Clay Mathematics Proceedings
Volume 12, 2010

Hopf Subalgebras of Rooted Trees from Dyson-Schwinger Equations

Loïc Foissy

Abstract. We consider the combinatorial Dyson-Schwinger equation $X = B^+(f(X))$ in the Connes-Kreimer Hopf algebra of rooted trees $\mathcal{H}$, where B^+ is the operator of grafting on a root, and f a formal series. The unique solution X of this equation generates a graded subalgebra $\mathcal{H}_f$ of $\mathcal{H}$. We characterize here all the formal series f such that $\mathcal{H}_f$ is a Hopf subalgebra. We obtain in this way a 2-parameter family of Hopf subalgebras of $\mathcal{H}$, organized into three isomorphism classes:

(1) A first (degenerate) one, restricted to a polynomial ring in one variable.
(2) A second one, restricted to the Hopf subalgebra of ladders, isomorphic to the Hopf algebra of symmetric functions.
(3) A last (infinite) one, which gives a family of isomorphic Hopf subalgebras of $\mathcal{H}$. These Hopf algebras can be seen as the coordinate ring of the group G of formal diffeomorphisms of the line tangent to the identity: in other terms, we obtain a family of embeddings of the Faà di Bruno Hopf algebra in $\mathcal{H}$.

In the second and the third cases, $\mathcal{H}_f$ is the graded dual of the enveloping algebra of a graded, connected Lie algebra $\mathfrak{g}$, such that the homogeneous components $\mathfrak{g}_n$ of $\mathfrak{g}$ are 1-dimensional when $n \geq 1$. Under a condition of commutativity, we prove that there exist three such Lie algebras:

(1) The Faà di Bruno Lie algebra, that is to say the Lie algebra of the group of formal diffeomorphisms G.
(2) The Lie algebra of corollas.
(3) A third one.

Embeddings in $\mathcal{H}$ of the dual of the enveloping algebra of the first case are given by the Dyson-Schwinger equations. For the second case, such an embedding is given by the subalgebra generated by corollas. We also describe an embedding in $\mathcal{H}$ for the third case.

Contents

2010 *Mathematics Subject Classification.* Primary 16T05. Secondary 81T15, 81T18.

Introduction

The Connes-Kreimer algebra $\mathcal{H}$ of rooted trees was introduced in [**8**]. This graded Hopf algebra is commutative, non-cocommutative, and is given a linear basis by the set of rooted forests. A particularly important operator of $\mathcal{H}$ is the grafting on a root B^+, which satisfies the following equation:

$$\Delta \circ B^+(x) = B^+(x) \otimes 1 + (\mathrm{Id} \otimes B^+) \circ \Delta(x).$$

In other words, B^+ is a 1-cocycle for the Cartier-Quillen cohomology of coalgebras. Moreover, the couple $(\mathcal{H}, B^+)$ satisfies a universal property; see Theorem 3 of the present text.

We consider here a family of subalgebras of $\mathcal{H}$, associated to the combinatorial Dyson-Schwinger equation [**1, 9, 10**]:

$$X = B^+(f(X)),$$

where $f(h) = \sum p_n h^n$ is a formal series such that $p_0 = 1$, and X is an element of the completion of $\mathcal{H}$ for the topology given by the gradation of $\mathcal{H}$. This equation admits a unique solution $X = \sum x_n$, where x_n is, for all $n \geq 1$, a linear span of rooted trees of weight n, inductively given by

$$\left\{ \begin{array}{rcl} x_1 &=& p_0 \bullet, \\ x_{n+1} &=& \displaystyle\sum_{k=1}^{n} \sum_{a_1+\cdots+a_k=n} p_k B^+(x_{a_1} \cdots x_{a_k}). \end{array} \right.$$

We denote by $\mathcal{H}_f$ the subalgebra of $\mathcal{H}$ generated by the x_n's.

For the usual Dyson-Schwinger equation, $f(h) = (1-h)^{-1}$. It turns out that, in this case, $\mathcal{H}_f$ is a Hopf subalgebra. This is not the case in general; we characterise here the formal series $f(h)$ such that $\mathcal{H}_f$ is Hopf. Namely, $\mathcal{H}_f$ is a Hopf subalgebra of $\mathcal{H}$ if and only if there exists $(\alpha, \beta) \in K^2$, such that $f(h) = 1$ if $\alpha = 0$, or $f(h) = e^{\alpha h}$ if $\beta = 0$, or $f(h) = (1 - \alpha\beta h)^{-\frac{1}{\beta}}$ if $\alpha\beta \neq 0$. We obtain in this way a two-parameter family $\mathcal{H}_{\alpha,\beta}$ of Hopf subalgebras of $\mathcal{H}$ and we explicitly describe a system of generators of these algebras. In particular, if $\alpha = 0$, then $\mathcal{H}_{\alpha,\beta} = K[\bullet]$; if $\alpha \neq 0$, then $\mathcal{H}_{\alpha,\beta} = \mathcal{H}_{1,\beta}$.

The Hopf algebra $\mathcal{H}_{\alpha,\beta}$ is commutative, graded and connected. By the Milnor-Moore theorem [**11**], its dual is the enveloping algebra of a Lie algebra $\mathfrak{g}_{\alpha,\beta}$. Computing this Lie algebra, we find three isomorphism classes of $\mathcal{H}_{\alpha,\beta}$'s:

(1) $\mathcal{H}_{0,1}$, equal to $K[\bullet]$.
(2) $\mathcal{H}_{1,-1}$, the subalgebra of ladders, isomorphic to the Hopf algebra of symmetric functions.
(3) The $\mathcal{H}_{1,\beta}$'s, with $\beta \neq -1$, isomorphic to the Faà di Bruno Hopf algebra.

Note that non-commutative versions of these results are presented in [**6**].

In particular, if $\mathcal{H}_{\alpha,\beta}$ is non-cocommutative, it is isomorphic to the Faà di Bruno Hopf algebra. We try to explain this fact in the third section of this text. The dual Lie algebra $\mathfrak{g}_{\alpha,\beta}$ satisfies the following properties:

(1) $\mathfrak{g}_{\alpha,\beta}$ is graded and connected.

(2) The homogeneous component $\mathfrak{g}(n)$ of degree n of $\mathfrak{g}$ is 1-dimensional for all $n \geq 1$.

Moreover, if $\mathcal{H}_{\alpha,\beta}$ is non-cocommutative, then $[\mathfrak{g}(1), \mathfrak{g}(n)] \neq (0)$ if $n \geq 2$. Such a Lie algebra will be called a FdB Lie algebra. We prove here that there exist, up to isomorphism, only three FdB Lie algebras:

(1) The Faà di Bruno Lie algebra, which is the Lie algebra of the group of formal diffeomorphisms tangent to the identity at 0.
(2) The Lie algebra of corollas.
(3) A third Lie algebra.

In particular, with a stronger condition of non-commutativity, a FdB Lie algebra is isomorphic to the Faà di Bruno Lie algebra, and this result can be applied to all $\mathcal{H}_{1,\beta}$'s when $\beta \neq -1$. The dual of the enveloping algebras of the two other FdB Lie algebras can also be embedded in $\mathcal{H}$, using corollas for the second, giving in a certain way a limit of $\mathcal{H}_{1,\beta}$ when β goes to ∞, and the third one with a different construction.

Notation. We denote by K a commutative field of characteristic zero.

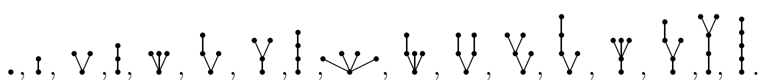
1. The Hopf algebra of rooted trees and Dyson-Schwinger equations

1.1. The Connes-Kreimer Hopf algebra. Let us first recall the construction of the Connes-Kreimer Hopf algebra of rooted trees.

Definition 1. [**13, 14**]

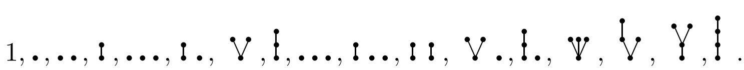
(1) A *rooted tree* is a finite graph, connected and without loops, with a special vertex called the *root*.
(2) The *weight* of a rooted tree is the number of its vertices.
(3) The set of rooted trees will be denoted by $\mathcal{T}$.

Examples. The rooted trees of weight ≤ 5 are



The Connes-Kreimer Hopf algebra of rooted trees $\mathcal{H}$ was introduced in [**2**]. As an algebra, $\mathcal{H}$ is the free associative, commutative, unitary algebra generated by the elements of $\mathcal{T}$. In other terms, a K-basis of $\mathcal{H}$ is given by rooted forests, that is to say not necessarily connected graphs F such that each connected component of F is a rooted tree. The set of rooted forests will be denoted by $\mathcal{F}$. The product of $\mathcal{H}$ is given by the concatenation of rooted forests, and the unit is the empty forest, denoted by 1.

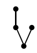
Examples. The rooted forests of weight ≤ 4 are

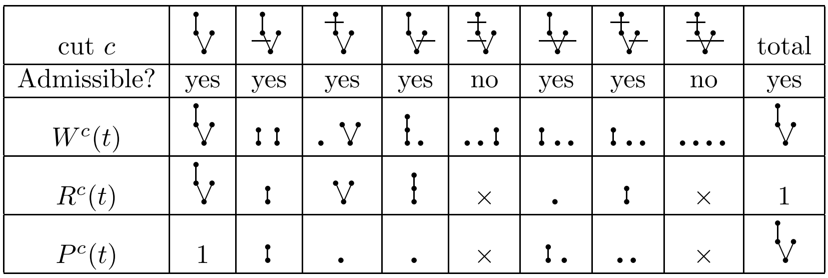


In order to make $\mathcal{H}$ a bialgebra, we now introduce the notion of cut of a tree t. A *non-total cut* c of a tree t is a choice of edges of t. Deleting the chosen edges, the cut makes t into a forest, denoted by $W^c(t)$. The cut c is *admissible* if any oriented path[1] in the tree meets at most one cut edge. For such a cut, the tree of $W^c(t)$

[1] The edges of the tree are oriented from the root to the leaves.

which contains the root of t is denoted by $R^c(t)$ and the product of the other trees of $W^c(t)$ is denoted by $P^c(t)$. We also add the total cut, which is by convention an admissible cut such that $R^c(t) = 1$ and $P^c(t) = W^c(t) = t$. The set of admissible cuts of t is denoted by $\mathrm{Adm}_*(t)$. Note that the empty cut of t is admissible; we denote $\mathrm{Adm}(t) = \mathrm{Adm}_*(t) - \{\text{empty cut, total cut}\}$.

Example. Let us consider the rooted tree $t =$ . As it has 3 edges, it has 2^3 non-total cuts.

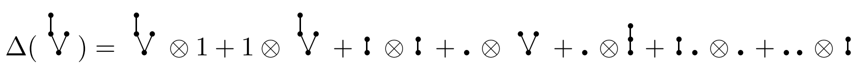


The coproduct of $\mathcal{H}$ is defined as the unique algebra morphism from $\mathcal{H}$ to $\mathcal{H}\otimes\mathcal{H}$ such that, for all rooted tree $t \in \mathcal{T}$,

$$\Delta(t) = \sum_{c \in \mathrm{Adm}_*(t)} P^c(t) \otimes R^c(t) = t \otimes 1 + 1 \otimes t + \sum_{c \in \mathrm{Adm}(t)} P^c(t) \otimes R^c(t).$$

As $\mathcal{H}$ is the polynomial algebra generated by $\mathcal{T}$, this makes sense.

Example.



THEOREM 2. [**2**] *With this coproduct, $\mathcal{H}$ is a bialgebra. The counit of $\mathcal{H}$ is given by*

$$\varepsilon : \left\{ \begin{array}{rcl} \mathcal{H} & \longrightarrow & K \\ F \in \mathcal{F} & \longrightarrow & \delta_{1,F}. \end{array} \right.$$

The antipode is the algebra endomorphism defined for all $t \in \mathcal{T}$ by

$$S(t) = - \sum_{c \text{ non-total cut of } t} (-1)^{n_c} W^c(t),$$

where n_c is the number of cut edges in c.

1.2. Gradation of $\mathcal{H}$ and completion. We grade $\mathcal{H}$ by declaring the forests of weight n homogeneous of degree n. We denote by $\mathcal{H}(n)$ the homogeneous component of $\mathcal{H}$ of degree n. Then $\mathcal{H}$ is a graded bialgebra, that is to say

(1) For all $i, j \in \mathbb{N}$, $\mathcal{H}(i)\mathcal{H}(j) \subseteq \mathcal{H}(i+j)$.
(2) For all $k \in \mathbb{N}$, $\Delta(\mathcal{H}(k)) \subseteq \sum_{i+j=k} \mathcal{H}(i) \otimes \mathcal{H}(j)$.

We define, for all $x, y \in \mathcal{H}$,

$$\begin{cases} \mathrm{val}(x) &= \max\left\{ n \in \mathbb{N} \mid x \in \bigoplus_{k \geq n} \mathcal{H}(k) \right\}, \\ d(x,y) &= 2^{-\mathrm{val}(x-y)}, \end{cases}$$

with the convention $2^{-\infty} = 0$. Then d is a distance on $\mathcal{H}$. The metric space $(\mathcal{H}, d)$ is not complete; its completion will be denoted by $\widehat{\mathcal{H}}$. As a vector space,

$$\widehat{\mathcal{H}} = \prod_{n \in \mathbb{N}} \mathcal{H}(n).$$

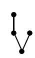
The elements of $\widehat{\mathcal{H}}$ will be denoted $\sum x_n$, where $x_n \in \mathcal{H}(n)$ for all $n \in \mathbb{N}$. The product $m : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}$ is homogeneous of degree 0, so is continuous. So it can be extended from $\widehat{\mathcal{H}} \otimes \widehat{\mathcal{H}}$ to $\widehat{\mathcal{H}}$, which is then an associative, commutative algebra. Similarly, the coproduct of $\mathcal{H}$ can be extended as a map

$$\Delta : \widehat{\mathcal{H}} \longrightarrow \mathcal{H} \widehat{\otimes} \mathcal{H} = \prod_{i,j \in \mathbb{N}} \mathcal{H}(i) \otimes \mathcal{H}(j).$$

Let $f(h) = \sum p_n h^n \in K[[h]]$ be any formal series, and let $X = \sum x_n \in \widehat{\mathcal{H}}$, such that $x_0 = 0$. The series of $\widehat{\mathcal{H}}$ of terms $p_n X^n$ is Cauchy, so converges. Its limit will be denoted by $f(X)$. In other words, $f(X) = \sum y_n$, with

$$y_n = \sum_{k=1}^{n} \sum_{a_1 + \cdots + a_k = n} p_k x_{a_1} \cdots x_{a_k}.$$

Remark. If $f(h) \in K[[h]]$, $g(h) \in K[[h]]$, without constant terms, and $X \in \widehat{\mathcal{H}}$, without constant terms, it is easy to show that $(f \circ g)(X) = f(g(X))$.

1.3. 1-cocycle of $\mathcal{H}$ and Dyson-Schwinger equations. We define the operator $B^+ : \mathcal{H} \longrightarrow \mathcal{H}$, sending a forest $t_1 \cdots t_n$ to the tree obtained by grafting $t_1, \cdots, t_n$ to a common root. For example, $B^+($$) =$ . This operator satisfies the following relation: for all $x \in \mathcal{H}$,

$$\Delta \circ B^+(x) = B^+(x) \otimes 1 + (\mathrm{Id} \otimes B^+) \circ \Delta(x). \tag{1}$$

This means that B^+ is a 1-cocycle for a certain cohomology, namely the Cartier-Quillen cohomology for coalgebras, the notion dual to the Hochschild cohomology [**2**]. Moreover, $(\mathcal{H}, B^+)$ satisfies the following universal property:

THEOREM 3 (Universal property). *Let A be a commutative algebra and let $L : A \longrightarrow A$ be a linear map.*

(1) *There exists a unique algebra morphism $\phi : \mathcal{H} \longrightarrow A$, such that $\phi \circ B^+ = L \circ \phi$.*
(2) *If moreover A is a Hopf algebra and L satisfies (1), then ϕ is a Hopf algebra morphism.*

The operator B^+ is homogeneous of degree 1, so is continuous. As a consequence, it can be extended as an operator $B^+ : \widehat{\mathcal{H}} \longrightarrow \widehat{\mathcal{H}}$. This operator still satisfies (1).

DEFINITION 4. [**1, 9, 10**] Let $f \in K[[h]]$. The *Dyson-Schwinger equation* associated to f is

$$X = B^+(f(X)), \tag{2}$$

where X is an element of $\widehat{\mathcal{H}}$, without constant term.

PROPOSITION 5. *The Dyson-Schwinger equation associated to the formal series $f(h) = \sum p_n h^n$ admits a unique solution $X = \sum x_n$, inductively defined by*

$$\left\{\begin{array}{rcl} x_0 &=& 0, \\ x_1 &=& p_0 \bullet, \\ x_{n+1} &=& \displaystyle\sum_{k=1}^{n} \sum_{a_1+\cdots+a_k=n} p_k B^+(x_{a_1} \cdots x_{a_k}). \end{array}\right.$$

PROOF. It is enough to identify the homogeneous components of the two members of (2). □

DEFINITION 6. The subalgebra of $\mathcal{H}$ generated by the homogeneous components x_n of the unique solution X of the Dyson-Schwinger equation (2) associated to f will be denoted by $\mathcal{H}_f$.

The aim of this text is to give a necessary and sufficient condition on f for $\mathcal{H}_f$ to be a Hopf subalgebra of $\mathcal{H}$.

Remarks.

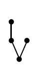
(1) If $f(0) = 0$, the unique solution of (2) is 0. As a consequence, $\mathcal{H}_f = K$ is a Hopf subalgebra.
(2) For all $\alpha \in K$, if $X = \sum x_n$ is the solution of the Dyson-Schwinger equation associated to f, the unique solution of the Dyson-Schwinger equation associated to αf is $\sum \alpha^n x_n$. As a consequence, if $\alpha \neq 0$, $\mathcal{H}_f = \mathcal{H}_{\alpha f}$. We shall then suppose in the sequel that $p_0 = 1$. In this case, $x_1 = \bullet$.

Examples.

(1) We take $f(h) = 1 + h$. Then $x_1 = \bullet$, $x_2 =$, $x_3 =$, $x_4 =$ . More generally, x_n is the ladder with n vertices, that is to say $(B^+)^n(1)$ (Definition 7). As a consequence, for all $n \geq 1$,

$$\Delta(x_n) = \sum_{i+j=n} x_i \otimes x_j.$$

So $\mathcal{H}_{1+h}$ is Hopf. Moreover, it is cocommutative.
(2) We take $f(h) = 1 + h + h^2 + 2h^3$. Then

$x_1 = \bullet$,
$x_2 =$,
$x_3 =$  $+$,
$x_4 = 2$  $+ 2$  $+$  $+$.

Hence

$$\begin{aligned}
\Delta(x_1) &= x_1 \otimes 1 + 1 \otimes x_1, \\
\Delta(x_2) &= x_2 \otimes 1 + 1 \otimes x_2 + x_1 \otimes x_1, \\
\Delta(x_3) &= x_3 \otimes 1 + 1 \otimes x_3 + x_1^2 \otimes x_1 + 3x_1 \otimes x_2 + x_2 \otimes x_1, \\
\Delta(x_4) &= x_4 \otimes 1 + 1 \otimes x_4 + 10x_1^2 \otimes x_2 + x_1^3 \otimes x_1 + 3x_2 \otimes x_2
\end{aligned}$$

$+2x_1x_2 \otimes x_1 + x_3 \otimes x_1 + x_1 \otimes (8$  $+5$ $)$,

so $\mathcal{H}_f$ is not Hopf.

We shall need later these two families of rooted trees:

Definition 7. Let $n \geq 1$.

(1) The *ladder* l_n of weight n is the rooted tree $(B^+)^n(1)$. For example,

$l_1 = \bullet$, $l_2 =$, $l_3 =$ , $l_4 =$ .

(2) The *corolla* c_n of weight n is the rooted tree $B^+(\bullet^{n-1})$. For example,

$c_1 = \bullet$, $c_2 =$, $c_3 =$ , $c_4 =$ .

The following lemma is an immediate corollary of proposition 5:

Lemma 8. *The coefficient of the ladder of weight n in x_n is p_1^{n-1}. The coefficient of the corolla of weight n in x_n is p_{n-1}.*

Using (1):

Lemma 9. *For all $n \geq 1$,*

(1) $\Delta(l_n) = \sum_{i=0}^{n} l_i \otimes l_{n-i}$, *with the convention $l_0 = 1$.*

(2) $\Delta(c_n) = c_n \otimes 1 + \sum_{i=0}^{n-1} \binom{n-1}{i} \bullet^i \otimes c_{n-i}$.

2. Formal series giving Hopf subalgebras

2.1. Statement of the main theorem. The aim of this section is to prove the following result:

Theorem 10. *Let $f(h) \in K[[h]]$, such that $f(0) = 1$. The following assertions are equivalent:*

1. *$\mathcal{H}_f$ is a Hopf subalgebra of $\mathcal{H}$.*
2. *There exists $(\alpha, \beta) \in K^2$ such that $(1 - \alpha\beta h) f'(h) = \alpha f(h)$.*
3. *There exists $(\alpha, \beta) \in K^2$ such that $f(h) = 1$ if $\alpha = 0$, or $f(h) = e^{\alpha h}$ if $\beta = 0$, or $f(h) = (1 - \alpha\beta h)^{-\frac{1}{\beta}}$ if $\alpha\beta \neq 0$.*

It is an easy exercise to prove that the second and the third statements are equivalent.

2.2. Proof of (1) $\Longrightarrow$ (2)**.** We suppose that $\mathcal{H}_f$ is Hopf.

LEMMA 11. *Let us suppose that* $p_1 = 0$*. Then* $f(h) = 1$*, so* (2) *holds with* $\alpha = 0$.

PROOF. Let us suppose that $p_n \neq 0$ for a certain $n \geq 2$. Let us choose a minimal n. Then $x_1 = \bullet$, $x_2 = \cdots = x_n = 0$, and $x_{n+1} = p_n c_{n+1}$. So

$$\Delta(x_{n+1}) = x_{n+1} \otimes 1 + 1 \otimes x_{n+1} + \sum_{i=1}^{n} \binom{n}{i} p_n \bullet^i \otimes c_{n+1-i} \in \mathcal{H}_f \otimes \mathcal{H}_f.$$

In particular, for $i = n-1$, $c_2 = \ddot{\mathbf{l}} \in \mathcal{H}_f$, so $x_2 \neq 0$: contradiction. $\square$

We now assume that $p_1 \neq 0$. Let $Z_\bullet : \mathcal{H} \longrightarrow K$, defined by $Z_\bullet(F) = \delta_{\bullet,F}$ for all $F \in \mathcal{F}$. This map $Z_\bullet$ is homogeneous of degree -1, so is continuous and can be extended to a map $Z_\bullet : \widehat{\mathcal{H}} \longrightarrow K$. We put $(Z_\bullet \otimes \mathrm{Id}) \circ \Delta(X) = \sum y_n$, where X is the unique solution of (2). A direct computation shows that y_n can be computed by induction with

$$\begin{cases} y_0 &= 1, \\ y_{n+1} &= \displaystyle\sum_{k=1}^{n} \sum_{a_1+\cdots+a_k=n} (k+1) p_{k+1} B^+(x_{a_1} \cdots x_{a_k}) \\ & \quad + \displaystyle\sum_{k=1}^{n} \sum_{a_1+\cdots+a_k=n} k p_k B^+(y_{a_1} x_{a_2} \cdots x_{a_k}). \end{cases}$$

As $\mathcal{H}_f$ is Hopf, $y_n \in \mathcal{H}_f$ for all $n \in \mathbb{N}$. Moreover, y_n is a linear span of rooted trees of weight n, so is a multiple of x_n; we put $y_n = \alpha_n x_n$.

Let us consider the coefficient of the ladder of weight n in y_n. By lemma 8, this is $\alpha_n p_1^{n-1}$. So, for all $n \geq 1$,

$$p_1^n \alpha_{n+1} = 2 p_1^{n-1} p_2 + p_1^n \alpha_n.$$

As $\alpha_1 = p_1$, for all $n \geq 1$, $\alpha_n = p_1 + 2\dfrac{p_2}{p_1}(n-1)$. Let us consider the coefficient of the corolla of weight n in y_n. By lemma 8, this is $\alpha_n p_n$. So, for all $n \geq 1$,

$$\alpha_n p_n = (n+1) p_{n+1} + n p_n p_1.$$

Summing all these relations, putting $\alpha = p_1$ and $\beta = 2\dfrac{p_2}{p_1} - 1$, we obtain the differential equation $(1 - \alpha\beta h) f'(h) = f(h)$, so (2) holds.

2.3. Proof of (2) $\Longrightarrow$ (1)**.** Let us suppose (2) or, equivalently, (3). We now write $\mathcal{H}_{\alpha,\beta}$ instead of $\mathcal{H}_f$. We first give a description of the x_n's.

DEFINITION 12.

(1) Let $F \in \mathcal{F}$. The coefficient s_F is inductively computed by

$$\begin{cases} s_\bullet &= 1, \\ s_{t_1^{a_1} \cdots t_k^{a_k}} &= a_1! \cdots a_k! \, s_{t_1}^{a_1} \cdots s_{t_k}^{a_k}, \\ s_{B^+\left(t_1^{a_1} \cdots t_k^{a_k}\right)} &= a_1! \cdots a_k! \, s_{t_1}^{a_1} \cdots s_{t_k}^{a_k}, \end{cases}$$

where $t_1, \cdots, t_k$ are distinct elements of $\mathcal{T}$.

(2) Let $F \in \mathcal{F}$. The coefficient e_F is inductively computed by

$$\left\{\begin{array}{rcl} e_{\bullet} &=& 1, \\ e_{t_1^{a_1}\cdots t_k^{a_k}} &=& \dfrac{(a_1+\cdots+a_k)!}{a_1!\cdots a_k!} e_{t_1}^{a_1}\cdots e_{t_k}^{a_k}, \\ e_{B^+(t_1^{a_1}\cdots t_k^{a_k})} &=& \dfrac{(a_1+\cdots+a_k)!}{a_1!\cdots a_k!} e_{t_1}^{a_1}\cdots e_{t_k}^{a_k}, \end{array}\right.$$

where $t_1, \cdots, t_k$ are distinct elements of $\mathcal{T}$.

Remarks.

(1) The coefficient s_F is the number of symmetries of F, that is to say the number of graph automorphisms of F respecting the roots.
(2) The coefficient e_F is the number of embeddings of F in the plane, that is to say the number of planar forests whose underlying rooted forest is F.

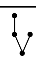
We now give β-equivalents of these coefficients. For all $k \in \mathbb{N}^*$, we put $[k]_\beta = 1+\beta(k-1)$ and $[k]_\beta! = [1]_\beta \cdots [k]_\beta$. We then inductively define $[s_F]_\beta$ and $[e_F]_\beta$ for all $F \in \mathcal{F}$ by

$$\left\{\begin{array}{rcl} [s_{\bullet}]_\beta &=& 1, \\ [s_{t_1^{a_1}\cdots t_k^{a_k}}]_\beta &=& [a_1]_\beta!\cdots[a_k]_\beta![s_{t_1}]_\beta^{a_1}\cdots[s_{t_k}]_\beta^{a_k}, \\ [s_{B^+(t_1^{a_1}\cdots t_k^{a_k})}]_\beta &=& [a_1]_\beta!\cdots[a_k]_\beta![s_{t_1}]_\beta^{a_1}\cdots[s_{t_k}]_\beta^{a_k}, \end{array}\right.$$

$$\left\{\begin{array}{rcl} [e_{\bullet}] &=& 1, \\ [e_{t_1^{a_1}\cdots t_k^{a_k}}]_\beta &=& \dfrac{[a_1+\cdots+a_k]_\beta!}{[a_1]_\beta!\cdots[a_k]_\beta!}[e_{t_1}]_\beta^{a_1}\cdots[e_{t_k}]_\beta^{a_k}, \\ [e_{B^+(t_1^{a_1}\cdots t_k^{a_k})}]_\beta &=& \dfrac{[a_1+\cdots+a_k]_\beta!}{[a_1]_\beta!\cdots[a_k]_\beta!}[e_{t_1}]_\beta^{a_1}\cdots[e_{t_k}]_\beta^{a_k}, \end{array}\right.$$

where $t_1, \cdots, t_k$ are distinct elements of $\mathcal{T}$. In particular, $[s_t]_1 = s_t$ and $[e_t]_1 = e_t$, whereras $[s_t]_0 = 1$ and $[e_t]_0 = 1$ all $t \in \mathcal{T}$.

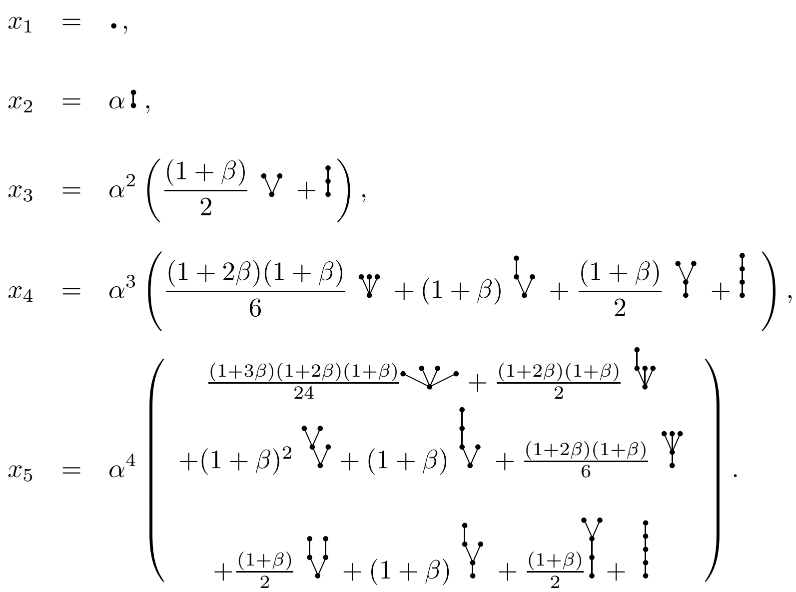
Examples.

t	s_t	$[s_t]_\beta$	e_t	$[e_t]_\beta$
$\bullet$	1	1	1	1
	1	1	1	1
	2	$(1+\beta)$	1	1
	1	1	1	1
	6	$(1+\beta)(1+2\beta)$	1	1
	1	1	2	$(1+\beta)$
	2	$(1+\beta)$	1	1
	1	1	1	1

PROPOSITION 13. *For all $n \in \mathbb{N}^*$, in $\mathcal{H}_{\alpha,\beta}$,*

$$x_n = \alpha^{n-1} \sum_{t \in \mathcal{T},\, \mathrm{weight}(t)=n} \frac{[s_t]_\beta [e_t]_\beta}{s_t} t.$$

Examples.



PROOF. For any $t \in \mathcal{T}$, we denote by b_t the coefficient of t in $x_{\text{weight}(t)}$. Then $b_\bullet = 1$. The formal series $f(h)$ is given by

$$f(h) = \sum_{n=0}^{\infty} \alpha^n \frac{[n]_\beta!}{n!} h^n.$$

If $t = B^+(t_1^{a_1} \cdots t_k^{a_k})$, where $t_1, \cdots, t_k$ are distinct elements of $\mathcal{T}$, then

$$b_t = \alpha^{a_1+\cdots+a_k} \frac{[a_1+\cdots+a_k]_\beta!}{(a_1+\cdots+a_k)!} \frac{(a_1+\cdots+a_k)!}{a_1!\cdots a_k!} b_{t_1}^{a_1} \cdots b_{t_k}^{a_k}.$$

The result comes from an easy induction. $\square$

As a consequence, $\mathcal{H}_{0,\beta} = K[\bullet]$, so $\mathcal{H}_{0,\beta}$ is a Hopf subalgebra. Moreover, $\mathcal{H}_{\alpha,\beta} = \mathcal{H}_{1,\beta}$ if $\alpha \neq 0$. So we can restrict ourselves to the case $\alpha = 1$. In order to ease the notation, we put $n_t = s_t e_t$ and $[n_t]_\beta = [s_t]_\beta [e_t]_\beta$ for all $t \in \mathcal{T}$. Then

$$\begin{cases} n_\bullet &= 1, \\ n_{B^+(t_1\cdots t_k)} &= k! n_{t_1} \cdots n_{t_k}, \end{cases}$$

$$\begin{cases} [n_\bullet]_\beta &= 1, \\ [n_{B^+(t_1\cdots t_k)}]_\beta &= [k]_\beta! [n_{t_1}]_\beta \cdots [n_{t_k}]_\beta. \end{cases}$$

As a consequence, an easy induction proves that

$$n_t = \prod_{s \text{ vertex of } t} (\text{fertility of } s)!, \qquad [n_t]_\beta = \prod_{s \text{ vertex of } t} [\text{fertility of } s]_\beta!.$$

We shall use the following result, proved in [**5, 7**]:

LEMMA 14. *For all forests $F \in \mathcal{F}$, $G, H \in \mathcal{T}$, denote by $n(F,G;H)$ the coefficient of $F \otimes G$ in $\Delta(H)$, and by $n'(F,G;H)$ the number of graftings of the trees of F over G giving the tree H. Then $n'(F,G;H) s_H = n(F,G;H) s_F s_G$.*

LEMMA 15. *Let $k, n \in \mathbb{N}^*$. We put, in $K[X_1, \cdots, X_n]$, $S = X_1 + \cdots + X_n$. Then*

$$\sum_{\alpha_1+\cdots+\alpha_n=k} \prod_{i=1}^{n} \frac{X_i(X_i+1)\cdots(X_i+\alpha_i-1)}{\alpha_i} = \frac{S(S+1)\cdots(S+k-1)}{k!}.$$

PROOF. By induction on k, see [**6**]. □

PROPOSITION 16. *If $\alpha = 1$,*

$$\Delta(X) = X \otimes 1 + \sum_{n=1}^{\infty} (1-\beta X)^{-n(1/\beta+1)+1} \otimes x_n.$$

So $\mathcal{H}_{1,\beta}$ is a Hopf subalgebra.

PROOF. As for all $n \geq 1$, x_n is a linear span of trees, we can write

$$\Delta(X) = X \otimes 1 + \sum_{F \in \mathcal{F},\, t \in \mathcal{T}} a_{F,t} F \otimes t.$$

Then, if $F \in \mathcal{F}$, $G \in \mathcal{T}$,

$$a_{F,G} = \sum_{H \in \mathcal{T}} \frac{[n_H]_\beta}{s_H} n(F, G; H) = \sum_{H \in \mathcal{T}} \frac{[n_H]_\beta}{s_F s_G} n'(F, G; H).$$

We put $F = t_1 \cdots t_k$, and we denote by $s_1, \cdots, s_n$ the vertices of the tree G, of respective fertility $f_1, \cdots, f_n$. Let us consider a grafting of F over G, such that α_i trees of F are grafted on the vertex s_i. Then $\alpha_1 + \cdots + \alpha_n = k$. Denoting by H the result of this grafting,

$$[n_H]_\beta = [n_G]_\beta [n_{t_1}]_\beta \cdots [n_{t_k}]_\beta \frac{[f_1+\alpha_1]_\beta!}{[f_1]_\beta!} \cdots \frac{[f_n+\alpha_n]_\beta!}{[f_n]_\beta!}.$$

Moreover, the number of such graftings is $\dfrac{k!}{\alpha_1! \cdots \alpha_n!}$. So, with lemma 15, putting $x_i = f_i + 1/\beta$ and $s = x_1 + \cdots + x_n$,

$$\begin{aligned}
a_{F,G} &= \sum_{\alpha_1+\cdots+\alpha_n=k} \frac{k!}{\alpha_1! \cdots \alpha_k!} \frac{1}{s_F s_G} [n_G]_\beta \prod_{i=1}^{k} [n_{t_i}]_\beta \frac{[f_1+\alpha_i]_\beta!}{[f_i]_\beta!}\\
&= \frac{k![n_G]_\beta!}{s_G s_F} \left(\prod_{i=1}^{k} [n_{t_i}]_\beta\right) \sum_{\alpha_1+\cdots+\alpha_n=k} \prod_{i=1}^{n} \frac{(1+f_i\beta)\cdots(1+(f_i+\alpha_i-1)\beta)}{\alpha_i!}\\
&= \frac{k![n_G]_\beta!}{s_G s_F} \left(\prod_{i=1}^{k} [n_{t_i}]_\beta\right) \sum_{\alpha_1+\cdots+\alpha_n=k} \prod_{i=1}^{n} \beta^{\alpha_i} \frac{x_i(x_i+1)\cdots(x_i+\alpha_i+1)}{\alpha_i!}\\
&= \frac{k![n_G]_\beta!}{s_G s_F} \left(\prod_{i=1}^{k} [n_{t_i}]_\beta\right) \beta^k \sum_{\alpha_1+\cdots+\alpha_n=k} \prod_{i=1}^{n} \frac{x_i(x_i+1)\cdots(x_i+\alpha_i+1)}{\alpha_i!}\\
&= \frac{k![n_G]_\beta!}{s_G s_F} \left(\prod_{i=1}^{k} [n_{t_i}]_\beta\right) \beta^k \frac{s(s+1)\cdots(s+k-1)}{k!}.
\end{aligned}$$

Moreover, as G is a tree, $s = f_1 + \cdots + f_n + n/\beta = n - 1 + n/\beta = n(1+1/\beta) - 1$.

We now write $F = t_1 \cdots t_k = u_1^{a_1} \cdots u_l^{a_l}$, where $u_1, \cdots, u_l$ are distinct elements of $\mathcal{T}$. Then

$$s_F = s_{u_1}^{a_1} \cdots s_{u_l}^{a_l} a_1! \cdots a_l!,$$

so

$$\frac{k![n_{t_1}]_\beta \cdots [n_{t_k}]_\beta}{s_F} = \frac{(a_1 + \cdots + a_l)!}{a_1! \cdots a_l!} \left(\frac{[n_{t_1}]_\beta}{s_{t_1}}\right)^{a_1} \cdots \left(\frac{[n_{t_l}]_\beta}{s_{t_l}}\right)^{a_l}.$$

As a conclusion, putting $Q_k(S) = \dfrac{S(S+1)\cdots(S+k-1)}{k!}$,

$$\begin{aligned}
\Delta(X) &= \sum_{n \geq 1} \sum_{t_1^{a_1} \cdots t_l^{a_l} \in \mathcal{F}} \frac{(a_1 + \cdots + a_l)!}{a_1! \cdots a_l!} \beta^{a_1 + \cdots + a_l} Q_{a_1 + \cdots + a_l}(n(1 + 1/\beta) - 1) \\
&\qquad \left(\frac{[n_{t_1}]_\beta}{s_{t_1}} t_1\right)^{a_1} \cdots \left(\frac{[n_{t_l}]_\beta}{s_{t_l}} t_l\right)^{a_l} \otimes \left(\sum_{\substack{G \in \mathcal{T} \\ \text{weight}(G) = n}} \frac{[n_G]_\beta!}{s_G} G \right) + X \otimes 1 \\
&= X \otimes 1 + \sum_{n=1}^{\infty} (1 - \beta X)^{-n(1/\beta + 1) + 1} \otimes x_n.
\end{aligned}$$

So $\Delta(X) \in \mathcal{H} \widehat{\otimes} \mathcal{H}$. Projecting on the homogeneous component of degree n, we obtain $\Delta(x) \in \mathcal{H} \otimes \mathcal{H}$, so $\mathcal{H}_{1,\beta}$ is a Hopf subalgebra. □

Remarks.

(1) For $(\alpha, \beta) = (1, 0)$, $f(h) = e^h$ and for all $n \in \mathbb{N}$, $x_n = \displaystyle\sum_{\substack{t \in \mathcal{T} \\ \text{weight}(t) = n}} \frac{1}{s_t} t$.

(2) For $(\alpha, \beta) = (1, 1)$, $f(h) = (1 - h)^{-1}$ and for all $n \in \mathbb{N}$, $x_n = \displaystyle\sum_{\substack{t \in \mathcal{T} \\ \text{weight}(t) = n}} e_t t$.

(3) For $(\alpha, \beta) = (1, -1)$, $f(h) = 1 + h$ and, as $[i]_{-1} = 0$ if $i \geq 2$, for all $n \in \mathbb{N}^*$, x_n is the ladder of weight n.

2.4. What is $\mathcal{H}_{\alpha,\beta}$? If $\alpha = 0$, then $\mathcal{H}_{0,\beta} = K[\bullet]$. If $\alpha \neq 0$, then obviously $\mathcal{H}_{\alpha,\beta} = \mathcal{H}_{1,\beta}$; let us suppose that $\alpha = 1$. The Hopf algebra $\mathcal{H}_{1,\beta}$ is graded, connected and commutative. Dually, its graded dual $\mathcal{H}_{1,\beta}^*$ is a graded, connected, cocommutative Hopf algebra. By the Milnor-Moore theorem [**11**], it is isomorphic to the enveloping algebra of the Lie algebra of its primitive elements. We now denote this Lie algebra by $\mathfrak{g}_{1,\beta}$. The dual of $\mathfrak{g}_{1,\beta}$ is identified with the quotient space

$$\text{coPrim}(\mathcal{H}_{1,\beta}) = \frac{\mathcal{H}_{1,\beta}}{(1) \oplus \text{Ker}(\varepsilon)^2},$$

and the transposition of the Lie bracket is the Lie cobracket δ induced by

$$(\varpi \otimes \varpi) \circ (\Delta - \Delta^{op}),$$

where ϖ is the canonical projection on $\mathrm{coPrim}(\mathcal{H}_{1,\beta})$. As $\mathcal{H}_{1,\beta}$ is the polynomial algebra generated by the x_n's, a basis of $\mathrm{coPrim}(\mathcal{H}_{1,\beta})$ is $(\varpi(x_n))_{n\in\mathbb{N}^*}$. By Proposition 16,

$$\begin{aligned}(\varpi\otimes\varpi)\circ\Delta(X) &= (\varpi\otimes\varpi)\left(\sum_{n=1}^{\infty}(1-\beta X)^{-n(1/\beta+1)+1}\otimes x_n\right)\\ &= \sum_{n\geq 1}(n(1+\beta)-\beta)\,\varpi(X)\otimes\varpi(x_n).\end{aligned}$$

Projecting on the homogeneous component of degree k,

$$(\varpi\otimes\varpi)\circ\Delta(x_k)=\sum_{i+j=k}^{k}(j(1+\beta)-\beta)\,\varpi(x_i)\otimes\varpi(x_j).$$

As a consequence,

$$\delta(\varpi(x_k))=\sum_{i+j=k}(1+\beta)(j-i)\varpi(x_i)\otimes\varpi(x_j).$$

Dually, the Lie algebra $\mathfrak{g}_{1,\beta}$ has the dual basis $(Z_n)_{n\geq 1}$, with bracket given by

$$[Z_i,Z_j]=(1+\beta)(j-i)Z_{i+j}.$$

So, if $\beta\neq -1$, this Lie algebra is isomorphic to the Faà di Bruno Lie algebra $\mathfrak{g}_{\mathrm{FdB}}$, which has a basis $(f_n)_{n\geq 1}$, and whose bracket defined by $[f_i,f_j]=(j-i)f_{i+j}$. So $\mathcal{H}_{1,\beta}$ is isomorphic to the Hopf algebra $\mathcal{U}(\mathfrak{g}_{\mathrm{FdB}})^*$, namely the Faà di Bruno Hopf algebra [**3**], coordinate ring of the group of formal diffeomorphisms of the line tangent to Id, that is to say

$$G_{\mathrm{FdB}}=\left(\left\{\sum a_nh^n\in K[[h]]\mid a_0=0,\ a_1=1\right\},\circ\right).$$

THEOREM 17. (1) *If $\alpha\neq 0$ and $\beta\neq -1$, $\mathcal{H}_{\alpha,\beta}$ is isomorphic to the Faà di Bruno Hopf algebra.*
(2) *If $\alpha\neq 0$ and $\beta=-1$, $\mathcal{H}_{\alpha,\beta}$ is isomorphic to the Hopf algebra of symmetric functions.*
(3) *If $\alpha=0$, $\mathcal{H}_{\alpha,\beta}=K[\bullet]$.*

Remark. If β and $\beta'\neq -1$, then $\mathcal{H}_{1,\beta}$ and $\mathcal{H}_{1,\beta'}$ are isomorphic but are not equal, as shown by considering x_3.

3. FdB Lie algebras

In the preceding section, we considered Hopf subalgebras of $\mathcal{H}$, generated in each degree by a linear span of trees. Their graded dual is then the enveloping algebra of a Lie algebra $\mathfrak{g}$, graded, with Poincaré-Hilbert formal series

$$\frac{h}{1-h}=\sum_{n=1}^{\infty}h^n.$$

Under a hypothesis of commutativity, we show that such a $\mathfrak{g}$ is isomorphic to the Faà di Bruno Lie algebra, so the considered Hopf subalgebra is isomorphic to the Faà di Bruno Hopf algebra.

Remark. The proofs of this section were completed using MuPAD pro 4. The notebook of the computations can be found at [**4**].

3.1. Definitions and first properties.

DEFINITION 18. Let $\mathfrak{g}$ be an $\mathbb{N}$-graded Lie algebra. For all $n \in \mathbb{N}$, we denote by $\mathfrak{g}(n)$ the homogeneous component of degree n of $\mathfrak{g}$. We shall say that $\mathfrak{g}$ is FdB if

(1) $\mathfrak{g}$ is connected, that is to say $\mathfrak{g}(0) = (0)$.
(2) For all $i \in \mathbb{N}^*$, $\mathfrak{g}(i)$ is one-dimensional.
(3) For all $n \geq 2$, $[\mathfrak{g}(1), \mathfrak{g}(n)] \neq (0)$.

Let $\mathfrak{g}$ be a FdB Lie algebra. For all $i \in \mathbb{N}^*$, we fix a non-zero element Z_i of $\mathfrak{g}(i)$. By conditions (1) and (2), $(Z_i)_{i \geq 1}$ is a basis of $\mathfrak{g}$. By homogeneity of the bracket of $\mathfrak{g}$, for all $i, j \geq 1$, there exists an element $\lambda_{i,j} \in K$, such that

$$[Z_i, Z_j] = \lambda_{i,j} Z_{i+j}.$$

The Jacobi relation gives, for all $i, j, k \geq 1$,

$$\lambda_{i,j}\lambda_{i+j,k} + \lambda_{j,k}\lambda_{j+k,i} + \lambda_{k,i}\lambda_{k+i,j} = 0. \tag{3}$$

Moreover, by antisymmetry, $\lambda_{j,i} = -\lambda_{i,j}$ for all $i, j \geq 1$. Condition (3) is expressed by $\lambda_{1,j} \neq 0$ for all $j \neq 1$.

LEMMA 19. *Up to a change of basis, we can suppose that $\lambda_{1,j} = 1$ for all $j \geq 2$ and that $\lambda_{2,3} \in \{0, 1\}$.*

PROOF. We define a family of scalars by

$$\left\{ \begin{array}{rcl} \alpha_1 &=& 1, \\ \alpha_2 &\neq& 0, \\ \alpha_n &=& \lambda_{1,2} \cdots \lambda_{1,n-1}\alpha_2 \text{ if } n \geq 3. \end{array} \right.$$

By condition (3), all these scalars are non-zero. We put $Z'_i = \alpha_i Z_i$. Then, for all $j \geq 2$,

$$[Z'_1, Z'_j] = \alpha_j \lambda_{1,j} Z_{1+j} = \frac{\alpha_j \lambda_{1,j}}{\alpha_{j+1}} Z'_{1+j} = Z'_{1+j}.$$

So, replacing the Z_i's by the Z'_i's, we can suppose that $\lambda_{1,j} = 1$ if $j \geq 2$.

Let us suppose now that $\lambda_{2,3} \neq 0$. We then choose

$$\alpha_2 = \frac{\lambda_{1,3}\lambda_{1,4}}{\lambda_{2,3}}.$$

Then

$$[Z'_2, Z'_3] = \frac{\lambda_{2,3}\alpha_2\alpha_3}{\alpha_5} Z'_5 = \frac{\lambda_{2,3}\alpha_2\lambda_{1,2}\alpha_2}{\lambda_{1,2}\lambda_{1,3}\lambda_{1,4}\alpha_2} Z'_5 = Z'_5.$$

So, replacing the Z_i's by the Z'_i's, we can suppose that $\lambda_{2,3} = 1$. $\square$

LEMMA 20. *If $i, j \geq 2$, $\lambda_{i,j} = \displaystyle\sum_{k=0}^{i-2} \binom{i-2}{k} (-1)^k \lambda_{2,j+k}$.*

PROOF. Let us write (3) with $i = 1$,

$$\lambda_{1,j}\lambda_{j+1,k} + \lambda_{j,k}\lambda_{j+k,1} + \lambda_{k,1}\lambda_{k+1,j} = 0.$$

If $j, k \geq 2$, then $\lambda_{1,j} = -\lambda_{k,1} = -\lambda_{j+k,1} = 1$, so

$$\lambda_{k+1,j} = \lambda_{k,j} - \lambda_{k,j+1}. \tag{4}$$

If $k = 2$, this gives the announced formula for $i = 3$.

Let us prove the result by induction on i. This is obvious for $i = 2$ and done for $i = 3$. Let us assume the result at rank $i - 1$. Then, by (4),

$$\begin{array}{rcl}
\lambda_{i,j} & = & \lambda_{i-1,j} - \lambda_{i-1,j+1} \\
& = & \displaystyle\sum_{k=0}^{i-3} \binom{i-3}{k} (-1)^k \lambda_{2,j+k} - \sum_{k=0}^{i-3} \binom{i-3}{k} (-1)^k \lambda_{2,j+1+k} \\
& = & \displaystyle\sum_{k=0}^{i-3} \binom{i-3}{k} (-1)^k \lambda_{2,j+k} + \sum_{k=1}^{i-2} \binom{i-3}{k-1} (-1)^k \lambda_{2,j+k} \\
& = & \displaystyle\lambda_{2,j} + \sum_{k=1}^{i-3} \binom{i-2}{k} (-1)^k \lambda_{2,j+k} + (-1)^{i-2} \lambda_{2,j+i-2} \\
& = & \displaystyle\sum_{k=0}^{i-2} \binom{i-2}{k} (-1)^k \lambda_{2,j+k}.
\end{array}$$

So the result is true for all $i \geq 2$. $\square$

As a consequence, the $\lambda_{i,j}$'s are entirely determined by the $\lambda_{2,j}$'s. We can improve this result, using the following lemma:

Lemma 21. *For all $k \geq 2$,* $\displaystyle\lambda_{2,2k} = \frac{1}{2k-3} \sum_{l=0}^{2k-4} \binom{2k-2}{l} (-1)^l \lambda_{2,l+3}.$

Proof. Let us write the relation of Lemma 20 for $(i,j) = (3, 2k)$ and $(i,j) = (2k, 3)$,

$$\begin{array}{rcl}
\lambda_{3,2k} & = & \lambda_{2,2k} - \lambda_{2,2k+1}, \\
\lambda_{2k,3} & = & \displaystyle\sum_{l=0}^{2k-2} \binom{2k-2}{l} (-1)^l \lambda_{2,3+l} \\
& = & \displaystyle\lambda_{2,2k+1} - (2k-2)\lambda_{2,2k} + \sum_{l=0}^{2k-4} \binom{2k-2}{l} (-1)^l \lambda_{2,3+l}.
\end{array}$$

Summing these two relations,

$$-(2k-3)\lambda_{2,2k} + \sum_{l=0}^{2k-4} \binom{2k-2}{l} (-1)^l \lambda_{2,3+l} = 0.$$

This gives the announced result. $\square$

As a consequence, the $\lambda_{i,j}$'s are entirely determined by the $\lambda_{2,j}$'s, with j odd. In order to ease the notation, we put $\mu_j = \lambda_{2,j}$ for all j odd. Then, for example,

$$\left\{\begin{array}{rcl}
\lambda_{2,4} & = & \mu_3, \\
\lambda_{2,6} & = & 2\mu_5 - \mu_3, \\
\lambda_{2,8} & = & 3\mu_7 - 5\mu_5 + 3\mu_3, \\
\lambda_{2,10} & = & 4\mu_9 - 14\mu_7 + 28\mu_5 - 17\mu_3, \\
\lambda_{2,12} & = & 5\mu_{11} - 30\mu_9 + 126\mu_7 - 255\mu_5 + 155\mu_3.
\end{array}\right.$$

Moreover, we showed that we can assume $\mu_3 = 0$ or 1.

Remark. The coefficient $\lambda_{2,2k+4}$ is then a linear span of coefficients μ_{2i+3}, $0 \leq i \leq k$. We put, for all $k \in \mathbb{N}$,

$$\lambda_{2,2k+4} = \sum_{i=0}^{k} a_{k,i}\mu_{2i+3}.$$

We can prove inductively the following results:

(1) For all $k \in \mathbb{N}$, $a_{k,k} = k + 1$.
(2) For all $k \geq 1$, $a_{k,k-1} = -\frac{1}{4}\binom{2k+2}{3}$. Up to the sign, this is the sequence $A000330$ of [**12**] (pyramidal numbers).
(3) For all $k \geq 2$, $a_{k,k-2} = \frac{1}{2}\binom{2k+2}{5}$. This is the sequence $A053132$ of [**12**].
(4) The sequence $(-a_{k,0})$ is the sequence of signed Genocchi numbers, $A001469$ in [**12**].

It seems that for all $i \leq k$,

$$a_{k,k-i} = \frac{2^{2i+2}-1}{i+1}B_{2i+2}\binom{2k+2}{2i+1},$$

where the B_{2n}'s are the Bernoulli numbers (see sequence $A002105$ of [**12**]).

3.2. Case where $\mu_3 = 1$. In this case:

Lemma 22. *Suppose that $\mu_3 = 1$. Then $\mu_5 = 1$ or $\dfrac{9}{10}$.*

Proof. By relation (3) for $(i, j, k) = (2, 3, 4)$,

$$5\mu_5 - 3\mu_7 + \mu_5\mu_7 - 3 = 0.$$

If $\mu_5 = 3$, we obtain $12 = 0$, absurd. So $\mu_7 = -\dfrac{5\mu_5 - 3}{\mu_5 - 3}$. By relation (3) for $(i, j, k) = (2, 3, 6)$,

$$\frac{-2}{\mu_5 - 3}\left((2\mu_5 - 4)\mu_5\mu_9 + 3 - 7\mu_5 + \mu_5^2 - 5\mu_5^3\right) = 0.$$

If $\mu_5 = 0$, we obtain $2 = 0$, absurd. If $\mu_5 = 2$, we obtain $66 = 0$, absurd. So

$$\mu_9 = -\frac{3 - 7\mu_5 + \mu_5^2 - 5\mu_5^3}{(2\mu_5 - 4)\mu_5}.$$

Writing relation (3) for $(i, j, k) = (3, 4, 5)$,

$$-\frac{9(\mu_5 - 1)^5(10\mu_5 - 9)}{\mu_5(\mu_5 - 2)(\mu_5 - 3)^2} = 0.$$

So $\mu_5 = 1$ or $\mu_5 = \dfrac{9}{10}$. □

Proposition 23. *Let us suppose that $\mu_3 = \mu_5 = 1$. Then*

$$\begin{cases} \lambda_{1,j} & = & 1 \text{ if } j \geq 2, \\ \lambda_{2,j} & = & 1 \text{ if } j \geq 3, \\ \lambda_{i,j} & = & 0 \text{ if } i, j \geq 3. \end{cases}$$

PROOF. Let us first prove inductively on j that $\lambda_{2,j} = 1$ if $j \geq 3$. This is immediate if $j = 3$ or 5 and comes from $\lambda_{2,4} = \mu_3$ for $j = 4$. Let us suppose that $\lambda_{2,j} = 1$ for $3 \leq j < n$. If $n = 2k$ is even, then

$$\lambda_{2,2k} = \frac{1}{2k-3} \sum_{l=0}^{2k-4} \binom{2k-2}{l} (-1)^l = 1 + \frac{1}{2k-3} \sum_{l=0}^{2k-2} \binom{2k-2}{l} (-1)^l = 1.$$

If $n = 2k+1$ is odd, write relation (3) for $(i,j,k) = (2,3,2k-2)$,

$$\begin{aligned} \lambda_{2,3}\lambda_{5,2k-2} + \lambda_{3,2k-2}\lambda_{2k+1,2} + \lambda_{2k-2,2}\lambda_{2k,3} &= 0, \\ \sum_{l=0}^{3} \binom{3}{l} (-1)^l \lambda_{2,2k-2+l} - \lambda_{2,2k-2} + \lambda_{2,2k-1} + \lambda_{2,2k-2}(\lambda_{2,2k} - \lambda_{2,2k+1}) &= 0, \\ \lambda_{2,2k-2} - 3\lambda_{2,2k-1} + 3\lambda_{2,2k} - \lambda_{2,2k+1} + 1 - \lambda_{2,2k+1} &= 0, \\ 1 - 3 + 3 - 2\lambda_{2,2k+1} + 1 &= 0, \end{aligned}$$

so $\lambda_{2,2k+1} = 1$. Finally, if $i, j \geq 3$, $\lambda_{i,j} = \displaystyle\sum_{k=0}^{i-2} \binom{i-2}{k} (-1)^k = 0$. □

LEMMA 24. *For all $N \geq 2$,*

$$S_N = \sum_{l=0}^{N} \binom{N}{l} (-1)^l \frac{(l+1)}{(l+2)(l+3)} = \frac{N-1}{(N+3)(N+2)(N+1)}.$$

PROOF. Indeed,

$$\begin{aligned} S_N &= \sum_{l=0}^{N} \frac{N!}{(l+3)!(N-l)!} (-1)^l (l+1)^2 \\ &= \frac{1}{(N+3)(N+2)(N+1)} \sum_{j=3}^{N+3} \binom{N+3}{j} (-1)^j (j-2)^2 \\ &= \frac{1}{(N+3)(N+2)(N+1)} \sum_{j=0}^{N+3} \binom{N+3}{j} (-1)^j (j-2)^2 \\ &\quad - \frac{1}{(N+3)(N+2)(N+1)} (4 - (N+3)) \\ &= 0 + \frac{N-1}{(N+3)(N+2)(N+1)}. \end{aligned}$$

□

PROPOSITION 25. *Let us suppose that $\mu_3 = 1$ and $\mu_5 = \frac{9}{10}$. Then, for all $i, j \geq 1$,*

$$\lambda_{i,j} = \frac{6(i-j)(i-2)!(j-2)!}{(i+j-2)!}.$$

PROOF. We first prove that $\lambda_{2,n} = \dfrac{6(n-2)}{(n-1)n}$. This is immediate for $n = 1$, 2, 3, 4, 5. Let us assume the result for all $j < n$, with $n \geq 6$. If $n = 2k$ is even, using

Lemma 20,

$$\lambda_{2,2k}=\frac{6}{2k-3}\sum_{l=0}^{2k-4}\binom{2k-2}{l}(-1)^l\frac{l+1}{(l+2)(l+3)}.$$

Then Lemma 24 gives the result. If $n=2k+3$ is odd, let us write the relation (3) with $(i,j,k)=(2,3,2k)$,

$$\lambda_{2,3}\lambda_{5,2k}+\lambda_{3,2k}\lambda_{2k+3,2}+\lambda_{2k,2}\lambda_{2k+2,3}=0.$$

So, with relation (4)

$$\begin{array}{rcl}\lambda_{2,2k}-3\lambda_{2,2k+1}+3\lambda_{2,2k+2}-\lambda_{2,2k+3}&&\\ -\lambda_{2,2k+3}(\lambda_{2,2k}-\lambda_{2,2k+1})+\lambda_{2,2k}(\lambda_{2,2k+2}-\lambda_{2,2k+3})&=&0,\\ \lambda_{2,2k+3}(-1-2\lambda_{2,2k}+\lambda_{2,2k+1})+\lambda_{2,2k}-3\lambda_{2,2k+1}&&\\ +3\lambda_{2,2k+2}+\lambda_{2,2k}\lambda_{2,2k+2}&=&0,\\ -\lambda_{2,2k+3}\dfrac{(2k+3)(k-1)(2k+5)}{k(2k+1)(2k-1)}+\dfrac{3(2k+5)(k-1)}{k(k+1)(2k-1)}&=&0,\end{array}$$

which implies the result.

Let us now prove the result by induction on i. This is immediate if $i=1$, and the first part of this proof for $i=2$. Let us assume the result at rank i. Then, by relation (4),

$$\begin{array}{rcl}\lambda_{i+1,j}&=&\lambda_{i,j}-\lambda_{i,j+1}\\ &=&6\dfrac{(j-i)(i-2)!(j-2)!}{(i+j-2)!}-6\dfrac{(j+1-i)(i-2)!(j-1)!}{(i+j-1)!}\\ &=&6\dfrac{(i-1)!(j-2)!(j+1-i)}{(i+j-1)!}.\end{array}$$

So the result is true for all i,j. □

3.3. Case where $\mu_3=0$.

In this case:

Proposition 26. *If $\mu_3=0$, then $\lambda_{i,j}=0$ for all $i,j\geq 2$.*

Proof. We first prove that $\mu_5=0$. If not, by (3) for $(i,j,k)=(2,3,4)$, $\mu_5\mu_7=0$, so $\mu_7=0$. By (3) with $(i,j,k)=(2,3,7)$, $-5\mu_5(28\mu_5+4\mu_9)=0$, so $\mu_9=-7\mu_5$. By (3) with $(i,j,k)=(3,4,5)$, $-36\mu_5^2=0$: contradiction. So $\mu_5=0$.

Let us then prove that all the μ_{2k+1}'s, $k\geq 1$, are zero. We assume that $\mu_3=\mu_5=\cdots=\mu_{2k-1}=0$, and $\mu_{2k+1}\neq 0$, with $l\geq 3$. By lemma 21, $\lambda_{2,2}=\cdots=\lambda_{2,2k-1}=\lambda_{2,2k}=0$ and $\lambda_{2,2k+1}\neq 0$. By relation (3) for $(i,j,k)=(2,3,n)$, combined with (4),

$$\begin{array}{rcl}\lambda_{2,3}\lambda_{5,n}+\lambda_{3,n}\lambda_{n+3,2}+\lambda_{n,2}\lambda_{n+2,3}&=&0,\\ -(\lambda_{2,n}-\lambda_{2,n+1})\lambda_{2,n+3}+\lambda_{2,n}(\lambda_{2,n+2}-\lambda_{2,n+3})&=&0.\end{array}$$

For $n=2k$, this gives $\lambda_{2,2k+1}\lambda_{2,2k+3}=0$, so $\lambda_{2,2k+3}=0$. For $n=2k+2$,

$$\lambda_{2,2k+2}(\lambda_{2,2k+4}-2\lambda_{2,2k+5})=0. \tag{5}$$

By Lemma 21,

$$\begin{aligned}\lambda_{2,2k+2} &= \frac{1}{2k-1}\binom{2k}{2k-2}\lambda_{2,2k+1}\\ &= k\lambda_{2,2k+1},\\ \lambda_{2,2k+4} &= \frac{1}{2k+1}\left(\binom{2k+2}{2k}\lambda_{2,2k+3}-\binom{2k+2}{2k-1}\lambda_{2,2k+2}+\binom{2k+2}{2k-2}\lambda_{2,2k+1}\right)\\ &= -\frac{k(k+1)(2k+1)}{6}\lambda_{2,2k+1}.\end{aligned}$$

With (5),

$$\lambda_{2,2k+5}=-\frac{k(k+1)(2k+1)}{12}\lambda_{2,2k+1}.$$

By relation (3) for $(i,j,k)=(3,4,2k)$,

$$\lambda_{3,4}\lambda_{7,2k}+\lambda_{4,2k}\lambda_{4+2k,3}+\lambda_{2k,3}\lambda_{2k+3,4}=0.$$

Moreover, using Lemma 20,

$$\left\{\begin{array}{rcl}\lambda_{3,4}&=&\lambda_{2,4}-\lambda_{2,5}=0,\\ \lambda_{4,2k}&=&\lambda_{2,2k}-2\lambda_{2,2k+1}+\lambda_{2,2k+2},\\ \lambda_{3,4+2k}&=&\lambda_{2,4+2k}-\lambda_{2,5+2k},\\ \lambda_{2k,3}&=&-\lambda_{2,2k}+\lambda_{2,2k+1},\\ \lambda_{3+2k,4}&=&-\lambda_{2,3+2k}+2\lambda_{2,4+2k}-\lambda_{2,5+2k}.\end{array}\right.$$

This gives

$$\frac{\lambda_{2,2k+1}^2k(3k-11)(2k+1)(k+1)}{12}=0,$$

so $\lambda_{2,2k+1}=\mu_{2k+1}=0$: contradiction. So all the μ_{2k+1}, $k\geq 1$, are zero. By Lemma 21, the $\lambda_{2,i}$'s, $i\geq 2$, are zero. By Lemma 20, the $\lambda_{i,j}$'s, $i,j\geq 2$, are zero. □

THEOREM 27. *Up to isomorphism, there are three* FdB *Lie algebras:*

(1) *The Faà di Bruno Lie algebra $\mathfrak{g}_{\mathrm{FdB}}$, with basis $(e_i)_{i\geq 1}$, and the bracket given by $[e_i,e_j]=(j-i)e_{i+j}$ for all $i,j\geq 1$.*
(2) *The corolla Lie algebra $\mathfrak{g}_c$, with basis $(e_i)_{i\geq 1}$, and the bracket given by $[e_1,e_j]=e_{j+1}$ and $[e_i,e_j]=0$ for all $i,j\geq 2$.*
(3) *Another Lie algebra $\mathfrak{g}_3$, with basis $(e_i)_{i\geq 1}$, and the bracket given by $[e_1,e_i]=e_{i+1}$, $[e_2,e_j]=e_{j+2}$, and $[e_i,e_j]=0$ for all $i\geq 2$, $j\geq 3$.*

PROOF. We have first to prove that these are indeed Lie algebras: this is done by direct computations. Let $\mathfrak{g}$ be a FdB Lie algebra. We showed that three cases are possible:

(1) $\mu_3=1$ and $\mu_5=\frac{9}{10}$. By Proposition 25, putting $e_i=\frac{Z_i}{6(i-2)!}$ if $i\geq 2$ and $e_1=Z_1$, we obtain the Faà di Bruno Lie algebra.
(2) $\mu_3=\mu_5=1$. By Proposition 23, we obtain the third Lie algebra.
(3) $\mu_3=0$. By Proposition 26, we obtain the corolla Lie algebra.

□

COROLLARY 28. *Let $\mathfrak{g}$ be a* FdB *Lie algebra, such that if i and j are two distinct elements of $\mathbb{N}^*$, then $[\mathfrak{g}(i),\mathfrak{g}(j)]\neq(0)$. Then $\mathfrak{g}$ is isomorphic to the Faà di Bruno Lie algebra.*

4. Dual of enveloping algebras of FdB Lie algebras

We realized in the first section the Faà di Bruno Hopf algebra, the dual of the enveloping algebra of the Faà di Bruno Lie algebra, as a Hopf subalgebra of $\mathcal{H}$. We now give a similar result for the two other FdB Lie algebra.

4.1. The corolla Lie algebra.

DEFINITION 29. We denote by $\mathcal{H}_c$ the subalgebra of $\mathcal{H}$ generated by the corollas.

PROPOSITION 30. *$\mathcal{H}_c$ is a graded Hopf subalgebra of $\mathcal{H}$. Its dual is isomorphic to the enveloping algebra of the corolla Lie algebra.*

PROOF. The subalgebra $\mathcal{H}_c$, being generated by homogeneous elements, is graded. By Lemma 9, $\mathcal{H}_c$ is a Hopf subalgebra of $\mathcal{H}$. As it is commutative, its dual is the enveloping algebra of the Lie algebra $\mathrm{Prim}(\mathcal{H}_c^*)$. The dual of this Lie algebra is the Lie coalgebra $\mathrm{coPrim}(\mathcal{H}_c) = \dfrac{\mathcal{H}_c}{(1) \oplus \mathrm{Ker}(\varepsilon)^2}$, with cobracket δ induced by $(\varpi \otimes \varpi) \circ (\Delta - \Delta^{op})$. As $\mathcal{H}_c$ is generated by the corollas, a basis of $\mathrm{coPrim}(\mathcal{H}_c)$ is $(\varpi(c_n))_{n \geq 1}$. Moreover, if $n \geq 1$,

$$\begin{array}{rcl} (\varpi \otimes \varpi) \circ \Delta(c_n) & = & \varpi(c_1) \otimes \varpi(c_{n-1}), \\ \delta(c_n) & = & \varpi(c_1) \otimes \varpi(c_{n-1}) - \varpi(c_{n-1}) \otimes \varpi(c_1). \end{array}$$

Let $(Z_n)_{n \geq 1}$ be the basis of $\mathrm{Prim}(\mathcal{H}_c^*)$, the dual of the basis $(\varpi(c_n))_{n \geq 1}$. By duality, for all $i, j \in \mathbb{N}^*$, such that $i \neq j$,

$$[Z_i, Z_j] = \left\{ \begin{array}{l} Z_{1+j} \text{ if } i = 1, \\ -Z_{i+1} \text{ if } j = 1, \\ 0 \text{ otherwise.} \end{array} \right.$$

So $\mathrm{Prim}(\mathcal{H}_c^*)$ is isomorphic to the corolla Lie algebra, via the isomorphism

$$\left\{ \begin{array}{rcl} \mathfrak{g}_c & \longrightarrow & \mathrm{Prim}(\mathcal{H}_c^*) \\ e_i & \longrightarrow & Z_i. \end{array} \right.$$

Dually, $\mathcal{H}_c$ is isomorphic to $\mathcal{U}(\mathfrak{g}_c)^*$. $\square$

Remark. We work in $K[\mathcal{T}][\beta]$. The generators of $\mathcal{H}_{1,\beta}$ then satisfy

$$x_{n+1} = \frac{[n]_\beta!}{n!} c_{n+1} + \mathcal{O}(\beta^{n-2}).$$

Note that the degree of $[n]_\beta!$ in β is $n-1$. So

$$\lim_{\beta \to \infty} \frac{n!}{[n]_\beta!} x_{n+1} = c_{n+1}.$$

In this sense, the Hopf algebra $\mathcal{H}_c$ is the limit of $\mathcal{H}_{1,\beta}$ when β goes to infinity.

4.2. The third FdB Lie algebra.

We consider the following element of $\widehat{\mathcal{H}}$:

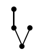
$$Y = B^+\left(\exp\left(\overset{\bullet}{\underset{\bullet}{|}} - \frac{1}{2} \bullet^2 + \bullet\right)\right) = \sum_{n \geq 1} y_n.$$

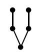
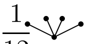
For example

$y_1 =$

$y_2 =$,

$y_3 =$,

$y_4 =$  $- \frac{1}{3}$  ,

$y_5 = \frac{1}{2}$  $- \frac{1}{12}$  .

Definition 31. We denote by $\mathcal{H}_3$ the subalgebra of $\mathcal{H}$ generated by the y_n's.

Proposition 32. *$\mathcal{H}_3$ is a graded Hopf subalgebra of $\mathcal{H}$. Its dual is isomorphic to the enveloping algebra of the third* FdB *Lie algebra.*

Proof. The subalgebra $\mathcal{H}_3$, being generated by homogeneous elements, is graded. An easy computation proves that $X =$ $- \frac{1}{2}$  $^2 +$ is a primitive element of $\mathcal{H}$. As a consequence, in $\widehat{\mathcal{H}}$, by (1),

$$\begin{aligned}
\Delta(X) &= X \otimes 1 + 1 \otimes X,\\
\Delta(\exp(X)) &= \exp(X \otimes 1 + 1 \otimes X)\\
&= \exp(X \otimes 1)\exp(1 \otimes X)\\
&= (\exp(X) \otimes 1)(1 \otimes \exp(X))\\
&= \exp(X) \otimes \exp(X),\\
\Delta(Y) &= \Delta \circ B^+(\exp(X))\\
&= Y \otimes 1 + \exp(X) \otimes Y.
\end{aligned}$$

Moreover, $X = y_2 - \frac{1}{2}y_1^2 + y_1 \in \mathcal{H}_3$, so taking the homogeneous component of degree n of $\Delta(Y)$, we obtain

$$\Delta(y_n) = y_n \otimes 1 + \sum_{k=1}^{n} \sum_{l=1}^{n-k} \sum_{a_1+\cdots+a_l=n-k} \frac{1}{l!} x_{a_1} \cdots x_{a_l} \otimes y_k,$$

where $x_1 =$ $= y_1$, $x_2 =$ $- \frac{1}{2}$  $= y_2 - \frac{1}{2}y_1^2$ and $x_i = 0$ if $i \geq 3$, so $\Delta(y_n) \in \mathcal{H}_3 \otimes \mathcal{H}_3$ and $\mathcal{H}_3$ is a Hopf subalgebra of $\mathcal{H}$. As it is commutative, its dual is the enveloping algebra of the Lie algebra $\mathrm{Prim}(\mathcal{H}_3^*)$. The dual of this Lie algebra is the Lie coalgebra $\mathrm{coPrim}(\mathcal{H}_3) = \dfrac{\mathcal{H}_3}{(1) \oplus \mathrm{Ker}(\varepsilon)^2}$, with cobracket δ induced by $(\varpi \otimes \varpi) \circ (\Delta - \Delta^{op})$. As $\mathcal{H}_3$ is generated by the y_n's, a basis of $\mathrm{coPrim}(\mathcal{H}_3)$ is $(\varpi(y_n))_{n \geq 1}$. Moreover,

$$\begin{aligned}
(\varpi \otimes \varpi) \circ \Delta(Y) &= \varpi(\exp(X)) \otimes \varpi(Y)\\
&= \varpi(X) \otimes \varpi(Y)\\
&= (\varpi(y_2) + \varpi(y_1)) \otimes \varpi(Y).
\end{aligned}$$

Taking the homogeneous component of degree n, with the convention $y_{-1} = y_0 = 0$,

$$\begin{array}{rcl}(\varpi \otimes \varpi) \circ \Delta(y_n) & = & \varpi(y_2) \otimes \varpi(y_{n-2}) + \varpi(y_1) \otimes \varpi(y_{n-1}),\\ \delta(\varpi(y_n)) & = & \varpi(y_2) \otimes \varpi(y_{n-2}) + \varpi(y_1) \otimes \varpi(y_{n-1})\\ & & -\varpi(y_{n-2}) \otimes \varpi(y_2) - \varpi(y_{n-1}) \otimes \varpi(y_1).\end{array}$$

Let $(Z_n)_{n\geq 1}$ be the basis of $Prim(\mathcal{H}_3^*)$ dual to the basis $(\varpi(c_n))_{n\geq 1}$. By duality, for all $i, j \in \mathbb{N}^*$, such that $i \geq 2$ and $j \geq 3$,

$$\left\{\begin{array}{rcl}[Z_1, Z_i] & = & Z_{1+j},\\ [Z_2, Z_j] & = & Z_{2+j},\\ [Z_i, Z_j] & = & 0.\end{array}\right.$$

So $Prim(\mathcal{H}_3^*)$ is isomorphic to third FdB Lie algebra, via the morphism

$$\left\{\begin{array}{rcl}\mathfrak{g}_3 & \longrightarrow & Prim(\mathcal{H}_c^*)\\ e_i & \longrightarrow & Z_i.\end{array}\right.$$

Dually, $\mathcal{H}_3$ is isomorphic to $\mathcal{U}(\mathfrak{g}_3)^*$. □

References

[1] Christoph Bergbauer and Dirk Kreimer, *Hopf algebras in renormalization theory: locality and Dyson-Schwinger equations from Hochschild cohomology*, IRMA Lect. Math. Theor. Phys., vol. 10, Eur. Math. Soc., Zürich, 2006, arXiv:hep-th/0506190.

[2] Alain Connes and Dirk Kreimer, *Hopf algebras, renormalization and noncommutative geometry*, Comm. Math. Phys **199** (1998), no. 1, 203–242, arXiv:hep-th/9808042.

[3] Héctor Figueroa and José M. Gracia-Bondia, *On the antipode of Kreimer's Hopf algebra*, Modern Phys. Lett. A **16** (2001), no. 22, 1427–1434, arXiv:hep-th/9912170.

[4] Loïc Foissy, *page personnelle*, http://loic.foissy.free.fr/pageperso/publications.html.

[5] ———, *Les algèbres de Hopf des arbres enracinés, II*, Bull. Sci. Math. **126** (2002), no. 4, 1249–288, arXiv:math.QA/0105212.

[6] ———, *Faà di Bruno subalgebras of the Hopf algebra of planar trees from combinatorial Dyson-Schwinger equations*, Advances in Mathematics **218** (2008), 136–162, ArXiv:0707.1204.

[7] Michael E. Hoffman, *Combinatorics of rooted trees and Hopf algebras*, Trans. Amer. Math. Soc. **355** (2003), no. 9, 3795–3811.

[8] Dirk Kreimer, *Combinatorics of (perturbative) Quantum Field Theory*, Phys. Rep. **4–6** (2002), 387–424, arXiv:hep-th/0010059.

[9] ———, *Dyson-Schwinger equations: from Hopf algebras to number theory*, Universality and renormalization, Fields Inst. Commun., no. 50, Amer. Math. Soc., Providence, RI, 2007, arXiv:hep-th/0609004.

[10] Dirk Kreimer and Karen Yeats, *An étude in non-linear Dyson-Schwinger equations*, Nuclear Phys. B Proc. Suppl. **160** (2006), 116–121, arXiv:hep-th/0605096.

[11] John W. Milnor and John C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264.

[12] N. J. A Sloane, *On-line encyclopedia of integer sequences*, http://www.research.att.com/ ˜njas/ sequences/ Seis.html.

[13] Richard P. Stanley, *Enumerative combinatorics. Vol. 1.*, Cambridge Studies in Advanced Mathematics, no. 49, Cambridge University Press, Cambridge, 1997.

[14] ———, *Enumerative combinatorics. Vol. 2.*, Cambridge Studies in Advanced Mathematics, no. 62, Cambridge University Press, Cambridge, 1999.

Laboratoire de Mathématiques EA4535, Université de Reims, Moulin de la Housse - BP 1039, 51687 REIMS Cedex 2 France

E-mail address: `loic.foissy@univ-reims.fr`