Bidendriform bialgebras, trees, and free quasi-symmetric functions

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ABSTRACT: we introduce bidendriform bialgebras, which are bialgebras such that both product and coproduct can be split into two parts satisfying good compatibilities. For example, the Malvenuto-Reutenauer Hopf algebra and the non-commutative Connes-Kreimer Hopf algebras of planar decorated rooted trees are bidendriform bialgebras. We prove that all connected bidendriform bialgebras are generated by their primitive elements as a dendriform algebra (bidendriform Milnor-Moore theorem) and then is isomorphic to a Connes-Kreimer Hopf algebra. As a corollary, the Hopf algebra of Malvenuto-Reutenauer is isomorphic to the Connes-kreimer Hopf algebra of planar rooted trees decorated by a certain set. We deduce that the Lie algebra of its primitive elements is free in characteristic zero (G. Duchamp, F. Hivert and J.-Y. Thibon conjecture).

RESUME : nous introduisons les bigèbres bidendriformes, qui sont des bigèbres dont le produit et le coproduit peuvent être scindés en deux avec de bonnes compatibilités. Par exemple, l'algèbre de Hopf de Malvenuto-Reutenauer et les algèbres de Hopf non-commutative de Connes-Kreimer sur les arbres plans enracinés décorés sont des bigèbres bidendriformes. Nous montrons que toute bigèbre bidendriforme connexe est engendrée par ses éléments primitifs comme algèbre dendriforme (version bidendriforme du théorème de Milnor-Moore) et qu'elle est alors isomorphe à une algèbre de Hopf de Connes-Kreimer. En conséquence, l'algèbre de Hopf de Malvenuto-Reutenauer est isomorphe à l'algèbre de Connes-Kreimer des arbres plans enracinés décorés par un certain ensemble. On en déduit que l'algèbre de Lie de ses éléments primitifs est libre en caractéristique zéro (conjecture de G. Duchamp, F. Hivert et J.-Y. Thibon).

Introduction

The Hopf algebra **FQSym** of Malvenuto-Reutenauer, also called Hopf algebra of free quasi-symmetric functions ([5, 18, 19, 20]) has certain interesting properties. For example, it is known that it is free as an algebra and cofree as a coalgebra; it has a non-degenerate Hopf pairing; it can be given a structure of dendriform algebra. The Hopf algebras $\mathcal{H}^{\mathcal{D}}$ of planar rooted trees, introduced in [7, 8, 9] have a lot of similar properties: they are free as algebras and cofree as coalgebras; they have a non-degenerate Hopf pairing (although not so explicit as the Malvenuto-Reutenauer algebra's); they also are dendriform algebras. So a natural question is: are these two objects isomorphic? More precisely, is there a set $\mathcal{D}$ of decorations such that $\mathcal{H}^{\mathcal{D}}$ is isomorphic to **FQSym**?

In order to answer positively this question, we study more in details dendriform algebras. This notion is introduced in [15] and is studied in [1, 16, 19]. A dendriform algebra A is a (non unitary) associative algebra, such that the product can be split into two parts $\prec$ and $\succ$ (left

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and right products),with good compatibilities (which mean that $(A, \prec, \succ)$ is a bimodule over itself). The notion of dendriform bialgebra or Hopf dendriform algebra is introduced in [18] and studied in [25, 26]. These are dendriform algebra with a coassociative coproduct $\tilde{\Delta}$, satisfying good relations with $\prec$ and $\succ$. We introduce here the notion of bidendriform bialgebra (section 2): a bidendriform bialgebra A is a dendriform bialgebra such that the coproduct $\tilde{\Delta}$ can be split into two parts $\Delta_\prec$ and $\Delta_\succ$ (left and right coproducts), such that $(A, \Delta_\succ, \Delta_\prec)$ is a bimodule over itself. There are also compatibilities between the left and right coproducts and the left and right products. As this set of axioms is self-dual, the dual of a finite-dimensional bidendriform bialgebra is also a bidendriform bialgebra. An example of bidendriform bialgebra is **FQSym** (section 4), or, more precisely, its augmentation ideal: as **FQSym** is a dendriform algebra and it is self dual as a Hopf algebra, we can also split the coproduct into two parts. Fortunately, the left and right coproducts defined in this way satisfy the wanted compatibilities with the left and right products.

We would like to give $\mathcal{H}^\mathcal{D}$ a structure of bidendriform algebra too. The method used for **FQSym** fails here: the left and right coproducts defined by duality, denoted here by $\Delta'_\prec$ and $\Delta'_\succ$, do not satisfy the compatibilities with the left and right products. Hence, we have to proceed in a different way. For this, in the same way as [17], we consider the category of dendriform algebras and give it a tensor product $\overline{\otimes}$ (section 3). This tensor product is a little bit different of the usual one, as dendriform algebras are not unitary objects: we have to add a copy of both algebras to their tensor products. We also define the notion of dendriform module over a dendriform algebra. Then the notions of dendriform bialgebra and bidendriform bialgebra become more clear: the coproduct of a dendriform bialgebra A has to be a morphism of dendriform algebras from A to $A\overline{\otimes}A$; the left and right coproducts of a bidendriform bialgebra A have to be morphism of dendriform modules from A to $A\overline{\otimes}A$. Now, as the dendriform algebra $\mathcal{H}^\mathcal{D}$ is freely generated by the elements $\bullet_d$ as a dendriform algebra, it is possible to define a unique structure of bidendriform bialgebra over $\mathcal{H}^\mathcal{D}$ (or, more exactly, on its augmentation ideal $\mathcal{A}^\mathcal{D}$) by $\Delta_\prec(\bullet_d) = \Delta_\succ(\bullet_d) = 0$ (theorem 31).

Let us now study more precisely the notion of bidendriform bialgebra. For a given bidendriform bialgebra A, we consider primitive elements of A, that is to say elements which vanish under both $\Delta_\prec$ and $\Delta_\succ$. We say that A is connected if, for every element a, the various iterated coproducts all vanish on a for a great enough rank. Then, if A is connected, we prove that A is generated as a dendriform algebra by its primitive elements (theorem 21). This theorem can be seen as a bidendriform version of the Milnor-Moore theorem ([22]), which says that a cocommutative, connected Hopf algebra is generated by its primitive elements (in characteristic zero).

To precise this result, we consider the bidendriform bialgebra $\mathcal{A}^\mathcal{D}$. We prove that its space of primitive elements is reduced to the space of its generators: the elements $\bullet_d$. In other terms, the triple $(coDend, Dend, Vect)$ is a good triple of operads, with the language of [14]. This implies that every connected bidendriform bialgebra is freely generated by its primitive elements, so is isomorphic to a $\mathcal{H}^\mathcal{D}$ for a well chosen $\mathcal{D}$ (theorem 39). A similar result is proved in [13] for preLie algebras.

We apply this rigidity result to **FQSym**: then, for a certain $\mathcal{D}$, **FQSym** is isomorphic to $\mathcal{H}^\mathcal{D}$ as a bidendriform bialgebra, and hence as a Hopf algebra. This allows us to answer a conjecture of [5]: if the characteristic of the base field is zero, then the Lie algebra of primitive elements of **FQSym** is free (corollary 40), as we already proved this result for $\mathcal{H}^\mathcal{D}$ in [8].

Thanks. I am grateful to Ralf Holtkamp for suggestions which greatly improve section 3.

Notations. K is a commutative field of any characteristic. If V is a K-vector space, we denote by $\mathcal{L}(V)$ the space of K-linear endomorphisms of V. If V and W are K-vector spaces, we denote by $\mathcal{L}(V, W)$ the space of K-linear applications from V to W.

1 Dendriform and codendriform bialgebras

1.1 Dendriform algebras and coalgebras

Definition 1 (See [1, 15, 16, 17, 19]). A dendriform algebra is a family $(A, \prec, \succ)$ such that:

1. A is a K-vector space and:

$$\prec: \left\{ \begin{array}{rcl} A \otimes A & \longrightarrow & A \\ a \otimes b & \longrightarrow & a \prec b, \end{array} \right. \quad \Bigg| \quad \succ: \left\{ \begin{array}{rcl} A \otimes A & \longrightarrow & A \\ a \otimes b & \longrightarrow & a \succ b. \end{array} \right.$$

2. For all $a, b, c \in A$:

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c), \tag{1}$$
$$(a \succ b) \prec c = a \succ (b \prec c), \tag{2}$$
$$(a \prec b + a \succ b) \succ c = a \succ (b \succ c). \tag{3}$$

Remark. If A is a dendriform algebra, we put:

$$m : \left\{ \begin{array}{rcl} A \otimes A & \longrightarrow & A \\ a \otimes b & \longrightarrow & ab = a \prec b + a \succ b. \end{array} \right.$$

Then $(1) + (2) + (3)$ is equivalent to the fact that m is associative. Hence, a dendriform algebra is a special (non unitary) associative algebra.

By duality, we obtain the notion of dendriform coalgebra:

Definition 2 A dendriform coalgebra is a family $(C, \Delta_\prec, \Delta_\succ)$ such that:

1. C is a K-vector space and:

$$\Delta_\prec : \left\{ \begin{array}{rcl} C & \longrightarrow & C \otimes C \\ a & \longrightarrow & \Delta_\prec(a) = a'_\prec \otimes a''_\prec, \end{array} \right. \quad \Bigg| \quad \Delta_\succ : \left\{ \begin{array}{rcl} C & \longrightarrow & C \otimes C \\ a & \longrightarrow & \Delta_\succ(a) = a'_\succ \otimes a''_\succ. \end{array} \right.$$

2. for all $a \in C$:

$$(\Delta_\prec \otimes Id) \circ \Delta_\prec(a) = (Id \otimes \Delta_\prec + Id \otimes \Delta_\succ) \circ \Delta_\prec(a), \tag{4}$$
$$(\Delta_\succ \otimes Id) \circ \Delta_\prec(a) = (Id \otimes \Delta_\prec) \circ \Delta_\succ(a), \tag{5}$$
$$(\Delta_\prec \otimes Id + \Delta_\succ \otimes Id) \circ \Delta_\succ(a) = (Id \otimes \Delta_\succ) \circ \Delta_\succ(a). \tag{6}$$

Remarks.

1. If C is a dendriform coalgebra, we put:

$$\tilde{\Delta} : \left\{ \begin{array}{rcl} C & \longrightarrow & C \otimes C \\ a & \longrightarrow & \tilde{\Delta}(a) = \Delta_\prec(a) + \Delta_\succ(a) = a' \otimes a''. \end{array} \right.$$

   Then $(4) + (5) + (6)$ is equivalent to the fact that $\tilde{\Delta}$ is coassociative. Hence, a dendriform coalgebra is a special (non counitary) coassociative coalgebra.

2. If $A = \bigoplus A_n$ is a $\mathbb{N}$-graded dendriform algebra, such that its homogeneous parts are finite-dimensional, then $(A^{*g}, \prec^*, \succ^*)$ is a $\mathbb{N}$-graded dendriform coalgebra (A^{*g} is the graded dual of A, that is to say $A^{*g} = \bigoplus A_n^* \subseteq A^*$).

3. In the same way, if C is a $\mathbb{N}$-graded dendriform coalgebra, such that its homogeneous parts are finite-dimensional, then $(C^{*g}, \Delta_\prec^*, \Delta_\succ^*)$ is a $\mathbb{N}$-graded dendriform algebra. (In fact, for any dendriform coalgebra C, the whole linear dual C^* is a dendriform algebra).

Definition 3 Let A be a dendriform coalgebra. We put:

$$\begin{array}{rcl} Prim_{coAss}(A) &=& \{a \in A \,/\, \tilde{\Delta}(a) = 0\},\\ Prim_{\prec}(A) &=& \{a \in A \,/\, \Delta_{\prec}(a) = 0\},\\ Prim_{\succ}(A) &=& \{a \in A \,/\, \Delta_{\succ}(a) = 0\},\\ Prim_{coDend}(A) &=& Prim_{\succ}(A) \cap Prim_{\prec}(A) \;=\; \{a \in A \,/\, \tilde{\Delta}(a) = \Delta_{\prec}(a) = \Delta_{\succ}(a) = 0\}. \end{array}$$

1.2 Dendriform and codendriform bialgebras

Definition 4 (See [17, 18, 25, 26]). A dendriform bialgebra is a family $(A, \prec, \succ, \tilde{\Delta})$ such that:

1. $(A, \prec, \succ)$ is a dendriform algebra.
2. $(A, \tilde{\Delta})$ is a coassociative (non counitary) coalgebra.
3. The following compatibilities are satisfied: for all $a, b \in A$,

$$\tilde{\Delta}(a \prec b) = a'b' \otimes a'' \prec b'' + a' \otimes a'' \prec b + a'b \otimes a'' + b' \otimes a \prec b'' + b \otimes a, \quad (7)$$
$$\tilde{\Delta}(a \succ b) = a'b' \otimes a'' \succ b'' + a' \otimes a'' \succ b + ab' \otimes b'' + b' \otimes a \succ b'' + a \otimes b. \quad (8)$$

Remarks.

1. (7) + (8) is equivalent to: for all $a \in A$,

$$\tilde{\Delta}(ab) = a'b' \otimes a''b'' + a' \otimes a''b + ab' \otimes b'' + a'b \otimes a'' + b' \otimes ab'' + a \otimes b + b \otimes a. \quad (9)$$

   If A is a dendriform bialgebra, we put $\overline{A} = A \oplus K$, which is given a structure of associative algebra and coassociative coalgebra in the following way: for all $a, b \in A$,

$$1.a = a, \qquad a.1 = a, \qquad 1.1 = 1, \qquad a.b = ab \text{ (product in } A);$$
$$\Delta(1) = 1 \otimes 1, \qquad \Delta(a) = 1 \otimes a + a \otimes 1 + \tilde{\Delta}(a).$$

   Then (9) means that $\overline{A}$ is a bialgebra. A dendriform bialgebra is then the augmentation ideal of a special bialgebra.

2. Another interpretation of (7) and (8) will be given in section 3.

By duality, we define the notion of codendriform bialgebra:

Definition 5 A codendriform bialgebra is a family $(A, m, \Delta_{\prec}, \Delta_{\succ})$ such that:

1. $(A, \Delta_{\prec}, \Delta_{\succ})$ is a dendriform coalgebra.
2. (A, m) is an associative (non unitary) algebra.
3. The following compatibilities are satisfied: for all $a, b \in A$,

$$\Delta_{\succ}(ab) = a'b'_{\succ} \otimes a''b''_{\succ} + a' \otimes a''b + ab'_{\succ} \otimes b''_{\succ} + b'_{\succ} \otimes ab''_{\succ} + a \otimes b, \quad (10)$$
$$\Delta_{\prec}(ab) = a'b'_{\prec} \otimes a''b''_{\prec} + a'b \otimes a'' + ab'_{\prec} \otimes b''_{\prec} + b'_{\prec} \otimes ab''_{\prec} + b \otimes a. \quad (11)$$

Remarks.

1. (10)+(11) is equivalent to (9). Hence, if A is a codendriform bialgebra, as before $\overline{A} = A \oplus K$ is given a structure of bialgebra. A codendriform bialgebra is then the augmentation ideal of a special bialgebra.
2. If A is a $\mathbb{N}$-graded codendriform bialgebra, such that its homogeneous parts are finite-dimensional, then $(A^{*g}, \Delta_{\prec}^*, \Delta_{\succ}^*, m^*)$ is a $\mathbb{N}$-graded dendriform bialgebra.
3. In the same way, if A is a $\mathbb{N}$-graded dendriform bialgebra, such that its homogeneous parts are finite-dimensional, then $(A^{*g}, \tilde{\Delta}^*, \prec^*, \succ^*)$ is a $\mathbb{N}$-graded codendriform bialgebra.

1.3 Free dendriform algebras

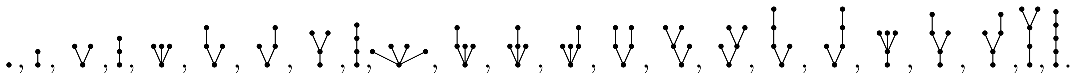
Let us recall here the construction of the Connes-Kreimer Hopf algebra of planar decorated rooted trees (See [7, 8, 9] for more details). It is a non-commutative version of the Connes-Kreimer Hopf algebra of rooted trees for Renormalisation ([4, 10, 11, 12]).

Definition 6

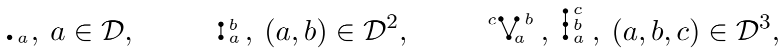
1. A *rooted tree* t is a finite graph, without loops, with a special vertex called *root* of t. A *planar rooted tree* t is a rooted tree with an imbedding in the plane. The *weight* of t is the number of its vertices. the set of planar rooted trees will be denoted by $\mathbb{T}$.
2. Let $\mathcal{D}$ be a nonempty set. A planar rooted tree decorated by $\mathcal{D}$ is a planar tree with an application from the set of its vertices into $\mathcal{D}$. The set of planar rooted trees decorated by $\mathcal{D}$ will be denoted by $\mathbb{T}^{\mathcal{D}}$.

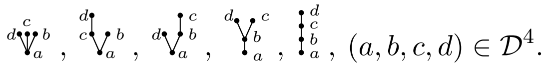
Examples.

1. Planar rooted trees with weight smaller than 5:



2. Planar rooted trees decorated by $\mathcal{D}$ with weight smaller than 4:



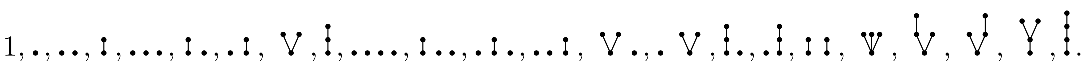


The algebra $\mathcal{H}$ (denoted by $\mathcal{H}_{P,R}$ in [7, 8]) is the free associative (non commutative) K-algebra generated by the elements of $\mathbb{T}$. Monomials in planar rooted trees in this algebra are called planar rooted forests. The set of planar rooted forests will be denoted by $\mathbb{F}$. Note that $\mathbb{F}$ is a basis of $\mathcal{H}$.

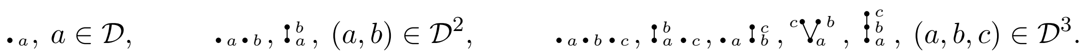
In the same way, if $\mathcal{D}$ is a nonempty set, the algebra $\mathcal{H}^{\mathcal{D}}$ (denoted by $\mathcal{H}^{\mathcal{D}}_{P,R}$ in [7, 8]) is the free associative (non commutative) K-algebra generated by the elements of $\mathbb{T}^{\mathcal{D}}$. Monomials in planar rooted trees decorated by $\mathcal{D}$ in these algebra are called planar rooted forests decorated by $\mathcal{D}$. The set of planar rooted forests decorated by $\mathcal{D}$ will be denoted by $\mathbb{F}^{\mathcal{D}}$. Note that $\mathbb{F}^{\mathcal{D}}$ is a basis of $\mathcal{H}^{\mathcal{D}}$. if $\mathcal{D}$ is reduced to a single element, then $\mathbb{T}^{\mathcal{D}}$ can be identified with $\mathbb{T}$ and $\mathcal{H}^{\mathcal{D}}$ can be identified with $\mathcal{H}$.

Examples.

1. Planar rooted forests of weight smaller than 4:



2. Planar rooted forests decorated by $\mathcal{D}$ of weight smaller than 3:



We now describe the Hopf algebra structure of $\mathcal{H}^{\mathcal{D}}$. Let $t \in \mathbb{T}^{\mathcal{D}}$. An *admissible cut* of t is a nonempty cut such that every path in the tree meets at most one edge which is cut by c. The set of admissible cut of t is denoted by $\mathcal{A}dm(t)$. An admissible cut c of t sends t to a couple $(P^c(t), R^c(t)) \in \mathbb{F}^{\mathcal{D}} \times \mathbb{T}^{\mathcal{D}}$, such that $R^c(t)$ is the connected component of the root of

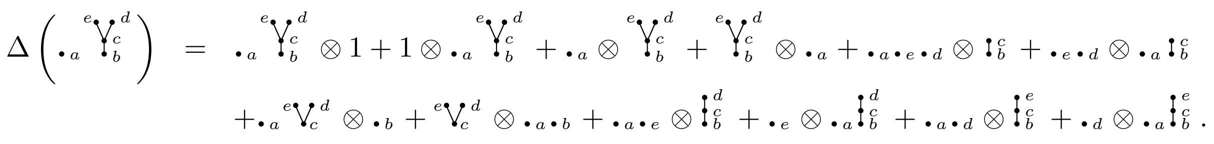
t after the application of c, and $P^c(t)$ is the planar forest of the other connected components (in the same order). Moreover, if c_v is the empty cut of t, we put $P^{c_v}(t) = 1$ et $R^{c_v}(t) = t$. We define the *total cut* of t as a cut c_t such that $P^{c_t}(t) = t$ and $R^{c_t}(t) = 1$. We then put $\mathcal{A}dm_*(t) = \mathcal{A}dm(t) \cup \{c_v, c_t\}$.

We now take $F \in \mathbb{F}^{\mathcal{D}}$, $F \neq 1$. There exists $t_1, \ldots, t_n \in \mathbb{T}^{\mathcal{D}}$, such that $F = t_1 \ldots t_n$. An *admissible cut* of F is a n-uple $(c_1, \ldots, c_n)$ such that $c_i \in \mathcal{A}dm_*(t_i)$ for all i. If all c_i's are empty (resp. total), then c is called the empty cut of F (resp. the total cut of F). The set of admissible cuts of F except the empty and the total cut is denoted by $\mathcal{A}dm(F)$. The set of all admissible cuts of F is denoted by $\mathcal{A}dm_*(F)$. For $c = (c_1, \ldots, c_n) \in \mathcal{A}dm_*(F)$, we put $P^c(F) = P^{c_1}(t_1) \ldots P^{c_n}(t_n)$ and $R^c(F) = R^{c_1}(t_1) \ldots R^{c_n}(t_n)$.

The coproduct $\Delta : \mathcal{H}^{\mathcal{D}} \longrightarrow \mathcal{H}^{\mathcal{D}} \otimes \mathcal{H}^{\mathcal{D}}$ is defined in the following way: for all $F \in \mathbb{F}^{\mathcal{D}}$,

$$\Delta(F) = F \otimes 1 + 1 \otimes F + \sum_{c \in \mathcal{A}dm(F)} P^c(F) \otimes R^c(F) = \sum_{c \in \mathcal{A}dm_*(F)} P^c(F) \otimes R^c(F).$$

Example. If $a, b, c, d, e \in \mathcal{D}$:



The counit ε is given by:

$$\varepsilon : \left\{ \begin{array}{rcl} \mathcal{H}^{\mathcal{D}} & \longrightarrow & K \\ F \in \mathbb{F}^{\mathcal{D}} & \longrightarrow & \delta_{F,1}. \end{array} \right.$$

We proved in [8], see also [9], that $\mathcal{H}^{\mathcal{D}}$ is isomorphic to the free dendriform algebra generated by $\mathcal{D}$, which is described in [18, 25] in terms of planar binary trees. So the augmentation ideal $\mathcal{A}^{\mathcal{D}}$ of $\mathcal{H}^{\mathcal{D}}$ inherits a structure of dendriform algebra, also described in [8] with the help of another basis of $\mathcal{H}^{\mathcal{D}}$, introduced by duality. Hence, $\mathcal{A}^{\mathcal{D}}$ is freely generated by the $\bullet_d$'s, $d \in \mathcal{D}$, as a dendriform algebra. Here is an example of a computation of a product $\prec$ in $\mathcal{A}^{\mathcal{D}}$. For all $x \in \mathcal{A}^{\mathcal{D}}$,

$$\bullet_d \prec x = B_d^+(x),$$

where $B_d^+ : \mathcal{H}^{\mathcal{D}} \longrightarrow \mathcal{H}^{\mathcal{D}}$ is the linear application which send a forest $t_1 \ldots t_n$ to the planar decorated tree obtained by grafting $t_1, \ldots, t_n$ on a common root decorated by d. (This comes from the description of $\prec$ in terms of graftings in [8] and proposition 36 of [7]).

As $\mathcal{H}^{\mathcal{D}}$ is self-dual ([7]), $\mathcal{A}^{\mathcal{D}}$ is given a structure of codendriform bialgebra. The description in [8] of the left and right products in the dual basis of forests allows us to describe this structure with the following definition:

Definition 7 Let $F = t_1 \ldots t_n \in \mathbb{F}^{\mathcal{D}}$, $F \neq 1$. The set $\mathcal{A}dm'_{\prec}(F)$ is the set of cuts $(c_1, \ldots, c_n) \in \mathcal{A}dm(F)$ such that c_n is the total cut of t_n if F is not a single tree, and $\emptyset$ otherwise. The set $\mathcal{A}dm'_{\succ}(F)$ is $\mathcal{A}dm(F) - \mathcal{A}dm'_{\prec}(F)$.

The dendriform coalgebra structure of $\mathcal{A}^{\mathcal{D}}$ is then given in the following way: For all $F \in \mathbb{F}^{\mathcal{D}} - \{1\}$,

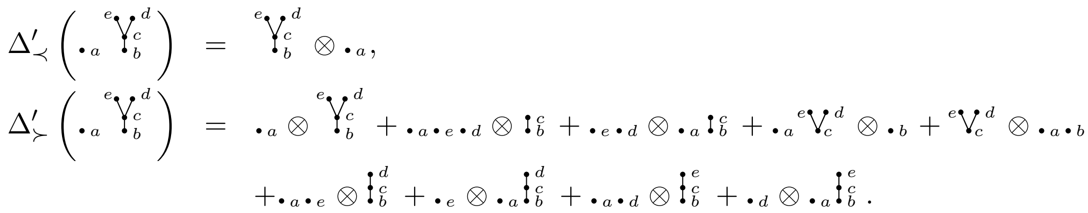
$$\Delta'_{\prec}(F) = \sum_{c \in \mathcal{A}dm_{\prec}(F)} P^c(F) \otimes R^c(F), \qquad \Delta'_{\succ}(F) = \sum_{c \in \mathcal{A}dm_{\succ}(F)} P^c(F) \otimes R^c(F).$$

The product of $\mathcal{A}^{\mathcal{D}}$ is the product induced by the product of $\mathcal{H}^{\mathcal{D}}$.

Examples.

1. If $t \in \mathbb{T}^{\mathcal{D}}$, $\Delta'_{\prec}(t) = 0$.

2. If $a, b, c, d, e \in \mathcal{D}$.



Moreover, $\mathcal{H}^{\mathcal{D}}$ can be graded. A set $\mathcal{D}$ is said to be graded when it is given an application $|.| : \mathcal{D} \longrightarrow \mathbb{N}$. We denote $\mathcal{D}_n = \{d \in \mathcal{D} \;/\; |d| = n\}$. We then put, for all $F \in \mathbb{F}^{\mathcal{D}}$:

$$|F| = \sum_{s \in vert(F)} |\text{decoration of } s|,$$

where $vert(F)$ is the set of vertices of F. Then $\mathcal{H}^{\mathcal{D}}$ is given a graded Hopf algebra structure and $\mathcal{A}^{\mathcal{D}}$ is given a graded (co)dendriform bialgebra structure by putting, for all $F \in \mathbb{F}^{\mathcal{D}}$, F homogeneous of degree $|F|$. When $\mathcal{D}_0$ is empty and $\mathcal{D}_n$ is finite for all n, then the homogeneous parts of $\mathcal{A}^{\mathcal{D}}$ are finite-dimensional and $(\mathcal{A}^{\mathcal{D}})_0 = (0)$. Moreover, if those conditions occur, we have the following result:

Proposition 8 *We consider the following formal series:*

$$D(X) = \sum_{n=1}^{+\infty} card(\mathcal{D}_n) X^n, \qquad R(X) = \sum_{n=0}^{+\infty} dim(\mathcal{H}_n^{\mathcal{D}}) X^n.$$

Then $R(X) = \dfrac{1 - \sqrt{1 - 4D(X)}}{2D(X)}$.

Proof. Similar to the proof of theorem 75 of [6]. $\Box$

Remark. We do not use here the Loday-Ronco presentation of free dendriform algebras with planar trees. It is although possible to work directly with the Loday-Ronco setting.

2 Bidendiform bialgebras

2.1 Definition

We now introduce the notion of bidendriform bialgebra. A bidendriform bialgebra is both a dendriform bialgebra and a codendriform bialgebra, with some compatibilities.

Definition 9 A bidendriform bialgebra is a family $(A, \prec, \succ, \Delta_{\prec}, \Delta_{\succ})$ such that:

1. $(A, \prec, \succ)$ is a dendriform algebra.

2. $(A, \Delta_{\prec}, \Delta_{\succ})$ is a dendriform coalgebra.

3. The following compatibilities are satisfied: for all $a, b \in A$,

$$\begin{aligned}
\Delta_{\succ}(a \succ b) &= a'b'_{\succ} \otimes a'' \succ b''_{\succ} + a' \otimes a'' \succ b + b'_{\succ} \otimes a \succ b''_{\succ} + ab'_{\succ} \otimes b''_{\succ} + a \otimes b, && (12)\\
\Delta_{\succ}(a \prec b) &= a'b'_{\succ} \otimes a'' \prec b''_{\succ} + a' \otimes a'' \prec b + b'_{\succ} \otimes a \prec b''_{\succ}, && (13)\\
\Delta_{\prec}(a \succ b) &= a'b'_{\prec} \otimes a'' \succ b''_{\prec} + ab'_{\prec} \otimes b''_{\prec} + b'_{\prec} \otimes a \succ b''_{\prec}, && (14)\\
\Delta_{\prec}(a \prec b) &= a'b'_{\prec} \otimes a'' \prec b''_{\prec} + a'b \otimes a'' + b'_{\prec} \otimes a \prec b''_{\prec} + b \otimes a. && (15)
\end{aligned}$$

Remarks.

1. $(13)+(15)$ and $(12)+(14)$ are equivalent to (7) and (8), so a bidendriform bialgebra is a special dendriform bialgebra.

2. $(12)+(13)$ and $(14)+(15)$ are equivalent to (10) and (11), so a bidendriform bialgebra is a special codendriform bialgebra.

3. If A is a graded bidendriform bialgebra, such that its homogeneous parts are finite-dimensional, then $(A^{*g}, \Delta_\prec^*, \Delta_\succ^*, \prec^*, \succ^*)$ is also a graded bidendriform bialgebra, as the transposes of (12) and (15) are themselves, and (13) and (14) are transposes from each other.

2.2 Primitive elements

We here give several results which will be useful to prove theorem 21.

First part. Let A be a dendriform coalgebra. We define inductively:

$$\Delta_\prec^0 = Id, \qquad \Delta_\prec^1 = \Delta_\prec, \qquad \Delta_\prec^n = \left(\Delta_\prec \otimes Id^{\otimes(n-1)}\right) \circ \Delta_\prec^{n-1}.$$

For all $n \in \mathbb{N}$, $\Delta_\prec^n : A \longrightarrow A^{\otimes(n+1)}$.

Lemma 10 *For all $n \in \mathbb{N}^*$, $i \in \{2, \ldots, n\}$,*

$$\left(Id^{\otimes(i-1)} \otimes \tilde{\Delta} \otimes Id^{\otimes(n-i)}\right) \circ \Delta_\prec^{n-1} = \Delta_\prec^n.$$

Proof. First, note that by (4):

$$\begin{array}{rcl}((Id \otimes \tilde{\Delta}) \circ \Delta_\prec) \otimes Id^{\otimes(n-i)} &=& ((\Delta_\prec \otimes Id) \circ \Delta_\prec) \otimes Id^{\otimes(n-i)} \\ &=& \left(\Delta_\prec \otimes Id^{\otimes(n-i+1)}\right) \circ \left(\Delta_\prec \otimes Id^{\otimes(n-i)}\right).\end{array}$$

Hence :

$$\begin{array}{rcl}\left(Id^{\otimes(i-1)} \otimes \tilde{\Delta} \otimes Id^{\otimes(n-i)}\right) \circ \Delta_\prec^{n-1} &=& \left(\Delta_\prec \otimes Id^{\otimes(n-1)}\right) \circ \ldots \circ \left(\Delta_\prec \otimes Id^{\otimes(n-i+2)}\right) \\ && \circ \left(((Id \otimes \tilde{\Delta}) \circ \Delta_\prec) \otimes Id^{\otimes(n-i)}\right) \circ \Delta_\prec^{n-i} \\ &=& \left(\Delta_\prec \otimes Id^{\otimes(n-1)}\right) \circ \ldots \circ \left(\Delta_\prec \otimes Id^{\otimes(n-i+2)}\right) \\ && \left(\Delta_\prec \otimes Id^{\otimes(n-i+1)}\right) \circ \left(\Delta_\prec \otimes Id^{\otimes(n-i)}\right) \circ \Delta_\prec^{n-i} \\ &=& \Delta_\prec^n. \ \Box\end{array}$$

Lemma 11 *Let $a \in A$, such that $\Delta_\prec^n(a) = 0$. Then $\Delta_\prec^{n-1}(a) \in Prim_\prec(A) \otimes Prim_{coAss}(A)^{\otimes(n-1)}$.*

Proof. By definition of $\Delta_\prec^n$, $\Delta_\prec^{n-1}(a)$ vanishes under $\Delta_\prec \otimes Id^{\otimes(n-1)}$, so belongs to $Prim_\prec(A) \otimes A^{\otimes(n-1)}$. Moreover, by lemma 10, if $i \geq 2$, $\Delta_\prec^{n-1}(a)$ vanishes under $Id^{\otimes(i-1)} \otimes \tilde{\Delta} \otimes Id^{\otimes(n-i)}$, so belongs to $A^{\otimes(i-1)} \otimes Prim_{coAss}(A) \otimes A^{\otimes(n-i)}$. $\Box$

Suppose now that A is a bidendriform bialgebra. Let $a_1, \ldots, a_n \in A$. We define inductively:

$$\begin{array}{rcl}\omega(a_1) &=& a_1, \\ \omega(a_1, a_2) &=& a_2 \prec a_1, \\ \omega(a_1, \ldots, a_n) &=& a_n \prec \omega(a_1, \ldots, a_{n-1}).\end{array}$$

Lemma 12 *Let $a_1 \in Prim_{\prec}(A)$ and $a_2,\ldots,a_n \in Prim_{coAss}(A)$. Let $k \in \mathbb{N}$. Then:*

$$\Delta_{\prec}^k(\omega(a_1,\ldots,a_n)) = \sum_{1\leq i_1<i_2<\ldots<i_k<n} \omega(a_1,\ldots,a_{i_1}) \otimes \ldots \otimes \omega(a_{i_k+1},\ldots,a_n).$$

In particular:

$$\left\{ \begin{array}{rcl} \Delta_{\prec}^{n-1}(\omega(a_1,\ldots,a_n)) &=& a_1 \otimes \ldots \otimes a_n, \\ \Delta_{\prec}^{k}(\omega(a_1,\ldots,a_n)) &=& 0 \text{ if } k \geq n. \end{array} \right.$$

Proof. By induction on k. It is immediate for $k=0$. Let us show the result for $k=1$ by induction on n. It is obvious for $n=1$. We suppose that $n \geq 2$. As $a_n \in Prim_{coAss}(A)$, we have, by (15):

$$\begin{array}{rcl} \Delta_{\prec}(\omega(a_1,\ldots,a_n)) &=& \omega(a_1,\ldots,a_{n-1})'_{\prec} \otimes a_n \prec \omega(a_1,\ldots,a_{n-1})''_{\prec} + \omega(a_1,\ldots,a_{n-1}) \otimes a_n \\ &=& \displaystyle\sum_{1\leq i<n-2} \omega(a_1,\ldots,a_i) \otimes a_n \prec \omega(a_{i+1},\ldots,a_{n-1}) + \omega(a_1,\ldots,a_{n-1}) \otimes a_n \\ &=& \displaystyle\sum_{1\leq i<n-2} \omega(a_1,\ldots,a_i) \otimes \omega(a_{i+1},\ldots,a_{n-1},a_n) + \omega(a_1,\ldots,a_{n-1}) \otimes a_n \\ &=& \displaystyle\sum_{1\leq i<n-1} \omega(a_1,\ldots,a_i) \otimes \omega(a_{i+1},\ldots,a_n). \end{array}$$

We suppose that the result is true at rank k. Then:

$$\begin{array}{rl} & \Delta_{\prec}^{k+1}(\omega(a_1,\ldots,a_n)) \\ = & \displaystyle\sum_{1\leq i_1<i_2<\ldots<i_k<n} \Delta_{\prec}(\omega(a_1,\ldots,a_{i_1})) \otimes \omega(a_{i_1+1},\ldots a_{i_2}) \otimes \ldots \otimes \omega(a_{i_k+1},\ldots,a_n) \\ = & \displaystyle\sum_{1\leq i_1<i_2<\ldots<i_{k+1}<n} \omega(a_1,\ldots,a_{i_1}) \otimes \ldots \otimes \omega(a_{i_{k+1}+1},\ldots,a_n). \Box \end{array}$$

For all $a \in A$, we put $N_{\prec}(a) = \inf\{n \in \mathbb{N} \,/\, \Delta_{\prec}^{n-1}(a) = 0\} \in \mathbb{N} \cup \{+\infty\}$.

Lemma 13 *Let $a \in A$, such that $N_{\prec}(a)$ is finite. Then a can be written as a linear span of terms $\omega(a_1,\ldots,a_n)$, $n \leq N_{\prec}(a)$, $a_1 \in Prim_{\prec}(A)$, $a_2,\ldots,a_n \in Prim_{coAss}(A)$.*

Proof. By induction on $n = N_{\prec}(a)$. If $n=0$, then $a=0$. If $n=1$, then $a \in Prim_{\prec}(A)$ and the result is obvious. Suppose $n \geq 2$. By lemma 11, we then put:

$$\Delta_{\prec}^{n-1}(a) = \sum a_1 \otimes \ldots \otimes a_n \in Prim_{\prec}(A) \otimes Prim_{coAss}(A)^{\otimes(n-1)}.$$

By lemma 12, $\Delta_{\prec}^{n-1}\left(a - \sum \omega(a_1,\ldots,a_n)\right) = 0$. The induction hypothesis applied to $a - \sum \omega(a_1,\ldots,a_n)$ gives the result. $\Box$

Second part. Suppose that A is a dendriform coalgebra. We define inductively:

$$\tilde{\Delta}^0 = Id, \qquad \tilde{\Delta}^1 = \tilde{\Delta}, \qquad \tilde{\Delta}^n = (\underbrace{Id \otimes \ldots \otimes Id}_{n-1} \otimes \tilde{\Delta}) \circ \tilde{\Delta}^{n-1}.$$

For all $n \in \mathbb{N}$, $\tilde{\Delta}^n : A \longrightarrow A^{\otimes(n+1)}$. Note that $\tilde{\Delta}^n$ could be defined for any coassociative coalgebra $(A, \tilde{\Delta})$.

Lemma 14 *Let $a \in Prim_{\prec}(A)$. Then for all $n \in \mathbb{N}$, $\tilde{\Delta}^n(a) \in A^{\otimes n} \otimes Prim_{\prec}(A)$.*

Proof. It is obvious if $n=0$. Suppose $n\geq 1$. By (5):

$$\begin{array}{rcl}(Id\otimes\Delta_{\prec})\circ\tilde{\Delta} &=& (Id\otimes\Delta_{\prec})\circ\Delta_{\prec}+(Id\otimes\Delta_{\prec})\circ\Delta_{\succ}\\ &=& (Id\otimes\Delta_{\prec})\circ\Delta_{\prec}+(\Delta_{\succ}\otimes Id)\circ\Delta_{\prec}\\ &=& (Id\otimes\Delta_{\prec}+\Delta_{\succ}\otimes Id)\circ\Delta_{\prec}.\end{array}$$

By coassociativity of $\tilde{\Delta}$, we have:

$$\begin{array}{rcl}(Id^{\otimes n}\otimes\Delta_{\prec})\circ\tilde{\Delta}^n(a) &=& (\tilde{\Delta}^{n-1}\otimes Id\otimes Id)\circ(Id\otimes\Delta_{\prec})\circ\tilde{\Delta}(a)\\ &=& (\tilde{\Delta}^{n-1}\otimes Id\otimes Id)\circ(Id\otimes\Delta_{\prec}+\Delta_{\succ}\otimes Id)\circ\Delta_{\prec}(a)\\ &=& 0.\end{array}$$

Hence, $\tilde{\Delta}^n(a)\in A^{\otimes n}\otimes Prim_{\prec}(A)$. $\square$

Lemma 15 *Let $a\in A$, such that $\tilde{\Delta}^n(a)=0$. Then $\tilde{\Delta}^{n-1}(a)\in Prim_{coAss}(A)^{\otimes n}$. Moreover, if $a\in Prim_{\prec}(a)$, then $\tilde{\Delta}^{n-1}(a)\in Prim_{coAss}(A)^{\otimes(n-1)}\otimes Prim_{coDend}(A)$.*

Proof. By coassociativity of $\tilde{\Delta}$, for all $i\in\{1,\ldots,n\}$:

$$(Id^{\otimes(i-1)}\otimes\tilde{\Delta}\otimes Id^{\otimes(n-i)})\circ\tilde{\Delta}^{n-1}(a)=\tilde{\Delta}^n(a)=0,$$

so $\tilde{\Delta}^{n-1}(a)\in Prim_{coAss}(A)^{\otimes n}$. The second assertion comes from lemma 14. $\square$

Suppose now that A is a bidendriform bialgebra. Let $a_1,\ldots,a_n\in A$. We define inductively:

$$\begin{array}{rcl}\omega'(a_1) &=& a_1,\\ \omega'(a_1,a_2) &=& a_1\succ a_2,\\ \omega'(a_1,\ldots,a_n) &=& \omega'(a_1,\ldots,a_{n-1})\succ a_n.\end{array}$$

Lemma 16 *Let $a_1,\ldots,a_n\in Prim_{coAss}(A)$. Let $k\in\mathbb{N}$. Then:*

$$\tilde{\Delta}^k(\omega'(a_1,\ldots,a_n))=\sum_{1\leq i_1<i_2<\ldots<i_k<n}\omega'(a_1,\ldots,a_{i_1})\otimes\ldots\otimes\omega'(a_{i_k+1},\ldots,a_n).$$

In particular:

$$\left\{\begin{array}{rcl}\tilde{\Delta}^{n-1}(\omega'(a_1,\ldots,a_n)) &=& a_1\otimes\ldots\otimes a_n,\\ \tilde{\Delta}^k(\omega'(a_1,\ldots,a_n)) &=& 0 \text{ if } k\geq n.\end{array}\right.$$

Moreover, if $a_n\in Prim_{coDend}(A)$, then $\omega'(a_1,\ldots,a_n)\in Prim_{\prec}(A)$.

Proof. By induction on k. It is immediate if $k=0$. Let us show the result for $k=1$ by induction on n. It is obvious for $n=1$. Suppose $n\geq 2$. As $a_n\in Prim_{coAss}(A)$, by (8):

$$\begin{array}{rcl}\tilde{\Delta}(\omega'(a_1,\ldots,a_n)) &=& \omega'(a_1,\ldots,a_{n-1})'\otimes\omega'(a_1,\ldots,a_{n-1})''\succ a_n+\omega(a_1,\ldots,a_{n-1})\otimes a_n,\\ &=& \displaystyle\sum_{1\leq i<n-2}\omega'(a_1,\ldots,a_i)\otimes\omega'(a_{i+1},\ldots,a_{n-1})\succ a_n+\omega'(a_1,\ldots,a_{n-1})\otimes a_n\\ &=& \displaystyle\sum_{1\leq i<n-2}\omega'(a_1,\ldots,a_i)\otimes\omega'(a_{i+1},\ldots,a_{n-1},a_n)+\omega'(a_1,\ldots,a_{n-1})\otimes a_n\\ &=& \displaystyle\sum_{1\leq i<n-1}\omega'(a_1,\ldots,a_i)\otimes\omega'(a_{i+1},\ldots,a_n).\end{array}$$

Suppose that the result is true at rank k. Then:

$$\begin{array}{rcl} & & \tilde{\Delta}^{k+1}(\omega'(a_1,\ldots,a_n)) \\ &=& \displaystyle\sum_{1\leq i_1<i_2<\ldots<i_k<n} \tilde{\Delta}(\omega'(a_1,\ldots,a_{i_1}))\otimes\omega'(a_{i_1+1},\ldots a_{i_2})\otimes\ldots\otimes\omega'(a_{i_k+1},\ldots,a_n) \\ &=& \displaystyle\sum_{1\leq i_1<i_2<\ldots<i_{k+1}<n} \omega'(a_1,\ldots,a_{i_1})\otimes\ldots\otimes\omega'(a_{i_{k+1}+1},\ldots,a_n). \end{array}$$

Suppose $a_n \in Prim_{\prec}(A)$. By (14), by putting $x=\omega'(a_1,\ldots,a_{n-1})$, $\Delta_{\prec}(\omega'(a_1,\ldots,a_n)) = \Delta_{\prec}(x \succ a_n)=0$. So $\omega'(a_1,\ldots,a_n)\in Prim_{\prec}(A)$. $\Box$

For all $a\in A$, we put $N(a)=\inf\{n\in\mathbb{N} \,/\, \tilde{\Delta}^{n-1}(a)=0\}\in\mathbb{N}\cup\{+\infty\}$.

Lemma 17 *Let $a\in Prim_{\prec}(A)$, such that $N(a)$ is finite. Then a can be written as a linear span of terms $\omega'(a_1,\ldots,a_n)$, $n\leq N(a)$, $a_1,\ldots,a_{n-1}\in Prim_{coAss}(A)$, $a_n\in Prim_{coDend}(A)$.*

Proof. Induction on $n=N(a)$. If $n=0$, then $a=0$. If $n=1$, then $a\in Prim_{coDend}(A) = Prim_{coAss}(A)\cap Prim_{\prec}(A)$. Suppose that $n\geq 2$ and put, by lemma 15:

$$\tilde{\Delta}^{n-1}(a)=\sum a_1\otimes\ldots\otimes a_n \in Prim_{coAss}(A)^{\otimes(n-1)}\otimes Prim_{coDend}(A).$$

We put $a'=a-\sum\omega'(a_1,\ldots,a_n)$. By lemma 16, $a'\in Prim_{\prec}(A)$ and $N(a')<n$. Hence, a' satisfies the induction hypothesis, so the result is true for a. $\Box$

2.3 Connected bidendriform bialgebras

We prove in this paragraph that if A is a connected (definition 18) bidendriform bialgebra, it is generated by its primitive elements (theorem 21).

Let C be a dendriform coalgebra. We define inductively:

$$\begin{array}{rcl} \mathcal{P}_C(0) &=& \{Id_C\}\subseteq\mathcal{L}(C), \\ \mathcal{P}_C(1) &=& \{\Delta_{\prec},\,\Delta_{\succ}\}\subseteq\mathcal{L}(C,C^{\otimes 2}), \\ \mathcal{P}_C(n) &=& \left\{\left(Id^{\otimes(i-1)}\otimes\Delta_{\prec}\otimes Id^{\otimes(n-i)}\right)\circ P \,/\, P\in\mathcal{P}_C(n-1),\ 1\leq i\leq n\right\} \\ && \cup\left\{\left(Id^{\otimes(i-1)}\otimes\Delta_{\succ}\otimes Id^{\otimes(n-i)}\right)\circ P \,/\, P\in\mathcal{P}_C(n-1),\ 1\leq i\leq n\right\}\subseteq\mathcal{L}(C,C^{\otimes(n+1)}). \end{array}$$

Definition 18

1. Let C be a dendriform coalgebra. It is said *connected* if, for all $a\in C$, there exists $n_a\in\mathbb{N}$, such that for all $P\in\mathcal{P}_C(n_a)$, $P(a)=0$.
2. Let C be a connected dendriform coalgebra. For all $a\in C$, we put:

$$deg_p(a)=\inf\{n\in\mathbb{N} \,/\, \forall P\in\mathcal{P}_C(n),\ P(a)=0\}\in\mathbb{N}.$$

3. Let C be a connected dendriform coalgebra. For all $n\in\mathbb{N}$, we put:

$$C^{\leq n}=\{a\in C \,/\, deg_p(a)\leq n\}.$$

Then $C^{\leq 0}=(0)$, $C^{\leq 1}=Prim_{coDend}(C)$, and $(C^{\leq n})_{n\in\mathbb{N}}$ is a increasing filtration.

Remarks.

1. For all $n \in \mathbb{N}$, $\Delta_\prec^n \in \mathcal{P}_C(n)$ et $\tilde{\Delta}^n \in Vect(\mathcal{P}_C(n))$. Hence, if C is connected, then for all $a \in C$, $N_\prec(a)$ and $N(a)$ are finite and smaller than $deg_p(a)$.

2. Let C be a $\mathbb{N}$-graded dendriform coalgebra, such that $C_0 = (0)$ (the homogeneous parts of C may not be finite-dimensional). Then C is connected, as, for all $a \in C_n$, for all $P \in \mathcal{P}(n)$:
$$P(a) \in \bigoplus_{k_1+\ldots+k_{n+1}=n} C_{k_1} \otimes \ldots \otimes C_{k_{n+1}} = (0).$$
Moreover, for all $a \in C$, $deg_p(a) \leq |a|$, where $|a|$ is the degree of a for the gradation of C.

Lemma 19 *Let C be a connected dendriform coalgebra. For all $n \geq 1$:*
$$\begin{array}{rcl}\Delta_\prec(C^{\leq n}) & \subseteq & C^{\leq n-1} \otimes C^{\leq n-1},\\ \Delta_\succ(C^{\leq n}) & \subseteq & C^{\leq n-1} \otimes C^{\leq n-1},\\ \tilde{\Delta}(C^{\leq n}) & \subseteq & C^{\leq n-1} \otimes C^{\leq n-1}.\end{array}$$

Proof. Let $a \in C^{\leq n}$. Let $P \in \mathcal{P}_C(n-1)$. Then $(P \otimes Id) \circ \Delta_\prec \in \mathcal{P}_C(n)$, so:
$$(P \otimes Id) \circ \Delta_\prec(a) = 0.$$
Hence, $\Delta_\prec(a) \in C^{\leq n-1} \otimes C$. In the same way, we obtain $\Delta_\prec(a) \in C \otimes C^{\leq n-1}$ by considering $(Id \otimes P) \circ \Delta_\prec$, so $\Delta_\prec(a) \in C^{\leq n-1} \otimes C^{\leq n-1}$. The procedure is the same for $\Delta_\succ(a)$, and the result for $\tilde{\Delta}(a)$ is obtained by addition. $\Box$

The following lemma is now immediate:

Lemma 20 *Let C be a connected dendriform coalgebra and $k, n \in \mathbb{N}^*$. Then:*
$$\Delta_\prec^k\left(C^{\leq n}\right) \subseteq \left(C^{\leq n-1}\right)^{\otimes(k+1)}, \qquad \tilde{\Delta}^k\left(C^{\leq n}\right) \subseteq \left(C^{\leq n-1}\right)^{\otimes(k+1)}.$$

Theorem 21 *Let A be a connected (as a dendriform coalgebra) bidendriform bialgebra. Then A is generated (as a dendriform algebra) by $Prim_{coDend}(A)$.*

Proof. Let B be the dendriform subalgebra of A generated by $Prim_{coDend}(A)$. Let $a \in A$. We denote $deg_p(a) = n$. Let us show that $a \in B$ by induction on n. If $n = 0$, then $a = 0 \in B$. If $n = 1$, then $a \in Prim_{coDend}(A) \subseteq B$. Suppose $n \geq 2$. As A is connected, by remark 1 after definition 18, $N_\prec(a) = k$ is finite and smaller than n. By lemma 13, we can suppose $a = \omega(a_1, \ldots, a_k)$, $k \leq n$, $a_1 \in Prim_\prec(A)$, $a_2, \ldots, a_k \in Prim_{coAss}(A)$. We have two different cases.

1. If $k \geq 2$, by lemma 20, $\Delta_\prec^{k-1}(\omega(a_1, \ldots, a_k)) = a_1 \otimes \ldots \otimes a_k \in \left(A^{\leq n-1}\right)^{\otimes k}$, so, for all i, $deg_p(a_i) < n$, so $a_i \in B$. Hence, $a \in B$.

2. If $k = 1$, then $a \in Prim_\prec(A)$. As A is connected, by remark 1 after definition 18, $N(a) = l$ is finite ans smaller than n. By lemma 17, we can suppose $a = \omega'(b_1, \ldots, b_l)$, $l \leq n$, $b_1, \ldots, b_{l-1} \in Prim_{coAss}(A)$, $b_l \in Prim_{coDend}(A)$. We have two different cases.

   (a) If $l \geq 2$, by lemma 20, $\tilde{\Delta}^{l-1}(\omega'(b_1, \ldots, b_l)) = b_1 \otimes \ldots \otimes b_l \in \left(A^{\leq n-1}\right)^{\otimes l}$, so, for all i, $deg_p(b_i) < n$, so $b_i \in B$. Hence, $a \in B$.

   (b) If $l = 1$, then $a = b_1 \in Prim_{coDend}(A) \subseteq B$. $\Box$

2.4 Projections on $Prim_{\prec}(A)$ and $Prim_{coDend}(A)$

As in [25, 26], we here define an eulerian idempotent for connected bidendriform bialgebras.

Let A be a bidendriform bialgebra. We put:

$$A^{D2} = Vect(x \prec y,\ x \succ y \,/\, x, y \in A).$$

We define $m_{\prec}^{n} : A^{\otimes n} \longrightarrow A$ inductively:

$$\begin{array}{rcl} m_{\prec}^{1}(a_1) &=& a_1, \\ m_{\prec}^{2}(a_1 \otimes a_2) &=& a_2 \prec a_1, \\ m_{\prec}^{n}(a_1 \otimes \ldots \otimes a_n) &=& m_{\prec}^{n-1}(a_2 \otimes \ldots \otimes a_n) \prec a_1. \end{array}$$

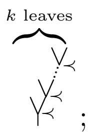
Proposition 22 *Let A be a bidendriform bialgebra such that for all $a \in A$, $N_{\prec}(a)$ is finite. We consider the following application:*

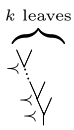
$$T_1 : \left\{ \begin{array}{rcl} A & \longrightarrow & A \\ a & \longrightarrow & \displaystyle\sum_{k=1}^{+\infty} (-1)^{k+1} m_{\prec}^{k} \circ \Delta_{\prec}^{k-1}(a). \end{array} \right.$$

Then T_1 is a projection on $Prim_{\prec}(A)$. Moreover, for all $a \in A$, $T_1(a) = a + A^{D2}$.

Proof. Remark first that T_1 is well defined, as for all $a \in A$, $\Delta_{\prec}^{k-1}(a) = 0$ for k great enough. Let us show that $T_1(a) \in Prim_{\prec}(A)$ for all a. By lemma 13, we can suppose $a = \omega(a_1, \ldots, a_n)$, avec $a_1 \in Prim_{\prec}(A)$, $a_2, \ldots, a_n \in Prim_{coAss}(A)$. If $n = 1$, then $T_1(a) = a_1 \in Prim_{\prec}(A)$. Suppose $n \geq 2$.

We consider the following binary trees: for all $k \in \mathbb{N}$,

$t_k^{(d)} =$ 

$;\ t_k^{(g)} =$ 

.

Then $m_{\prec}^{k}$ and ω can be graphically represented in the following way:

$$\left\{ \begin{array}{rcl} \omega(x_1, \ldots, x_k) &=& t_k^{(d)}.x_k \otimes \ldots \otimes x_1, \\ m_{\prec}^{k}(x_1 \otimes \ldots \otimes x_k) &=& t_k^{(g)}.x_k \otimes \ldots \otimes x_1. \end{array} \right.$$

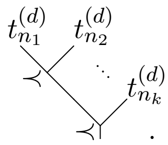
By lemma 12, we have:

$$\Delta_{\prec}^{k-1}(\omega(a_1, \ldots, a_n)) = \sum_{n_1 + \ldots + n_k = n} t_{n_1}^{(d)}.(a_{n_1} \otimes \ldots \otimes a_1) \otimes \ldots \otimes t_{n_k}^{(d)}.(a_n \otimes \ldots \otimes a_{n_1 + \ldots + n_{k-1} + 1}).$$

(The n_i's are positive, non zero integers). Then:

$$T_1(\omega(a_1, \ldots, a_n)) = \sum_{k=1}^{+\infty} \sum_{n_1 + \ldots + n_k = n} (-1)^{k+1} t_{n_1, \ldots, n_k}.(a_n \otimes \ldots \otimes a_1),$$

where the n_i's are positive, non zero integers and $t_{n_1, \ldots, n_k}$ is the following tree:



.

As $t_{1,n_2,\ldots,n_k} = t_{1+n_2,\ldots,n_k}$, we obtain:

$$\begin{array}{rcl} T_1(\omega(a_1,\ldots,a_n)) &=& \displaystyle\sum_{k=1}^{+\infty} \sum_{\substack{n_1+\ldots+n_k=n,\\ n_1\geq 2}} (-1)^{k+1} t_{n_1,\ldots,n_k}.(a_n\otimes\ldots\otimes a_1)\\ && \displaystyle +\sum_{k=1}^{+\infty} \sum_{n_1+\ldots+n_k=n-1} (-1)^{k+2} t_{1,n_1,\ldots,n_k}.(a_n\otimes\ldots\otimes a_1)\\ &=& \displaystyle\sum_{k=1}^{+\infty} \sum_{\substack{n_1+\ldots+n_k=n,\\ n_1\geq 2}} (-1)^{k+1} t_{n_1,\ldots,n_k}.(a_n\otimes\ldots\otimes a_1)\\ && \displaystyle -\sum_{k=1}^{+\infty} \sum_{\substack{n'_1+\ldots+n_k=n,\\ n'_1\geq 2}} (-1)^{k+1} t_{n'_1,\ldots,n_k}.(a_n\otimes\ldots\otimes a_1) \text{ (where } n'_1=1+n_1)\\ &=& 0. \end{array}$$

Hence, $T_1(a) \in Prim_\prec(A)$ for all $a \in A$. Moreover, if $a \in Prim_\prec(A)$, $T_1(a) = a$. So T_1 is a projection on $Prim_\prec(A)$. Finally, for all $a \in A$, we have:

$$T_1(a) = a + \sum_{k=2}^{+\infty}(-1)^{k+1} m_\prec^k \circ \Delta_\prec^{k-1}(a) = a + A^{D2}. \square$$

We define $m_\succ^n : A^{\otimes n} \longrightarrow A$ inductively:

$$\begin{array}{rcl} m_\succ^1(a_1) &=& a_1,\\ m_\succ^2(a_1\otimes a_2) &=& a_1 \succ a_2,\\ m_\succ^n(a_1\otimes\ldots\otimes a_n) &=& a_1 \succ m_\succ^{n-1}(a_2\otimes\ldots\otimes a_n). \end{array}$$

With lemmas 14 and 16, we can show the following proposition in the same way as proposition 22:

Proposition 23 *Let A be a bidendriform bialgebra such that for all $a \in Prim_\prec(A)$, $N(a)$ is finite. We consider the following application:*

$$T_2 : \left\{ \begin{array}{rcl} Prim_\prec(A) & \longrightarrow & A\\ a & \longrightarrow & \displaystyle\sum_{k=1}^{+\infty}(-1)^{k+1} m_\succ^k \circ \tilde\Delta^{k-1}(a). \end{array}\right.$$

Then T_2 is a projector from $Prim_\prec(A)$ into $Prim_{coDend}(A)$. Moreover, for all $a \in A$, $T_2(a) = a + A^{D2}$.

Corollary 24 *Let A be a bidendriform bialgebra such that for all $a \in A$, $N_\prec(a)$ is finite and for all $A \in Prim_\prec(A)$, $N(a)$ is finite (for example, A is connected). We consider the application $T = T_2 \circ T_1 : A \longrightarrow A$. Then T is a projection from A into $Prim_{coDend}(A)$. Moreover, for all $a \in A$, $T(a) = a + A^{D2}$. So $A = Prim_{coDend}(A) + A^{D2}$.*

Proof. Immediate, by composition. $\square$

Corollary 25 *Let A be a $\mathbb{N}$-graded bidendriform bialgebra, such that the homogeneous part of A are finite-dimensional and $A_0 = (0)$. Then $A = Prim_{coDend}(A) \oplus A^{D2}$.*

Proof. Then A is connected (remark 2 after definition 18). By corollary 24, $A = Prim_{coDend}(A) + A^{D2}$. In the same way, the bidendriform bialgebra A^{*g} is connected. We then have $A^{*g} = Prim_{coDend}(A^{*g}) + (A^{*g})^{D2}$. By taking the orthogonal:

$$(0) = (A^{*g})^\perp = Prim_{coDend}(A^{*g})^\perp \cap ((A^{*g})^{D2})^\perp = A^{D2} \cap Prim_{coDend}(A).$$

So $A = Prim_{coDend}(A) \oplus A^{D2}$. $\square$

3 Tensor product and dendriform modules

3.1 Tensor product of dendriform algebras

We here recall how the category of dendriform algebras can be given a structure of tensor category (see [17]). As dendriform algebras are not objects with unit, we have to extend the usual tensor product in order to obtain a copy of A and B in the tensor product of A and B.

Definition 26 Let A, B be two vector spaces. Then:

$$A\overline{\otimes}B = (A \otimes B) \oplus (K \otimes B) \oplus (A \otimes K).$$

Let A be a dendriform algebra. We extend $\prec, \succ: A \otimes A \longrightarrow A$ in applications $\prec, \succ: A\overline{\otimes}A \longrightarrow A$ in the following way : for all $a \in A$,

$$a \prec 1 = a, \qquad a \succ 1 = 0, \qquad 1 \prec a = 0, \qquad 1 \succ a = a.$$

Note that (1)-(3) are now satisfied on $A\overline{\otimes}A\overline{\otimes}A$. Moreover, the product of A is now defined from $A\overline{\otimes}A$ to A. We extend it in an application from $(A \oplus K) \otimes (A \oplus K) = (A\overline{\otimes}A) \oplus (K \otimes K)$ to $A \oplus K$ by putting $1.1 = 1$.

Proposition 27 *Let A, B be two dendriform algebras. Then $A\overline{\otimes}B$ is given a structure of dendriform algebra in the following way: for all $a, a' \in A \cup K$, $b, b' \in B \cup K$,*

$$\begin{array}{rcl}(a \otimes b) \prec (a' \otimes b') &=& a.a' \otimes b \prec b' \text{ if } b \text{ or } b' \in B,\\ (a \otimes 1) \prec (a' \otimes 1) &=& a \prec a' \otimes 1\,;\\ (a \otimes b) \succ (a' \otimes b') &=& a.a' \otimes b \succ b' \text{ if } b \text{ or } b' \in B,\\ (a \otimes 1) \succ (a' \otimes 1) &=& a \succ a' \otimes 1.\end{array}$$

Proof. see [17]. □

Remark. $A \otimes K$ is a dendriform subalgebra of $A\overline{\otimes}B$ which is isomorphic to A and $K \otimes B$ is a dendriform subalgebra of $A\overline{\otimes}B$ which is isomorphic to B.

Proposition 28 1. *Let A, B, C be unitary dendriform algebras. Then the following application is an isomorphism of dendriform algebras:*

$$\left\{\begin{array}{rcl}(A\overline{\otimes}B)\overline{\otimes}C &\longrightarrow& A\overline{\otimes}(B\overline{\otimes}C)\\ (a \otimes b) \otimes c &\longrightarrow& a \otimes (b \otimes c).\end{array}\right.$$

2. *Let A, A', B, B' be unitary dendriform algebras and $\phi : A \longrightarrow A'$, $\psi : B \longrightarrow B'$ be morphisms of unitary dendriform algebras. We then define:*

$$\phi\overline{\otimes}\psi : \left\{\begin{array}{rcl}A\overline{\otimes}B &\longrightarrow& A'\overline{\otimes}B'\\ a \otimes b &\longrightarrow& \phi(a) \otimes \psi(b),\\ a \otimes 1 &\longrightarrow& \phi(a) \otimes 1,\\ 1 \otimes b &\longrightarrow& 1 \otimes \psi(b),\end{array}\right.$$

for all $a \in A$, $b \in B$. Then $\phi\overline{\otimes}\psi$ is a morphism of dendriform algebras.

In other terms, the category of unitary dendriform algebras is a tensor category with $\overline{\otimes}$.

Proof. Direct computations. □

We can now reformulate the axioms of bidendriform bialgebras. Let $(A, \prec, \succ)$ be a dendriform algebra and let $\Delta_\prec, \Delta_\succ : A \longrightarrow A \otimes A$. We put $\tilde{\Delta} = \Delta_\prec + \Delta_\succ$. We then define:

$$\overline{\Delta}_\prec : \left\{ \begin{array}{rcl} A & \longrightarrow & A \overline{\otimes} A \\ a & \longrightarrow & \Delta_\prec(a) + a \otimes 1, \end{array} \right.$$

$$\overline{\Delta}_\succ : \left\{ \begin{array}{rcl} A & \longrightarrow & A \overline{\otimes} A \\ a & \longrightarrow & \Delta_\succ(a) + 1 \otimes a, \end{array} \right. \tag{16}$$

$$\Delta : \left\{ \begin{array}{rcl} A & \longrightarrow & A \overline{\otimes} A \\ a & \longrightarrow & \tilde{\Delta}(a) + a \otimes 1 + 1 \otimes a. \end{array} \right.$$

We extend these applications on $A \oplus K$ to $A \overline{\otimes} A$ by putting $\overline{\Delta}_\prec(1) = \overline{\Delta}_\succ(1) = 0$ and $\Delta(1) = 1 \otimes 1$. Note that $\Delta = \overline{\Delta}_\prec + \overline{\Delta}_\succ$ on A. We can easily prove the following assertions:

1. (4)-(6) for $\Delta_\prec$ and $\Delta_\succ$ are equivalent to (4)-(6) for $\overline{\Delta}_\prec$ and $\overline{\Delta}_\succ$.
2. (12)-(15) is equivalent to: for all $a, b \in A$,
$$\begin{array}{rclrcl} \overline{\Delta}_\succ(a \succ b) & = & \Delta(a) \succ \overline{\Delta}_\succ(b), & \overline{\Delta}_\prec(a \succ b) & = & \Delta(a) \succ \overline{\Delta}_\prec(b), \\ \overline{\Delta}_\succ(a \prec b) & = & \Delta(a) \prec \overline{\Delta}_\succ(b), & \overline{\Delta}_\prec(a \prec b) & = & \Delta(a) \prec \overline{\Delta}_\prec(b). \end{array}$$
3. (8)-(7) is equivalent to: for all $a, b \in A$,
$$\Delta(a \succ b) = \Delta(a) \succ \Delta(b), \qquad \Delta(a \prec b) = \Delta(a) \prec \Delta(b).$$
In other terms, (8)-(7) is equivalent to $\Delta : A \longrightarrow A \overline{\otimes} A$ is a morphism of dendriform algebras.

3.2 Dendriform modules

(See for example [21] for complements on operads). We denote by $\mathcal{D}end = (\mathcal{D}end(n))_{n \in \mathbb{N}^*}$ the operad of dendriform algebras (we consider here non-Σ-operads, that is to say there is no action of the symmetric groups). In other words, $\mathcal{D}end$ is the operad generated by $\prec$ and $\succ \in \mathcal{D}end(2)$ and the following relations:

$$\begin{array}{rcl} \prec \circ (\prec, I) & = & \prec \circ (I, \prec + \succ), \\ \prec \circ (\succ, I) & = & \succ \circ (I, \prec), \\ \succ \circ (\prec + \succ, I) & = & \succ \circ (I, \succ). \end{array}$$

Let A be a dendriform algebra. A dendriform module over A is a vector space M together with applications, for all $n \in \mathbb{N}^*$:

$$\left\{ \begin{array}{rcl} \mathcal{D}end(n) \otimes A^{\otimes(n-1)} \otimes M & \longrightarrow & M \\ p \otimes (a_1 \otimes \ldots \otimes a_{n-1}) \otimes m & \longrightarrow & p.(a_1 \otimes \ldots \otimes a_{n-1} \otimes m), \end{array} \right.$$

satisfying the same associativity relations and unit relation as those for $\mathcal{D}end$-algebras. In other terms, a dendriform module over A is a vector space M together with two applications:

$$\dashv : \left\{ \begin{array}{rcl} A \otimes M & \longrightarrow & M \\ a \otimes m & \longrightarrow & a \dashv m, \end{array} \right. \qquad \vdash : \left\{ \begin{array}{rcl} A \otimes M & \longrightarrow & M \\ a \otimes m & \longrightarrow & a \vdash m, \end{array} \right.$$

with the following compatibilities: for all $a, b \in A$, $m \in M$,

$$(a \prec b) \dashv m = a \dashv (b \dashv m + b \vdash m), \tag{17}$$

$$(a \succ b) \dashv m = a \vdash (b \dashv m), \tag{18}$$

$$(a \prec b + a \succ b) \vdash m = a \vdash (b \vdash m). \tag{19}$$

(We have $a \dashv m = \prec .(a \otimes m)$ and $a \vdash m = \succ .(a \otimes m)$).

Remarks.

1. If M is a dendriform module over A, then $\dashv + \vdash$ gives M a structure of left module over the associative algebra $(A, \prec + \succ)$ by (17)+(18)+(19). This action will be denoted by $a.m = a \dashv m + a \vdash m$.
2. A is a dendriform module over itself with $\dashv = \prec$ and $\vdash = \succ$.
3. Suppose that A is a dendriform bialgebra. Then, as Δ is a morphism of dendriform algebras, $A\overline{\otimes}A$ is given a structure of a dendriform module over A with, for all $a \in A$, $\sum b \otimes c \in A\overline{\otimes}A$:
$$a \dashv \left(\sum b \otimes c\right) = \sum \Delta(a) \prec (b \otimes c), \qquad a \vdash \left(\sum b \otimes c\right) = \sum \Delta(a) \succ (b \otimes c).$$
Moreover, Δ is a morphism of dendriform modules.
4. Let $(A, \prec, \succ, \tilde{\Delta})$ be a dendriform bialgebra and let $\Delta_\prec, \Delta_\succ : A \longrightarrow A \otimes A$. Observe that (12)-(15) is equivalent to: $\overline{\Delta}_\prec, \overline{\Delta}_\succ$ are morphisms of dendriform modules.

Proposition 29 *Let A be a free dendriform algebra generated by a subspace V and let M be a dendriform module over A. For any linear application $\phi : V \longrightarrow M$, there exists a unique morphism of dendriform modules $\Phi : A \longrightarrow M$, such that $\Phi_{|V} = \phi$.*

Proof.
Unicity. Because V generates A as a dendriform algebra, and hence as a dendriform module.
Existence. As A is freely generated by V, we can suppose that $A = \bigoplus_{n\geq 1} \mathcal{D}end(n) \otimes V^{\otimes n}$. We then define:
$$\Phi : \left\{ \begin{array}{rcl} A & \longrightarrow & M \\ p \otimes (v_1 \otimes \ldots \otimes v_n) & \longrightarrow & p.(v_1 \otimes \ldots \otimes v_{n-1} \otimes \phi(v_n)) \end{array} \right.$$
for all $p \in \mathcal{D}end(n)$, $v_1, \ldots, v_n \in V$. Then Φ fits the asked conditions. $\Box$

Proposition 30 *Let $(A, \prec, \succ, \tilde{\Delta})$ be a dendriform bialgebra, with applications $\Delta_\prec, \Delta_\succ : A \longrightarrow A \otimes A$. We extend them in applications $A \oplus K \longrightarrow A\overline{\otimes}A$ as in (16). We suppose the following conditions:*

1. *$(A, \Delta_\prec, \Delta_\succ)$ satisfies relations (4), (5) and (6) on a set of generators of the dendriform algebra A. Moreover, $\Delta_\prec + \Delta_\succ = \tilde{\Delta}$ on this set of generators.*
2. *$\Delta_\prec$ and $\Delta_\succ$ are morphisms of A-dendriform modules.*

Then A is a bidendriform bialgebra.

Proof. Let us first show relation (4). We put $X = \{a \in A \,/\, a \text{ satisfies } (4))\}$. We have:
$$\begin{array}{rcl} X & = & Ker\left((\Delta_\prec \otimes Id) \circ \Delta_\prec - (Id \otimes \Delta_\prec + Id \otimes \Delta_\succ) \circ \Delta_\prec\right) \\ & = & Ker\left((\overline{\Delta}_\prec \overline{\otimes} Id) \circ \overline{\Delta}_\prec - (Id \overline{\otimes} \overline{\Delta}_\prec + Id \overline{\otimes} \overline{\Delta}_\succ) \circ \overline{\Delta}_\prec\right), \end{array}$$
which is the kernel of a certain morphism of dendriform modules from A into $A\overline{\otimes}A\overline{\otimes}A$ by hypothesis 2. So X is a dendriform submodule of A, hence a dendriform subalgebra of A. As it contains a set of generators by hypothesis 1, it is A. We can prove (5), (6), and the fact that $\tilde{\Delta} = \Delta_\prec + \Delta_\succ$ on the whole A in the same way. As the hypothesis 2 is a reformulation of axioms (12)-(15), A is a bidendriform bialgebra. $\Box$

3.3 Bidendriform structure on $\mathcal{A}^{\mathcal{D}}$

Unfortunately, $(\mathcal{A}^{\mathcal{D}}, \prec, \succ, \Delta'_{\prec}, \Delta'_{\succ})$ is not a bidendriform bialgebra. For example, for $a = b = \bullet_d$:

$$a \prec b = \begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}{}^{d}_{d}, \qquad \Delta'_{\prec}(a \prec b) = 0,$$

whereas $a'b'_{\prec} \otimes a'' \prec b''_{\prec} + a'b \otimes a'' + b'_{\prec} \otimes a \prec b''_{\prec} + b \otimes a = \bullet_d \otimes \bullet_d$. So (15) is not satisfied.

Theorem 31 *There is a unique structure of bidendriform bialgebra (with the already known dendriform bialgebra structure) on $\mathcal{A}^{\mathcal{D}}$ such that for all $d \in \mathcal{D}$, $\Delta_{\prec}(\bullet_d) = \Delta_{\succ}(\bullet_d) = 0$. Hence, $(\mathcal{A}^{\mathcal{D}}, \prec, \succ, \Delta_{\prec}, \Delta_{\succ})$ is a bidendriform bialgebra, which induces the structure of Hopf algebra of $\mathcal{H}^{\mathcal{D}}$ already described.*

Proof. We use the notations of proposition 30.

Unicity. As the $\bullet_d$'s generate $\mathcal{A}^{\mathcal{D}}$, there is at most one way to extend $\overline{\Delta}_{\prec}$ and $\overline{\Delta}_{\succ}$ to A as morphisms of dendriform modules, with notations of proposition 30.

Existence. By proposition 29, we can extend $\overline{\Delta}_{\prec}$ and $\overline{\Delta}_{\succ}$ to A as morphisms of dendriform modules. So the second condition of proposition 30 is satisfied. The first one is trivially satisfied on the set of generators $\{\bullet_d \ / \ d \in \mathcal{D}\}$. $\Box$

When $\mathcal{D}$ is a graded set, this structure obviously respects the gradation of $\mathcal{A}^{\mathcal{D}}$:

Corollary 32 *If $\mathcal{D}$ is a graded set, then $\mathcal{A}^{\mathcal{D}}$ is a graded bidendriform bialgebra.*

Proposition 33 *Let $F \in \mathbb{F}^{D}$, $F \neq 1$. We consider the set $\mathcal{A}dm_{\prec}(F)$ of admissible cuts of F satisfying one of these two conditions:*

1. *c cuts one of the edges which are on the path from the root of the last tree of F to the leave which is at most on the right of F.*
2. *c cuts totally the last tree of F if F is not a single tree.*

Then:

$$\Delta_{\prec}(F) \;=\; \sum_{c \in \mathcal{A}dm_{\prec}(F)} P^c(F) \otimes R^c(F). \tag{20}$$

Prove. We denote by $F'_{\ll} \otimes F''_{\ll}$ the second member of (20). Let us prove the result by induction on $n = weight(F)$. If $n = 1$, then $F = \bullet_d$ and the result is obvious. If $n \geq 2$, we have two possible cases:

1. $F = GH$, with $weight(G), weight(H) < n$. By a study of $\mathcal{A}dm_{\prec}(F)$, we easily have:

$$\begin{aligned} F'_{\ll} \otimes F''_{\ll} &= G'H'_{\ll} \otimes G''H''_{\ll} + G'H \otimes G'' + GH'_{\ll} \otimes H''_{\ll} + GH'_{\ll} \otimes H''_{\ll} + H \otimes G \\ &= G'H'_{\prec} \otimes G''H''_{\prec} + G'H \otimes G'' + GH'_{\prec} \otimes H''_{\prec} + GH'_{\prec} \otimes H''_{\prec} + H \otimes G, \end{aligned}$$

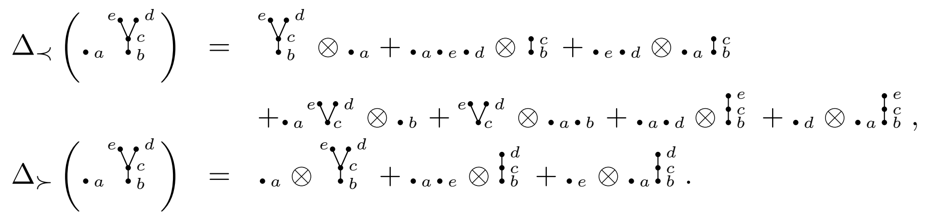
   by the induction hypothesis on H. By (11), this is equal to $\Delta_{\prec}(F)$.

2. $F = B_d^+(G)$, $weight(G) = n - 1$. By a study of $\mathcal{A}dm_{\prec}(G)$, we easily have:

$$\begin{aligned} F'_{\ll} \otimes F''_{\ll} &= G'_{\ll} \otimes B_d^+(G''_{\ll}) + G \otimes \bullet_d \\ &= G'_{\prec} \otimes B_d^+(G''_{\prec}) + G \otimes \bullet_d \\ &= G'_{\prec} \otimes \bullet_d \prec G''_{\prec} + G \otimes \bullet_d, \end{aligned}$$

   by the induction hypothesis on H. As $F = \bullet_d \prec G$, by (15) for $a = \bullet_d$, this is equal to $\Delta_{\prec}(F)$. $\Box$

Example.



3.4 Primitive elements of $\mathcal{A}^{\mathcal{D}}$ and universal property

We here prove that a connected bidendriform bialgebra is isomorphic to $\mathcal{A}^{\mathcal{D}}$ for a certain set $\mathcal{D}$.

Proposition 34 *$Prim_{coDend}(\mathcal{A}^{\mathcal{D}}) = Vect(\bullet_d \;/\; d \in \mathcal{D})$.*

Proof. In a immediate way, $\bullet_d \in Prim_{coDend}(\mathcal{A}^{\mathcal{D}})$. Let us show the other inclusion.

First case. Suppose that $\mathcal{D}$ is finite. We graduate $\mathcal{D}$ by putting all its elements in degree 1. Then $\mathcal{A}^{\mathcal{D}}$ is graded, with $\mathcal{A}^{\mathcal{D}}_0 = (0)$ and $\mathcal{A}^{\mathcal{D}}_n$ finite-dimensional for all n. By corollary 25, $\mathcal{A}^{\mathcal{D}} = Prim_{coDend}(\mathcal{A}^{\mathcal{D}}) \oplus (\mathcal{A}^{\mathcal{D}})^{D2}$. As $\mathcal{A}^{\mathcal{D}}$ is freely generated by the $\bullet_d$'s, $Vect(\bullet_d \;/\; d \in \mathcal{D}) \oplus (\mathcal{A}^{\mathcal{D}})^{D2} = \mathcal{A}^{\mathcal{D}}$, and this implies that $Prim_{coDend}(\mathcal{A}^{\mathcal{D}}) \subseteq Vect(\bullet_d \;/\; d \in \mathcal{D})$.

General case. Let $p \in Prim_{coDend}(\mathcal{A}^{\mathcal{D}})$. There exists a finite subset $\mathcal{D}'$ of $\mathcal{D}$, such that $p \in \mathcal{A}^{\mathcal{D}'}$. Then $p \in Vect(\bullet_d \;/\; d \in \mathcal{D}') \subseteq Vect(\bullet_d \;/\; d \in \mathcal{D})$. $\Box$

Theorem 35 *Let A be a bidendriform bialgebra. For all $d \in \mathcal{D}$, let $p_d \in Prim_{coDend}(A)$. There exists a unique morphism of bidendriform bialgebras:*

$$\Psi : \left\{ \begin{array}{rcl} \mathcal{A}^{\mathcal{D}} & \longrightarrow & A \\ \bullet_d & \longrightarrow & p_d. \end{array} \right.$$

Moreover, we have:

1. *If the p_d's are linearly independant, then Ψ is monic.*
2. *if the dendriform coalgebra A is connected and if the family $(p_d)_{d\in\mathcal{D}}$ linearly generates $Prim_{coDend}(A)$, then Ψ is epic.*
3. *if the dendriform coalgebra A is connected and if the family $(p_d)_{d\in\mathcal{D}}$ is a linear basis of $Prim_{coDend}(A)$, then Ψ is an isomorphism.*

Proof. As $\mathcal{A}^{\mathcal{D}}$ is freely generated by the $\bullet_d$'s, Ψ defines a unique morphism of dendriform algebras. As $\bullet_d$ and p_d are both primitive, Ψ is a morphism of bidendriform bialgebras. We now prove the three assertions.

1. We graduate $\mathcal{D}$ by putting all its elements of degree 1. Suppose that $Ker(\Psi) \neq (0)$. Let $x \in Ker(\Psi)$, non-zero, of minimal degree n. So Ψ is monic on $\mathcal{A}^{\mathcal{D}}_{<n} = \mathcal{A}^{\mathcal{D}}_0 \oplus \ldots \oplus \mathcal{A}^{\mathcal{D}}_{n-1}$. Moreover, $\Delta_{\prec}(x) \in \mathcal{A}^{\mathcal{D}}_{<n} \otimes \mathcal{A}^{\mathcal{D}}_{<n}$ and $(\Psi \otimes \Psi) \circ \Delta_{\prec}(x) = \Delta_{\prec}(\Psi(x)) = 0$. By injectivity, $\Delta_{\prec}(x) = 0$. In the same way, $\Delta_{\succ}(x) = 0$. So $x \in Prim_{coDend}(\mathcal{A}^{\mathcal{D}}) = Vect(\bullet_d \;/\; d \in \mathcal{D})$ (proposition 34). As the p_d's are linearly independant, $Ker(\Psi) \cap Vect(\bullet_d \;/\; d \in \mathcal{D}) = (0)$: contradiction. So Ψ is monic.
2. Then $Im(\Psi)$ is a dendriform subalgebra of A which contains $Prim_{coDend}(A)$. As A is connected, $Prim_{coDend}(A)$ generates A (theorem 21), so Ψ is epic.
3. Comes from 1 and 2. $\Box$

This corollary is immediate:

Corollary 36 *1. Let A be a connected bidendriform bialgebra. Let $(p_d)_{d\in\mathcal{D}}$ be a basis of $Prim_{coDend}(A)$. Then the morphism $\Psi : \mathcal{A}^{\mathcal{D}} \longrightarrow A$ described in theorem 35 is an isomorphism of bidendriform bialgebras.*

2. Let A be a $\mathbb{N}$-graded bidendriform bialgebra which is connected as a dendriform coalgebra. Let $(p_d)_{d\in\mathcal{D}}$ be a basis of $Prim_{coDend}(A)$ made of homogeneous elements. Then $\mathcal{D}$ is given a gradation by putting $|d| = |p_d|$. Then the morphism $\Psi : \mathcal{A}^{\mathcal{D}} \longrightarrow A$ described in theorem 35 is an isomorphism of graded bidendriform bialgebras.

Corollary 37 *Let A be a $\mathbb{N}$-graded bidendriform bialgebra such that $A_0 = (0)$ and A_n is finite-dimensional for all $n \in \mathbb{N}$. We consider the following formal series:*

$$P(X) = \sum_{n=1}^{+\infty} dim(Prim_{coDend}(A)_n)X^n, \qquad R(X) = \sum_{n=1}^{+\infty} dim(A_n)X^n.$$

Then $P(X) = \dfrac{R(X)}{(1+R(X))^2}$.

Proof. Immediate if $A = (0)$. Suppose that $A \neq (0)$ (so $P(X) \neq 0$). By corollary 36, we can suppose $A = \mathcal{A}^{\mathcal{D}}$. By proposition 34, $P(X) = D(X)$ of proposition 8. Hence:

$$\begin{aligned}
& R(X)+1 = \frac{1-\sqrt{1-4P(X)}}{2P(X)} \\
\Rightarrow\ & 2P(X)(R(X)+1) - 1 = -\sqrt{1-4P(X)} \\
\Rightarrow\ & 4P(X)^2(R(X)+1)^2 + 1 - 4P(X)(R(X)+1) = 1 - 4P(X) \\
\Rightarrow\ & P(X)(R(X)+1)^2 - R(X) - 1 = -1 \\
\Rightarrow\ & P(X) = \frac{R(X)}{(R(X)+1)^2}. \ \Box
\end{aligned}$$

4 Application to the Hopf algebra FQSym

4.1 Recalls

(See [3, 5, 20]). The algebra **FQSym** is the vector space generated by the elements $(\mathbf{F}_u)_{u\in\mathbb{S}}$, where $\mathbb{S}$ is the disjoint union of the symmetric groups S_n ($n \in \mathbb{N}$). Its product and its coproduct are given in the following way: for all $u \in S_n$, $v \in S_m$, by putting $u = (u_1 \ldots u_n)$,

$$\begin{aligned}
\Delta(\mathbf{F}_u) &= \sum_{i=0}^{n} \mathbf{F}_{st(u_1\ldots u_i)} \otimes \mathbf{F}_{st(u_{i+1}\ldots u_n)}, \\
\mathbf{F}_u.\mathbf{F}_v &= \sum_{\zeta\in sh(n,m)} \mathbf{F}_{(u\times v).\zeta^{-1}},
\end{aligned}$$

where $sh(n,m)$ is the set of (n,m)-shuffles, and st is the standardisation. Its unit is $1 = \mathbf{F}_\emptyset$, where $\emptyset$ is the unique element of S_0. Moreover, **FQSym** is a $\mathbb{N}$-graded Hopf algebra, by putting $|\mathbf{F}_u| = n$ if $u \in S_n$.

Examples.

$$\begin{aligned}
\mathbf{F}_{(1\,2)}\mathbf{F}_{(1\,2\,3)} &= \mathbf{F}_{(1\,2\,3\,4\,5)} + \mathbf{F}_{(1\,3\,2\,4\,5)} + \mathbf{F}_{(1\,3\,4\,2\,5)} + \mathbf{F}_{(1\,3\,4\,5\,2)} + \mathbf{F}_{(3\,1\,2\,4\,5)} \\
&\quad + \mathbf{F}_{(3\,1\,4\,2\,5)} + \mathbf{F}_{(3\,1\,4\,5\,2)} + \mathbf{F}_{(3\,4\,1\,2\,5)} + \mathbf{F}_{(3\,4\,1\,5\,2)} + \mathbf{F}_{(3\,4\,5\,1\,2)},
\end{aligned}$$

$$\begin{aligned}
\Delta\left(\mathbf{F}_{(1\,2\,5\,4\,3)}\right) &= 1 \otimes \mathbf{F}_{(1\,2\,5\,4\,3)} + \mathbf{F}_{(1)} \otimes \mathbf{F}_{(1\,4\,3\,2)} + \mathbf{F}_{(1\,2)} \otimes \mathbf{F}_{(3\,2\,1)} \\
&\quad + \mathbf{F}_{(1\,2\,3)} \otimes \mathbf{F}_{(2\,1)} + \mathbf{F}_{(1\,2\,4\,3)} \otimes \mathbf{F}_{(1)} + \mathbf{F}_{(1\,2\,5\,4\,3)} \otimes 1.
\end{aligned}$$

4.2 Bidendriform structure on FQSym

Let $(\mathbf{FQSym})_+ = Vect(\mathbf{F}_u \,/\, u \in S_n,\ n \geq 1)$ be the augmentation ideal of **FQSym**. We define $\prec, \succ, \Delta_\prec$ and $\Delta_\succ$ on $(\mathbf{FQSym})_+$ in the following way: for all $u \in S_n$, $v \in S_m$, by putting $u = (u_1 \ldots u_n)$,

$$\begin{aligned}
\mathbf{F}_u \prec \mathbf{F}_v &= \sum_{\substack{\zeta \in sh(n,m)\\ \zeta^{-1}(n+m)=n}} \mathbf{F}_{(u\times v).\zeta^{-1}},\\
\mathbf{F}_u \succ \mathbf{F}_v &= \sum_{\substack{\zeta \in sh(n,m)\\ \zeta^{-1}(n+m)=n+m}} \mathbf{F}_{(u\times v).\zeta^{-1}},\\
\Delta_\prec(\mathbf{F}_u) &= \sum_{i=u^{-1}(n)}^{n-1} \mathbf{F}_{st(u_1\ldots u_i)} \otimes \mathbf{F}_{st(u_{i+1}\ldots u_n)},\\
\Delta_\succ(\mathbf{F}_u) &= \sum_{i=1}^{u^{-1}(n)-1} \mathbf{F}_{st(u_1\ldots u_i)} \otimes \mathbf{F}_{st(u_{i+1}\ldots u_n)}.
\end{aligned}$$

Examples.

$$\begin{aligned}
\mathbf{F}_{(1\,2)} \prec \mathbf{F}_{(1\,2\,3)} &= \mathbf{F}_{(1\,3\,4\,5\,2)} + \mathbf{F}_{(3\,1\,4\,5\,2)} + \mathbf{F}_{(3\,4\,1\,5\,2)} + \mathbf{F}_{(3\,4\,5\,1\,2)},\\
\mathbf{F}_{(1\,2)} \succ \mathbf{F}_{(1\,2\,3)} &= \mathbf{F}_{(1\,2\,3\,4\,5)} + \mathbf{F}_{(1\,3\,2\,4\,5)} + \mathbf{F}_{(1\,3\,4\,2\,5)} + \mathbf{F}_{(3\,1\,2\,4\,5)} + \mathbf{F}_{(3\,1\,4\,2\,5)} + \mathbf{F}_{(3\,4\,1\,2\,5)},
\end{aligned}$$

$$\begin{aligned}
\Delta_\prec\left(\mathbf{F}_{(1\,2\,5\,4\,3)}\right) &= \mathbf{F}_{(1\,2\,3)} \otimes \mathbf{F}_{(2\,1)} + \mathbf{F}_{(1\,2\,4\,3)} \otimes \mathbf{F}_{(1)},\\
\Delta_\succ\left(\mathbf{F}_{(1\,2\,5\,4\,3)}\right) &= \mathbf{F}_{(1)} \otimes \mathbf{F}_{(1\,4\,3\,2)} + \mathbf{F}_{(1\,2)} \otimes \mathbf{F}_{(3\,2\,1)}.
\end{aligned}$$

Theorem 38 *$((\mathbf{FQSym})_+, \prec, \succ, \Delta_\prec, \Delta_\succ)$ is a connected bidendriform bialgebra.*

Proof. The structure of dendriform bialgebra is already introduced in [26], so we already have (1)-(3), and (12)+(14), (13)+(15). We consider the symmetric non-degenerate pairing on $(\mathbf{FQSym})_+$ defined in [5, 20] by $< \mathbf{F}_\sigma, \mathbf{F}_\tau >= \delta_{\sigma,\tau^{-1}}$. Note that $\Delta_\prec, \Delta_\succ$ are the transposes of $\prec, \succ$ for this pairing. So $((\mathbf{FQSym})_+, \Delta_\prec, \Delta_\succ)$ is a codendriform coalgebra. So we already have (4)-(6), and (12)+(13), (14)+(15). It is then enough to prove (13). We have:

$$\begin{aligned}
\Delta_\succ\left(\mathbf{F}_{(u_1\ldots u_n)} \prec \mathbf{F}_{(v_1\ldots v_m)}\right) &= \Delta_\succ\left(\sum_{1\leq j_1<\ldots<j_m<n+m} \mathbf{F}_{\underbrace{(u_1\ldots v_1+n\ldots v_m+n\ldots u_n)}_{v_i+n \text{ in position } j_i}}\right)\\
&= \sum \sum_{1\leq j_1<\ldots<j_m<n+m} \mathbf{F}_{st(u_1\ldots)} \otimes \mathbf{F}_{st(\ldots m+n\ldots u_n)}\\
&= \underbrace{\left(\mathbf{F}_{(u_1\ldots u_n)}\right)'\left(\mathbf{F}_{(v_1\ldots v_n)}\right)'_\succ \otimes \left(\mathbf{F}_{(u_1\ldots u_n)}\right)'' \prec \left(\mathbf{F}_{(v_1\ldots v_n)}\right)''_\succ}_{\text{terms with } u_i\text{'s and } v_j\text{'s on both sides on the } \otimes}\\
&\quad + \underbrace{\left(\mathbf{F}_{(u_1\ldots u_n)}\right)' \otimes \left(\mathbf{F}_{(u_1\ldots u_n)}\right)'' \prec \mathbf{F}_{(v_1\ldots v_n)}}_{\text{terms with all the } v_i\text{'s on the right on the } \otimes}\\
&+ \underbrace{\left(\mathbf{F}_{(v_1\ldots v_n)}\right)'_\succ \otimes \mathbf{F}_{(u_1\ldots u_n)} \prec \left(\mathbf{F}_{(v_1\ldots v_n)}\right)''_\succ}_{\text{terms with all the } u_i\text{'s the right of the } \otimes}.
\end{aligned}$$

So (13) is satisfied, and $(\mathbf{FQSym})_+$ is a bidendriform bialgebra. Moreover, it is $\mathbb{N}$-graded, with $((\mathbf{FQSym})_+)_0 = (0)$, so it is connected. $\Box$

Remark. It is of course possible to prove (4)-(6) directly. For example, the two members of (4) for $a=\mathbf{F}_u$, with $u=(u_1\ldots u_n)$, are equal to:

$$\sum_{\sigma^{-1}(n)\leq i<j<n} \mathbf{F}_{st(u_1\ldots u_i)}\otimes \mathbf{F}_{st(u_{i+1}\ldots u_j)}\otimes \mathbf{F}_{st(u_{j+1}\ldots u_n)}.$$

For (5), we obtain:

$$\sum_{1\leq i<\sigma^{-1}(n)\leq j<n} \mathbf{F}_{st(u_1\ldots u_i)}\otimes \mathbf{F}_{st(u_{i+1}\ldots u_j)}\otimes \mathbf{F}_{st(u_{j+1}\ldots u_n)}.$$

And for (6):

$$\sum_{1\leq i<j<\sigma^{-1}(n)} \mathbf{F}_{st(u_1\ldots u_i)}\otimes \mathbf{F}_{st(u_{i+1}\ldots u_j)}\otimes \mathbf{F}_{st(u_{j+1}\ldots u_n)}.$$

Moreover, $(\mathbf{FQSym})_+$ is graded and connected, with:

$$R(X)=\sum_{n=1}^{+\infty} dim(\mathbf{FQSym}_n)X^n=\sum_{n=1}^{+\infty} n!X^n.$$

By corollary 37:

$$P(X)=\sum_{n=1}^{+\infty} dim(Prim_{coDend}((\mathbf{FQSym})_+)_n)X^n=\sum_{n=1}^{+\infty} p_nX^n=\frac{R(X)}{(R(X)+1)^2}.$$

We obtain then:

n	1	2	3	4	5	6	7	8	9	10	11	12
p_n	1	0	1	6	39	284	2 305	20 682	203 651	2 186 744	25 463 925	319 989 030

Remark. Let $\Psi:\mathcal{A}\longrightarrow(\mathbf{FQSym})_+$ be the unique morphism of bidendriform bialgebras which send $\bullet$ to $\mathbf{F}_{(1)}$. It coincides with the morphism $\overline{\overline{\Phi^*}}$ of [19]. We obtain by theorem 35 that it is monic.

By corollary 36:

Theorem 39 *Let $\mathcal{D}$ be a graded set such that $\mathcal{D}_0=\emptyset$ and for all $n\geq 1$, $card(\mathcal{D}_n)=p_n$. Then $(\mathbf{FQSym})_+$ and $\mathcal{A}^{\mathcal{D}}$ are isomorphic as graded bidendriform bialgebras. Hence,* **FQSym** *and $\mathcal{H}^{\mathcal{D}}$ are isomorphic as graded Hopf algebras.*

We can now prove the conjecture 3.8 of [5]:

Corollary 40 *If the field K is of characteristic 0, the Lie algebra $Prim_{coAss}(\mathbf{FQSym})$ is free.*

Proof. By proposition 141 of [6], the Lie algebra $Prim_{coAss}(\mathcal{H}^{\mathcal{D}})$ is free. □

We use notations of [5]; the basis $(V_\sigma)_{\sigma\in\mathbb{S}}$ is defined by duality and contains a basis of $Prim_{coAss}(\mathbf{FQSym})$. Then these elements give a basis of primitive elements of $(\mathbf{FQSym})_+$ of degree ≤ 4:

$$\mathbf{V}_1 = \mathbf{F}_1,$$

$$\mathbf{V}_{231} = \mathbf{F}_{231}-\mathbf{F}_{132},$$

$$
\begin{aligned}
\mathbf{V}_{3142} &= \mathbf{F}_{3142} - \mathbf{F}_{2143}, \\
\mathbf{V}_{2431} &= \mathbf{F}_{2431} - \mathbf{F}_{1432}, \\
\mathbf{V}_{2341} &= \mathbf{F}_{2341} - \mathbf{F}_{1342}, \\
\mathbf{V}_{3241} &= \mathbf{F}_{1243} - \mathbf{F}_{1342} - \mathbf{F}_{2143} + \mathbf{F}_{3241}, \\
\mathbf{V}_{3412} - \mathbf{V}_{2413} &= \mathbf{F}_{3412} - \mathbf{F}_{2413}, \\
\mathbf{V}_{3421} - \mathbf{V}_{2413} &= \mathbf{F}_{1423} - \mathbf{F}_{2413} - \mathbf{F}_{1432} + \mathbf{F}_{3421}.
\end{aligned}
$$

Remark. The same can be done for other Hopf algebras which contain the Malvenuto-Reutenauer Hopf algebra. The first example is the Hopf algebra **PQSym** of parking functions of Novelli and Thibon ([23, 24]). As it is shown in [24], **PQSym** is a bidendriform bialgebra, so is isomorphic to $\mathcal{H}^{\mathcal{D}}$, where $\mathcal{D}$ is a certain graded set. A second example is the Hopf algebra of uniform block permutations of Aguiar and Orellana ([2]). Hence, it is isomorphic to $\mathcal{H}^{\mathcal{D}}$, where $\mathcal{D}$ is a certain graded set, with the following values:

n	1	2	3	4	5	6	7	8	9
$card(\mathcal{D}_n)$	1	1	7	72	962	16 135	330 624	8 117 752	235 133 003

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