Combinatorial Dyson-Schwinger equations

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Feynman graphs

1. A finite number of possible edges.
2. A finite number of possible vertices.
3. A finite number of possible external edges (external structure).

To each external structure is associated a formal series in the Feynman graphs, called the propagator.
Feynman graphs and Dyson-Schwinger equations

Reformulation with trees

Results

Combinatorial Dyson-Schwinger systems

Equations with several 1-cocycles

Examples in QED
Propagators in QED

\[ \begin{align*}
\text{\(\cdots\)} & = \sum_{n \geq 1} x^n \left( \sum_{\gamma \in (n)} s_{\gamma \gamma} \right) \\
\text{\(\cdots\)} & = -\sum_{n \geq 1} x^n \left( \sum_{\gamma \in (n)} s_{\gamma \gamma} \right) \\
\text{\(\cdots\)} & = -\sum_{n \geq 1} x^n \left( \sum_{\gamma \in (n)} s_{\gamma \gamma} \right)
\end{align*} \]
How to describe the propagators?

The algebra $H_{DF}$ is the free commutative algebra generated by the Feynman graphs of a given QFT.

For any primitive Feynman graph $\gamma$, one defines the insertion operator $B_\gamma$ over $H_{DF}$. This operators associates to a graph $G$ the sum (with symmetry coefficients) of the insertions of $G$ into $\gamma$.

The propagators then satisfy a system of equations involving the insertion operators, called systems of Dyson-Schwinger equations.
Example

In QED:

\[ B \left( \begin{array}{c}
\text{\includegraphics[width=1cm]{example1}}
\end{array} \right) = \frac{1}{2} \begin{array}{c}
\text{\includegraphics[width=1cm]{example2}}
\end{array} + \frac{1}{2} \begin{array}{c}
\text{\includegraphics[width=1cm]{example3}}
\end{array} \]

\[ B \left( \begin{array}{c}
\text{\includegraphics[width=1cm]{example4}}
\end{array} \right) = \frac{1}{3} \begin{array}{c}
\text{\includegraphics[width=1cm]{example5}}
\end{array} + \frac{1}{3} \begin{array}{c}
\text{\includegraphics[width=1cm]{example6}}
\end{array} + \frac{1}{3} \begin{array}{c}
\text{\includegraphics[width=1cm]{example7}}
\end{array} \]
Then:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{Feynman graph}
\end{array}
\end{array}
\end{align*}
= \sum_{\gamma \in \Gamma} x^{\vert \gamma \vert} B_{\gamma} \left( \begin{array}{c}
\frac{(1 + \cdots)^{1+2\vert \gamma \vert}}{(1 + \cdots)^{2\vert \gamma \vert}} \end{array} \right)
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{Feynman graph}
\end{array}
\end{array}
\end{align*}
= -xB \left( \begin{array}{c}
\frac{(1 + \cdots)^{2}}{(1 + \cdots)^{2}}
\end{array} \right)
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{Feynman graph}
\end{array}
\end{array}
\end{align*}
= -xB \left( \begin{array}{c}
\frac{(1 + \cdots)^{2}}{(1 + \cdots)(1 + \cdots)^{2}}
\end{array} \right)
\]
Let $A$ be a vector space. The tensor square of $A$ is a space $A \otimes A$ with a bilinear product $\otimes : A \times A \longrightarrow A \otimes A$ with a universal property. If $(e_i)_{i \in I}$ is a basis of $A$, then $(e_i \otimes e_j)_{i,j \in I}$ is a basis $A \otimes A$.

If $A$ is an associative algebra, its (bilinear) product becomes a linear map $m : A \otimes A \longrightarrow A$, sending $e_i \otimes e_j$ over $e_i.e_j$. The associativity is given by the following commuting square:
Dualizing the associativity axiom, we obtain the coassociativity axiom: a coalgebra is a vector space $C$ with a map $\Delta : C \rightarrow C \otimes C$ such that:

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
\downarrow \Delta & & \downarrow \text{Id} \otimes \Delta \\
C \otimes C & \xrightarrow{\Delta \otimes \text{Id}} & C \otimes C \otimes C
\end{array}
\]

A Hopf algebra is both an algebra and a coalgebra, with the compatibility:

\[\Delta(xy) = \Delta(x)\Delta(y).\]

(And a technical condition of existence of an antipode).
Examples

- If $G$ is a group, $KG$ is a Hopf algebra, with $\Delta(x) = x \otimes x$ for all $x \in G$.
- If $\mathfrak{g}$ is a Lie algebra, its enveloping algebra is a Hopf algebra, with $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g}$.
- The algebra of Feynman graphs $H_{DF}$ is a graded Hopf algebra. For example:

\[
\Delta(\begin{array}{c}
\text{\includegraphics{example1.png}}
\end{array}) = \begin{array}{c}
\text{\includegraphics{example2.png}}
\end{array} \otimes 1 + 1 \otimes \begin{array}{c}
\text{\includegraphics{example3.png}}
\end{array} + \begin{array}{c}
\text{\includegraphics{example4.png}}
\end{array} \otimes \begin{array}{c}
\text{\includegraphics{example5.png}}
\end{array}.
\]
Question

For a given system of Dyson-Schwinger equations \((S)\), is the subalgebra generated by the homogeneous components of \((S)\) a Hopf subalgebra?
Proposition

The operators $B_\gamma$ satisfy: for all $x \in H_{DF}$,

$$\Delta \circ B_\gamma(x) = B_\gamma(x) \otimes 1 + (\text{Id} \otimes B_\gamma) \circ \Delta(x).$$

This relation allows to lift any system of Dyson-Schwinger equation to the Hopf algebra of decorated rooted trees.

We first treat the case of a single equation with a single 1-cocycle.
The Hopf algebra of rooted trees $H_R$ (or Connes-Kreimer Hopf algebra) is the free commutative algebra generated by the set of rooted trees. The set of rooted forests is a linear basis of $H_R$:

$$1, \ldots, 1, \ldots, 1, \ldots, 1, \ldots, \ldots, \ldots, \ldots, \ldots, V, \ldots, V, \ldots, V, \ldots, V, \ldots, V, \ldots, V,$$
The coproduct is given by admissible cuts:

\[ \Delta(t) = \sum_{c \text{ admissible cut}} P^c(t) \otimes R^c(t). \]

<table>
<thead>
<tr>
<th>cut</th>
<th>( W^c(t) )</th>
<th>( R^c(t) )</th>
<th>( P^c(t) )</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Admissible ?</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>( W^c(t) )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>( R^c(t) )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>( P^c(t) )</td>
<td>1</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
</tbody>
</table>

\[ \Delta(\checkmark) = \checkmark \otimes 1 + 1 \otimes \checkmark + \checkmark \otimes \checkmark + \ldots \checkmark + \ldots \checkmark + \ldots \checkmark + \ldots \checkmark. \]
The grafting operator of $H_R$ is the map $B^+: H_R \to H_R$, associating to a forest $t_1 \ldots t_n$ the tree obtained by grafting $t_1, \ldots, t_n$ on a common root. For example:

$$B^+(\ldots) = \begin{array}{c} \bullet \\
\end{array}.$$ 

**Proposition**

For all $x \in H_R$:

$$\Delta \circ B^+(x) = B^+(x) \otimes 1 + (Id \otimes B^+) \circ \Delta(x).$$
Universal property

Let $A$ be a commutative Hopf algebra and let $L : A \longrightarrow A$ such that for all $a \in A$:

$$\Delta \circ L(a) = L(a) \otimes 1 + (Id \otimes L) \circ \Delta(a).$$

Then there exists a unique morphism Hopf algebra morphism $\phi : H_R \longrightarrow A$ with $\phi \circ B^+ = L \circ \phi$. 
Let $f(h) \in K[[h]]$. The combinatorial Dyson-Schwinger equations associated to $f(h)$ is:

$$X = B^+(f(X)),$$

where $X$ lives in the completion of $H_R$.

This equation has a unique solution $X = \sum x_n$, with:

$$
\begin{cases}
  x_1 &= p_0 \\
  x_{n+1} &= \sum_{k=1}^{n} \sum_{a_1+\ldots+a_k=n} p_k B^+(x_{a_1} \ldots x_{a_k}),
\end{cases}
$$

where $f(h) = p_0 + p_1 h + p_2 h^2 + \ldots$
Examples

- If \( f(h) = 1 + h \):

  \[
  X = . + \v + \v + \v + \v + \v + \cdots
  \]

- If \( f(h) = (1 - h)^{-1} \):

  \[
  X = . + \v + \v + \v + 2 \v + \v + \v + \v + 3 \v + \v + 2 \v + \v + 2 \v + \v + \v + \v + \v + \v + \cdots
  \]
Let \( f(h) \in K[[h]] \). The homogeneous components of the unique solution of the combinatorial Dyson-Schwinger equation associated to \( f(h) \) generate a subalgebra of \( H_R \) denoted by \( H_f \).

**\( H_f \) is not always a Hopf subalgebra**

For example, for \( f(h) = 1 + h + h^2 + 2 h^3 + \cdots \), then:

\[
X = \cdot + 1 + V + 1 + 2 V + 2 \overset{\vee}{V} + \overset{\vee}{V} + 1 + \cdots
\]

So:

\[
\Delta(x_4) = x_4 \otimes 1 + 1 \otimes x_4 + (10x_1^2 + 3x_2) \otimes x_2
\]

\[
+ (x_1^3 + 2x_1x_2 + x_3) \otimes x_1 + x_1 \otimes (8 \overset{\vee}{V} + 5 \overset{\wedge}{1} ).
\]
Theorem

Let \( f(h) \in K[[h]] \), with \( f(0) = 1 \). The following assertions are equivalent:

1. \( H_f \) is a Hopf subalgebra of \( H_R \).
2. There exists \( (\alpha, \beta) \in K^2 \) such that \( (1 - \alpha \beta h)f'(h) = \alpha f(h) \).
3. There exists \( (\alpha, \beta) \in K^2 \) such that \( f(h) = 1 \) if \( \alpha = 0 \) or \( f(h) = e^{\alpha h} \) if \( \beta = 0 \) or \( f(h) = (1 - \alpha \beta h)^{-\frac{1}{\beta}} \) if \( \alpha \beta \neq 0 \).
Idea of the proof.

1 $\implies$ 2. We put $f(h) = 1 + p_1 h + p_2 h^2 + \cdots$. If $H_f$ is a Hopf subalgebra, for all $n \geq 1$ there exists a scalar $\alpha_n$ such that:

$$(Z \otimes \text{Id}) \circ \Delta(x_{n+1}) = \alpha_n x_n.$$

Considering the coefficient of $(B^+)^n(1)$, we obtain:

$$p_1^{n-1} \alpha_n = 2(n - 1)p_1^{n-1} p_2 + p_1^n.$$

Considering the coefficient of $B^+ (. \ ns^{-1})$, we obtain:

$$\alpha_n p_n = (n + 1)p_{n+1} + np_n p_1.$$

We put $\alpha = p_1$ and $\beta = 2 \frac{p_2}{p_1} - 1$, then:

$$(1 - \alpha \beta h)f'(h) = \alpha f(h).$$
2. \implies 1. By the description of $X$:

\begin{align*}
x_1 &= . , \\
x_2 &= \alpha \cdot , \\
x_3 &= \alpha^2 \left( \frac{1 + \beta}{2} \mathcal{Y} + \mathcal{I} \right), \\
x_4 &= \alpha^3 \left( \frac{(1 + 2\beta)(1 + \beta)}{6} \mathcal{W} + (1 + \beta) \mathcal{V} + \frac{1 + \beta}{2} \mathcal{Y} + \mathcal{I} \right), \\
x_5 &= \alpha^4 \left( \frac{(1 + 3\beta)(1 + 2\beta)(1 + \beta)}{24} \mathcal{W} + \frac{(1 + 2\beta)(1 + \beta)}{2} \mathcal{V} + \frac{(1 + \beta)^2}{2} \mathcal{V} + (1 + \beta) \mathcal{V} + \frac{(1 + 2\beta)(1 + \beta)}{6} \mathcal{Y} + \mathcal{I} \right).
\end{align*}
Particular cases

- If \((\alpha, \beta) = (1, -1)\), \(x_n = (B^+)^n(1)\) for all \(n\) (ladder of degree \(n\)).
- If \((\alpha, \beta) = (1, 1)\),

\[
x_n = \sum_{|t|=n} \#\{\text{embeddings of } t \text{ in the plane}\} \cdot t.
\]
- If \((\alpha, \beta) = (1, 0)\),

\[
x_n = \sum_{|t|=n} \frac{1}{\#\{\text{symmetries of } t\}} \cdot t.
\]
Hence, we have a family of Hopf subalgebras $H_{(\alpha, \beta)}$ of $H_R$ indexed by $(\alpha, \beta)$.

- If $\alpha = 0$, $H_{(\alpha, \beta)} = K[.]$.

- If $\alpha \neq 0$, by the Cartier-Quillen-Milnor-Moore theorem, $H^*_{(\alpha, \beta)}$ is an enveloping algebra. Its Lie algebra has a basis $(Z_i)_{i \geq 1}$ and for all $i, j$:

$$[Z_i, Z_j] = (\beta + 1)(j - i)Z_{i+j}.$$
Theorem

If $\alpha \neq 0$ and $\beta = -1$, $H(\alpha, \beta)$ is isomorphic to the Hopf algebra of symmetric functions.

If $\alpha \neq 0$ and $\beta \neq -1$, $H(\alpha, \beta)$ is isomorphic to the Faà di Bruno Hopf algebra. In other words, $H(\alpha, \beta)$ is the coordinate ring of the group of formal diffeomorphisms of the line that are tangent to the identity:

$$G = \left( \{ f(h) = h + a_1 h^2 + \ldots \mid a_1, a_2, \ldots \in K \}, \circ \right).$$
In the case of Dyson-Schwinger systems, we have to use trees with decorated vertices. The combinatorial Dyson-Schwinger systems have the form:

\[
(S) : \left\{\begin{array}{ll}
X_1 &= B_1^+(f_1(X_1, \ldots, X_n)) \\
&\vdots \\
X_n &= B_n^+(f_n(X_1, \ldots, X_n)),
\end{array}\right.
\]

where \( f_1, \ldots, f_n \in K[[h_1, \ldots, h_n]] - K \), with constant terms equal to 1. Such a system has a unique solution \((X_1, \ldots, X_n) \in H_{\{1, \ldots, n\}}\). The subalgebra generated by the homogeneous components of the \( X_i \)'s is denoted by \( H_(S) \).

If this subalgebra is Hopf, we shall say that the system is Hopf.
Description of two families of Dyson-Schwinger systems:
1. Fundamental systems,
2. Cyclic systems.

Four operations on Dyson-Schwinger systems:
1. Change of variables,
2. Concatenation,
3. Dilatation,
4. Extension.
Main theorem

Let \((S)\) be Hopf combinatorial Dyson-Schwinger system. Then \((S)\) is obtained from the concatenation of fundamental or cyclic systems with the help of a change of variables, a dilatation and a finite number of extensions.
Fundamental systems

Let $\beta_1, \ldots, \beta_k \in K$. The following system is an example of a fundamental system:

\[
\begin{align*}
X_i &= B_i \left( (1 - \beta_i X_i) \prod_{j=1}^{k} (1 - \beta_j X_j)^{-\frac{1+\beta_j}{\beta_j}} \prod_{j=k+1}^{n} (1 - X_j)^{-1} \right) \\
& \quad \text{if } i \leq k, \\
X_i &= B_i \left( (1 - X_i) \prod_{j=1}^{k} (1 - \beta_j X_j)^{-\frac{1+\beta_j}{\beta_j}} \prod_{j=k+1}^{n} (1 - X_j)^{-1} \right) \\
& \quad \text{if } i > k.
\end{align*}
\]
Cyclic systems

The following systems are cyclic: if \( n \geq 2 \),

\[
\begin{align*}
X_1 &= B_1^+(1 + X_2), \\
X_2 &= B_2^+(1 + X_3), \\
&\vdots \\
X_n &= B_n^+(1 + X_1).
\end{align*}
\]
Change of variables

Let \((S)\) be the following system:

\[
(S) : \begin{cases}
    X_1 = B_1^+(f_1(X_1, \ldots, X_n)) \\
    \vdots \\
    X_n = B_n^+(f_n(X_1, \ldots, X_n)).
\end{cases}
\]

If \((S)\) is Hopf, then for all family \((\lambda_1, \ldots, \lambda_n)\) of non-zero scalars, this system is Hopf:

\[
(S) : \begin{cases}
    X_1 = B_1^+(f_1(\lambda_1 X_1, \ldots, \lambda_n X_n)) \\
    \vdots \\
    X_n = B_n^+(f_n(\lambda_1 X_1, \ldots, \lambda_n X_n)).
\end{cases}
\]
Let \((S)\) and \((S')\) be the following systems:

\[
\begin{align*}
(S) : & \quad \begin{cases} 
X_1 = B_1^+(f_1(X_1, \ldots, X_n)) \\
\vdots \\
X_n = B_n^+(f_n(X_1, \ldots, X_n)).
\end{cases} \\
(S') : & \quad \begin{cases} 
X_1 = B_1^+(g_1(X_1, \ldots, X_m)) \\
\vdots \\
X_m = B_m^+(g_m(X_1, \ldots, X_m)).
\end{cases}
\end{align*}
\]
Concatenation

The following system is Hopf if, and only if, the \((S)\) and \((S')\) are Hopf:

\[
\begin{align*}
X_1 &= B_1^+(f_1(X_1, \ldots, X_n)) \\
\vdots \\
X_n &= B_n^+(f_n(X_1, \ldots, X_n)) \\
X_{n+1} &= B_{n+1}^+(g_1(X_{n+1}, \ldots, X_{n+m})) \\
\vdots \\
X_{n+m} &= B_{n+m}^+(g_m(X_{n+1}, \ldots, X_{n+m})).
\end{align*}
\]

This property leads to the notion of connected (or indecomposable) system.
Extension

Let \((S)\) be the following system:

\[
\begin{align*}
X_1 &= B_1^+(f_1(X_1, \ldots, X_n)) \\
\vdots \\
X_n &= B_n^+(f_n(X_1, \ldots, X_n)).
\end{align*}
\]

\((S)\):

Then \((S')\) is an extension of \((S)\):

\[
\begin{align*}
X_1 &= B_1^+(f_1(X_1, \ldots, X_n)) \\
\vdots \\
X_n &= B_n^+(f_n(X_1, \ldots, X_n)) \\
X_{n+1} &= B_{n+1}^+(1 + a_1 X_1).
\end{align*}
\]

\((S')\):
Iterated extensions

\[
(S) : \begin{cases}
X_1 & = B_1 \left( (1 - \beta X_1)^{-\frac{1}{\beta}} \right), \\
X_2 & = B_2 (1 + X_1), \\
X_3 & = B_3 (1 + X_1), \\
X_4 & = B_4 (1 + 2X_2 - X_3), \\
X_5 & = B_5 (1 + X_4).
\end{cases}
\]
Dilatation

\((S')\) is a dilatation of \((S)\):

\[
(S) : \quad \begin{cases} 
X_1 &= B_1^+(f(X_1, X_2)), \\
X_2 &= B_2^+(g(X_1, X_2)), 
\end{cases}
\]

\[
(S') : \quad \begin{cases} 
X_1 &= B_1^+(f(X_1 + X_2 + X_3, X_4 + X_5)), \\
X_2 &= B_2^+(f(X_1 + X_2 + X_3, X_4 + X_5)), \\
X_3 &= B_3^+(f(X_1 + X_2 + X_3, X_4 + X_5)), \\
X_4 &= B_4^+(g(X_1 + X_2 + X_3, X_4 + X_5)), \\
X_5 &= B_5^+(g(X_1 + X_2 + X_3, X_4 + X_5)).
\end{cases}
\]
We now consider equations of the form:

\[(E) : X = \sum_{j \in J} B_j \left( f^{(j)}(X) \right) ,\]

where:

- \( J \) is a set.
- For all \( j \), \( B_j \) is a 1-cocycle of a certain degree.
- For all \( j \), \( f^{(j)} \) is a formal series such that \( f^{(j)}(0) = 1 \).
Lemma

Let us assume that $(E)$ is Hopf. If $B_i$ and $B_j$ have the same degree, then $f^{(i)} = f^{(j)}$ have the same degree.

This allows to assume that $J \subseteq \mathbb{N}^*$. 
Main theorem

Under these hypothesis, if there is at most one constant $f(j)$, there exists $\lambda, \mu \in K$ such that:

\[ (E) : X = \begin{cases} 
\sum_{j \in J} B_j \left((1 - \mu X)^{-\frac{\lambda j}{\mu}}+1\right) & \text{if } \mu \neq 0, \\
\sum_{j \in J} B_j \left( e^{j \lambda X} \right) & \text{if } \mu = 0. 
\end{cases} \]

Example

For $\lambda = 1$ and $\mu = -1$, the following equation gives a Hopf subalgebra:

\[ X = \sum_{n \geq 1} B_n \left((1 + X)^{n+1}\right). \]
Main theorem

If there are at least two constant \( f(j) \), there exists \( \alpha \in K \), and \( m \in \mathbb{N} \) such that:

\[
(E) : \quad X = \sum_{j \in J \cap m\mathbb{N}} B_j (1 + \alpha X) + \sum_{j \in J \setminus m\mathbb{N}} B_j (1).
\]
We now consider systems of the form:

\[
(S) : \begin{cases}
    X_1 &= \sum_{i \in J_1} B_{1,i}^+(f_{1,i}(X_1, \ldots, X_n)) \\
    &\vdots \\
    X_n &= \sum_{i \in J_n} B_{n,i}^+(f_{n,i}(X_1, \ldots, X_n)),
\end{cases}
\]

where for all \( k, i \), \( B_{k,i} \) is a 1-cocycle of degree \( i \).
Theorem

We assume that $1 \in J_k$ for all $k$. Then (S) is entirely determined by $f_{1,1}, \ldots, f_{n,1}$. 
Fundamental system

\[
\begin{cases}
X_i = \sum_{q \in J_i} B_{i,q} \left( (1 - \beta_i X_i) \prod_{j=1}^{k} (1 - \beta_j X_j)^{-\frac{1+\beta_j}{\beta_j}} q \prod_{j=k+1}^{n} (1 - X_j)^{-q} \right) \\
X_i = \sum_{q \in J_i} B_{i,q} \left( (1 - X_i) \prod_{j=1}^{k} (1 - \beta_j X_j)^{-\frac{1+\beta_j}{\beta_j}} q \prod_{j=k+1}^{n} (1 - X_j)^{-q} \right)
\end{cases}
\]

if \( i \leq k \),

if \( i > k \).
For example, we choose $n = 3$, $k = 2$, $\beta_1 = -1/3$, $\beta_2 = 1$, $J_1 = \mathbb{N}^*$, $J_2 = J_3 = \{1\}$. After a change of variables $h_1 \rightarrow 3h_1$, we obtain:

\[
\begin{cases}
X_1 &= \sum_{k \geq 1} B_{1,k} \left( \frac{(1 + X_1)^{1+2k}}{(1 - X_2)^{2k}(1 - X_3)^k} \right), \\
X_2 &= B_2 \left( \frac{(1 + X_1)^2}{(1 - X_2)(1 - X_3)} \right), \\
X_3 &= B_3 \left( \frac{(1 + X_1)^2}{1 - X_2} \right).
\end{cases}
\]

This is the example of the introduction, with $X_1 = \bigcirc$, $X_2 = \bigcirc\rightarrow\bigcirc$, $X_3 = \bigcirc\bigcirc\sim\bigcirc\bigcirc$. 
Cyclic systems

\[
\begin{align*}
X_1 &= \sum_{j \in I_1} B_{1,j} \left( 1 + X_{1+j} \right), \\
& \vdots \\
X_n &= \sum_{j \in I_1} B_{n,j} \left( 1 + X_{n+j} \right).
\end{align*}
\]

\[(S) : \]

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