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The Infinitesimal Hopf Algebra and the Operads of Planar Forests

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We introduce two operads based on the set of planar forests. With its usual product and two other products defined by different types of graftings, the algebra of planar rooted trees $\mathcal{H}$ becomes an algebra over these operads. The compatibility with the infinitesimal coproduct of $\mathcal{H}$ and these structures is studied. As an application, an inductive way of computing the dual basis of $\mathcal{H}$ for its infinitesimal pairing is given. Moreover, three Cartier–Quillen–Milnor–Moore theorems are given for the operads of planar forests and a rigidity theorem for one of them.

Introduction

The Connes–Kreimer Hopf algebra of rooted trees, introduced in [1, 7–9], is a commutative, noncocommutative Hopf algebra, its coproduct being given by admissible cuts of trees. A noncommutative version, the Hopf algebra of planar rooted trees, is introduced in [4, 6]. We furthemore introduced in [5] an infinitesimal version of this object, replacing admissible cuts by left-admissible cuts: this last object is here denoted by $\mathcal{H}$. Similarly, with the Hopf case, $\mathcal{H}$ is a self-dual object and it owns a nondegenerate, symmetric Hopf pairing, denoted by $\langle -, - \rangle$. This pairing is related to a partial order on the set of planar rooted forests, making it isomorphic to the Tamari poset. As a consequence, $\mathcal{H}$ is given

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a dual basis denoted by $(f_F)_{F\in\mathbf{F}}$, indexed by the set $\mathbf{F}$ of planar forest. In particular, the sub-family $(f_t)_{t\in\mathbf{T}}$ indexed by the set of planar rooted trees $\mathbf{T}$ is a basis of the space of primitive elements of $\mathcal{H}$.

The aim of this text is to introduce two structures of operad on the space of planar forests. We introduce two (nonsymmetric) operads $\mathbb{P}_\searrow$ and $\mathbb{P}_\nearrow$ defined in the following way:

(1) $\mathbb{P}_\searrow$ is generated by m and $\searrow\in\mathbb{P}_\searrow(2)$, with relations:

$$\begin{cases} m\circ(\searrow,I)=\searrow\circ(I,m),\\ m\circ(m,I)=m\circ(I,m),\\ \searrow\circ(m,I)=\searrow\circ(I,\searrow).\end{cases}$$

(2) $\mathbb{P}_\nearrow$ is generated by m and $\nearrow\in\mathbb{P}_\nearrow(2)$, with relations:

$$\begin{cases} m\circ(\nearrow,I)=\nearrow\circ(I,m),\\ m\circ(m,I)=m\circ(I,m),\\ \nearrow\circ(\nearrow,I)=\nearrow\circ(I,\nearrow).\end{cases}$$

Note that these operads are not the operad of algebras with two compatible associative products of [2], which is also described in terms of planar rooted trees. We then introduce two products on $\mathcal{H}$ or on its augmentation ideal $\mathcal{M}$, denoted by $\nearrow$ and $\searrow$. The product $F\nearrow G$ consists of grafting F on the left leaf of G and the product $F\searrow G$ consists of grafting F on the left root of G. Together with its usual product m, $\mathcal{M}$ becomes both a $\mathbb{P}_\searrow$- and a $\mathbb{P}_\nearrow$-algebra. More precisely, $\mathcal{M}$ is the free $\mathbb{P}_\searrow$- and $\mathbb{P}_\nearrow$-algebra generated by a single element $\bullet$. As a consequence, $\mathbb{P}_\searrow$ and $\mathbb{P}_\nearrow$ inherit a combinatorial representation using planar forests, with composition iteratively described using the products $\searrow$ and $\nearrow$.

We then give several applications of these operadic structures. For example, the antipode of $\mathcal{H}$ is described in terms of the operad $\mathbb{P}_\searrow$. We show how to compute elements f_ts, with $t\in\mathbf{T}$, using the action of $\mathbb{P}_\searrow$, and the elements f_Fs, $F\in\mathbf{F}$ from the preceding ones using the action of $\mathbb{P}_\nearrow$. Combining all these results, it is possible to compute by induction the basis $(f_F)_{F\in\mathbf{F}}$.

We finally study the compatibilities of products m, $\nearrow$, $\searrow$, the coproduct $\tilde\Delta$ and the coproduct $\tilde\Delta_\nearrow$ dual of $\nearrow$. This leads to the definition of two types of $\mathbb{P}_\nearrow$-bialgebras, and one type of $\mathbb{P}_\searrow$-bialgebras. Each type then defines a suboperad of $\mathbb{P}_\nearrow$ or $\mathbb{P}_\searrow$ corresponding to primitive elements of $\mathcal{M}$, which are explicitly described:

(1) The first one is a free operad, generated by the element $\overset{\bullet}{\bullet} - \bullet\bullet \in \mathbb{P}_\nearrow(2)$. As a consequence, the space of primitive elements of $\mathcal{H}$ admits a basis $(p_t)_{t \in \mathbf{T}_b}$ indexed by the set of planar binary trees. The link with the basis $(f_t)_{t \in \mathbf{T}}$ is given with the help of the Tamari order.

(2) The second one admits a combinatorial representation in terms of planar rooted trees. It is generated by the corollas $c_n \in \mathbb{P}_\nearrow(n)$, $n \geq 2$, with the following relations: for all $k, l \geq 2$:

$$c_k \circ (c_l, \underbrace{I, \ldots, I}_{k-1 \text{ times}}) = c_l \circ (\underbrace{I, \ldots, I}_{l-1 \text{ times}}, c_k).$$

(3) The third one admits a combinatorial representation in terms of planar rooted trees, and is freely generated by $\overset{\bullet}{\bullet} \in \mathbb{P}_\searrow(2)$.

We also give the definition of a double $\mathbb{P}_\nearrow$-bialgebra, combining the two types of $\mathbb{P}_\nearrow$-bialgebras already introduced. We then prove a rigidity theorem: any double $\mathbb{P}_\nearrow$-bialgebra connected as a coalgebra is isomorphic to a decorated version of $\mathcal{M}$.

This text is organized as follows: the first section gives several recalls on the infinitesimal Hopf algebra of planar rooted trees and its pairing. The two products $\searrow$ and $\nearrow$ are introduced in Section 2, as well as the combinatorial representation of the two associated operads. The applications to the computation of $(f_F)_{F \in \mathbf{F}}$ is given in Section 3. Section 4 is devoted to the study of the suboperads of primitive elements and the last section deals with the rigidity theorem for double $\mathbb{P}_\nearrow$-bialgebras.

Notations.

(1) We shall denote by K a commutative field, of any characteristic. Every vector space, algebra, coalgebra, etc. will be taken over K.

(2) Let (A, Δ, ε) be a counitary coalgebra. Let $1 \in A$, nonzero, such that $\Delta(1) = 1 \otimes 1$. We then define the noncounitary coproduct:

$$\tilde{\Delta} : \begin{cases} Ker(\varepsilon) \longrightarrow Ker(\varepsilon) \otimes Ker(\varepsilon) \\ a \qquad\quad\; \longrightarrow \tilde{\Delta}(a) = \Delta(a) - a \otimes 1 - 1 \otimes a. \end{cases}$$

We shall use the Sweedler notations $\Delta(a) = a^{(1)} \otimes a^{(2)}$ and $\tilde{\Delta}(a) = a' \otimes a''$.

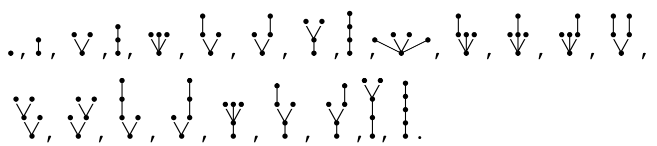
1 Planar Rooted Forests and Their Infinitesimal Hopf Algebra

We here recall some results and notations of [5].

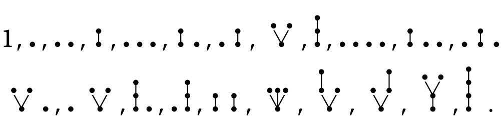
1.1 Planar trees and forests

(1) The set of planar trees is denoted by **T**, and the set of planar forests is denoted by **F**. The weight of a planar forest is the number of its vertices. For all $n \in \mathbb{N}$, we denote by $\mathbf{F}(n)$ the set of planar forests of weight n.

Examples. Planar rooted trees of weight ≤ 5:



Planar rooted forests of weight ≤ 4:



(2) The algebra $\mathcal{H}$ is the free associative, unitary algebra generated by **T**. As a consequence, a linear basis of $\mathcal{H}$ is given by **F**, and its product is given by the concatenation of planar forests.

(3) We shall also need two partial orders and a total order on the set $\mathrm{Vert}(F)$ of vertices of $F \in \mathbf{F}$, defined in [4, 5]. We put $F = t_1 \dots t_n$, and let s, s' be two vertices of F.

(a) We shall say that $s \geq_{\text{high}} s'$ if there exists a path from s' to s in F, the edges of F being oriented from the roots to the leaves. Note that $\geq_{\text{high}}$ is a partial order, whose Hasse graph is the forest F.

(b) If s and s' are not comparable for $\geq_{\text{high}}$, we shall say that $s \geq_{\text{left}} s'$ if one of these assertions is satisfied:

(i) s is a vertex of t_i and s' is a vertex of t_j, with $i < j$.

(ii) s and s' are vertices of the same t_i, and $s \geq_{\text{left}} s'$ in the forest obtained from t_i by deleting its root.

This defines the partial order $\geq_{\text{left}}$ for all forests F, by induction on the weight.

(c) We shall say that $s \geq_{h,l} s'$ if $s \geq_{\text{high}} s'$ or $s \geq_{\text{left}} s'$. This defines a total order on the vertices of F.

1.2 Infinitesimal Hopf algebra of planar forests

(1) Let $F \in \mathbf{F}$. An *admissible cut* is a nonempty cut of certain edges and trees of F, such that each path in a noncut tree of F meets at most one cut edge. The set of admissible cuts of F will be denoted by $\mathrm{Adm}(F)$. If c is an admissible cut of F, the forest of the vertices that are over the cuts of c will be denoted by $P^c(t)$ (branch of the cut c), and the remaining forest will be denoted by $R^c(t)$ (trunk of the cut). An admissible cut of F will be said to be *left-admissible* if, for all vertices x and y of F, $x \in P^c(F)$ and $x \leq_{\mathrm{left}} y$ imply that $y \in P^c(F)$. The set of left-admissible cuts of F will be denoted by $\mathrm{Adm}^l(F)$.

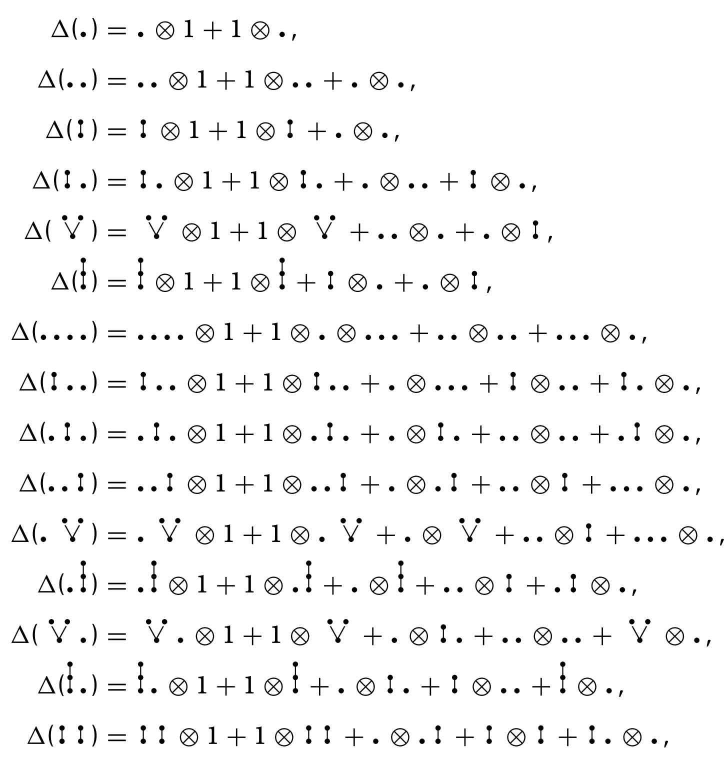
(2) $\mathcal{H}$ is given a coproduct by the following formula: for all $F \in \mathbf{F}$:

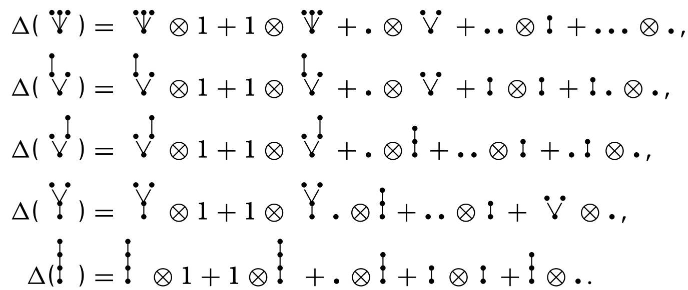
$$\Delta(F) = \sum_{c \in \mathcal{A}\mathrm{dm}^l(F)} P^c(F) \otimes R^c(F) + F \otimes 1 + 1 \otimes F.$$

Then $(\mathcal{H}, \Delta)$ is an infinitesimal bialgebra, that is to say: for all $x, y \in \mathcal{H}$,

$$\Delta(xy) = (x \otimes 1)\Delta(y) + \Delta(x)(1 \otimes y) - x \otimes y.$$

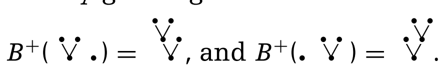
Examples.





We proved in [5] that $\mathcal{H}$ is an infinitesimal Hopf algebra, that is to say, it has an antipode S. This antipode satisfies $S(1) = 1$, $S(t) \in \mathrm{Prim}(\mathcal{H})$ for all $t \in \mathbf{T}$, and $S(F) = 0$ for all $F \in \mathbf{F} - (\mathbf{T} \cup \{1\})$.

1.3 Pairing on $\mathcal{H}$

(1) We define the operator $B^+ : \mathcal{H} \longrightarrow \mathcal{H}$, which associates, to a forest $F \in \mathbf{F}$, the tree obtained by grafting the roots of the trees of F on a common root. For example, .

(2) The application γ is defined by

$$\gamma : \begin{cases} \mathcal{H} \longrightarrow \mathcal{H} \\ t_1 \dots t_n \in \mathbf{F} \longrightarrow \delta_{t_1,\bullet}\, t_2 \dots t_n. \end{cases}$$

(3) There exists a unique pairing $\langle -, - \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow K$, satisfying:
   (i) $\langle 1, x \rangle = \varepsilon(x)$ for all $x \in \mathcal{H}$.
   (ii) $\langle xy, z \rangle = \langle y \otimes x, \Delta(z) \rangle$ for all $x, y, z \in \mathcal{H}$.
   (iii) $\langle B^+(x), y \rangle = \langle x, \gamma(y) \rangle$ for all $x, y \in \mathcal{H}$.
   Moreover,
   (iv) $\langle -, - \rangle$ is symmetric and nondegenerate.
   (v) If x and y are homogeneous of different weights, $\langle x, y \rangle = 0$.
   (vi) $\langle S(x), y \rangle = \langle x, S(y) \rangle$ for all $x, y \in \mathcal{H}$.
   This pairing admits a combinatorial interpretation using the partial orders $\geq_{\mathrm{left}}$ and $\geq_{\mathrm{high}}$ and is related to the Tamari order on planar binary trees; see [5].

(4) We denote by $(f_F)_{F \in \mathbf{F}}$ the dual basis of the basis of forests for the pairing $\langle -, - \rangle$. In other terms, for all $F \in \mathbf{F}$, f_F is defined by $\langle f_F, G \rangle = \delta_{F,G}$, for all forest $G \in \mathbf{F}$. The family $(f_t)_{t \in \mathbf{T}}$ is a basis of the space $\mathrm{Prim}(\mathcal{H})$ of primitive elements of $\mathcal{H}$.

2 The Operads of Forests and Graftings

2.1 A few recalls on non-Σ-operads

(1) We shall work here with non-Σ-operads [12]. Recall that such an object is a family $\mathbb{P} = (\mathbb{P}(n))_{n\in\mathbb{N}}$ of vector spaces, together with a composition for all $n, k_1, \ldots, k_n \in \mathbb{N}$:

$$\begin{cases} \mathbb{P}(n) \otimes \mathbb{P}(k_1) \otimes \cdots \otimes \mathbb{P}(k_n) \longrightarrow \mathbb{P}(k_1 + \cdots + k_n) \\ p \otimes p_1 \otimes \cdots \otimes p_n \longrightarrow p \circ (p_1, \ldots, p_n). \end{cases}$$

The following associativity condition is satisfied: for all $p \in \mathbb{P}(n)$, $p_1 \in \mathbb{P}(k_1)$, …, $p_n \in \mathbb{P}(k_n)$, $p_{1,1}, \ldots, p_{n,k_n} \in \mathbb{P}$,

$$\begin{aligned} &(p \circ (p_1, \ldots, p_n)) \circ (p_{1,1}, \ldots, p_{1,k_1}, \ldots, p_{n,1}, \ldots, p_{n,k_n}) \\ &\quad = p \circ (p_1 \circ (p_{1,1}, \ldots, p_{1,k_1}), \ldots, p_n \circ (p_{n,1}, \ldots, p_{n,k_n})). \end{aligned}$$

Moreover, there exists a unit element $I \in \mathbb{P}(1)$, satisfying: for all $p \in \mathbb{P}(n)$,

$$\begin{cases} p \circ (I, \ldots, I) = p, \\ I \circ p = p. \end{cases}$$

An operad is a non-Σ-operad $\mathbb{P}$ with a right action of the symmetric group S_n on $\mathbb{P}(n)$ for all n, satisfying a certain compatibility with the composition.

(2) Let $\mathbb{P}$ be a non-Σ-operad. A $\mathbb{P}$-algebra is a vector space A, together with an action of $\mathbb{P}$:

$$\begin{cases} \mathbb{P}(n) \otimes A^{\otimes n} \longrightarrow A \\ p \otimes a_1 \otimes \cdots \otimes a_n \longrightarrow p.(a_1, \ldots, a_n), \end{cases}$$

satisfying the following compatibility: for all $p \in \mathbb{P}(n)$, $p_1 \in \mathbb{P}(k_1)$, …, $p_n \in \mathbb{P}(k_n)$, for all $a_{1,1}, \ldots, a_{n,k_n} \in A$,

$$\begin{aligned} &(p \circ (p_1, \ldots, p_n)).(a_{1,1}, \ldots, a_{1,k_1}, \ldots, a_{n,1} \ldots, a_{n,k_n}) \\ &\quad = p.(p_1.(a_{1,1}, \ldots, a_{1,k_1}), \ldots, p_n.(a_{n,1}, \ldots, a_{n,k_n})). \end{aligned}$$

Moreover, $I.a = a$ for all $a \in A$.

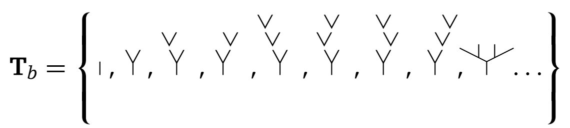
In particular, if V is a vector space, the free $\mathbb{P}$-algebra generated by V is

$$F_{\mathbb{P}}(V) = \bigoplus_{n\in\mathbb{N}} \mathbb{P}(n) \otimes V^{\otimes n},$$

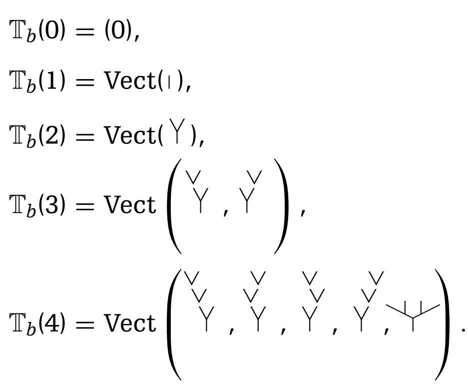
with the action of $\mathbb{P}$ given by

$$\begin{aligned} &p.((p_1 \otimes a_{1,1} \otimes \ldots \otimes a_{1,k_1}), \ldots, (p_n \otimes a_{n,1} \otimes \ldots \otimes a_{n,k_n})) \\ &\qquad = (p \circ (p_1, \ldots, p_n)) \otimes a_{1,1} \otimes \ldots \otimes a_{1,k_1} \otimes \ldots \otimes a_{n,1} \otimes \ldots \otimes a_{n,k_n}. \end{aligned}$$

(3) Let $\mathbf{T}_b$ be the set of planar binary trees:



For all $n \in \mathbb{N}$, $\mathbb{T}_b(n)$ is the vector space generated by the elements of $\mathbf{T}_b$ with n leaves:



The family of vector spaces $\mathbb{T}_b$ is given a structure of non-Σ-operad by graftings on the leaves. More precisely, if $t, t_1, \ldots, t_n \in \mathbf{T}_b$, t with n leaves, then $t \circ (t_1, \ldots, t_n)$ is the binary tree obtained by grafting t_1 on the first leaf of t, t_2 on the second leaf of t, and so on (note that the leaves of t are ordered from left to right). The unit is $|$.

It is known that $\mathbb{T}_b$ is the free non-Σ-operad generated by $\curlyvee \in \mathbb{T}_b(2)$. Similarly, given elements $m_1, \ldots, m_k$ in $\mathbb{P}(2)$, it is possible to describe the free non-Σ-operad $\mathbb{P}$ generated by these elements in terms of planar binary trees whose internal vertices are decorated by $m_1, \ldots, m_k$.

2.2 Presentations of the operads of forests

Definition 1.

(1) $\mathbb{P}_{\searrow}$ is the non-Σ-operad generated by m and $\searrow \in \mathbb{P}_{\searrow}(2)$, with relations:

$$\begin{cases} m \circ (\searrow, I) = \searrow \circ (I, m), \\ m \circ (m, I) = m \circ (I, m), \\ \searrow \circ (m, I) = \searrow \circ (I, \searrow). \end{cases}$$

(2) $\mathbb{P}_{\nearrow}$ is the non-Σ-operad generated by m and $\nearrow \in \mathbb{P}_{\nearrow}(2)$, with relations:

$$\begin{cases} m \circ (\nearrow, I) = \nearrow \circ (I, m), \\ m \circ (m, I) = m \circ (I, m), \\ \nearrow \circ (\nearrow, I) = \nearrow \circ (I, \nearrow). \end{cases}$$

Remark. We shall prove in [3] that these quadratic operads are Koszul.

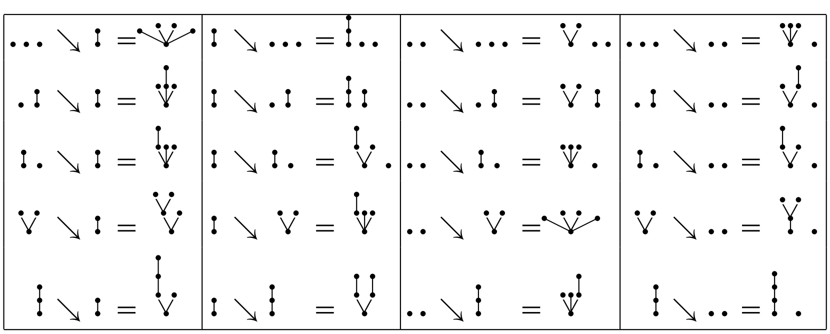
2.3 Grafting on the root

Let $F, G \in \mathbf{F} - \{1\}$. We put $G = t_1 \dots t_n$ and $t_1 = B^+(G_1)$. We define

$$F \searrow G = B^+(FG_1)t_2 \dots t_n.$$

In other terms, F is grafted on the root of the first tree of G, on the left. In particular, $F \searrow \bullet = B^+(F)$.

Examples.



Obviously, $\searrow$ can be inductively defined in the following way: for $F, G, H \in \mathbf{F} - \{1\}$,

$$\begin{cases} F \searrow \bullet &= B^+(F), \\ F \searrow (GH) &= (F \searrow G)H \\ F \searrow B^+(G) &= B^+(FG). \end{cases}$$

We denote by $\mathcal{M}$ the augmentation ideal of $\mathcal{H}$, that is to say, the vector space generated by the elements of $\mathbf{F} - \{1\}$. We extend $\searrow : \mathcal{M} \otimes \mathcal{M} \longrightarrow \mathcal{M}$ by linearity.

Proposition 2. For all $x, y, z \in \mathcal{M}$:

$$x \searrow (yz) = (x \searrow y)z, \tag{1}$$

$$x \searrow (y \searrow z) = (xy) \searrow z. \tag{2}$$

Proof. We can restrict ourselves to $x, y, z \in \mathbf{F} - \{1\}$. Then (1) is immediate. In order to prove (2), we put $z = B^+(z_1)z_2$, $z_1, z_2 \in \mathbf{F}$. Then

$$x \searrow (y \searrow z) = x \searrow (B^+(yz_1)z_2) = B^+(xyz_1)z_2 = (xy) \searrow (B^+(z_1)z_2) = (xy) \searrow z,$$

which proves (2). ∎

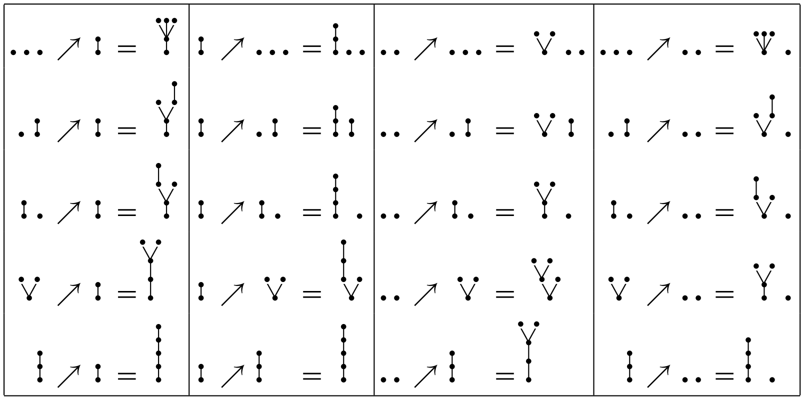
Corollary 3. $\mathcal{M}$ is given a graded $\mathbb{P}_{\searrow}$-algebra structure by its products m and by $\searrow$.

Proof. Immediate, by Proposition 2. ∎

2.4 Grafting on the left leaf

Let $F, G \in \mathbf{F}$. Suppose that $G \neq 1$. Then $F \nearrow G$ is the planar forest obtained by grafting F on the leave of G, which is at most on the left. For $G = 1$, we put $F \nearrow 1 = F$. In particular, $F \nearrow \bullet = B^+(F)$.

Examples.



In an obvious way, $\nearrow$ can be inductively defined in the following way: for F, G, $H \in \mathbf{F}$,

$$\begin{cases} F \nearrow 1 & = F, \\ F \nearrow (GH) & = (F \nearrow G)H \text{ if } G \neq 1, \\ F \nearrow B^+(G) & = B^+(F \nearrow G). \end{cases}$$

We extend $\nearrow$: $\mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}$ by linearity.

Proposition 4.

(1) For all $x, z \in \mathcal{H}$, $y \in \mathcal{M}$:

$$x \nearrow (yz) = (x \nearrow y)z. \tag{3}$$

(2) For all $x, y, z \in \mathcal{H}$:

$$x \nearrow (y \nearrow z) = (x \nearrow y) \nearrow z.$$

So $(\mathcal{H}, \nearrow)$ is an associative algebra, with unitary element 1.

Proof. Note that (3) is immediate for $x, y, z \in \mathbf{F}$, with $y \neq 1$. This implies the first point. In order to prove the second point, we consider:

$$Z = \{z \in \mathcal{H} \ / \ \forall x, y \in \mathcal{H},\ x \nearrow (y \nearrow z) = (x \nearrow y) \nearrow z\}.$$

Let us first prove that $1 \in Z$: for all $x, y \in \mathcal{H}$,

$$x \nearrow (y \nearrow 1) = x \nearrow y = (x \nearrow y) \nearrow 1.$$

Let $z_1, z_2 \in Z$. Let us show that $z_1 z_2 \in Z$. By linearity, we can separate the proof into two cases:

(1) $z_1 = 1$. Then it is obvious.
(2) $\varepsilon(z_1) = 0$. Let $x, y \in \mathcal{H}$. By the first point,

$$\begin{aligned} x \nearrow (y \nearrow (z_1 z_2)) &= x \nearrow ((y \nearrow z_1) z_2)) \\ &= (x \nearrow (y \nearrow z_1)) z_2 \\ &= ((x \nearrow y) \nearrow z_1) z_2 \\ &= (x \nearrow y) \nearrow (z_1 z_2). \end{aligned}$$

So Z is a subalgebra of $\mathcal{H}$. Let us show that it is stable by B^+. Let $z \in Z$, $x, y \in \mathcal{H}$. Then

$$\begin{aligned} x \nearrow (y \nearrow B^+(z)) &= x \nearrow B^+(y \nearrow z) \\ &= B^+(x \nearrow (y \nearrow z)) \\ &= B^+((x \nearrow y) \nearrow z) \\ &= (x \nearrow y) \nearrow B^+(z). \end{aligned}$$

So Z is a subalgebra of $\mathcal{H}$, stable by B^+. Hence, $Z = \mathcal{H}$. ■

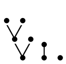
Remarks.

(1) Equation (3) is equivalent to: for any $x, y, z \in \mathcal{H}$,

$$x \nearrow (yz) - \varepsilon(y) x \nearrow z = (x \nearrow y) z - \varepsilon(y) xz.$$

(2) Let $F \in \mathbf{F} - \{1\}$. There exists a unique family $(\bullet F_1, \ldots, \bullet F_n)$ of elements of $\mathbf{F}$ such that

$$F = (\bullet F_1) \nearrow \ldots \nearrow (\bullet F_n).$$

For example,  $= (\bullet\bullet) \nearrow (\bullet\bullet) \nearrow$ ($\bullet$  $\bullet$). As a consequence, $(\mathcal{H}, \nearrow)$ is freely generated by $\bullet\mathbf{F}$ as an associative algebra.

Corollary 5. $\mathcal{M}$ is given a graded $\mathbb{P}_{\nearrow}$-algebra structure by its product m and by $\nearrow$.

Proof. Immediate, by Proposition 4. ■

2.5 Dimensions of $\mathbb{P}_\searrow$ and $\mathbb{P}_\nearrow$

We now compute the dimensions of $\mathbb{P}_\searrow(n)$ and $\mathbb{P}_\nearrow(n)$ for all n and deduce that $\mathcal{M}$ is the free $\mathbb{P}_\searrow$- and $\mathbb{P}_\nearrow$-algebra generated by $\centerdot$.

Notation. We denote by r_n the number of planar rooted forests and we put $R(X) = \sum_{n=1}^{+\infty} r_n X^n$. It is well known [4, 15] that $R(X) = \frac{1-2X-\sqrt{1-4X}}{2X}$. The coefficients r_n are the Catalan numbers; see sequence A000108 of [13].

Proposition 6. For $\stackrel{?}{\to} \in \{\searrow, \nearrow\}$ and all $n \in \mathbb{N}^*$, in the $\mathbb{P}_{\stackrel{?}{\to}}$-algebra $\mathcal{M}$:

$$\mathbb{P}_{\stackrel{?}{\to}}(n).(\centerdot, \ldots, \centerdot) = \mathrm{Vect}(\text{planar forests of weight } n).$$

As a consequence, $\mathcal{M}$ is generated as a $\mathbb{P}_{\stackrel{?}{\to}}$-algebra by $\centerdot$.

Proof. $\subseteq$. Immediate, as $\mathcal{M}$ is a graded $\mathbb{P}_{\stackrel{?}{\to}}$-algebra.
$\supseteq$. Induction on n. For $n = 1$, $I.(\centerdot) = \centerdot$. For $n \geq 2$, two cases are possible.

(1) $F = F_1 F_2$, $\mathrm{weight}(F_i) = n_i < n$. By the induction hypothesis, there exists $p_1, p_2 \in \mathbb{P}_{\stackrel{?}{\to}}$, such that $F_1 = p_1.(\centerdot, \ldots, \centerdot)$ and $F_2 = p_2.(\centerdot, \ldots, \centerdot)$. Then $(m \circ (p_1, p_2)).(\centerdot, \ldots, \centerdot) = m.(F_1, F_2) = F_1 F_2$.

(2) $F \in \mathbf{T}$. Let us put $F = B^+(G)$. Then there exists $p \in \mathbb{P}_{\stackrel{?}{\to}}$, such that $p.(\centerdot, \ldots, \centerdot) = G$. Then

$$\begin{cases} (\searrow \circ (p, I)).(\centerdot, \ldots, \centerdot) = G \searrow \centerdot = F, \\ (\nearrow \circ (p, I)).(\centerdot, \ldots, \centerdot) = G \nearrow \centerdot = F. \end{cases}$$

Hence, in both cases, $F \in \mathbb{P}_{\stackrel{?}{\to}}(n).(\centerdot, \ldots, \centerdot)$. ∎

Corollary 7. For all $\stackrel{?}{\to} \in \{\searrow, \nearrow\}$, $n \in \mathbb{N}^*$, $\dim(\mathbb{P}_{\stackrel{?}{\to}}(n)) \geq r_n$.

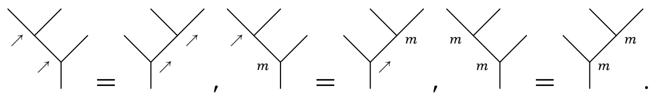
Proof. Because we proved the surjectivity of the following application:

$$ev_{\stackrel{?}{\to}} : \begin{cases} \mathbb{P}_{\stackrel{?}{\to}}(n) \longrightarrow \mathrm{Vect}(\text{planar forests of weight } n) \\ p \qquad\;\; \longrightarrow p.(\centerdot, \ldots, \centerdot). \end{cases}$$

∎

Lemma 8. For all $\stackrel{?}{\to} \in \{\searrow, \nearrow\}$, $n \in \mathbb{N}^*$, $\dim(\mathbb{P}_{\stackrel{?}{\to}}(n)) \leq r_n$.

Proof. We prove it for $\stackrel{?}{\to} = \nearrow$. Let us fix $n \in \mathbb{N}^*$. Then $\mathbb{P}_\nearrow(n)$ is linearly generated by planar binary trees whose internal vertices are decorated by m and $\nearrow$. The following relations hold:

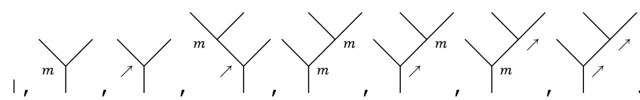


In the sequel of the proof, we shall say that such a tree is *admissible* if it satisfies the following conditions:

(1) For each internal vertex s decorated by m, the left child of s is a leaf.

(2) For each internal vertex s decorated by $\nearrow$, the left child of s is a leaf or is decorated by m.

For example, here are the admissible trees with one, two, or three leaves:



The preceding relations imply that $\mathbb{P}_{\nearrow}(n)$ is linearly generated by admissible trees with n leaves. So $\dim(\mathbb{P}_{\nearrow}(n))$ is smaller than a_n, the number of admissible trees with n leaves. For $n \geq 2$, we denote by b_n the number of admissible trees with n leaves whose root is decorated by m, and by c_n the number of admissible trees with n leaves whose root is decorated by $\nearrow$. We also put $b_1 = 1$ and $c_1 = 0$. Finally, we define:

$$A(X) = \sum_{n\geq 1} a_n X^n, \qquad B(X) = \sum_{n\geq 1} b_n X^n, \qquad C(X) = \sum_{n\geq 1} c_n X^n.$$

Immediately, $A(X) = B(X) + C(X)$. Every admissible tree with $n \geq 2$ leaves whose root is decorated by m is of the form $m \circ (I, t)$, where t is an admissible tree with $n-1$ leaves. Hence, $B(X) = XA(X) + X$. Moreover, every admissible tree with $n \geq 2$ leaves whose root is decorated by $\nearrow$ is of the form $\nearrow \circ(t_1, t_2)$, where t_1 is an admissible tree with k leaves whose eventual root is decorated by m and t_2 an admissible tree with $n-k$ leaves $(1 \leq k \leq n-1)$. Hence, for all $n \geq 2$, $c_n = \sum_{k=1}^{n-1} b_k a_{n-k}$, so $C(X) = B(X)A(X)$. As a conclusion,

$$\begin{cases} A(X) = B(X) + C(X), \\ B(X) = XA(X) + X, \\ C(X) = B(X)A(X). \end{cases}$$

So, $A(X) = XA(X) + X + B(X)A(X) = XA(X) + X + XA(X)^2 + XA(X)$, and

$$XA(X)^2 + (2X-1)A(X) + X = 0.$$

As $a_1 = 1$:

$$A(X) = \frac{1 - 2X - \sqrt{1 - 4X}}{2X} = R(X).$$

So, for all $n \geq 1$, $\dim(\mathbb{P}_{\nearrow}(n)) \leq a_n = r_n$. The proof is similar for $\mathbb{P}_{\searrow}$. ■

As immediate consequences:

Theorem 9. For $\overset{?}{\rightarrow} \in \{\searrow, \nearrow\}$, $n \in \mathbb{N}^*$, $\dim(\mathbb{P}_{\overset{?}{\rightarrow}}(n)) = r_n$. Moreover, the following application is bijective:

$$ev_{\overset{?}{\rightarrow}} : \begin{cases} \mathbb{P}_{\overset{?}{\rightarrow}}(n) \longrightarrow \text{Vect(planar forests of weight } n) \subseteq \mathcal{M} \\ p \qquad\quad \longrightarrow p.(\bullet, \ldots, \bullet). \end{cases}$$

Corollary 10.

(1) $(\mathcal{M}, m, \searrow)$ is the free $\mathbb{P}_{\searrow}$-algebra generated by $\bullet$.

(2) $(\mathcal{M}, m, \nearrow)$ is the free $\mathbb{P}_{\nearrow}$-algebra generated by $\bullet$.

2.6 A combinatorial description of the composition

Let $\overset{?}{\rightarrow} \in \{\searrow, \nearrow\}$. We identify $\mathbb{P}_{\overset{?}{\rightarrow}}$ and the vector space of nonempty planar forests via Theorem 9. In other terms, we identify $F \in \mathbf{F}(n)$ and $ev_{\overset{?}{\rightarrow}}^{-1}(F) \in \mathbb{P}_{\overset{?}{\rightarrow}}(n)$.

Notations.

(1) In order to distinguish the compositions in $\mathbb{P}_{\searrow}$ and $\mathbb{P}_{\nearrow}$, we now denote:

  (a) $F \circ\!\!\searrow (F_1, \ldots, F_n)$ the composition of $\mathbb{P}_{\searrow}$,

  (b) $F \circ\!\!\nearrow (F_1, \ldots, F_n)$ the composition of $\mathbb{P}_{\nearrow}$.

(2) In order to distinguish the action of the operads $\mathbb{P}_{\searrow}$ and $\mathbb{P}_{\nearrow}$ on $\mathcal{M}$, we now denote:

  (a) $F \bullet\!\!\searrow (x_1, \ldots, x_n)$ the action of $\mathbb{P}_{\searrow}$ on $\mathcal{M}$,

  (b) $F \bullet\!\!\nearrow (x_1, \ldots, x_n)$ the action of $\mathbb{P}_{\nearrow}$ on $\mathcal{M}$.

Our aim in this paragraph is to describe the compositions of $\mathbb{P}_{\searrow}$ and $\mathbb{P}_{\nearrow}$ in terms of forests. We shall prove the following result:

Theorem 11.

(1) The composition of $\mathbb{P}_{\searrow}$ in the basis of planar forests can be inductively defined in this way:

$$\begin{cases} \tun \circ\!\!\!\searrow (H) &= H,\\ B^+(F)\circ\!\!\!\searrow (H_1,\ldots,H_{n+1}) &= (F\circ\!\!\!\searrow (H_1,\ldots,H_n)) \searrow H_{n+1},\\ FG\circ\!\!\!\searrow (H_1,\ldots,H_{n_1+n_2}) &= F\circ\!\!\!\searrow (H_1,\ldots,H_{n_1})G\circ\!\!\!\searrow (H_{n_1+1},\ldots,H_{n_1+n_2}). \end{cases}$$

(2) The composition of $\mathbb{P}_{\nearrow}$ in the basis of planar forests can be inductively defined in this way:

$$\begin{cases} \tun \circ\!\!\!\nearrow (H) &= H,\\ B^+(F)\circ\!\!\!\nearrow (H_1,\ldots,H_{n+1}) &= (F\circ\!\!\!\nearrow (H_1,\ldots,H_n)) \nearrow H_{n+1},\\ FG\circ\!\!\!\nearrow (H_1,\ldots,H_{n_1+n_2}) &= F\circ\!\!\!\nearrow (H_1,\ldots,H_{n_1})G\circ\!\!\!\nearrow (H_{n_1+1},\ldots,H_{n_1+n_2}). \end{cases}$$

Examples. Let $F_1, F_2, F_3 \in \mathbf{F} - \{1\}$.

$\tun\tun \circ\!\!\!\nearrow (F_1,F_2) = F_1F_2$	$\tun\tun \circ\!\!\!\searrow (F_1,F_2) = F_1F_2$
$\tdeux \circ\!\!\!\nearrow (F_1,F_2) = F_1 \nearrow F_2$	$\tdeux \circ\!\!\!\searrow (F_1,F_2) = F_1 \searrow F_2$
$\tun\tun\tun \circ\!\!\!\nearrow (F_1,F_2,F_3) = F_1F_2F_3$	$\tun\tun\tun \circ\!\!\!\searrow (F_1,F_2,F_3) = F_1F_2F_3$
$\tun\tdeux \circ\!\!\!\nearrow (F_1,F_2,F_3) = F_1(F_2 \nearrow F_3)$	$\tun\tdeux \circ\!\!\!\searrow (F_1,F_2,F_3) = F_1(F_2 \searrow F_3)$
$\tdeux\tun \circ\!\!\!\nearrow (F_1,F_2,F_3) = (F_1 \nearrow F_2)F_3$	$\tdeux\tun \circ\!\!\!\searrow (F_1,F_2,F_3) = (F_1 \searrow F_2)F_3$
$\ttroisun \circ\!\!\!\nearrow (F_1,F_2,F_3) = (F_1F_2) \nearrow F_3$	$\ttroisun \circ\!\!\!\searrow (F_1,F_2,F_3) = (F_1F_2) \searrow F_3$
$\ttroisdeux \circ\!\!\!\nearrow (F_1,F_2,F_3) = (F_1 \nearrow F_2) \nearrow F_3$	$\ttroisdeux \circ\!\!\!\searrow (F_1,F_2,F_3) = (F_1 \searrow F_2) \searrow F_3$

Proposition 12. Let $\overset{?}{\rightarrow} \in \{\searrow, \nearrow\}$.

(1) $\tun$ is the unit element of $\mathbb{P}_{\overset{?}{\rightarrow}}$.

(2) $\tun\tun = m$ in $\mathbb{P}_{\overset{?}{\rightarrow}}(2)$. Consequently, in $\mathbb{P}_{\overset{?}{\rightarrow}}$, $\tun\tun \circ (F,G) = FG$ for all $F, G \in \mathbf{F} - \{1\}$.

(3) Let $F, G \in \mathbf{F}$. In $\mathbb{P}_{\overset{?}{\rightarrow}}$, $\tdeux = \overset{?}{\rightarrow}$. Consequently, $\tdeux \overset{?}{\circ\!\!\!\rightarrow} (F,G) = F\overset{?}{\rightarrow}G$ for all $F, G \in \mathbf{F} - \{1\}$.

Proof.

(1) Indeed, $ev_{\overset{?}{\rightarrow}}(\tun) = \tun = ev_{\overset{?}{\rightarrow}}(I)$. Hence, $\tun = I$.

(2) By definition, $ev_{\overset{?}{\to}}(\bullet\bullet) = \bullet\bullet = ev_{\overset{?}{\to}}(m)$. So $\bullet\bullet = m$ in $\mathbb{P}_{\overset{?}{\to}}(2)$. Moreover, for all $F, G \in \mathbf{F} - \{1\}$:

$$
\begin{aligned}
ev_{\overset{?}{\to}}(FG) = FG \\
&= m \overset{?}{\bullet} (F, G) \\
&= m \overset{?}{\bullet} (F \overset{?}{\bullet} (\bullet, \ldots, \bullet), G \overset{?}{\bullet} (\bullet, \ldots, \bullet)) \\
&= \left(m \overset{?}{\ominus} (F, G) \right) \overset{?}{\bullet} (\bullet, \ldots, \bullet) \\
&= ev_{\overset{?}{\to}}(m \overset{?}{\ominus} (F, G)).
\end{aligned}
$$

So $FG = m \overset{?}{\ominus} (F, G) = \bullet\bullet \overset{?}{\ominus} (F, G)$.

(3) Indeed, $ev_{\overset{?}{\to}}(\overset{\bullet}{\underset{\bullet}{|}}) = \bullet \overset{?}{\to} \bullet = ev_{\overset{?}{\to}}(\overset{?}{\to})$. So $\overset{\bullet}{\underset{\bullet}{|}} = \overset{?}{\to}$ in $\mathbb{P}_{\overset{?}{\to}}(2)$. Moreover,

$$
\begin{aligned}
ev_{\overset{?}{\to}}(F \overset{?}{\to} G) = F \overset{?}{\to} G \\
&= \overset{?}{\to} \overset{?}{\bullet} (F, G) \\
&= \overset{?}{\to} \overset{?}{\bullet} (F \overset{?}{\bullet} (\bullet, \ldots, \bullet), G \overset{?}{\bullet} (\bullet, \ldots, \bullet)) \\
&= (\overset{?}{\to} \overset{?}{\ominus} (F, G)).(\bullet, \ldots, \bullet) \\
&= ev_{\overset{?}{\to}}(\overset{?}{\to} \overset{?}{\ominus} (F, G)).
\end{aligned}
$$

So, $F \overset{?}{\to} G = \overset{?}{\to} \overset{?}{\ominus} (F, G) = \overset{\bullet}{\underset{\bullet}{|}} \overset{?}{\ominus} (F, G)$. ■

Proposition 13.

(1) Let $F, G \in \mathbf{F}$, different from 1, of respective weights n_1 and n_2. Let $H_{1,1}, \ldots, H_{1,n_1}$ and $H_{2,1}, \ldots, H_{2,n_2} \in \mathbf{F} - \{1\}$. Let $\overset{?}{\to} \in \{\searrow, \nearrow\}$. Then, in $\mathbb{P}_{\overset{?}{\to}}$:

$$(FG) \overset{?}{\ominus} (H_{1,1}, \ldots, H_{1,n_1}, H_{2,1}, \ldots, H_{2,n_2}) = F \overset{?}{\ominus} (H_{1,1}, \ldots, H_{1,n_1}) G \overset{?}{\ominus} (H_{2,1}, \ldots, H_{2,n_2}).$$

(2) Let $F \in \mathbf{F}$, of weight $n \geq 1$. Let $H_1, \ldots, H_{n+1} \in \mathbf{F}$. In $\mathbb{P}_{\overset{?}{\to}}$:

$$B^+(F) \overset{?}{\ominus} (H_1, \ldots, H_{n+1}) = (F \overset{?}{\ominus} (H_1, \ldots, H_n)) \overset{?}{\to} H_{n+1}.$$

Proof.

(1) Indeed, in $\mathbb{P}_{\overset{?}{\to}}$:

$$
\begin{aligned}
&(FG) \overset{?}{\ominus} (H_{1,1}, \ldots, H_{1,n_1}, H_{2,1}, \ldots, H_{2,n_2}) \\
&\quad = (m \overset{?}{\ominus} (F, G)) \overset{?}{\ominus} (H_{1,1}, \ldots, H_{1,n_1}, H_{2,1}, \ldots, H_{2,n_2})
\end{aligned}
$$

$$= m \overset{?}{\ominus} (F \overset{?}{\ominus} (H_{1,1}, \dots, H_{1,n_1}), G \overset{?}{\ominus} (H_{2,1}, \dots, H_{2,n_2}))$$
$$= F \overset{?}{\ominus} (H_{1,1}, \dots, H_{1,n_1}) G \overset{?}{\ominus} (H_{2,1}, \dots, H_{2,n_2})).$$

(2) In $\mathbb{P}_{\overset{?}{\to}}$:

$$\begin{aligned}
B^+(F) \overset{?}{\ominus} (H_1, \dots, H_{n+1}) &= (F \overset{?}{\to} \bullet) \overset{?}{\ominus} (H_1, \dots, H_{n+1}) \\
&= (\begin{smallmatrix}\bullet\\\bullet\end{smallmatrix} \overset{?}{\ominus} (F, \bullet)) \overset{?}{\ominus} (H_1, \dots, H_{n+1}) \\
&= \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix} \overset{?}{\ominus} (F \overset{?}{\ominus} (H_1, \dots, H_n), \bullet \overset{?}{\ominus} (H_{n+1})) \\
&= \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix} \overset{?}{\ominus} (F \overset{?}{\ominus} (H_1, \dots, H_n), H_{n+1}) \\
&= (F \overset{?}{\ominus} (H_1, \dots, H_n)) \overset{?}{\to} H_{n+1}. \qquad \blacksquare
\end{aligned}$$

Combining Propositions 12 and 13, we obtain Theorem 11.

3 Applications to the Infinitesimal Hopf Algebra $\mathcal{H}$

3.1 Antipode of $\mathcal{H}$

Here we give a description of the antipode of $\mathcal{H}$ in terms of the action $\searrow$ of the operad $\mathbb{P}_{\searrow}$.

Notations. For all $n \in \mathbb{N}^*$, we denote $l_n = (B^+)^n(1) \in \mathbf{F}(n)$. For example,

$$l_1 = \bullet,\ l_2 = \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix},\ l_3 = \begin{smallmatrix}\bullet\\\bullet\\\bullet\end{smallmatrix},\ l_4 = \begin{smallmatrix}\bullet\\\bullet\\\bullet\\\bullet\end{smallmatrix},\ l_5 = \begin{smallmatrix}\bullet\\\bullet\\\bullet\\\bullet\\\bullet\end{smallmatrix} \dots$$

Lemma 14. Let $t \in \mathbf{T}$. There exists a unique $k \in \mathbb{N}^*$, and a unique family $(t_2 \dots, t_k) \in \mathbf{T}^{k-1}$ such that

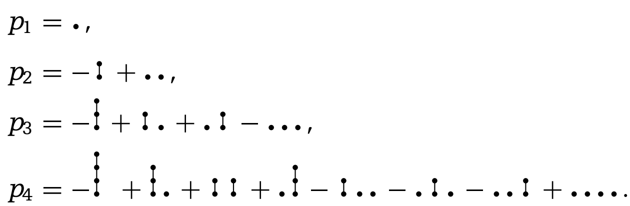
$$t = l_k \searrow (\bullet, t_2, \dots, t_k).$$

Proof. Induction on the weight n of t. If $n = 1$, then $t = \bullet$, so $k = 1$ and the family is empty. We suppose the result true at all rank $< n$. We put $t = B^+(s_1 \dots s_m)$. Necessarily, $t_k = B^+(s_2 \dots s_m)$ and $l_{n-1} \searrow (\bullet, t_2, \dots, t_{k-1}) = s_1$. We conclude with the induction hypothesis on s_1. $\blacksquare$

Example.

Definition 15. For all $n \in \mathbb{N}^*$, we put $p_n = \sum_{k=1}^{n} \sum_{\substack{a_1+\cdots+a_k=n \\ \forall i,\, a_i>0}} (-1)^k l_{a_1} \dots l_{a_k}$.

Examples.



Remark that p_n is in fact the antipode of l_n in $\mathcal{H}$. It is also the antipode of l_n in the noncommutative Connes–Kreimer Hopf algebra of planar trees [4].

Corollary 16. Let $t \in \mathbf{T}$, written under the form $t = l_k \searrow (t_1, \dots, t_k)$, with $t_1 = \bullet$. Then

$$S(t) = p_k \searrow (t_1, \dots, t_k).$$

Proof. Corollary of Proposition 15 of [5], observing that left cuts are cuts on edges from the root of t_i to the root of t_{i+1} in t, for $i = 1, \dots, n-1$. ■

3.2 Inverse of the application γ

Proposition 17. The restriction $\gamma : \mathrm{Prim}(\mathcal{H}) \longrightarrow \mathcal{H}$ is bijective.

Proof. By Proposition 21 of [5]:

$$\gamma_{|\mathrm{Prim}(\mathcal{H})} : \begin{cases} \mathrm{Prim}(\mathcal{H}) & \longrightarrow \mathcal{H} \\ f_{B^+(F)}\ (F \in \mathbf{F}) & \longrightarrow f_F. \end{cases}$$

So this restriction is clearly bijective. ■

We shall denote $\gamma^{-1}_{|\mathrm{Prim}(\mathcal{H})} : \mathcal{H} \longrightarrow \mathrm{Prim}(\mathcal{H})$ the inverse of this restriction. Then, for all $F \in \mathbf{F}$, $\gamma^{-1}_{|\mathrm{Prim}(\mathcal{H})}(f_F) = f_{B^+(F)}$. Our aim is to express $\gamma^{-1}_{|\mathrm{Prim}(\mathcal{H})}$ in the basis of forests.

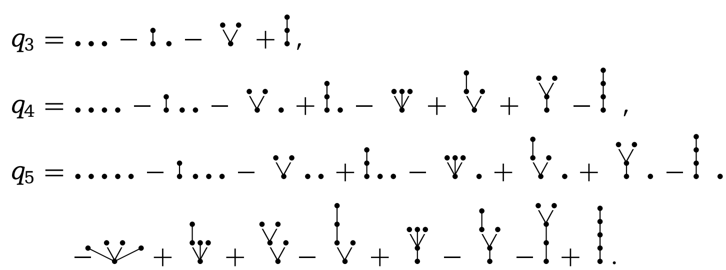
We define inductively a sequence $(q_n)_{n\in\mathbb{N}^*}$ of elements of $\mathbb{P}_{\searrow}$:

$$\begin{cases} q_1 &= \bullet \in \mathbb{P}_{\searrow}(1), \\ q_2 &= \bullet\bullet - \overset{\bullet}{\underset{\bullet}{|}} \in \mathbb{P}_{\searrow}(2), \\ q_{n+1} &= (\bullet\bullet - \overset{\bullet}{\underset{\bullet}{|}}) \mathbin{\circ\!\!\!\searrow} (q_n, \bullet) \in \mathbb{P}_{\searrow}(n+1) \text{ for } n \geq 1. \end{cases}$$

For all $F \in \mathbf{F}$, $\bullet\bullet \mathbin{\circ\!\!\!\searrow} (F, \bullet) = F\bullet$ and $\overset{\bullet}{\underset{\bullet}{|}} \mathbin{\circ\!\!\!\searrow} (F, \bullet) = B^+(F)$. So, q_n can also be defined in the following way:

$$\begin{cases} q_1 &= \bullet \in \mathbb{P}_{\searrow}(1), \\ q_{n+1} &= q_n\bullet - B^+(q_n) \in \mathbb{P}_{\searrow}(n+1) \text{ for } n \geq 1. \end{cases}$$

Examples.



Lemma 18. Let $F \in \mathbf{F} - \{1\}$, and $t \in \mathbf{T}$. Then, in $\mathcal{H}$:

$$\Delta(F \searrow t) = (F \searrow t) \otimes 1 + 1 \otimes (F \searrow t) + F' \otimes F'' \searrow t + Ft' \otimes t'' + F \otimes t.$$

Proof. The nonempty and nontotal left-admissible cuts of the tree $F \searrow t$ are

- The cut on the edges relating F to t. For this cut c, $P^c(F \searrow t) = F$ and $R^c(F \searrow t) = t$.
- Cuts acting only on edges of F or on edges relating F to t, at the exception of the preceding case. For such a cut, there exists a unique nonempty, nontotal left-admissible cut c' of F, such that $P^c(F \searrow t) = P^{c'}(F)$ and $R^c(F \searrow t) = R^{c'}(F) \searrow t$.
- Cuts acting on edges of t. Then necessarily $F \subseteq P^c(F \searrow t)$. For such a cut, there exists a unique nonempty, nontotal left-admissible cut c' of t, such that $P^c(F \searrow t) = FP^{c'}(t)$ and $R^c(F \searrow t) = R^{c'}(t)$.

Summing these cuts, we obtain the announced compatibility. ∎

Proposition 19. Let $F = t_1 \dots t_n \in \mathbf{F}$. Then

$$\gamma^{-1}_{|\mathrm{Prim}(\mathcal{H})}(F) = q_{n+1} \bullet\!\!\searrow (\bullet, t_1, \dots, t_n).$$

Proof. *First step.* Let us show the following property: for all $x \in \mathrm{Prim}(\mathcal{H})$, $t \in \mathbf{T}$, $q_2 \bullet\!\!\searrow (x, t)$ is primitive. By Lemma 18, using the linearity in F:

$$\begin{aligned} \Delta(x \searrow t) &= (x \searrow t) \otimes 1 + 1 \otimes (x \searrow t) + x \otimes t + xt' \otimes t'', \\ \Delta(xt) &= xt \otimes 1 + 1 \otimes xt + x \otimes t + xt' \otimes t''. \end{aligned}$$

Hence,

$$\Delta(q_2 \bullet\!\!\searrow (x, t)) = \Delta(xt - x \searrow t) = (xt - x \searrow t) \otimes 1 + 1 \otimes (xt - x \searrow t).$$

Second step. Let us show that for all $x \in \mathrm{Prim}(\mathcal{H})$, $t_1, \dots, t_n \in \mathbf{T}$, $q_{n+1} \bullet\!\!\searrow (x, t_1, \dots, t_n) \in \mathrm{Prim}(\mathcal{H})$ by induction on n. This is obvious for $n = 0$, as $q_1 \bullet\!\!\searrow (x) = x$. Suppose the result at rank $n - 1$. Then

$$\begin{aligned} q_{n+1} \bullet\!\!\searrow (x, t_1, \dots, t_n) &= (q_2 \circ\!\!\searrow (q_n, I)) \bullet\!\!\searrow (x, t_1, \dots, t_n) \\ &= q_2 \bullet\!\!\searrow (\underbrace{q_n \bullet\!\!\searrow (x, t_1, \dots, t_{n-1})}_{\in \mathrm{Prim}(\mathcal{H})}, t_n) \ \in \ \mathrm{Prim}(\mathcal{H}) \end{aligned}$$

by the first step. As the tree $\bullet$ is primitive, we deduce that, for all forest $F = t_1 \dots t_n \in \mathbf{F}$, $q_{n+1} \bullet\!\!\searrow (\bullet, t_1, \dots, t_n) \in \mathrm{Prim}(\mathcal{H})$.

Third step. Let us show that for all $x, y \in \mathcal{M}$, $\gamma(q_2 \bullet\!\!\searrow (x, y)) = \gamma(x)y$. We can limit ourselves to $x, y \in \mathbf{F} - \{1\}$. Then $q_2 \bullet\!\!\searrow (x, y) = xy - x \searrow y$. Moreover, by definition of $\searrow$, $x \searrow y$ is a forest whose first tree is not equal to $\bullet$. Hence, $\gamma(q_2 \bullet\!\!\searrow (x, y)) = \gamma(xy) - 0 = \gamma(x)y$.

Last step. Let us show by induction on n that $\gamma(q_{n+1} \bullet\!\!\searrow (\bullet, t_1, \dots, t_n)) = t_1 \dots t_n$. As $q_1 \bullet\!\!\searrow (\bullet) = \bullet$, this is obvious if $n = 0$. Let us suppose the result true at all rank $n - 1$. By the third step:

$$\begin{aligned} \gamma(q_{n+1} \bullet\!\!\searrow (\bullet, t_1, \dots, t_n)) &= \gamma(q_2 \bullet\!\!\searrow (q_n \bullet\!\!\searrow (\bullet, t_1, \dots, t_{n-1}), t_n)) \\ &= \gamma(q_n \bullet\!\!\searrow (\bullet, t_1, \dots, t_{n-1}))t_n \\ &= t_1 \dots t_n. \end{aligned}$$

Consequently, $x = q_{n+1} \bullet\!\!\searrow (\bullet, t_1, \dots, t_n) \in \mathrm{Prim}(\mathcal{H})$, and satisfies $\gamma(x) = t_1 \dots t_n$, which proves Proposition 19. ∎

Examples. Let $t_1, t_2, t_3 \in \mathbf{T}$.

$$\begin{aligned}
\gamma^{-1}_{|\mathrm{Prim}(\mathcal{H})}(t_1) &= \centerdot t_1 - \centerdot \searrow t_1,\\
\gamma^{-1}_{|\mathrm{Prim}(\mathcal{H})}(t_1 t_2) &= \centerdot t_1 t_2 - (\centerdot \searrow t_1) t_2 - (\centerdot t_1) \searrow t_2 + (\centerdot \searrow t_1) \searrow t_2,\\
\gamma^{-1}_{|\mathrm{Prim}(\mathcal{H})}(t_1 t_2 t_3) &= \centerdot t_1 t_2 t_3 - (\centerdot \searrow t_1) t_2 t_3 - (\centerdot t_1) \searrow t_2 t_3 + (\centerdot \searrow t_1) \searrow t_2 t_3 - (\centerdot t_1 t_2) \searrow t_3\\
&\quad + (\centerdot \searrow t_1 t_2) \searrow t_3 + ((\centerdot t_1) \searrow t_2) \searrow t_3 - ((\centerdot \searrow t_1) \searrow t_2) \searrow t_3.
\end{aligned}$$

3.3 Elements of the dual basis

Lemma 20. For all $x, y \in \mathcal{H}$, $\Delta(x \nearrow y) = x \nearrow y^{(1)} \otimes y^{(2)} + x^{(1)} \otimes x^{(2)} \nearrow y - x \otimes y$. In other terms, $(\mathcal{H}, \nearrow, \Delta)$ is an infinitesimal Hopf algebra.

Proof. We restrict to $x = F \in \mathbf{F} - \{1\}$, $y = G \in \mathbf{F} - \{1\}$. The nonempty and nontotal left-admissible cuts of the tree $F \nearrow G$ are

- The cut on the edges relating F to G. For this cut c, $P^c(F \nearrow G) = F$ and $R^c(F \nearrow G) = G$.
- Cuts acting only on edges of F or on edges relating F to G, at the exception of the preceding case. For such a cut, there exists a unique nonempty, nontotal left-admissible cut c' of F, such that $P^c(F \nearrow G) = P^{c'}(F)$ and $R^c(F \nearrow G) = R^{c'}(F) \nearrow G$.
- Cuts acting on edges of G. Then necessarily $F \subseteq P^c(F \nearrow G)$. For such a cut, there exists a unique nonempty, nontotal left-admissible cut c' of t, such that $P^c(F \nearrow G) = F \nearrow P^{c'}(G)$ and $R^c(F \nearrow G) = R^{c'}(G)$.

Summing these cuts, we obtain, denoting $\Delta(F) = F \otimes 1 + 1 \otimes F + F' \otimes F''$ and $\Delta(G) = G \otimes 1 + 1 \otimes G + G' \otimes G''$:

$$\begin{aligned}
\tilde{\Delta}(F \nearrow G) &= (F \nearrow G) \otimes 1 + 1 \otimes (F \nearrow G) + F \otimes G + F' \otimes F'' \nearrow G + F \nearrow G' \otimes G''\\
&= (F \otimes 1) \nearrow \Delta(G) + \Delta(F) \nearrow (1 \otimes G) - F \otimes G.
\end{aligned}$$

So $(\mathcal{H}, \nearrow, \Delta)$ is an infinitesimal bialgebra. As it is graded and connected, it has an antipode. ■

Proposition 21. Let $F = t_1 \dots t_n \in \mathbf{F}$. Then $f_F = f_{t_n} \nearrow \dots \nearrow f_{t_1}$.

Proof. *First step.* We show the following result: for all $F \in \mathbf{F}$, $t \in \mathbf{T}$, $f_F \nearrow f_t = f_{tF}$. We proceed by induction on the weight n of F. If $n = 0$, then $F = 1$ and the result is obvious. We now suppose that the result is true at all rank $< n$. Let be $G \in \mathbf{F}$, and let us prove that

$\langle f_F \nearrow f_t, G\rangle = \delta_{tF,G}$. Three cases are possible.

(1) $G = 1$. Then $\langle f_F \nearrow f_t, G\rangle = \langle f_F \nearrow f_t, 1\rangle = \varepsilon(f_F \nearrow f_t) = 0 = \delta_{tF,G}$.

(2) $G = G_1 G_2$, $G_i \neq 1$. Then, by Lemma 20:

$$\begin{aligned}
\langle f_F \nearrow f_t, G\rangle &= \langle \Delta(f_F \nearrow f_t), G_2 \otimes G_1\rangle \\
&= \sum_{F_1 F_2 = F} \langle f_{F_2} \otimes f_{F_1} \nearrow f_t, G_2 \otimes G_1\rangle \\
&\quad + \langle f_F \nearrow f_t \otimes 1 + f_F \nearrow 1 \otimes f_t, G_2 \otimes G_1\rangle - \langle f_F \otimes f_t, G_2 \otimes G_1\rangle \\
&= \sum_{\substack{F_1 F_2 = F, \\ \mathrm{weight}(F_1) < n}} \langle f_{F_2} \otimes f_{F_1} \nearrow f_t, G_2 \otimes G_1\rangle + \langle 1 \otimes f_F \nearrow f_t, G_2 \otimes G_1\rangle \\
&\quad + \langle f_F \nearrow f_t \otimes 1, G_2 \otimes G_1\rangle + \langle f_F \otimes f_t, G_2 \otimes G_1\rangle \\
&\quad - \langle f_F \otimes f_t, G_2 \otimes G_1\rangle \\
&= \sum_{\substack{F_1 F_2 = F, \\ \mathrm{weight}(F_1) < n}} \langle f_{F_2} \otimes f_{tF_1}, G_2 \otimes G_1\rangle \\
&= \sum_{\substack{F_1 F_2 = F, \\ \mathrm{weight}(F_1) < n}} \delta_{F_2, G_2} \delta_{tF_1, G_1} \\
&= \delta_{tF,G}.
\end{aligned}$$

(3) $G = B^+(G_1)$. Note that $f_F \nearrow f_t$ is a linear span of forests $H_1 \nearrow H_2$, with H_1, $H_2 \neq 1$. By definition of $\nearrow$, the first tree of such a forest is not $\bullet$. Hence, $\gamma(f_F \nearrow f_t) = 0$ and

$$\langle f_F \otimes f_t, G\rangle = \langle \gamma(f_F \otimes f_t), G_1\rangle = 0 = \delta_{tF,G},$$

as $tF \notin \mathbf{T}$ because $F \neq 1$.

Second step. We now prove Proposition 21 by induction on n. It is obvious for $n = 1$. Suppose the result true at all rank $n - 1$. By the first step:

$$f_{t_1 \ldots t_n} = f_{t_2 \ldots t_n} \nearrow f_{t_1} = (f_{t_n} \nearrow \ldots \nearrow f_{t_2}) \nearrow f_{t_1} = f_{t_n} \nearrow \ldots \nearrow f_{t_2} \nearrow f_{t_1},$$

using the induction hypothesis for the second equality. ■

Remarks.

(1) As an immediate corollary, because $\nearrow$ is associative, for all forests $F_1, \ldots, F_k \in \mathbf{F}$, $f_{F_1 \ldots F_k} = f_{F_k} \nearrow \ldots \nearrow f_{F_1}$.

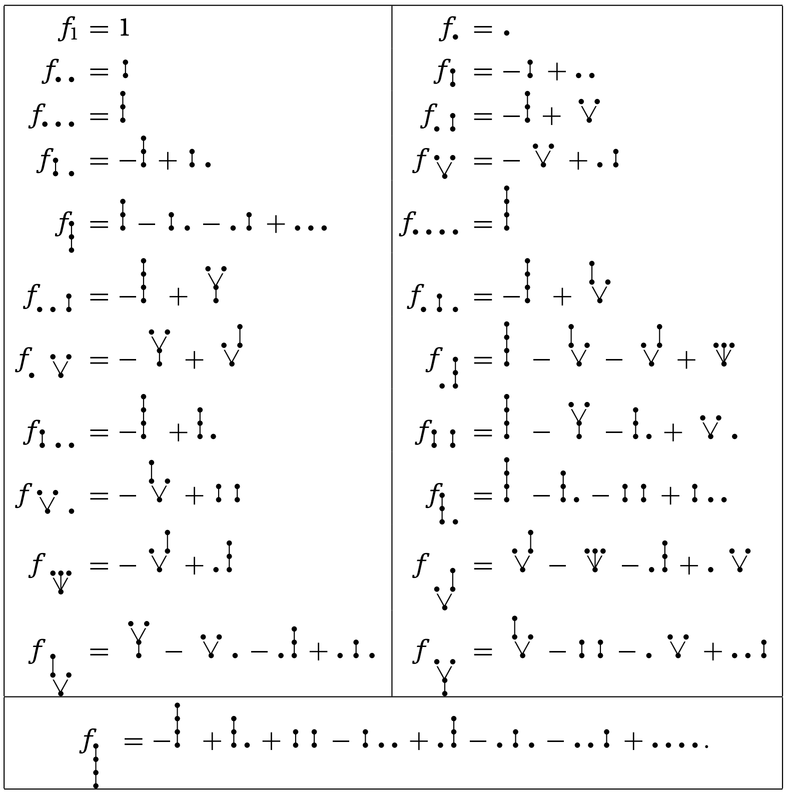
(2) In terms of operads, Proposition 21 can be rewritten in the following way:

Corollary 22. Let $F_1, \ldots, F_n \in \mathbf{F}$. Then $f_{F_1 \ldots F_n} = l_n {\circ\!\!\!\nearrow} (f_{F_n}, \ldots, f_{F_1})$.

Remark. Hence, the dual basis $(f_F)_{F\in\mathbf{F}}$ can be inductively computed, using Proposition 21 of [5], together with Propositions 19 and 21 of the present text:

$$\begin{cases} f_1 &= 1, \\ f_{t_1\ldots t_n} &= f_{t_n} \nearrow \ldots \nearrow f_{t_1}, \\ f_{B^+(t_1\ldots t_n)} &= \gamma_{|\mathrm{Prim}(\mathcal{H})}^{-1}(f_{t_1\ldots t_n}). \end{cases}$$

For example,



4 Primitive Suboperads

4.1 Compatibilities between products and coproducts

We define another coproduct $\Delta_\nearrow$ on $\mathcal{H}$ in the following way: for all $x, y, z \in \mathcal{H}$,

$$\langle \Delta_\nearrow(x), y \otimes z\rangle = \langle x, z \nearrow y\rangle.$$

Lemma 23. For all forests $F \in \mathbf{F}$, $\Delta_\nearrow(F) = \sum_{\substack{F_1,F_2\in\mathbf{F}\\ F_1F_2=F}} F_1 \otimes F_2$.

Proof. Let $F, G, H \in \mathbf{F}$. Then

$$\begin{aligned}\langle \Delta_{\nearrow}(F), f_G \otimes f_H\rangle &= \langle F, f_H \nearrow f_G\rangle \\ &= \langle F, f_{GH}\rangle \\ &= \delta_{F,GH} \\ &= \sum_{\substack{F_1, F_2 \in \mathbf{F} \\ F_1 F_2 = F}} \langle F_1 \otimes F_2, f_G \otimes f_H\rangle.\end{aligned}$$

As $(f_F)_{F\in\mathbf{F}}$ is a basis of $\mathcal{H}$ and $\langle -, -\rangle$ is nondegenerate, this proves the result. ■

Remark. As a consequence, the elements of $\mathbf{T}$ are primitive for this coproduct.

We now have defined three products, namely, m, $\nearrow$, and $\searrow$, and two coproducts, namely, $\tilde{\Delta}$ and $\tilde{\Delta}_{\nearrow}$, on $\mathcal{M}$, obtained from Δ and $\Delta_{\nearrow}$ by subtracting their primitive parts. The following properties sum up the different compatibilities.

Proposition 24. For all $x, y \in \mathcal{M}$:

$$\tilde{\Delta}(xy) = (x \otimes 1)\tilde{\Delta}(y) + \tilde{\Delta}(x)(1 \otimes y) + x \otimes y, \tag{4}$$

$$\tilde{\Delta}(x \nearrow y) = (x \otimes 1) \nearrow \tilde{\Delta}(y) + \tilde{\Delta}(x) \nearrow (1 \otimes y) + x \otimes y, \tag{5}$$

$$\tilde{\Delta}_{\nearrow}(xy) = (x \otimes 1)\tilde{\Delta}_{\nearrow}(y) + \tilde{\Delta}_{\nearrow}(x)(1 \otimes y) + x \otimes y, \tag{6}$$

$$\tilde{\Delta}_{\nearrow}(x \nearrow y) = (x \otimes 1) \nearrow \tilde{\Delta}_{\nearrow}(y), \tag{7}$$

$$\tilde{\Delta}_{\nearrow}(x \searrow y) = (x \otimes 1) \searrow \tilde{\Delta}_{\nearrow}(y). \tag{8}$$

Proof. The compatibility between $\nearrow$ or $\searrow$ and $\tilde{\Delta}_{\nearrow}$ remains to be considered. Let $F, G \in \mathbf{F} - \{1\}$. We put $G = t_1 \dots t_n$, where the t_is are trees. Then $F \nearrow G = (F \nearrow t_1)t_2 \dots t_n$, and $F \nearrow t_1$ is a tree. Hence

$$\begin{aligned}\tilde{\Delta}_{\nearrow}(F \nearrow G) &= \sum_{i=1}^{n-1} (F \nearrow t_1)t_2 \dots t_i \otimes t_{i+1} \dots t_n \\ &= \sum_{i=1}^{n-1} F \nearrow (t_1 t_2 \dots t_i) \otimes t_{i+1} \dots t_n \\ &= (F \otimes 1) \nearrow \tilde{\Delta}_{\nearrow}(G).\end{aligned}$$

The proof is similar for $F \searrow G$. So all these compatibilities are satisfied. ■

Remark. There is no similar compatibility between $\tilde{\Delta}$ and $\searrow$. In particular, Lemma 19 is not true if $t \in \mathbf{F} \setminus \mathbf{T}$.

This justifies the following definitions:

Definition 25.

(1) A $\mathbb{P}_{\nearrow}$-bialgebra of type 1 is a family $(A, m, \nearrow, \tilde{\Delta})$, such that
  (a) $(A, m, \nearrow)$ is a $\mathbb{P}_{\nearrow}$-algebra.
  (b) $(A, \tilde{\Delta})$ is a coassociative, noncounitary coalgebra.
  (c) Compatibilities (4) and (5) are satisfied.

(2) A $\mathbb{P}_{\nearrow}$-bialgebra of type 2 is a family $(A, m, \nearrow, \tilde{\Delta}_{\nearrow})$, such that
  (a) $(A, m, \nearrow)$ is a $\mathbb{P}_{\nearrow}$-algebra.
  (b) $(A, \tilde{\Delta}_{\nearrow})$ is a coassociative, noncounitary coalgebra.
  (c) Compatibilities (6) and (7) are satisfied.

(3) A $\mathbb{P}_{\searrow}$-bialgebra is a family $(A, m, \searrow, \tilde{\Delta}_{\nearrow})$, such that
  (a) $(A, m, \searrow)$ is a $\mathbb{P}_{\searrow}$-algebra.
  (b) $(A, \tilde{\Delta}_{\nearrow})$ is a coassociative, noncounitary coalgebra.
  (c) Compatibilities (6) and (8) are satisfied.

Example. The augmentation ideal $\mathcal{M}$ of the infinitesimal Hopf algebra of trees $\mathcal{H}$ is both a $\mathbb{P}_{\nearrow}$-infinitesimal bialgebra of type 1 and 2, and also a $\mathbb{P}_{\searrow}$-infinitesimal bialgebra.

If A is a bialgebra of such a type, we denote by $\mathrm{Prim}(A)$ the kernel of the coproduct. We deduce the definition of the following suboperads:

Definition 26. Let $n \in \mathbb{N}$. We put:

$$\begin{cases} \mathbb{PRIM}^{(1)}_{\nearrow}(n) = \left\{ p \in \mathbb{P}_{\nearrow}(n) \;/\; \begin{array}{c} \text{For all } A,\ \mathbb{P}_{\nearrow}\text{-infinitesimal bialgebra of type 1,} \\ \text{and for } a_1, \ldots, a_n \in \mathrm{Prim}(A), \\ p.(a_1, \ldots, a_n) \in \mathrm{Prim}(A). \end{array} \right\}, \\ \\ \mathbb{PRIM}^{(2)}_{\nearrow}(n) = \left\{ p \in \mathbb{P}_{\nearrow}(n) \;/\; \begin{array}{c} \text{For all } A,\ \mathbb{P}_{\nearrow}\text{-infinitesimal bialgebra of type 2,} \\ \text{and for } a_1, \ldots, a_n \in \mathrm{Prim}_{\nearrow}(A), \\ p.(a_1, \ldots, a_n) \in \mathrm{Prim}_{\nearrow}(A). \end{array} \right\}, \\ \\ \mathbb{PRIM}_{\searrow}(n) = \left\{ p \in \mathbb{P}_{\searrow}(n) \;/\; \begin{array}{c} \text{For all } A,\ \mathbb{P}_{\searrow}\text{-infinitesimal bialgebra,} \\ \text{and for } a_1, \ldots, a_n \in \mathrm{Prim}_{\nearrow}(A), \\ p.(a_1, \ldots, a_n) \in \mathrm{Prim}_{\nearrow}(A). \end{array} \right\}. \end{cases}$$

We identify $\mathbb{P}_\nearrow(n)$ and $\mathbb{P}_\searrow(n)$ with the homogeneous component of weight n of $\mathcal{M}$. We put $Prim(\mathcal{M}) = Ker(\tilde{\Delta})$ and $Prim_\nearrow(\mathcal{M}) = Ker(\tilde{\Delta}_\nearrow)$. We obtain:

Proposition 27.

(1) For all $n \in \mathbb{N}$:

$$\mathbb{PRIM}^{(1)}_\nearrow(n) = \left\{p \in \mathbb{P}_\nearrow(n) \ / \ p \bullet\!\!\nearrow (\bullet, \ldots, \bullet) \in \mathrm{Prim}(\mathcal{M})\right\} = \mathbb{P}_\nearrow(n) \cap \mathrm{Prim}(\mathcal{M}).$$

(2) For all $n \in \mathbb{N}$:

$$\mathbb{PRIM}^{(2)}_\nearrow(n) = \left\{p \in \mathbb{P}_\nearrow(n) \ / \ p \bullet\!\!\nearrow (\bullet, \ldots, \bullet) \in \mathrm{Prim}_\nearrow(\mathcal{M})\right\} = \mathbb{P}_\nearrow(n) \cap \mathrm{Prim}_\nearrow(\mathcal{M}).$$

(3) For all $n \in \mathbb{N}$:

$$\mathbb{PRIM}_\searrow(n) = \left\{p \in \mathbb{P}_\searrow(n) \ / \ p \bullet\!\!\searrow (\bullet, \ldots, \bullet) \in \mathrm{Prim}_\nearrow(\mathcal{M})\right\} = \mathbb{P}_\searrow(n) \cap \mathrm{Prim}_\nearrow(\mathcal{M}).$$

Proof. As $\mathcal{M}$ is a $\mathbb{P}_\nearrow$-infinitesimal bialgebra, by definition,

$$\mathbb{PRIM}^{(1)}_\nearrow(n) \subseteq \left\{p \in \mathbb{P}_\nearrow(n) \ / \ p \bullet\!\!\nearrow (\bullet, \ldots, \bullet) \in \mathrm{Prim}(\mathcal{M})\right\}.$$

Moreover, $\left\{p \in \mathbb{P}_\nearrow(n) \ / \ p \bullet\!\!\nearrow (\bullet, \ldots, \bullet) \in \mathrm{Prim}(\mathcal{M})\right\} = \mathbb{P}_\nearrow(n) \cap \mathrm{Prim}(\mathcal{M})$, as, for all $p \in \mathbb{P}_\nearrow(n)$, $p \bullet\!\!\nearrow (\bullet, \ldots, \bullet) = p \in \mathcal{M}$.

We now show that $\left\{p \in \mathbb{P}_\nearrow(n) \ / \ p \bullet\!\!\nearrow (\bullet, \ldots, \bullet) \in \mathrm{Prim}(\mathcal{M})\right\} \subseteq \mathbb{PRIM}^{(1)}_\nearrow(n)$. We take $p \in \mathbb{P}_\nearrow(n)$, such that $p \bullet\!\!\nearrow (\bullet, \ldots, \bullet) \in \mathrm{Prim}(\mathcal{M})$. Let $\mathcal{D} = \{1, \ldots, n\}$ and let A be the free $\mathbb{P}_\nearrow$-algebra generated by $\mathcal{D}$ (with a unit). It can be described as the associative algebra $\mathcal{H}^{\mathcal{D}}$ generated by the set of planar rooted trees decorated by $\mathcal{D}$, and can be given a structure of $\mathbb{P}_\nearrow$-infinitesimal bialgebra. As $\mathcal{M}$ is freely generated by $\bullet$ as a $\mathbb{P}_\nearrow$-algebra, there exists a unique morphism of $\mathbb{P}_\nearrow$-algebras from $\mathcal{M}$ to $\mathcal{M}^{\mathcal{D}}$, augmentation ideal of $\mathcal{H}^{\mathcal{D}}$:

$$\xi : \begin{cases} \mathcal{M} \longrightarrow \mathcal{M}^{\mathcal{D}} \\ \bullet \quad \longrightarrow \bullet_1 + \cdots + \bullet_n. \end{cases}$$

As $\bullet \in \mathrm{Prim}(\mathcal{M})$ and $\bullet_1 + \cdots + \bullet_n \in \mathrm{Prim}(A)$, ξ is a $\mathbb{P}_\nearrow$-infinitesimal bialgebra morphism from $\mathcal{M}$ to $\mathcal{M}^{\mathcal{D}}$. So, $\xi(p \bullet\!\!\nearrow (\bullet, \ldots, \bullet)) \in \mathrm{Prim}(A)$.

Let $F \in A$ be a forest, and $s_1 \geq_{h,l} \ldots \geq_{h,l} s_k$ its vertices. For all $i \in \{1, \ldots, k\}$, we put d_i the decoration of s_i. The *decoration word* associated to F is the word $d_1 \ldots d_n$. It belongs to $M(\mathcal{D})$, the free monoid generated by the elements of $\mathcal{D}$. For all $w \in M(\mathcal{D})$, Let A_w be the subspace of A generated by forests whose decoration word is w. This defines a $M(\mathcal{D})$-gradation of A, as a $\mathbb{P}_\nearrow$-infinitesimal bialgebra of type 1.

Consider the projection $\pi_{1,\ldots,n}$ onto $A_{1,\ldots,n}$. We get:

$$\begin{aligned}\pi_{1,\ldots,n}\circ\xi(p\bullet\!\!\nearrow(\bullet,\ldots,\bullet)) \in \mathrm{Prim}(A),\qquad&\\ &=\pi_{1,\ldots,n}(p\bullet\!\!\nearrow(\xi(\bullet),\ldots,\xi(\bullet)))\\ &=\pi_{1,\ldots,n}(p\bullet\!\!\nearrow(\bullet_1+\cdots+\bullet_n,\ldots,\bullet_1+\cdots+\bullet_n))\\ &=p\bullet\!\!\nearrow(\bullet_1,\ldots,\bullet_n).\end{aligned}$$

So $p\bullet\!\!\nearrow(\bullet_1,\ldots,\bullet_n)\in\mathrm{Prim}(A)$.

Let B be a $\mathbb{P}_\nearrow$-infinitesimal bialgebra and let $a_1,\ldots,a_n\in\mathrm{Prim}(B)$. As $\mathcal{M}^{\mathcal{D}}$ is freely generated by the $\bullet_i$ s, there exists a unique morphism of $\mathbb{P}_\nearrow$-algebras:

$$\chi:\begin{cases}A\longrightarrow B\\ \bullet_i\longrightarrow a_i.\end{cases}$$

As the $\bullet_i$ and the a_is are primitive, χ is a $\mathbb{P}$-infinitesimal bialgebra morphism. So

$$\xi(p\bullet\!\!\nearrow(\bullet_1,\ldots,\bullet_n))=p.(\xi(\bullet_1),\ldots,\xi(\bullet_n))=p.(a_1,\ldots,a_n)\in\chi(\mathrm{prim}(\mathcal{M}^{\mathcal{D}}))\subseteq\mathrm{Prim}(A).$$

Hence, $p\in\mathbb{PRIM}^{(1)}_\nearrow(n)$. The proof is similar for $\mathbb{PRIM}^{(2)}_\nearrow$ and $\mathbb{PRIM}_\searrow$. ■

4.2 Suboperad $\mathbb{PRIM}^{(1)}_\nearrow$

Lemma 28. We define inductively the following elements of $\mathbb{P}_\nearrow$:

$$\begin{cases}q_1=\bullet,\\ q_{n+1}=(\bullet\bullet-\overset{\bullet}{\bullet})\circ\!\!\nearrow(q_n,\bullet)\;=\;q_n\bullet-B^+(q_n),\ \text{for } n\geq 1.\end{cases}$$

Then, for all $n\geq 1$, q_n belongs to $\mathbb{PRIM}^{(1)}_\nearrow$. Moreover, for all $x_1,\ldots,x_n\in\mathrm{Prim}(\mathcal{M})$:

$$\gamma(q_n\bullet\!\!\nearrow(x_1,\ldots,x_n))=\gamma(x_1)x_2\ldots x_n.$$

Remark. These q_ns are the same as the q_ns defined in Section 3.2.

Proof. Let us remark that $f_{\overset{\bullet}{\bullet}}=\bullet\bullet-\overset{\bullet}{\bullet}\in\mathrm{Prim}(\mathcal{M})$. By Proposition 27, $\bullet\bullet-\overset{\bullet}{\bullet}\in\mathbb{PRIM}^{(1)}_\nearrow(2)$. As $\mathbb{PRIM}^{(1)}_\nearrow$ is a suboperad of $\mathbb{P}_\nearrow$, it follows that all the q_ns belong to $\mathbb{PRIM}^{(1)}_\nearrow(n)$.

Let $x_1,\ldots,x_n\in\mathrm{Prim}(\mathcal{M})$. Let us show that $\gamma(q_n\bullet\!\!\nearrow(x_1,\ldots,x_n))=\gamma(x_1)x_2\ldots x_n$ by induction on n. If $n=1$, this is immediate. For $n=2$, $q_2\bullet\!\!\nearrow(x_1x_2)=x_1x_2-x_1\nearrow x_2$. Moreover, $x_1\nearrow x_2$ is a linear span of forests whose first tree is not $\bullet$. So

$\gamma(q_2 \bullet\!\!\nearrow (x_1, x_2)) = \gamma(x_1 x_2) - 0 = \gamma(x_1)x_2$.

Suppose now that the result is true at rank $n-1$. Then

$$
\begin{aligned}
q_n \bullet\!\!\nearrow (x_1, \ldots, x_n) &= q_2 \bullet\!\!\nearrow (\underbrace{q_{n-1} \bullet\!\!\nearrow (x_1, \ldots, x_{n-1})}_{\in \mathrm{Prim}(\mathcal{M})}, x_n),\\
\gamma(q_n \bullet\!\!\nearrow (x_1, \ldots, x_n)) &= \gamma(q_2 \bullet\!\!\nearrow (q_{n-1} \bullet\!\!\nearrow (x_1, \ldots, x_{n-1}), x_n))\\
&= \gamma(q_{n-1} \bullet\!\!\nearrow (x_1, \ldots, x_{n-1}))x_n\\
&= \gamma(x_1)x_2 \ldots x_n.
\end{aligned}
$$

So the result holds for all $n \geq 1$. ■

Theorem 29. The non-Σ-operad $\mathbb{PRIM}^{(1)}_{\nearrow}$ is freely generated by $\overset{\bullet}{\underset{\bullet}{|}} - \bullet\bullet$.

Proof. Let us first show that the family $(q_n)_{n\geq 1}$ generates $\mathbb{PRIM}^{(1)}_{\nearrow}$. Let $\mathbb{P}$ be the suboperad of $\mathbb{PRIM}^{(1)}_{\nearrow}$ generated by the q_ns. Let us prove by induction on k that $\mathbb{PRIM}^{(1)}_{\nearrow}(k) = \mathbb{P}(k)$. If $k = 1$, $\mathbb{P}(1) = \mathbb{PRIM}^{(1)}_{\nearrow}(1) = K\bullet$. Suppose the result at all ranks $\leq k-1$. By the rigidity theorem for infinitesimal bialgebra of [11], a basis of $\mathcal{H}$ is $(f_{t_1} \ldots f_{t_n})_{t_1 \ldots t_n \in \mathbf{F}}$, so a basis of $\mathrm{Prim}(\mathcal{M})$ is

$$\left(\gamma^{-1}_{\mathrm{Prim}(\mathcal{H})}(f_{t_1} \ldots f_{t_n})\right)_{t_1 \ldots t_n \in \mathbf{F}}.$$

So, a basis of $\mathbb{PRIM}^{(1)}_{\nearrow}(k)$ is $(\gamma^{-1}_{\mathrm{Prim}(\mathcal{H})}(f_{t_1} \ldots f_{t_n}))_{\substack{t_1 \ldots t_n \in \mathbf{F}\\ \mathrm{weight}(t_1 \ldots t_n) = k-1}}$. By Lemma 28,

$$\gamma^{-1}_{\mathrm{Prim}(\mathcal{H})}(f_{t_1} \ldots f_{t_n}) = q_{n+1} \bullet\!\!\nearrow (\bullet, f_{t_1}, \ldots f_{t_n}).$$

By the induction hypothesis, the f_{t_i}s belong to $\mathbb{P}$. So

$$\gamma^{-1}_{\mathrm{Prim}(\mathcal{H})}(f_{t_1} \ldots f_{t_n}) = q_{n+1} \bullet\!\!\nearrow (\bullet, f_{t_1}, \ldots f_{t_n}) \in \mathbb{P}(n).$$

So $\mathbb{PRIM}^{(1)}_{\nearrow} = \mathbb{P}$.

Moreover, if we denote by $\mathbb{P}'$ the suboperad of $\mathbb{PRIM}^{(1)}_{\nearrow}$ generated by q_2, then, immediately, $\mathbb{P}' \subseteq \mathbb{P}$. Finally, by induction on n, $q_n \in \mathbb{P}'(n)$ for all $n \geq 1$ and $\mathbb{P} \subseteq \mathbb{P}'$. So $\mathbb{P}' = \mathbb{P} = \mathbb{PRIM}^{(1)}_{\nearrow}$ is generated by q_2.

Let $\mathbb{P}_{q_2}$ be the non-Σ-operad freely generated by q_2. There is a non-Σ-operad epimorphism:

$$\Psi : \begin{cases} \mathbb{P}_{q_2} \longrightarrow \mathbb{PRIM}^{(1)}_{\nearrow} \\ q_2 \longrightarrow q_2. \end{cases}$$

The dimension of $\mathbb{P}_{q_2}(n)$ is the number of planar binary rooted trees with n leaves, that is to say the Catalan number $c_n = \frac{(2n-2)!}{(n-1)!n!}$. On the other side, the dimension of $\mathbb{PRIM}^{(1)}_{\nearrow}(n)$ is the number of planar rooted trees with n vertices, that is to say c_n. So Ψ is an isomorphism. ■

In other terms, in the language of [10]:

Theorem 30. The triple of operads $(\mathbb{A}ss, \mathbb{P}^{\Sigma}_{\nearrow}, \mathbb{FREE}_2)$, where $\mathbb{P}^{\Sigma}_{\nearrow}$ is the symmetrization of $\mathbb{P}_{\nearrow}$ and $\mathbb{FREE}_2$, is the free operad generated by an element in $\mathbb{FREE}_2(2)$, is a good triple of operads.

Remark. Note that if A is a $\mathbb{P}_{\nearrow}$-bialgebra of type 1, then $(A, m, \tilde{\Delta})$ is a nonunitary infinitesimal bialgebra. Hence, if $(K \oplus A, m, \Delta)$ has an antipode S, then $-S$ is an eulerian idempotent for A.

4.3 Another basis of Prim($\mathcal{H}$)

Recall that $\mathbb{T}_b$ is freely generated (as a non-Σ-operad) by $\curlyvee$. In particular, if $t_1, t_2 \in \mathbf{T}_b$, we denote:

$$t_1 \vee t_2 = \curlyvee \circ (t_1, t_2).$$

Every element $t \in \mathbf{T}_b - \{\mid\}$ can be uniquely written as $t = t^l \vee t^r$.

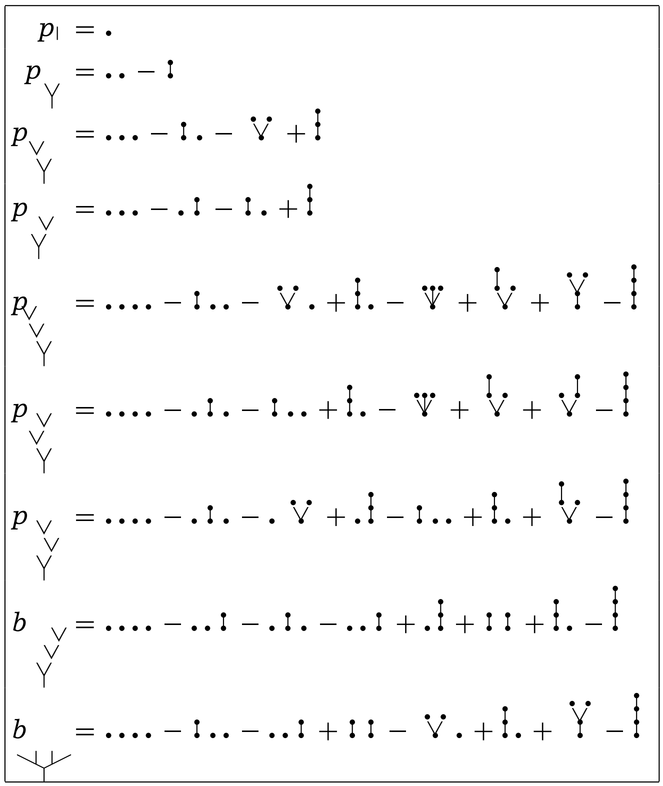
There exists a morphism of operads:

$$\Theta : \begin{cases} \mathbf{T}_b \longrightarrow \mathbb{P}_{\nearrow} \\ \curlyvee \longrightarrow \bullet\bullet - \overset{\bullet}{\bullet}. \end{cases}$$

By Theorem 29, Θ is injective and its image is $\mathbb{PRIM}^{(1)}_{\nearrow}$. So, we obtain a basis of $\mathbb{PRIM}^{(1)}_{\nearrow}$ indexed by $\mathbf{T}_b$, given by $p_t = \Theta(t)$. It is also a basis of Prim($\mathcal{M}$), which can be inductively computed by

$$\begin{cases} p_{|} & = \bullet, \\ p_{t_1 \vee t_2} & = (\bullet\bullet - \mathfrak{l}\,) \not\nearrow (p_{t_1}, p_{t_2}) \;=\; p_{t_1} p_{t_2} - p_{t_1} \nearrow p_{t_2}. \end{cases}$$

Examples.

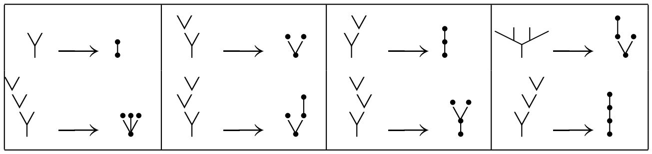


4.4 From the basis $(f_t)_{t \in \mathbf{T}}$ to the basis $(p_t)_{t \in \mathbf{T}_b}$

We define inductively the application $\kappa : \mathbf{T}_b \longrightarrow \mathbf{T}$ in the following way:

$$\kappa : \begin{cases} \mathbf{T}_b & \longrightarrow \mathbf{T} \\ | & \longrightarrow \bullet, \\ t_1 \vee t_2 & \longrightarrow \kappa(t_2) \searrow \kappa(t_1). \end{cases}$$

Examples.



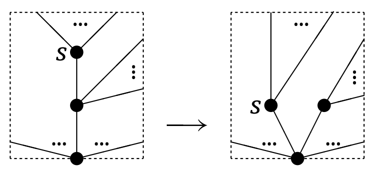
It is easy to show that κ is bijective, with inverse given by

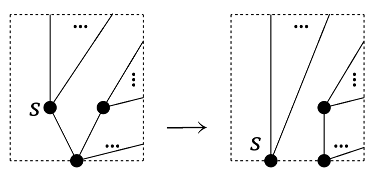
$$\kappa^{-1}:\begin{cases}\mathbf{T} & \longrightarrow \mathbf{T}_b\\ \bullet & \longrightarrow \mid,\\ B^+(s_1\ldots s_m) & \longrightarrow \kappa^{-1}(B^+(s_2\ldots s_m))\vee\kappa^{-1}(s_1).\end{cases}$$

Let us recall the partial order $\leq$, defined in [5], on the set $\mathbf{F}$ of planar forests, making it isomorphic to the Tamari poset.

Definition 31. Let $F\in\mathbf{F}$.

(1) An *admissible transformation* on F is a local transformation of F of one of the following types (the part of F that is not in the frame remains unchanged):

First kind: 

Second kind: 

(2) Let F and $G\in\mathbf{F}$. We shall say that $F\leq G$ if there exists a finite sequence $F_0,\ldots,F_k$ of elements of $\mathbf{F}$ such that:
   (a) For all $i\in\{0,\ldots,k-1\}$, F_{i+1} is obtained from F_i by an admissible transformation.
   (b) $F_0=F$.
   (c) $F_k=G$.

The aim of this section is to prove the following result:

Theorem 32. Let $t \in \mathbf{T}_b$. Then $p_t = \sum_{\substack{s \in \mathbf{T} \\ s \leq \kappa(t)}} f_s$.

Proof. By induction on the number n of leaves of t. If $n = 1$, then $t = |$ and $p_| = \bullet = f_\bullet$. Suppose the result at all ranks $\leq n-1$. As p_t is primitive, we can put

$$p_t = \sum_{s \in \mathbf{T}} a_s f_s.$$

Write $t = t_1 \vee t_2$. By the induction hypothesis,

$$p_{t_1} = \sum_{\substack{s_1 \in \mathbf{T} \\ s_1 \leq \kappa(t_1)}} f_{s_1} \quad \text{and} \quad p_{t_2} = \sum_{\substack{s_2 \in \mathbf{T} \\ s_2 \leq \kappa(t_2)}} f_{s_2}.$$

As $t = t_1 \vee t_2$, $p_t = (\bullet\bullet - \vdots) \not\nearrow (p_{t_1}, p_{t_2}) = p_{t_1} p_{t_2} - p_{t_1} \nearrow p_{t_2}$. So, for all $s \in \mathbf{T}$, as s is primitive for $\Delta_\nearrow$,

$$\begin{aligned} a_s &= \langle p_t, s \rangle \\ &= \langle p_{t_1} p_{t_2} - p_{t_1} \nearrow p_{t_2}, s \rangle \\ &= \langle p_{t_2} \otimes p_{t_1}, \Delta(s) - \Delta_\nearrow(s) \rangle \\ &= \langle p_{t_2} \otimes p_{t_1}, \Delta(s) \rangle \\ &= \sum_{\substack{s_1 \in \mathbf{T} \\ s_1 \leq \kappa(t_1)}} \sum_{\substack{s_2 \in \mathbf{T} \\ s_2 \leq \kappa(t_2)}} \langle f_{s_2} \otimes f_{s_1}, \Delta(s) \rangle. \end{aligned}$$

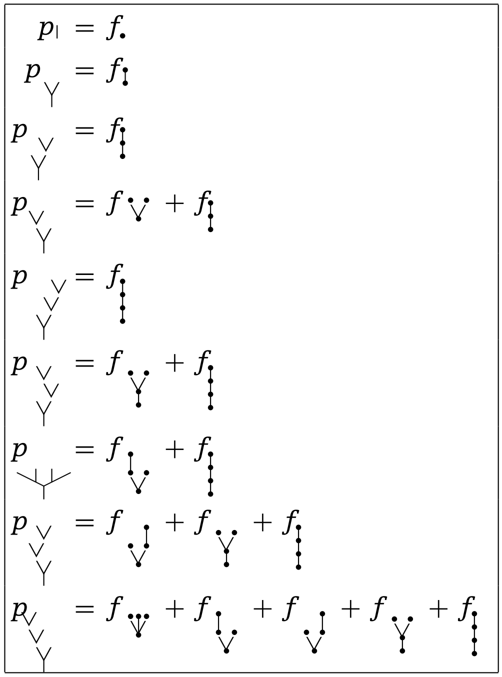
So a_s is the number of left-admissible cuts c of s, such that $P^c(s) \leq \kappa(t_2)$ and $R^c(s) \leq \kappa(t_1)$.

Suppose that $a_s \neq 0$. Then, there exists a left-admissible cut c of s, such that $P^c(s) \leq \kappa(t_2)$ and $R^c(s) \leq \kappa(t_1)$. As s is a tree, $s \leq \kappa(t_2) \searrow \kappa(t_1) = \kappa(t)$. Moreover, by considering the degree of $P^c(s)$, this cut c is unique, so $a_s = 1$. Reciprocally, if $s \leq \kappa(t)$, if c is the unique left-admissible cut such that weight$(P^c(s)) =$ weight(t_2), then $P^c(s) \leq \kappa(t_2)$ and $R^c(s) \leq \kappa(t_1)$. So $a_s \neq 0$. Hence, $(s \leq \kappa(t)) \Longrightarrow (a_s \neq 0) \Longrightarrow (a_s = 1) \Longrightarrow (s \leq \kappa(t))$. This proves Theorem 32. ∎

Let μ be the Mbius function of the poset $\mathbf{F}$ [14, 15]. By the Mbius inversion formula:

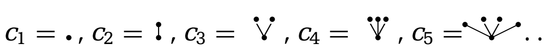
Corollary 33. Let $s \in \mathbf{T}$. Then $f_s = \sum_{t \in T_b,\, \kappa(t) \leq s} \mu(\kappa(t), s) p_t$.

Examples.



4.5 Suboperad $\mathbb{PRIM}^{(2)}_{\nearrow}$

For all $n \in \mathbb{N}$, we put $c_{n+1} = B^+(\bullet^n)$. In other terms, c_{n+1} is the corolla tree with $n+1$ vertices, or equivalently with n leaves.

Examples. 

Lemma 34. The set $\mathbf{T}$ is a basis of the operad $\mathbb{PRIM}^{(2)}_{\nearrow}$. As an operad, $\mathbb{PRIM}^{(2)}_{\nearrow}$ is generated by the c_ns, $n \geq 2$. Moreover, for all $k, l \geq 2$,

$$c_k \not\nearrow (c_l, \underbrace{\bullet, \ldots, \bullet}_{k-1 \text{ times}}) = c_l \not\nearrow (\underbrace{\bullet, \ldots, \bullet}_{l-1 \text{ times}}, c_k).$$

Proof. The operad $\mathbb{PRIM}^{(2)}_{\nearrow}$ is identified with $\mathrm{Prim}_{\nearrow}(\mathcal{M})$ by Proposition 27. So $\mathrm{Prim}_{\nearrow}(\mathcal{M})$ is equal to $\mathrm{Vect}(\mathbf{T})$. Let $\mathbb{P}$ be the suboperad of $\mathbb{PRIM}^{(2)}_{\nearrow}$ generated by the corollas. Let $t \in \mathbf{T}$, of weight n. Let us prove that $t \in \mathbb{P}$ by induction on n. If $n = 1$, then $t = \bullet \in \mathbb{P}$. If $n \geq 2$, we can suppose that $t = B^+(t_1 \dots t_k)$, with $t_1, \dots, t_k \in \mathbb{P}$. Then, by Theorem 11:

$$c_{k+1} \not\nearrow (t_1, \dots, t_k, \bullet) = (\bullet^{\,k} \not\nearrow (t_1, \dots, t_k)) \nearrow \bullet = (t_1 \dots t_k) \nearrow \bullet = B^+(t_1 \dots t_k) = t.$$

So $t \in \mathbb{P}$. Hence, $\mathbb{P} = \mathbb{PRIM}^{(2)}_{\nearrow}$.

Let $k, l \geq 2$. Then, by Theorem 11:

$$\begin{aligned} c_k \not\nearrow (c_l, \bullet, \dots, \bullet) &= (\bullet^{\,k-1} \not\nearrow (c_l, \bullet, \dots, \bullet)) \nearrow \bullet \\ &= (c_l \bullet^{\,k-2}) \nearrow \bullet \\ &= B^+(c_l \bullet^{\,k-2}) \\ &= B^+(B^+(\bullet^{\,l-1}) \bullet^{\,k-2}). \end{aligned}$$

On the other hand,

$$\begin{aligned} c_l \not\nearrow (\bullet, \dots, \bullet, c_k) &= (\bullet^{\,l-1} \not\nearrow (\bullet, \dots, \bullet)) \nearrow c_k \\ &= (\bullet^{\,l-1}) \nearrow c_k \\ &= (\bullet^{\,l-1}) \nearrow B^+(c^{k-1}) \\ &= B^+(((\bullet^{\,l-1}) \nearrow \bullet) \bullet^{\,k-2}) \\ &= B^+(B^+(\bullet^{\,l-1}) \bullet^{\,k-2}). \end{aligned}$$

So, $c_k \not\nearrow (c_l, \bullet, \dots, \bullet) = c_l \not\nearrow (\bullet, \dots, \bullet, c_k)$. ∎

Definition 35. The operad $\mathbb{T}$ is the non-Σ-operad generated by the elements $c_n \in \mathbb{T}(n)$, for $n \geq 2$, and the following relations: for all $k, l \geq 2$,

$$c_k \circ (c_l, \underbrace{I, \dots, I}_{k-1 \text{ times}}) = c_l \circ (\underbrace{I, \dots, I}_{l-1 \text{ times}}, c_k).$$

In other terms, a $\mathbb{T}$-algebra A has a family of n-multilinear products $[., \dots, .] : A^{\otimes n} \longrightarrow A$ for all $n \geq 2$, with the associativity condition

$$[[a_1, \dots, a_l], a_{l+1}, \dots, a_{l+k}] = [a_1, \dots, a_{l-1}, [a_l, \dots, a_{l+k}]].$$

In particular, $[., .]$ is associative.

Theorem 36. The operads $\mathbb{T}$ and $\mathbb{PRIM}^{(2)}_{\nearrow}$ are isomorphic.

Proof. By Lemma 34, there is an epimorphism of operads:

$$\begin{cases} \mathbb{T} \longrightarrow \mathbb{PRIM}^{(2)}_{\nearrow} \\ c_n \longrightarrow c_n. \end{cases}$$

In order to prove this is an isomorphism, it is enough to prove that $\dim(\mathbb{T}(n)) \leq dim(\mathbb{PRIM}^{(2)}_{\nearrow}(n))$ for all $n \geq 2$. By Lemma 34, $\dim(\mathbb{PRIM}^{(2)}_{\nearrow}(n))$ is the nth Catalan number. Because of the defining relations, $\mathbb{T}(n)$ is generated as a vector space by elements of the form $c_l \circ (I, b_2, \ldots, b_l)$, with $b_i \in \mathbb{T}(n_i)$, such that $n_1 + \cdots + n_l = n - 1$. Hence, we define inductively the following subsets of the free non-Σ-operad generated by the c_ns, $n \geq 2$:

$$X(n) = \begin{cases} \{I\} \text{ if } n = 1, \\ \bigcup_{l=2}^{n} \bigcup_{i_2+\cdots+i_l=n-1} c_l \circ (I, X(i_2), \ldots, X(i_l)) \text{ if } n \geq 2. \end{cases}$$

Then the images of the elements of $X(n)$ linearly generate $\mathbb{T}(n)$, so $\dim(\mathbb{T}(n)) \leq card(X(n))$ for all n. We now put $a_n = card(X(n))$ and prove that a_n is the nth Catalan number. We denote by $A(h)$ their generating formal series. Then

$$\begin{cases} a_1 = 1, \\ a_n = \sum_{l=2}^{n} \sum_{i_2+\cdots+i_l=n-1} a_{i_1} \ldots a_{i_l} \text{ if } n \geq 2. \end{cases}$$

In terms of generating series,

$$A(h) - a_1 h = h \frac{A(x)}{1 - A(x)}.$$

So $A(h)^2 - A(h) + h = 0$. As $A(h) = 1$,

$$A(h) = \frac{1 - \sqrt{1 - 4h}}{2}.$$

So a_n is the nth Catalan number for all $n \geq 1$. ∎

In other terms:

Theorem 37. The triple of operads $(\mathbb{A}ss, \mathbb{P}^{\Sigma}_{\nearrow}, \mathbb{T}^{\Sigma})$ is a good triple of operads.

Remark. Note that if A is a $\mathbb{P}_{\nearrow}$-bialgebra of type 2, then $(A, m, \tilde{\Delta}_{\nearrow})$ is a nonunitary infinitesimal bialgebra. Hence, if $(K \oplus A, m, \Delta_{\nearrow})$ has an antipode $S_{\nearrow}$, then $-S_{\nearrow}$ is an eulerian idempotent for A.

4.6 Suboperad $\mathbb{PRIM}_{\searrow}$

Lemma 38. The set $\mathbf{T}$ is a basis of the operad $\mathbb{PRIM}_{\searrow}$. As an operad, $\mathbb{PRIM}_{\searrow}$ is generated by $\tdeux$.

Proof. Let $\mathbb{P}$ be the suboperad of $\mathbb{PRIM}_{\searrow}$ generated by $\tdeux$. Let $t \in \mathbf{T}$, of weight n. Let us prove that $t \in \mathbb{P}$ by induction on n. If $n = 1$ or 2, this is obvious. If $n \geq 2$, suppose that $t = B^+(t_1 \dots t_k)$. By the induction hypothesis, t_1 and $B^+(t_2 \dots t_k)$ belong to $\mathbb{P}$. Then

$$t = t_1 \searrow B^+(t_2 \dots t_k) = \tdeux \searrow (t_1, B^+(t_2 \dots t_k)).$$

So $t \in \mathbb{P}$. ■

Theorem 39. The non-Σ-operad $\mathbb{PRIM}_{\searrow}$ is freely generated by $\tdeux$.

Proof. Similar as the proof of Theorem 29. ■

In other terms:

Theorem 40. The triple of operads $(\mathbb{A}ss, \mathbb{P}^{\Sigma}_{\searrow}, \mathbb{F}_2)$, where $\mathbb{F}_2$ is the free operad generated by an element in $\mathbb{F}_2(2)$, is a good triple of operads.

Remark. Note that if A is a $\mathbb{P}_{\searrow}$-bialgebra, then $(A, m, \tilde{\Delta})$ is a nonunitary infinitesimal bialgebra. Hence, if $(K \oplus A, m, \Delta)$ has an antipode S, then $-S$ is an eulerian idempotent for A.

5 A Rigidity Theorem for $\mathbb{P}_{\nearrow}$-Algebras

5.1 Double $\mathbb{P}_{\nearrow}$-infinitesimal bialgebras

Definition 41. A double $\mathbb{P}_{\nearrow}$-infinitesimal bialgebra is a family $(A, m, \nearrow, \tilde{\Delta}, \tilde{\Delta}_{\nearrow})$, where $m, \nearrow : A \otimes A \longrightarrow A$, $\tilde{\Delta}, \tilde{\Delta}_{\nearrow} : A \longrightarrow A \otimes A$, with the following compatibilities:

(1) $(A, m, \nearrow)$ is a (nonunitary) $\mathbb{P}_{\nearrow}$-algebra.

(2) For all $x \in A$:

$$\begin{cases} (\tilde{\Delta} \otimes Id) \circ \tilde{\Delta}(x) &= (Id \otimes \tilde{\Delta}) \circ \tilde{\Delta}(x), \\ (\tilde{\Delta}_{\nearrow} \otimes Id) \circ \tilde{\Delta}_{\nearrow}(x) &= (Id \otimes \tilde{\Delta}_{\nearrow}) \circ \tilde{\Delta}_{\nearrow}(x), \\ (\tilde{\Delta} \otimes Id) \circ \tilde{\Delta}_{\nearrow}(x) &= (Id \otimes \tilde{\Delta}_{\nearrow}) \circ \tilde{\Delta}(x). \end{cases}$$

In other terms, $(A, \tilde{\Delta}^{cop}, \tilde{\Delta}_{\nearrow}^{cop})$ is a $\mathbb{P}_{\nearrow}$-coalgebra.

(3) $(A, m, \nearrow, \tilde{\Delta})$ is a $\mathbb{P}_{\nearrow}$-bialgebra of type 1.

(4) $(A, m, \nearrow, \tilde{\Delta}_{\nearrow})$ is a $\mathbb{P}_{\nearrow}$-bialgebra of type 2.

Remark. If $(A, m, \nearrow, \tilde{\Delta}, \tilde{\Delta}_{\nearrow})$ is a graded double $\mathbb{P}_{\nearrow}$-infinitesimal bialgebra, with finite-dimensional homogeneous components, then its graded dual $(A^*, \tilde{\Delta}^{*,op}, \tilde{\Delta}_{\nearrow}^{*,op}, m^{*,cop}, \nearrow^{*,cop})$ also is.

Theorem 42. $(\mathcal{M}, m, \nearrow, \tilde{\Delta}, \tilde{\Delta}_{\nearrow})$ is a double $\mathbb{P}$-infinitesimal bialgebra.

Proof. We already know that $(\mathcal{M}, m, \nearrow)$ is a $\mathbb{P}_{\nearrow}$-algebra. Moreover, $(\mathcal{M}, \tilde{\Delta}^{cop}, \tilde{\Delta}_{\nearrow}^{cop})$ is isomorphic to $(\mathcal{M}^*, m^*, \nearrow^*)$ via the pairing $\langle -,-\rangle$, so it is a $\mathbb{P}_{\nearrow}$-coalgebra. It is already known that $(\mathcal{M}, m, \tilde{\Delta})$ and $(\mathcal{M}, \nearrow, \tilde{\Delta})$ are infinitesimal bialgebras. As $(\mathcal{M}, \nearrow, \tilde{\Delta})$ is isomorphic to $(\mathcal{M}^{op}, m^{op}, \tilde{\Delta}_{\nearrow}^{cop})$ via the pairing $\langle -,-\rangle$, it is also an infinitesimal bialgebra. So all the needed compatibilities are satisfied. ■

Remarks.

(1) Via the pairing $\langle -,-\rangle$, $\mathcal{M}$ is isomorphic to its graded dual as a double $\mathbb{P}_{\nearrow}$-infinitesimal bialgebra. As a consequence, as $\mathcal{M}$ is the free $\mathbb{P}_{\nearrow}$-algebra generated by $\bullet$, then $\mathcal{M}^{cop}$ is also the cofree $\mathbb{P}_{\nearrow}$-coalgebra cogenerated by $\bullet$.

(2) All these results can be easily extended to infinitesimal Hopf algebras of decorated planar rooted trees; in other terms, to every free $\mathbb{P}_{\nearrow}$-algebras.

Lemma 43. In the double-infinitesimal $\mathbb{P}_{\nearrow}$-algebra $\mathcal{M}$, $Ker(\tilde{\Delta}) \cap Ker(\tilde{\Delta}_{\nearrow}) = \mathrm{Vect}(\bullet)$.

Proof. $\supseteq$. Obvious.

$\subseteq$. Let $x \in Ker(\tilde{\Delta}) \cap Ker(\tilde{\Delta}_{\nearrow})$. Then $\tilde{\Delta}_{\nearrow}(x) = 0$, so x is a linear span of trees. We can write:

$$x = \sum_{t \in \mathbf{T}} a_t t.$$

Consider the terms in $\mathcal{M} \otimes \bullet$ of $\tilde{\Delta}(x)$. We get $\sum_{t \in \mathbf{T} - \{\bullet\}} a_t B^-(t) \otimes \bullet = 0$, where $B^-(t)$ is the forest obtained by deleting the root of t. So, if $t \neq \bullet$, then $a_t = 0$. So $x \in \mathrm{vect}(\bullet)$. ■

Remark. This lemma can be extended to any free $\mathbb{P}_{\nearrow}$-algebra: if V is a vector space, then the free $\mathbb{P}_{\nearrow}$-algebra $F_{\mathbb{P}_{\nearrow}}(V)$ generated by V is given a structure of double $\mathbb{P}_{\nearrow}$-infinitesimal bialgebra by $\tilde{\Delta}(v) = \tilde{\Delta}_{\nearrow}(v) = 0$ for all $v \in V$. In this case, $Ker(\tilde{\Delta}) \cap Ker(\tilde{\Delta}_{\nearrow}) = V$ for $F_{\mathbb{P}_{\nearrow}}(V)$.

5.2 Connected double $\mathbb{P}_{\nearrow}$-infinitesimal bialgebras

Notations. Let A be a double $\mathbb{P}_{\nearrow}$-infinitesimal bialgebra. The iterated coproducts will be denoted in the following way: for all $n \in \mathbb{N}$,

$$\tilde{\Delta}^n : \begin{cases} A \longrightarrow A^{\otimes(n+1)} \\ a \longrightarrow a^{(1)} \otimes \ldots \otimes a^{(n+1)}, \end{cases}$$

$$\tilde{\Delta}^n_{\nearrow} : \begin{cases} A \longrightarrow A^{\otimes(n+1)} \\ a \longrightarrow a^{(1)}_{\nearrow} \otimes \ldots \otimes a^{(n+1)}_{\nearrow}. \end{cases}$$

Definition 44. Let A be a double $\mathbb{P}_{\nearrow}$-infinitesimal bialgebra. It will be called *connected* if, for any $a \in A$, every iterated coproduct $A \longrightarrow A^{\otimes(n+1)}$ vanishes on a for a great enough n.

Theorem 45. Let A be a connected double $\mathbb{P}_{\nearrow}$-infinitesimal bialgebra. Then A is isomorphic to the free $\mathbb{P}_{\nearrow}$-algebra generated by $\mathrm{Prim}(A) = Ker(\tilde{\Delta}) \cap Ker(\tilde{\Delta}_{\nearrow})$ as a double $\mathbb{P}_{\nearrow}$-infinitesimal bialgebra.

Proof. *First step.* We shall use the results on infinitesimal Hopf algebras of [5]. We show that $A = \mathrm{Prim}(A) + A.A + A \nearrow A$. As $(A, \nearrow, \tilde{\Delta})$ is a connected nonunitary infinitesimal bialgebra, it (or more precisely its unitarization) has an antipode $S_{\nearrow}$, defined by

$$S_{\nearrow} : \begin{cases} A \longrightarrow A \\ a \longrightarrow \sum_{i=0}^{\infty} (-1)^{i+1} a^{(1)} \nearrow \ldots \nearrow a^{(i+1)}. \end{cases}$$

As $(A, \tilde{\Delta})$ is connected, this makes sense. Moreover, $-S_{\nearrow}$ is the projector on $Ker(\tilde{\Delta})$ in the direct sum $A = Ker(\tilde{\Delta}) \oplus A \nearrow A$.

In the same order of ideas, as $(A, m, \tilde{\Delta}_{\nearrow})$ is a connected infinitesimal bialgebra, we can define its antipode $S^{\nearrow}$ by

$$S^{\nearrow} : \begin{cases} A \longrightarrow A \\ a \longrightarrow \sum_{i=0}^{\infty} (-1)^{i+1} a^{(1)}_{\nearrow} \ldots a^{(i+1)}_{\nearrow}, \end{cases}$$

and $-S^{\nearrow}$ is the projector on $Ker(\tilde{\Delta}_{\nearrow})$ in the direct sum $A = Ker(\tilde{\Delta}_{\nearrow}) \oplus A.A$.

Let $a \in A$, $b \in Ker(\tilde{\Delta}_{\nearrow})$. Then $\tilde{\Delta}_{\nearrow}(a \nearrow b) = (a \otimes 1)\tilde{\Delta}_{\nearrow}(b) = 0$. So $A \nearrow Ker(\tilde{\Delta}_{\nearrow})$ is a subset of $Ker(\tilde{\Delta}_{\nearrow})$. Moreover, if $\tilde{\Delta}_{\nearrow}(a) = 0$, then $(Id \otimes \tilde{\Delta}_{\nearrow}) \circ \tilde{\Delta}(a) = (\tilde{\Delta} \otimes Id) \circ \tilde{\Delta}_{\nearrow}(a) = 0$. So $\tilde{\Delta}(a) \in A \otimes Ker(\tilde{\Delta}_{\nearrow})$. As a consequence, if $n \geq 1$,

$$\tilde{\Delta}^n(a) = (\tilde{\Delta}^{n-1} \otimes Id) \circ \tilde{\Delta}(a) \in A^{\otimes n} \otimes Ker(\tilde{\Delta}_{\nearrow}).$$

Hence, for all $n \in \mathbb{N}$, $\tilde{\Delta}^n(Ker(\tilde{\Delta}_\nearrow)) \in A^{\otimes n} \otimes Ker(\tilde{\Delta}_\nearrow)$. Finally, we deduce that $S_\nearrow(Ker(\tilde{\Delta}_\nearrow)) \subseteq Ker(\tilde{\Delta}_\nearrow)$.

Let $a \in A$. Then $S^\nearrow(a) \in Ker(\tilde{\Delta}_\nearrow)$ and $S_\nearrow \circ S^\nearrow(a) \in Ker(\tilde{\Delta}) \cap Ker(\tilde{\Delta}_\nearrow) = \mathrm{Prim}(A)$ by the preceding point. Moreover,

$$\begin{aligned} S^\nearrow(a) &= -a + A.A, \\ S_\nearrow \circ S^\nearrow(a) &= -S^\nearrow(a) + A \nearrow A, \\ S_\nearrow \circ S^\nearrow(a) &= a + A.A + A \nearrow A. \end{aligned}$$

This proves the first step.

Second step. As A is connected, it classically inherits a filtration of $\mathbb{P}_\nearrow$-algebra given by the kernels of the iterated coproducts. We denote by deg_p the associated degree. In particular, for all $x \in A$, $deg_p(x) \leq 1$ if, and only if, $x \in \mathrm{Prim}(A)$. Let B be the $\mathbb{P}_\nearrow$-subalgebra of A generated by $\mathrm{Prim}(A)$. Let $a \in A$, let us show that $a \in B$ by induction on $n = deg_p(a)$. If $n \leq 1$, then $a \in \mathrm{Prim}(A) \subseteq B$. Suppose that the result is true at all ranks $\leq n-1$. Then, by the first step, we can write

$$a = b + \sum_i a_i b_i + \sum_j c_j d_j,$$

with $b \in \mathrm{Prim}(A)$, $a_i, b_i, c_j, d_j \in A$. Because of the filtration, we can suppose that $deg_p(a_i)$, $deg_p(b_i)$, $deg_p(c_j)$, $deg_p(d_j) < n$. By the induction hypothesis, they belong to B, so $a \in B$.

Last step. So, there is an epimorphism of $\mathbb{P}_\nearrow$-algebras:

$$\phi : \begin{cases} F_{\mathbb{P}_\nearrow}(\mathrm{Prim}(A)) \longrightarrow A \\ a \in \mathrm{Prim}(A) \longrightarrow a, \end{cases}$$

where $F_{\mathbb{P}_\nearrow}(\mathrm{Prim}(A))$ is the free $\mathbb{P}_\nearrow$-algebra generated by $\mathrm{Prim}(A)$. As the elements of $\mathrm{Prim}(A)$ are primitive both in A and $F_{\mathbb{P}_\nearrow}(\mathrm{Prim}(A))$, this is a morphism of double $\mathbb{P}_\nearrow$-infinitesimal bialgebras. Suppose that it is not monic. Take then $x \in Ker(\phi)$, nonzero, of minimal degree. Then it is primitive, so belongs to $\mathrm{Prim}(A)$ (Lemma 43). Hence, $\phi(a) = a = 0$: this is a contradiction. So ϕ is a bijection. ■

In other terms:

Corollary 46. The triple of operads $((\mathbb{P}^{\Sigma}_{\nearrow})^{op}, \mathbb{P}^{\Sigma}_{\nearrow}, \mathbb{VECT})$ is a good triple. Here, $\mathbb{VECT}$ denotes the operad of vector spaces:

$$\mathbb{VECT}(k) = \begin{cases} KI & \text{if } k = 1, \\ 0 & \text{if } k \neq 1, \end{cases}$$

where I is the unit of $\mathbb{VECT}$.

We have also showed that $S_{\nearrow} \circ S^{\nearrow}$ is the projection on $\mathrm{Prim}(A)$ in the direct sum $A = \mathrm{Prim}(A) \oplus (A.A + A \nearrow A)$. So $S_{\nearrow} \circ S^{\nearrow}$ is the eulerian idempotent.

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