Classification of systems of Dyson-Schwinger equations in the Hopf algebra of decorated rooted trees

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ABSTRACT. We consider systems of combinatorial Dyson-Schwinger equations (briefly, SDSE) $X_1 = B_1^+(F_1(X_1, \ldots, X_N))$, ..., $X_N = B_N^+(F_N(X_1, \ldots, X_N))$ in the Connes-Kreimer Hopf algebra $\mathcal{H}_I$ of rooted trees decorated by $I = \{1, \ldots, N\}$, where B_i^+ is the operator of grafting on a root decorated by i, and $F_1, \ldots, F_N$ are non-constant formal series. The unique solution $X = (X_1, \ldots, X_N)$ of this equation generates a graded subalgebra $\mathcal{H}_{(S)}$ of $\mathcal{H}_I$. We characterise here all the families of formal series $(F_1, \ldots, F_N)$ such that $\mathcal{H}_{(S)}$ is a Hopf subalgebra. More precisely, we define three operations on SDSE (change of variables, dilatation and extension) and give two families of SDSE (cyclic and fundamental systems), and prove that any SDSE (S) such that $\mathcal{H}_{(S)}$ is Hopf is the concatenation of several fundamental or cyclic systems after the application of a change of variables, a dilatation and iterated extensions.

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Contents

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Introduction

The Connes-Kreimer Hopf algebra of rooted trees is introduced in [14] and studied in [2, 3, 5, 6, 7, 8, 13, 18]. This graded, commutative, non-cocommutative Hopf algebra is generated by the set of rooted trees. We shall work here with a decorated version $\mathcal{H}_{\mathcal{D}}$ of this algebra, where $\mathcal{D}$ is a finite, non-empty set, replacing rooted trees by rooted trees with vertices decorated by the elements of $\mathcal{D}$. This algebra has a family of operators $(B_d^+)_{d\in\mathcal{D}}$ indexed by $\mathcal{D}$, where B_d^+ sends a forest F to the rooted tree obtained by grafting the trees of F on a common root decorated by d. These operators satisfy the following equation: for all $x \in \mathcal{H}_{\mathcal{D}}$,

$$\Delta \circ B_d^+(x) = B_d^+(x) \otimes 1 + (Id \otimes B_d^+) \circ \Delta(x).$$

As explained in [6], this means that B_d^+ is a 1-cocycle for a certain cohomology of coalgebras, dual to the Hochschild cohomology.

We are interested here in systems of combinatorial Dyson-Schwinger equations (briefly, SDSE), that is to say, if the set of decorations is $\{1,\ldots,N\}$, a system (S) of the form:

$$\left\{\begin{array}{rcl} X_1 &=& B_1^+(F_1(X_1,\ldots,X_N)),\\ &\vdots& \\ X_N &=& B_N^+(F_N(X_1,\ldots,X_N)), \end{array}\right.$$

where $F_1,\ldots,F_N \in K[[h_1,\ldots,h_N]]$ are formal series in N indeterminates. These systems (in a Feynman graph version) are used in Quantum Field Theory, as it is explained in [1, 15, 16]. They possess a unique solution, which is a family of N formal series in rooted trees, or equivalently elements of a completion of $\mathcal{H}_{\mathcal{D}}$. The homogeneous components of these elements generate a subalgebra $\mathcal{H}_{(S)}$ of $\mathcal{H}_{\mathcal{D}}$. Our problem here is to determine Hopf SDSE, that is to say SDSE (S) such that $\mathcal{H}_{(S)}$ is a Hopf subalgebra of $\mathcal{H}_{\mathcal{D}}$. In the case of a single combinatorial Dyson-Schwinger equation, this question has been answered in [9].

In order to answer this, we first associate an oriented graph to any SDSE, reflecting the dependence of the different X_i's; more precisely, the vertices of $G_{(S)}$ are the elements of I, and there is an edge from i to j if F_i depends on h_j. We shall say that (S) is connected if $G_{(S)}$ is connected. Noting that any SDSE is the disjoint union of several connected SDSE, we can restrict our study to connected SDSE. We introduce three operations on Hopf SDSE:

- Change of variables, which replaces h_i by $\lambda_i h_i$ for all $i \in I$, where $\lambda_i \neq 0$ for all i. This operation replaces $\mathcal{H}_{(S)}$ by an isomorphic Hopf algebra and does not change $G_{(S)}$.
- Dilatation, which replaces each vertex of $G_{(S)}$ by several vertices. This operation increases the number of vertices. For example, consider:

$$(S):\left\{\begin{array}{rcl} X_1 &=& B_1^+(f(X_1,X_2)),\\ X_2 &=& B_2^+(g(X_1,X_2)),\end{array}\right.$$

where $f,g \in K[[h_1,h_2]]$; then the following SDSE is a dilatation of (S):

$$(S'):\left\{\begin{array}{rcl} X_1 &=& B_1^+(f(X_1+X_2+X_3,X_4+X_5)),\\ X_2 &=& B_2^+(f(X_1+X_2+X_3,X_4+X_5)),\\ X_3 &=& B_3^+(f(X_1+X_2+X_3,X_4+X_5)),\\ X_4 &=& B_4^+(g(X_1+X_2+X_3,X_4+X_5)),\\ X_5 &=& B_5^+(g(X_1+X_2+X_3,X_4+X_5)),\end{array}\right.$$

- Extension, which adds a vertex 0 to $G_{(S)}$ with an affine formal series. This operation increases the number of vertices by 1. For example, consider:

$$(S):\left\{\begin{array}{rcl} X_1 &=& B_1^+(f(X_1,X_2)),\\ X_2 &=& B_2^+(f(X_1,X_2)),\end{array}\right.$$

where $f \in K[[h_1,h_2]]$ and $a,b \in K$; then the following SDSE is an extension of (S):

$$(S'):\left\{\begin{array}{rcl} X_0 &=& B_0^+(1+aX_1+bX_2),\\ X_1 &=& B_1^+(f(X_1,X_2)),\\ X_2 &=& B_2^+(f(X_1,X_2)),\end{array}\right.$$

We then introduce two families of Hopf SDSE:

- Cycles, which are SDSE such that the associated graph is an oriented graph and all the formal series of the system are affine; see theorem 28. For example, the following system is a 4-cycle:

$$\left\{\begin{array}{rcl} X_1 &=& B_1^+(1+X_2),\\ X_2 &=& B_2^+(1+X_3),\\ X_3 &=& B_3^+(1+X_4),\\ X_4 &=& B_4^+(1+X_1).\end{array}\right.$$

The associated oriented graph is:

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 4 \arrow[u] & 3 \arrow[l] \end{tikzcd}$$

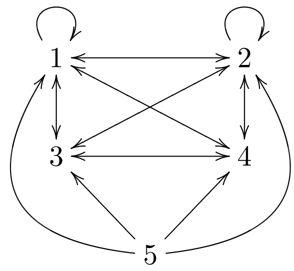
- Fundamental SDSE, described in theorem 30. Here is an example of fundamental SDSE:

$$\left\{\begin{array}{rcl} X_1 &=& B_1^+\left(f_{\beta_1}(X_1)f_{\frac{\beta_2}{1+\beta_2}}((1+\beta_2)h_2)(1-h_3)^{-1}(1-h_4)^{-1}\right),\\ X_2 &=& B_2^+\left(f_{\frac{\beta_1}{1+\beta_1}}(X_1)f_{\beta_2}(h_2)(1-h_3)^{-1}(1-h_4)^{-1}\right),\\ X_3 &=& B_3^+\left(f_{\frac{\beta_1}{1+\beta_1}}((1+\beta_1)X_1)f_{\frac{\beta_2}{1+\beta_2}}((1+\beta_2)h_2)(1-h_4)^{-1}\right),\\ X_4 &=& B_4^+\left(f_{\frac{\beta_1}{1+\beta_1}}((1+\beta_1)X_1)f_{\frac{\beta_2}{1+\beta_2}}((1+\beta_2)h_2)(1-h_3)^{-1}\right),\\ X_5 &=& B_5^+\left(f_{\frac{\beta_1}{1+\beta_1}}((1+\beta_1)X_1)f_{\frac{\beta_2}{1+\beta_2}}((1+\beta_2)h_2)(1-h_3)^{-1}(1-h_4)^{-1}\right),\end{array}\right.$$

where $\beta_1, \beta_2 \in K - \{-1\}$ and, for all $\beta \in K$, f_β is the following formal series:

$$f_\beta(h) = \sum_{k=0}^{\infty} \frac{(1+\beta)\cdots(1+(k-1)\beta)}{k!} h^k.$$

The associated oriented graph is:



The main result of this paper is theorem 14, which says that any connected Hopf SDSE is obtained by a dilatation and a finite number of iterated extensions of a cycle or a fundamental SDSE.

Let us now give a few explanations on the way this result is obtained. An important tool is given by a family indexed by I^2 of scalar sequences $\left(\lambda_n^{(i,j)}\right)_{n\geq 1}$ associated to any Hopf SDSE. They allow to reconstruct the coefficients of the formal series of (S), as explained in proposition 19. Particular cases of possible sequence $\left(\lambda_n^{(i,j)}\right)_{n\geq 1}$ are affine sequences, up to a finite number of terms: this leads to the notion of level of a vertex. It is shown that level decreases along the oriented paths of $G_{(S)}$ (proposition 21), and this implies the following alternative if (S) is connected: any vertex is of finite level or no vertex is of finite level. In particular, any vertex of a fundamental SDSE is of finite level, whereas no vertex of a cycle is of finite level.

We then consider two special families of SDSE:

- We first assume that the graph associated to (S) does not contain any vertex related to itself. This case includes cycles and their dilatations (called multicycles), and a special case of fundamental SDSE called quasi-complete SDSE. We show, using graph-theoretical considerations and the coefficients $\lambda_n^{(i,j)}$, that under an hypothesis of symmetry, they are the only possibilities.

- We then assume that any vertex of (S) has an ascendant related to itself. We then prove that (S) is fundamental.

This results are then unified in corollary 48. It says that any Hopf SDSE with a connected graph contains a multicycle or a a fundamental SDSE (S_0) and is obtained from (S_0) by adding repeatedly a finite number of vertices. This result is precised for the multicycle case in theorem 49 and for the fundamental case in theorem 50. The compilation of these results then proves theorem 14.

This text is organised as follows: the first section gives some recalls on the structure of Hopf algebra of $\mathcal{H}_{\mathcal{D}}$ and on the pre-Lie product on $\mathfrak{g}_{(S)} = Prim\left(\mathcal{H}_{(S)}^*\right)$. In the second section are given the definitions of SDSE and their different operations: change of variables, dilatation and extension. The main theorem of the text is also stated in this section. The following section introduces the coefficients $\lambda_n^{(i,j)}$ and their properties, especially their link with the pre-Lie product of $\mathfrak{g}_{(S)}$. The level of a vertex is defined in the fourth section, which also contains lemmas on vertices of level 0, 1 or ≥ 2, before that fundamental and multicyclic SDSE are introduced in the fifth section. The next section contains preliminary results about graphs with no self-dependent vertices or such that any vertex is the descendant of a self-dependent vertex, and the main theorem is finally proved in the seventh section.

Notations. We denote by K a commutative field of characteristic zero. All vector spaces, algebras, coalgebras, Hopf algebras, etc. will be taken over K.

1 Preliminaries

1.1 Decorated rooted trees

Definition 1 [19, 20]

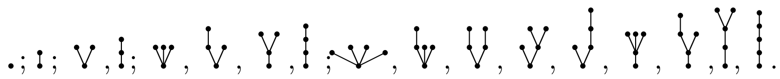
1. A *rooted tree* t is a finite graph, without loops, with a special vertex called the *root* of t. The *weight* of t is the number of its vertices. The set of rooted trees will be denoted by $\mathcal{T}$.

2. Let $\mathcal{D}$ be a non-empty set. A *rooted tree decorated by* $\mathcal{D}$ is a rooted tree with an application from the set of its vertices into $\mathcal{D}$. The set of rooted trees decorated by $\mathcal{D}$ will be denoted by $\mathcal{T}_{\mathcal{D}}$.

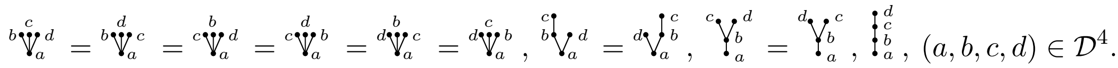
3. Let $i \in \mathcal{D}$. The set of rooted trees decorated by $\mathcal{D}$ with root decorated by i will be denoted by $\mathcal{T}_{\mathcal{D}}^{(i)}$.

Examples.

1. Rooted trees with weight smaller than 5:



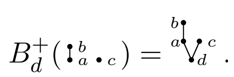
2. Rooted trees decorated by $\mathcal{D}$ with weight smaller than 4:



Definition 2

1. We denote by $\mathcal{H}_{\mathcal{D}}$ the polynomial algebra generated by $\mathcal{T}_{\mathcal{D}}$.

2. Let $t_1, \ldots, t_n$ be elements of $\mathcal{T}_{\mathcal{D}}$ and let $d \in \mathcal{D}$. We denote by $B_d^+(t_1 \ldots t_n)$ the rooted tree obtained by grafting $t_1, \ldots, t_n$ on a common root decorated by d. This map B_d^+ is extended in an operator from $\mathcal{H}_{\mathcal{D}}$ to $\mathcal{H}_{\mathcal{D}}$.

For example, 

1.2 Hopf algebras of decorated rooted trees

In order to make $\mathcal{H}_{\mathcal{D}}$ a bialgebra, we now introduce the notion of cut of a tree $t \in \mathcal{T}_{\mathcal{D}}$. A *non-total cut* c of a tree t is a choice of edges of t. Deleting the chosen edges, the cut makes t into a forest denoted by $W^c(t)$. The cut c is *admissible* if any oriented path in the tree meets at most one cut edge. For such a cut, the tree of $W^c(t)$ which contains the root of t is denoted by $R^c(t)$ and the product of the other trees of $W^c(t)$ is denoted by $P^c(t)$. We also add the total cut, which is by convention an admissible cut such that $R^c(t) = 1$ and $P^c(t) = W^c(t) = t$. The set of admissible cuts of t is denoted by $Adm_*(t)$. Note that the empty cut of t is admissible; we put $Adm(t) = Adm_*(t) - \{\text{empty cut, total cut}\}$.

example. Let $a, b, c, d \in \mathcal{D}$ and let us consider the rooted tree $t = B_d^+(B_b^+(\bullet_a)\bullet_c)$. As it as 3 edges, it has 2^3 non-total cuts.

cut c	$\emptyset$	$\{bd\}$	$\{ab\}$	$\{cd\}$	$\{ab, bd\}$	$\{bd, cd\}$	$\{ab, cd\}$	$\{ab, bd, cd\}$	total
Admissible?	yes	yes	yes	yes	no	yes	yes	no	yes
$W^c(t)$	$B_d^+(B_b^+(\bullet_a)\bullet_c)$	$B_b^+(\bullet_a)B_d^+(\bullet_c)$	$\bullet_a B_d^+(\bullet_b\bullet_c)$	$B_d^+(B_b^+(\bullet_a))\bullet_c$	$\bullet_a\bullet_b B_d^+(\bullet_c)$	$B_b^+(\bullet_a)\bullet_c\bullet_d$	$\bullet_a B_d^+(\bullet_b)\bullet_c$	$\bullet_a\bullet_b\bullet_c\bullet_d$	$B_d^+(B_b^+(\bullet_a)\bullet_c)$
$R^c(t)$	$B_d^+(B_b^+(\bullet_a)\bullet_c)$	$B_d^+(\bullet_c)$	$B_d^+(\bullet_b\bullet_c)$	$B_d^+(B_b^+(\bullet_a))$	$\times$	$\bullet_d$	$B_d^+(\bullet_b)$	$\times$	1
$P^c(t)$	1	$B_b^+(\bullet_a)$	$\bullet_a$	$\bullet_c$	$\times$	$B_b^+(\bullet_a)\bullet_c$	$\bullet_a\bullet_c$	$\times$	$B_d^+(B_b^+(\bullet_a)\bullet_c)$

The coproduct of $\mathcal{H}_\mathcal{D}$ is defined as the unique algebra morphism from $\mathcal{H}_\mathcal{D}$ to $\mathcal{H}_\mathcal{D} \otimes \mathcal{H}_\mathcal{D}$ such that for all rooted tree $t \in \mathcal{T}_\mathcal{D}$:

$$\Delta(t) = \sum_{c \in Adm_*(t)} P^c(t) \otimes R^c(t) = t \otimes 1 + 1 \otimes t + \sum_{c \in Adm(t)} P^c(t) \otimes R^c(t).$$

As $\mathcal{H}_\mathcal{D}$ is the free associative commutative unitary algebra generated by $\mathcal{T}_\mathcal{D}$, this makes sense. This coproduct makes $\mathcal{H}_\mathcal{D}$ a Hopf algebra. Although it won't play any role in this text, we recall that the antipode S is the unique algebra automorphism of $\mathcal{H}_\mathcal{D}$ such that for all $t \in \mathcal{T}_\mathcal{D}$:

$$S(t) = -\sum_{c \text{ cut of } t} (-1)^{n_c} W_c(t),$$

where n_c is the number of cut edges of c.

Example.

$$\Delta(B_d^+(B_b^+(\bullet_a)\bullet_c)) = B_d^+(B_b^+(\bullet_a)\bullet_c) \otimes 1 + 1 \otimes B_d^+(B_b^+(\bullet_a)\bullet_c) + B_b^+(\bullet_a) \otimes B_d^+(\bullet_c) + \bullet_a \otimes B_d^+(\bullet_b\bullet_c) + \bullet_c \otimes B_d^+(B_b^+(\bullet_a)) + B_b^+(\bullet_a)\bullet_c \otimes \bullet_d + \bullet_a\bullet_c \otimes B_d^+(\bullet_b).$$

A study of admissible cuts shows the following result:

Proposition 3 *For all $d \in \mathcal{D}$, for all $x \in \mathcal{H}_\mathcal{D}$:*

$$\Delta \circ B_d^+(x) = B_d^+(x) \otimes 1 + (Id \otimes B_d^+) \circ \Delta(x).$$

Remarks.

1. In other words, B_d^+ is a 1-cocycle for a certain cohomology of coalgebras, see [6].
2. If $t \in \mathcal{T}_\mathcal{D}^{(i)}$, then $\Delta(t) - t \otimes 1 \in \mathcal{H}_\mathcal{D} \otimes \mathcal{T}_\mathcal{D}^{(i)}$.

1.3 Gradation of $\mathcal{H}_\mathcal{D}$ and completion

We grade $\mathcal{H}_\mathcal{D}$ by declaring the forests with n vertices homogeneous of degree n. We denote by $\mathcal{H}_\mathcal{D}(n)$ the homogeneous component of $\mathcal{H}_\mathcal{D}$ of degree n. Then $\mathcal{H}_\mathcal{D}$ is a graded bialgebra, that is to say:

- For all $i, j \in \mathbb{N}$, $\mathcal{H}_\mathcal{D}(i)\mathcal{H}(j) \subseteq \mathcal{H}_\mathcal{D}(i+j)$.
- For all $k \in \mathbb{N}$, $\Delta(\mathcal{H}_\mathcal{D}(k)) \subseteq \sum_{i+j=k} \mathcal{H}_\mathcal{D}(i) \otimes \mathcal{H}_\mathcal{D}(j)$.

We define, for all $x \in \mathcal{H}_{\mathcal{D}}$:

$$val(x) = \max \left\{ n \in \mathbb{N} \mid x \in \bigoplus_{k \geq n} \mathcal{H}_{\mathcal{D}}(k) \right\}.$$

We then put, for all $x, y \in \mathcal{H}_{\mathcal{D}}$, $d(x,y) = 2^{-val(x-y)}$, with the convention $2^{-\infty} = 0$. Then d is a distance on $\mathcal{H}_{\mathcal{D}}$. The metric space $(\mathcal{H}_{\mathcal{D}}, d)$ is not complete; its completion will be denoted by $\widehat{\mathcal{H}_{\mathcal{D}}}$. As a vector space:

$$\widehat{\mathcal{H}_{\mathcal{D}}} = \prod_{n \in \mathbb{N}} \mathcal{H}_{\mathcal{D}}(n).$$

The elements of $\widehat{\mathcal{H}_{\mathcal{D}}}$ will be denoted by $\sum x_n$, where $x_n \in \mathcal{H}_{\mathcal{D}}(n)$ for all $n \in \mathbb{N}$. The product $m : \mathcal{H}_{\mathcal{D}} \otimes \mathcal{H}_{\mathcal{D}} \longrightarrow \mathcal{H}_{\mathcal{D}}$ is homogeneous of degree 0, so is continuous: it can be extended from $\widehat{\mathcal{H}_{\mathcal{D}}} \otimes \widehat{\mathcal{H}_{\mathcal{D}}}$ to $\widehat{\mathcal{H}_{\mathcal{D}}}$, which is then an associative, commutative algebra. Similarly, the coproduct of $\mathcal{H}_{\mathcal{D}}$ can be extended as a map:

$$\Delta : \widehat{\mathcal{H}_{\mathcal{D}}} \longrightarrow \mathcal{H}_{\mathcal{D}} \widehat{\otimes} \mathcal{H}_{\mathcal{D}} = \prod_{i,j \in \mathbb{N}} \mathcal{H}_{\mathcal{D}}(i) \otimes \mathcal{H}_{\mathcal{D}}(j).$$

Let $f(h) = \sum p_n h^n \in K[[h]]$ be any formal series, and let $X = \sum x_n \in \widehat{\mathcal{H}_{\mathcal{D}}}$, such that $x_0 = 0$. The series of $\widehat{\mathcal{H}_{\mathcal{D}}}$ of terms $p_n X^n$ is Cauchy, so converges. Its limit will be denoted by $f(X)$. In other words, $f(X) = \sum y_n$, with:

$$\left\{ \begin{array}{rcl} y_0 &=& p_0, \\ y_n &=& \displaystyle\sum_{k=1}^{n} \sum_{a_1 + \cdots + a_k = n} p_k x_{a_1} \cdots x_{a_k} \text{ if } n \geq 1. \end{array} \right.$$

1.4 Pre-Lie structure on the dual of $\mathcal{H}_{\mathcal{D}}$

By the Cartier-Quillen-Milnor-Moore theorem [17], the graded dual $\mathcal{H}_{\mathcal{D}}^*$ of $\mathcal{H}_{\mathcal{D}}$ is an enveloping algebra. Its Lie algebra $Prim(\mathcal{H}_{\mathcal{D}}^*)$ has a basis $(f_t)_{t \in \mathcal{T}_{\mathcal{D}}}$ indexed by $\mathcal{T}_{\mathcal{D}}$:

$$f_t : \left\{ \begin{array}{rcl} \mathcal{H}_{\mathcal{D}} & \longrightarrow & K \\ t_1 \ldots t_n & \longrightarrow & \left\{ \begin{array}{l} 0 \text{ if } n \neq 1, \\ \delta_{t,t_1} \text{ if } n = 1. \end{array} \right. \end{array} \right.$$

Recall that a pre-Lie algebra (or equivalently a Vinberg algebra or a left-symmetric algebra) is a couple $(A, \star)$, where $\star$ is a bilinear product on A such that for all $x, y, z \in A$:

$$(x \star y) \star z - x \star (y \star z) = (y \star x) \star z - y \star (x \star z).$$

Pre-Lie algebras are Lie algebras, with bracket given by $[x, y] = x \star y - y \star x$.

The Lie bracket of $Prim(\mathcal{H}_{\mathcal{D}}^*)$ is induced by a pre-Lie product $\star$ given in the following way: if $f, g \in Prim(\mathcal{H}_{\mathcal{D}}^*)$, $f \star g$ is the unique element of $Prim(\mathcal{H}_{\mathcal{D}}^*)$ such that for all $t \in \mathcal{T}_{\mathcal{D}}$,

$$(f \star g)(t) = (f \otimes g) \circ (\pi \otimes \pi) \circ \Delta(t),$$

where π is the projection on $Vect(\mathcal{T}^{\mathcal{D}})$ which vanishes on the forests which are not trees. In other words, if $t, t' \in \mathcal{T}_{\mathcal{D}}$:

$$f_t \star f_{t'} = \sum_{t'' \in \mathcal{T}_{\mathcal{D}}} n(t, t'; t'') f_{t''},$$

where $n(t, t'; t')$ is the number of admissible cuts c of t'' such that $P^c(t'') = t$ and $R^c(t'') = t'$. It is proved that $(prim(\mathcal{H}_{\mathcal{D}}^*), \star)$ is the free pre-Lie algebra generated by the $\bullet_d$'s, $d \in \mathcal{D}$: see [3, 4]. Note that $\mathcal{H}_{\mathcal{D}}^*$ is isomorphic to the Grossman-Larson Hopf algebra of rooted trees [10, 11, 12].

2 Definitions and properties of SDSE

2.1 Unique solution of an SDSE

Definition 4 Let I be a finite, non-empty set, and let $F_i \in K[[h_j, j \in I]]$ be a non-constant formal series for all $i \in I$. *The system of Dyson-Schwinger combinatorial equations* (briefly, the SDSE) associated to $(F_i)_{i\in I}$ is:

$$\forall i \in I,\ X_i = B_i^+(f_i(X_j, j \in I)),$$

where $X_i \in \widehat{\mathcal{H}_I}$ for all $i \in I$.

In order to ease the notation, we shall often assume that $I = \{1, \ldots, N\}$ in the proofs, without loss of generality.

Notations. We assume here that $I = \{1, \ldots, N\}$.

1. Let (S) be an SDSE. We shall denote, for all $i \in I$:

$$F_i = \sum_{p_1,\cdots,p_N} a^{(i)}_{(p_1,\cdots,p_N)} h_1^{p_1} \cdots h_N^{p_N}.$$

2. Let $1 \leq j \leq N$. We put $\varepsilon_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ where the 1 is in position j. We shall denote, for all $i \in I$, $a^{(i)}_j = a^{(i)}_{\varepsilon_j}$; for all $j, k \in I$, $a^{(i)}_{j,k} = a^{(i)}_{\varepsilon_j+\varepsilon_k}$, and so on.

Remark. We assume that there is no constant F_i. Indeed, if $F_i \in K$, then X_i is a multiple of $\bullet_i$. We shall always avoid this degenerated case in all this text.

Proposition 5 *Let (S) be an SDSE. Then it admits a unique solution* $(X_i)_{i\in I} \in \left(\widehat{\mathcal{H}_I}\right)^I$.

Proof. We assume here that $I = \{1, \ldots, N\}$. If $(X_1, \cdots, X_N)$ is a solution of S, then X_i is a linear (infinite) span of rooted trees with a root decorated by i. We denote:

$$X_i = \sum_{t \in \mathcal{T}_I^{(i)}} a_t t.$$

These coefficients are uniquely determined by the following formulas: if

$$t = B_i^+\left(t_{1,1}^{p_{1,1}} \cdots t_{1,q_1}^{p_{1,q_1}} \cdots t_{N,1}^{p_{N,1}} \cdots t_{N,q_N}^{p_{N,q_N}}\right),$$

where the $t_{i,j}$'s are different trees, such that the root of $t_{i,j}$ is decorated by i for all $i \in I$, $1 \leq j \leq q_i$, then:

$$a_t = \left(\prod_{i=1}^N \frac{(p_{i,1} + \cdots + p_{i,q_i})!}{p_{i,1}! \cdots p_{i,q_i}!}\right) a^{(i)}_{(p_{1,1}+\cdots+p_{1,q_1},\cdots,p_{N,1}+\cdots+p_{N,q_N})} a_{t_{1,1}}^{p_{1,1}} \cdots a_{t_{N,q_N}}^{p_{N,q_N}}. \tag{1}$$

So (S) has a unique solution. $\square$

Definition 6 Let (S) be an SDSE and let $X = (X_i)_{i\in I}$ be its unique solution. The subalgebra of $\mathcal{H}_I$ generated by the homogeneous components $X_i(k)$'s of the X_i's will be denoted by $\mathcal{H}_{(S)}$. If $\mathcal{H}_{(S)}$ is Hopf, the system (S) will be said to be Hopf.

2.2 Graph associated to an SDSE

We associate a oriented graph to each SDSE in the following way:

Definition 7 Let (S) be an SDSE.

1. We construct an oriented graph $G_{(S)}$ associated to (S) in the following way:
   - The vertices of $G_{(S)}$ are the elements of I.
   - There is an edge from i to j if, and only if, $\dfrac{\partial F_i}{\partial h_j} \neq 0$.
2. If $\dfrac{\partial F_i}{\partial h_i} \neq 0$, the vertex i will be said to be *self-dependent*. In other words, if i is self-dependent, there is a loop from i to itself in $G_{(S)}$.
3. If $G_{(S)}$ is connected, we shall say that (S) is *connected.*

Remark. If (S) is not connected, then (S) is the union of SDSE $(S_1), \cdots, (S_k)$ with disjoint sets of indeterminates , so $\mathcal{H}_{(S)} \approx \mathcal{H}_{(S_1)} \otimes \cdots \otimes \mathcal{H}_{(S_k)}$. As a corollary, (S) is Hopf if, and only if, for all j, (S_j) is Hopf.

Let (S) be an SDSE and let $G_{(S)}$ be the associated graph. Let i and j be two vertices of $G_{(S)}$. We shall say that j is a direct descendant of i (or i is a direct ascendant of j) if there is an oriented edge from i to j; we shall say that j is a descendant of i (or i is an ascendant of j) if there is an oriented path from i to j. We shall write "$i \longrightarrow j$" for "j is a direct descendant of i".

2.3 Operations on Hopf SDSE

Proposition 8 (change of variables) *Let (S) be the SDSE associated to $(F_i(h_j, j \in I))_{i\in I}$. Let λ_i and μ_i be non-zero scalars for all $i \in I$. The system (S) is Hopf if, and only if, the SDSE system (S') associated to $(\mu_i F_i(\lambda_j h_j,\ j \in J))_{i\in I}$ is Hopf.*

Proof. We assume that $I = \{1, \ldots, N\}$. We consider the following morphism:

$$\phi : \left\{ \begin{array}{rcl} \mathcal{H}_I & \longrightarrow & \mathcal{H}_I \\ F \in \mathcal{F} & \longrightarrow & (\mu_1\lambda_1)^{n_1(F)} \cdots (\mu_N\lambda_N)^{n_N(F)} F, \end{array} \right.$$

where $n_i(F)$ is the number of vertices of F decorated by i. Then ϕ is a Hopf algebra automorphism and for all i, $\phi \circ B_i^+ = \mu_i\lambda_i B_i^+ \circ \phi$. Moreover, if we put $Y_i = \frac{1}{\lambda_i}\phi(X_i)$ for all i:

$$\begin{array}{rcl} Y_i & = & \dfrac{1}{\lambda_i}\phi \circ B_i^+(F_i(X_1, \cdots, X_N)) \\ & = & \dfrac{1}{\lambda_i}\mu_i\lambda_i B_i^+(F_i(\phi(X_1), \cdots, \phi(X_N))) \\ & = & \mu_i B_i^+(F_i(\lambda_1 Y_1, \cdots, \lambda_N Y_N)). \end{array}$$

So $(Y_1, \cdots, Y_N)$ is the solution of the system (S'). Moreover, ϕ sends $\mathcal{H}_{(S)}$ onto $\mathcal{H}_{(S')}$. As ϕ is a Hopf algebra automorphism, $\mathcal{H}_{(S)}$ is a Hopf subalgebra of $\mathcal{H}_I$ if, and only if, $\mathcal{H}_{(S')}$ is. $\square$

Remark. A change of variables does not change the graph associated to (S).

Proposition 9 (restriction) *Let (S) be the SDSE associated to $(F_i(h_j,\ j \in I))_{i\in I}$ and let $I' \subseteq I$, non-empty. Let (S') be the SDSE associated to $\left(F_i(h_j, j \in I)_{|h_j=0,\, \forall j \notin I'}\right)_{i\in I'}$. If (S) is Hopf, then (S') also is.*

Proof. We consider the epimorphism ϕ of Hopf algebras from $\mathcal{H}_I$ to $\mathcal{H}_{I'}$, obtained by sending the forests with at least a vertex decorated by an element which is not in I' to zero. Then ϕ sends $\mathcal{H}_{(S)}$ to $\mathcal{H}_{(S')}$. As ϕ is a morphism of Hopf algebras, if $\mathcal{H}_{(S)}$ is a Hopf subalgebra of $\mathcal{H}_I$, $\mathcal{H}_{(S')}$ is a Hopf subalgebra of $\mathcal{H}_{I'}$. $\Box$

Remark. The restriction to a subset of vertices I' changes $G_{(S)}$ into the graph obtained by deleting all the vertices $j \notin I'$ and all the edges related to these vertices.

Proposition 10 (dilatation) *Let (S) be the system associated to $(F_i)_{i\in I}$ and (S') be a system associated to a family $(F'_j)_{j\in J}$, such that there exists a partition $J = \bigcup_{i\in I} J_i$, with the following property: for all $i \in I$, for all $x \in I_i$,*

$$F'_x = F_i\left(\sum_{y\in I_j} h_y,\ j \in I\right).$$

Then (S) is Hopf, if, and only if, (S') is Hopf. We shall say that (S') is a dilatation of (S).

Proof. We assume here that $I = \{1, \ldots, N\}$.

$\Longrightarrow$. Let us assume that (S) is Hopf. For all $i \in I$, we can then write:

$$\Delta(X_i) = \sum_{n\geq 0} P_n^{(i)}(X_1, \cdots, X_N) \otimes X_i(n),$$

with the convention $X_i(0) = 1$. Let $\phi : \mathcal{H}_I \longrightarrow \mathcal{H}_{I'}$ be the morphism of Hopf algebras such that, for all $1 \leq i \leq N$:

$$\phi \circ B_i^+ = \sum_{j\in I_i} B_j^+ \circ \phi.$$

Then, immediately, for all $1 \leq i \leq N$:

$$\phi(X_i) = \sum_{j\in I_i} X'_j.$$

As a consequence:

$$\sum_{j\in I_i} \Delta(X'_j) = \sum_{j\in I_i}\sum_{n\geq 0} P_n^{(i)}\left(\sum_{k\in I_1} X'_k, \cdots, \sum_{k\in I_N} X'_k\right) \otimes X'_j(n).$$

Conserving the terms of the form $F \otimes t$, where t is a tree with root decorated by j, for all $j \in I_i$:

$$\Delta(X'_j) = \sum_{n\geq 0} P_n^{(i)}\left(\sum_{k\in I_1} X'_k, \cdots, \sum_{k\in I_N} X'_k\right) \otimes X'_j(n).$$

So (S') is Hopf.

$\Longleftarrow$. By restriction, choosing an element in each I_i, if (S') is Hopf, then (S) is Hopf. $\Box$

Remark. If (S') is a dilatation of (S), then the set of vertices J of the graph $G_{(S')}$ associated to (S') admits a partition indexed by the vertices of $G_{(S)}$, and there is an edge from $x \in J_i$ to $y \in J_j$ in $G_{(S')}$ if, and only if, there is an edge from i to j in $G_{(S)}$.

Example. Let $f, g \in K[[h_1, h_2]]$. Let us consider the following SDSE:

$$(S): \left\{ \begin{array}{rcl} X_1 & = & B_1^+(f(X_1, X_2)), \\ X_2 & = & B_2^+(g(X_1, X_2)), \end{array} \right.$$

$$(S'): \left\{ \begin{array}{rcl} X_1 & = & B_1^+(f(X_1 + X_2 + X_3, X_4 + X_5)), \\ X_2 & = & B_2^+(f(X_1 + X_2 + X_3, X_4 + X_5)), \\ X_3 & = & B_3^+(f(X_1 + X_2 + X_3, X_4 + X_5)), \\ X_4 & = & B_4^+(g(X_1 + X_2 + X_3, X_4 + X_5)), \\ X_5 & = & B_5^+(g(X_1 + X_2 + X_3, X_4 + X_5)). \end{array} \right.$$

Then (S') is a dilatation of (S).

Proposition 11 (extension) *Let (S) be the SDSE associated to $(F_i)_{i \in I}$. Let $0 \notin I$ and let (S') be associated to $(F_i)_{i \in I \cup \{0\}}$, with:*

$$F_0 = 1 + \sum_{i \in I} a_i^{(0)} h_i.$$

Then (S') is Hopf if, and only if, the two following conditions hold:

1. *(S) is Hopf.*
2. *For all $i, j \in I^{(0)} = \left\{ j \in I \,/\, a_j^{(0)} \neq 0 \right\}$, $F_i = F_j$.*

If these two conditions hold, we shall say that (S') is an extension of (S).

Proof. We assume here that $I = \{1, \ldots, N\}$.
$\Longrightarrow$. Let us assume that (S') is Hopf. By restriction, (S) is Hopf. Moreover:

$$X_0 = B_0^+ \left(1 + \sum_{i=1}^N a_i^{(0)} X_i \right) = \bullet_0 + \sum_{i=1}^N a_i^{(0)} B_0^+ \circ B_i^+(f_i(X_1, \cdots, X_N)).$$

As $\mathcal{H}_{(S')}$ is a graded Hopf subalgebra, the projection on $\mathcal{H}_{\{0, \cdots, N\}} \otimes \mathcal{H}_{\{0, \cdots, N\}}(2)$ gives:

$$\sum_{i=1}^N a_i^{(0)} F_i(X_1, \cdots, X_N) \otimes \mathbf{I}^{\,i}_{\,0} \in \mathcal{H}_{(S')} \widehat{\otimes} \mathcal{H}_{(S')}.$$

So this is of the form:

$$P \otimes X_0(2) = P \otimes \left(\sum_{i=1}^N a_i^{(0)} \mathbf{I}^{\,i}_{\,0} \right),$$

for a certain $P \in \widehat{\mathcal{H}_{(S')}}$. As the $\mathbf{I}^{\,i}_{\,0}$'s, $i \in I$, are linearly independent, we obtain that for all i, j, $a_i^{(0)} F_i(X_1, \cdots, X_N) = a_i^{(0)} P$ for all i, and this implies the second item.

$\Longleftarrow$. As (S) is Hopf, we can put for all $1 \leq i \leq N$:

$$\Delta(X_i) = X_i \otimes 1 + \sum_{k=1}^{+\infty} P_k^{(i)} \otimes X_i(k),$$

where $P_n^{(i)}$ is an element of the completion of $\mathcal{H}_{(S)}$. By the second hypothesis, if $i,j \in I$, as $F_i = F_j$, $P_n^{(i)} = P_n^{(j)}$. We then denote by P_n the common value of $P_n^{(i)}$ for all $i \in I$. So:

$$\begin{aligned}
\Delta(X_0) &= \bullet_0 \otimes 1 + 1 \otimes \bullet_0 + \sum_{i=1}^N a_i^{(0)} \Delta \circ B_0^+(X_i)\\
&= X_0 \otimes 1 + 1 \otimes X_0 + \sum_{i=1}^N a_i^{(0)}(1+X_i) \otimes \bullet_0 + \sum_{i=1}^N \sum_{j=1}^\infty a_i^{(0)} P_j^{(i)} \otimes B_0^+(X_i(j))\\
&= X_0 \otimes 1 + 1 \otimes X_0 + \sum_{i=1}^N a_i^{(0)}(1+X_i) \otimes \bullet_0 + \sum_{i=1}^N \sum_{j=1}^\infty a_i^{(0)} P_j \otimes B_0^+(X_i(j))\\
&= X_0 \otimes 1 + 1 \otimes X_0 + \sum_{i=1}^N a_i^{(0)}(1+X_i) \otimes \bullet_0 + \sum_{i=1}^N P_j \otimes B_0^+\left(\sum_{j=1}^\infty a_i^{(0)} X_i(j)\right)\\
&= X_0 \otimes 1 + 1 \otimes X_0 + \sum_{i=1}^N a_i^{(0)}(1+X_i) \otimes \bullet_0 + \sum_{i=1}^N P_j \otimes X_0(j+1).
\end{aligned}$$

This belongs to the completion of $\mathcal{H}_{(S')} \otimes \mathcal{H}_{(S')}$, so (S') is Hopf. $\Box$

Remarks.

1. If (S) is an extension of (S'), then $G_{(S)}$ is obtained from $G_{(S')}$ by adding a non-self-dependent vertex with no ascendant.
2. If $I^{(0)}$ is reduced to a single element, then condition 2 is empty.

Definition 12 Let (S) a Hopf SDSE and let $i \in I$. We shall say that i is an *extension vertex* if, denoting by J the set of descendants of i, the restriction of (S) to $J \cup \{i\}$ is an extension of the restriction of (S) to J.

2.4 Constant terms of the formal series

Lemma 13 *Let (S) be an Hopf SDSE. If $F_i(0,\cdots,0)=0$, then $X_i = 0$.*

Proof. If $F_i(0,\cdots,0)=0$, then the homogeneous component of degree 1 of X_i is zero, so $\bullet_i \notin \mathcal{H}_{(S)}$. Considering the terms of the form $F \otimes \bullet_i$ in $\Delta(X_i)$, we obtain:

$$F_i(X_j,\ j \in I) \otimes \bullet_i \in \mathcal{H}_{(S)} \otimes \mathcal{H}_{(S)}.$$

As $\bullet_i \notin \mathcal{H}_{(S)}$, necessarily $F_i(X_j,\ j\in I)=0$, so $X_i = 0$. $\Box$

As a consequence, if $F_i(0,\cdots,0)=0$, then $\mathcal{H}_{(S)} = \mathcal{H}_{(S')}$, where (S') is the restriction of (S) to $I - \{i\}$. Using a change of variables, we shall always suppose in the sequel that for all i, $F_i(0,\cdots,0)=1$.

2.5 Main theorem

Notations. For all $\beta \in K$, we put:

$$f_\beta(h) = \sum_{k=0}^{+\infty} \frac{(1+\beta)\cdots(1+\beta(k-1))}{k!} h^k = \begin{cases} (1-\beta h)^{-\frac{1}{\beta}} \text{ if } \beta \neq 0,\\ e^h \text{ if } \beta = 0. \end{cases}$$

The main aim of this text is to prove the following result:

Theorem 14 *Let (S) be a connected SDSE. It is Hopf if and only if one of the following assertion holds:*

1. *(Extended multicyclic SDSE). The set I admits a partition $I = I_{\overline{1}} \cup \cdots \cup I_{\overline{N}}$ indexed by the elements of $\mathbb{Z}/N\mathbb{Z}$, $N \geq 2$, with the following conditions:*
   - *For all $i \in I_{\overline{k}}$:*
   $$F_i = 1 + \sum_{j \in I_{\overline{k+1}}} a_j^{(i)} h_j.$$
   - *If i and i' have a common direct ascendant in $G_{(S)}$, then $F_i = F_{i'}$ (so i and i' have the same direct descendants).*

2. *(Extended fundamental SDSE). There exists a partition:*
$$I = \left(\bigcup_{i \in I_0} J_i\right) \cup \left(\bigcup_{i \in J_0} J_i\right) \cup K_0 \cup I_1 \cup J_1 \cup I_2,$$
with the following conditions:
   - *K_0, I_1, J_1, I_2 can be empty.*
   - *The set of indices $I_0 \cup J_0$ is not empty.*
   - *For all $i \in I_0 \cup J_0$, J_i is not empty.*

   Up to a change of variables:

   (a) *For all $i \in I_0$, there exists $\beta_i \in K$, such that for all $x \in J_i$:*
   $$F_x = f_{\beta_i}\left(\sum_{y \in J_i} h_y\right) \prod_{j \in I_0 - \{i\}} f_{\frac{\beta_j}{1+\beta_j}}\left((1+\beta_j)\sum_{y \in J_j} h_y\right) \prod_{j \in J_0} f_1\left(\sum_{y \in J_j} h_y\right).$$

   (b) *For all $i \in J_0$, for all $x \in J_i$:*
   $$F_x = \prod_{j \in I_0} f_{\frac{\beta_j}{1+\beta_j}}\left((1+\beta_j)\sum_{y \in J_j} h_y\right) \prod_{j \in J_0 - \{i\}} f_1\left(\sum_{y \in J_j} h_y\right).$$

   (c) *For all $i \in K_0$:*
   $$F_i = \prod_{j \in I_0} f_{\frac{\beta_j}{1+\beta_j}}\left((1+\beta_j)\sum_{y \in J_j} h_y\right) \prod_{j \in J_0} f_1\left(\sum_{y \in J_j} h_y\right).$$

   (d) *For all $i \in I_1$, there exist $\nu_i \in K$ and a family of scalars $\left(a_j^{(i)}\right)_{j \in I_0 \cup J_0 \cup K_0}$, with $(\nu_i \neq 1)$ or $(\exists j \in I_0,\ a_j^{(i)} \neq 1 + \beta_j)$ or $(\exists j \in J_0,\ a_j^{(i)} \neq 1)$ or $(\exists j \in K_0,\ a_j^{(i)} \neq 0)$. Then, if $\nu_i \neq 0$:*
   $$F_i = \frac{1}{\nu_i} \prod_{j \in I_0} f_{\frac{\beta_j}{\nu_i a_j^{(i)}}}\left(\nu_i a_j^{(i)} \sum_{y \in J_j} h_y\right) \prod_{j \in J_0} f_{\frac{1}{\nu_i a_j^{(i)}}}\left(\nu_i a_j^{(i)} \sum_{y \in J_j} h_y\right) \prod_{j \in K_0} f_0\left(\nu_i a_j^{(i)} h_j\right) + 1 - \frac{1}{\nu_i}.$$
   If $\nu_i = 0$:
   $$F_i = -\sum_{j \in I_0} \frac{a_j^{(i)}}{\beta_j} \ln\left(1 - \sum_{y \in J_j} h_y\right) - \sum_{j \in J_0} a_j^{(i)} \ln\left(1 - \sum_{y \in J_j} h_y\right) + \sum_{j \in K_0} a_j^{(i)} h_j + 1.$$

(e) For all $i \in J_1$, there exists $\nu_i \in K - \{0\}$ and a family of scalars $\left(a_j^{(i)}\right)_{j \in I_0 \cup J_0 \cup K_0 \cup I_1}$, with the three following conditions:

- *$I_1^{(i)} = \{j \in I_1 \,/\, a_j^{(i)} \neq 0\}$ is not empty.*
- *For all $j \in I_1^{(i)}$, $\nu_j = 1$.*
- *For all $j, k \in I_1^{(i)}$, $F_j = F_k$. In particular, we put $b_t^{(i)} = a_t^{(j)}$ for any $j \in I_1^{(i)}$, for all $t \in I_0 \cup J_0 \cup K_0$.*

Then:

$$\begin{aligned} F_i &= \frac{1}{\nu_i} \prod_{j \in I_0} f_{\frac{\beta_j}{b_j^{(i)}-1-\beta_j}} \left(\left(b_j^{(i)} - 1 - \beta_j\right) \sum_{y \in J_j} h_y \right) \prod_{j \in J_0} f_{\frac{\beta_j}{b_j^{(i)}-1}} \left(\left(b_j^{(i)} - 1\right) \sum_{y \in J_j} h_y \right) \\ & \prod_{j \in K_0} f_0\left(b_j^{(i)} h_j\right) + \sum_{j \in I_1^{(i)}} a_j^{(i)} h_1 + 1 - \frac{1}{\nu_i}. \end{aligned}$$

(f) $I_2 = \{x_1, \ldots, x_m\}$ and for all $1 \leq k \leq m$, there exist a set:

$$I^{(x_k)} \subseteq \left(\bigcup_{i \in I_0} J_i \right) \cup \left(\bigcup_{i \in J_0} J_i \right) \cup K_0 \cup I_1 \cup J_1 \cup \{x_1, \ldots, x_{k-1}\}$$

and a family of non-zero scalars $\left(a_j^{(x_k)}\right)_{j \in I^{(x_k)}}$ such that for all $i, j \in I^{(x_k)}$, $F_i = F_j$. Then:

$$F_{x_k} = 1 + \sum_{j \in I^{(x_k)}} a_j^{(x_k)} h_j.$$

Here is the graph of a system of an extended multicyclic SDSE, with $N = 5$. The different subset of the partition are indicated by the different colours. the multicycle corresponds to the five boxes. An arrow between two boxes means that all vertices of the boxes are related by an arrow.

Here is the graph of an extended fundamental SDSE. The vertices in J_i, with $i \in I_0$, are green. There are two elements in I_0, one with $\beta_i = -1$ (light green vertices) and one with $\beta_i \neq -1$ (dark green vertex). There are two elements in J_0, corresponding to light blue and dark blue vertices. The unique element of K_0 is red; the unique element of I_1 is yellow; the unique element of J_1 is orange; the dark vertices are the elements of I_2. An arrow between two boxes means that all vertices of the boxes are related by an arrow.

For example, the SDSE associated to the following formal series has such a graph:

$$
\begin{array}{rcl}
F_1 & = & f_\beta(h_1)f_1(h_4+h_5)f_1(h_6+h_7+h_8)\\
F_2=F_3 & = & (1+h_2+h_3)f_{\frac{\beta}{1+\beta}}((1+\beta)h_1)f_1(h_4+h_5)f_1(h_6+h_7+h_8)\\
F_4=F_5 & = & f_{\frac{\beta}{1+\beta}}((1+\beta)h_1)f_1(h_6+h_7+h_8)\\
F_6=F_7=F_8 & = & f_{\frac{\beta}{1+\beta}}((1+\beta)h_1)f_1(h_4+h_5)\\
F_9 & = & f_{\frac{\beta}{1+\beta}}((1+\beta)h_1)f_1(h_4+h_5)f_1(h_6+h_7+h_8)\\
F_{10} & = & \dfrac{1}{\nu}f_{\frac{\beta}{\nu a_1^{(10)}}}\left(\nu a_1^{(10)}h_1\right)f_{\frac{-1}{\nu a_2^{(10)}}}\left(\nu a_2^{(10)}(h_2+h_3)\right)f_{\frac{1}{\nu a_4^{(10)}}}\left(\nu a_4^{(10)}(h_4+h_5)\right)\\
& & f_{\frac{1}{\nu a_6^{(10)}}}\left(\nu a_6^{(10)}(h_6+h_7+h_8)\right)f_0\left(\nu a_9^{(10)}h_9\right)+1-\dfrac{1}{\nu},\\
F_{11} & = & \dfrac{1}{\nu'}f_{\frac{\beta}{a_1^{(10)}-1-\beta}}\left(\left(a_1^{(10)}-1-\beta\right)h_1\right)f_{\frac{-1}{a_2^{(10)}}}\left(a_2^{(10)}(h_2+h_3)\right)\\
& & f_{\frac{1}{a_4^{(10)}-1}}\left(\left(a_4^{(10)}-1\right)(h_4+h_5)\right)f_{\frac{1}{a_6^{(10)}-1}}\left(\left(a_6^{(10)}-1\right)(h_6+h_7+h_8)\right)\\
& & f_0\left(a_9^{(10)}h_9\right)+a_{10}^{(11)}h_{10}+1-\dfrac{1}{\nu'},\\
F_{12}=F_{13} & = & 1+a_{10}^{(12)}h_{10},\\
F_{14} & = & 1+a_{13}^{(14)}h_{13},\\
F_{15} & = & 1+a_{12}^{(15)}h_{12}+a_{13}^{(15)}h_{13},\\
F_{16} & = & 1+a_{15}^{(16)}h_{15},\\
F_{17} & = & 1+a_2^{(17)}h_2,\\
F_{18} & = & 1+a_{17}^{(18)}h_{17},\\
F_{19} & = & 1+a_{17}^{(19)}h_{17},
\end{array}
$$

where $\beta\neq -1$, $\nu,\nu'\neq 0$, and the coefficients $a_j^{(i)}$ are non-zero.

3 Characterisation and properties of Hopf SDSE

3.1 Subalgebras of $\mathcal{H}_\mathcal{D}$ generated by spans of trees

Let us fix a non-empty set $\mathcal{D}$.

Lemma 15 *Let V be a subspace of $Vect(\mathcal{T}_\mathcal{D})$ and let us consider the subalgebra A of $\mathcal{H}_\mathcal{D}$ generated by V. Recall that for all $d \in \mathcal{D}$, $f_{\bullet_d}$ is the following linear map:*

$$f_{\bullet_d} : \left\{ \begin{array}{rcl} \mathcal{H}_\mathcal{D} & \longrightarrow & K \\ t_1 \cdots t_n & \longrightarrow & \delta_{t_1 \cdots t_n, \bullet_d}. \end{array} \right.$$

Then A is a Hopf subalgebra if, and only if, the two following assertions are both satisfied:

1. *For all $d \in \mathcal{D}$, $(f_{\bullet_d} \otimes Id) \circ \Delta(V) \subseteq V + K$.*
2. *For all $d \in \mathcal{D}$, $(Id \otimes f_{\bullet_d}) \circ \Delta(V) \subseteq A$.*

Proof. $\Longrightarrow$. If A is Hopf, then $\Delta(V) \subseteq A \otimes A$. As $V \subseteq Vect(\mathcal{T}_\mathcal{D})$, $\Delta(V) \subseteq \mathcal{H} \otimes (Vect(\mathcal{T}_\mathcal{D}) + K)$. So:

$$\Delta(V) \subseteq (A \otimes A) \cap (\mathcal{H} \otimes (Vect(\mathcal{T}_\mathcal{D}) + K)) = A \otimes (V \oplus K).$$

This implies both assertions.

$\Longleftarrow$. We use here Sweedler's notations: $\Delta(a) = a' \otimes a''$ and $(\Delta \otimes Id) \circ \Delta(a) = a' \otimes a'' \otimes a'''$ for all $a \in A$.

First step. Let us consider the following subspace of $Prim(\mathcal{H}_\mathcal{D}^*)$:

$$B = \{ f \in Prim(\mathcal{H}_\mathcal{D}^*) \; / \; (f \otimes Id) \circ \Delta(V) \subseteq V + K \}.$$

By hypothesis 1, $f_{\bullet_d} \in B$ for all $d \in \mathcal{D}$. We recall here that $\star$ is the pre-Lie product of $Prim(\mathcal{H}_\mathcal{D}^*)$. Let f and $g \in B$. For all $v \in V$:

$$(f \star g \otimes Id) \circ \Delta(v) = f \circ \pi(v') g \circ \pi(v'') v'''.$$

As $f \in B$, $f \circ \pi(v') v'' \in V + K$. As $g \in B$, $f \circ \pi(v') g \circ \pi(v'') v''' \in V + K$. So $f \star g \in B$, and B is a sub-pre-Lie algebra of $Prim(\mathcal{H}_\mathcal{D}^*)$. As $Prim(\mathcal{H}_\mathcal{D}^*)$ is generated as a pre-Lie algebra by the $f_{\bullet_d}$'s, $B = Prim(\mathcal{H}_\mathcal{D}^*)$.

Second step. Let us consider the following subspace of $\mathcal{H}_\mathcal{D}^*$:

$$B' = \{ f \in \mathcal{H}_\mathcal{D}^* \; / \; (f \otimes Id) \circ \Delta(A) \subseteq A \}.$$

Let $f \in Prim(\mathcal{H}_\mathcal{D}^*)$. By the first step, for all $v_1, \cdots, v_n \in V$:

$$(f \otimes Id) \circ \Delta(v_1 \cdots v_n) = f(v_1' \cdots v_n') v_1'' \cdots v_n'' = \sum_{i=1}^{n} v_1 \cdots f(v_i') v_i'' \cdots v_n \in A,$$

so $Prim(\mathcal{H}_\mathcal{D}^*) \subseteq B'$. Let $f, g \in B'$. For all $a \in A$:

$$(fg \otimes Id) \circ \Delta(a) = f(a') g(a'') a'''.$$

As $f \in B'$, $f(a') a'' \in A$. As $g \in B'$, $f(a') g(a'') a''' \in A$. So B' is a subalgebra of $\mathcal{H}_\mathcal{D}^*$. As it contains $Prim(\mathcal{H}_\mathcal{D}^*)$, it is equal to $\mathcal{H}_\mathcal{D}^*$. So:

$$\Delta(A) \subseteq \mathcal{H}_\mathcal{D} \otimes A + \bigcap_{f \in \mathcal{H}_\mathcal{D}^*} Ker(f) \otimes \mathcal{H}_\mathcal{D} = \mathcal{H}_\mathcal{D} \otimes A.$$

Third step. Let us consider the following subspace of $Prim(\mathcal{H}_\mathcal{D}^*)$:

$$C = \{f \in Prim(\mathcal{H}_\mathcal{D}^*) \,/\, (Id \otimes f) \circ \Delta(V) \subseteq A\}.$$

By the second hypothesis, $f_{\bullet_d} \in B$ for all $d \in \mathcal{D}$. Let us take f and $g \in C$. For all $v \in V$:

$$(Id \otimes (f \star g)) \circ \Delta(v) = v' f \circ \pi(v'') g \circ \pi(v''').$$

As $g \in C$, $v' g \circ \pi(v'') \in A$. Let us denote:

$$v' \circ \pi(v'') = \sum v_1 \cdots v_n,$$

where $v_1, \ldots, v_n$ are elements of V. Then:

$$v' f \circ \pi(v'') g \circ \pi(v''') = \sum v_1' \cdots v_n' f \circ \pi(v_1'' \cdots v_n'') g \circ \pi(v''').$$

By the second step, as $V \subseteq Vect(\mathcal{T}_\mathcal{D})$:

$$\Delta(V) \subseteq (\mathcal{H}_\mathcal{D} \otimes A) \cap (\mathcal{H}_\mathcal{D} \otimes (Vect(\mathcal{T}_\mathcal{D}) + K)) = \mathcal{H}_\mathcal{D} \otimes (V + K).$$

So:

$$\sum v_1' \cdots v_n' \otimes \pi(v_1'' \cdots v_n'') = \sum \sum_{i=1}^{n} v_1 \cdots v_i' \cdots v_n \otimes \pi(v_i'').$$

Finally:

$$(Id \otimes (f \star g)) \circ \Delta(v) = \sum \sum_{i=1}^{n} v_1 \cdots v_i' \cdots v_n \otimes f \circ \pi(v_i'').$$

As $f \in B'$, this belongs to A. So $f \star g \in B'$. As at the end of the first step, we conclude that $B' = Prim(\mathcal{H}_\mathcal{D}^*)$.

Last step. As in the second step, we conclude that for all $f \in \mathcal{H}_\mathcal{D}^*$, $(Id \otimes f) \circ \Delta(A) \subseteq A$. So $\Delta(A) \subseteq A \otimes \mathcal{H}_\mathcal{D}$, and $\Delta(A) \subseteq (\mathcal{H}_\mathcal{D} \otimes A) \cap (A \otimes \mathcal{H}_\mathcal{D}) = A \otimes A$. So A is a Hopf subalgebra. $\Box$

3.2 Definition of the structure coefficients

Proposition 16 *Let (S) be an SDSE. It is Hopf if, and only if, for all $i, j \in I$, for all $n \geq 1$, there exists a scalar $\lambda_n^{(i,j)}$ such that for all $t' \in \mathcal{T}_i(n)$:*

$$\sum_{t \in \mathcal{T}_i(n+1)} n_j(t, t') a_t = \lambda_n^{(i,j)} a_{t'},$$

where $n_j(t, t')$ is the number of leaves l of t decorated by j such that the cut of l gives t'.

Proof. $\Longrightarrow$. Let us assume that (S) is Hopf. Then $\mathcal{H}_{(S)}$ is a Hopf subalgebra of $\mathcal{H}_I$. Let us use lemma 15, with $V = Vect(X_i(n),\ i \in I,\ n \geq 1)$. So $(f_{\bullet_j} \otimes Id) \circ \Delta(X_i(n+1))$ belongs to $\mathcal{H}_{(S)}$, and is a linear span of trees of degree n with a root decorated by i, so is a multiple of $X_i(n)$. We then denote:

$$(f_{\bullet_j} \otimes Id) \circ \Delta(X_i(n+1)) = \lambda_n^{(i,j)} X_i(n) = \sum_{t' \in \mathcal{T}(n)} \lambda_n^{(i,j)} a_{t'} t'.$$

By definition of the coproduct Δ:

$$(f_{\bullet_j} \otimes Id) \circ \Delta(X_i(n+1)) = \sum_{t \in \mathcal{T}(n+1),\, t' \in \mathcal{T}(n)} n_j(t, t') a_t t'.$$

The result is proved by identifying the coefficients in the basis $\mathcal{T}(n)$ of these two expressions of $(f_{\bullet_j} \otimes Id) \circ \Delta(X_i(n+1))$.

$\Longleftarrow$. Let us prove that both conditions of lemma 15 are satisfied, with the same V as before. By hypothesis, for all $i,j \in I$, for all $n \geq 2$, $(f_{\bullet_j} \otimes Id) \circ \Delta(X_i(n)) = \lambda_{n-1}^{(i,j)} X_i(n-1) \in V$. Moreover, $(f_{\bullet_j} \otimes Id) \circ \Delta(X_i(1)) = \delta_{i,j} \in K$, so the first condition is satisfied. For the second one:

$$(Id \otimes f_{\bullet_j}) \circ \Delta(X_i) = (Id \otimes f_{\bullet_j}) \circ \Delta(B_i^+(F_i(X_j,\, j \in I))) = F_i(X_j,\, j \in I) \in \mathcal{H}_{(S)}.$$

So $\mathcal{H}_{(S)}$ is a Hopf subalgebra of $\mathcal{H}_I$. $\square$

3.3 Properties of the coefficients $\lambda_n^{(i,j)}$

The coefficients $\lambda_n^{(i,j)}$'s are entirely determined by the $a_j^{(i)}$'s and $a_{j,k}^{(i)}$'s, and determine the other coefficients of the F_i's, as shown by the following result:

Lemma 17 *Let us assume that (S) is Hopf, with $I = \{1, \ldots, N\}$. Let us fix $i \in I$.*

1. *For all sequence $i = i_1 \longrightarrow \cdots \longrightarrow i_n$ of vertices of $G_{(S)}$:*
$$\lambda_n^{(i,j)} = a_j^{(i_n)} + \sum_{p=1}^{n-1} (1 + \delta_{j,i_{p+1}}) \frac{a_{j,i_{p+1}}^{(i_p)}}{a_{i_{p+1}}^{(i_p)}}.$$
In particular, $\lambda_1^{(i,j)} = a_j^{(i)}$.

2. *For all $p_1, \cdots, p_N \in \mathbb{N}$:*
$$a_{(p_1,\cdots,p_j+1,\cdots,p_N)}^{(i)} = \frac{1}{p_j+1} \left(\lambda_{p_1+\cdots+p_N+1}^{(i,j)} - \sum_{l \in I} p_l a_j^{(l)} \right) a_{(p_1,\cdots,p_N)}^{(i)}.$$

Proof. 1. Let us consider a sequence $i_1, \cdots, i_n$ of elements of I, such that $i_1 = i$ and for all $1 \leq p \leq n-1$, $a_{i_{p+1}}^{(i_p)} \neq 0$. By definition of $\lambda_n^{(i,j)}$:

$$\begin{array}{rcl}
\lambda_n^{(i,j)} a_{\substack{i_n\\ i_{n-1}\\ \vdots\\ i_2\\ i_1}} & = & a_{\substack{j\\ i_n\\ i_{n-1}\\ \vdots\\ i_2\\ i_1}} + (1+\delta_{j,i_n}) a_{\substack{j\ \ i_n\\ \vee\, i_{n-1}\\ \vdots\\ i_2\\ i_1}} + \sum_{p=1}^{n-2} a_{\substack{\ \ \ i_n\\ \ \ \ \vdots\\ j\ \ i_{p+1}\\ \vee\, i_p\\ \vdots\\ i_1}}, \\
\lambda_n^{(i,j)} a_{i_2}^{(i_1)} \cdots a_{i_n}^{(i_{n-1})} & = & a_{i_2}^{(i_1)} \cdots a_{i_n}^{(i_{n-1})} a_j^{(i_n)} + (1+\delta_{j,i_n}) a_{i_2}^{(i_1)} \cdots a_{i_n,j}^{(i_{n-1})} \\
& & + \sum_{p=1}^{n-2} (1+\delta_{j,i_{p+1}}) a_{i_2}^{(i_1)} \cdots a_{j,i_{p+1}}^{(i_p)} a_{i_{p+2}}^{(i_{p+1})} \cdots a_{i_n}^{(i_{n-1})}, \\
\lambda_n^{(i,j)} & = & a_j^{(i_n)} + \sum_{p=1}^{n-1} (1+\delta_{j,i_{p+1}}) \frac{a_{j,i_{p+1}}^{(i_p)}}{a_{i_{p+1}}^{(i_p)}}.
\end{array}$$

This proves the first point of the lemma.

2. Let us now fix $p_1, \cdots, p_N \in \mathbb{N}$. By definition, for $t' = B_i^+(\bullet_1{}^{p_1} \cdots \bullet_N{}^{p_N})$:

$$\begin{array}{rcl}
\lambda_{p_1+\cdots+p_N+1}^{(i,j)} a_{B_i^+(\bullet_1{}^{p_1} \cdots \bullet_N{}^{p_N})} & = & (p_j+1) a_{B_i^+(\bullet_1{}^{p_1} \cdots \bullet_j{}^{p_j+1} \cdots \bullet_N{}^{p_N})} \\
& & + \sum_{l=1}^{N} a_{B_i^+(\bullet_1{}^{p_1} \cdots \bullet_l{}^{p_l-1} \cdots \bullet_N{}^{p_N} \substack{j\\ l})}, \\
\lambda_{p_1+\cdots+p_N+1}^{(i,j)} a_{(p_1,\cdots,p_N)}^{(i)} & = & (p_j+1) a_{(p_1,\cdots,p_j+1,\cdots,p_N)}^{(i)} + \sum_{l=1}^{N} p_l a_{(p_1,\cdots,p_N)}^{(i)} a_j^{(l)}.
\end{array}$$

This proves the second point of the lemma. $\square$

Remarks.

1. As a consequence of the second point, if (S) is Hopf and if $a^{(i)}_{(p_1,\cdots,p_N)} = 0$, then $a^{(i)}_{(l_1,\cdots,l_N)} = 0$ if $l_1 \geq p_1, \cdots, l_N \geq p_N$. In particular, as there is no constant F_i, for all i, there exists a j such that $a_j^{(i)} \neq 0$.
2. So the sequences considered in the first point of lemma 17 always exist.
3. Moreover, for all vertices i, j of $G_{(S)}$, $i \rightarrow j$ if and only if $a_j^{(i)} \neq 0$.
4. Finally, for all $i \in I$, for all $p \geq 1$, $X_i(p) \neq 0$.

Proposition 18 *Let (S) be a Hopf SDSE.*

1. *Let i, j be vertices of $G_{(S)}$, such that j is not a descendant of i. Then for all $n \geq 1$:*
$$\lambda_n^{(i,j)} = 0.$$
2. *Let (S) be a Hopf SDSE with set of vertices I and let (S') be a Hopf SDSE with set of vertices J. Then (S') is a dilatation of (S) if, and only if, J admits a partition indexed by the elements of I and for all $i, j \in I$, for all $x \in J_i$, $y \in J_j$, for all $n \geq 1$:*
$$\lambda_n^{(i,j)} = \lambda_n^{(x,y)}.$$
3. *Let $i \in I$ such that:*
$$F_i = 1 + \sum_{j \in I} a_j^{(i)} h_j.$$
Then for all direct descendant i' of i, for all j, for all $n \geq 1$:
$$\lambda_{n+1}^{(i,j)} = \lambda_n^{(i',j)}.$$
As a consequence, if i', i'' are two direct descendants of i, $F_{i'} = F_{i''}$.

Proof. 1. Let us consider a sequence $i = i_1, \cdots, i_n$ of elements of I such that $a_{i_{k+1}}^{(i_k)} \neq 0$ for all $1 \leq k \leq n-1$. Then j is not a direct descendant of $i_1, \cdots, i_n$, so $a_j^{(i_n)} = 0$ and $a_{j,i_{k+1}}^{(i_k)} = 0$ for all k. By lemma 17, $\lambda_n^{(i,j)} = 0$.

2. $\Longrightarrow$. From lemma 17-1, choosing an element x_i in J_i for all $i \in I$.
$\Longleftarrow$. Let us consider the dilatation (S'') of (S) corresponding to the partition of J. Then the coefficients $\lambda_n^{(i,j)}$ of (S') and (S'') are equal, so by lemma 17-2, $(S') = (S'')$.

3. Let us consider a sequence $i, i' = i_1, \cdots, i_n$ of elements of I such that $a_{i_{k+1}}^{(i_k)} \neq 0$ for all $1 \leq k \leq n-1$. By hypothesis on i, $a_{j,i'}^{(i)} = 0$. By lemma 17-1:
$$\lambda_{n+1}^{(i,j)} = a_j^{(i_n)} + 0 + \sum_{k=1}^{n-1} (1 + \delta_{j,i_{k+1}}) \frac{a_{j,i_{k+1}}^{(i_k)}}{a_{i_{k+1}}^{(i_k)}} = \lambda_n^{(i',j)}.$$
So, if i' and i'' are two direct descendants of i, for all $k \in I$, for all $n \geq 1$, $\lambda_n^{(i',k)} = \lambda_n^{(i'',k)}$. By lemma 17-2, $F_{i'} = F_{i''}$. $\square$

Proposition 19 *Let (S) be an SDSE, with $I=\{1,\ldots,N\}$. It is Hopf if, and only if, the two following conditions are satisfied:*

1. *There exist scalars $\lambda_n^{(i,j)}$ satisfying, for all $1\leq i,j\leq N$, for all $(p_1,\cdots,p_N)\in\mathbb{N}^N$:*

$$a^{(i)}_{(p_1,\cdots,p_j+1,\cdots,p_N)}=\frac{1}{p_j+1}\left(\lambda^{(i,j)}_{p_1+\cdots+p_N+1}-\sum_{l\in I}p_l a_j^{(l)}\right)a^{(i)}_{(p_1,\cdots,p_N)}.$$

2. *For all $p\geq 1$, for all $i,j,d_1,\cdots,d_p\in I$, such that $a^{(i)}_{(p_1,\cdots,p_N)}\neq 0$ where p_i is the number of d_p's equal to i, for all $n_1,\cdots,n_p\geq 1$:*

$$\lambda^{(i,j)}_{n_1+\cdots+n_p+1}-a_j^{(i)}=\lambda^{(i,j)}_{p+1}-a_j^{(i)}+\sum_{l\in I}\left(\lambda^{(d_l,j)}_{n_l}-a_j^{(d_l)}\right).$$

Proof. *Preliminary step.* Let us assume the first point and let $t'\in\mathcal{T}_\mathcal{D}^{(i)}$. We use the following notations:

$$t'=B_i^+\left(\prod_{s\in\mathcal{T}_\mathcal{D}}s^{r_s}\right).$$

We also denote, for all $j\in I$:

$$p_j=\sum_{s\in\mathcal{T}_\mathcal{D}^{(j)}}r_s.$$

Then, by (1):

$$a_{t'}=\frac{\displaystyle\prod_{j=1}^N p_j!}{\displaystyle\prod_{s\in\mathcal{T}_\mathcal{D}}r_s!}a^{(i)}_{(p_1,\cdots,p_N)}\prod_{s\in\mathcal{T}_\mathcal{D}}a_s^{r_s}.$$

Hence:

$$\begin{aligned}\sum_{t\in\mathcal{T}_\mathcal{D}^{(i)}}n_j(t,t')a_t &= n_j\left(B_i^+\left(\bullet_j\prod_{s\in\mathcal{T}_\mathcal{D}}s^{r_s}\right),t'\right)a_{B_i^+(\bullet_j\prod s^{r_s})}\\
&\quad+\sum_{\substack{s_1,s_2\in\mathcal{T}_\mathcal{D}\\ r_{s_2}\geq 1}}(r_{s_1}+1)n_j(s_1,s_2)a_{B_i^+\left(\frac{s_1}{s_2}\prod s^{r_s}\right)}\\
&=(r_{\bullet_j}+1)\frac{(p_j+1)\displaystyle\prod_{j=1}^N p_j!}{(r_{\bullet_j}+1)\displaystyle\prod_{s\in\mathcal{T}_\mathcal{D}}r_s!}a^{(i)}_{(p_1,\cdots,p_j+1,\cdots,p_N)}a_{\bullet_j}\prod_{s\in\mathcal{T}_\mathcal{D}}a_s^{r_s}\\
&\quad+\sum_{s_1,s_2\in\mathcal{T}_\mathcal{D}}(r_{s_1}+1)n_j(s_1,s_2)\frac{r_{s_2}}{r_{s_1}+1}a_{t'}\frac{a_{s_1}}{a_{s_2}}\\
&=(p_j+1)\frac{a^{(i)}_{(p_1,\cdots,p_j+1,\cdots,p_N)}}{a^{(i)}_{(p_1,\cdots,p_N)}}a_{t'}+\sum_{s_1,s_2\in\mathcal{T}_\mathcal{D}}n_j(s_1,s_2)r_{s_2}a_{t'}\frac{a_{s_1}}{a_{s_2}}\\
&=\left(\lambda^{(i,j)}_{p_1+\cdots+p_N+1}-\sum_{l=1}^N p_j a_j^{(l)}+\sum_{\substack{s_1,s_2\in\mathcal{T}_\mathcal{D}\\ r_{s_2}>0}}n_j(s_1,s_2)r_{s_2}\frac{a_{s_1}}{a_{s_2}}\right)a_{t'}.\end{aligned}$$

$\Longrightarrow$. Let us assume that (S) is Hopf. We already prove the existence of the scalars $\lambda_n^{(i,j)}$. We obtain from the preceding computation:

$$\lambda^{(i,j)}_{weight(t')} a_{t'} = \left(\lambda^{(i,j)}_{p_1+\cdots+p_N+1} - \sum_{l=1}^{N} p_j a_j^{(l)} + \sum_{s_2 \in \mathcal{T}_\mathcal{D}} r_{s_2} \lambda^{(d(s_2),j)}_{weight(s_2)} \right) a_{t'},$$

where $d(s_2)$ is the decoration of the root of s_2. Let us choose p, $i, j, d_1, \cdots, d_p$, $n_1, \cdots, n_p$ as in the hypotheses of the proposition. Let us choose for all $1 \leq j \leq p$ a tree s_j with root decorated by d_j, of weight n_j, such that $a_{s_j} \neq 0$: this always exists (for example take a convenient ladder). Let us take $t' = B_i^+(s_1 \cdots s_p)$. Then $a_{t'} \neq 0$ because $a^{(i)}_{(p_1,\cdots,p_N)} \neq 0$, so:

$$\lambda^{(i,j)}_{n_1+\cdots+n_p+1} = \lambda^{(i,j)}_{p+1} + \sum_{l=1}^{p} \left(\lambda^{(d_l,j)}_{n_l} - a_j^{(d_l)} \right).$$

$\Longleftarrow$. Let us show the condition of proposition 16 by induction on the weight n of t'. For $n = 1$, then $t' = \bullet_i$. Then, by hypothesis on the $a^{(i)}_{(p_1,\cdots,p_N)}$, $a_j^{(i)} = \lambda_1^{(i,j)}$. So:

$$\sum_{t \in \mathcal{T}_i(n+1)} n_j(t,t') a_t = \mathrm{I}^{j}_{i} = a_j^{(i)} = \lambda_1^{(i,j)} a_{\bullet_i}.$$

Let us assume the result for all tree of weight $< n$. The preceding computation then gives:

$$\sum_{t \in \mathcal{T}_\mathcal{D}^{(i)}} n_j(t,t') a_t = \left(\lambda^{(i,j)}_{p_1+\cdots+p_N+1} - \sum_{l=1}^{N} p_j a_j^{(l)} + \sum_{\substack{s_1, s_2 \in \mathcal{T}_\mathcal{D} \\ r_{s_2} > 0}} n_j(s_1, s_2) r_{s_2} \frac{a_{s_1}}{a_{s_2}} \right) a_{t'}.$$

The induction hypothesis and the condition on the coefficients $\lambda_n^{(i,j)}$ then give that this is equal to $\lambda^{(i,j)}_{weight(t')+1} a_{t'}$. So $\mathcal{H}_{(S)}$ is a Hopf subalgebra of $\mathcal{H}_I$. $\Box$

4 Level of a vertex

The second item of proposition 19-2 is immediately satisfied if there exist scalars b_j and $a_j^{(i)}$ such that $\lambda_n^{(i,j)} = b_j(n-1) + a_j^{(i)}$ for all $n \geq 1$ and all $i, j \in I$. This motivates the definition of the level of a vertex.

4.1 Definition of the level

Definition 20 Let (S) be a Hopf SDSE, and let i be a vertex of $G_{(S)}$. It will be said to be of *level* $\leq M$ if for all vertex j, there exist scalar $b_j^{(i)}$, $\tilde{a}_j^{(i)}$, such that for all $n > M$:

$$\lambda_n^{(i,j)} = b_j^{(i)}(n-1) + \tilde{a}_j^{(i)}.$$

The vertex i will be said to be of *level* M if it is of level $\leq M$ and not of level $\leq M - 1$.

Remark. In order to prove that i is of level $\leq M$, it is enough to consider the j's which are descendants of i. Indeed, if j is not a descendant of i, by proposition 18-1, $\lambda_n^{(i,j)} = 0$ for all $n \geq 1$.

Proposition 21 *Let (S) be a Hopf SDSE, i a vertex of $G_{(S)}$ and j a direct descendant of $G_{(S)}$.*

1. *i has level 0 or 1 if, and only if, j as level 0.*

2. *Let $M \geq 2$. Then i has level M if, and only if, j has level $M-1$.*

Moreover, if this holds, then for all $k \in I$, $b_k^{(i)} = b_k^{(j)}$.

Proof. Let $i \in G_{(S)}$ and j be a direct descendant of i. As (S) is Hopf, let us use the second point of proposition 19, with $k=1$ and $d_1 = j$. Then for all l, for all $n \geq 1$, as $a_j^{(i)} \neq 0$:

$$\lambda_{n+1}^{(i,l)} = \lambda_2^{(i,l)} + \lambda_n^{(j,l)} - a_l^{(j)}.$$

So for all $M \geq 1$, i is of level $\leq M$ if, and only if, j is of level $\leq M-1$. Moreover, if this holds, then $b_k^{(i)} = b_k^{(j)}$ for all k.

The first point is a reformulation of the preceding result for $M=1$. Let us assume that $M \geq 2$. If i is of level M, then j is of level $\leq M-1$. If j is of level $\leq M-2$, then i is of level $\leq M-1$: contradiction. So j is of level $M-1$. The converse is proved in the same way. $\Box$

Corollary 22 *Let (S) be a connected Hopf SDSE. Then if one of the vertices of $G_{(S)}$ is of finite level, then all vertices of $G_{(S)}$ are of finite level. Moreover, the coefficients $b_j^{(i)}$ depend only of j. They will now be denoted by b_j.*

Proposition 18-1 immediately implies the following result:

Lemma 23 *Let (S) be a connected Hopf SDSE and let j be a vertex of $G_{(S)}$ of finite level. If there exists a vertex i in $G_{(S)}$ which is not a descendant of j, then $b_j = 0$.*

4.2 Vertices of level 0

Let (S) be a Hopf SDSE with $I = \{1, \ldots, N\}$, and let us assume that i is a vertex of level 0. In this case, the coefficients $a_{(p_1,\cdots,p_N)}^{(i)}$ satisfy an induction of the following form:

$$\left\{ \begin{array}{rcl} a_{(0,\cdots,0)}^{(i)} &=& 1, \\ a_{(p_1,\cdots,p_j+1,\cdots,p_N)}^{(i)} &=& \displaystyle\frac{1}{p_j+1}\left(\lambda_j + \sum_{l=1}^N \mu_j^{(l)} p_l\right) a_{(p_1,\cdots,p_N)}^{(i)}. \end{array} \right.$$

In order to ease the notation, we shall write $a_{(p_1,\cdots,p_N)}$ instead of $a_{(p_1,\cdots,p_N)}^{(i)}$ and F instead of F_i in this section.

Lemma 24 *Under the preceding hypothesis:*

1. *Let us denote $J = \{j \in I \mid \lambda_j = 0\}$. There exists a partition $I = I_1 \cup \cdots \cup I_M \cup J$, and scalars $\beta_1, \cdots, \beta_M$, such that for all $i, j \in I \setminus J = I_1 \cup \cdots \cup I_M$:*

$$\mu_i^{(j)} = \left\{ \begin{array}{l} 0 \text{ if } i,j \text{ do not belong to the same } I_l, \\ \lambda_i \beta_l \text{ if } i,j \in I_l. \end{array} \right.$$

2. *Moreover $F(h_1, \cdots, h_N) = \displaystyle\prod_{p=1}^M f_{\beta_p}\left(\sum_{l \in I_p} \lambda_l h_l\right)$.*

Proof. Let us fix $i \neq j$. Then:

$$\begin{aligned}
& a_{(p_1,\cdots,p_i+1,\cdots,p_j+1,\cdots,p_N)} \\
= & \frac{1}{p_i+1}\left(\lambda_i+\mu_i^{(j)}+\sum_{l=1}^N \mu_i^{(l)} p_l\right) a_{(p_1,\cdots,p_j+1,\cdots,p_N)} \\
= & \frac{1}{(p_i+1)(p_j+1)}\left(\lambda_i+\mu_i^{(j)}+\sum_{l=1}^N \mu_i^{(l)} p_l\right)\left(\lambda_j+\sum_{l=1}^N \mu_j^{(l)} p_l\right) a_{(p_1,\cdots,p_N)}, \\
= & \frac{1}{(p_i+1)(p_j+1)}\left(\lambda_j+\mu_j^{(i)}+\sum_{l=1}^N \mu_j^{(l)} p_l\right)\left(\lambda_i+\sum_{l=1}^N \mu_i^{(l)} p_l\right) a_{(p_1,\cdots,p_N)}.
\end{aligned}$$

For $(p_1,\cdots,p_N)=(0,\cdots,0)$, as $a_{(0,\cdots,0)}=1$:

$$\mu_i^{(j)}\lambda_j=\mu_j^{(i)}\lambda_i. \tag{2}$$

For $(p_1,\cdots,p_N)=\varepsilon_k$, we obtain:

$$\left(\lambda_i+\mu_i^{(j)}+\mu_i^{(k)}\right)\left(\lambda_j+\mu_j^{(k)}\right)\lambda_k=\left(\lambda_j+\mu_j^{(i)}+\mu_j^{(k)}\right)\left(\lambda_i+\mu_i^{(k)}\right)\lambda_k.$$

So, if $\lambda_k \neq 0$:

$$\mu_i^{(j)}\mu_j^{(k)}=\mu_j^{(i)}\mu_i^{(k)}. \tag{3}$$

If $\lambda_k=0$, it is not difficult to prove inductively that $a_{(p_1,\cdots,p_N)}=0$ if $p_k>0$, so F is an element of $K[[h_1,\cdots,h_{k-1},h_{k+1},\cdots,h_N]]$. Hence, up to a restriction to $I\setminus J$, we can suppose that all the λ_k's are non-zero. We then put $\nu_i^{(j)}=\frac{\mu_i^{(j)}}{\lambda_i}$ for all i,j. Then (2) and (3) become: for all i,j,k,

$$\begin{aligned}
\nu_i^{(j)} &= \nu_j^{(i)}, &(4)\\
\nu_i^{(j)}\left(\nu_i^{(k)}-\nu_j^{(k)}\right) &= 0. &(5)
\end{aligned}$$

Let $1\leq i,j\leq N$. We shall say that $i\,\mathcal{R}\,j$ if $i=j$ or if $\nu_i^{(j)}\neq 0$. Let us show that $\mathcal{R}$ is an equivalence. By (4), it is clearly symmetric. Let us assume that $i\mathcal{R}j$ and $j\mathcal{R}k$. If $i=j$ or $j=k$ or $i=k$, then $i\,\mathcal{R}\,k$. If i,j,k are distinct, then $\nu_i^{(j)}\neq 0$ and $\nu_j^{(k)}\neq 0$. By (5), $\nu_i^{(k)}=\nu_j^{(k)}\neq 0$, so $i\,\mathcal{R}\,k$. We denote by $I_1,\cdots,I_M$ the equivalence classes of $\mathcal{R}$.

Let us assume that $i\,\mathcal{R}\,j$, $i\neq j$. Then $\nu_i^{(j)}\neq 0$, so for all k, $\nu_j^{(k)}=\nu_i^{(k)}$. In particular, $\nu_j^{(i)}=\nu_i^{(i)}=\nu_i^{(j)}=\nu_j^{(j)}$. So, finally, there exists a family of scalars $(\beta_i)_{1\leq i\leq M}$, such that:

- If $i,j\in I_l$, then $\nu_i^{(j)}=\beta_l$, and $\mu_i^{(j)}=\lambda_i\beta_l$.
- If i and j are not in the same I_l, then $\nu_i^{(j)}=\mu_i^{(j)}=0$.

An easy induction then proves:

$$a_{(p_1,\cdots,p_N)}=\frac{\lambda_1^{p_1}\cdots\lambda_N^{p_N}}{p_1!\cdots p_N!}\prod_{p=1}^M(1+\beta_p)\cdots\left(1+\beta_p\left(\sum_{l\in I_p}p_l-1\right)\right).$$

This implies the assertion on F. $\square$

4.3 Vertices of level 1

Let us now assume that i is of level 1. Then, up to a restriction to i and its direct descendants, the coefficients $a^{(i)}_{(p_1,\cdots,p_N)} = a_{(p_1,\cdots,p_N)}$ satisfy an induction of the form:

$$\left\{\begin{array}{rcl} a^{(i)}_{(0,\cdots,0)} &=& 1,\\ a^{(i)}_{\varepsilon_j} &=& a^{(i)}_j,\\ a^{(i)}_{(p_1,\cdots,p_j+1,\cdots,p_N)} &=& \dfrac{1}{p_j+1}\left(\lambda_j+\displaystyle\sum_{l=1}^N \mu_j^{(l)} p_l\right) a^{(i)}_{(p_1,\cdots,p_N)} \text{ if } (p_1,\cdots,p_N)\neq(0,\cdots,0). \end{array}\right.$$

In order to ease the notation, we shall write $a_{(p_1,\cdots,p_N)}$ instead of $a^{(i)}_{(p_1,\cdots,p_N)}$ and F instead of F_i in this section.

Lemma 25 *Under the preceding hypothesis, one of the following assertions holds:*

1. *There exists a partition $I = I_1\cup\cdots\cup I_M\cup J$, scalars $\beta_1,\cdots,\beta_M$, a non-zero scalar ν such that:*
$$F(h_1,\cdots,h_N) = \frac{1}{\nu}\prod_{p=1}^M f_{\beta_p}\left(\sum_{l\in I_p}\nu a_l h_l\right) + \sum_{l\in J} a_l h_l + 1 - \frac{1}{\nu}.$$
2. *There exists a partition $\{1,\cdots,N\} = I_1\cup\cdots\cup I_M\cup J$, scalars ν_p for $1\leq p\in M$, such that:*
$$F(h_1,\cdots,h_N) = 1 - \sum_{p=1}^M \frac{1}{\nu_p}\ln\left(1-\nu_p\sum_{l\in I_p} a_l h_l\right) + \sum_{l\in J} a_l h_l.$$

Proof. Let us compute $a_{j,k}$ in two different ways:

$$\left(\lambda_j+\mu_j^{(k)}\right) a_k = \left(\lambda_k+\mu_k^{(j)}\right) a_j.$$

In other words:

$$\left|\begin{array}{cc} \lambda_j+\mu_j^{(k)} & a_j\\ \lambda_k+\mu_k^{(j)} & a_k \end{array}\right| = 0. \tag{6}$$

Let us take $J = \{j \,/\, \forall k,\ \lambda_j+\mu_j^{(k)} = 0\}$. Let us consider an element $j\in J$. Then an easy induction proves that for all $(p_1,\cdots,p_N)$ such that $p_1+\cdots+p_N\geq 2$ and $p_j\geq 1$, $a_{(p_1,\cdots,p_N)} = 0$. As a consequence:

$$F(h_1,\cdots,h_N) = F(h_1,\cdots,h_{j-1},0,h_{j+1},\cdots,h_N) + a_j h_j.$$

So:

$$F = \tilde{F}(h_i,\ i\notin J) + \sum_{j\in J} a_j h_j.$$

We now assume that, up to a restriction, $J=\emptyset$. Let us choose an i and let us put $b_{(p_1,\cdots,p_N)} = (p_i+1)a_{(p_1,\cdots,p_i+1,\cdots,p_N)}$. Then, for all $j\in I$, for all $(p_1,\cdots,p_N)$:

$$b_{(p_1,\cdots,p_j+1,\cdots,p_N)} = \frac{1}{p_j+1}\left(\lambda_j+\mu_j^{(i)}+\sum_{l=1}^N \mu_j^{(l)} p_l\right) b_{(p_1,\cdots,p_N)}.$$

We deduce from lemma 24 that there exist a partition $I = I_1\cup\cdots\cup I_M$ and scalars $\beta_1,\ldots,\beta_M$, such that:

$$\mu_j^{(l)} = \left\{\begin{array}{l} 0 \text{ if } j,l \text{ are not in the same } I_k,\\ \left(\lambda_j+\mu_j^{(i)}\right)\beta_k \text{ if } j,l\in I_k. \end{array}\right.$$

So $\mu_j^{(i)}$ does not depend on i such that $\mu_j^{(i)} \neq 0$. So there exist scalars μ_j such that:

$$\mu_j^{(l)} = \left\{ \begin{array}{l} 0 \text{ if } j,l \text{ are not in the same } I_k, \\ (\lambda_j + \mu_j)\,\beta_k \text{ if } j,l \in I_k. \end{array} \right.$$

1. Let us assume that $M \geq 2$. Let us choose $j \in I_1$. Then for all $k \in I_2 \cup \cdots \cup I_M$, (6) gives:

$$\left| \begin{array}{cc} \lambda_j & a_j \\ \lambda_k & a_k \end{array} \right| = 0.$$

   We denote $I_2 \cup \cdots \cup I_k = \{i_1, \cdots, i_M\}$. We proved that the vectors $(\lambda_j, \lambda_{i_1}, \cdots, \lambda_{i_M})$ and $(a_j, a_{i_1}, \cdots, a_{i_M})$ are colinear. Choosing then a $j \in I_2$, we obtain that there exists a scalar ν, such that $(\lambda_i)_{i\in I} = \nu (a_i)_{i \in I}$. Two cases are possible.

   (a) If $\nu \neq 0$, putting $a'_{(p_1,\cdots,p_N)} = \nu a_{(p_1,\cdots,p_N)}$ if $(p_1,\cdots,p_N) \neq (0,\cdots,0)$ and $a'_{(0,\cdots,0)}$, then the family $\left(a'_{(p_1,\cdots,p_N)}\right)$ satisfies the hypothesis of lemma 24. As a consequence, $F(h_1,\cdots,h_N)$ satisfies the first case.

   (b) If $\nu = 0$, then we put, for all j, $\mu_j = \nu'_j a_j$. By (6), for j and k in the same I_l, $\nu'_j = \nu'_k$ if j and k are in the same I_l: this common value is now denoted ν_l. It is then not difficult to prove that:

$$F(h_1,\cdots,h_N) = 1 - \sum_{p=1}^{M} \frac{1}{\nu_p} \ln\left(1 - \nu_p \sum_{l \in I_p} a_l h_l\right).$$

   This is a second case.

2. Let us assume that $M = 1$. Then $(\lambda_j + \mu_j)\beta_1 = \mu_j^{(i)}$ for all $i,j \in I$.

   (a) Let us suppose that $\beta_1 \neq 1$. Then, for all $j,k \in I$ $\mu_j = \frac{\beta_1}{1-\beta_1}\lambda_j$. So, for all j, $\lambda_j + \mu_j = \frac{1}{1-\beta_1}\lambda_j$. So (6) implies that $(\lambda_j)_{j\in I}$ and $(a_j)_{j \in I}$ are colinear. As in 1.(a), this is a first case.

   (b) Let us assume that $\beta_1 = 1$. So $\lambda_j = 0$ for all j. As in 1.(b), this is a second case.

$\square$

4.4 Vertices of level ≥ 2

Lemma 26 *Let (S) be a Hopf SDSE and let i be a vertex of $G_{(S)}$. We suppose that there exists a vertex j, such that:*

- *j is a descendant of i.*
- *All oriented path from i to j are of length ≥ 3.*

Then $F_i = 1 + \sum_{i \longrightarrow l} a_l^{(i)} h_l$.

Proof. We assume here that $I = \{1,\ldots,N\}$. Let L be the minimal length of the oriented paths from i to j. By hypothesis, $L \geq 3$. Then the homogeneous component of degree $L+1$ of X_i contains trees with a leave decorated by j, and all these trees are ladders (that is to say trees with no ramification). By proposition 16, if $t' \in \mathcal{T}_{\mathcal{D}}^{(i)}(L)$:

$$\lambda_L^{(i,j)} a_{t'} = \sum_{t \in \mathcal{T}_{\mathcal{D}}^{(i)}(L+1)} n_j(t,t') a_t.$$

For a good-chosen ladder t', the second member is non-zero, so $\lambda_L^{(i,j)}$ is non-zero. If t' is not a ladder, the second member is 0, so $a_{t'} = 0$. As a conclusion, $X_i(L)$ is a linear span of ladders. Considering its coproduct, for all $p \leq L$, $X_i(p)$ is a linear span of ladders. In particular, $X_i(3)$ is a linear span of ladders. But:

$$X_i(3) = \sum_{l,m} a_l^{(i)} a_m^{(l)} \mathbf{I}_{\substack{m\\l\\i}} + \sum_{l \leq m} a_{l,m}^{(i)}\, {}^{l}\mathbf{V}_{i}{}^{m},$$

so $a_{l,m}^{(i)} = 0$ for all l, m. Hence, F_i contains only terms of degree ≤ 1. $\square$

Remark. This lemma can be applied with $i = j$, if i is not a self-dependent vertex.

Proposition 27 *Let (S) be a Hopf SDSE and let i be a vertex of $G_{(S)}$ of level ≥ 2. Then i is an extension vertex.*

Proof. We denote by M the level of i. By proposition 21, all the descendants of i are of level $\leq M-1$, so i is not a descendant of itself.

Let M be the level of i and let us assume that $M \geq 3$. Let j be a direct descendant of i, k be a direct descendant of j, l be a direct descendant of k. Then j has level $M-1$, k has level $M-2$, l has level $M-3$. So in the graph of the restriction to $\{i,j,k,l\}$ is:

$$\begin{tikzcd} i \arrow[r] & j \arrow[r] & k \arrow[r] & l & \text{or} & i \arrow[r] & j \arrow[r] & k \arrow[r] & l \arrow[loop right] \end{tikzcd}$$

The result is then deduced from lemma 26.

Let us now assume that i is of level 2 and is not an extension vertex. Let j be a direct descendant of i and k be a direct descendant of j. By proposition 21, j is of level 1 and k is of level 0, so k is not a direct descendant of i. The graph of the restriction of (S) to $\{i,j,k\}$ is:

$$\begin{tikzcd} i \arrow[r] & j \arrow[r] & k & \text{or} & i \arrow[r] & j \arrow[r] & k \arrow[loop right] \end{tikzcd}$$

First step. Let us first prove that there exists a direct descendant j of i such that $a_{j,j}^{(i)} \neq 0$. Let us assume that this is not true. As i is not an extension vertex, there exist $j, j' \in I$ such that $a_{j,j'}^{(i)} \neq 0$, $j \neq j'$. Let k be a direct descendant of j. Considering the different levels, the graph associated to the restriction to $\{i,j,j',k\}$ is:

$$\begin{tikzcd}[column sep=small] & i \arrow[dl] \arrow[dr] & & \text{or} & & i \arrow[dl] \arrow[dr] & & \text{or} & & i \arrow[dl] \arrow[dr] & & \text{or} & & i \arrow[dl] \arrow[dr] & \\ j \arrow[dr] & & j' \arrow[dl] & & j \arrow[dr] & & j' \arrow[dl] & & j \arrow[dr] & & j' & & j \arrow[dr] & & j' \\ & k & & & & k \arrow[loop right] & & & & k & & & & k \arrow[loop right] & \end{tikzcd}$$

Up to a change of variables, we put:

$$F_i(0, \cdots, 0, h_j, 0, \cdots, 0, h_{j'}, 0, \cdots, 0) = 1 + h_j + h_{j'} + b h_j h_{j'} + \mathcal{O}(h^3).$$

Then by proposition 16, $\lambda_2^{(i,j)} a_{\mathbf{I}^{j}_{i}} = 2a_{{}^{j}\mathbf{V}_{i}{}^{j}} + a_{\mathbf{I}_{\substack{j\\j\\i}}} = 0$, so $\lambda_2^{(i,j)} = 0$. On the other hand, $\lambda_2^{(i,j)} a_{\mathbf{I}^{j'}_{i}} = a_{{}^{j}\mathbf{V}_{i}{}^{j'}} + a_{\mathbf{I}_{\substack{j\\j'\\i}}} = b$, so $0 = b$: this contradicts $a_{j,j'}^{(i)} \neq 0$.

Second step. Let us consider a vertex j such that $a_{j,j}^{(i)} \neq 0$. Up to a change of variables, we can assume that $a_j^{(i)} = 1$ and that for all direct descendant k of j, $a_k^{(j)} = 1$. By lemma 23, $b_i = b_j = 0$. So, as i is of level 2, there exist scalars a, b, such that:

$$\lambda_n^{(i,j)} = \begin{cases} 1 \text{ if } n = 1, \\ a \text{ if } n = 2, \\ b \text{ if } n \geq 3. \end{cases}$$

Then proposition 19-1 implies:

$$F_i(0,\cdots,0,h_j,0,\cdots,0) = 1 + h_j + \frac{a}{2!}h_j^2 + \frac{ab}{6}h_j^3 + \mathcal{O}(h_j^4).$$

By hypothesis, $a \neq 0$. Moreover, by proposition 16, $b = \lambda_3^{(i,j)} a_{\begin{smallmatrix} k\\ j\\ i\end{smallmatrix}} = a \; = a$. So:

$$F_i(0,\cdots,0,h_j,0,\cdots,0) = 1 + h_j + \frac{a}{2!}h_j^2 + \frac{a^2}{6}h_j^3 + \mathcal{O}(h_j^4).$$

As j has level 1, we put:

$$\lambda_n^{(j,k)} = \left\{ \begin{array}{c} a_k^{(j)} = 1 \text{ if } n = 1, \\ c(n-1) + d \text{ if } n \geq 2, \end{array} \right.$$

where $c(= b_k)$ and d are scalars. From proposition 19-1:

$$F_j(0,\cdots,0,h_k,0,\cdots,0) = 1 + h_k + \frac{c+d}{2!}h_k^2 + \frac{(c+d)(2c+d)}{6}h_k^3 + \mathcal{O}(h_k^4).$$

Moreover, $\lambda_3^{(i,k)} a_j \; = a$, so $\lambda_3^{(i,k)} \frac{a}{2} = a$ and $\lambda_3^{(i,k)} = 2$. Then $\lambda_3^{(i,k)} a_{\begin{smallmatrix} k\\ j\\ i\end{smallmatrix}} = 2a$, so $c + d = 2$. Similarly, using , we obtain $\lambda_4^{(i,k)} = 3$. Using , we obtain:

$$3\frac{c+d}{2} = 3\frac{(c+d)(2c+d)}{6}.$$

As $c + d = 2$, $2c + d = 3$, so $c = d = 1$ and $\lambda_n^{(j,k)} = n$ for all $n \geq 2$. As $\lambda_1^{(j,k)} = 1$, $\lambda_n^{(j,k)} = n$ for all $n \geq 1$.

Let now $l \in I$ which is not a direct descendant of j and let k be a direct descendant of j. For all $n \geq 1$:

$$\lambda_n^{(j,l)} = \lambda_n^{(j,l)} a_{B_j^+(\bullet_k{}^{n-1})} = a_{B_j^+(\bullet_k{}^{n-1} \begin{smallmatrix} l\\ k\end{smallmatrix})} = (n-1)a_l^{(k)}.$$

We proved that for any vertex l of $G_{(S)}$, for all $n \geq 1$:

$$\lambda_n^{(j,l)} = \left\{ \begin{array}{l} n \text{ if } l \text{ is a direct descendant of } j, \\ a_l^{(k)}(n-1) \text{ if } l \text{ is not a direct descendant of } j, \end{array} \right.$$

where k is any direct descendant of j. This proves that j has level 0, so i has level 1: contradiction. So i is an extension vertex. $\square$

5 Examples of Hopf SDSE

5.1 cycles and multicycles

Notation. We denote by $l(i_1,\cdots,i_n)$ the ladder with decorations, from the root to the leave, $i_1,\cdots,i_n$. In other words:

$$l(i_1,\cdots,i_p) = B_{i_1}^+ \circ \cdots \circ B_{i_n}^+(1) = \begin{smallmatrix} i_n\\ i_{n-1}\\ \vdots\\ i_2\\ i_1\end{smallmatrix}.$$

Theorem 28 *Let $N \geq 2$. The SDSE associated to the following formal series is Hopf:*

$$\left\{ \begin{array}{rcl} F_1 &=& 1 + h_2, \\ &\vdots& \\ F_{N-1} &=& 1 + h_N, \\ F_N &=& 1 + h_1. \end{array} \right.$$

Proof. We identify $\{1, \cdots, N\}$ and $\mathbb{Z}/N\mathbb{Z}$, via the bijection $i \longrightarrow \bar{i}$. Then, for all $n \geq 1$ and for all $1 \leq i \leq N$, $X_{\bar{i}}(n) = l(\bar{i}, \cdots \overline{i+n-1})$. As a consequence:

$$\Delta(X_{\bar{i}}) = X_{\bar{i}} \otimes 1 + 1 \otimes X_{\bar{i}} + \sum_{p=1}^{+\infty} X_{\overline{i+p}} \otimes X_{\bar{i}}(p).$$

So $\mathcal{H}_{(S)}$ is Hopf. $\square$

Note that the graph $G_{(S)}$ associated to such a system is an oriented cycle of length N, with only non-self-dependent vertices.

Definition 29 Let (S) be a Hopf SDSE. It will be said to be *multicyclic* if, up to change of variable, it is a dilatation of a system described in theorem 28.

The graph of a multicyclic SDSE will be called a multicycle. In other term, a N-multicycle ($N \geq 2$) is such that the set I of its vertices admits a partition $I = I_{\bar{1}} \cup \cdots \cup I_{\overline{N}}$ indexed by the elements of $\mathbb{Z}/N\mathbb{Z}$, such that the direct descendants of a vertex i in $I_{\bar{j}}$ are the elements of $I_{\overline{j+1}}$ for all $j \in \mathbb{Z}/N\mathbb{Z}$. Moreover, up to a change of variables, for all $i \in G_{(S)}$:

$$F_i = 1 + \sum_{i \longrightarrow l} h_l.$$

Here is an example of a 5-multicycle:

Note that if $N = 2$, $G_{(S)}$ is a complete bipartite graph, that is to say that the set of vertices of $G_{(S)}$ admits a partition into two parts, and for all vertices i and j, there is an edge from i to j if, and only if, i and j are not in the same part of the partition.

5.2 Fundamental SDSE

Theorem 30 *Let I be a set with a partition $I = I_0 \cup J_0 \cup K_0 \cup I_1 \cup J_1$, such that:*

- *I_0, J_0, K_0, I_1, J_1 can be empty.*
- *$I_0 \cup J_0$ is not empty.*

The SDSE defined in the following way is Hopf:

1. *For all $i \in I_0$, there exists $\beta_i \in K$, such that:*

$$F_i = f_{\beta_i}(h_i) \prod_{j \in I_0 - \{i\}} f_{\frac{\beta_j}{1+\beta_j}}((1+\beta_j)h_j) \prod_{j \in J_0} f_1(h_j).$$

2. *For all $i \in J_0$:*

$$F_i = \prod_{j \in I_0} f_{\frac{\beta_j}{1+\beta_j}}((1+\beta_j)h_j) \prod_{j \in J_0 - \{i\}} f_1(h_j).$$

3. *For all $i \in K_0$:*
$$F_i = \prod_{j \in I_0} f_{\frac{\beta_j}{1+\beta_j}}((1+\beta_j)h_j) \prod_{j \in J_0} f_1(h_j).$$

4. *For all $i \in I_1$, there exist $\nu_i \in K$, a family of scalars $(a_j^{(i)})_{j \in I_0 \cup J_0 \cup K_0}$, such that $(\nu_i \neq 1)$ or $(\exists j \in I_0,\ a_j^{(i)} \neq 1+\beta_j)$ or $(\exists j \in J_0,\ a_j^{(i)} \neq 1)$ or $(\exists j \in K_0,\ a_j^{(i)} \neq 0)$. Then, if $\nu_i \neq 0$:*
$$F_i = \frac{1}{\nu_i} \prod_{j \in I_0} f_{\frac{\beta_j}{\nu_i a_j^{(i)}}} \left(\nu_i a_j^{(i)} h_j\right) \prod_{j \in J_0} f_{\frac{1}{\nu_i a_j^{(i)}}} \left(\nu_i a_j^{(i)} h_j\right) \prod_{j \in K_0} f_0 \left(\nu_i a_j^{(i)} h_j\right) + 1 - \frac{1}{\nu_i}.$$
If $\nu_i = 0$:
$$F_i = -\sum_{j \in I_0} \frac{a_j^{(i)}}{\beta_j} \ln(1-h_j) - \sum_{j \in J_0} a_j^{(i)} \ln(1-h_j) + \sum_{j \in K_0} a_j^{(i)} h_j + 1.$$

5. *For all $i \in J_1$, there exists $\nu_i \in K - \{0\}$, a family of scalars $(a_j^{(i)})_{j \in I_0 \cup J_0 \cup K_0 \cup I_1}$, with the following conditions:*
   - *$I_1^{(i)} = \{j \in I_1 \ / \ a_j^{(i)} \neq 0\}$ is not empty.*
   - *For all $j \in I_1^{(i)}$, $\nu_j = 1$.*
   - *For all $j, k \in I_1^{(i)}$, $F_j = F_k$. In particular, we put $b_t^{(i)} = a_t^{(j)}$ for any $j \in I_1^{(i)}$, for all $t \in I_0 \cup J_0 \cup K_0$.*

   Then:
$$\begin{aligned} F_i &= \frac{1}{\nu_i} \prod_{j \in I_0} f_{\frac{\beta_j}{b_j^{(i)}-1-\beta_j}} \left(\left(b_j^{(i)} - 1 - \beta_j\right) h_j\right) \prod_{j \in J_0} f_{\frac{1}{b_j^{(i)}-1}} \left(\left(b_j^{(i)} - 1\right) h_j\right) \prod_{j \in K_0} f_0 \left(b_j^{(i)} h_j\right) \\ &\quad + \sum_{j \in I_1^{(i)}} a_j^{(i)} h_1 + 1 - \frac{1}{\nu_i}. \end{aligned}$$

Proof. In order to simplify the notation, we assume that $I = \{1, \ldots, N\}$. We shall use proposition 19 with, for all $i, j \in I$:
$$\lambda_n^{(i,j)} = \begin{cases} a_j^{(i)} \text{ if } n = 1, \\ \tilde{a}_j^{(i)} + b_j(n-1) \text{ if } n \geq 2, \end{cases}$$
the coefficients being given in the following arrays:

1. $a_i^{(j)}$:

$i \setminus j$	$\in I_0$	$\in J_0$	$\in K_0$	$\in I_1$	$\in J_1$
$\in I_0$	$(1+\beta_i) - \delta_{i,j}\beta_i$	$1+\beta_i$	$1+\beta_i$	$a_i^{(j)}$	$\frac{b_i^{(j)}-1-\beta_i}{\nu_j}$
$\in J_0$	1	$1-\delta_{i,j}$	1	$a_i^{(j)}$	$\frac{b_i^{(j)}-1}{\nu_j}$
$\in K_0$	0	0	0	$a_i^{(j)}$	$\frac{b_i^{(j)}}{\nu_j}$
$\in I_1$	0	0	0	0	$a_i^{(j)}$
$\in J_1$	0	0	0	0	0

2. $\tilde{a}_i^{(j)}$:

$i \setminus j$	$\in I_0$	$\in J_0$	$\in K_0$	$\in I_1$	$\in J_1$
$\in I_0$	$(1+\beta_i)-\delta_{i,j}\beta_i$	$1+\beta_i$	$1+\beta_i$	$\nu_j a_i^{(j)}$	$b_i^{(j)}-1-\beta_i$
$\in J_0$	1	$1-\delta_{i,j}$	1	$\nu_j a_i^{(j)}$	$b_i^{(j)}-1$
$\in K_0$	0	0	0	$\nu_j a_i^{(j)}$	$b_i^{(j)}$
$\in I_1$	0	0	0	0	0
$\in J_1$	0	0	0	0	0

3. b_j:

j	$\in I_0$	$\in J_0$	$\in K_0$	$\in I_1$	$\in J_1$
b_j	$1+\beta_j$	1	0	0	0

The second item of proposition 19 is immediate. Let us prove for example the first item for $i \in J_1$ and $j \in I_0$. Let us fix $(p_1, \ldots, p_N) \in \mathbb{N}^N - \{(0, \ldots, 0)\}$.

$$\begin{aligned}
&\lambda^{(i,j)}_{p_1+\ldots+p_N+1} - \sum_l a_j^{(l)} p_l \\
&= b_j^{(i)} - 1 - \beta_j - (1+\beta_j)\sum_{l=1}^N p_l - \sum_{l \in I_0 \cup J_0 \cup K_0} (1+\beta_j)p_l + \beta_j p_j - \sum_{l \in I_1 \cup J_1} a_j^{(l)} p_l \\
&= b_j^{(i)} - 1 - \beta_j + \beta_j p_j + \sum_{l \in I_1 \cup J_1} \left(1+\beta_j - a_j^{(l)}\right) p_l.
\end{aligned}$$

If there exists $l \in (I_1 \cup J_1) - I_1^{(i)}$, such that $p_l \neq 0$, then $a^{(i)}_{(p_1,\ldots,p_j+1,\ldots,p_N)} = a^{(i)}_{(p_1,\ldots,p_N)} = 0$ and then the result is immediate. We now suppose that $p_l = 0$ for all $l \in (I_1 \cup J_1) - I_1^{(i)}$. Then:

$$\begin{aligned}
\lambda^{(i,j)}_{p_1+\ldots+p_N+1} - \sum_l a_j^{(l)} p_l &= b_j^{(i)} - 1 - \beta_j + \beta_j p_j + \sum_{l \in I_1^{(i)}} \left(1+\beta_j - a_j^{(l)}\right) p_l \\
&= b_j^{(i)} - 1 - \beta_j + \beta_j p_j + \left(1+\beta_j - b_j^{(i)}\right) \sum_{l \in I_1^{(i)}} p_l.
\end{aligned}$$

1. If $\sum_{l \in I_1^{(i)}} p_l = 0$, then:

$$a^{(i)}_{(p_1,\ldots,p_j+1,\ldots,p_N)} = \left(b_j^{(i)} - 1 - \beta_j p_j\right) \frac{a^{(i)}_{(p_1,\ldots,p_N)}}{p_j+1}.$$

   The first item of proposition 19 is immediate.

2. If $\sum_{l \in I_1^{(i)}} p_l = 1$, then $a^{(i)}_{(p_1,\ldots,p_j+1,\ldots,p_N)} = 0$ and $\lambda^{(i,j)}_{p_1+\ldots+p_N+1} - \sum_l a_j^{(l)} p_l = 0$. So the first item of proposition 19 holds.

3. If $\sum_{l \in I_1^{(i)}} p_l \geq 2$, then $a^{(i)}_{(p_1,\ldots,p_j+1,\ldots,p_N)} = a^{(i)}_{(p_1,\ldots,p_N)} = 0$, so the result is immediate.

The other cases are proved in the same way, so this SDSE is Hopf. □

Remarks.

1. For all $\lambda \neq 0$:
$$f_{\frac{\beta}{\lambda}}(\lambda h) = \sum_{k=0}^{\infty} \frac{\lambda(\lambda+\beta)\cdots(\lambda+(k-1)\beta)}{k!} h^k.$$
The second side of this formula is equal to 1 if $\lambda = 0$. So, formulas defining the SDSE of theorem 30 are always defined.

2. The vertices of $I_0 \cup J_0 \cup K_0$ are of level 0. A vertex i of I_1 is of level 0 if $\nu_i = 1$; otherwise, it is of level 1. The vertices of J_1 are of level 1.

Definition 31

1. A Hopf SDSE will be said to be *fundamental* if, up to a change of variables, it is the dilatation of a system of theorem 30.
2. A fundamental Hopf SDSE (S) will be said to be *abelian* if for any vertex $i \in I$, $b_i = 0$.

Remark. In other words, (S) is abelian if $J_0 = \emptyset$ and if for any $i \in I_0$, $\beta_i = -1$. Then, for all $i \in K_0$, $F_i = 1$. As there is no constant F_i, we obtain $K_0 = \emptyset$.

A particular case is obtained when $I = J_0$. Then we obtain the following systems:

Theorem 32 *Let I be a finite subset which is not a singleton. The SDSE associated to the following formal series is Hopf:*
$$F_i = \prod_{j \neq i} (1 - h_j)^{-1}, \text{ for all } i \in I.$$

The graph associated to such an SDSE is a complete graph with only non-self-dependent vertices, that is to say that there is an edge from i to j in $G_{(S)}$if, and only if, $i \neq j$. In particular, if $N = 2$, $G_{(S)}$ is $1 \longleftrightarrow 2$, as for the SDSE of theorem 28 with $N = 2$.

Definition 33 Let (S) be a Hopf SDSE. It will be said to be *quasi-complete* if, up to change of variable, it is a dilatation of one of the systems described in theorem 32.

The graphs associated to quasi-complete SDSE shall be called *quasi-complete*. A quasi-complete graph G has only non-self-dependent vertices; there exists a partition $I = I_1 \cup \cdots \cup I_M$ of the set I of vertices of $G_{(S)}$ such that, for all $x, y \in I$, there is an edge from x to y if, and only if, x and i are not in the same I_i. In particular, quasi complete graphs with $M = 2$ are complete bipartite graphs. Moreover, if (S) is quasi-complete, up to a change of variables, for all $x \in I_i$:
$$F_x = \prod_{j \neq i} \left(1 - \sum_{y \in I_j} h_y\right)^{-1}.$$
Here is an example of a 2-quasi-complete graph and a 3-quasi-complete graph:

Another particular case is the following: assume that $I = I_0$ and that $\beta_x = -1$ for all $x \in I_0$. Then, for all $x \in I$, $F_x = 1 + h_x$. Note that $G_{(S)}$ is not connected if $|I| \geq 2$, and this is the only case where $G_{(S)}$ is not connected. The dilatation of such an SDSE will be called *a non-connected* *fundamental SDSE*. For such an SDSE, the set of indices I admits a partition $I = I_1 \cup \cdots \cup I_M$ ($M \geq 2$) and up to a change of variables, for all $1 \leq i \leq M$, for all $x \in I_i$:

$$F_x = 1 + \sum_{y \in I_i} h_y.$$

Remark. Note that a dilatation replacing $x \in K_0 \cup I_1 \cup J_1$ by a set J_x in a system of theorem 30 also gives a system of theorem 30. The same remark applies when the dilatation replaces $x \in I_0$, with $\beta_x = 0$, by a set J_x. So we shall always assume that the dilatation giving a fundamental SDSE from an SDSE of theorem 30 satisfies $J_x = \{x\}$ for any $x \in K_0 \cup I_1 \cup J_1$ and for any $x \in I_0$ such that $\beta_x = 0$.

6 Two families of Hopf SDSE

We here first give characterisations of multicyclic and quasi-complete SDSE. We then consider Hopf SDSE such that any vertex is a descendant of a self-dependent vertex. We prove that such an SDSE is fundamental. The results of this section will be used to prove the main theorem 14.

6.1 A lemma on non-self-dependent vertices

Lemma 34 *Let (S) be a Hopf SDSE and let $i \in I$ such that $a_i^{(i)} = 0$. Let j, k and $l \in I$ such that $a_j^{(i)} \neq 0$, $a_k^{(j)} \neq 0$ and $a_l^{(i)} \neq 0$. Then $a_k^{(i)} \neq 0$ or $a_k^{(l)} \neq 0$.*

Proof. Let us assume that $a_k^{(i)} = 0$. As $a_j^{(i)} \neq 0$, $j \neq k$. As $a_k^{(i)} = 0$, $a_{{}^{j}\vee^{k}_{i}} = a_{j,k}^{(i)} = 0$. Then, from proposition 16, $a_j^{(i)} \lambda_2^{(i,k)} = \lambda_2^{(i,k)} a_{\substack{j\\i}} = a_{\substack{k\\j\\i}} + a_{{}^{j}\vee^{k}_{i}} = a_j^{(i)} a_k^{(j)} + 0$; hence, $\lambda_2^{(i,k)} = a_k^{(j)}$. Moreover, As $a_l^{(i)} \neq 0$, $l \neq k$. Then, by proposition 16, $a_l^{(i)} \lambda_2^{(i,k)} = \lambda_2^{(i,k)} a_{\substack{l\\i}} = a_{\substack{k\\l\\i}} + a_{{}^{l}\vee^{k}_{i}} = a_l^{(i)} a_k^{(l)} + 0$, so $\lambda_2^{(i,k)} = a_k^{(l)}$. Hence, $a_k^{(l)} = a_k^{(j)} \neq 0$. $\square$

Remark. In other words, if (S) is Hopf, then, in $G_{(S)}$:

$$\begin{tikzcd} i \arrow[r] \arrow[d] & j \arrow[d] \\ l & k \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} i \arrow[r] \arrow[d] & j \arrow[d] \\ l \arrow[r] & k \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} i \arrow[r] \arrow[d] \arrow[dr] & j \arrow[d] \\ l & k \end{tikzcd} .$$

A special case is given by $i = k$:

$$\begin{tikzcd} i \arrow[r, leftrightarrow] \arrow[d] & j \\ l & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} i \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & j \\ l & \end{tikzcd} .$$

6.2 Symmetric Hopf SDSE

Proposition 35 *Let (S) be a Hopf SDSE, such that $G_{(S)}$ is a N-multicycle with $N \geq 3$. Then (S) is a multicyclic SDSE.*

Proof. Let $I = I_{\overline{1}} \cup \cdots \cup I_{\overline{N}}$ be the partition of the set of vertices of the multicycle $G_{(S)}$. As $N \geq 3$, for all $i \in I$, by lemma 26 with $i = j$:

$$F_i = 1 + \sum_{i \longrightarrow j} a_j^{(i)} h_j.$$

Let $j, j' \in I_{\overline{m}}$. Then any $i \in I_{\overline{m-1}}$ is a direct ascendant of j and j'. By proposition 18-3, $F_j = F_{j'}$. In particular, for $k \in I_{\overline{m+1}}$, $a_k^{(j)} = a_k^{(j')}$. We apply the change of variables sending h_k to $\frac{1}{a_k^{(j)}} h_k$ if $k \in I_{\overline{m+1}}$, where j is any element of $I_{\overline{m}}$. Then, for any $j \in I_{\overline{m}}$:

$$F_j = 1 + \sum_{k \in I_{\overline{m+1}}} h_k.$$

So (S) is multicyclic. $\square$

Proposition 36 *Let (S) be a Hopf SDSE, such that $G_{(S)}$ is M-quasi-complete graph ($M \geq 2$). Then (S) is a 2-multicyclic or a quasi-complete SDSE.*

Proof. First, let us choose two vertices $x \to y$ in $G_{(S)}$. Then $y \to x$ in $G_{(S)}$, and by proposition 16, $\lambda_2^{(y,y)} a_{\substack{x\\y}} = a_{\substack{y\\x\\y}} + a_{{}^{y}\vee^{x}_{y}}$, so $\lambda_2^{(y,y)} a_x^{(y)} = a_x^{(y)} a_y^{(x)} + 0$, and $a_y^{(x)} = \lambda_2^{(y,y)}$ depends only on y. So, up to a change of variables, we can suppose that all the $a_y^{(x)}$'s are equal to 0 or 1. We first study three preliminary cases.

First preliminary case. Let us assume that $G_{(S)} = 1 \longleftrightarrow 2$. We put:

$$F_1(h_2) = \sum_{i=0}^{\infty} a_i h_2^i, \qquad F_2(h_1) = \sum_{i=0}^{\infty} b_i h_1^i,$$

with $a_1 = b_1 = 1$. Then $\lambda_3^{(1,1)} = \lambda_3^{(1,1)} a_{\substack{1\\2\\1}} = 2a_{{}^{1}\vee^{1}_{\substack{2\\1}}} = 2b_2$. On the other hand, $\lambda_3^{(1,1)} a_{{}^{2}\vee^{2}_{1}} = 2a_{{}^{2}\vee^{\substack{1\\2}}_{1}}$, so $2a_2 b_2 = 2a_2$: $a_2 = 0$ or $b_2 = 1$. Similarly, $b_2 = 0$ or $a_2 = 1$. So $a_2 = b_2 = 0$ or 1. In the first case, $F_1(h_2) = 1 + h_2$ and $F_2(h_1) = 1 + h_1$. In the second case, let us apply lemma 17-1 with $(i_1, \cdots, i_n) = (1, 2, 1, 2, \cdots)$. If $n = 2k$ is even, we obtain $\lambda_n^{(1,2)} = 2 + 2(k-1) = 2k = n$. If $n = 2k+1$ is odd, $\lambda_n^{(1,2)} = 1 + 2k = n$. So $\lambda_n^{(1,2)} = n$ for all $n \geq 1$. By proposition 19-1, for all $n \geq 1$, $a_{n+1} = a_n$. So for all $n \geq 0$, $a_n = 1$ and $F_1(h_2) = (1-h_2)^{-1}$. Similarly, $F_2(h_1) = (1-h_1)^{-1}$.

Second preliminary case. Let us suppose that $G_{(S)}$ is the following graph (which is 3-quasi-complete):

$$\begin{tikzcd} 1 \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & 2 \\ & 3 \arrow[ur, leftrightarrow] & \end{tikzcd}$$

We put:

$$\left\{ \begin{array}{rcl} F_1(h_2, h_3) & = & 1 + h_2 + h_3 + a_2 h_2^2 + a_3 h_3^2 + a' h_2 h_3 + \mathcal{O}(h^3), \\ F_2(h_1, h_3) & = & 1 + h_1 + h_3 + b_1 h_1^2 + b_3 h_3^2 + b' h_1 h_3 + \mathcal{O}(h^3), \\ F_3(h_1, h_2) & = & 1 + h_1 + h_2 + c_1 h_1^2 + c_2 h_2^2 + c' h_1 h_2 + \mathcal{O}(h^3). \end{array} \right.$$

By restriction, using the first preliminary case, restricting to $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$,$a_2 = b_1$, $a_3 = c_1$ and $b_3 = c_2$ and all these elements are in $\{0,1\}$. Moreover, by proposition 16, $\lambda_2^{(1,2)} a_{\substack{2\\1}} = 2a_{{}^{2}\vee^{2}_{1}}$, so $\lambda_2^{(1,2)} = 2a_2$. On the other hand, $\lambda_2^{(1,2)} a_{\substack{3\\1}} = a_{\substack{2\\3\\1}} + a_{{}^{2}\vee^{3}_{1}}$, so $\lambda_2^{(1,2)} = 1 + a'$. Hence, $1 + a' = 2a_2$. By symmetry, we obtain $1 + a' = 2a_3$, so $a_2 = a_3$. Similarly, $b_1 = b_3$ and $c_1 = c_2$, so $a_2 = a_3 = b_1 = b_3 = c_1 = c_2 = 0$ or 1.

If they are all equal to 0, then $a' = -1$. Then $\lambda_3^{(3,1)} a_{\substack{2\\1\\3}} = a_{\substack{1\\2\\1\\3}}$, so $\lambda_3^{(3,1)} = 1$. Moreover, $\lambda_3^{(3,1)} a_{\substack{1\\2\\3}} = a_{{}^{1}\vee^{\substack{1\\2}}_{3}}$, so $\lambda_3^{(3,1)} = -1$: this is a contradiction, so $a_2 = a_3 = b_1 = b_3 = c_1 = c_2 = 1$, and $a' = 1$. Similarly, $b' = 1$ and $c' = 1$. As in the first preliminary case, using lemma 17-1, we prove that $\lambda_n^{(i,j)} = n$ if $i \neq j$ for all $n \geq 1$, and then that $F_1(h_2, h_3) = (1-h_2)^{-1}(1-h_3)^{-1}$. Similarly, $F_2(h_1, h_3) = (1-h_1)^{-1}(1-h_3)^{-1}$ and $F_3(h_1, h_2) = (1-h_1)^{-1}(1-h_2)^{-1}$.

Third preliminary case. We now consider the 2-quasi-complete graph with three vertices $1 \longleftrightarrow 2 \longleftrightarrow 3$. Then $I_1 = \{1, 3\}$ and $I_2 = \{2\}$. We put:

$$F_2(h_1, h_3) = 1 + h_1 + h_3 + a_{(2,0)}h_1^2 + a_{(0,2)}h_3^2 + a_{(1,1)}h_1h_3 + \mathcal{O}(h^3).$$

Restricting to $\{1, 2\}$, by the first preliminary case, we obtain $F_1(h_2) = 1 + h_2$ or $F_1(h_2) = (1-h_2)^{-1}$.

1. Let us assume that $F_1(h_2) = 1 + h_2$. Then by the first case, $F_2(h_1, 0) = 1 + h_1$, so $a_{(2,0)} = 0$. Moreover, $\lambda_2^{(2,1)} a_{\substack{1\\2}} = 0$, so $\lambda_2^{(2,1)} a_{\substack{3\\2}} = a_{\substack{1\ \ 3\\ \vee\\ 2}}$: $a_{(1,1)} = 0$. Then $\lambda_2^{(2,3)} a_{\substack{1\\2}} = a_{\substack{1\ \ 3\\ \vee\\ 2}}$, so $\lambda_2^{(2,3)} = a_{(1,1)} = 0$, and $\lambda_2^{(2,3)} a_{\substack{3\\2}} = 2a_{\substack{3\ \ 3\\ \vee\\ 2}}$: $a_{(0,2)} = 0$. As a consequence, $F_2(h_1, h_3) = 1 + h_2 + h_3$. Restricting to $2 \longleftrightarrow 3$, by the first point, $F_3(h_2) = 1 + h_2$.

2. Let us assume that $F_1(h_2) = (1-h_2)^{-1}$. Then $F_2(h_1, 0) = (1-h_2)^{-1}$ by the first point, so $a_{(0,2)} = 1$. By the first preliminary case, this implies that $F_2(0, h_3) = (1-h_3)^{-1}$ and $F_3(h_2) = (1-h_2)^{-1}$. Similarly with the first case, we prove that $\lambda_n^{(2,i)} = n$ if $i = 1$ or 3 for all $n \geq 1$. By proposition 19-1:

$$a_{(m+1,n)} = \frac{m+n+1}{m+1} a_{(m,n)}, \qquad a_{(m,n+1)} = \frac{m+n+1}{n+1} a_{(m,n)}.$$

   An easy induction proves that $a_{(m,n)} = \binom{m+n}{m}$ for all m, n, so $F_2(h_1, h_3) = (1-h_1-h_3)^{-1}$.

We separate the proof of the general case into two subcases.

General case, first subcase. $M = 2$. We put $I_1 = \{x_1, \cdots, x_r\}$ and $I_2 = \{y_1, \cdots, y_s\}$. For $x_i \in I_1$, we put:

$$F_{x_p} = \sum_{(q_1, \cdots, q_s)} a^{(x_p)}_{(q_1, \cdots, q_s)} h_{y_1}^{q_1} \cdots h_{y_s}^{q_s}.$$

Restricting to the vertices x_p and y_q, by the first preliminary case, two cases are possible.

1. $a^{(x_p)}_{y_q, y_q} = 0$. Then, by the third preliminary case, restricting to x_p, y_q and $y_{q'}$, for all y_q, $y_{q'}$, $a^{(x_p)}_{y_q, y_{q'}} = 0$. So:

$$F_{x_p} = 1 + \sum_q h_{y_q}.$$

2. $\lambda_n^{(x_p, y_q)} = n$ for all $n \geq 1$. Using proposition 19-1, we obtain:

$$a^{(x_p)}_{(q_1, \cdots, q_m+1, \cdots, q_s)} = \frac{1 + q_1 + \cdots + q_s}{q_m + 1} a^{(x_p)}_{(q_1, \cdots, q_s)}.$$

   An easy induction proves:

$$a^{(x_p)}_{(q_1, \cdots, q_s)} = \frac{(q_1 + \cdots + q_s)!}{q_1! \cdots q_s!}.$$

   So:

$$F_{x_p} = \left(1 - \sum_q h_{y_q}\right)^{-1}.$$

A similar result holds for the y_q's. So, we prove that for any vertex i of $G_{(S)}$, one of the following holds:

1. $F_i = 1 + \sum_{i \longrightarrow j} h_j$.

2. $F_i = \left(1 - \sum_{i \longrightarrow j} h_j\right)^{-1}$.

Moreover, by the first preliminary case, if i and j are related, they satisfy both (a) or both (b). As the graph is connected, every vertex satisfies (a) or every vertex satisfies (b).

General case, second subcase. $M \geq 3$. Let us fix $i \in G$ and let us denote $y_1, \cdots, y_q$ its direct descendants. Restricting to the vertices i and y_j, two cases are possible.

1. $a^{(i)}_{y_j,y_j} = 0$. As $M \geq 3$, with a good choice of $y_{j'}$, we can restrict to the second preliminary case, and we obtain $a^{(i)}_{y_j,y_j} = 1$: contradiction. So this case is impossible.

2. $\lambda_n^{(x,y_j)} = n$ for all $n \geq 1$. Using proposition 19-1, we obtain, similarly with the case $M = 2$, if $i \in I_p$:
$$F_i = \prod_{q \neq p} \left(1 - \sum_{l \in h_q} h_l\right)^{-1}.$$

So (S) is quasi-complete. $\square$

Definition 37

1. Let G be a graph. We shall say that G is *symmetric* if it has only non-self-dependent vertices and if, for $i \neq j$, there is an edge from i to j if, and only if, there is an edge from j to i.

2. Let (S) be an SDSE. We shall say that (S) is *symmetric* if $G_{(S)}$ is symmetric.

Theorem 38 *Let (S) be a connected symmetric Hopf SDSE. Then (S) is 2-multicyclic or quasi-complete.*

Proof. By proposition 36, it is enough to prove that $G_{(S)}$ is a M-quasi-complete graph, with $M \geq 2$. Let us consider a maximal quasi-complete subgraph G' of $G_{(S)}$. This exists, as $G_{(S)}$ contains quasi-complete subgraphs (for example, two related vertices). Let us assume that $G' \neq G_{(S)}$. As $G_{(S)}$ is connected, there exists a vertex $i \in G_{(S)}$, related to a vertex of G'. Let us put $I' = I'_1 \cup \cdots I'_M$ be the partition of the set of vertices of G'.

First, if i is related to a vertex j of I'_p, it is related to any vertex of I'_p. Indeed, let j' be another vertex of I'_p and let $k \in I'_q$, $q \neq p$. By lemma 34, j' is related to i. As $G_{(S)}$ is symmetric, i is related to j'.

Let us assume that i is not related to at least two I_p's. Let us take k, l in G', in two different I_p's, not related to i. By the first step, j, k and l are in different I_p's, so are related. By lemma 34, k or l is related to i. As $G_{(S)}$ is symmetric, then i is related to k or l: contradiction. So i is not related to at most one I_p's.

As a conclusion:

1. If i is related to every I_p's, by the first step i is related to every vertices of G', so $G' \cup \{i\}$ is an $M+1$-quasi-complete graph, with partition $I_1 \cup \cdots \cup I_M \cup \{x\}$: this contradicts the maximality of G'.

2. If i is related to every I_p's but one, we can suppose up to a reindexation that i is not related to I_M. Then, by the first step, i is related to every vertices of $I_1 \cup \cdots \cup I_{M-1}$. So $G' \cup \{x\}$ is an M-quasi-complete graph, with partition $I_1 \cup \cdots \cup (I_M \cup \{x\})$: this contradicts the maximality of G'.

In both cases, this is a contradiction, so $G_{(S)} = G'$ is quasi-complete. $\square$

6.3 Formal series of a self-dependent vertex

Let (S) be a Hopf SDSE, and let us assume that i is a self-dependent vertex of $G_{(S)}$. Up to a change of variables, we can suppose that $a_j^{(i)} = 0$ or 1 for all j. In particular, we assume that $a_i^{(i)} = 1$.

Lemma 39 *Under these hypotheses, i is of level 0 and for all $j \in I$, $b_j = (1+\delta_{i,j})a_{i,j}^{(i)}$.*

Proof. We apply lemma 17-1, with $i_k = i$ for all i. We obtain, for all $n \geq 1$:

$$\lambda_n^{(i,j)} = a_j^{(i)} + (1+\delta_{i,j})(n-1)\frac{a_{i,j}^{(i)}}{a_i^{(i)}}.$$

So this proves the assertion. $\square$

Remark. So all the descendants of i are also of level 0.

Lemma 40 *Under the former hypotheses, there exists a partition $I = I_1 \cup \cdots \cup I_M \cup J$ (J eventually empty), with $i \in I_1$, such that the coefficients $a_j^{(k)}$ are given in the following array:*

$j \setminus k$	I_1	I_2	I_3	$\cdots$	I_M	J
I_1	1	β_1+1	$\cdots$	$\cdots$	β_1+1	$*$
I_2	$\vdots$	$1-\beta_2$	1	$\cdots$	1	$\vdots$
I_3	$\vdots$	1	$1-\beta_3$	$\ddots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	1	$\vdots$
I_M	1	1	$\cdots$	1	$1-\beta_M$	$\vdots$
J	0	$\cdots$	$\cdots$	$\cdots$	0	$*$

Moreover, for all $j \in I_1$:

$$F_j = \prod_{p=1}^{M} f_{\beta_p}\left(\sum_{l \in I_p} h_l\right).$$

Finally, the coefficients $\lambda_n^{(j,k)}$ are given by $\lambda_n^{(j,k)} = b_k(n-1) + a_k^{(j)}$ for all $n \geq 1$ with:

k	I_1	I_2	$\cdots$	I_M	J
b_k	β_1+1	1	$\cdots$	1	0

Proof. We can apply lemma 24 with $\lambda_j = a_j^{(i)}$ and $\mu_j^{(l)} = -a_j^{(l)} + (1+\delta_{i,j})\,a_{i,j}^{(i)}$. Then $I = I_1 \cup \cdots I_M \cup J$, such that $-a_j^{(k)} + (1+\delta_{i,j})\,a_{i,j}^{(i)}$ is given for all j,k by the array:

$j \setminus k$	I_1	I_2	$\cdots$	I_M	J
I_1	β_1	0	$\cdots$	0	$*$
I_2	0	β_2	$\ddots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	0	$\vdots$
I_M	0	$\cdots$	0	β_M	$\vdots$
J	0	$\cdots$	$\cdots$	0	$*$

We assume that $i \in I_1$, without loss of generality. For the row $j \in J$, the result comes from the following observation: let $j,k \in I$ such that $a_j^{(i)} = 0$ and $a_k^{(i)} \neq 0$, then, by proposition 19-1:

$$a_{j,k}^{(i)} = \left(a_j^{(i)} - a_j^{(k)} + a_{i,j}^{(i)}\right) a_k^{(i)} = 0.$$

As $a_j^{(i)}=0$, then $a_{i,j}^{(i)}=0$, so $a_j^{(k)}=0$.

Lemma 24 also gives:

$$F_i=\prod_{p=1}^{k} f_{\beta_p}\left(\sum_{l\in I_p} h_l\right).$$

So $(1+\delta_{i,j})a_{i,j}^{(i)}=\beta_1+1$ if $j\in I_1$, 1 if $j\in I_2\cup\cdots\cup I_M$, and 0 if $j\in J$. So $a_j^{(k)}$ is given by for all j,k by the indicated array. We obtain in lemma 39 that:

$$b_k=\left\{\begin{array}{l}\beta_1+1 \text{ if } k\in I_1,\\ 1 \text{ if } k\in I_2\cup\cdots\cup I_M,\\ 0 \text{ if } k\in J.\end{array}\right.$$

As a conclusion, if $j\in I_1$, then for all $1\leq k\leq N$, $a_k^{(j)}=a_k^{(i)}$ and $\lambda_n^{(j,k)}=\lambda_n^{(i,k)}$ for all $n\geq 1$. By proposition 19, $F_i=F_j$. □

6.4 Hopf SDSE generated by self-dependent vertices

Proposition 41 *Let (S') be a Hopf SDSE, and let i be a self-dependent vertex of $G_{(S')}$. Let (S) be the restriction of (S') to i and all its descendants. Then (S) is fundamental, with $K_0=I_1=J_1=\emptyset$.*

Proof. We use the notations of lemma 40. Note that if i,j are in the same I_k, then $\lambda_n^{(i,k)}=\lambda_n^{(j,k)}$ for all $n\geq 1$, for all $k\in I$. So, by proposition 18-2 the Hopf SDSE formed by i and its descendant is the dilatation of a system with the following coefficients $\lambda_n^{(j,k)}$:

$j\setminus k$	1	2	3	$\cdots$	M
1	$(\beta_1+1)(n-1)+1$	n	$\cdots$	$\cdots$	n
2	$(\beta_1+1)n$	$n-\beta_2$	n	$\cdots$	n
3	$\vdots$	n	$n-\beta_3$	$\ddots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	n
M	$(\beta_1+1)n$	n	$\cdots$	n	$n-\beta_M$

with $i=1$. We already proved in lemma 40 that:

$$F_1=\prod_{j=1}^{M} f_{\beta_j}(h_j).$$

If $j\neq 1$, for all $(k_1,\cdots,k_M)$:

$$\begin{aligned}
a_{(k_1+1,\cdots,k_M)}^{(j)} &= \left((\beta_1+1)\sum_{l=1}^{M} k_l+\beta_1+1-(\beta_1+1)\sum_{l=1}^{M} k_l-k_1\right)\frac{a_{(k_1,\cdots,k_M)}^{(j)}}{k_1+1}\\
&= (\beta_1+1+\beta_1 k_1)\frac{a_{(k_1,\cdots,k_M)}^{(j)}}{k_1+1},\\
a_{(k_1,\cdots,k_j+1,\cdots,k_M)}^{(j)} &= \left(\sum_{l=1}^{M} k_l+1-\beta_j-\sum_{l=1}^{M} k_l+\beta_j k_j\right)\frac{a_{(k_1,\cdots,k_M)}^{(j)}}{k_j+1}\\
&= (1-\beta_j+\beta_j k_j)\frac{a_{(k_1,\cdots,k_M)}^{(j)}}{k_j+1}.
\end{aligned}$$

If $l \neq 1$ and $l \neq j$:

$$a^{(j)}_{(k_1,\cdots,k_l+1,\cdots,k_M)} = \left(\sum_{l=1}^{M} k_l - \sum_{l=1}^{M} k_l + \beta_l k_l\right) \frac{a^{(j)}_{(k_1,\cdots,k_M)}}{k_l+1} = (1+\beta_l k_l)\frac{a^{(j)}_{(k_1,\cdots,k_M)}}{k_l+1}.$$

So, if $j \neq 1$:

$$F_j = f_{\frac{\beta_1}{1+\beta_1}}((1+\beta_1)h_1) f_{\frac{\beta_j}{1-\beta_j}}((1-\beta_j)h_j) \prod_{k\neq 1,j} f_{\beta_k}(h_k).$$

Let us put $I'_0 = \{j \geq 2 \,/\, \beta_j \neq 1\}$ and $J'_0 = \{j \geq 2 \,/\, \beta_j = 1\}$. Then, after the change of variables $h_j \longrightarrow \frac{1}{1-\beta_j} h_j$ for all $j \in I'_0$:

$$\left\{\begin{array}{rcl}
F_1 &=& f_{\beta_1}(h_1) \displaystyle\prod_{j\in I'_0} f_{\beta_j}\left(\frac{1}{1-\beta_j}h_j\right) \prod_{j\in J'_0} f_1(h_j),\\
F_j &=& f_{\frac{\beta_1}{1+\beta_1}}((1+\beta_1)h_1) f_{\frac{\beta_j}{1-\beta_j}}(h_j) \displaystyle\prod_{j\in I'_0-\{j\}} f_{\beta_j}\left(\frac{1}{1-\beta_j}h_j\right) \prod_{j\in J'_0} f_1(h_j) \text{ if } j \in I'_0,\\
F_j &=& f_{\frac{\beta_1}{1+\beta_1}}((1+\beta_1)h_1) \displaystyle\prod_{j\in I'_0} f_{\beta_j}\left(\frac{1}{1-\beta_j}h_j\right) \prod_{j\in J'_0-\{j\}} f_1(h_j) \text{ if } j \in J'_0.
\end{array}\right.$$

Putting $\gamma_j = \frac{\beta_j}{1-\beta_j}$ for all $j \in I_0$, then, as $\beta_j = \frac{\gamma_j}{1+\gamma_j}$ and $1-\beta_j = \frac{1}{1+\gamma_j}$:

$$\left\{\begin{array}{rcl}
F_1 &=& f_{\beta_1}(h_1) \displaystyle\prod_{j\in I'_0} f_{\frac{\gamma_j}{1+\gamma_j}}((1+\gamma_j)h_j) \prod_{j\in J'_0} f_1(h_j),\\
F_j &=& f_{\frac{\beta_1}{1+\beta_1}}((1+\beta_1)h_1) f_{\gamma_j}(h_j) \displaystyle\prod_{j\in I'_0-\{j\}} f_{\frac{\gamma_j}{1+\gamma_j}}((1+\gamma_j)h_j) \prod_{j\in J'_0} f_1(h_j) \text{ if } j \in I'_0,\\
F_j &=& f_{\frac{\beta_1}{1+\beta_1}}((1+\beta_1)h_1) \displaystyle\prod_{j\in I'_0} f_{\frac{\gamma_j}{1+\gamma_j}}((1+\gamma_j)h_j) \prod_{j\in J'_0-\{j\}} f_1(h_j) \text{ if } j \in J'_0.
\end{array}\right.$$

So this a fundamental system, with $I_0 = \{1\} \cup I'_0$ and $J_0 = J'_0$. □

Corollary 42 *Let (S) be a connected Hopf SDSE such that any vertex of $G_{(S)}$ is the descendant of a self-dependent vertex. Then (S) is fundamental, with $K_0 = I_1 = J_1 = \emptyset$.*

Proof. Let x be a self-dependent vertex of (S). Then the system formed by x and its descendants is fundamental. We then put $I_0^{(x)}$ and $J_0^{(x)}$ the partition of the set formed by x and its descendants. We separate $I_0^{(x)}$ into two parts:

$$I_{0,1} = \left\{y \in I_0^{(x)} \,/ \beta_y \neq -1\right\},\ I_{0,2} = \left\{y \in I_0^{(x)} \,/ \beta_y = -1\right\}.$$

Then, after elimination of an eventual dilatation by restriction, the direct descendants of $x \in I_{0,2}^{(x)}$ are x, the elements of $I_{0,1}^{(x)}$ and $J_0^{(x)}$; the direct descendants of $x \in I_{0,1}^{(x)}$ are the elements of $I_{0,1}^{(x)}$ and $J_0^{(x)}$; the direct descendants of $x \in J_0^{(x)}$ are the elements of $I_{0,1}^{(x)}$ and the elements of $J_0^{(x)}$ except x. Let us consider the following cases:

1. If there exists a vertex x, such that $J_0^{(x)} \neq \emptyset$, then, as $G_{(S)}$ is connected, for any self-dependent vertex y, $J_0^{(y)} = J_0^{(x)}$. As a consequence, for any self-dependent vertex y, $I_{0,1}^{(x)} = I_{0,1}^{(y)}$. We then deduce that (S) is fundamental, with $J_0 = J_0^{(x)}$ for any self-dependent vertex x.

2. If for any self-dependent vertex x, $J_0^{(x)} = \emptyset$, and if there is a self-dependent vertex x such that $I_{0,2}^{(x)} \neq \emptyset$, then by connectivity of $G_{(S)}$, for any self-dependent vertex y, $I_{0,2}^{(y)} = I_{0,2}^{(x)}$ and $I_{0,1}^{(y)} = \{y\}$, or $I_{0,2}^{(y)}$ is empty if $y \in I_{0,2}^{(x)}$. Then (S) is a fundamental, with $J_0 = \emptyset$.

3. If for any self-dependent vertex x, $J_0^{(x)}=\emptyset=I_{0,2}^{(x)}$. Then by connectivity, $I=I_{0,1}^{(x)}$ for any self-dependent vertex. So (S) is fundamental, with $J_0=\emptyset$.

In all cases, (S) is fundamental. $\Box$

7 The structure theorem of Hopf SDSE

7.1 Connecting vertices

Definition 43 Let (S) be an SDSE and let $i\in G_{(S)}$.

1. We denote by $G_{(S)}^{(i)}$ is the subgraph of $G_{(S)}$ formed by i and all its descendants.

2. The vertex i is a *connecting vertex* of $G_{(S)}$ if $G_{(S)}^{(i)}-\{i\}$ is not connected.

Lemma 44 *Let (S) be a Hopf SDSE and let $i\in G_{(S)}$ be a connecting vertex. Then (i is the descendant of a self-dependent vertex) or (i belongs to a symmetric subgraph of $G_{(S)}$) or (i is not self-dependent and relates several components of a non-connected fundamental SDSE).*

Proof. *First step.* If i is self-dependent, it is a descendant of itself and the conclusion holds. Let us assume that i is not self-dependent. Let $G_1,\cdots,G_M$ be the connected components of $G_{(S)}^{(i)}-\{i\}$ ($M\geq 2$). Let $x_p\in G_p$ be a direct descendant of i for all p. Let x'_p be a direct descendant of x_p. Then $x'_p\in G_p$. Choosing $q\neq j$ and applying lemma 34, there is an edge from i to x'_p. Iterating this process, we deduce that any vertex of $G_{(S)}^{(i)}-\{i\}$ is a direct descendant of i. If i is the direct descendant of a vertex $j\in G_{(S)}^{(i)}-\{i\}$, then i is included in the symmetric subgraph $i\longleftrightarrow j$ of $G_{(S)}^{(i)}$, so the conclusion holds.

Second step. Let us now assume that i is not the direct descendant of any $j\in G_{(S)}^{(i)}-\{i\}$. Let $n\geq 2$, $j\in G_p$, and let $i\to x_2\to\cdots\to x_n$ in $G_{(S)}^{(i)}$, where $x_2,\cdots,x_n\in G_q$, $p\neq q$. Then, as i is not related to any x_l, $\lambda_n^{(i,j)}a^{(i)}_{l(i,x_2,\cdots,x_n)}=a_{B_i^+(\bullet_j\, l(x_2,\cdots,x_n))}$, so $\lambda_n^{(i,j)}=\frac{a^{(i)}_{j,x_2}}{a^{(i)}_{x_2}}$, and $\lambda_n^{(i,j)}$ does not depend on n: we put $\lambda_n^{(i,j)}=\lambda_j$ for all $j\in G-\{i\}$, $n\geq 2$. In other words, i has level ≤ 1, and $b_j=0$ for all j.

Third step. In order to simplify the writing of the proof, up to a reindexation, we shall suppose that $i=0$ and the vertices of $G_{(S)}^{(0)}-\{0\}$ are the elements of $\{1,\cdots,N\}$. By a change of variables, we can suppose that $a_j^{(0)}=1$ for all $1\leq j\leq N$. By the second step, we can use lemma 25, with $\mu_j^{(l)}=-a_j^{(l)}$ for all $1\leq j,l\leq N$ and $\lambda_j=a_{j,k}^{(0)}$ for all j,k in two different connected components of $G_{(S)}^{(0)}-\{0\}$.

1. In the first case, we obtain the following values for $a_j^{(k)}$ and λ_j:

$j\setminus k$	I_1	I_2	$\cdots$	I_M	J
I_1	$-\nu\beta_1$	0	$\cdots$	0	$-\nu$
I_2	0	$-\nu\beta_2$	$\ddots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	0	$\vdots$
I_M	0	$\cdots$	0	$-\nu\beta_M$	$-\nu$
J	0	$\cdots$	$\cdots$	0	0

j	I_1	$\cdots$	I_M	J
λ_j	ν	$\cdots$	ν	0

As there are no vertices with no descendants, necessarily $\nu \neq 0$ and $\beta_p \neq 0$ for all p. For the same reason, $I_1 \cup \cdots \cup I_M = \emptyset$ is impossible. If $J \neq \emptyset$, then any vertex of J is related to every vertex of $I_1 \cup \cdots \cup I_M$, so $G_{(S)}^{(0)} - \{0\}$ is connected: impossible, as 0 is a connected vertex. So $J = \emptyset$, and 0 connects several totally self-dependent subgraphs.

2. In the second case, we obtain the following values for $a_j^{(k)}$ and λ_j:

$j \setminus k$	I_1	I_2	$\cdots$	I_M	J
I_1	$-\nu_1$	0	$\cdots$	0	0
I_2	0	$-\nu_2$	$\ddots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	0	$\vdots$
I_M	0	$\cdots$	0	$-\nu_M$	0
J	0	$\cdots$	$\cdots$	0	0

j	I_1	$\cdots$	I_M	J
λ_j	0	$\cdots$	0	0

As there are no vertices with no descendants, $J = \emptyset$ and $\nu_l \neq 0$ for all l.

Moreover, as $b_j = 1 + \beta_j = 0$ for all $j \geq 1$, 0 connects several components of a non-connected fundamental SDSE. $\square$

7.2 Structure of connected Hopf SDSE

Lemma 45 *Let (S) be a Hopf SDSE containing a multicycle with set of vertices $I = I_{\overline{1}} \cup \cdots \cup I_{\overline{M}}$, Then any non-self-dependent vertex of $G_{(S)}$ has direct descendants in at most one $I_{\overline{k}}$.*

Proof. Let us assume that the vertex 0 of $G_{(S)}$ have a direct descendant $x \in I_{\overline{k}}$ and $y \in I_{\overline{l}}$ with $\overline{k} \neq \overline{l}$. Then lemma 34 implies that any direct descendant of x is a direct descendant of 0, so 0 has also a direct descendant in $I_{\overline{k+1}}$. Similarly, 0 has a direct descendant in $I_{\overline{l+1}}$. Iterating this process, 0 has direct descendants in all the $I_{\overline{i}}$'s. Up to a restriction, the situation is the following:

$$\begin{tikzcd}
0 \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrrr] & & & & \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & N \arrow[llll, bend left]
\end{tikzcd}$$

Moreover, for all $1 \leq i \leq k$, $F_i(h_{i+1}) = 1 + h_{i+1}$, with the convention $h_{N+1} = h_1$.

We first assume $M \geq 3$. In order to ease the notation, we do not write the index ${}^{(0)}$ in the sequel of the proof. By proposition 16, $\lambda_2^{(0,2)} a_{\substack{1\\0}} = a_{\substack{2\\1\\0}} + a_{1 \bigvee_0 2}$, so $\lambda_2^{(0,2)} = 1 + \frac{a_{1,2}}{a_1}$. On the other hand, $\lambda_2^{(0,2)} a_{\substack{2\\0}} = 2a_{2 \bigvee_0 2}$, so $\lambda_2^{(0,2)} = 2\frac{a_{2,2}}{a_2}$. Hence:

$$1 + \frac{a_{1,2}}{a_1} = 2\frac{a_{2,2}}{a_2}.$$

Moreover, $\lambda_3^{(0,2)} a_{\substack{3\\2\\0}} = a_{2 \bigvee_0 \substack{3\\2}}$, so $\lambda_3^{(0,2)} = 2\frac{a_{2,2}}{a_2}$. On the other hand, $\lambda_3^{(0,2)} a_{\substack{2\\1\\0}} = a_{2 \bigvee_0 \substack{2\\1}}$, so $\lambda_3^{(0,2)} = \frac{a_{1,2}}{a_1}$. Hence:

$$\frac{a_{1,2}}{a_1} = 2\frac{a_{2,2}}{a_2} = 1 + \frac{a_{1,2}}{a_1}.$$

This is a contradiction.

Let us now prove the result for $N = 2$. We assume that there exists a Hopf SDSE with the graph:

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 1 \arrow[rr, leftrightarrow] & & 2 \end{tikzcd}$$

and such that $F_1 = 1 + h_2$ and $F_2 = 1 + h_1$. We write:

$$F_0 = \sum_{i,j} a_{(i,j)} h_1^i h_2^j,$$

with $a_{(1,0)}$ and $a_{(0,1)}$ non-zero. Then $\lambda_2^{(0,1)} a_{\substack{1\\0}} = 2a_{{}^{1}\vee^{1}_{0}}$, so $\lambda_2^{(0,1)} = \frac{2a_{(2,0)}}{a_{(1,0)}}$. On the other hand, $\lambda_2^{(0,1)} a_{\substack{2\\0}} = a_{{}^{1}\vee^{2}_{0}} + a_{\substack{1\\2\\0}}$, so $\lambda_2^{(0,1)} = \frac{a_{(1,1)}}{a_{(0,1)}} + 1$. We obtain:

$$\frac{2a_{(2,0)}}{a_{(1,0)}} = \frac{a_{(1,1)}}{a_{(0,1)}} + 1.$$

Moreover, $\lambda_3^{(0,1)} a_{\substack{2\\1\\0}} = a_{{}^{1}\vee^{\substack{2\\1}}_{0}} + a_{\substack{1\\2\\1\\0}}$, so $\lambda_3^{(0,1)} = \frac{2a_{(2,0)}}{a_{(1,0)}} + 1$. On the other hand, $\lambda_3^{(0,1)} a_{\substack{1\\2\\0}} = 2a_{{}^{1}\vee^{\substack{1\\2}}_{0}}$, so $\lambda_3^{(0,1)} = \frac{a_{(1,1)}}{a_{(0,1)}}$. So:

$$\frac{a_{(1,1)}}{a_{(0,1)}} + 1 = \frac{2a_{(2,0)}}{a_{(1,0)}} = \frac{a_{(1,1)}}{a_{(0,1)}} - 1.$$

This is a contradiction. $\square$

Lemma 46 *Let (S) be a Hopf SDSE, such that any vertex of $G_{(S)}$ has a direct ascendant. Let i be a vertex of $G_{(S)}$. Then (i is a descendant of a self-dependent vertex) or (i belongs to a multicycle of $G_{(S)}$) or (i belongs to a symmetric subgraph of $G_{(S)}$).*

Proof. Let us first prove that i is the descendant of a vertex of a cycle of $G_{(S)}$. As any vertex has a direct ascendant, it is possible to define inductively a sequence $(x_l)_{l\geq 0}$ of vertices of $G_{(S)}$, such that $x_0 = i$ and x_{l+1} is a direct ascendant of x_l for all l. As $G_{(S)}$ is finite, there exists $0 \leq l < m$, such that $x_l = x_m$. Then $x_l \leftarrow x_{l+1} \leftarrow \cdots \leftarrow x_{m-1} \leftarrow x_m = x_l$ is a cycle of $G_{(S)}$, and i is a descendant of any vertex of this cycle.

Let $G' = x_1 \rightarrow \cdots \rightarrow x_s \rightarrow x_1$ be a cycle such that i is a descendant of a vertex of G', chosen with a minimal s. As s is minimal, there are no edges from x_l to x_m in $G_{(S)}$ if $m \neq l + 1$, with the convention $x_{s+1} = x_1$. The situation is the following:

$$\begin{tikzcd} x_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & x_s \arrow[ll, bend right] & \\ y_1 \arrow[r] & \cdots \arrow[r] & y_{t-1} \arrow[r] & i \end{tikzcd}$$

Three cases are possible:

1. If $s = 1$, then i is the descendant of a self-dependent vertex.

2. If $s = 2$, the situation is the following:

$$\begin{tikzcd} x_1 \arrow[r, leftrightarrow] \arrow[d] & x_2 & & \\ y_1 \arrow[r] & \cdots \arrow[r] & y_{t-1} \arrow[r] & i \end{tikzcd}$$

By minimality of s, there are no self-dependent vertex in $\{x_1, x_2, y_1, \cdots, y_{t-1}, i\}$. Applying repeatedly lemma 34, there is an edge from y_1 to x_1, then from y_2 to y_1, $\cdots$, then from i to y_{t-1}. So i belongs to a symmetric subgraph of $G_{(S)}$.

3. If $s \geq 3$, then the subgraph formed by $x_1, \cdots, x_s$ is a multicycle. Let G' be a maximal multicycle of length s of G, such that i is a descendant of a vertex of G'. We denote by I' the set of vertices of G'. Let us assume that $i \notin G'$. There exists $x_1 \rightarrow y_1 \rightarrow \cdots \rightarrow y_{t-1} \rightarrow y_t = i$ in G, with $t \leq 1$, and $x_1 \in I'$. Up to a reindexation, we can assume that $x_1 \in I'_{\overline{1}}$. By lemma 34, y_1 is the direct descendant of any vertex of $I_{\overline{1}}$ and the direct ascendant of any vertex of $I_{\overline{3}}$. By lemma 45, y_1 is not the direct ascendant of any vertex of $I'_{\overline{k}}$ if $\overline{k} \neq \overline{3}$. So $I' \cup \{x\} = I'_{\overline{1}} \cup \left(I'_{\overline{2}} \cup \{i\}\right) \cup \cdots \cup I'_{\overline{s}}$ gives a multicycle of length s, such that i is a descendant of a vertex of $I' \cup \{i\}$: this contradicts the maximality of G'. So $i \in I'$.

□

By the preceding study of Hopf symmetric SDSE:

Corollary 47 *Let (S) be connected Hopf SDSE, such that any vertex of $G_{(S)}$ has a direct ascendant. Then (any vertex of $G_{(S)}$ is the descendant of a self-dependent vertex, so (S) is fundamental) or ((S) is quasi-complete, so (S) is fundamental) or ((S) is multicyclic).*

Corollary 48 *Let (S) be a connected Hopf SDSE. Then there exists a sequence $(G_i)_{0 \leq i \leq k}$ of subgraphs of $G_{(S)}$, such that:*

- *The system (S_0) associated to the F_i's, $i \in G_0$, is fundamental or is multicyclic.*
- $G_k = G_{(S)}$.
- *For all $0 \leq i \leq k-1$, G_{i+1} is obtained from G_i by adding a non-self-dependent vertex without any ascendant in G_i.*

If G_0 is fundamental, any vertex is of finite level. If G_0 is multicyclic, no vertex is of finite level.

Proof. *First step.* Let us first prove the following (weaker) result: if (S) is a Hopf SDSE, there exists a sequence $(G_i)_{0 \leq i \leq k}$ of subgraphs of $G_{(S)}$, such that:

- G_0 is the disjoint union of several fundamental systems or is multicyclic.
- $G_k = G_{(S)}$.
- For all $0 \leq i \leq k-1$, G_{i+1} is obtained from G_i by adding a non-self-dependent vertex without any ascendant in G_i.

Let us proceed by induction on N. If $N = 1$, then $G_{(S)} = G_0$ is formed by a single vertex which is necessarily self-dependent, so (S) is fundamental. Let us assume the induction hypothesis at rank $\leq N-1$. If any vertex of $G_{(S)}$ has an ascendant, then by corollary 47, we can take $G_{(S)} = G_0$. If it is not the case, let us take i being a vertex with no ascendant. The induction hypothesis can be applied to the components of $G_{(S)} - \{i\}$. We complete the sequence $(G_0, \cdots, G_k)$ given in this way by $G_{k+1} = G_{(S)}$.

As a consequence, the set of descendants of any self-dependent vertex, every symmetric subgraph, every multicycle of $G_{(S)}$ is included in G_0.

Second step. Let us assume that $G_{(S)}$ is connected. If G_0 is connected, then it is fundamental or multicyclic. If it is not, let us assume that it is not a non-connected abelian fundamental SDSE. So one of the components H of G_0 is not a fundamental abelian SDSE with $I = I_0$. Then for a good choice of i, the vertex added to G_{i-1} to obtain G_i is a connecting vertex, connecting a subgraph containing H and other subgraphs. By the first step, as it does not belong to G_0, this vertex is not the descendant of a self-dependent vertex and does not belong to a symmetric subgraph. By construction, it does not connect several components of a non-connected fundamental SDSE: this is a contradiction with lemma 44. So G_0 is of the announced form. $\square$

7.3 Connected Hopf SDSE with a multicycle

Let us precise the structure of connected Hopf SDSE containing a multicycle.

Theorem 49 *Let (S) be a connected Hopf SDSE containing a N-multicyclic SDSE. Then I admits a partition $I = I_{\overline{1}} \cup \cdots \cup I_{\overline{N}}$, with the following conditions:*

1. *If $x \in I_{\overline{k}}$, its direct descendants are all in $I_{\overline{k+1}}$.*
2. *If x and x' have a common direct ascendant, then they have the same direct descendants.*

Moreover, for all $x \in I$:

$$F_x = 1 + \sum_{x \longrightarrow y} a_y^{(x)} h_y.$$

If x and x' have a common direct ascendant, then $F_x = F_{x'}$. Such an SDSE will be called an extended multicyclic SDSE.

Proof. We use the notations of corollary 48. We proceed by induction on k. If $k = 0$, (S) is a multicycle and the result is immediate. Let us assume the result at rank $k-1$ and let (S') be the restriction of (S) to all the vertices except the last one, denoted by x. By the induction hypothesis, the set of its vertices admits a partition $I' = I'_{\overline{1}} \cup \cdots \cup I'_{\overline{N}}$, with the required conditions. Let us first prove that all the direct descendants of x are in the same $I'_{\overline{m}}$. Let $y \in I_{\overline{k}}$ and $z \in I_{\overline{l}}$ be two direct descendants of x, with $\overline{k} \neq \overline{l}$. Let $y' \in I_{\overline{k+1}}$ be a direct descendant of y and $z' \in I_{\overline{l+1}}$ be a direct descendant of z. Lemma 34 implies that x is a direct ascendant of z' and y', as y can't be a direct ascendant of z' and z can't be a direct ascendant of y' because $\overline{k} \neq \overline{l}$. So we can replace y by y' and z by z'. Iterating the process, we can assume that y and z are in the multicycle: this contradicts lemma 45. So the direct descendants of x are all in $I_{\overline{m}}$ for a good m. We then take $I_{\overline{l}} = I'_{\overline{l}}$ if $\overline{l} \neq \overline{m-1}$ and $I_{\overline{m-1}} = I'_{\overline{m-1}} \cup \{x\}$ and this proves the first assertion on $G_{(S)}$.

We now prove the assertion on F_x. We separate the proof into two subcases. Let us first assume $M \geq 3$. There is an oriented path $x \rightarrow x_{\overline{m}} \rightarrow \cdots \rightarrow x_{\overline{m+M-1}}$, with $x_{\overline{i}} \in I'_{\overline{i}}$ for all i. Moreover, there is no shorter oriented path from x to $x_{\overline{m+M-1}}$. As $M \geq 3$, from lemma 26:

$$F_x = 1 + \sum_{x \longrightarrow y} a_y^{(x)} h_y.$$

Let us secondly assume that $M = 2$. Let $1, \ldots, p$ be the direct descendants of x and let 0 be a direct descendant of 1. Then as $1, \ldots, p$ are in the same part of the partition of I', they are not direct descendants of 1. Let us first restrict to $\{x, 1, 0\}$. By proposition 16, $\lambda_3^{(x,0)} a_{\substack{0\\1\\x}} = 0$ as $a_{0,0}^{(1)} = 0$ by the induction hypothesis, $\lambda_3^{(x,0)} = 0$. Moreover, $0 = \lambda_3^{(x,0)} a_{\substack{1\ \ 1\\ \vee\\ x}} = a_{\substack{\quad 0\\ 1\ \ 1\\ \vee\\ x}}$, so $a_{1,1}^{(x)} = 0$. Similarly, $a_{2,2}^{(x)} = \cdots = a_{p,p}^{(x)} = 0$. Let us now take $1 \leq i < j \leq p$. Then $\lambda_2^{(x,i)} a_{\substack{i\\x}} = 0$, so $\lambda_2^{(x,i)} = 0$ and $0 = \lambda_2^{(x,i)} a_{\substack{j\\x}} = a_{\substack{i\\j\\x}}$, so $a_{i,j}^{(x)} = 0$. As a conclusion, F_x is of the required form.

Proposition 18-3 implies that $F_x = F_{x'}$ if x and x' have a common ascendant, and this implies the second assertion on $G_{(S)}$. $\square$

Remark. In particular, the vertex added to G_i in order to obtain G_{i+1} is an extension vertex. By proposition 11, any such SDSE is Hopf.

7.4 Connected Hopf SDSE with finite levels

We now prove the following theorem:

Theorem 50 *Let (S) be a connected Hopf SDSE, such that any vertex of (S) has a finite level. Then (S) is obtained from a fundamental system by a finite number (possibly 0) of extensions. Such an SDSE will be called an extended fundamental SDSE.*

Proof. Let (S) be a connected Hopf SDSE, such that any vertex of (S) is of finite level. We use notations of corollary 48. We shall proceed by induction on k. If $k=0$, then $S=S_0$ and the result is obvious. Let us now assume the result at rank $k-1$. By the induction hypothesis, the system (S') associated to G_{k-1} is a dilatation of a system of theorem 30. Moreover, G is obtained from G_{k-1} by adding a vertex with all its direct descendants in G_{k-1}. Let us denote by 0 this vertex. We separate the proof into three cases.

First case. Let us assume that 0 is of level 0. Then all the direct descendants of 0 are of level 0, so are in $I_0\cup J_0\cup I_1$, and $\nu_x=1$ for all direct descendants of x in J_i with $i\in I_1$. Moreover, for all $x\in I$, $\lambda_n^{(0,x)}=b_x(n-1)+a_x^{(0)}$.

Let us take $x,y\in I$. Using proposition 19-1 into two different ways:

$$a_{x,y}^{(0)}=\left(b_y+a_y^{(0)}-a_y^{(x)}\right)a_x^{(0)}=\left(b_x+a_x^{(0)}-a_x^{(y)}\right)a_y^{(0)}.$$

So, for all $x,y\in I$:

$$\left(b_y-a_y^{(x)}\right)a_x^{(0)}=\left(b_x-a_x^{(y)}\right)a_y^{(0)}. \tag{7}$$

If x and y are in the same I_i with $i\in I_0\cup J_0$, then $b_y-a_y^{(x)}=b_x-a_x^{(y)}\neq 0$, so $a_x^{(0)}=a_y^{(0)}$ and for all $n\geq 1$, $\lambda_n^{(0,x)}=\lambda_n^{(0,y)}$. Hence, up to a restriction, we can assume that there is no dilatations on (S').

Let $i\in I_1$. If $\nu_i\neq 1$, we already know that $a_i^{(0)}=0$. Let us assume $\nu_i=1$ and let us choose $j\in I_0\cup J_0\cup K_0$, such that $a_j^{(i)}\neq b_j$. Then $b_i=a_i^{(j)}=0$, so (7) gives $\left(b_j-a_j^{(i)}\right)a_i^{(0)}=0$. So $a_i^{(0)}=0$ for all $i\in I_1$. So the direct descendants of 0 are all in $I_0\cup J_0\cup K_0$. Using proposition 19-1 with $i\in I_0\cup J_0\cup K_0$:

$$\begin{aligned}
a^{(0)}_{(p_1,\cdots,p_i+1,\cdots,p_N)}&=\left(a_i^{(0)}+b_i(p_1+\cdots+p_N)-\sum_{j\in I_0\cup J_0\cup K_0-\{i\}}b_ip_j-a_i^{(i)}p_i\right)\frac{a^{(0)}_{(p_1,\cdots,p_N)}}{p_i+1}\\
&=\left(a_i^{(0)}+\left(b_i-a_i^{(i)}\right)p_i\right)\frac{a^{(0)}_{(p_1,\cdots,p_N)}}{p_i+1}.
\end{aligned}$$

So:

$$F_0=\prod_{i\in I_0}f_{\frac{\beta_i}{a_i^{(0)}}}\left(a_i^{(0)}h_i\right)\prod_{i\in J_0}f_{\frac{1}{a_i^{(0)}}}\left(a_i^{(0)}h_i\right)\prod_{i\in K_0}f_0\left(a_i^{(0)}h_i\right).$$

So (S) is a system of theorem 30, with $0\in K_0\cup I_1$.

Second case. Let us assume that 0 is of level 1 and is not an extension vertex. Then all the direct descendants of 0 are of level 0, so are in $I_0\cup J_0\cup I_1$, and $\nu_x=1$ for all direct descendants of x in I_1. Moreover, for all $i\in I$, $\lambda_1^{(0,i)}=a_i^{(0)}$ and $\lambda_n^{(0,i)}=b_i(n-1)+\tilde{a}_i^{(0)}$ if $n\geq 2$.

First item. Let us assume that $a_i^{(0)}=0$. Then by proposition 19-1:

$$\begin{aligned}
a^{(0)}_{(p_1,\cdots,1,\cdots,p_N)}&=\left(\tilde{a}_i^{(0)}+b_i(p_1+\cdots+p_N)-\sum_{j=1}^N a_i^{(j)}p_j\right)a^{(0)}_{(p_1,\cdots,0,\cdots,p_N)}\\
0&=\left(\tilde{a}_i^{(0)}-\sum_{j\in I_1}a_i^{(j)}p_j\right)a^{(0)}_{(p_1,\cdots,0,\cdots,p_N)}.
\end{aligned}$$

If there is a $j \in I_0 \cup J_0 \cup K_0$, such that $a_j^{(0)} \neq 0$, then for $(p_1, \cdots, p_N) = \varepsilon_j$, we obtain $\tilde{a}_i^{(0)} = 0$. If it is not the case, as 0 is not an extension vertex, there exists $j, k \in I_1$, $a_{j,k}^{(0)} \neq 0$ (so $a_j^{(0)} \neq 0$ and $a_k^{(0)} \neq 0$). Then, for $(p_1, \cdots, p_N) = \varepsilon_j$, $(p_1, \cdots, p_N) = \varepsilon_k$, and $(p_1, \cdots, p_N) = \varepsilon_j + \varepsilon_k$, we obtain:

$$\tilde{a}_i^{(0)} + a_i^{(j)} = \tilde{a}_i^{(0)} + a_i^{(k)} = \tilde{a}_i^{(0)} + a_i^{(j)} + a_i^{(k)} = 0.$$

So $\tilde{a}_i^{(0)} = 0$. So in all cases, $\tilde{a}_i^{(0)} = 0$. Moreover, for $(p_1, \cdots, p_N) = \varepsilon_j$ for any $j \in I_1$, we obtain $a_i^{(j)} a_j^{(0)} = 0$. As a conclusion, we proved:

1. For all $i \in I$, $\left(a_i^{(0)} = 0\right) \Longrightarrow \left(\tilde{a}_i^{(0)} = 0\right)$.

2. Let us put $I_1^{(0)} = \left\{i \in I_1 \,/\, a_i^{(0)} \neq 0\right\}$. Then for $i \in I$, such that $a_i^{(0)} = 0$, for all $j \in I_1^{(0)}$, $a_i^{(j)} = 0$.

Second item. Let us take $i, j \in I$. Using proposition 19-1 into two different ways:

$$a_{i,j}^{(0)} = \left(b_j + \tilde{a}_j^{(0)} - a_j^{(i)}\right) a_i^{(0)} = \left(b_i + \tilde{a}_i^{(0)} - a_i^{(j)}\right) a_j^{(0)}. \tag{8}$$

Let us take $i, j \in I_1$. Then $a_j^{(i)} = a_i^{(j)} = b_i = b_j = 0$, so (8) gives:

$$\tilde{a}_j^{(0)} a_i^{(0)} = \tilde{a}_i^{(0)} a_j^{(0)}.$$

So $\left(\tilde{a}_i^{(0)}\right)_{i \in I_1}$ and $\left(a_i^{(0)}\right)_{i \in I_1}$ are colinear. By the first item, we deduce that there exists a scalar $\nu \in K$, such that for all $i \in I_1$, $\tilde{a}_i^{(0)} = \nu a_i^{(0)}$. Let us now take $i, j \in I_0 \cup J_0 \cup K_0$, with $i \neq j$. Then $b_i = a_i^{(j)}$ and $b_j = a_j^{(i)}$, so (8) gives:

$$\tilde{a}_j^{(0)} a_i^{(0)} = \tilde{a}_i^{(0)} a_j^{(0)}.$$

So $\left(\tilde{a}_i^{(0)}\right)_{i \in I_0 \cup J_0 \cup K_0}$ and $\left(a_i^{(0)}\right)_{i \in I_0 \cup J_0 \cup K_0}$ are colinear. By the first item, we deduce that there exists a scalar $\nu' \in K$, such that for all $i \in I_0 \cup J_0 \cup K_0$, $\tilde{a}_i^{(0)} = \nu' a_i^{(0)}$. Let us now take $i \in I_0 \cup J_0 \cup K_0$ and $j \in I_1$. Then $b_j = a_j^{(i)} = 0$, so $\nu a_j^{(0)} a_i^{(0)} = \left(b_i + \nu' a_i^{(0)} - a_i^{(j)}\right) a_j^{(0)}$. In other words:

$$\forall i \in I_0 \cup J_0 \cup K_0,\ \forall j \in I_1,\ (\nu - \nu') a_i^{(0)} a_j^{(0)} = (b_i - a_i^{(j)}) a_j^{(0)}. \tag{9}$$

Third item. Let us assume that $I_1^{(0)} = \emptyset$. Then all the direct descendants of 0 are in $I_0 \cup J_0 \cup K_0$. Moreover, if $i \in I_0 \cup J_0 \cup K_0$:

$$\begin{aligned} a_{(p_1, \cdots, p_{i+1}, \cdots, p_N)}^{(0)} &= \left(\nu a_i^{(0)} + b_i(p_1 + \cdots + p_N) - \sum_{j \in I_0 \cup J_0 \cup K_0 - \{i\}} b_i p_j - a_i^{(i)} p_i\right) \frac{a_{(p_1, \cdots, p_N)}^{(0)}}{p_i + 1} \\ &= \left(\nu a_i^{(0)} + \left(b_i - a_i^{(i)}\right) p_i\right) \frac{a_{(p_1, \cdots, p_N)}^{(0)}}{p_i + 1}. \end{aligned}$$

It is then not difficult to show that (S) is a system of theorem 30, with $0 \in I_1$.

Fourth item. Let us assume that $\nu = \nu'$. Let $j \in I_1$. If $\nu_j \neq 1$, then we already know that $a_j^{(0)} = 0$. If $\nu_j = 1$, then for a good choice of i, $b_i - a_i^{(j)} \neq 0$ in (9), so $a_j^{(0)} = 0$: then $I_1^{(0)} = \emptyset$, and the result is proved in the third item.

Fifth item. Let us assume that $I_1^{(0)} \neq \emptyset$. By the preceding item, $\nu \neq \nu'$. Let us take $j \in I_1^{(0)}$. By (9), for all $i \in I_0 \cup J_0 \cup K_0$, $a_i^{(j)} = b_i - (\nu - \nu')a_i^{(0)}$ does not depend of j. As a consequence, $F_j = F_k$ for all $j, k \in I_1^{(0)}$. We put $b_i^{(0)} = a_i^{(j)}$ for all $i \in I_0 \cup J_0 \cup K_0$, where j is any element of $I_1^{(0)}$. Let us use proposition 19-1. For all $i \in I_0 \cup J_0 \cup K_0$, if $(p_1, \cdots, p_N) \neq (0, \cdots, 0)$:

$$a^{(0)}_{(p_1,\cdots,p_i+1,\cdots,p_N)} = \left(\nu' a_i^{(0)} + \left(b_i - a_i^{(i)}\right) p_i + (\nu - \nu')a_i^{(0)} \sum_{j \in I_1^{(0)}} p_j \right) \frac{a^{(0)}_{(p_1,\cdots,p_N)}}{p_i + 1}.$$

For all $j \in I_1^{(0)}$, if $(p_1, \cdots, p_N) \neq (0, \cdots, 0)$:

$$a^{(0)}_{(p_1,\cdots,p_i+1,\cdots,p_N)} = \nu a_i^{(0)} \frac{a^{(0)}_{(p_1,\cdots,p_N)}}{p_i + 1}.$$

Let us fix $i \in I_0 \cup J_0 \cup K_0$ and $j \in I_1^{(0)}$. Then:

$$\begin{aligned}
a_{i,i}^{(0)} &= \left(\nu' a_i^{(0)} + b_i - a_i^{(i)}\right) a_i^{(0)}, \\
a_{i,i,j}^{(0)} &= \nu a_i^{(0)} a_j^{(0)} \left(\nu' a_i^{(0)} + b_i - a_i^{(i)}\right), \\
a_{i,j}^{(0)} &= \nu a_i^{(0)} a_j^{(0)}, \\
a_{i,i,j}^{(0)} &= \nu a_i^{(0)} a_j^{(0)} \left(\nu' a_i^{(0)} + b_i - a_i^{(i)} + (\nu - \nu')a_i^{(0)}\right).
\end{aligned}$$

Identifying the two expressions of $a_{i,i,j}^{(0)}$, as $\nu \neq \nu'$ and $a_j^{(0)} \neq 0$, we obtain $\nu \left(a_i^{(0)}\right)^2 = 0$. If for all $i \in I_0 \cup J_0 \cup K_0$, $a_i^{(0)} = 0$, then by the second item, for all $j \in I_1^{(0)}$, $a_i^{(j)} = 0$, then $F_j = 1$; this is impossible. So there is an $i \in I_0 \cup J_0 \cup K_0$, such that $a_i^{(0)} \neq 0$. As a consequence, $\nu = 0$. So $\nu' \neq 0$, and we then easily obtain that:

$$\begin{aligned}
F_0 = {} & \frac{1}{\nu'} \prod_{i \in I_0} f_{\frac{\beta_i}{b_i^{(0)} - 1 - \beta_i}} \left(\left(b_i^{(0)} - 1 - \beta_i \right) h_i \right) \prod_{i \in J_0} f_{\frac{1}{b_i^{(0)} - 1}} \left(\left(b_i^{(0)} - 1 \right) h_i \right) \prod_{i \in I_0} f_0 \left(b_i^{(0)} h_i \right) \\
& + \sum_{i \in I_1^{(0)}} a_i^{(0)} h_i + 1 - \frac{1}{\nu'}.
\end{aligned}$$

So (S) is a system of theorem 30, with $0 \in J_1$.

Third case. 0 is a vertex of level ≥ 2. By proposition 27, it is an extension vertex. □

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