these Hopf algebras. Let $\mathcal{D}$ be a non-empty set. We consider the set $\mathbf{T}_R^{\mathcal{D}}$ of rooted trees decorated by $\mathcal{D}$ and the set $\mathbf{T}^{\mathcal{D}}$ of planar rooted trees decorated by $\mathcal{D}$. For example:

1. Rooted trees decorated by $\mathcal{D}$, of weight ≤ 4:

$$\tdun{a},\ a \in \mathcal{D}, \quad \tddeux{a}{b},\ (a,b) \in \mathcal{D}^2, \quad \tdtroisun{a}{b}{c} = \tdtroisun{b}{a}{c},\ \tdtroisdeux{a}{b}{c},\ (a,b,c) \in \mathcal{D}^3,$$

$$\tdquatreun{a}{b}{c}{d} = \tdquatreun{b}{a}{c}{d} = \ldots,\ \tdquatredeux{a}{b}{c}{d},\ \tdquatrequatre{a}{b}{c}{d} = \tdquatrequatre{b}{a}{c}{d},\ \tdquatrecinq{a}{b}{c}{d},\ (a,b,c,d) \in \mathcal{D}^4.$$

2. Planar rooted trees decorated by $\mathcal{D}$, of weight ≤ 4:

$$\tdun{a},\ a \in \mathcal{D}, \quad \tddeux{a}{b},\ (a,b) \in \mathcal{D}^2, \quad \tdtroisun{a}{b}{c},\ \tdtroisdeux{a}{b}{c},\ (a,b,c) \in \mathcal{D}^3,$$

$$\tdquatreun{a}{b}{c}{d},\ \tdquatredeux{a}{b}{c}{d},\ \tdquatretrois{a}{b}{c}{d},\ \tdquatrequatre{a}{b}{c}{d},\ \tdquatrecinq{a}{b}{c}{d},\ (a,b,c,d) \in \mathcal{D}^4.$$

The algebra $\mathcal{H}_R^{\mathcal{D}}$ is the free associative commutative unitary K-algebra generated by $\mathbf{T}_R^{\mathcal{D}}$, and the algebra $\mathcal{H}^{\mathcal{D}}$ is the free associative unitary K-algebra generated by $\mathbf{T}^{\mathcal{D}}$. Their monomial sets are respectively denoted by $\mathbf{F}_R^{\mathcal{D}}$ (rooted forests decorated by $\mathcal{D}$) and by $\mathbf{F}^{\mathcal{D}}$ (planar rooted forests decorated by $\mathcal{D}$).

Examples. Planar rooted forests decorated by $\mathcal{D}$ of weight ≤ 3:

$$1, \quad \tdun{a},\ a \in \mathcal{D}, \quad \tdun{a}\tdun{b},\ \tddeux{a}{b},\ (a,b) \in \mathcal{D}^2,$$

$$\tdun{a}\tdun{b}\tdun{c},\ \tddeux{a}{b}\tdun{c},\ \tdun{a}\tddeux{b}{c},\ \tdtroisun{a}{b}{c},\ \tdtroisdeux{a}{b}{c},\ (a,b,c) \in \mathcal{D}^3.$$

Both are given a coproduct using admissible cuts. For example, let $a,b,c \in \mathcal{D}$. Both in $\mathcal{H}_R^{\mathcal{D}}$ and $\mathcal{H}^{\mathcal{D}}$:

$$\begin{array}{rcl}
\Delta(\tdun{a}) & = & \tdun{a} \otimes 1 + 1 \otimes \tdun{a}, \\
\Delta(\tddeux{a}{b}) & = & \tddeux{a}{b} \otimes 1 + 1 \otimes \tddeux{a}{b} + \tdun{a} \otimes \tdun{b}, \\
\Delta(\tdtroisun{a}{b}{c}) & = & \tdtroisun{a}{b}{c} \otimes 1 + 1 \otimes \tdtroisun{a}{b}{c} + \tdun{a}\tdun{b} \otimes \tdun{c} + \tdun{a} \otimes \tddeux{b}{c} + \tdun{b} \otimes \tddeux{a}{c}, \\
\Delta(\tdtroisdeux{a}{b}{c}) & = & \tdtroisdeux{a}{b}{c} \otimes 1 + 1 \otimes \tdtroisdeux{a}{b}{c} + \tddeux{a}{b} \otimes \tdun{c} + \tdun{a} \otimes \tddeux{b}{c}.
\end{array}$$

Moreover, $\mathcal{H}^{\mathcal{D}}$ is also a self-dual Hopf algebra. The isomorphism is defined using the following linear maps:

$$B_d^+ : \left\{ \begin{array}{rcl} \mathcal{H}^{\mathcal{D}} & \longrightarrow & \mathcal{H}^{\mathcal{D}} \\ t_1 \ldots t_n & \longrightarrow & \text{the tree obtained by grafting } t_1, \ldots, t_n \\ & & \text{on a common root decorated by } d. \end{array} \right.$$

$$\gamma_d : \left\{ \begin{array}{rcl} \mathcal{H}^{\mathcal{D}} & \longrightarrow & \mathcal{H}^{\mathcal{D}} \\ t_1 \ldots t_n & \longrightarrow & \left\{ \begin{array}{rl} 0 & \text{if } \ t_n \neq \tdun{d}, \\ t_1 \ldots t_{n-1} & \text{if } \ t_n = \tdun{d}. \end{array} \right. \end{array} \right.$$

For example, if $a,b,c,d \in \mathcal{D}$, $B_d^+(\tdun{a}\tddeux{b}{c}) = \tdquatretrois{a}{b}{c}{d}$, $B_d^+(\tddeux{b}{c}\tdun{a}) = \tdquatredeux{c}{b}{a}{d}$, and $\gamma_d(\tddeux{a}{b}\tdun{c}) = \delta_{c,d}\tddeux{a}{b}$.

This isomorphism also gives a non degenerate symmetric pairing on $\mathcal{H}^{\mathcal{D}}$ and the dual basis of the basis $\mathbf{F}^{\mathcal{D}}$ is denoted by $(e_F)_{F \in \mathbf{F}^{\mathcal{D}}}$.

Chapter 2

Dendriform structures

This chapter is devoted to the study of dendriform structures over $\mathcal{H}$ or $\mathcal{H}^{\mathcal{D}}$ and the consequences on other combinatorial Hopf algebras, such as for example the Malvenuto-Reutenauer algebra of permutations.

The notion of dendriform algebra is introduced in [Lod01], in an operadic context: the operad of dendriform algebras is the Koszul dual of the quadratic operad of dialgebras. A dendriform algebra is an associative algebra with an associativity splitting, that is to say its product is split into a sum $\prec + \succ$, with good compatibilities for $\prec$ and $\succ$. The free dendriform algebra on one generator is the Loday-Ronco algebra of planar binary trees; an alternative description is the augmentation ideal of $\mathcal{H}$, using a description of the product in the dual basis $(e_F)_{F\in\mathbf{F}}$ in terms of graftings.

Other examples of dendriform algebras are known, such as the shuffle algebra (commutative dendriform or Zinbiel algebra) or the Malvenuto-Reutenauer algebra **FQSym** (or algebra of free quasi-symmetric functions). All these exemples are Hopf dendriform algebras [Lod04], and the dendriform Milnor-Moore-Cartier-Quillen theorem [Cha02, Ron01] insures that they are enveloping dendriform algebras of brace algebras. We are here especially interested in **FQSym**. In particular, a question is to know if **FQSym** is a free dendrifrom algebra.

In order to answer to this, we introduce the notion of bidendriform bialgebra, and prove a rigidity theorem. A bidendriform algebra is both a dendriform algebra and coalgebra, with some compatibilities between products and coproducts. It is also a dendriform Hopf algebra, as well as its dual. The rigidity theorem insures that any bidendriform bialgebra, connected as a dendriform coalgebra, is free. In the context of triple of operads, this gives a good triple $(coDend, Dend, Vect)$. As a corollary, any connected bidendriform bialgebra is self-dual, and its Lie algebra is free (this comes from results about $\mathcal{H}^{\mathcal{D}}$). When applied to **FQSym**, this machinery proves the freeness as a dendriform algebra. It can also be applied, for example, to the Hopf algebra of parking functions.

So the space $Prim_{coDend}(\mathbf{FQSym})$ of its dendriform primitive elements freely generates **FQSym** as a dendriform algebra, and freely generates the space $Prim_{coAss}(\mathbf{FQSym})$ of its primitive elements as a brace algebra. We give combinatorial bases of these spaces. They are indexed by a family of partially decorated planar trees; the associated elements of **FQSym** are inductively computed with the help of the brace structure of $Prim_{coAss}(\mathbf{FQSym})$, and with the insertion of $n+1$ into a permutation $\sigma \in S_n$ at any place.

2.1 Dendriform algebras

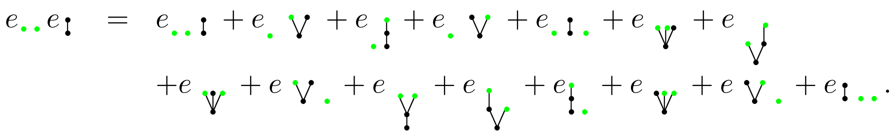
2.1.1 Free dendriform algebras

The notion of dendriform algebra is introduced and studied in [Agu04, Lod01, Lod02, Lod04, LR02]. Namely, a dendriform algebra is an associative, non unitary algebra (A, m) such that the product m can be written as $m = \prec + \succ$, with the following axioms: for all $a, b, c \in A$,

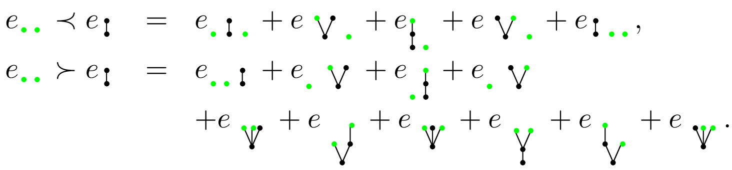
$$\begin{array}{rcl} (a \prec b) \prec c & = & a \prec (b \prec c + b \succ c), \\ (a \succ b) \prec c & = & a \succ (b \prec c), \\ (a \prec b + a \succ b) \succ c & = & a \succ (b \succ c). \end{array}$$

In other terms, $(A, \prec, \succ)$ is a bimodule over the associative algebra $(A, \prec + \succ)$.

J.L. Loday and M. Ronco gave a description of the free dendriform algebra on one generator in terms of binary planar trees [LR98]. An alternative description is given in terms of planar rooted forests: the augmentation ideal of the dual of $\mathcal{H}$ is given a dendriform structure. Recall that the product of two elements of the dual basis $(e_F)_{F \in \mathbf{F}}$ is given by graftings, for example:



The left and right products are given by separating the terms following the last tree of the forests:



Free dendriform algebras on N generators are described in terms of planar decorated rooted forests in a similar way.

2.1.2 Dendriform Hopf algebra of free quasi-symmetric functions

Another example of dendriform algebra is the Malvenuto-Reutenauer Hopf algebra of permutations, also known as the Hopf algebra of free quasi-symmetric functions and here denoted by **FQSym** [AS05, DHT00, MR95]. The algebra **FQSym** is the vector space generated by the elements $(\mathbf{F}_u)_{u \in \mathbb{S}}$, where $\mathbb{S}$ is the disjoint union of the symmetric groups S_n, $n \in \mathbb{N}$. Its product and its coproduct are given in the following way: for all $u \in S_m$, $v \in S_n$, by putting $u = (u_1 \ldots u_m)$,

$$\begin{array}{rcl} \Delta(\mathbf{F}_u) & = & \displaystyle\sum_{i=0}^{m} \mathbf{F}_{st(u_1 \ldots u_i)} \otimes \mathbf{F}_{st(u_{i+1} \ldots u_m)}, \\ \mathbf{F}_u.\mathbf{F}_v & = & \displaystyle\sum_{\zeta \in sh(m,n)} \mathbf{F}_{(u \times v).\zeta^{-1}}, \end{array}$$

where $sh(m,n)$ is the set of (m,n)-shuffles, and st is the standardisation. Its unit is $1 = \mathbf{F}_\emptyset$, where $\emptyset$ is the unique element of S_0. Moreover, **FQSym** is a $\mathbb{N}$-graded Hopf algebra, by putting $|\mathbf{F}_u| = n$ if $u \in S_n$.

Examples.

$$\begin{array}{rcl} \mathbf{F}_{(1\,2)}\mathbf{F}_{(1\,2\,3)} & = & \mathbf{F}_{(1\,2\,3\,4\,5)} + \mathbf{F}_{(1\,3\,2\,4\,5)} + \mathbf{F}_{(1\,3\,4\,2\,5)} + \mathbf{F}_{(1\,3\,4\,5\,2)} + \mathbf{F}_{(3\,1\,2\,4\,5)} \\ & & + \mathbf{F}_{(3\,1\,4\,2\,5)} + \mathbf{F}_{(3\,1\,4\,5\,2)} + \mathbf{F}_{(3\,4\,1\,2\,5)} + \mathbf{F}_{(3\,4\,1\,5\,2)} + \mathbf{F}_{(3\,4\,5\,1\,2)}, \end{array}$$

$$\begin{array}{rcl} \Delta\left(\mathbf{F}_{(1\,2\,5\,4\,3)}\right) & = & 1 \otimes \mathbf{F}_{(1\,2\,5\,4\,3)} + \mathbf{F}_{(1)} \otimes \mathbf{F}_{(1\,4\,3\,2)} + \mathbf{F}_{(1\,2)} \otimes \mathbf{F}_{(3\,2\,1)} \\ & & + \mathbf{F}_{(1\,2\,3)} \otimes \mathbf{F}_{(2\,1)} + \mathbf{F}_{(1\,2\,4\,3)} \otimes \mathbf{F}_{(1)} + \mathbf{F}_{(1\,2\,5\,4\,3)} \otimes 1. \end{array}$$

Its augmentation ideal is given a dendriform structure, with:

$$\left\{\begin{array}{rcl}\mathbf{F}_u \prec \mathbf{F}_v &=& \displaystyle\sum_{\substack{\zeta\in sh(n,m)\\ \zeta^{-1}(n+m)=n}} \mathbf{F}_{(u\times v).\zeta^{-1}},\\ \mathbf{F}_u \succ \mathbf{F}_v &=& \displaystyle\sum_{\substack{\zeta\in sh(n,m)\\ \zeta^{-1}(n+m)=n+m}} \mathbf{F}_{(u\times v).\zeta^{-1}}.\end{array}\right.$$

Examples.

$$\begin{array}{rcl}\mathbf{F}_{(1\,2)} \prec \mathbf{F}_{(1\,2\,3)} &=& \mathbf{F}_{(1\,3\,4\,5\,2)}+\mathbf{F}_{(3\,1\,4\,5\,2)}+\mathbf{F}_{(3\,4\,1\,5\,2)}+\mathbf{F}_{(3\,4\,5\,1\,2)},\\ \mathbf{F}_{(1\,2)} \succ \mathbf{F}_{(1\,2\,3)} &=& \mathbf{F}_{(1\,2\,3\,4\,5)}+\mathbf{F}_{(1\,3\,2\,4\,5)}+\mathbf{F}_{(1\,3\,4\,2\,5)}+\mathbf{F}_{(3\,1\,2\,4\,5)}+\mathbf{F}_{(3\,1\,4\,2\,5)}+\mathbf{F}_{(3\,4\,1\,2\,5)}.\end{array}$$

A conjecture of [DHT00] was that the Lie algebra of primitive elements of **FQSym** is free. We proved this conjecture using the dendriform structure of **FQSym**, as explained in the sequel.

2.2 Bidendriform Hopf algebras

2.2.1 Definition and rigidity theorem

we now split the coproduct Δ into two parts.

Definition 9 A bidendriform bialgebra is a family $(A,\prec,\succ,\Delta_\prec,\Delta_\succ)$ such that:

1. $(A,\prec,\succ)$ is a dendriform algebra.
2. $(A,\Delta_\prec,\Delta_\succ)$ is a dendriform coalgebra.
3. The following compatibilities are satisfied: for all $a,b\in A$,

$$\left\{\begin{array}{rcl}\Delta_\succ(a\succ b) &=& a'b'_\succ\otimes a''\succ b''_\succ + a'\otimes a''\succ b + b'_\succ\otimes a\succ b''_\succ + ab'_\succ\otimes b''_\succ + a\otimes b,\\ \Delta_\succ(a\prec b) &=& a'b'_\succ\otimes a''\prec b''_\succ + a'\otimes a''\prec b + b'_\succ\otimes a\prec b''_\succ,\\ \Delta_\prec(a\succ b) &=& a'b'_\prec\otimes a''\succ b''_\prec + ab'_\prec\otimes b''_\prec + b'_\prec\otimes a\succ b''_\prec,\\ \Delta_\prec(a\prec b) &=& a'b'_\prec\otimes a''\prec b''_\prec + a'b\otimes a'' + b'_\prec\otimes a\prec b''_\prec + b\otimes a.\end{array}\right.$$

A possible reformulation of these compatibiliy axioms is that $\Delta_\prec$ and $\Delta_\succ$ are morphisms of dendriform modules over A, with a convenient dendriform structure over $A\overline{\otimes}A=(A\otimes K)\oplus(A\otimes A)\oplus(K\otimes A)$. As a consequence:

Theorem 10 *There is a unique structure of bidendriform bialgebra on the free dendriform algebra $\mathcal{H}^\mathcal{D}$ generated by $\mathcal{D}$ such that for all $d\in\mathcal{D}$, $\Delta_\prec(\bullet_d)=\Delta_\succ(\bullet_d)=0$. Hence, $(\mathcal{H}^\mathcal{D},\prec,\succ,\Delta_\prec,\Delta_\succ)$ is a bidendriform bialgebra, which induces the structure of Hopf algebra of $\mathcal{H}^\mathcal{D}$ already described.*

The key of the structure of bidendriform bialgebra is the following Cartier-Quillen-Milnor-Moore theorem:

Theorem 11 *Let A be a connected (as a dendriform coalgebra) bidendriform bialgebra. Then A is isomorphic as a bidendriform bialgebra to the free dendriform algebra generated by the space of dendriform primitive elements $Prim_{coDend}(A)=Ker(\Delta_\prec)\cap Ker(\Delta_\succ)$.*

The proof uses the following iterated products and coproducts, inductively defined:

$$\begin{cases} \Delta_{\prec}^{0} &= Id, \\ \Delta_{\prec}^{1} &= \Delta_{\prec}, \\ \Delta_{\prec}^{n} &= \left(\Delta_{\prec} \otimes Id^{\otimes(n-1)}\right) \circ \Delta_{\prec}^{n-1}, \end{cases}$$

$$\begin{cases} \tilde{\Delta}^{0} &= Id, \\ \tilde{\Delta}^{1} &= \tilde{\Delta}, \\ \tilde{\Delta}^{n} &= (Id^{\otimes(n-1)} \otimes \tilde{\Delta}) \circ \tilde{\Delta}^{n-1}, \end{cases}$$

$$\begin{cases} \omega(a_1) &= a_1, \\ \omega(a_1, a_2) &= a_2 \prec a_1, \\ \omega(a_1, \ldots, a_n) &= a_n \prec \omega(a_1, \ldots, a_{n-1}), \end{cases}$$

$$\begin{cases} \omega'(a_1) &= a_1, \\ \omega'(a_1, a_2) &= a_1 \succ a_2, \\ \omega'(a_1, \ldots, a_n) &= \omega'(a_1, \ldots, a_{n-1}) \succ a_n. \end{cases}$$

Note that $\tilde{\Delta} = \Delta_{\prec} + \Delta_{\succ}$ is a coassociative coproduct. A triple induction proves that A is generated by $Prim_{coDend}(A)$. The idea is to "destroy" the elements of A into elements of smaller degree using the iterated coproducts and then "reconstruct" them with the iterated products.

As a consequence, A is the image of the free dendrifrom algebra generated by the space $V = Prim_{coDend}(A)$ by an epimorphism of bidendrifom bialgebras. As the space of codendriform primitive elements of the free dendriform algebra generated by V is V, this epimorphism is an isomorphism.

2.2.2 Corollaries

This helps to prove the freeness conjecture of the Lie algebra of primitive elements of **FQSym**. Indeed, **FQSym** is given a structure of bidendriform bialgebra by:

$$\begin{cases} \Delta_{\prec}(\mathbf{F}_u) &= \displaystyle\sum_{i=u^{-1}(n)}^{n-1} \mathbf{F}_{st(u_1 \ldots u_i)} \otimes \mathbf{F}_{st(u_{i+1} \ldots u_n)}, \\ \Delta_{\succ}(\mathbf{F}_u) &= \displaystyle\sum_{i=1}^{u^{-1}(n)-1} \mathbf{F}_{st(u_1 \ldots u_i)} \otimes \mathbf{F}_{st(u_{i+1} \ldots u_n)}. \end{cases}$$

For example:

$$\begin{aligned} \Delta_{\prec}\left(\mathbf{F}_{(1\,2\,5\,4\,3)}\right) &= \mathbf{F}_{(1\,2\,3)} \otimes \mathbf{F}_{(2\,1)} + \mathbf{F}_{(1\,2\,4\,3)} \otimes \mathbf{F}_{(1)}, \\ \Delta_{\succ}\left(\mathbf{F}_{(1\,2\,5\,4\,3)}\right) &= \mathbf{F}_{(1)} \otimes \mathbf{F}_{(1\,4\,3\,2)} + \mathbf{F}_{(1\,2)} \otimes \mathbf{F}_{(3\,2\,1)}. \end{aligned}$$

By theorem 11, **FQSym** is freely generated by the space of its dendriform primitive elements as a bidendriform algebra, so is isomorphic to $\mathcal{H}^{\mathcal{D}}$ as a Hopf algebra, for a certain graded set $\mathcal{D}$. As a consequence, $Prim(\mathbf{FQSym})$ is isomorphic to $Prim(\mathcal{H}^{\mathcal{D}})$ as a Lie algebra, so is free by [4].

Other bidendriform Hopf algebras are known.The first example is the Hopf algebra **PQSym** of parking functions of Novelli and Thibon [NT07a, NT07b]. As it is shown in [NT07b], **PQSym** is a bidendriform bialgebra, so is isomorphic to a certain $\mathcal{H}^{\mathcal{D}}$. Other examples are the Hopf algebra of uniform block permutations of Aguiar and Orellana [AO08] and free tridendriform algebras.

2.3 Generators of the Malvenuto-Reutenauer algebra

So **FQSym** is isomorphic to $\mathcal{H}^{\mathcal{D}}$ as a Hopf algebra, where $\mathcal{D}$ is a certain graded set. A manipulation of formal series give the first values of $p_n = card(\mathcal{D}_n)$:

n	1	2	3	4	5	6	7	8	9	10	11	12
p_n	1	0	1	6	39	284	2 305	20 682	203 651	2 186 744	25 463 925	319 989 030

It is possible to give a combinatorial description of a convenient $\mathcal{D}$, as explained in the sequel.

2.3.1 Brace algebras

Definition 12 [Agu04, Cha02, Ron01] A brace algebra is a couple $(A, \langle\rangle)$ where A is a vector space and $\langle\rangle$ is a family of operators $A^{\otimes n} \longrightarrow A$ defined for all $n \geq 2$:

$$\left\{\begin{array}{rcl} A^{\otimes n} & \longrightarrow & A \\ a_1 \otimes \ldots \otimes a_n & \longrightarrow & \langle a_1, \ldots, a_{n-1}; a_n \rangle, \end{array}\right.$$

with the following compatibilities: for all $a_1, \ldots, a_m$, $b_1, \ldots, b_n$, $c \in A$,

$$\langle a_1, \ldots, a_m; \langle b_1, \ldots, b_n; c\rangle\rangle = \sum \langle \langle A_0, \langle A_1; b_1\rangle, A_2, \langle A_3; b_2\rangle, A_4, \ldots, A_{2n-2}, \langle A_{2n-1}; b_n\rangle, A_{2n}; c\rangle,$$

where this sum runs over partitions of the ordered set $\{a_1, \ldots, a_n\}$ into (possibly empty) consecutive intervals $A_0 \sqcup \ldots \sqcup A_{2n}$. We use the convention $\langle a \rangle = a$ for all $a \in A$.

By the dendriform Milnor-Moore theorem [Cha02, Ron01], if $A = (A, \prec, \succ, \tilde{\Delta})$ is a dendriform Hopf algebra, then the space of its primitive elements $Prim_{coAss}(A) = Ker(\tilde{\Delta})$ inherits a structure of brace algebra given by:

$$\langle p_1, \ldots, p_{n-1}; p_n \rangle$$

$$= \sum_{i=0}^{n-1} (-1)^{n-1-i} \left(p_1 \prec (p_2 \prec (\ldots \prec p_i) \ldots) \succ p_n \prec (\ldots (p_{i+1} \succ p_{i+2}) \succ \ldots) \succ p_{n-1}\right).$$

Moreover, if A is freely generated as a dendriform algebra by a subspace $V \subseteq Prim_{coAss}(A)$, then $Prim_{coAss}(A)$ is freely generated as a brace algebra by V.

As a consequence, the free brace algebra generated by a set $\mathcal{D}$ admits a basis indexed by the set $T^{\mathcal{D}}$ of planar rooted trees decorated by $\mathcal{D}$. For example:

$$\begin{array}{rcl} Brace(\mathcal{D})_1 & = & Vect(e_{\bullet_a},\ a \in \mathcal{D}), \\ Brace(\mathcal{D})_2 & = & Vect(e_{\,{}^{b}_{a}},\ a, b \in \mathcal{D}), \\ Brace(\mathcal{D})_3 & = & Vect(e_{{}^{c}\mathrm{V}^{b}_{a}},\ e_{{}^{c}_{b}{}_{a}},\ a, b, c \in \mathcal{D}), \\ Brace(\mathcal{D})_4 & = & Vect(e_{{}^{d\,c\,b}\mathrm{V}_{a}},\ e_{{}^{d}_{c}\mathrm{V}^{b}_{a}},\ e_{{}^{d}\mathrm{V}^{c}_{b\,a}},\ e_{{}^{d}\mathrm{V}^{c}_{b}{}_{a}},\ e_{{}^{d}_{c}{}_{b}{}_{a}},\ a, b, c, d \in \mathcal{D}), \ldots \end{array}$$

The brace bracket satisfies, for all $t_1, \ldots, t_{n-1} \in T^{\mathcal{D}}$, $d \in \mathcal{D}$:

$$\langle e_{t_1}, \ldots, e_{t_{n-1}}; e_{\bullet_d}\rangle = e_{B_d^+(t_{n-1} \ldots t_1)}.$$

For example, if $a, b, c, d \in \mathcal{D}$, $\langle e_{\bullet_a}, e_{\,{}^{c}_{b}}; e_{\bullet_d}\rangle = e_{{}^{c}_{b}\mathrm{V}^{a}_{d}}$.

2.3.2 Applications

As **FQSym** is the free dendriform algebra generated by $Prim_{coDend}(\mathbf{FQSym})$, the brace algebra $Prim_{coAss}(\mathbf{FQSym})$ is freely generated by $Prim_{coDend}(\mathbf{FQSym})$. So primitive elements in the dendriform sense of **FQSym** allow to construct primitive elements in the associative sense of **FQSym**. In the other sense, we can construct elements of degree n of $Prim_{coDend}(\mathbf{FQSym})$ from elements of $Prim_{coAss}(\mathbf{FQSym})$ of degree $n-1$ in the following way:

Proposition 13 *Let $i \in \mathbb{N}^*$. We define $\Phi_i : \mathbf{FQSym} \longrightarrow \mathbf{FQSym}$ in the following way: for all $n \in \mathbb{N}$, for all $\sigma = (\sigma_1, \ldots, \sigma_n) \in S_n$,*

$$\Phi_i(\mathbf{F}_\sigma) = \left\{ \begin{array}{rl} 0 & \text{if} \quad i \geq n, \\ \mathbf{F}_{(\sigma_1,\ldots,\sigma_i,n+1,\sigma_{i+1},\ldots,\sigma_n)} & \text{if} \quad i < n. \end{array} \right.$$

Let $n \geq 2$. The following application is bijective:

$$\Phi : \left\{ \begin{array}{rcl} (Prim_{coAss}(\mathbf{FQSym})_{n-1})^{n-2} & \longrightarrow & Prim_{coDend}(\mathbf{FQSym})_n \\ (p_1, \ldots, p_{n-2}) & \longrightarrow & \Phi_1(p_1) + \ldots + \Phi_{n-2}(p_{n-2}) \end{array} \right.$$

Proof. An easy computation using the combinatorial description of the coproducts of **FQSym** shows that Φ is well-defined. It is clearly injective. The surjectivity comes from the following formula, proved with manipulations of formal series:

$$\dim(Prim_{coDend}(\mathbf{FQSym})(n)) = (n-2)\dim(Prim_{coAss}(\mathbf{FQSym})(n-1)).$$

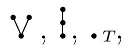
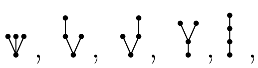
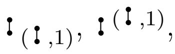
So Φ is bijective. $\Box$

Using the description of free brace algebras in terms of planar rooted trees and the preceding theorem, we deduce the following combinatorial basis of $Prim_{coAss}(\mathbf{FQSym})$, with the help of the following set of planar, partially decorated, rooted trees $\mathbb{T}(n)$:

1. $\mathbb{T}(0)$ is the set of non decorated planar trees. The weight of an element of $\mathbb{T}(0)$ is the number of its vertices.
2. Suppose that $\mathbb{T}(n)$ is defined. Then $\mathbb{T}(n+1)$ is the set of planar trees defined by :
   (a) The elements of $\mathbb{T}(n+1)$ are partially decorated planar trees.
   (b) The vertices of the elements of $\mathbb{T}(n+1)$ can eventually be decorated by a pair (t, k), with $t \in \mathbb{T}(n)$ and k an integer in $\{1, \ldots, n-1\}$.
   (c) The weight of an element of $\mathbb{T}(n)$ is the sum of the number of its vertices and of the weights of the trees of $\mathbb{T}(n)$ that appear in its decorations.

Inductively, for all $n \in \mathbb{N}$, $\mathbb{T}(n) \subseteq \mathbb{T}(n+1)$. We put $\mathbb{T} = \bigcup_{n \in \mathbb{N}} \mathbb{T}(n)$.

Examples.

1. Elements of $\mathbb{T}$ of weight 1: $\bullet$.
2. Elements of $\mathbb{T}$ of weight 2: .
3. Elements of $\mathbb{T}$ of weight 3: , with $T = ($$, 1)$.
4. Elements of $\mathbb{T}$ of weight 4:
   (a) 
   (b) 

(c) $\bullet_{(\mathrm{V},1)}$, $\bullet_{(\mathrm{V},2)}$, $\bullet_{(\overset{\bullet}{\mathrm{I}},1)}$, $\bullet_{(\overset{\bullet}{\mathrm{I}},2)}$, $\bullet_{(\bullet_{(\mathrm{I},1)},1)}$, $\bullet_{(\bullet_{(\mathrm{I},1)},2)}$.

We can then define a basis $(p_t)_{i\in\mathbb{T}}$ of $Prim_{coAss}(\mathbf{FQSym})$ inductively in the following way:

1. $p_\bullet = \mathbf{F}_{(1)}$.
2. If $t = \bullet_{(t',i)}$, then $p_t = \Phi_i(p_{t'})$.
3. If t is not a single root, let $t_1, \ldots, t_{n-1}$ be the children of its roots, from left to right, and t_n its root. Then $p_t = \langle p_{t_{n-1}}, \ldots, p_{t_1}; p_{t_n}\rangle$.

Combining the preceding results:

Theorem 14 *$(p_t)_{t\in\mathbb{T}}$ is a basis of $Prim_{coAss}(\mathbf{FQSym})$. A basis of $Prim_{coDend}(\mathbf{FQSym})$ is given by the p_t's, where t is a single root.*

Examples.

1. $p_\bullet = \mathbf{F}_{(1)}$.
2. $p_{\mathrm{I}} = -\mathbf{F}_{(21)} + \mathbf{F}_{(12)}$.
3. (a) $p_{\bullet_{(\mathrm{I},1)}} = -\mathbf{F}_{(231)} + \mathbf{F}_{(132)}$.
   (b) $p_{\mathrm{V}} = \mathbf{F}_{(231)} - \mathbf{F}_{(132)} - \mathbf{F}_{(312)} + \mathbf{F}_{(213)}$.
   (c) $p_{\overset{\bullet}{\mathrm{I}}} = \mathbf{F}_{(321)} - \mathbf{F}_{(231)} - \mathbf{F}_{(213)} + \mathbf{F}_{(123)}$.
4. (a) $p_{\Psi} = -\mathbf{F}_{(2341)} + \mathbf{F}_{(1342)} + \mathbf{F}_{(3142)} + \mathbf{F}_{(3412)} - \mathbf{F}_{(2143)} - \mathbf{F}_{(2413)} - \mathbf{F}_{(4213)} + \mathbf{F}_{(3214)}$.
   (b) $p_{\mathrm{V}_{(left)}} = -\mathbf{F}_{(2431)} - \mathbf{F}_{(4231)} + \mathbf{F}_{(2341)} + \mathbf{F}_{(3241)} + \mathbf{F}_{(1432)} + \mathbf{F}_{(4132)} + \mathbf{F}_{(4312)} - \mathbf{F}_{(1342)} - \mathbf{F}_{(3142)} - \mathbf{F}_{(3412)} - \mathbf{F}_{(3214)} + \mathbf{F}_{(2314)}$.
   (c) $p_{\mathrm{V}_{(right)}} = -\mathbf{F}_{(3241)} + \mathbf{F}_{(2341)} + \mathbf{F}_{(2143)} + \mathbf{F}_{(2413)} + \mathbf{F}_{(4213)} - \mathbf{F}_{(1243)} - \mathbf{F}_{(1423)} - \mathbf{F}_{(4123)} - \mathbf{F}_{(2314)} - \mathbf{F}_{(3214)} + \mathbf{F}_{(1324)} + \mathbf{F}_{(3124)}$.
   (d) $p_{\mathrm{Y}} = -\mathbf{F}_{(3421)} + \mathbf{F}_{(2431)} + \mathbf{F}_{(4231)} - \mathbf{F}_{(3241)} + \mathbf{F}_{(2314)} - \mathbf{F}_{(1324)} - \mathbf{F}_{(3124)} + \mathbf{F}_{(2134)}$.
   (e) $p_{\vdots} = -\mathbf{F}_{(4321)} + \mathbf{F}_{(3421)} + \mathbf{F}_{(3241)} - \mathbf{F}_{(2341)} + \mathbf{F}_{(3214)} - \mathbf{F}_{(2314)} - \mathbf{F}_{(2134)} + \mathbf{F}_{(1234)}$.
   (f) $p_{\mathrm{I}_{(\mathrm{I},1)}} = \mathbf{F}_{(2341)} + \mathbf{F}_{(2431)} + \mathbf{F}_{(4231)} - 2\mathbf{F}_{(1342)} - \mathbf{F}_{(1432)} - \mathbf{F}_{(4132)} - \mathbf{F}_{(3142)} - \mathbf{F}_{(3412)} + \mathbf{F}_{(1243)} + \mathbf{F}_{(2143)} + \mathbf{F}_{(2413)}$.
   (g) $p_{\mathrm{I}(\mathrm{I},1)} = \mathbf{F}_{(3421)} - \mathbf{F}_{(2431)} - \mathbf{F}_{(2314)} + \mathbf{F}_{(1324)}$.
   (h) $p_{\bullet_{(\mathrm{V},1)}} = \mathbf{F}_{(2431)} - \mathbf{F}_{(1432)} - \mathbf{F}_{(3412)} + \mathbf{F}_{(2413)}$.
   (i) $p_{\bullet_{(\mathrm{V},2)}} = \mathbf{F}_{(2341)} - \mathbf{F}_{(1342)} - \mathbf{F}_{(3142)} + \mathbf{F}_{(2143)}$.
   (j) $p_{\bullet_{(\overset{\bullet}{\mathrm{I}},1)}} = \mathbf{F}_{(3421)} - \mathbf{F}_{(2431)} - \mathbf{F}_{(2413)} + \mathbf{F}_{(1423)}$.
   (k) $p_{\bullet_{(\overset{\bullet}{\mathrm{I}},2)}} = \mathbf{F}_{(3241)} - \mathbf{F}_{(2341)} - \mathbf{F}_{(2143)} + \mathbf{F}_{(1243)}$.
   (l) $p_{\bullet_{(\bullet_{(\mathrm{I},1)},1)}} = -\mathbf{F}_{(2431)} + \mathbf{F}_{(1432)}$.
   (m) $p_{\bullet_{(\bullet_{(\mathrm{I},1)},2)}} = -\mathbf{F}_{(2341)} + \mathbf{F}_{(1342)}$.

Chapter 3

Combinatorial Dyson-Schwinger equations

We work in this chapter in the commutative setting. We consider a family of subalgebras of $\mathcal{H}_R$, generated by a combinatorial Dyson-Schwinger equation [BK06, Kre07, KY06]:

$$X = B^+(f(X)),$$

where $f(h) = \sum a_k h^k$ is a formal series such that $a_0 = 1$. All this makes sense in a completion of $\mathcal{H}_R$, where this equation admits a unique solution $X = \sum x_k$, which coefficients are inductively defined by:

$$\left\{ \begin{array}{rcl} x_1 & = & \bullet, \\ x_{n+1} & = & \displaystyle\sum_{k=1}^{n} \sum_{\alpha_1+\ldots+\alpha_k=n} a_k B^+(x_{\alpha_1}\ldots x_{\alpha_k}), \end{array} \right.$$

A classical example of Dyson-Schwinger equation is given by $f(h) = (1-h)^{-1}$. We characterise the formal series such that the associated subalgebra is Hopf: we obtain a two-parameters family $\mathcal{H}_{\alpha,\beta}$ of Hopf subalgebras of $\mathcal{H}_R$ and we explicitely describe the system of generator of these algebras.

We then characterise the isomorphism classes of $\mathcal{H}_{\alpha,\beta}$. We obtain three classes:

1. $\mathcal{H}_{0,1}$, equal to $K[\bullet]$.

2. $\mathcal{H}_{1,-1}$, the subalgebra of ladders, isomorphic to the Hopf algebra of symmetric functions.

3. The $\mathcal{H}_{1,\beta}$'s, with $\beta \neq -1$, isomorphic to the Faà di Bruno Hopf algebra.

A similar result holds in the non commutative case obtained by replacing $\mathcal{H}_R$ par $\mathcal{H}$, as explained in [9].

In order to understand why, up to two degenerate cases, we only obtain Faà di Bruno Hopf algebras, we introduce the notion of FdB Lie algebra: a Lie algebra is FdB if it is graded and connected, with 1-dimensional homogeneous components of degre > 0. By the Milnor-Moore theorem, the Hopf algebras obtained by Dyson-Schwinger equations are duals of enveloping algebras, and the corresponding Lie algebras are FdB. We also assume a condition of non commutativity, satisfied up to degenerate cases. We prove that there are only three such Lie algebras; with a stronger non commutativity condition, there is only one, the Faà di Bruno Lie algebra.

We end this chapter with examples of systems of Dyson-Schwinger equations, generalization of the former study in decorated cases.

3.1 Definitions

3.1.1 Completion of $\mathcal{H}_R$

Recall that $\mathcal{H}_R$ is graded by putting the forests of weight n homogeneous of degree n. We denote by $\mathcal{H}_R(n)$ the homogeneous component of $\mathcal{H}_R$ of degree n. We define, for all $x, y \in \mathcal{H}_R$:

$$\left\{ \begin{array}{rcl} val(x) & = & \max\left\{ n \in \mathbb{N} \,/\, x \in \bigoplus_{k \geq n} \mathcal{H}_R(k) \right\}, \\ d(x,y) & = & 2^{-val(x-y)}, \end{array} \right.$$

with the convention $2^{-\infty} = 0$. Then d is a distance on $\mathcal{H}_R$. The metric space $(\mathcal{H}_R, d)$ is not complete: its completion will be denoted by $\widehat{\mathcal{H}_R}$. As a vector space:

$$\widehat{\mathcal{H}_R} = \prod_{n \in \mathbb{N}} \mathcal{H}_R(n).$$

The elements of $\widehat{\mathcal{H}_R}$ will be denoted $\sum x_n$, where $x_n \in \mathcal{H}_R(n)$ for all $n \in \mathbb{N}$. The product $m : \mathcal{H}_R \otimes \mathcal{H}_R \longrightarrow \mathcal{H}_R$ is homogeneous of degree 0, so is continuous. So it can be extended from $\widehat{\mathcal{H}_R} \otimes \widehat{\mathcal{H}_R}$ to $\widehat{\mathcal{H}_R}$, which is then an associative, commutative algebra. Similarly, the coproduct of $\mathcal{H}_R$ can be extended in a map:

$$\Delta : \widehat{\mathcal{H}_R} \longrightarrow \mathcal{H}_R \widehat{\otimes} \mathcal{H}_R = \prod_{i,j \in \mathbb{N}} \mathcal{H}_R(i) \otimes \mathcal{H}_R(j).$$

Let $f(h) = \sum p_n h^n \in K[[h]]$ be any formal series, and let $X = \sum x_n \in \widehat{\mathcal{H}_R}$, such that $x_0 = 0$. The series of $\widehat{\mathcal{H}_R}$ of term $p_n X^n$ is Cauchy, so converges. Its limit will be denoted by $f(X)$. In other terms, $f(X) = \sum y_n$, with:

$$y_n = \sum_{k=1}^{n} \sum_{a_1 + \ldots + a_k = n} p_k x_{a_1} \ldots x_{a_k}.$$

3.1.2 Combinatorial Dyson-Schwinger equation

Definition 15 [BK06, Kre07, KY06]. Let $f(h) \in K[[h]]$. The *Dyson-Schwinger equation* associated to $f(h)$ is:

$$X = B^+(f(X)), \tag{3.1}$$

where X is an element of $\widehat{\mathcal{H}_R}$, without constant term.

This equation admits a unique solution in $\widehat{\mathcal{H}_R}$:

Proposition 16 *We put $f(h) = \sum p_n h^n$. The Dyson-Schwinger equation associated to $f(h)$ admits a unique solution $X = \sum x_n$, inductively defined by:*

$$\left\{ \begin{array}{rcl} x_0 & = & 0, \\ x_1 & = & p_0 \bullet, \\ x_{n+1} & = & \displaystyle\sum_{k=1}^{n} \sum_{a_1 + \ldots + a_k = n} p_k B^+(x_{a_1} \ldots x_{a_k}). \end{array} \right.$$

Definition 17 The subalgebra of $\mathcal{H}_R$ generated by the homogeneous components x_n's of the unique solution X of the Dyson-Schwinger equation (3.1) associated to $f(h)$ will be denoted by $\mathcal{H}_f$. If $\mathcal{H}_f$ is a Hopf subalgebra of $\mathcal{H}_R$, we shall say that the Dyson-Schwinger equation (3.1) is Hopf.

Remark. If $p_0 = 0$, then $X = 0$ and $\mathcal{H}_f = K$. We now assume that $f(0) \neq 0$. Up to a rescaling, we assume that $f(0) = 1$.

Examples.

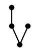
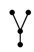
1. We take $f(h) = 1 + h$. Then $x_1 = \bullet$, $x_2 =$, $x_3 =$, $x_4 =$. More generally, x_n is the ladder with n vertices, that is to say $(B^+)^n(1)$. As a consequence, for all $n \geq 1$:
$$\Delta(x_n) = \sum_{i+j=n} x_i \otimes x_j.$$
So $\mathcal{H}_{1+h}$ is Hopf. Moreover, it is cocommutative.

2. We take $f(h) = 1 + h + h^2 + 2h^3 + \mathcal{O}(h^4)$. Then:

$x_1 = \bullet$,

$x_2 =$,

$x_3 =$  $+$,

$x_4 = 2$  $+ 2$  $+$  $+$ .

Hence:

$$\Delta(x_4) = x_4 \otimes 1 + 1 \otimes x_4 + 10x_1^2 \otimes x_2 + x_1^3 \otimes x_1 + 3x_2 \otimes x_2 + 2x_1x_2 \otimes x_1 + x_3 \otimes x_1 + x_1 \otimes (8\,\cdot\,+5\,\cdot),$$

where the trees in the last term are, in order,  and ,

so $\mathcal{H}_f$ is not Hopf.

3.2 Characterization of Hopf Dyson-Schwinger equations

The formal series such that $\mathcal{H}_f$ is Hopf are given by the following theorem:

Theorem 18 *Let $f(h) \in K[[h]]$, such that $f(0) = 1$. The following assertions are equivalent:*

1. *$\mathcal{H}_f$ is a Hopf subalgebra of $\mathcal{H}_R$.*
2. *There exists $(\alpha, \beta) \in K^2$, such that $(1 - \alpha\beta h)f'(h) = \alpha f(h)$.*
3. *There exists $(\alpha, \beta) \in K^2$, such that $f(h) = 1$ if $\alpha = 0$, or $f(h) = e^{\alpha h}$ if $\beta = 0$, or $f(h) = (1 - \alpha\beta h)^{-\frac{1}{\beta}}$ if $\alpha\beta \neq 0$.*

It is an easy exercise to prove that the second and third points are equivalent. We shall give in this section a sketch of the proof of the equivalence $1 \Longleftrightarrow 2$.

3.2.1 Proof of $1 \Longrightarrow 2$

We suppose that $\mathcal{H}_f$ is Hopf. If $p_1 = 0$, then it is not difficult to see that $f(h) = 1$, so 2 holds with $\alpha = 0$. We now assume that $p_1 \neq 0$. Let $Z_\bullet : \mathcal{H}_R \longrightarrow K$, defined by $Z_\bullet(F) = \delta_{\bullet,F}$ for all forest F. This map $Z_\bullet$ is homogeneous of degree -1, so is continuous and can be extended in a map $Z_\bullet : \widehat{\mathcal{H}_R} \longrightarrow K$. We put $(Z_\bullet \otimes Id) \circ \Delta(X) = \sum y_n$, where X is the unique solution of (3.1). A direct computation shows that y_n can be computed by induction with:

$$\begin{cases} y_0 &= 1, \\ y_{n+1} &= \displaystyle\sum_{k=1}^{n} \sum_{a_1+\ldots+a_k=n} (k+1)p_{k+1}B^+(x_{a_1}\ldots x_{a_k}) \\ &\quad + \displaystyle\sum_{k=1}^{n} \sum_{a_1+\ldots+a_k=n} kp_kB^+(y_{a_1}x_{a_2}\ldots x_{a_k}). \end{cases}$$

As $\mathcal{H}_f$ is Hopf, $y_n \in \mathcal{H}_f$ for all $n \in \mathbb{N}$. Moreover, y_n is a linear span of rooted trees of weight n, so is a multiple of x_n: we put $y_n = \alpha_n x_n$.

Let us consider the coefficient of $(B^+)^n(1)$ (ladder of weight n) in y_n. By a direct computation, this is $\alpha_n p_1^{n-1}$. So, for all $n \geq 1$:

$$p_1^n \alpha_{n+1} = 2p_1^{n-1} p_2 + p_1^n \alpha_n.$$

As $\alpha_1 = p_1$, for all $n \geq 1$, $\alpha_n = p_1 + 2\frac{p_2}{p_1}(n-1)$. Let us consider the coefficient of $B^+(\bullet^{n-1})$ (corolla of weight n) in y_n. By a direct computation, this is $\alpha_n p_n$. So, for all $n \geq 1$:

$$\alpha_n p_n = (n+1)p_{n+1} + n p_n p_1.$$

Summing all these relations, putting $\alpha = p_1$ and $\beta = 2\frac{p_2}{p_1} - 1$, we obtain the differential equation $(1 - \alpha\beta h) f'(h) = f(h)$, so 2 holds.

3.2.2 Proof of $2 \Longrightarrow 1$

Let us suppose 2 or, equivalently, 3. We now write $\mathcal{H}_{\alpha,\beta}$ instead of $\mathcal{H}_f$. We first give a description of the x_n's, using the following combinatorial coefficients:

Definition 19

1. Let $F \in \mathbf{F}_R$. The coefficient s_F is inductively defined by:
$$\left\{\begin{array}{rcl} s_\bullet &=& 1, \\ s_{t_1^{a_1}\ldots t_k^{a_k}} &=& a_1!\ldots a_k! s_{t_1}^{a_1}\ldots s_{t_k}^{a_k}, \\ s_{B^+\left(t_1^{a_1}\ldots t_k^{a_k}\right)} &=& a_1!\ldots a_k! s_{t_1}^{a_1}\ldots s_{t_k}^{a_k}, \end{array}\right.$$
where $t_1, \ldots, t_k$ are distinct elements of $\mathbf{T}_R$.

2. Let $F \in \mathbf{F}_R$. The coefficient m_F is inductively defined by:
$$\left\{\begin{array}{rcl} m_\bullet &=& 1, \\ m_{t_1^{a_1}\ldots t_k^{a_k}} &=& \dfrac{(a_1+\ldots+a_k)!}{a_1!\ldots a_k!} m_{t_1}^{a_1}\ldots m_{t_k}^{a_k}, \\ m_{B^+\left(t_1^{a_1}\ldots t_k^{a_k}\right)} &=& \dfrac{(a_1+\ldots+a_k)!}{a_1!\ldots a_k!} m_{t_1}^{a_1}\ldots m_{t_k}^{a_k}, \end{array}\right.$$
where $t_1, \ldots, t_k$ are distinct elements of $\mathbf{T}_R$.

Remarks.

1. The coefficient s_F is the number of symmetries of F, that is to say the number of graph automorphisms of F respecting the roots.
2. The coefficient m_F is the number of embeddings of F in the plane, that is to say the number of planar forests which underlying rooted forest is F.

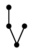
We now give β-equivalents of these coefficients. For all $k \in \mathbb{N}^*$, we put $[k]_\beta = 1 + \beta(k-1)$ and $[k]_\beta! = [1]_\beta \ldots [k]_\beta$. Note that it is not the standard definition of β-integer. We then inductively define $[s_F]_\beta$ and $[m_F]_\beta$ for all $F \in \mathbf{F}_R$ by:

$$\left\{\begin{array}{rcl} [s_\bullet]_\beta &=& 1, \\ [s_{t_1^{a_1}\ldots t_k^{a_k}}]_\beta &=& [a_1]_\beta!\ldots [a_k]_\beta! [s_{t_1}]_\beta^{a_1}\ldots [s_{t_k}]_\beta^{a_k}, \\ [s_{B^+\left(t_1^{a_1}\ldots t_k^{a_k}\right)}]_\beta &=& [a_1]_\beta!\ldots [a_k]_\beta! [s_{t_1}]_\beta^{a_1}\ldots [s_{t_k}]_\beta^{a_k}, \end{array}\right.$$

$$\left\{\begin{array}{rcl} [m_\bullet] &=& 1, \\ [m_{t_1^{a_1}\ldots t_k^{a_k}}]_\beta &=& \dfrac{[a_1+\ldots+a_k]_\beta!}{[a_1]_\beta!\ldots [a_k]_\beta!} [m_{t_1}]_\beta^{a_1}\ldots [m_{t_k}]_\beta^{a_k}, \\ [m_{B^+\left(t_1^{a_1}\ldots t_k^{a_k}\right)}]_\beta &=& \dfrac{[a_1+\ldots+a_k]_\beta!}{[a_1]_\beta!\ldots [a_k]_\beta!} [m_{t_1}]_\beta^{a_1}\ldots [m_{t_k}]_\beta^{a_k}, \end{array}\right.$$

where $t_1, \ldots, t_k$ are distinct elements of $\mathbf{T}_R$. In particular, $[s_t]_1 = s_t$ and $[m_t]_1 = m_t$, whereras $[s_t]_0 = 1$ and $[m_t]_0 = 1$ all $t \in \mathbf{T}_R$.

Examples.

t	s_t	$[s_t]_\beta$	m_t	$[m_t]_\beta$
$\bullet$	1	1	1	1
	1	1	1	1
	2	$(1+\beta)$	1	1
	1	1	1	1
	6	$(1+\beta)(1+2\beta)$	1	1
	1	1	2	$(1+\beta)$
	2	$(1+\beta)$	1	1
	1	1	1	1

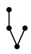
Proposition 20 *For all $n \in \mathbb{N}^*$, in $\mathcal{H}_{\alpha,\beta}$,* $x_n = \alpha^{n-1} \sum_{t \in \mathbf{T}_R,\, weight(t)=n} \frac{[s_t]_\beta [m_t]_\beta}{s_t} t.$

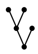
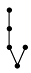
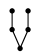
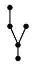
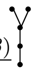
Examples.

$x_1 = \bullet$,

$x_2 = \alpha$ ,

$x_3 = \alpha^2 \left(\frac{(1+\beta)}{2} \right.$  $+$  $\left. \right)$,

$x_4 = \alpha^3 \left(\frac{(1+2\beta)(1+\beta)}{6} \right.$  $+ (1+\beta)$  $+ \frac{(1+\beta)}{2}$  $+$ $\left. \right)$,

$x_5 = \alpha^4 \left(\frac{(1+3\beta)(1+2\beta)(1+\beta)}{24} \right.$  $+ \frac{(1+2\beta)(1+\beta)}{2}$ 
$+ (1+\beta)^2$  $+ (1+\beta)$  $+ \frac{(1+2\beta)(1+\beta)}{6}$ 
$+ \frac{(1+\beta)}{2}$  $+ (1+\beta)$  $+ \frac{(1+\beta)}{2}$  $+$ $\left. \right)$.

Direct technical computations using this description prove the following proposition:

Proposition 21 *If $\alpha = 1$,* $\Delta(X) = X \otimes 1 + \sum_{n=1}^{\infty} (1 - \beta X)^{-n(1/\beta+1)+1} \otimes x_n.$

So $\mathcal{H}_{1,\beta}$ is a Hopf subalgebra.

Remarks. We obtain the following particular cases:

1. For $(\alpha, \beta) = (1, 0)$, $f(h) = e^h$ and for all $n \in \mathbb{N}$, $x_n = \sum_{\substack{t \in \mathbf{T}_R \\ weight(t)=n}} \frac{1}{s_t} t.$

2. For $(\alpha,\beta)=(1,1)$, $f(h)=(1-h)^{-1}$ and for all $n\in\mathbb{N}$, $x_n=\sum\limits_{\substack{t\in\mathbf{T}_R\\ weight(t)=n}} m_t t.$

3. For $(\alpha,\beta)=(1,-1)$, $f(h)=1+h$ and, as $[i]_{-1}=0$ if $i\geq 2$, for all $n\in\mathbb{N}^*$, x_n is the ladder of weight n.

3.3 What is $\mathcal{H}_{\alpha,\beta}$?

3.3.1 Lie algebra associated to $\mathcal{H}_{\alpha,\beta}$

If $\alpha=0$, then $\mathcal{H}_{0,\beta}=K[\bullet]$. If $\alpha\neq 0$, then obviously $\mathcal{H}_{\alpha,\beta}=\mathcal{H}_{1,\beta}$: let us suppose that $\alpha=1$. The Hopf algebra $\mathcal{H}_{1,\beta}$ is graded, connected and commutative. Dually, its graded dual $\mathcal{H}_{1,\beta}^*$ is a graded, connected, cocommutative Hopf algebra. By the Milnor-Moore theorem, it is isomorphic to the enveloping algebra of the Lie algebra of its primitive elements. We now denote this Lie algebra by $\mathfrak{g}_{1,\beta}$. The dual of $\mathfrak{g}_{1,\beta}$ is identified with the quotient space:

$$coPrim(\mathcal{H}_{1,\beta})=\frac{\mathcal{H}_{1,\beta}}{(1)\oplus Ker(\varepsilon)^2},$$

and the transposition of the Lie bracket is the Lie cobracket δ induced by:

$$(\varpi\otimes\varpi)\circ(\Delta-\Delta^{op}),$$

where ϖ is the canonical projection on $coPrim(\mathcal{H}_{1,\beta})$. As $\mathcal{H}_{1,\beta}$ is the polynomial algebra generated by the x_n's, a basis of $coPrim(\mathcal{H}_{1,\beta})$ is $(\varpi(x_n))_{n\in\mathbb{N}^*}$. Proposition 21 gives:

$$\delta(\varpi(x_k))=\sum_{i+j=k}(1+\beta)(j-i)\varpi(x_i)\otimes\varpi(x_j).$$

Dually, the Lie algebra $\mathfrak{g}_{1,\beta}$ has a basis $(Z_n)_{n\geq 1}$, dual of the basis $(\varpi(x_n))_{n\in\mathbb{N}^*}$, with bracket given by:

$$[Z_i,Z_j]=(1+\beta)(j-i)Z_{i+j}.$$

So $\mathfrak{g}_{1,-1}$ is abelian. If $\beta\neq-1$, $\mathfrak{g}_{1,\beta}$ is isomorphic to the Faà di Bruno Lie algebra $\mathfrak{g}_{FdB}$, which has a basis $(f_n)_{n\geq 1}$, and its bracket defined by $[f_i,f_j]=(j-i)f_{i+j}$. So $\mathcal{H}_{1,\beta}$ is isomorphic to the Hopf algebra $\mathcal{U}(\mathfrak{g}_{FdB})^*$, namely the Faà di Bruno Hopf algebra [FGB01], coordinate ring of the group of formal diffeomorphisms of the line tangent to Id, that is to say:

$$G_{FdB}=\left(\left\{\sum a_nh^n\in K[[h]]\ /\ a_0=0,\ a_1=1\right\},\circ\right).$$

Theorem 22 *1. If $\alpha\neq 0$ and $\beta\neq-1$, $\mathcal{H}_{\alpha,\beta}$ is isomorphic to the Faà di Bruno Hopf algebra.*

2. If $\alpha\neq 0$ and $\beta=-1$, $\mathcal{H}_{\alpha,\beta}$ is isomorphic to the Hopf algebra of symmetric functions.

3. If $\alpha=0$, $\mathcal{H}_{\alpha,\beta}=K[\bullet]$.

Remark. If β and $\beta'\neq-1$, then $\mathcal{H}_{1,\beta}$ and $\mathcal{H}_{1,\beta'}$ are isomorphic but are not equal.

3.3.2 FdB Lie algebras

So, up to degenerate cases, we always obtain an embedding of the Faà di Bruno Hopf algebra into $\mathcal{H}_R$. We would like to explain this fact. In any case, the Hopf algebra we obtained is the graded dual of the enveloping algebra of a graded, connected Lie algebra, such that any homogeneous component of degree ≥ 1 is one-dimensional. This fact (with a condition of non commutativity) motivates the following definition:

Definition 23 Let $\mathfrak{g}$ be a $\mathbb{N}$-graded Lie algebra. For all $n \in \mathbb{N}$, we denote by $\mathfrak{g}(n)$ the homogeneous component of degree n of $\mathfrak{g}$. We shall say that $\mathfrak{g}$ is *FdB* if:

1. $\mathfrak{g}$ is connected, that is to say $\mathfrak{g}(0) = (0)$.
2. For all $i \in \mathbb{N}^*$, $\mathfrak{g}$ is one-dimensional.
3. For all $n \geq 2$, $[\mathfrak{g}(1), \mathfrak{g}(n)] \neq (0)$.

A study of cases, using technical computations with MuPAD pro 4, proves the following theorem:

Theorem 24 *Up to an isomorphism, there are three FdB Lie algebras:*

1. *The Faà di Bruno Lie algebra $\mathfrak{g}_{FdB}$, with basis $(e_i)_{i\geq 1}$, and the bracket given by $[e_i, e_j] = (j - i)m_{i+j}$ for all $i, j \geq 1$.*
2. *The corolla Lie algebra $\mathfrak{g}_c$, with basis $(e_i)_{i\geq 1}$, and the bracket given by $[e_1, e_j] = e_{j+1}$ and $[e_i, e_j] = 0$ for all $i, j \geq 2$.*
3. *Another Lie algebra $\mathfrak{g}_3$, with basis $(e_i)_{i\geq 1}$, and the bracket given by $[e_1, e_i] = e_{i+1}$, $[e_2, e_j] = e_{j+2}$, and $[e_i, e_j] = 0$ for all $i \geq 2$, $j \geq 3$.*

In particular, if $[\mathfrak{g}(i), \mathfrak{g}(j)] \neq (0)$ for all $i \neq j$:

Corollary 25 *Let $\mathfrak{g}$ be a FdB Lie algebra, such that if i and j are two distinct elements of $\mathbb{N}^*$, then $[\mathfrak{g}(i), \mathfrak{g}(j)] \neq (0)$. Then $\mathfrak{g}$ is isomorphic to the Faà di Bruno Lie algebra.*

3.3.3 Dual of enveloping algebras of Lie algebras of type 2 and 3

Through Dyson-Schwinger equations, we already obtained embeddings of the graded dual of the FdB Lie algebra of the first type into $\mathcal{H}_R$. We now consider the two other cases.

Definition 26 We denote by $\mathcal{H}_c$ the subalgebra of $\mathcal{H}_R$ generated by the corollas $B^+(\bullet^{n-1})$, $n \geq 1$.

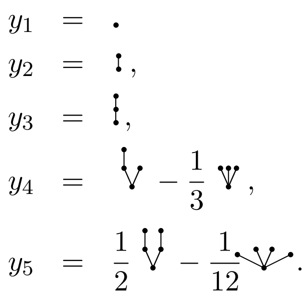
It is immediate to prove that $\mathcal{H}_C$ is a Hopf subalgebra of $\mathcal{H}_R$. Moreover, it is dual to the enveloping algebra of a FdB Lie algebra. A direct computation shows:

Proposition 27 *$\mathcal{H}_c$ is a graded Hopf subalgebra of $\mathcal{H}_R$. Its dual is isomorphic to the enveloping algebra of the corolla Lie algebra.*

We consider the following element of $\widehat{\mathcal{H}_R}$:

$$Y = B^+\left(\exp\left(\overset{\bullet}{\bullet} - \frac{1}{2}\bullet^2 + \bullet\right)\right) = \sum_{n\geq 1} y_n.$$

For example:



Definition 28 We denote by $\mathcal{H}_3$ the subalgebra of $\mathcal{H}_R$ generated by the y_n's.

Proposition 29 *$\mathcal{H}_3$ is a graded Hopf subalgebra of $\mathcal{H}_R$. Its dual is isomorphic to the enveloping algebra of the third FdB Lie algebra.*

Proof. The subalgebra $\mathcal{H}_3$, being generated by homogeneous elements, is graded. An easy computation proves that $X = \overset{\bullet}{\bullet} - \frac{1}{2}\bullet^2 + \bullet$ is a primitive element of $\mathcal{H}_R$. As a consequence, in $\widehat{\mathcal{H}_R}$:

$$\begin{array}{rcl} \Delta(X) & = & X \otimes 1 + 1 \otimes X, \\ \Delta(\exp(X)) & = & \exp(X) \otimes \exp(X), \\ \Delta(Y) & = & \Delta \circ B^+(\exp(X)) \\ & = & Y \otimes 1 + \exp(X) \otimes Y. \end{array}$$

Moreover, $X = y_2 - \frac{1}{2}y_1^2 + y_1 \in \mathcal{H}_3$, so taking the homogeneous component of degree n of $\Delta(Y)$, we obtain:

$$\Delta(y_n) = y_n \otimes 1 + \sum_{k=1}^{n} \sum_{l=1}^{n-k} \sum_{a_1+\ldots+a_l=n-k} \frac{1}{l!} x_{a_1} \ldots x_{a_l} \otimes y_k,$$

where $x_1 = \bullet = y_1$, $x_2 = \overset{\bullet}{\bullet} - \frac{1}{2}\bullet\bullet = y_2 - \frac{1}{2}y_1^2$ and $x_i = 0$ if $i \geq 3$, so $\Delta(y_n) \in \mathcal{H}_3 \otimes \mathcal{H}_3$ and $\mathcal{H}_3$ is a Hopf subalgebra of $\mathcal{H}_R$. As it is commutative, its dual is the enveloping algebra of the Lie algebra $Prim(\mathcal{H}_3^*)$. A direct computation shows that $Prim(\mathcal{H}_3^*)$ is isomorphic to third FdB Lie algebra. $\square$

3.4 Systems of Dyson-Schwinger equations

3.4.1 Definitions

Considering decorated rooted trees by $\{1, \ldots, N\}$ instead of rooted trees, we can extend the definition of Dyson-Schwinger equations to systems of Dyson-Schwinger equations:

Definition 30 A system of Dyson-Schwinger equations (briefly, a SDSE) is a system of the form:

$$\left\{ \begin{array}{rcl} X_1 & = & B_1^+(f_1(X_1, \ldots, X_N)), \\ & \vdots & \\ X_N & = & B_N^+(f_N(X_1, \ldots, X_N)), \end{array} \right.$$

where $f_1, \ldots, f_N \in K[[h_1, \ldots, h_N]] - K$, and $X_1, \ldots, X_N \in \overline{\mathcal{H}_R^{\{1,\ldots,N\}}}$.

Similarly with the case of a single equation:

Proposition 31 *Let (S) be a SDSE. Then it admits a unique solution $(X_1, \ldots, X_N) \in \left(\overline{\mathcal{H}_R^{\{1,\ldots,N\}}}\right)^N$.*

Definition 32 Let (S) be a SDSE. Let $X = (X_1, \ldots, X_N)$ be its unique solution. The subalgebra of $\mathcal{H}_N$ generated by the homogeneous components $X_i(k)$ of the X_i's will be denoted by $\mathcal{H}_{(S)}$. If $\mathcal{H}_{(S)}$ is Hopf, the system (S) will be said to be Hopf.

3.4.2 Operations on Hopf SDSE

It is possible to obtain new Hopf SDSE from old ones by certain operations:

Proposition 33 (change of variables) *Let (S) be the SDSE associated to $(f_1, \ldots, f_N) \in K[[h_1, \ldots, h_N]]^N$. Let $\lambda_1, \ldots, \lambda_N, \mu_1, \ldots, \mu_N$ be non-zero scalars. The system (S) is Hopf if, and only if, the SDSE system (S') associated to $(\mu_1 f_1(\lambda_1 h_1, \ldots, \lambda_N h_N), \ldots, \mu_N f_N(\lambda_1 h_1, \ldots, \lambda_N h_N))$ is Hopf.*

Proposition 34 (dilatation) *Let (S) be the system associated to $(f_1, \ldots, f_N)$ and (S') be a system associated to a family $(\tilde{f}_i)_{i\in I}$, such that there exists a partition $I = I_1 \cup \ldots \cup I_N$ of I, with the following property: for all $1 \leq i \leq N$, for all $j \in I_i$, with $h = (h_k)_{k\in I}$,*

$$\tilde{f}_j(h) = f_i\left(\sum_{k\in I_1} h_k, \ldots, \sum_{k\in I_N} h_k\right).$$

Then (S) is Hopf, if, and only if, (S') is Hopf.

Notations. For all $\beta \neq 0$, we put $g_\beta(h) = (1-\beta h)^{-\frac{1}{\beta}}$. We also put $g_0(h) = e^h$. In other terms, for any $\beta \in K$:

$$g_\beta(h) = \sum_{k=0}^{\infty} \frac{(1+\beta)\ldots(1+\beta(k-1))}{k!} h^k.$$

Example. As the Dyson-Schwinger equation associated to g_β is Hopf, the SDSE associated to $(g_\beta(h_1 + \ldots + h_N))_{1\leq i\leq N}$ is Hopf.

3.4.3 Examples of Hopf SDSE

We here give a few examples of Hopf SDSE of different natures:

Proposition 35 *1. Let $N \geq 2$. The SDSE associated to the following formal series is Hopf:*

$$\begin{cases} f_1 &= 1 + h_2, \\ \vdots & \vdots \\ f_{N-1} &= 1 + h_N, \\ f_N &= 1 + h_1. \end{cases}$$

2. Let $N \geq 2$. The SDSE associated to the following formal series is Hopf:

$$f_i = \prod_{j\neq i} (1-h_j)^{-1}, \text{ for all } 1 \leq i \leq N.$$

3. Let $N \geq 1$ and $\beta_1, \ldots, \beta_N \in K$. The SDSE associated to the following formal series are Hopf:

$$f_i = g_{\beta_i}(h_i) \prod_{j\neq j} g_{\frac{\beta_j}{1+\beta_j}}((1+\beta_j)h_j), \text{ for all } 1 \leq i \leq N.$$

The study of Hopf SDSE is slightly more complicated than the case of a single equation. The dependance of each formal series in the various indeterminates plays in particular an important role. We can all the same give the following result:

Theorem 36 *Let (S) be a Hopf SDSE such that, for all $1 \leq i, j \leq N$ $\frac{\partial f_i}{\partial h_j} \neq 0$. Then up to a change of variables, (S) is the dilatation of a Hopf SDSE of the third type of proposition 35, with $\beta_1 \ldots, \beta_N \neq -1$.*

Chapter 4

Quantization and quantum double of $\mathcal{H}^{\mathcal{D}}$

We extend in this chapter some aspects of the theory of quantum group theory to the settings of Hopf algebra of decorated planar trees. We treat $\mathcal{H}^{\mathcal{D}}$ like (the dual of) the enveloping algebra of the positive part $\mathfrak{g}^+$ of a semi-simple Lie algebra; both are graded, connected Hopf algebras; both can be quantized as braided Hopf algebra, giving self-dual objects respectively denoted by $\mathcal{H}_q^{\mathcal{D}}$ and $\mathcal{U}_q(\mathfrak{g}^+)$. In the quantum group case, a bosonization and a quantum double allows to construct $\mathcal{U}_g(\mathfrak{g})$ from $\mathcal{U}_q(\mathfrak{g}^+)$: we extend this construction to $\mathcal{H}_q^{\mathcal{D}}$ and obtain a Hopf algebra $D(\mathcal{H}_q^{\mathcal{D}})$.

We then introduce a category of highest weight modules over $D(\mathcal{H}_q^{\mathcal{D}})$ and describe the Verma and simple modules of this category. We introduce the notion of crystal basis of such a module, prove the existence and uniqueness of crystal basis for simple highest weight modules. Then a combinatorial process allows to give the decomposition of the tensor product of two such modules.

4.1 Quantization of $\mathcal{H}^{\mathcal{D}}$

We here fix a non-empty and finite set $\mathcal{D}$. Let $A = (a_{i,j})_{i,j\in\mathcal{D}}$ be a symmetric matrix with integer coefficients, and let $q \in K - \{0\}$.

Notations.

1. We put, for all $x, y \in \mathbb{Z}^{\mathcal{D}}$, $x.y = \displaystyle\sum_{i,j\in\mathcal{D}} x_i a_{i,j} y_j$.
2. For all $F \in \mathbf{F}^{\mathcal{D}}$, $|F| = (n_i(F))_{i\in\mathcal{D}}$, where $n_i(F)$ is the number of vertices of F decorated by i.

We now give $\mathcal{H}^{\mathcal{D}}$ a braiding c_q:

$$c_q : \left\{ \begin{array}{rcl} \mathcal{H}^{\mathcal{D}} \otimes \mathcal{H}^{\mathcal{D}} & \longrightarrow & \mathcal{H}^{\mathcal{D}} \otimes \mathcal{H}^{\mathcal{D}} \\ F \otimes G & \longrightarrow & q^{|F|.|G|} G \otimes F. \end{array} \right.$$

This braiding gives $\mathcal{H}^{\mathcal{D}} \otimes \mathcal{H}^{\mathcal{D}}$ a new product, namely $m_q = (m \otimes m) \circ (Id \otimes c_q \otimes Id)$. Let us consider the unique algebra morphism $\Delta_q : \mathcal{H}^{\mathcal{D}} \longrightarrow (\mathcal{H}^{\mathcal{D}} \otimes \mathcal{H}^{\mathcal{D}}, m_q)$ such that for all $x \in \mathcal{H}^{\mathcal{D}}$:

$$\Delta_q \circ B_d^+(x) = B_d^+(x) \otimes 1 + (Id \otimes B_d^+) \circ \Delta_q(x).$$

This Δ_q is a coassociative coproduct, and $(\mathcal{H}^{\mathcal{D}}, \Delta_q)$ is a braided Hopf algebra, denoted by $\mathcal{H}_q^{\mathcal{D}}$. This coproduct can be combinatorially defined with the help of admissible cuts: for all $t \in \mathbf{T}^{\mathcal{D}}$,

$$\Delta_q(t) = t \otimes 1 + 1 \otimes t + \sum_{\kappa \in Adm(t)} q^{a_\kappa} P^\kappa(t) \otimes R^\kappa(t),$$

where a_κ is a certain integer. The antipode of $\mathcal{H}_q^{\mathcal{D}}$ is denoted by T_q. It can be described in terms of cuts.

Example. If $i, j, k \in \mathcal{D}$:

$$\Delta_q(\ttroisun{k}{i}{j}) = \ttroisun{k}{i}{j} \otimes 1 + 1 \otimes \ttroisun{k}{i}{j} + \tun{i}\tun{j} \otimes \tun{k} + \tun{i} \otimes \tdeux{k}{j} + q^{a_{i,j}} \tun{j} \otimes \tdeux{k}{i}.$$

4.1.1 Duality

Similarly with the classical case, obtained for $q = 1$, $\mathcal{H}_q^{\mathcal{D}}$ is a self-dual braided Hopf algebra. More precisely, we defined a Hopf pairing denoted by $\langle -, - \rangle_q$.

Theorem 37 *There exists a unique Hopf pairing $\langle -, - \rangle_q : \mathcal{H}_q^{\mathcal{D}} \times \mathcal{H}_q^{\mathcal{D}} \longrightarrow K$ such that, for all $x, y \in \mathcal{H}_q^{\mathcal{D}}$, $d \in \mathcal{D}$, $\langle B_d^+(x), y \rangle_q = \langle x, \gamma_d(y) \rangle_q$. This pairing is homogeneous, symmetric, and non degenerate.*

The dual basis of the basis of forests will be denoted by $(e_F^q)_{F \in \mathbf{F}^{\mathcal{D}}}$.

Examples. Here, $\mathcal{D}$ is reduced to a single element d, and $a_{d,d} = 1$. As all the vertices of the forest of $\mathcal{H}^{\mathcal{D}}$ are decorated by the same element d, we do not write it. The values of the pairing $\langle -, - \rangle_q$ on the forests of weight ≤ 3 are given in the following arrays:

	$\tun$
$\tun$	1

	$\tun\tun$	$\tdeux$
$\tun\tun$	$1+q$	1
$\tdeux$	1	0

	$\tun\tun\tun$	$\tdeux\tun$	$\tun\tdeux$	$\ttroisun$	$\ttroisdeux$
$\tun\tun\tun$	$(1+q+q^2)(1+q)$	$1+q+q^2$	$1+q+q^2$	$1+q$	1
$\tdeux\tun$	$1+q+q^2$	q	1	1	0
$\tun\tdeux$	$1+q+q^23$	1	q	0	0
$\ttroisun$	$1+q$	1	0	0	0
$\ttroisdeux$	1	0	0	0	0

The first elements of the dual basis are given by:

$$\begin{array}{rcl}
e_{\tun} & = & \tun, \\
e_{\tdeux} & = & \tun\tun - (1+q)\tdeux, \\
e_{\tun\tun\tun} & = & \ttroisdeux, \\
e_{\tun\tdeux} & = & \dfrac{\tun\tdeux - \ttroisun - q^2\ttroisdeux}{q}, \\
e_{\tun\tun} & = & \tdeux, \\
e_{\tdeux\tun} & = & \ttroisun - (1+q)\ttroisdeux, \\
e_{\ttroisun} & = & \dfrac{q\tdeux\tun - \tun\tdeux + (1-q^2)\ttroisun + q(q-1)\ttroisdeux}{q}, \\
e_{\ttroisdeux} & = & \tun\tun\tun - (q+1)\tdeux\tun - q\tun\tdeux + (q-1)\ttroisun + (1+q+q^3)\ttroisdeux.
\end{array}$$

4.1.2 Drinfeld double of $\mathcal{H}_q^{\mathcal{D}}$

We put $\mathcal{C} = K[X_i^{\pm 1},\ i \in \mathcal{D}]$, $\mathcal{H}_q^+ = \mathcal{H}_q^{\mathcal{D}}$ and $\mathcal{H}_q^- = (\mathcal{H}_q^{\mathcal{D}})^{cop}$. Let us construct the quantum double [Kas95, KRT97] of $\mathcal{H}_q^{\mathcal{D}}$, here denoted by $D(\mathcal{H}_q^{\mathcal{D}})$:

Theorem 38 *The algebras $\mathcal{H}_q^-$, $\mathcal{H}_q^+$ and $\mathcal{C}$ are subalgebras of $D(\mathcal{H}_q^{\mathcal{D}})$. Moreover, the following application is an isomorphism of vector spaces:*

$$\left\{ \begin{array}{rcl} \mathcal{H}_q^- \otimes \mathcal{H}_q^+ \otimes \mathcal{C} & \longrightarrow & D(\mathcal{H}_q^{\mathcal{D}}) \\ x \otimes y \otimes X^\alpha & \longrightarrow & xyX^\alpha. \end{array} \right.$$

For all $\alpha, \beta \in \mathbb{Z}^{\mathcal{D}}$, and $x \in \mathcal{H}_q^-$, $y \in \mathcal{H}_q^+$, homogeneous:

$$X^\alpha x = q^{\alpha.|x|} x X^\alpha, \qquad X^\alpha y = q^{-\alpha.|y|} y X^\alpha,$$

$$yx = \sum\sum q^{-|x''|.|y'''|} q^{-|x'''|.|y''|} q^{-|x'''|.|y'''|} \langle x', T_q^{-1}(y''') \rangle_q \langle x''', y' \rangle_q \, x'' y'' X^{|x'''|-|y'''|}.$$

The coproduct is given in the following way, denoting $\Delta_q(x) = \sum x' \otimes x''$ and $\Delta_q(y) = \sum y' \otimes y''$:

$$\Delta(x) = \sum x'' \otimes x' X^{|x''|}, \qquad \Delta(y) = \sum y' X^{-|y''|} \otimes y'', \qquad \Delta(X^\alpha) = X^\alpha \otimes X^\alpha.$$

The antipode of $D(\mathcal{H}_q^{\mathcal{D}})$ is given by:

$$S(x) = T_q^{-1}(x) X^{-|x|}, \qquad S(y) = X^{|y|} T_q(y), \qquad S(X^\alpha) = X^{-\alpha}.$$

Its inverse is given by:

$$S^{-1}(x) = X^{-|x|} T_q(x), \qquad S^{-1}(y) = T_q^{-1}(y) X^{|y|}, \qquad S^{-1}(X^\alpha) = X^{-\alpha}.$$

4.2 Heighest weight modules over $D(\mathcal{H}_q^{\mathcal{D}})$

Playing the game of mimicking quantum enveloping algebras, we now introduce highest weight vectors and modules, and describe Verma modules.

4.2.1 Heighest weight vectors and Verma modules

Definition 39 *Let M be $D(\mathcal{H}_q^{\mathcal{D}})$-module, $v \in M$, and $\lambda = (\lambda_d)_{d \in \mathcal{D}} \in (K^*)^{\mathcal{D}}$.*

1. *We shall say that v is a highest weight vector of weight λ if:*
   (a) *for all $x \in \mathcal{H}_q^+$, $x.v = \varepsilon(x) v$.*
   (b) *for all $d \in \mathcal{D}$, $X_d.v = \lambda_d v$.*
2. *We shall say that M is a highest weight module if it is generated by highest weight vectors.*

Notations. Let $\alpha = (\alpha_d)_{d \in \mathcal{D}} \in \mathbb{Z}^{\mathcal{D}}$, $\lambda = (\lambda_d)_{d \in \mathcal{D}} \in (K^*)^{\mathcal{D}}$. We put $\lambda^\alpha = \prod_{d \in \mathcal{D}} \lambda_d^{\alpha_d}$. Then condition (b) is equivalent to condition: (b') for all $\alpha \in \mathbb{Z}^{\mathcal{D}}$, $X^\alpha.v = \lambda^\alpha v$.

We now give a description of the Verma modules:

Theorem 40 *Let $\lambda \in (K^*)^{\mathcal{D}}$. We put $M_\lambda = \mathcal{H}_q^{\mathcal{D}}$ as a vector space. For all $v \in V$, we put $\Delta_q^2(v) = \sum v' \otimes v'' \otimes v'''$ its two-times iterated coproduct in $\mathcal{H}_q^{\mathcal{D}} = M_\lambda$, with homogeneous v''s, v'''s and v''''s. The following formulas define a structure of $D(\mathcal{H}_q^{\mathcal{D}})$-module over M_λ: for all $x \in \mathcal{H}_q^-$, $y \in \mathcal{H}_q^+$, $\alpha \in \mathbb{Z}^{\mathcal{D}}$,*

$$\left\{ \begin{array}{rcl} x.v &=& xv, \\ X^\alpha.v &=& \lambda^\alpha q_{\alpha,|v|} v, \\ y.v &=& \sum q^{-(|v''|+|v'''|).|v'|} \lambda^{|v'''|-|v'|} \langle v''' T_q^{-1}(v'), y \rangle_q v''. \end{array} \right.$$

Moreover, the element $v_\lambda = 1$ is a highest vector of weight λ which generates M_λ.

Proof. It is enough to consider that the free $D(\mathcal{H}_q^{\mathcal{D}})$-module M generated by the element 1, and the relations $y.1 = \varepsilon(y)1$ for all $y \in \mathcal{H}_q^+$, $X^\alpha.1 = \lambda^\alpha 1$ for all $\alpha \in \mathbb{Z}^{\mathcal{D}}$. As $D(\mathcal{H}_q^{\mathcal{D}}) = \mathcal{H}_q^- \mathcal{H}_q^+ \mathcal{C}$, M can be identified, as a $\mathcal{H}_q^-$-module, with $\mathcal{H}_q^-$ via the application:

$$\left\{ \begin{array}{ccc} \mathcal{H}_q^- & \longrightarrow & M \\ x & \longrightarrow & x.1 \end{array} \right.$$

Hence, this induces a $D(\mathcal{H}_q^{\mathcal{D}})$-module structure on $\mathcal{H}_q^-$, such that 1 is a highest vector of weight λ. Direct computations give the different formulas for the action. $\square$

This proof also immediately gives the following result:

Proposition 41 *Let V be a $D(\mathcal{H}_q^{\mathcal{D}})$-module, $v \in V$ a highest weight vector of weight λ. There exists a unique morphism of $D(\mathcal{H}_q^{\mathcal{D}})$-modules from M_λ to V, sending v_λ to v.*

In other terms, the M_λ's are the Verma modules of $D(\mathcal{H}_q^{\mathcal{D}})$.

4.2.2 Simple highest weight modules

We now suppose the following condition:

(C_1) Let $\alpha \in \mathbb{Z}^{\mathcal{D}}$. If, for all $\beta \in \mathbb{Z}^{\mathcal{D}}$, $q^{\alpha.\beta} = 1$, then $\alpha = (0, \ldots, 0)$.

Example. We take the matrix $A = (a_{i,j})_{i,j \in \mathcal{D}}$ with integer coefficients, symmetric and invertible.

Proposition 42 *Let $\lambda \in (K^*)^{\mathcal{D}}$. The $D(\mathcal{H}_q^{\mathcal{D}})$-module M_λ has a unique simple quotient S_λ. Moreover, S_λ is a highest weight $D(\mathcal{H}_q^{\mathcal{D}})$-module of weight λ, generated by $u_\lambda = \overline{v}_\lambda$.*

Proof. *Preliminary step.* The vector space $M_\lambda = \mathcal{H}_q^{\mathcal{D}}$ is graded by the weight of forests. For any $\mathbb{N}$-graded $D(\mathcal{H}_q^{\mathcal{D}})$-module V, we denote by V' the submodule of V generated by the subspace:

$$\{v \in V \,/\, v \text{ is homogeneous of degree } \geq 1 \text{ and, for all } y \in \mathcal{H}_q^+,\ y.v = \varepsilon(y)v\}.$$

We define inductively $V^{(i)}$ by:

$$\left\{ \begin{array}{l} V^{(0)} = (0), \\ V^{(i+1)} \text{ contains } V^{(i)}, \text{ and } \dfrac{V^{(i+1)}}{V^{(i)}} = \left(\dfrac{V}{V^{(i)}}\right)'. \end{array} \right.$$

We consider the following submodule of M_λ:

$$N_\lambda = \bigcup_{n=1}^{+\infty} M_\lambda^{(n)}.$$

Then N_λ is the unique maximal submodule of M_λ. $\square$

Example. The base field K is given a trivial structure of $D(\mathcal{H}_q^{\mathcal{D}})$-module. Then, $1 \in K$ is a highest weight vector of weight $(1, \ldots, 1)$, so K is a simple quotient of $M_{(1,\ldots,1)}$. Hence, $S_{(1,\ldots,1)} \approx K$.

As in the quantum enveloping algebra case:

Corollary 43 *The simple highest weight modules are the S_λ's. Moreover, if $\lambda \neq \mu$, then S_λ and S_μ are non isomorphic.*

4.2.3 Contravariant form on S_λ

We here introduce a non degenerate form on the simple modules S_λ, in order to show that a tensor product of S_λ's is isomorphic to a direct sum of S_μ's.

Lemma 44 *The following application defines an involutive Hopf algebra morphism form $D(\mathcal{H}_q^{\mathcal{D}})$ to $D(\mathcal{H}_q^{\mathcal{D}})^{op}$:*

$$\theta : \left\{ \begin{array}{ccc} D(\mathcal{H}_q^{\mathcal{D}}) & \longrightarrow & D(\mathcal{H}_q^{\mathcal{D}}) \\ x \in \mathcal{H}_q^+ & \longrightarrow & T_q^{-1}(x)X^{-|x|} \in \mathcal{H}_q^- \mathcal{C}, \\ y \in \mathcal{H}_q^- & \longrightarrow & X^{|y|}T_q(y) \in \mathcal{C}\mathcal{H}_q^+, \\ X^\alpha \in \mathcal{C} & \longrightarrow & X^\alpha \in \mathcal{C}. \end{array} \right.$$

Definition 45 Let $\lambda \in (K^*)^{\mathcal{D}}$. We consider the following application:

$$L'_\lambda : \left\{ \begin{array}{ccc} \mathcal{H}_q^{\mathcal{D}} & \longrightarrow & \mathcal{H}_q^{\mathcal{D}} \\ x & \longrightarrow & \lambda^{2|x''|}x'T_q(x''). \end{array} \right.$$

We define a bilinear form on $S_\lambda \times S_\lambda$ by putting, for all $a, b \in \mathcal{H}_q^-$:

$$\langle a.u_\lambda, b.u_\lambda \rangle_\lambda = \langle L'_\lambda(a), b \rangle_q = \langle a, L'_\lambda(b) \rangle_q.$$

Direct computations prove the following proposition:

Proposition 46 *1. $\langle -, - \rangle_\lambda$ is symmetric and non-degenerate.*

2. For all $x \in D(\mathcal{H}_q^{\mathcal{D}})$, $v, w \in S_\lambda$, $\langle x.v, w \rangle_\lambda = \langle v, \theta(x).w \rangle_\lambda$.

Using this non-degenerate form, as in the quantum enveloping algebra case:

Theorem 47 *Let $\lambda, \mu \in (K^*)^{\mathcal{D}}$. Then $S_\lambda \otimes S_\mu$ is isomorphic to a direct sum of S_ν's.*

4.3 Copies of $\mathcal{U}_q(\mathfrak{sl}(2))$

In order to define a crystal basis, we need $D(\mathcal{H}_q^{\mathcal{D}})$ to be generated by a set of copies of $\mathcal{U}_q(\mathfrak{sl}(2))$. For this, we need $\mathcal{H}_q^{\mathcal{D}}$ to be primitively generated.

4.3.1 Generation of $\mathcal{H}_q^{\mathcal{D}}$ by primitive elements

We now suppose the following conditions:

(C_2) $K = k(q)$, where q is transcendental over k.

(C_3) $q_{i,j} = q^{a_{i,j}}$,with $(a_{i,j})_{i,j \in \mathcal{D}}$ a symmetric, invertible matrix, with integer coefficients.

Note that these conditions imply condition (C_1).

A condition for $\mathcal{H}_q^{\mathcal{D}}$ to be primitively generated is given by:

Theorem 48 *The following assertions are equivalent:*

1. $\mathcal{H}_q^{\mathcal{D}}$ is generated, as an algebra, by $Prim(\mathcal{H}_q^{\mathcal{D}})$.

2. The $a_{i,j}$'s are all > 0 or all < 0.

4.3.2 The Hopf subalgebras $\mathcal{U}_t$

In the sequel, we shall suppose the following conditions:

(C_4) For all i, j, $a_{i,j} > 0$.

(C_5) For all i, $a_{i,i}$ is even.

Then, conditions 1 and 2 of theorem 48 are satisfied.

Definition 49 Let $t \in \mathbf{T}^{\mathcal{D}}$. We put:

$$q_t = q^{-\frac{|t|\cdot|t|}{2}} \in k(q), \qquad F_t = e_t^q \in \mathcal{H}_q^-, \qquad K_t = X^{|t|} \in \mathcal{C}.$$

As $\langle -, - \rangle_q$ restricted to $Prim(\mathcal{H}_q^{\mathcal{D}}) \times Prim(\mathcal{H}_q^{\mathcal{D}})$ is non-degenerate, there exists a unique $E_t \in Prim(\mathcal{H}_q^+) = Prim(\mathcal{H}_q^{\mathcal{D}})$, such that for all $t' \in \mathbf{T}^{\mathcal{D}}$:

$$\langle F_{t'}, E_t \rangle_q = \frac{-1}{q_t - q_t^{-1}} \delta_{t',t}.$$

We denote by $\mathcal{U}_t$ the subalgebra of $D(\mathcal{H}_q^{\mathcal{D}})$ generated by E_t, F_t, K_t, K_t^{-1}.

Remark. We put $|t| = (\alpha_i)_{i \in \mathcal{D}}$. By symmetry of the $a_{i,j}$'s:

$$|t| \cdot |t| = \sum_{i \in \mathcal{D}} a_{i,i} \alpha_i^2 + 2 \sum_{i > j} a_{i,j} \alpha_i \alpha_j.$$

As the $a_{i,i}$'s are even, $|t| \cdot |t|$ is even, and this gives a sense to q_t.

From the definition of the product and coproduct of $D(\mathcal{H}_q^{\mathcal{D}})$:

Proposition 50 *The following relations are satisfied:*

$$\begin{array}{rcl}
K_t E_t &=& q_t^2 E_t K_t, \\
K_t F_t &=& q_t^{-2} F_t K_t, \\
[E_t, F_t] &=& \dfrac{K_t - K_t^{-1}}{q_t - q_t^{-1}}, \\
\Delta(E_t) &=& K_t^{-1} \otimes E_t + E_t \otimes 1, \\
\Delta(F_t) &=& 1 \otimes F_t + F_t \otimes K_t, \\
\Delta(K_t^{\pm 1}) &=& K_t^{\pm 1} \otimes K_t^{\pm 1}.
\end{array}$$

As a consequence, $\mathcal{U}_t$ is a Hopf subalgebra of $D(\mathcal{H}_q^{\mathcal{D}})$ isomorphic to $\mathcal{U}_{q_t}(\mathfrak{sl}(2))$.

By analogy with the quantum enveloping algebras, we introduce the following definition:

Definition 51

1. Let $\lambda \in (K^*)^{\mathcal{D}}$, such that for all $d \in \mathcal{D}$, there exists $a_d \in \mathbb{Z}$, satisfying $\lambda_d = q^{a_d}$. Let f be the group morphism:

$$f : \left\{ \begin{array}{rcl} \mathbb{Z}^{\mathcal{D}} & \longrightarrow & \mathbb{Z} \\ (n_d)_{d \in \mathcal{D}} & \longrightarrow & \displaystyle\sum_{d \in \mathcal{D}} a_d n_d. \end{array} \right.$$

   Then for all $\alpha \in \mathbb{Z}^{\mathcal{D}}$, $\lambda^\alpha = q^{f(\alpha)}$. In a shorter way, we shall denote $\lambda = q^f$.

2. For all $D(\mathcal{H}_q^{\mathcal{D}})$-module M, and all $f : \mathbb{Z}^{\mathcal{D}} \longrightarrow \mathbb{Z}$, we put:

$$M^f = \{v \in M \;/\; X^\alpha.v = q^{f(\alpha)}.v,\ \forall \alpha \in \mathbb{Z}^{\mathcal{D}}\}.$$

3. We shall say that $\lambda = q^f$ is a *dominant weight* if $f(\alpha) \in \mathbb{N}$ for all $\alpha \in \mathbb{N}^{\mathcal{D}}$.

The following proposition wil allow us to define crystal bases:

Proposition 52 *Let λ be a dominant weight. Then, for all $t \in \mathbf{T}^{\mathcal{D}}$:*

$$S_\lambda = \bigoplus_{n \in \mathbb{N}} F_t^n.Ker(E_t).$$

4.4 Crystal bases

Notations. $\mathcal{A}$ is the subring of $k(q) = K$ of rational functions in q with no pole at 0.

We now introduce the definition of a crystal basis. Roughly speaking, a crystal basis of a module M is a $\mathcal{A}$-form L of M, with a basis of L/qL, satisfying some compatibilities with the action of $D(\mathcal{H}_q^{\mathcal{D}})$. To such an object is attached a graph (the crystal), which allows to combinatorially decompose a tensor product.

4.4.1 Definition

Definition 53

1. Let M be a highest weight module over $D(\mathcal{H}_q^{\mathcal{D}})$. We shall say that M is *admissible* if it is isomorphic to a direct sum of S_{λ_i}'s, where the λ_i's are dominant weights. By proposition 52, for all $t \in \mathbf{T}^{\mathcal{D}}$:

$$M = \bigoplus_{n \in \mathbb{N}} F_t^n.Ker(E_t).$$

2. By analogy with the quantum enveloping algebras [Jos95, Kas90, Kas91, Lit95], we define:

$$\tilde{F}_t : \left\{ \begin{array}{ccc} M & \longrightarrow & M, \\ F_t^n.a & \longrightarrow & F_t^{n+1}.a, \end{array} \right. \qquad \tilde{E}_t : \left\{ \begin{array}{ccc} M & \longrightarrow & M \\ F_t^n.a & \longrightarrow & F_t^{n-1}.a, \end{array} \right.$$

where $a \in Ker(E_t)$, with the convention $F_t^{-1}.a = 0$.

3. We shall say that (L, B) is a *crystal basis* of M if:

   (a) L is a free sub-$\mathcal{A}$-module of M, and $M = L \otimes_{\mathcal{A}} k(q)$.

   (b) B is a basis of the k-vector space $\dfrac{L}{qL}$.

   (c) For all $t \in \mathbf{T}^{\mathcal{D}}$, L is stable under the action of $\tilde{F}_t$ and $\tilde{E}_t$.

   (d) For all $t \in \mathbf{T}^{\mathcal{D}}$, $\tilde{E}_t(B) \subseteq B \cup \{0\}$, $\tilde{F}_t(B) \subseteq B \cup \{0\}$.

   (e) $L = \bigoplus_{f \in (\mathbb{Z}^{\mathcal{D}})^*} L^f$ and $B = \coprod_{f \in (\mathbb{Z}^{\mathcal{D}})^*} B^f$, with $L^f = L \cap M^f$ and $B^f = B \cap \left(\dfrac{L^f}{qL^f}\right)$.

   (f) For all $b, b' \in B$, $t \in \mathbf{T}^{\mathcal{D}}$, $b = \tilde{F}_t b' \iff b' = \tilde{E}_t b$.

4. Let us assume that M has a crystal basis (L, B). The *crystal graph* Γ of M is the $\mathbf{T}^{\mathcal{D}}$-colored oriented graph whose vertices are the elements of B, with an edge colored by $t \in \mathbf{T}^{\mathcal{D}}$ from b to b' if, and only if, $b' = \tilde{F}_t b$. Moreover, the set of vertices of Γ is given an application wt taking its values in $(\mathbb{Z}^{\mathcal{D}})^*$, defined by $wt(b) = f$ for any $b \in B^f$.

4.4.2 Existence and uniqueness

Let $\lambda = q^f$ be a dominant weight, and let $L(\lambda)$ be the sub-$\mathcal{A}$-module of S_λ generated by the elements $\tilde{F}_{t_1} \ldots \tilde{F}_{t_n}(u_\lambda) = F_{t_1} \ldots F_{t_n}.u_\lambda$, with $t_1 \ldots t_n \in \mathbf{F}^{\mathcal{D}}$. Let $B(\lambda)$ be the image of these elements in $\dfrac{L(\lambda)}{qL(\lambda)}$, 0 excepted.

Theorem 54 *The couple $(L(\lambda), B(\lambda))$ is a crystal basis of S_λ.*

Remarks.

1. Let us suppose that for all $i \in I$, M_i admits a crystal basis (L_i, B_i). Then $\bigoplus_{i\in I} M_i$ admits a crystal basis (L, B), defined by $L = \bigoplus_{i\in I} L_i$ and $B = \coprod_{i\in I} B_i$. As a consequence, every admissible module admits a crystal basis.
2. Here is a description of the crystal graph of S_λ, where $\lambda = q^f$ is a dominant weight. The vertices are the forests $t_1 \ldots t_n$, with $f(|t_n|) \neq 0$. There exists a t-colored edge from F to G if, and only if, $G = tF$. Moreover, for all forest F which is a vertex of the crystal graph, $wt(F)(\alpha) = f(\alpha) + \alpha.|F|$ for all $\alpha \in \mathbb{Z}^{\mathcal{D}}$.

Uniqueness comes from the following proposition:

Proposition 55 *Let M be an admissible $D(\mathcal{H}_q^{\mathcal{D}})$-module. Let (L, B) be a crystal basis of M. Let $f \in (\mathbb{Z}^{\mathcal{D}})^*$, such that M^f is non-zero, and such that f minimalizes $n(f)$. Let N_1 be the submodule generated by M^f. By semi-simplicity, there exists a sub-module N_2 of M, such that $M = N_1 \oplus N_2$. For $i \in \{1, 2\}$, we put $L_i = L \cap N_i$ and $B_i = B \cap \dfrac{L_i}{qL_i}$. Then:*

1. *(L_i, B_i) is a crystal basis of N_i. Moreover, $L = L_1 \oplus L_2$ and $B = B_1 \coprod B_2$.*
2. *$(L_1, B_1) \approx (L(\lambda), B(\lambda))^{dim(M^f)}$, with $\lambda = q^f$.*

An induction using this last proposition proves:

Corollary 56 *Let M be an admissible $D(\mathcal{H}_q^{\mathcal{D}})$-module. Let (L, B) be a crystal basis of M. There exists an isomorphism $M \longrightarrow \bigoplus_{i\in I} S_{\lambda_i}$, sending (L, B) to $\left(\bigoplus_{i\in I} L(\lambda_i), \coprod_{i\in I} B(\lambda_i)\right)$.*

Note that the crystal graph of $\bigoplus S_{\lambda_i}$ allows to get the λ_i's, considering the values of the application wt on the set of vertices of the graph without incoming edges. Hence:

Corollary 57 *Let M and M' be two admissible $D(\mathcal{H}_q^{\mathcal{D}})$-modules.*

1. *M admits a crystal basis, unique up to an isomorphism.*
2. *M and M' are isomorphic if, and only if, they have isomorphic crystal graphs.*

4.4.3 Compatibility with the tensor product

Let M be an admissible $D(\mathcal{H}_q^{\mathcal{D}})$-module. Let (L, B) be a crystal basis of M. For all $t \in \mathbf{T}^{\mathcal{D}}$, $b \in B$, we put:

$$\left\{ \begin{array}{rcl} \varphi_t(b) & = & \max\{n \in \mathbb{N} \,/\, \tilde{F}_t^n(b) \neq 0\} \in \mathbb{N} \cup \{+\infty\}, \\ \epsilon_t(b) & = & \max\{n \in \mathbb{N} \,/\, \tilde{E}_t^n(b) \neq 0\} \in \mathbb{N} \cup \{+\infty\}. \end{array} \right.$$

Let us now describe the crystal of a tensor product:

Theorem 58 *Let M and M' be two admissible modules. Let (L, B) and (L', B') be crystal bases of M and M'.*

1. *$(L \otimes L', B \otimes B')$ is a crystal basis of $M \otimes M'$.*
2. *For all $b \in B$, $b' \in B'$,*

$$wt(b \otimes b') = wt(b) + wt(b').$$

$$\tilde{E}_t(b \otimes b') = \begin{cases} b \otimes \tilde{E}_t(b') & \text{if } \varphi_t(b') \geq \epsilon_t(b), \\ \tilde{E}_t(b) \otimes b' & \text{if } \varphi_t(b') < \epsilon_t(b). \end{cases}$$

$$\tilde{F}_t(b \otimes b') = \begin{cases} b \otimes \tilde{F}_t(b') & \text{if } \varphi_t(b') > \epsilon_t(b), \\ \tilde{F}_t(b) \otimes b' & \text{if } \varphi_t(b') \leq \epsilon_t(b). \end{cases}$$

4.4.4 Decomposition of a tensor product $S_\lambda \otimes S_\mu$

The study of the crystal of a tensor product allows to give a decomposition into simples:

Theorem 59 *Let $\lambda = q^f, \mu = q^g$ be two dominant weights. For all $\alpha \in \mathbb{N}^{\mathcal{D}}$, let a_α be the number of forests $t_1 \dots t_n \in \mathbf{F}^{\mathcal{D}}$ of degree α, such that $g(|t_1|)$ and $f(|t_n|)$ are non-zeros. Then:*

$$S_\lambda \otimes S_\mu \approx \bigoplus_{\alpha \in \mathbb{N}^{\mathcal{D}}} (S_{\lambda\mu\Lambda_\alpha})^{\oplus a_\alpha}.$$

Proof. We have to describe the crystal graph of $S_\lambda \otimes S_\mu$ with the help of theorem 58. It is enough to describe the vertices with no incoming edges.Let $b \otimes b'$ be such a vertex. We can choose $b = t_1 \dots t_n \in \mathbf{F}^{\mathcal{D}}$, $f(|t_n|) \neq 0$, $b' = s_1 \dots s_m \in \mathbf{F}^{\mathcal{D}}$, $g(|s_m|) \neq 0$. Let us suppose that $s_1 \dots s_m \neq 1$. Then $\tilde{F}_{t_{s_1}} b \otimes s_2 \dots s_m = b \otimes b'$ by theorem 58 (as then $\varphi_{s_1}(s_2 \dots s_m) = +\infty$), so there exists an edge decorated by s_1 going to $b \otimes b'$. So $s_1 \dots s_m = b' = 1$. Let us suppose $g(|t_1|) = 0$. Then $\varphi_{t_1}(1) = 0$ in S_μ, so $\tilde{F}_{t_{t_1}}(t_2 \dots t_n \otimes 1) = t_1 \dots t_n \otimes 1$: this is a contradiction. So $b \otimes b'$ is of the form $t_1 \dots t_n \otimes 1$, with $g(|t_1|) \neq 0$, $f(|t_n|) \neq 0$.

In the other sense, suppose that $b \otimes b'$ is of this form. Let us fix $t \in \mathbf{T}^{\mathcal{D}}$. Two cases are possible.

1. $g(|t|) \neq 0$. Then $\varphi_t(b') = \infty \geq \epsilon_t(b)$, so $\tilde{E}_t(b \otimes b') = b \otimes \tilde{E}_t(1) = 0$.
2. $g(|t|) = 0$. Then $\varphi_t(b') = 0$. Moreover, $t \neq t_1$ as $g(|t_1|) \neq 0$, so $\tilde{E}_t(b) = \tilde{E}_t(t_1 \dots t_n) = 0$, so $\epsilon_t(b) = 0$. Finally, $\tilde{E}_t(b \otimes b') = b \otimes \tilde{E}_t(1) = 0$.

Hence, the edges of the crystal graph of $S_\lambda \otimes S_\mu$ without incoming edges are the $t_1 \dots t_n \otimes 1$'s, with $f(|t_n|) \neq 0$, and $g(|t_1|) \neq 0$. As there are exactly a_α such elements of weight given by $wt(t_1 \dots t_n \otimes 1) = f + g + (\alpha.-)$, we proved the announced result. $\square$

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