Lie algebras associated to systems of Dyson-Schwinger equations

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ABSTRACT. We consider systems of combinatorial Dyson-Schwinger equations in the Connes-Kreimer Hopf algebra $\mathcal{H}_I$ of rooted trees decorated by a set I. Let $\mathcal{H}_{(S)}$ be the subalgebra of $\mathcal{H}_I$ generated by the homogeneous components of the unique solution of this system. If it is a Hopf subalgebra, we describe it as the dual of the enveloping algebra of a Lie algebra $\mathfrak{g}_{(S)}$ of one of the following types:

1. $\mathfrak{g}_{(S)}$ is an associative algebra of paths associated to a certain oriented graph.

2. Or $\mathfrak{g}_{(S)}$ is an iterated extension of the Faà di Bruno Lie algebra.

3. Or $\mathfrak{g}_{(S)}$ is an iterated extension of an infinite-dimensional abelian Lie algebra.

We also describe the character groups of $\mathcal{H}_{(S)}$.

Keywords. Systems of Dyson-Schwinger equations, Hopf algebras of decorated trees, pre-Lie algebras.

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Contents

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Introduction

The Connes-Kreimer Hopf algebra of (decorated) rooted trees $h^{\mathcal{D}}$ is introduced in [16] and studied in [2, 3, 5, 6, 7, 8, 14, 21]. For any element d of the set of decorations D, we define an operator B_d^+ of $\mathcal{H}^{\mathcal{D}}$, sending a forest F to the rooted tree obtained by grafting the trees of F on a common root decorated by d. This operator satisfies the following equation: for all $x \in \mathcal{H}_{\mathcal{D}}$,

$$\Delta \circ B_d^+(x) = B_d^+(x) \otimes 1 + (Id \otimes B_d^+) \circ \Delta(x).$$

As explained in [6], this means that B_d^+ is a 1-cocycle for a certain cohomology of coalgebras, dual to the Hochschild cohomology.

We now take $D = \{1, \ldots, N\}$ as a set of decorations. A system of combinatorial Dyson-Schwinger equations (briefly, an SDSE), is a system (S) of the form:

$$\left\{ \begin{array}{rcl} X_1 & = & B_1^+(F_1(X_1, \ldots, X_N)), \\ & \vdots & \\ X_N & = & B_N^+(F_N(X_1, \ldots, X_N)), \end{array} \right.$$

where $F_1, \ldots, F_N \in K[[h_1, \ldots, h_N]]$ are formal series in N indeterminates (see [1, 17, 18] for applications to Quantum Fields Theory). Such a system possesses a unique solution, which is a family of N formal series in rooted trees, or equivalently elements of a completion of $\mathcal{H}_{\mathcal{D}}$. The homogeneous components of these elements generate a subalgebra $\mathcal{H}_{(S)}$ of $\mathcal{H}_{\mathcal{D}}$. We determined in [10] the SDSE such that $\mathcal{H}_{(S)}$ is a Hopf subalgebra, generalizing the results of [9] for a single combinatorial Dyson-Schwinger equations. For this, we first associate an oriented graph to any SDSE, reflecting the dependence of the different X_i's; more precisely, the vertices of $G_{(S)}$ are the elements of I, and there is an edge from i to j if F_i depends on h_j. The SDSE is said to be connected if its associated graph $G_{(S)}$ is connected. We then introduced several operations on SDSE, especially change of variables (proposition 4 of the present paper) and two families of SDSE, namely fundamental and multicyclic SDSE, here described in theorem 6. For example, the following system is multicyclic:

$$\left\{ \begin{array}{rcl} X_1 & = & B_1^+(1 + X_2), \\ X_2 & = & B_2^+(1 + X_3), \\ X_3 & = & B_3^+(1 + X_4), \\ X_4 & = & B_4^+(1 + X_1). \end{array} \right.$$

The associated oriented graph is:

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 4 \arrow[u] & 3 \arrow[l] \end{tikzcd}$$

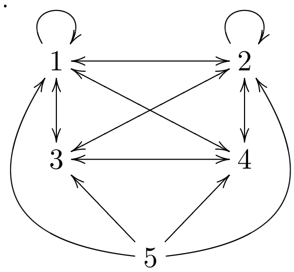
Let us take $\beta_1, \beta_2 \in K - \{-1\}$. For all $\beta \in K$, f_β is the following formal series:

$$f_\beta(h) = \sum_{k=0}^{\infty} \frac{(1+\beta)\cdots(1+(k-1)\beta)}{k!} h^k.$$

Here is an example of a fundamental SDSE:

$$\begin{cases} X_1 &= B_1^+\left(f_{\beta_1}(X_1) f_{\frac{\beta_2}{1+\beta_2}}((1+\beta_2)X_2)(1-X_3)^{-1}(1-X_4)^{-1}\right), \\ X_2 &= B_2^+\left(f_{\frac{\beta_1}{1+\beta_1}}(X_1) f_{\beta_2}(X_2)(1-X_3)^{-1}(1-X_4)^{-1}\right), \\ X_3 &= B_3^+\left(f_{\frac{\beta_1}{1+\beta_1}}((1+\beta_1)X_1) f_{\frac{\beta_2}{1+\beta_2}}((1+\beta_2)X_2)(1-X_4)^{-1}\right), \\ X_4 &= B_4^+\left(f_{\frac{\beta_1}{1+\beta_1}}((1+\beta_1)X_1) f_{\frac{\beta_2}{1+\beta_2}}((1+\beta_2)X_2)(1-X_3)^{-1}\right), \\ X_5 &= B_5^+\left(f_{\frac{\beta_1}{1+\beta_1}}((1+\beta_1)X_1) f_{\frac{\beta_2}{1+\beta_2}}((1+\beta_2)X_2)(1-X_3)^{-1}(1-X_4)^{-1}\right), \end{cases}$$

The associated oriented graph is:



The present paper is devoted to the description of the Hopf algebras $\mathcal{H}_{(S)}$. By the Cartier-Quillen-Milnor-Moore theorem, they are dual of enveloping algebra $\mathcal{U}(\mathfrak{g}_{(S)})$, and it turns out that $\mathfrak{g}_{(S)}$ is a pre-Lie algebra [4], that is to say it has a bilinear product $\star$ such that for all $f, g, h \in \mathfrak{g}_{(S)}$:

$$(f \star g) \star h - f \star (g \star h) = (g \star f) \star h - g \star (f \star h).$$

In our case, $\mathfrak{g}_{(S)}$ has a basis $(f_i(k))_{i \in I, k \geq 1}$ and by proposition 10 its pre-Lie product is given by:

$$f_j(l) \star f_i(k) = \lambda_k^{(i,j)} f_i(k+l),$$

where the coefficients $\lambda_k^{(i,j)}$ are described in proposition 8; the Lie bracket of $\mathfrak{g}_{(S)}$ is the antisymmetrisation of $\star$. The product $\star$ can be associative, for example in the multicyclic case. Then, up to a change of variables, $f_j(l) \star f_i(k) = f_i(k+l)$ if there is an oriented path of length k from i to j in the oriented graph associated to (S), or 0 otherwise; see proposition 15. The associative algebra $\mathfrak{g}_{(S)}$ can then be described using the graph $G_{(S)}$ associated to the studied SDSE.

The fundamental case is separated into two subcases. In the non-abelian case, the Lie algebra $\mathfrak{g}_{(S)}$ is described as an iterated semi-direct product of the Faà di Bruno Lie algebra by infinite dimensional modules; see theorems 20 and 21. Similarly, the character group of $\mathcal{H}_{(S)}$ is an iterated semi-direct product of the Faà di Bruno group of formal diffeomorphisms by modules of formal series:

$$Ch(\mathcal{H}_{(S)}) = G_m \rtimes (G_{m-1} \rtimes (\cdots G_2 \rtimes (G_1 \rtimes G_0) \cdots),$$

where G_0 is the Faà di Bruno group and $G_1, \ldots, G_{m-1}$ are isomorphic to direct sums of $(tK[[t]], +)$ as groups; see theorem 23. The second subcase is similar, replacing the Faà di Bruno Lie algebra by an abelian Lie algebra; see theorems 27 and 28 for the Lie algebra, and theorem 30 for the group of characters.

This text is organised as follows: the first section gives some recalls on the structure of Hopf algebra of $\mathcal{H}_{\mathcal{D}}$ and on the pre-Lie product on $\mathfrak{g}_{(S)} = Prim\left(\mathcal{H}^*_{(S)}\right)$. In the second section are recalled the definitions and properties of SDSE. The following section introduces the coefficients $\lambda_n^{(i,j)}$ and their properties, especially their link with the pre-Lie product of $\mathfrak{g}_{(S)}$. The next three sections deals with the description of the Lie algebra $\mathfrak{g}_{(S)}$ and the group $Ch\left(\mathcal{H}_{(S)}\right)$ when $\mathfrak{g}_{(S)}$ is associative, in the non-abelian, fundamental case and finally in the abelian, fundamental case.

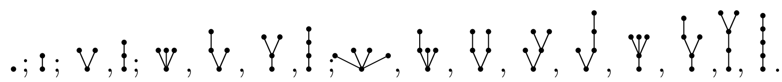
Notations. We denote by K a commutative field of characteristic zero. All vector spaces, algebras, coalgebras, Hopf algebras, etc. will be taken over K.

1 Preliminaries

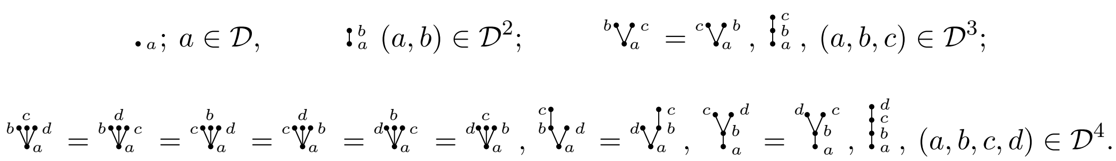
1.1 Hopf algebras of decorated rooted trees

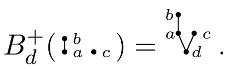
Let $\mathcal{D}$ be a non-empty set. We denote by $\mathcal{H}_{\mathcal{D}}$ the polynomial algebra generated by the set $\mathcal{T}_{\mathcal{D}}$ of rooted trees decorated by elements of $\mathcal{D}$. For example:

1. Rooted trees with 1, 2, 3,4 or 5 vertices:



2. Rooted trees decorated by $\mathcal{D}$ with 1, 2, 3 or 4 vertices:



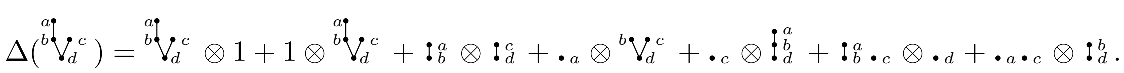
Let $t_1, \ldots, t_n$ be elements of $\mathcal{T}_{\mathcal{D}}$ and let $d \in \mathcal{D}$. We denote by $B_d^+(t_1 \ldots t_n)$ the rooted tree obtained by grafting $t_1, \ldots, t_n$ on a common root decorated by d. This map B_d^+ is extended in an operator from $\mathcal{H}_{\mathcal{D}}$ to $\mathcal{H}_{\mathcal{D}}$. For example, .

In order to make $\mathcal{H}_{\mathcal{D}}$ a bialgebra, we now introduce the notion of cut of a tree $t \in \mathcal{T}_{\mathcal{D}}$. A *non-total cut* c of a tree t is a choice of edges of t. Deleting the chosen edges, the cut makes t into a forest denoted by $W^c(t)$. The cut c is *admissible* if any oriented path in the tree meets at most one cut edge. For such a cut, the tree of $W^c(t)$ which contains the root of t is denoted by $R^c(t)$ and the product of the other trees of $W^c(t)$ is denoted by $P^c(t)$. We also add the total cut, which is by convention an admissible cut such that $R^c(t) = 1$ and $P^c(t) = W^c(t) = t$. The set of admissible cuts of t is denoted by $Adm_*(t)$. Note that the empty cut of t is admissible; we put $Adm(t) = Adm_*(t) - \{\text{empty cut, total cut}\}$.

The coproduct of $\mathcal{H}_{\mathcal{D}}$ is defined as the unique algebra morphism from $\mathcal{H}_{\mathcal{D}}$ to $\mathcal{H}_{\mathcal{D}} \otimes \mathcal{H}_{\mathcal{D}}$ such that for all rooted tree $t \in \mathcal{T}_{\mathcal{D}}$:

$$\Delta(t) = \sum_{c \in Adm_*(t)} P^c(t) \otimes R^c(t) = t \otimes 1 + 1 \otimes t + \sum_{c \in Adm(t)} P^c(t) \otimes R^c(t).$$

Example.



We grade $\mathcal{H}_\mathcal{D}$ by declaring the forests with n vertices homogeneous of degree n. We denote by $\mathcal{H}_\mathcal{D}(n)$ the homogeneous component of $\mathcal{H}_\mathcal{D}$ of degree n. Then $\mathcal{H}_\mathcal{D}$ is a graded bialgebra. The *completion* $\widehat{\mathcal{H}_\mathcal{D}}$ of $\mathcal{H}_\mathcal{D}$ is the vector space:

$$\widehat{\mathcal{H}_\mathcal{D}}=\prod_{n\in\mathbb{N}}\mathcal{H}_\mathcal{D}(n).$$

The elements of $\widehat{\mathcal{H}_\mathcal{D}}$ will be denoted by $\sum x_n$, where $x_n\in\mathcal{H}_\mathcal{D}(n)$ for all $n\in\mathbb{N}$.

Let $f(h)=\sum p_nh^n\in K[[h]]$ be any formal series, and let $X=\sum x_n\in\widehat{\mathcal{H}_\mathcal{D}}$, such that $x_0=0$. The series of $\widehat{\mathcal{H}_\mathcal{D}}$ of terms p_nX^n is Cauchy, so converges. Its limit will be denoted by $f(X)$. In other words, $f(X)=\sum y_n$, with:

$$\left\{\begin{array}{rcl} y_0&=&p_0,\\ y_n&=&\displaystyle\sum_{k=1}^n\sum_{a_1+\cdots+a_k=n}p_kx_{a_1}\cdots x_{a_k}\text{ if }n\geq 1.\end{array}\right.$$

1.2 Pre-Lie structure on the dual of $\mathcal{H}_\mathcal{D}$

By the Cartier-Quillen-Milnor-Moore theorem [20], the graded dual $\mathcal{H}_\mathcal{D}^*$ of $\mathcal{H}_\mathcal{D}$ is an enveloping algebra. Its Lie algebra $Prim(\mathcal{H}_\mathcal{D}^*)$ has a basis $(f_t)_{t\in\mathcal{T}_\mathcal{D}}$ indexed by $\mathcal{T}_D$:

$$f_t:\left\{\begin{array}{rcl}\mathcal{H}_\mathcal{D}&\longrightarrow&K\\ t_1\ldots t_n&\longrightarrow&\left\{\begin{array}{l}0\text{ if }n\neq 1,\\ \delta_{t,t_1}\text{ if }n=1.\end{array}\right.\end{array}\right.$$

Recall that a pre-Lie algebra (or equivalently a Vinberg algebra or a left-symmetric algebra) is a couple $(A,\star)$, where $\star$ is a bilinear product on A such that for all $x,y,z\in A$:

$$(x\star y)\star z-x\star(y\star z)=(y\star x)\star z-y\star(x\star z).$$

Pre-Lie algebras are Lie algebras, with bracket given by $[x,y]=x\star y-y\star x$.

The Lie bracket of $Prim(\mathcal{H}_\mathcal{D}^*)$ is induced by a pre-Lie product $\star$ given in the following way: if $f,g\in Prim(\mathcal{H}_\mathcal{D}^*)$, $f\star g$ is the unique element of $Prim(\mathcal{H}_\mathcal{D}^*)$ such that for all $t\in\mathcal{T}_\mathcal{D}$,

$$(f\star g)(t)=(f\otimes g)\circ(\pi\otimes\pi)\circ\Delta(t),$$

where π is the projection on $Vect(\mathcal{T}^\mathcal{D})$ which vanishes on the forests which are not trees. In other words, if $t,t'\in\mathcal{T}_\mathcal{D}$:

$$f_t\star f_{t'}=\sum_{t''\in\mathcal{T}_\mathcal{D}}n(t,t';t'')f_{t''},$$

where $n(t,t';t')$ is the number of admissible cuts c of t'' such that $P^c(t'')=t$ and $R^c(t'')=t'$. It is proved that $(prim(\mathcal{H}_\mathcal{D}^*),\star)$ is the free pre-Lie algebra generated by the $\bullet_d$'s, $d\in\mathcal{D}$: see [3, 4].

Note. The Hopf algebra $\mathcal{H}_\mathcal{D}^*$ is isomorphic to the Grossman-Larson Hopf algebra of rooted trees [11, 12, 13].

2 Recalls on SDSE

2.1 Unique solution of an SDSE

Definition 1 Let I be a finite, non-empty set, and let $F_i\in K[[h_j,j\in I]]$ be a non-constant formal series for all $i\in I$. The *system of Dyson-Schwinger combinatorial equations* (briefly, the SDSE) associated to $(F_i)_{i\in I}$ is:

$$\forall i\in I,\ X_i=B_i^+(f_i(X_j,j\in I)),$$

where $X_i\in\widehat{\mathcal{H}_I}$ for all $i\in I$.

In order to ease the notation, we shall often assume that $I = \{1, \ldots, N\}$ in the proofs, without loss of generality.

Notations. We assume here that $I = \{1, \ldots, N\}$.

1. Let (S) be an SDSE. We shall denote, for all $i \in I$, $F_i = \displaystyle\sum_{p_1,\cdots,p_N} a^{(i)}_{(p_1,\cdots,p_N)} h_1^{p_1} \cdots h_N^{p_N}$.

2. Let $1 \leq i, j \leq N$. We denote by $a^{(i)}_j$ the coefficient of h_j in F_i.

Proposition 2 *Let (S) be an SDSE. Then it admits a unique solution $(X_i)_{i\in I} \in \left(\widehat{\mathcal{H}_I}\right)^I$. We put $X_i = \displaystyle\sum_{t\in\mathcal{T}_I^{(i)}} a_t t$.*

Definition 3 Let (S) be an SDSE and let $X = (X_i)_{i\in I}$ be its unique solution. The subalgebra of $\mathcal{H}_I$ generated by the homogeneous components $X_i(k)$'s of the X_i's will be denoted by $\mathcal{H}_{(S)}$. If $\mathcal{H}_{(S)}$ is Hopf, the system (S) will be said to be Hopf.

We proved in [10] the following results:

Proposition 4 (change of variables) *Let (S) be the SDSE associated to $(F_i(h_j, j \in I))_{i\in I}$. Let λ_i and μ_i be non-zero scalars for all $i \in I$. The system (S) is Hopf if, and only if, the SDSE system (S') associated to $(\mu_i F_i(\lambda_j h_j,\ j \in J))_{i\in I}$ is Hopf.*

Moreover, a change of variables replace $\mathcal{H}_{(S)}$ by an isomorphic Hopf algebra.

2.2 Graph associated to an SDSE

We associate a oriented graph to each SDSE in the following way:

Definition 5 Let (S) be an SDSE.

1. We construct an oriented graph $G_{(S)}$ associated to (S) in the following way:
   - The vertices of $G_{(S)}$ are the elements of I.
   - There is an edge from i to j if, and only if, $\dfrac{\partial F_i}{\partial h_j} \neq 0$.

2. If $\dfrac{\partial F_i}{\partial h_i} \neq 0$, the vertex i will be said to be *self-dependent*. In other words, if i is self-dependent, there is a loop from i to itself in $G_{(S)}$.

3. If $G_{(S)}$ is connected, we shall say that (S) is *connected*.

Let (S) be an SDSE and let $G_{(S)}$ be the associated graph. Let i and j be two vertices of $G_{(S)}$. We shall say that j is a direct descendant of i (or i is a direct ascendant of j) if there is an oriented edge from i to j; we shall say that j is a descendant of i (or i is an ascendant of j) if there is an oriented path from i to j. We shall write "$i \longrightarrow j$" for "j is a direct descendant of i".

Remark. An change of variables does not change the graph $G_{(S)}$.

2.3 Classification of SDSE

The following result is proved in [10]:

Theorem 6 *Let (S) be a connected SDSE. It is Hopf if and only if, up to a change of variables, one of the following assertion holds:*

1. *(*Extended multicyclic SDSE*). The set I admits a partition $I = I_{\overline{1}} \cup \cdots \cup I_{\overline{N}}$ indexed by the elements of $\mathbb{Z}/N\mathbb{Z}$, $N \geq 2$, with the following conditions:*
   - *For all $i \in I_{\overline{k}}$:*
   $$F_i = 1 + \sum_{j \in I_{\overline{k+1}}} a_j^{(i)} h_j.$$
   - *If i and i' have a common direct ascendant in $G_{(S)}$, then $F_i = F_{i'}$ (so i and i' have the same direct descendants).*

2. *(*Extended fundamental SDSE*). There exists a partition:*
   $$I = \left(\bigcup_{i \in I_0} J_i\right) \cup \left(\bigcup_{i \in J_0} J_i\right) \cup K_0 \cup I_1 \cup J_1 \cup I_2,$$
   with the following conditions:
   - *K_0, I_1, J_1, I_2 can be empty.*
   - *The set of indices $I_0 \cup J_0$ is not empty.*
   - *For all $i \in I_0 \cup J_0$, J_i is not empty.*

   Up to a change of variables:

   (a) For all $x \in I_0$, there exists $\beta_x \in K$, such that for all $i \in J_x$:
   $$F_i = f_{\beta_x}\left(\sum_{j \in J_x} h_j\right) \prod_{y \in I_0 - \{x\}} f_{\frac{\beta_y}{1+\beta_y}}\left((1+\beta_y)\sum_{j \in J_y} h_j\right) \prod_{y \in J_0} f_1\left(\sum_{j \in J_y} h_j\right).$$

   (b) For all $x \in J_0$, for all $i \in J_x$:
   $$F_i = \prod_{y \in I_0} f_{\frac{\beta_y}{1+\beta_y}}\left((1+\beta_y)\sum_{j \in J_y} h_j\right) \prod_{y \in J_0 - \{x\}} f_1\left(\sum_{j \in J_y} h_j\right).$$

   (c) For all $i \in K_0$:
   $$F_i = \prod_{y \in I_0} f_{\frac{\beta_y}{1+\beta_y}}\left((1+\beta_y)\sum_{j \in J_y} h_j\right) \prod_{y \in J_0} f_1\left(\sum_{j \in J_y} h_j\right).$$

   (d) For all $i \in I_1$, there exist $\nu_i \in K$ and a family of scalars $\left(a_j^{(i)}\right)_{j \in I_0 \cup J_0 \cup K_0}$, with $(\nu_i \neq 1)$ or $(\exists j \in I_0,\ a_j^{(i)} \neq 1 + \beta_j)$ or $(\exists j \in J_0,\ a_j^{(i)} \neq 1)$ or $(\exists j \in K_0,\ a_j^{(i)} \neq 0)$. Then, if $\nu_i \neq 0$:
   $$F_i = \frac{1}{\nu_i} \prod_{y \in I_0} f_{\frac{\beta_y}{\nu_i a_y^{(i)}}}\left(\nu_i a_y^{(i)} \sum_{j \in J_y} h_j\right) \prod_{y \in J_0} f_{\frac{1}{\nu_i a_y^{(i)}}}\left(\nu_i a_y^{(i)} \sum_{j \in J_y} h_j\right) \prod_{j \in K_0} f_0\left(\nu_i a_j^{(i)} h_j\right) + 1 - \frac{1}{\nu_i}.$$

If $\nu_i = 0$:

$$F_i = -\sum_{y\in I_0} \frac{a_y^{(i)}}{\beta_y} \ln\left(1-\sum_{j\in J_y} h_j\right) - \sum_{y\in J_0} a_y^{(i)} \ln\left(1-\sum_{j\in J_y} h_j\right) + \sum_{j\in K_0} a_j^{(i)} h_j + 1.$$

(e) *For all $i \in J_1$, there exists $\nu_i \in K - \{0\}$ and a family of scalars $\left(a_j^{(i)}\right)_{j\in I_0\cup J_0\cup K_0\cup I_1}$, with the three following conditions:*

- $I_1^{(i)} = \{j \in I_1 \,/\, a_j^{(i)} \neq 0\}$ *is not empty.*
- *For all $j \in I_1^{(i)}$, $\nu_j = 1$.*
- *For all $j,k \in I_1^{(i)}$, $F_j = F_k$. In particular, we put $b_t^{(i)} = a_t^{(j)}$ for any $j \in I_1^{(i)}$, for all $t \in I_0 \cup J_0 \cup K_0$.*

Then:

$$\begin{aligned} F_i &= \frac{1}{\nu_i} \prod_{y\in I_0} f_{\frac{\beta_y}{b_y^{(i)}-1-\beta_y}} \left(\left(b_y^{(i)}-1-\beta_y\right)\sum_{j\in J_y} h_j\right) \prod_{y\in J_0} f_{\frac{\beta_y}{b_y^{(i)}-1}} \left(\left(b_y^{(i)}-1\right)\sum_{j\in J_y} h_j\right) \\ &\quad \prod_{j\in K_0} f_0\left(b_j^{(i)} h_j\right) + \sum_{j\in I_1^{(i)}} a_j^{(i)} h_1 + 1 - \frac{1}{\nu_i}. \end{aligned}$$

(f) *$I_2 = \{x_1, \ldots, x_m\}$ and for all $1 \leq k \leq m$, there exist a set:*

$$I^{(x_k)} \subseteq \left(\bigcup_{i\in I_0} J_i\right) \cup \left(\bigcup_{i\in J_0} J_i\right) \cup K_0 \cup I_1 \cup J_1 \cup \{x_1, \ldots, x_{k-1}\}$$

and a family of non-zero scalars $\left(a_j^{(x_k)}\right)_{j\in I^{(x_k)}}$ such that for all $i,j \in I^{(x_k)}$, $F_i = F_j$. Then:

$$F_{x_k} = 1 + \sum_{j\in I^{(x_k)}} a_j^{(x_k)} h_j.$$

The elements of I_2 will be called extension vertices. *If $I_2 = \emptyset$, we shall say that (S) is a* fundamental system.

Definition 7 An extended fundamental Hopf SDSE (S) will be said to be *abelian* if $J_0 = \emptyset$ and if for all $x \in I_0$, $\beta_x = -1$.

3 Structure coefficients of the pre-Lie agebra $\mathfrak{g}_{(S)}$

3.1 Definition of the structure coefficients

We here recall several results of [10].

Proposition 8 *Let (S) be a Hopf SDSE. For all $i,j \in I$, for all $n \geq 1$, there exists a scalar $\lambda_n^{(i,j)}$ such that for all $t' \in \mathcal{T}_i(n)$:*

$$\sum_{t\in\mathcal{T}_i(n+1)} n_j(t,t') a_t = \lambda_n^{(i,j)} a_{t'},$$

where $n_j(t,t')$ is the number of leaves l of t decorated by j such that the cut of l gives t'.

In the case of extended fundamental SDSE, the coefficients $\lambda_n^{(i,j)}$ are given, for all $i,j\notin I_2$, by:

$$\lambda_n^{(i,j)}=\left\{\begin{array}{l} a_j^{(i)} \text{ if } n=1,\\ \tilde{a}_j^{(i)}+b_j(n-1) \text{ if } n\geq 2,\end{array}\right.$$

the coefficients being given in the following arrays:

- $a_i^{(j)}$:

$i\setminus j$	$\in J_y,\ y\in I_0$	$\in J_y,\ y\in J_0$	$\in K_0$	$\in I_1$	$\in J_1$
$\in J_x,\ x\in I_0$	$(1+\beta_x)-\delta_{x,y}\beta_x$	$1+\beta_x$	$1+\beta_x$	$a_x^{(j)}$	$\frac{b_x^{(j)}-1-\beta_x}{\nu_j}$
$\in J_x,\ x\in J_0$	1	$1-\delta_{x,y}$	1	$a_x^{(j)}$	$\frac{b_x^{(j)}-1}{\nu_j}$
$\in K_0$	0	0	0	$a_i^{(j)}$	$\frac{b_i^{(j)}}{\nu_j}$
$\in I_1$	0	0	0	0	$a_i^{(j)}$
$\in J_1$	0	0	0	0	0

- $\tilde{a}_i^{(j)}$:

$i\setminus j$	$\in J_y,\ y\in I_0$	$\in J_y,\ y\in J_0$	$\in K_0$	$\in I_1$	$\in J_1$
$\in J_x,\ x\in I_0$	$(1+\beta_x)-\delta_{x,y}\beta_x$	$1+\beta_x$	$1+\beta_x$	$\nu_j a_x^{(j)}$	$b_x^{(j)}-1-\beta_x$
$\in J_x, x\in J_0$	1	$1-\delta_{x,y}$	1	$\nu_j a_i^{(j)}$	$b_i^{(j)}-1$
$\in K_0$	0	0	0	$\nu_j a_i^{(j)}$	$b_i^{(j)}$
$\in I_1$	0	0	0	0	0
$\in J_1$	0	0	0	0	0

- b_j:

j	$\in J_y,\ y\in I_0$	$\in J_y,\ y\in J_0$	$\in K_0$	$\in I_1$	$\in J_1$
b_j	$1+\beta_y$	1	0	0	0

If $i\notin I_2$ and $j\in I_2$, then $\lambda_n^{(i,j)}=0$ for all $n\geq 1$. Moreover, if $i\in I_2$, let i' be a direct descendant of i. Then for all $j\in I$, for all $n\geq 2$, $\lambda_n^{(i,j)}=\lambda_{n-1}^{(i',j)}$.

3.2 Prelie structure on $\mathcal{H}_{(S)}^*$

Let us consider a Hopf SDSE (S). Then $\mathcal{H}_{(S)}^*$ is the enveloping algebra of the Lie algebra $\mathfrak{g}_{(S)}=Prim\left(\mathcal{H}_{(S)}^*\right)$. By [19], it inherits from $Prim(\mathcal{H}_{\mathcal{D}}^*)$ a pre-Lie product given in the following way: for all $f,g\in G_{(S)}$, for all $x\in\mathcal{H}_{(S)}$, $f\star g$ is the unique element of $\mathfrak{g}_{(S)}$ such that for all $x\in vect(X_i(n)\ /\ i\in I, n\geq 1)$,

$$(f\star g)(x)=(f\otimes g)\circ(\pi\otimes\pi)\circ\Delta(x).$$

Let $(f_i(p))_{i\in I,p\geq 1}$ be the basis of $\mathfrak{g}_{(S)}$, dual of the basis $(X_i(p))_{i\in I,p\geq 1}$. By homogeneity of Δ, and as $\Delta(X_i(n))$ is a linear span of elements $-\otimes X_i(p)$, $0\leq p\leq n$, we obtain the existence of coefficients $a_{k,l}^{(i,j)}$ such that, for all $i,j\in I$, $k,l\geq 1$:

$$f_j(l)\star f_i(k)=a_{k,l}^{(i,j)}f_i(k+l).$$

By duality, $a_{k,l}^{(i,j)}$ is the coefficient of $X_j(l)\otimes X_i(k)$ in $\Delta(X_i(k+l))$, so is uniquely determined in the following way: for all $t'\in\mathcal{T}_{\mathcal{D}}^{(j)}(l)$, $t''\in\mathcal{T}_{\mathcal{D}}^{(i)}(k)$,

$$\sum_{t\in\mathcal{T}_{\mathcal{D}}^{(i)}(k+l)} n(t',t'';t)a_t=a_{k,l}^{(i,j)}a_{t'}a_{t''}.$$

Lemma 9 *For all* $t' \in \mathcal{T}_\mathcal{D}^{(j)}(l)$, $t'' \in \mathcal{T}_\mathcal{D}^{(i)}(k)$, $\displaystyle\sum_{t \in \mathcal{T}_\mathcal{D}^{(i)}(k+l)} n(t',t'';t)a_t = \lambda_k^{(i,j)} a_{t'}a_{t''}$.

Proof. By induction on k. If $k=1$, then $t'' = \bullet_i$, so:

$$\sum_{t \in \mathcal{T}_\mathcal{D}^{(i)}(k+l)} n(t',t'';t)a_t = a_{B_i^+(t'')} = a_j^{(i)} a_{t'} = \lambda_1^{(i,j)} a_{t'}a_{t''},$$

as $a_{t''}=1$. Let us assume the result at all rank $\leq k-1$. We put $t'' = B_i^+(\prod_{s \in \mathcal{T}_\mathcal{D}} s^{r_s})$. We put $p_j = \displaystyle\sum_{s \in \mathcal{T}_\mathcal{D}^{(j)}} r_s$ for all $j \in I$. Then:

$$\begin{aligned}
&\sum_{t \in \mathcal{T}_\mathcal{D}^{(i)}(k+l)} n(t',t'';t)a_t\\
=\;& n\left(t',t'',B_i^+\left(\bullet_j \prod_{s \in \mathcal{T}_\mathcal{D}} s^{r_s}\right)\right) a_{B_i^+(t' \prod s^{r_s})} + \sum_{\substack{s_1,s_2 \in \mathcal{T}_\mathcal{D}\\ r_{s_2}\geq 1}} (r_{s_1}+1) n(t',s_2;s_1) a_{B_i^+\left(\frac{s_1}{s_2}\prod s^{r_s}\right)}\\
=\;& (r_{t'+1}) \frac{(p_j+1)\prod_{j=1}^N p_j!}{(r_{t'+1})\prod_{s \in \mathcal{T}_\mathcal{D}} r_s!} a^{(i)}_{(p_1,\cdots,p_j+1,\cdots,p_N)} a_{t'} \prod_{s \in \mathcal{T}_\mathcal{D}} a_s^{r_s} + \sum_{s_1,s_2 \in \mathcal{T}_\mathcal{D}} (r_{s_1}+1) n_j(s_1,s_2) \frac{r_{s_2}}{r_{s_1}+1} a_{t''} \frac{a_{s_1}}{a_{s_2}}\\
=\;& (p_j+1) \frac{a^{(i)}_{(p_1,\cdots,p_j+1,\cdots,p_N)}}{a^{(i)}_{(p_1,\cdots,p_N)}} a_{t'}a_{t''} + \sum_{s_1,s_2 \in \mathcal{T}_\mathcal{D}} n_j(s_1,s_2) r_{s_2} \frac{a_{s_1}}{a_{s_2}} a_{t''}\\
=\;& \left(\lambda^{(i,j)}_{p_1+\cdots+p_N+1} - \sum_{l=1}^N p_j a_j^{(l)} + \sum_{\substack{s_1,s_2 \in \mathcal{T}_\mathcal{D}\\ r_{s_2}>0}} n_j(s_1,s_2) r_{s_2} \frac{a_{s_1}}{a_{s_2}}\right) a_{t'}a_{t''}\\
=\;& \left(\lambda^{(i,j)}_{p_1+\cdots+p_N+1} - \sum_{l=1}^N p_j a_j^{(l)} + \sum_{s_2 \in \mathcal{T}_\mathcal{D}} r_{s_2} \lambda^{(r(s_2),j)}_{|s_2|}\right) a_{t'}a_{t''},
\end{aligned}$$

using the induction hypothesis on s_2, denoting by $r(s_2)$ the decoration of the root of s_2. As $a_{t'} \neq 0$, $a^{(i)}_{(p_1,\cdots,p_n)} \neq 0$, proposition 19-3 of [10] implies:

$$\begin{aligned}
\lambda^{(i,j)}_{1+\sum r_s|s|} &= \lambda^{(i,j)}_{1+\sum r_s} + \sum_s r_s \left(\lambda^{(r(s),j)}_{|s|} - a_j^{(r(s))}\right)\\
\lambda^{(i,j)}_{|t''|} &= \lambda^{(i,j)}_{p_1+\cdots+p_N+1} + \sum_s r_s \lambda^{(r(s),j)}_{|s|} - \sum_l p_l a_j^{(l)}.
\end{aligned}$$

So the induction hypothesis is proved at rank n. $\Box$

Combining this lemma with the preceding observations:

Proposition 10 *Let (S) be a Hopf SDSE. The pre-Lie algebra $\mathfrak{g}_{(S)} = Prim\left(\mathcal{H}^*_{(S)}\right)$ has a basis $(f_i(k))_{i \in I, k \geq 1}$, and the pre-Lie product of two elements of this basis is given by:*

$$f_j(l) \star f_i(k) = \lambda_k^{(i,j)} f_i(k+l).$$

Remark. Let us consider a fundamental SDSE (S), with $I_1 = J_1 = I_2 = \emptyset$. Combining proposition 10 with the arrays following proposition 8, then for all $i, j \in I$, for all $k, l \geq 1$:

$$[f_j(l), f_i(k)] = b_j k f_i(k+l) - b_i l f_j(k+l).$$

We assume that $I = \{1, \ldots, N\}$. Let $\mathbf{W} = Der(K[x_1^{\pm 1}, \ldots, x_N^{\pm 1}])$ be the Lie algebra of derivations of a Laurent polynomial algebra; $\mathbf{W}$ is a Lie algebra of generalized-Witt type [15]. It is not difficult to show that there is a Lie algebra morphism:

$$\left\{ \begin{array}{rcl} \mathfrak{g}_{(S)} & \longrightarrow & \mathbf{W} \\ f_i(k) & \longrightarrow & b_i (x_1 \ldots x_N)^k x_i \frac{\partial}{\partial x_i}. \end{array} \right.$$

This morphism is injective if, and only if, $b_1, \ldots, b_N \neq 0$. If this holds, $\mathfrak{g}_{(S)}$ can be identified with a Lie subalgebra of the positive part of $\mathbf{W}$.

4 Lie algebra and group associated to $\mathcal{H}_{(S)}$, associative case

Let us consider a connected Hopf SDSE (S). We now study the pre-Lie algebra $\mathfrak{g}_{(S)}$ of proposition 10. We separate this study into three cases:

- *Associative case*: the pre-Lie algebra $\mathfrak{g}_{(S)}$ is associative. This holds in particular if (S) is an extended multicyclic SDSE.
- *Abelian case*: (S) is an extended fundamental, abelian SDSE, see definition 7.
- *Non-abelian case*: (S) is an extended fundamental, non-abelian SDSE.

We first treat the associative case.

4.1 Characterization of the associative case

Proposition 11 *Let (S) be a Hopf SDSE. Then the pre-Lie algebra $\mathfrak{g}_{(S)}$ is associative if, and only if, for all $i \in I$:*

$$F_i = 1 + \sum_{i \longrightarrow j} a_j^{(i)} h_j.$$

Proof. $\Longrightarrow$. Let us assume that $\star$ is associative. Let $i, j, k \in I$, let us show that $a_{j,k}^{(i)} = 0$. If $a_j^{(i)} = 0$ or $a_k^{(i)} = 0$, then $a_{j,k}^{(i)} = 0$. Let us suppose that $a_j^{(i)} \neq 0$ and $a_k^{(i)} \neq 0$. Then:

$$\begin{array}{rcl} 0 & = & (f_k(1) \star f_j(1)) \star f_i(1) - f_k(1) \star (f_j(1) \star f_i(1)) \\ & = & \left(\lambda_1^{(j,k)} \lambda_1^{(i,j)} - \lambda_1^{(i,j)} \lambda_2^{(i,k)} \right) f_i(3) \\ & = & \lambda_1^{(i,j)} \left(\lambda_1^{(j,k)} - \lambda_2^{(i,k)} \right) f_i(3) \\ & = & a_j^{(i)} \left(a_k^{(j)} - \lambda_2^{(i,k)} \right) f_i(3). \end{array}$$

So $\lambda_2^{(i,k)} = a_k^{(j)}$. Moreover, by proposition 8:

$$a_j^{(i)} a_k^{(j)} = \lambda_2^{(i,k)} a_{\begin{smallmatrix} j \\ i \end{smallmatrix}} = a_{\begin{smallmatrix} k \\ j \\ i \end{smallmatrix}} + (1 + \delta_{j,k}) a_{{}^{j}\vee_{i}{}^{k}} = a_j^{(i)} a_k^{(j)} + (1 + \delta_{j,k}) a_{j,k}^{(i)}.$$

So $a_{j,k}^{(i)} = 0$. As a consequence, $F_i = 1 + \sum_{i \longrightarrow j} a_j^{(i)} h_j$.

$\Longleftarrow$. Then $X_i(n)$ is a linear span of ladders of weight n for all $n \geq 1$, for all $i \in I$. As a consequence, if $x \in Vect(X_i(n) \,/\, i \in I, n \geq 1)$, for all $f, g \in \mathfrak{g}_{(S)}$, denoting $\Delta(x) = x' \otimes x''$ and $(\Delta \otimes Id) \circ \Delta(x) = x' \otimes x'' \otimes x'''$:

$$(f \star g)(x) = (f \otimes g) \circ (\pi \otimes \pi) \circ \Delta(x) = (f \otimes g) \circ \Delta(x) = f(x')g(x'').$$

So if $f, g, h \in G_{(S)}$, for all $x \in Vect(X_i(n) \,/\, i \in I, n \geq 1)$:

$$((f \star g) \star h)(x) = f(x')g(x'')h(x''') = (f \star (g \star h))(x).$$

So $(f \star g) \star h = f \star (g \star h)$: $\mathfrak{g}_{(S)}$ is an associative algebra. $\Box$

Corollary 12 *Let (S) be a connected Hopf SDSE. Then $\mathfrak{g}_{(S)}$ is associative if, and only if one of the following assertions holds:*

1. *(S) is an extended multicyclic SDSE.*
2. *(S) is an extended fundamental SDSE, with:*
   - *For all $i \in I_0$, $\beta_i = -1$.*
   - *J_0, K_0, I_1 and J_1 are empty.*

If the second assertion holds, then (S) is also an extended fundamental abelian SDSE, and another interpretation of $\mathfrak{g}_{(S)}$ can be given; see theorem 28.

4.2 An algebra associated to an oriented graph

Notations. Let G an oriented graph, $i, j \in G$, and $n \geq 1$. We shall denote $i \xrightarrow{n} j$ if there is an oriented path from i to j of length n in G.

Definition 13 Let G be an oriented graph, with set of vertices denoted by I. The associative, non-unitary algebra A_G is generated by $P_i(1)$, $i \in I$, and the following relations:

- If j is not a direct descendant of i in G, $P_j(1)P_i(1) = 0$.
- If $i_1 \to i_2 \to \cdots \to i_n$ and $i_1 \to i'_2 \to \cdots \to i'_n$ in G, then:

$$P_{i_n}(1) \cdots P_{i_2}(1)P_{i_1}(1) = P_{i'_n}(1) \cdots P_{i'_2}(1)P_{i_1}(1).$$

Let G be an oriented graph, and let $i \in I$ and $n \geq 1$. For any oriented path $i \to i_2 \to \cdots \to i_n$ in G, we denote $P_i(n) = P_{i_n}(1) \cdots P_{i_2}(1)P_i(1)$. If there is no such an oriented path, we put $P_i(n) = 0$. By definition of A_G (second family of relations), this does not depend of the choice of the path. Graphically, $P_i(n)$ should be seen as representing any path from the vertex i of length n.

Lemma 14 *Let G be an oriented graph. Then the $P_i(n)$'s, $i \in I$, $n \geq 1$, linearly generate A_G. Moreover, if $P_i(m)$ and $P_j(n)$ are non-zero, then:*

$$P_j(n)P_i(m) = \begin{cases} P_i(m+n) \text{ if } i \xrightarrow{m} j, \\ 0 \text{ if not.} \end{cases}$$

Proof. By the first relation, $P_i(n) = P_{i_n}(1) \cdots P_{i_2}(1)P_i(1) = 0$ if $(i, i_1, \ldots, i_n)$ is not an oriented path in G. So the $P_i(n)$'s, $i \in I$, $n \geq 1$, linearly generate A_G.

let us fix $P_i(m) = P_{i_m}(1) \cdots P_{i_2}(1)P_i(1)$ and $P_j(n) = P_{j_n}(1) \cdots P_{j_2}(1)P_j(1)$ both non-zero. If $i \xrightarrow{m} j$ we can choose $i_2, \ldots, i_m$ such that $i \to i_2 \to \cdots \to i_m \to j$. Then:

$$P_j(n)P_i(m) = P_{j_n}(1) \cdots P_{j_2}(1)P_j(1)P_{i_m}(1) \cdots P_{i_2}(1)P_i(1) = P_i(m+n).$$

If this is not the case, then j is not a direct descendant of i_m, so $P_j(1)P_{i_m}(1) = 0$ and $P_j(n)P_i(m) = 0$. $\Box$

Proposition 15 *Let G be an oriented graph.*

1. *The following conditions are equivalent:*

   (a) *The family $(P_i(n))_{i\in I, n\geq 1}$ is a basis of A_G.*

   (b) *All the $P_i(n)$ are non-zero.*

   (c) *The graph G satisfies the following conditions:*

   - *Any vertex of G has a direct descendant.*
   - *If two vertices of G have a common direct ascendant, then they have the same direct descendants.*

   (d) *The SDSE associated to the following formal series is Hopf:*

$$\forall i \in I,\ F_i = 1 + \sum_{i\rightarrow j} h_j.$$

2. *If this holds, then A_G is generated by $P_i(1)$, $i \in I$, and the following relations:*

   - *If j is not a direct descendant of i in G, $P_j(1)P_i(1) = 0$.*
   - *If $i \rightarrow j$ and $i \rightarrow k$ in G, then $P_j(1)P_i(1) = P_k(1)P_i(1)$.*

   The product of A_G is given by:

$$P_j(n)P_i(m) = \left\{ \begin{array}{l} P_i(m+n) \text{ if } i \stackrel{m}{\longrightarrow} j, \\ 0 \text{ if not.} \end{array} \right.$$

   Moreover, if (S) is the system of condition (d), $\mathfrak{g}_{(S)}$ is associative and isomorphic to A_G.

Proof. 1. $(a) \Longrightarrow (b)$ is obvious.

$(b) \Longrightarrow (c)$. Let us assume (b). Then for all $i \in I$, $P_i(2) \neq 0$, so there exists a j such that $i \rightarrow j$ in G: any vertex of G has a direct descendant. Let us assume $i \rightarrow j$ and $i \rightarrow j'$ in G. Let k be a direct descendant of j. Then $P_i(2) = P_j(1)P_i(i) = P_{j'}(1)P_i(1)$ and $P_i(3) = P_k(1)P_j(1)P_i(1) = P_k(1)P_i(2) \neq 0$, so $P_k(1)P_i(2) = P_k(1)P_{j'}(1)P_i(1) \neq 0$. As a consequence, $P_k(1)P_{j'}(1) \neq 0$ and k is a direct descendant of j'. By symmetry, the direct descendants of j' are also direct descendants of j: two direct descendants of a same vertex have the same direct descendants.

$(c) \Longrightarrow (d)$. Then for all $i \in I$, for all $n \geq 1$, $X_i(n) = \sum l(i, i_2, \cdots, i_n)$, where the sum runs on all oriented paths $i \rightarrow i_2 \rightarrow \cdots \longrightarrow i_n$ in $G_{(S)}$. So:

$$\Delta(X_i(n)) = \sum \sum_{k=0}^{n} l(i_{k+1}, \ldots, i_n) \otimes l(i, i_2, \cdots, i_k).$$

If $i \rightarrow i_2 \cdots \rightarrow i_k \rightarrow i_{k+1}$ and $i \rightarrow i'_2 \cdots \rightarrow i'_k \rightarrow i'_{k+1}$, the second condition on G implies that i_3 and i'_3 are direct descendants of i_2 and i'_2,..., i_{k+1} and i'_{k+1} are direct descendants of i_k and i'_k. So:

$$\Delta(X_i(n)) = \sum_{k=0}^{n} \sum_{\substack{i\rightarrow\cdots\rightarrow i_k,\\ i\stackrel{k}{\longrightarrow} i_{k+1},\\ i_{k+1}\rightarrow\cdots\rightarrow i_n}} l(i_{k+1}, \ldots, i_n) \otimes l(i, i_2, \cdots, i_k) = \sum_{k=0}^{n} \sum_{i\stackrel{k}{\longrightarrow} j} X_j(n-k) \otimes X_i(k).$$

So (S) is Hopf.

$(d) \Longrightarrow (a)$. Then, for all $i \in I$, for all $n \geq 1$, $X_i(n) = \sum l(i, i_2, \cdots, i_n)$, where the sum runs on all oriented paths $i \rightarrow i_2 \rightarrow \cdots \longrightarrow i_n$ in $G_{(S)}$. By proposition 11, $\mathfrak{g}_{(S)}$ is associative. Moreover, it is quite immediate to prove that in $\mathfrak{g}_{(S)}$:

- If j is not a direct descendant of i in G, $f_j(1)f_i(1)=0$.
- If $i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_n$ and $i_1 \rightarrow i'_2 \rightarrow \cdots \rightarrow i'_n$ in G, then:

$$f_{i_n}(1)\cdots f_{i_2}(1)f_{i_1}(1)=f_{i'_n}(1)\cdots f_{i'_2}(1)f_{i_1}(1)=f_{i_1}(n).$$

So there is a morphism of algebras from A_G to $\mathfrak{g}_{(S)}$, sending $P_i(1)$ to $f_i(1)$. This morphism sends $P_i(n)$ to $f_i(n)$. As the $f_i(n)$'s are linearly independent, so are the $P_i(n)$'s.

2. Let A'_G be the associative, non-unitary algebra generated by the relations of proposition 15-2. As these relation are immediatly satisfied in A_G, there is a unique morphism of algebras:

$$\Phi:\left\{\begin{array}{rcl} A'_G & \longrightarrow & A_G\\ P_i(1) & \longrightarrow & P_i(1). \end{array}\right.$$

Let $i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_n$ and $i_1 \rightarrow i'_2 \rightarrow \cdots \rightarrow i'_n$ in G. Let us prove that $P_{i_k}(1)\cdots P_{i_2}(1)P_{i_1}(1)=P_{i'_k}(1)\cdots P_{i'_2}(1)P_{i_1}(1)$ in A'_G by induction on k. For $k=2$, this is implied by the second family of relations defining A'_G. Let us assume the result at rank k. Then, both in A_G and A'_G:

$$P_{i_{k+1}}(1)P_{i_k}(1)\cdots P_{i_2}(1)P_{i_1}(1)=P_{i_{k+1}}(1)P_{i'_k}(1)\cdots P_{i'_2}(1)P_{i_1}(1).$$

This is equal to $P_i(k+1)$ in A_G, so is non-zero. As a consequence, $P_{i_{k+1}}(1)P_{i'_k}(1)\neq 0$ in A_G, so $i'_k \rightarrow i_{k+1}$ in G. By definition of A'_G, $P_{i_{k+1}}(1)P_{i'_k}(1)=P_{i'_{k+1}}(1)P_{i'_k}(1)$ in A'_G, so:

$$P_{i_{k+1}}(1)P_{i_k}(1)\cdots P_{i_2}(1)P_{i_1}(1)=P_{i'_{k+1}}(1)P_{i'_k}(1)\cdots P_{i'_2}(1)P_{i_1}(1).$$

So the relations defining A_G are also satisfied in A'_G, so there is a morphism of algebras:

$$\Psi:\left\{\begin{array}{rcl} A_G & \longrightarrow & A'_G\\ P_i(1) & \longrightarrow & P_i(1). \end{array}\right.$$

It is clear that Φ and Ψ are inverse isomorphisms of algebras. $\Box$

Corollary 16 *Let (S) a Hopf SDSE. If $\mathfrak{g}_{(S)}$ is associative, then the graph $G_{(S)}$ satisfies condition (c) of proposition 15 and $\mathfrak{g}_{(S)}$ is isomorphic to $A_{G_{(S)}}$.*

Proof. *First step.* Let i,j,k be vertices of $G_{(S)}$ and $n\geq 1$ such that $i \xrightarrow{n} j$ and $i \xrightarrow{n} k$. Let us prove that $F_j=F_k$ by induction on n. If $n=1$, by proposition 18-3 of [10], $F_j=F_k$. If $n\geq 2$, then there exists vertices of $G_{(S)}$ such that:

$$i\rightarrow j_1\rightarrow \ldots \rightarrow j_{n-1}\rightarrow j, \qquad i\rightarrow k_1\rightarrow \ldots \rightarrow k_{n-1}\rightarrow k.$$

The case $n=1$ implies that $F_{j_1}=F_{k_1}$, so $j_1 \xrightarrow{n-1} j$ and $j_1 \xrightarrow{n-1} k$. By the induction hypothesis, $F_j=F_k$. In other words, if $i \xrightarrow{n} j$ and $i \xrightarrow{n} k$, then $a_l^{(j)}=a_l^{(k)}$ for all $l\in I$.

Second step. Then, for all $i\in I$, for all $n\geq 1$:

$$X_i(n)=\sum a_{i_1}^{(i)}\cdots a_{i_n}^{(i_{n-1})}l(i,i_2,\cdots,i_n),$$

where the sum runs on all oriented paths $i\rightarrow i_2\rightarrow \cdots \longrightarrow i_n$ in $G_{(S)}$. The first step implies that $a_{i_1}^{(i)}\ldots a_{i_n}^{(i_{n-1})}$ depends only of i and n: we denote it by $a_n^{(i)}$. Then:

$$\begin{array}{rcl} X_i(n) & = & \sum a_n^{(i)}l(i,i_2,\cdots,i_n),\\ \Delta(X_i(n)) & = & \displaystyle\sum_{k+l=n}\sum_{i\xrightarrow{l}j}\frac{a_n^{(i)}}{a_l^{(i)}a_k^{(j)}}X_j(k)\otimes X_i(l). \end{array}$$

Dually, putting $p_i(n)=a_n^{(i)}f_i(n)$ for all $1\leq i\leq N$, $n\geq 1$, the pre-Lie product of $\mathfrak{g}_{(S)}$ is given by:

$$\begin{array}{rcl} f_j(n)\star f_i(m) &=& \left\{\begin{array}{l} \dfrac{a_{m+n}^{(i)}}{a_m^{(i)}a_n^{(j)}}f_i(m+n) \text{ if } i\xrightarrow{m} j,\\ 0 \text{ otherwise;}\end{array}\right.\\ p_j(n)\star p_i(m) &=& \left\{\begin{array}{l} p_i(m+n) \text{ if } i\xrightarrow{m} j,\\ 0 \text{ otherwise.}\end{array}\right.\end{array}$$

Last step. It is then clear that the associative algebra $\mathfrak{g}_{(S)}$ is generated by the $p_i(1)$, $i\in I$, and that these elements satisfy the relations defining $A_{G_{(S)}}$. So there is an epimorphism of algebras:

$$\Theta:\left\{\begin{array}{rcl} A_{G_{(S)}} &\longrightarrow& \mathfrak{g}_{(S)}\\ P_i(1) &\longrightarrow& p_i(1).\end{array}\right.$$

This morphism sends $P_i(n)$ to $p_i(n)$ for all $n\geq 1$. As the $p_i(n)$'s are a basis of $A_{G_{(S)}}$, the $P_i(n)$'s are linearly independent in $A_{G_{(S)}}$, so the graph $G_{(S)}$ satisfies condition (c) of proposition 15. Moreover, Θ is an isomorphism. $\Box$

4.3 Group of characters

The non-unitary, associative algebra $\mathfrak{g}_{(S)}$ is graded, with $p_i(k)$ homogeneous of degree k for all $k\geq 1$. Moreover, $\mathfrak{g}_{(S)}(0)=(0)$. The completion $\widehat{\mathfrak{g}_{(S)}}$ is then an associative non-unitary algebra. We add it a unit and obtain an associative unitary algebra $K\oplus\widehat{\mathfrak{g}_{(S)}}$. It is then not difficult to show that the following set is a subgroup of the units of $K\oplus\widehat{\mathfrak{g}_{(S)}}$:

$$G=\left\{1+\sum_{k\geq 1}x_k\mid \forall k\geq 1,\ x_k\in\mathfrak{g}_{(S)}(k)\right\}.$$

Proposition 17 *The group of characters $Ch\left(\mathcal{H}_{(S)}\right)$ is isomorphic to G.*

Proof. We put $V=Vect(X_i(k)|i\in I,k\geq 1)$. Let $g\in V^*$. Then g can be uniquely extended in a map $\widehat{g}$ from $\mathcal{H}_{(S)}$ to K by $g((1)+Ker(\varepsilon)^2)=(0)$, where ε is the counit of $\mathcal{H}_{(S)}$. Moreover, $\widehat{g}\in\widehat{\mathfrak{g}_{(S)}}$. This construction implies a bijection:

$$\Omega:\left\{\begin{array}{rcl} Ch\left(\mathcal{H}_{(S)}\right) &\longrightarrow& G\\ f &\longrightarrow& 1+\widehat{f_{|V}}.\end{array}\right.$$

Let $f_1,f_2\in Ch\left(\mathcal{H}_{(S)}\right)$. For all $x\in V$, we put $\Delta(x)=x\otimes 1+1\otimes x+x'\otimes x''$. As x is a linear span of ladders, $x'\otimes x''\in V\otimes V$. So:

$$\begin{array}{rcl}(f_1.f_2)(x) &=& (f_1\otimes f_2)\circ\Delta(x)\\ &=& f_1(x)+f_2(x)+f_1(x')f_1(x'')\\ &=& f_{1|V}(x)+f_{2|V}(x)+f_{1|V}(x')f_{2|V}(x'')\\ &=& \widehat{f_{1|V}}(x)+\widehat{f_{2|V}}(x)+\widehat{f_{1|V}}(x')\widehat{f_{2|V}}(x'')\\ &=& \widehat{f_{1|V}}(x)+\widehat{f_{2|V}}(x)+\left(\widehat{f_{1|V}}\star\widehat{f_{2|V}}\right)(x).\end{array}$$

So $\widehat{(f_1.f_2)_{|V}}=\widehat{f_{1|V}}+\widehat{f_{2|V}}+\widehat{f_{1|V}}\star\widehat{f_{2|V}}$. This implies that Ω is a group isomorphism. $\Box$

5 Lie algebra and group associated to $\mathcal{H}_{(S)}$, non-abelian case

5.1 Modules over the Faà di Bruno Lie algebra

Let $\mathfrak{g}_{FdB}$ be the Faà di Bruno Lie algebra. Recall that it has a basis $(e(k))_{k\geq 1}$, with bracket given by:

$$[e(k),e(l)]=(l-k)e(k+l).$$

The $\mathfrak{g}_{FdB}$-module V_0 has a basis $(f(k))_{k\geq 1}$, and the action of $\mathfrak{g}_{FdB}$ is given by:

$$e(k).f(l)=lf(k+l).$$

We can then construct a semi-direct product $V_0^M \triangleleft \mathfrak{g}_{FdB}$, described in the following proposition:

Proposition 18 *Let $M\in\mathbb{N}^*$. The Lie algebra $V_0^M \triangleleft \mathfrak{g}_{FdB}$ has a basis:*

$$\left(f^{(i)}(k)\right)_{1\leq i\leq M,\, k\geq 1}\cup (e(k))_{k\geq 1},$$

and its Lie bracket given by:

$$\left\{\begin{array}{rcl}[e(k),e(l)]&=&(l-k)e(k+l),\\ [e(k),f^{(i)}(l)]&=&lf^{(i)}(k+l),\\ [f^{(i)}(k),f^{(j)}(l)]&=&0.\end{array}\right.$$

We now take $\mathfrak{g}=V_0^{\oplus M}\triangleleft\mathfrak{g}_{FdB}$. We define a family of $\mathfrak{g}$-modules. Let $c\in K$ and $\upsilon=(\upsilon_1,\ldots,\upsilon_M)\in K^M$. The module $W_{c,\upsilon}$ has a basis $(g(k))_{k\geq 1}$, and the action of $\mathfrak{g}$ is given by:

$$\left\{\begin{array}{rcl}e(k).g(l)&=&(l+c)g(k+l),\\ f^{(i)}(k).g(l)&=&\upsilon_i g(k+l).\end{array}\right.$$

The semi-direct product is given in the following proposition:

Proposition 19 *Let $\mathfrak{g}$ be the Lie algebra $\left(W_{c_1,\upsilon^{(1)}}\oplus\ldots\oplus W_{c_N,\upsilon^{(N)}}\right)\triangleleft\left(V_0^M\triangleleft\mathfrak{g}_{FdB}\right)$. It has a basis:*

$$\left(g^{(j)}(k)\right)_{1\leq j\leq N,\, k\geq 1}\cup\left(f^{(i)}(k)\right)_{1\leq i\leq M,\, k\geq 1}\cup (e(k))_{k\geq 1},$$

and its bracket is given by:

$$\left\{\begin{array}{rcl}[e(k),e(l)]&=&(l-k)e(k+l),\\ [e(k),f^{(i)}(l)]&=&lf^{(i)}(k+l),\\ [e(k),g^{(i)}(l)]&=&(l+c'_i)g^{(i)}(k+l),\\ [f^{(i)}(k),f^{(j)}(l)]&=&0,\\ [f^{(i)}(k),g^{(j)}(l)]&=&\upsilon_i^{(j)}g^{(j)}(k+l),\\ [g^{(i)}(k),g^{(j)}(l)]&=&0.\end{array}\right.$$

Let us take $\mathfrak{g}$ as in this proposition. We define three families of modules over $\mathfrak{g}$:

1. Let $\nu=(\nu_1,\ldots,\nu_M)\in K^M$. The module $W'_{\nu,0}$ has a basis $(h(k))_{k\geq 1}$, and the action of $\mathfrak{g}$ is given by:

$$\left\{\begin{array}{rcl}e(k).g(l)&=&(l-1)h(k+l),\\ f^{(i)}(k).h(1)&=&\nu_i h(k+1),\\ f^{(i)}(k).h(l)&=&0\text{ if }l\geq 2,\\ g^{(i)}(k).h(l)&=&0.\end{array}\right.$$

2. Let $\nu=(\nu_1,\ldots,\nu_M)\in K^M$. The module $W'_{\nu,1}$ has a basis $(h(k))_{k\geq 1}$, and the action of $\mathfrak{g}$ is given by:
$$\left\{\begin{array}{rcl} e(k).h(1)&=&h(k+1),\\ e(k).h(l)&=&(l-1)h(k+l)\text{ if }l\geq 2,\\ f^{(i)}(k).h(1)&=&\nu_i h(k+1),\\ f^{(i)}(k).h(l)&=&0\text{ if }l\geq 2,\\ g^{(i)}(k).h(l)&=&0.\end{array}\right.$$

3. Let $c\in K$, $\nu=(\nu_1,\ldots,\nu_M)\in K^M$, $\mu=(\mu_1,\ldots,\mu_N)\in K^N$. The module $W''_{c,\nu,\mu}$ has a basis $(h(k))_{k\geq 1}$, and the action of $\mathfrak{g}$ is given by:
$$\left\{\begin{array}{rcl} e(k).h(l)&=&(l+c)h(k+l),\\ f^{(i)}(k).h(l)&=&\nu_i h(k+l),\\ g^{(i)}(k).h(1)&=&\mu_i h(k+1),\\ g^{(i)}(k).h(l)&=&0\text{ if }l\geq 2.\end{array}\right.$$

5.2 Description of the Lie algebra

Theorem 20 *Let us consider a fundamental non-abelian SDSE. Then $\mathfrak{g}_{(S)}$ has the following form:*
$$\mathfrak{g}_{(S)}\approx W\triangleleft\left(\left(W_{c_1,v^{(1)}}\oplus\ldots\oplus W_{c_N,v^{(N)}}\right)\triangleleft\left(V_0^M\triangleleft\mathfrak{g}_{FdB}\right)\right),$$
where W is a direct sum of $W'_{\nu,0}$, $W'_{\nu,1}$ and $W''_{c,\nu,\mu}$.

Proof. *First step.* We first consider a fundamental Hopf SDSE (S) such that $I_1=J_1=I_2=\emptyset$. The set J of the vertices of $G_{(S)}$ admits a partition $J=(J_x)_{x\in I_0}\cup(J_x)_{x\in J_0}\cup(J_x)_{x\in K_0}$. We put:
$$A=\{j\in J\ /\ b_j\neq 0\},\ B=\{j\in J\ /\ b_j=0\}.$$
In other terms, $i\in A$ if, and only if, ($i\in J_x$, with $x\in I_0$ such that $b_x\neq -1$) or ($i\in J_x$, with $x\in J_0$). As we are in the non-abelian case, $A\neq\emptyset$. Let us choose $i_x\in J_x$ for all $x\in I$, and $i_{x_0}\in A$. In order to enlighten the notations, we put $i_0=i_{x_0}$. We define, for all $k\geq 1$:
$$\left\{\begin{array}{rcl} p_{i_0}(k)&=&\dfrac{1}{b_{x_0}}f_{i_0}(k),\\ p_i(k)&=&\dfrac{1}{b_{x_0}}(f_i(k)-f_{i_0}(k))\text{ if }i\in J_{x_0}-\{i_0\},\\ p_{i_x}(k)&=&\dfrac{1}{b_x}f_i(k)-\dfrac{1}{b_{x_0}}f_{i_0}(k)\text{ if }x\neq x_0\text{ and }x\in A,\\ p_{i_x}(k)&=&f_i(k)\text{ if }x\in B,\\ p_i(k)&=&\dfrac{1}{b_x}(f_i(k)-f_{i_x}(k))\text{ if }i\in J_x-\{i_x\},\ x\neq x_0\text{ and }x\in A,\\ p_i(k)&=&f_i(k)-f_{i_x}(k)\text{ if }i\in J_x-\{i_x\},\ x\in B.\end{array}\right.$$

Then direct computations show that the Lie bracket of $\mathfrak{g}_{(S)}$ is given in the following way: for all $k,l\geq 1$,

- $[p_{i_0}(k),p_{i_0}(l)]=(l-k)p_{i_0}(k+l)$.
- For all $i\in I$, $[p_{i_0}(k),p_i(l)]=\left\{\begin{array}{l}(l+d_{x_0})p_i(k+l)\text{ if }i\in J_{x_0}-\{i_0\},\\ lp_i(k+l)\text{ if }i\notin J_{x_0}.\end{array}\right.$
- For all $i\in J_{x_0}-\{i_0\}$, for all $x\neq x_0$, $[p_{i_x}(k),p_i(l)]=\left\{\begin{array}{l}-d_{x_0}p_i(k+l)\text{ if }x\in A,\\ 0\text{ if }x\in B.\end{array}\right.$
- For all $x,x'\in I-\{x_0\}$, $[p_{i_x}(k),p_{i_{x'}}(l)]=0$.

- For all $x,x'\in I-\{x_0\}$, $i\in J_{x'}-\{i_{x'}\}$, $[p_{i_x}(k),p_i(l)]=\left\{\begin{array}{l}0 \text{ if } x\neq x',\\ d_x p_i(k+l) \text{ if } x=x'.\end{array}\right.$

- For all $x,x'\in I-\{x_0\}$, $i\in J_x-\{i_x\}$, $j\in J_{x'}-\{i_{x'}\}$, $[p_i(k),p_j(l)]=0$.

We used the following notations:

$$d_x=\left\{\begin{array}{l}\dfrac{-\beta_x}{1+\beta_x} \text{ if } x\in I_0,\ \beta_x\neq -1,\\ 1 \text{ if } x\in I_0,\ \beta_x=-1,\\ -1 \text{ if } x\in J_0,\\ 0 \text{ if } x\in K_0.\end{array}\right.$$

So the Lie algebra $\mathfrak{g}_{(S)}$ is isomorphic to:

$$\left(W^{|J_{x_0}|-1}_{d_{x_0},(-d_{x_0},\cdots,-d_{x_0},0,\cdots,0)}\oplus\bigoplus_{x\in I-\{x_0\}} W^{|I_x|-1}_{0,(0,\cdots,0,d_x,0,\cdots,0)}\right)\triangleleft\left(V_0^{|I|-1}\triangleleft\mathfrak{g}_{FdB}\right).$$

A basis adapted to this decomposition is:

$$(p_i(k))_{i\in J_{x_0}-\{i_0\},k\geq 1}\cup\left(\bigcup_{x\in I-\{x_0\}}(p_i(k))_{i\in J_x-\{i_x\},k\geq 1}\right)\cup\left(\bigcup_{x\in I-\{x_0\}}(p_{i_x}(k))_{k\geq 1}\right)\cup(p_{i_0}(k))_{k\geq 1}.$$

Second step. We now assume that $I_1\neq\emptyset$. Then the descendants of $j\in I_1$ form a system of the first step, so $\mathfrak{g}_{(S)}=W_{I_1}\triangleleft\mathfrak{g}_{(S_0)}$, where $W_{I_1}=Vect(f_j(k)\ /\ j\in I_1,k\geq 1\}$ and (S_0) is a restriction of (S) as in the first step. Let us fix $j\in I_1$ and let us consider the $\mathfrak{g}_{(S_0)}$-module $W_j=Vect(f_j(k)\ /\ k\geq 1)$. With the notations of the preceding step:

- $[p_{i_0}(k),f_j(l)]=\left(l-1+\frac{a^{(j)}_{i_0}}{b_{x_0}}\right)f_j(k+l)$ if $l=1$.

- $[p_{i_0}(k),f_j(l)]=\left(l-1+\nu_j\frac{a^{(j)}_{i_0}}{b_{x_0}}\right)f_j(k+l)$ if $l\geq 2$.

- $[p_{i_x}(k),f_j(l)]=\left(\frac{a^{(j)}_{i_x}}{b_x}-\frac{a^{(j)}_{i_0}}{b_{x_0}}\right)f_j(k+l)$ if $l=1$, $x\in A$.

- $[p_{i_x}(k),f_j(l)]=\nu_j\left(\frac{a^{(j)}_{i_x}}{b_x}-\frac{a^{(j)}_{i_0}}{b_{x_0}}\right)f_j(k+l)$ if $l\geq 2$, $x\in A$.

- $[p_{i_x}(k),f_j(l)]=a^{(j)}_{i_x}f_j(k+l)$ if $l=1$, $x\in B$.

- $[p_{i_x}(k),f_j(l)]=\nu_j a^{(j)}_{i_x}f_j(k+l)$ if $l\geq 2$, $x\in B$.

- $[p_i(x),f_j(l)]=0$ if i is not a i_x.

If $\nu_j\neq 0$, we put $p_j(k)=f_j(k)$ if $k\geq 2$ and $p_j(1)=\nu_j f_j(1)$. Then, for all l:

- $[p_{i_0}(k),p_j(l)]=\left(l-1+\nu_j\frac{a^{(j)}_{i_0}}{b_{x_0}}\right)p_j(k+l)$.

- $[p_{i_x}(k),p_j(l)]=\nu_j\left(\frac{a^{(j)}_{i_x}}{b_x}-\frac{a^{(j)}_{i_0}}{b_{x_0}}\right)p_j(k+l)$ if $x\in A$.

- $[p_{i_x}(k),p_j(l)]=\nu_j a^{(j)}_{i_x}p_j(k+l)$ if $x\in B$.

- $[p_i(x),p_j(l)]=0$ if i is not a i_x.

So W_j is a module $W_{c,\upsilon}$. If $\nu_j = 0$ and $a_{i_0}^{(j)} \neq 0$, we put $p_j(k) = f_j(k)$ if $k \geq 2$ and $p_j(1) = \frac{b_{x_0}}{a_{i_0}^{(j)}} f_j(1)$. Then:

- $[p_{i_0}(k), p_j(l)] = p_j(k+l)$ if $l = 1$.
- $[p_{i_0}(k), p_j(l)] = (l-1)p_j(k+l)$ if $l \geq 2$.
- $[p_{i_x}(k), f_j(l)] = \left(\frac{a_{i_x}^{(j)}}{b_x} - \frac{a_{i_0}^{(j)}}{b_{x_0}} \right) f_j(k+l)$ if $l = 1$, $x \in A$.
- $[p_{i_x}(k), f_j(l)] = 0$ if $l \geq 2$, $x \in A$.
- $[p_{i_x}(k), f_j(l)] = a_{i_x}^{(j)} f_j(k+l)$ if $l = 1$, $x \in B$.
- $[p_{i_x}(k), f_j(l)] = 0$ if $l \geq 2$, $x \in B$.
- $[p_i(x), p_j(l)] = 0$ if i is not a i_x.

So W_j is a module $W'_{\nu,1}$. If $\nu_j = 0$ and $a_{i_0}^{(j)} = 0$, we put $p_j(k) = f_j(k)$ for all $k \geq 1$. Then:

- $[p_{i_0}(k), p_j(l)] = (l-1)p_j(k+l)$.
- $[p_{i_x}(k), f_j(l)] = \left(\frac{a_{i_x}^{(j)}}{b_x} - \frac{a_{i_0}^{(j)}}{b_{x_0}} \right) f_j(k+l)$ if $l = 1$, $x \in A$.
- $[p_{i_x}(k), f_j(l)] = 0$ if $l \geq 2$, $x \in A$.
- $[p_{i_x}(k), f_j(l)] = a_{i_x}^{(j)} f_j(k+l)$ if $l = 1$, $x \in B$.
- $[p_{i_x}(k), f_j(l)] = 0$ if $l \geq 2$, $x \in B$.
- $[p_i(x), p_j(l)] = 0$ if i is not a i_x.

So W_j is a module $W'_{\nu,0}$.

Last step. We now consider vertices in J_1. If $j \in J_1$, then its descendants are vertices of the first step and i elements of I_1 such that $\nu_i = 1$. As before, $\mathfrak{g}_{(S)} = W_{J_1} \triangleleft \mathfrak{g}_{(S_1)}$, where $W_{J_1} = Vect(f_j(k) \,/\, j \in J_1, k \geq 1\}$ and (S_1) is a restriction of (S) as in the second step. Let us fix $j \in J_1$ and let us consider the $\mathfrak{g}_{(S_1)}$-module $W_j = Vect(f_j(k) \,/\, k \geq 1)$. As $\nu_j \neq 0$, putting $p_j(k) = f_j(k)$ if $k \geq 2$ and $p_j(1) = \nu_j f_j(1)$:

- $[p_{i_0}(k), p_j(l)] = \left(l - 1 + \nu_j \frac{a_{i_0}^{(j)}}{b_{x_0}} \right) p_j(k+l)$.
- $[p_{i_x}(k), p_j(l)] = \nu_j \left(\frac{a_{i_x}^{(j)}}{b_x} - \frac{a_{i_0}^{(j)}}{b_{x_0}} \right) p_j(k+l)$ if $x \in A$.
- $[p_{i_x}(k), p_j(l)] = \nu_j a_{i_x}^{(j)} p_j(k+l)$ if $x \in B$.
- $[p_i(k), p_j(l)] = \nu_j a_i^{(j)} p_j(k+l)$ if $l = 1$, $i \in I_1$, with $\nu_i = 1$.
- $[p_i(k), p_j(l)] = 0$ if $l \geq 2$, $i \in I_1$.
- $[p_i(x), p_j(l)] = 0$ if $i \notin I_1$ and is not a i_x.

So W_j is a module $W''_{c,\nu,\mu}$. $\square$

Theorem 21 *Let (S) be a connected, extended, fundamental, non-abelian SDSE. Then the Lie algebra $\mathfrak{g}_{(S)}$ is of the form:*

$$\mathfrak{g}_m \triangleleft (\mathfrak{g}_{m-1} \triangleleft (\cdots \mathfrak{g}_2 \triangleleft (\mathfrak{g}_1 \triangleleft \mathfrak{g}_0) \cdots),$$

where $\mathfrak{g}_0$ is the Lie algebra associated to the restriction of (S) to the vertices which are not extension vertices (so $\mathfrak{g}_0$ is described in theorem 20) and, for $j \geq 1$, $\mathfrak{g}_j$ is an abelian $(\mathfrak{g}_{j-1} \triangleleft (\cdots \mathfrak{g}_2 \triangleleft (\mathfrak{g}_1 \triangleleft \mathfrak{g}_0) \cdots)$-module having a basis $(h^{(j)}(k))_{k\geq 1}$.

Proof. The Lie algebra $\mathfrak{g}_j$ is the Lie algebra $Vect(f_{x_j}(k) \,/\, k \geq 1)$, where $J_2 = \{x_1, \ldots, x_m\}$, with the notations of theorem 6. □

5.3 Associated group

Let us now consider the character group $Ch\left(\mathcal{H}_{(S)}\right)$ of $\mathcal{H}_{(S)}$. In the preceding cases, $\mathfrak{g}_{(S)}$ contains a sub-Lie algebra isomorphic to the Faà di Bruno Lie algebra, so $Ch\left(\mathcal{H}_{(S)}\right)$ contains a subgroup isomorphic to the Faà di Bruno subgroup:

$$G_{FdB} = \{x + a_1 x^2 + a_2 x^3 + \cdots \mid \forall i,\ a_i \in K\},$$

with the product defined by $A(x).B(x) = B \circ A(x)$. Moreover, each modules earlier defined on $\mathfrak{g}_{FdB}$ corresponds to a module over G_{FdB} by exponentiation:

Definition 22

1. The module $\mathbb{V}_0$ is isomorphic to $yK[[y]]$ as a vector space, and the action of G_{FdB} is given by:
$$A(x).P(y) = P \circ A(y).$$
2. Let $G = \left(\mathbb{V}_0^{\oplus M}\right) \rtimes G_{FdB}$. Let $c \in K$, and $\upsilon = (\upsilon_1, \cdots, \upsilon_M) \in K^M$. Then $\mathbb{W}_{c,\upsilon}$ is $zK[[z]]$ as a vector space, and the action of G is given by:
$$(P_1(y), \cdots, P_M(y), A(x)).Q(z) = exp\left(\sum_{i=1}^{M} \upsilon_i P_i(z)\right)\left(\frac{A(z)}{z}\right)^c Q \circ A(z).$$
3. Let us consider the semi-direct product $G = \left(\mathbb{W}_{c_1,\varepsilon^{(1)}} \oplus \cdots \oplus \mathbb{W}_{c_N,\varepsilon^{(N)}}\right) \triangleleft \left(\mathbb{V}_0^{\oplus M} \triangleleft G_{FdB}\right)$.
   (a) Let $\nu = (\nu_1, \cdots, \nu_M) \in K^M$. Then $\mathbb{W}'_{\nu,0}$ is $tK[[t]]$ as a vector space, and for all $X = (Q_1(z), \cdots, Q_N(z), P_1(y), \cdots, P_M(y), A(x)) \in G$:
$$\begin{aligned} X.t &= \left(1 + \sum_{i=1}^{M} \nu_i P_i(t)\right) t, \\ X.R(t) &= \left(\frac{t}{A(t)}\right) R \circ A(t), \end{aligned}$$
   for all $R(t) \in t^2 K[[t]]$.
   (b) Let $\nu = (\nu_1, \cdots, \nu_M) \in K^M$. Then $\mathbb{W}'_{\nu,1}$ is $tK[[t]]$ as a vector space, and for all $X = (Q_1(z), \cdots, Q_N(z), P_1(y), \cdots, P_M(y), A(x)) \in G$:
$$\begin{aligned} X.t &= \left(1 + \sum_{i=1}^{M} \nu_i P_i(t)\right)\left(t + t\ln\left(\frac{A(t)}{t}\right)\right), \\ X.R(t) &= \left(\frac{t}{A(t)}\right) R \circ A(t), \end{aligned}$$
   for all $R(t) \in t^2 K[[t]]$.

(c) Let $c \in K$, $\nu = (\nu_1, \cdots, \nu_M) \in K^M$, $\mu = (\mu_1, \ldots, \mu_N) \in K^N$. Then $\mathbb{W}''_{c,\nu,\mu}$ is $tK[[t]]$ as a vector space, and for all $X = (Q_1(z), \cdots, Q_N(z), P_1(y), \cdots, P_M(y), A(x)) \in G$:

$$\begin{array}{rcl} X.t & = & \left(\frac{A(t)}{t}\right)^c exp\left(\sum_{i=1}^M \mu_i P_i(t)\right)\left(1+\sum_{i=1}^M \mu_i Q_i(t)\right) A(t),\\ X.R(t) & = & \left(\frac{t}{A(t)}\right)^c exp\left(\sum_{i=1}^M \mu_i P_i(t)\right) R \circ A(t), \end{array}$$

for all $R(t) \in t^2K[[t]]$.

Direct computations prove that they are indeed modules.

Theorem 23 *Let (S) be a connected Hopf SDSE in the non-abelian, fundamental case. Then the group $Ch\left(\mathcal{H}_{(S)}\right)$ is of the form:*

$$G_m \rtimes (G_{m-1} \rtimes (\cdots G_2 \rtimes (G_1 \rtimes G_0)\cdots)),$$

where G_0 is a semi-direct product of the form:

$$G_0 = \mathbb{W}' \rtimes (\mathbb{W} \rtimes (\mathbb{V} \rtimes G_{FdB})),$$

where $\mathbb{V}$ is a direct sum of modules $\mathbb{V}_0$, $\mathbb{W}$ a direct sum of modules $\mathbb{W}_{c,\upsilon}$, and $\mathbb{W}'$ a direct sum of modules $\mathbb{W}'_{\nu,0}$, $\mathbb{W}'_{\nu,1}$ and $\mathbb{W}''_{c,\nu,\mu}$. Moreover, for all $m \geq 1$, $G_m = (tK[[t]], +)$ as a group.

Proof. The group $Ch\left(\mathcal{H}_{(S)}\right)$ is isomorphic to the group of characters of $\mathcal{U}(\mathfrak{g})^*$, where $\mathfrak{g}$ is described in theorem 21. This implies that this group has a structure of semi-direct product as described in theorem 23. Let us consider the Hopf algebra $\mathcal{H}$ of coordinates of G_0. It is a graded Hopf algebra, and direct computations prove that its graded dual is the enveloping algebra of $\mathfrak{g}_0$ of theorem 21. So $\mathcal{H}$ is isomorphic to $\mathcal{H}_{(S_0)}$. $\Box$

6 Lie algebra and group associated to $\mathcal{H}_{(S)}$, abelian case

We now treat the abelian case. Recall that in this case, $J_0 = K_0 = \emptyset$ and, for all $i \in I_0$, $\beta_i = -1$.

6.1 Modules over an abelian Lie algebra

Let $\mathfrak{g}_{ab}$ be an abelian Lie algebra, with basis $\left(e^{(i)}(k)\right)_{1\leq i\leq M, k\geq 1}$. We define a family of modules over this Lie algebra:

Definition 24 Let $\upsilon = (\upsilon_1, \cdots, \upsilon_M) \in K^M$. Then V_υ has a basis $(f(k))_{k\geq 1}$, and the action of $\mathfrak{g}_{ab}$ is given by:

$$e^{(i)}(k).f(l) = \upsilon_i f(k+l).$$

We can then describe the semi-direct product:

Proposition 25 *Let $\mathfrak{g}$ be the Lie algebra $\left(\bigoplus_{i=1}^N V_{\upsilon^{(i)}}\right) \triangleleft \mathfrak{g}_{ab}$. It has a basis:*

$$(e^{(i)}(k))_{1\leq i\leq M, k\geq 1} \cup (f^{(i)}(k))_{1\leq i\leq N, k\geq 1},$$

and its Lie bracket is given by:

$$\left\{\begin{array}{rcl} [e^{(i)}(k), e^{(j)}(l)] & = & 0,\\ [e^{(i)}(k), f^{(j)}(l)] & = & \upsilon_i^{(j)} f^{(j)}(k+l),\\ [f^{(i)}(k), f^{(j)}(l)] & = & 0. \end{array}\right.$$

We now define two families of modules over such a Lie algebra.

Definition 26 Let $\mathfrak{g}$ be a Lie algebra of proposition 25.

1. Let $\nu = (\nu_1, \ldots, \nu_M) \in K^M$. The module W_ν has a basis $(g(k))_{k\geq 1}$, and the action of $\mathfrak{g}$ is given by:
$$\left\{\begin{array}{rcl} e^{(i)}(k).g(1) & = & \nu_i g(k+1), \\ e^{(i)}(k).g(l) & = & 0 \text{ if } l \geq 2, \\ f^{(i)}(k).g(l) & = & 0. \end{array}\right.$$

2. Let $\nu = (\nu_1, \ldots, \nu_M) \in K^M$ and $\mu = (\mu_1, \ldots, \mu_N) \in K^N$, such that for all $1 \leq i \leq M$, for all $1 \leq j \leq N$, $\mu_j \left(\nu_i - \upsilon_i^{(j)}\right) = 0$. The module $W'_{\nu,\mu}$ has a basis $(g(k))_{k\geq 1}$, and the action of $\mathfrak{g}$ is given by:
$$\left\{\begin{array}{rcl} e^{(i)}(k).g(l) & = & \nu_i g(k+l), \\ f^{(j)}(k).g(1) & = & \mu_j g(k+1), \\ f^{(j)}(k).g(l) & = & 0 \text{ if } l \geq 2. \end{array}\right.$$

Remark. The condition $\mu_j \left(\nu_i - \upsilon_i^{(j)}\right) = 0$ is necessary for $W'_{\nu,\mu}$ to be a $\mathfrak{g}$-module. Indeed:
$$\begin{array}{rcl} [e^{(i)}(k), f^{(j)}(l)].g(1) & = & \upsilon_i^{(j)} \mu_j g(k+l+1), \\ e^{(i)}(k).\left(f^{(j)}(l).g(1)\right) - f^{(j)}(l).\left(e^{(i)}(k).g(1)\right) & = & \mu_j \nu_i g(k+l+1). \end{array}$$

6.2 Description of the Lie algebra

We here consider a connected Hopf SDSE (S) in the abelian case.

Theorem 27 *Let us consider a Hopf SDSE of abelian fundamental type, with no extension vertices. Then $\mathfrak{g}_{(S)}$ has the following form:*
$$\mathfrak{g}_{(S)} \approx W \lhd \left((V_{\upsilon^{(1)}} \oplus \ldots \oplus V_{\upsilon^{(N)}}) \lhd \mathfrak{g}_{ab}\right),$$
where W is a direct sum of W_ν and $W'_{\nu,\mu}$.

Proof. *First step.* We first consider an abelian Hopf SDSE such that $J_0 = K_0 = I_1 = J_1 = I_2 = \emptyset$. For all $x \in I_0$, let us fix $i_x \in J_x$. We put $p_{i_x}(k) = f_{i_x}(k)$ and $p_i(k) = f_i(k) - f_{i_x}(k)$ if $i \in J_x - \{i_x\}$. Then direct computations show that:

- $[p_{i_x}(k), p_{i_{x'}(l)}] = 0$.
- $[p_{i_x}(k), p_j(l)] = \delta_{x,x'} p_j(k+l)$ if $j \in J_{x'} - \{i_{x'}\}$.
- $[p_i(k), p_j(l)] = 0$ if i, j are not i_x's.

So $\mathfrak{g}_{(S)} \approx \left(\bigoplus_{x\in I_0} V_{(0,\ldots,0,1,0,\ldots,0)}^{\oplus |J_x|-1}\right) \lhd \mathfrak{g}_{ab}$, where $\mathfrak{g}_{ab} = Vect(p_{i_x}(k) \,/\, x \in I_0,\ k \geq 1)$.

Second step. We now assume that $I_1 \neq \emptyset$. Then the descendants of $j \in I_1$ form a system as in the first step, so $\mathfrak{g}_{(S)} = W_{I_1} \lhd \mathfrak{g}_{(S_0)}$, where $W_{I_1} = Vect(f_j(k) \,/\, j \in I_1, k \geq 1\}$ and (S_0) is the restriction of (S) to the regular vertices. Let us fix $j \in I_1$ and let us consider the $\mathfrak{g}_{(S_0)}$-module $W_j = Vect(f_j(k) \,/\, k \geq 1)$. With the notations of the preceding step:

- $[p_{i_x}(k), f_j(l)] = a_{i_x}^{(j)} f_j(k+l)$ if $l = 1$.
- $[p_{i_x}(k), f_j(l)] = \nu_j a_{i_x}^{(j)} f_j(k+l)$ if $l \geq 2$.

- $[p_i(x), f_j(l)] = 0$ if i is not a i_x.

If $\nu_j \neq 0$, we put $p_j(k) = f_j(k)$ if $k \geq 2$ and $p_j(1) = \nu_j f_j(1)$. Then, for all l:

- $[p_{i_x}(k), f_j(l)] = \nu_j a_{i_x}^{(j)} f_j(k+l)$.
- $[p_i(x), f_j(l)] = 0$ if i is not a i_x.

So W_j is a module V_υ. If $\nu_j = 0$, we put $p_j(k) = f_j(k)$ for all $k \geq 1$. Then:

- $[p_{i_x}(k), f_j(l)] = a_{i_x}^{(j)} f_j(k+l)$ if $l = 1$.
- $[p_{i_x}(k), f_j(l)] = 0$ if $l \geq 2$.
- $[p_i(x), f_j(l)] = 0$ if i is not a i_x.

So W_j is a module W_ν.

Last step. We now consider vertices in J_1. If $j \in J_1$, then its descendants are vertices of the first step and vertices in I_1 such that $\nu_i = 1$. As before, $\mathfrak{g}_{(S)} = W_{J_1} \triangleleft \mathfrak{g}_{(S_1)}$, where $W_{J_1} = Vect(f_j(k) \,/\, j \in J_1, k \geq 1\}$ and (S_1) is the restriction of (S) to the regular vertices and the vertices of I_1. Let us fix $j \in J_1$ and let us consider the $\mathfrak{g}_{(S_1)}$-module $W_j = Vect(f_j(k)/k \geq 1)$. As $\nu_j \neq 0$, putting $p_j(k) = f_j(k)$ if $k \geq 2$ and $p_j(1) = \nu_j f_j(1)$:

- $[p_{i_x}(k), p_j(l)] = \nu_j a_{i_x}^{(j)} p_j(k+l)$.
- $[p_i(k), p_j(l)] = 0$ if $i \in J_x - \{i_x\}$.
- $[p_i(k), p_j(l)] = \nu_j a_i^{(j)} p_j(k+l)$ if $l = 1$ and $i \in I_1$.
- $[p_i(k), p_j(l)] = 0$ if $l \geq 2$ and $i \in I_1$.

So W_j is a module $W'_{\nu,\mu}$. □

Theorem 28 *Let (S) be a connected Hopf SDSE in the non-abelian, fundamental case. Then the Lie algebra $\mathfrak{g}_{(S)}$ is of the form:*

$$\mathfrak{g}_m \triangleleft (\mathfrak{g}_{m-1} \triangleleft (\cdots \mathfrak{g}_2 \triangleleft (\mathfrak{g}_1 \triangleleft \mathfrak{g}_0) \cdots),$$

where $\mathfrak{g}_0$ is the Lie algebra associated to the restriction of (S) to the non-extension vertices (so is described in theorem 27), and, for $j \geq 1$, $\mathfrak{g}_j$ is an abelian $(\mathfrak{g}_{j-1} \triangleleft (\cdots \mathfrak{g}_2 \triangleleft (\mathfrak{g}_1 \triangleleft \mathfrak{g}_0) \cdots)$-module having a basis $(h^{(j)}(k))_{k \geq 1}$.

Proof. Similar to the proof of theorem 20. □

6.3 Associated group

Let us now consider the character group $Ch\left(\mathcal{H}_{(S)}\right)$ of $\mathcal{H}_{(S)}$. In the preceding cases, $\mathfrak{g}_{(S)}$ contains an abelian sub-Lie algebra $\mathfrak{g}_{ab}$, so $Ch\left(\mathcal{H}_{(S)}\right)$ contains a subgroup isomorphic to the group:

$$G_{ab} = \left\{ \left(a_1^{(i)} x + a_2^{(i)} x^2 + \cdots \right)_{1 \leq i \leq M}, \mid \forall 1 \leq i \leq M, \forall k \geq 1,\ a_k^{(i)} \in K \right\},$$

with the product defined by $(A^{(i)}(x))_{i \in I}.(B^{(i)}(x))_{i \in I} = (A^{(i)}(x) + B^{(i)}(x) + A^{(i)}(x)B^{(i)}(x))_{i \in I}$. Note that G_{ab} is isomorphic to the following subgroup of the following group of the units of the ring $K[[x]]^M$:

$$G_1 = \left\{ \begin{pmatrix} 1+xf_1(x) \\ \vdots \\ 1+xf_M(x) \end{pmatrix} \mid f_1(x), \ldots, f_M(x) \in K[[x]] \right\}.$$

The isomorphism is given by:

$$\left\{ \begin{array}{rcl} G_{ab} & \longrightarrow & G_1 \\ \left(a_1^{(i)}x + a_2^{(i)}x^2 + \cdots\right)_{1\leq i\leq M} & \longrightarrow & \begin{pmatrix} 1+a_1^{(1)}x+a_2^{(1)}x^2+\ldots \\ \vdots \\ 1+a_1^{(M)}x+a_2^{(M)}x^2+\ldots \end{pmatrix}. \end{array} \right.$$

Moreover, each modules earlier defined on $\mathfrak{g}_{ab}$ corresponds to a module over G_{ab} by exponentiation, as explained in the following definition:

Definition 29

1. Let $\upsilon = (\upsilon_1, \ldots, \upsilon_M) \in K^M$. The module $\mathbb{V}_\upsilon$ is isomorphic to $yK[[y]]$ as a vector space, and the action of G_{ab} is given by:

$$(A^{(i)}(x))_{1\leq i\leq M}.P(y) = exp\left(\sum_{i=1}^{M} \upsilon_i A^{(i)}(y)\right) P(y).$$

2. Let us consider the semi-direct product $G = \left(\bigoplus_{i=1}^{N} \mathbb{V}_{\upsilon^{(i)}}\right) \triangleleft G_{ab}$.

   (a) Let $\nu = (\nu_1, \ldots, \nu_M) \in K^M$. The module $\mathbb{W}_\nu$ is $zK[[z]]$ as a vector space, and the action of G is given in the following way: for all $X = (P_1(y), \ldots, P_N(y), A_1(x), \ldots, A_m(x)) \in G$,

$$\left\{ \begin{array}{rcl} X.z & = & \left(1 + \sum_{i=1}^{M} \nu_i A_i(z)\right) z, \\ X.z^2R(z) & = & z^2R(z), \end{array} \right.$$

   for all $R(z) \in K[[z]]$.

   (b) Let $\nu = (\nu_1, \ldots, \nu_M) \in K^M$ and $\mu = (\mu_1, \ldots, \mu_N) \in K^N$, such that for all $1 \leq i \leq M$, for all $1 \leq j \leq N$, $\mu_j \left(\nu_i - \upsilon_i^{(j)}\right) = 0$. The module $\mathbb{W}'_{\nu,\mu}$ is $zK[[z]]$ as a vector space, and the action of G is given in the following way: for all $X = (P_1(y), \ldots, P_N(y), A_1(x), \ldots, A_m(x)) \in G$,

$$\left\{ \begin{array}{rcl} X.z & = & exp\left(\sum_{i=1}^{M} \nu_i A_i(z)\right)\left(1 + \sum_{i=1}^{N} \mu_i P_i(z)\right) z, \\ X.z^2R(z) & = & exp\left(\sum_{i=1}^{M} \nu_i A_i(z)\right) z^2R(z), \end{array} \right.$$

   for all $R(z) \in K[[z]]$.

Direct computations prove that they are indeed modules. The condition $\mu_j \left(\nu_i - \upsilon_i^{(j)}\right) = 0$ is necessary for $\mathbb{W}'_{\nu,\mu}$ to be a module. Indeed:

$$\begin{array}{rcl} A_i(x).(P_j(y).z) & = & (exp(\nu_i A_i(z)) + \mu_j exp(\nu_i A_i(z))P_j(z))\, z, \\ (A_i(x)P_j(y)).z & = & \left(exp(\upsilon_i^{(j)} A_i(y))P_j(y)A_i(x)\right).z \\ & = & \left(1 + exp(\upsilon_i^{(j)} A_i(z))P_j(z)\right) z + (exp(\nu_i A_i(z)) - 1)z \\ & = & \left(exp(\nu_i A_i(z)) + \mu_j exp(\upsilon_i^{(j)} A_i(z))P_j(z)\right) z. \end{array}$$

Theorem 30 *Let (S) be a connected Hopf SDSE in the abelian case. Then the group $Ch\left(\mathcal{H}_{(S)}\right)$ is of the form:*

$$G_N \rtimes (G_{N-1} \rtimes (\cdots G_2 \rtimes (G_1 \rtimes G_0)\cdots)),$$

where G_0 is a semi-direct product of the form:

$$G_0 = \mathbb{W} \rtimes (\mathbb{V} \rtimes G_{ab}),$$

where $\mathbb{V}$ is a direct sum of modules $\mathbb{V}_\upsilon$, and $\mathbb{W}$ a direct sum of modules $\mathbb{W}_\nu$ and $\mathbb{W}'_{\nu,\mu}$. Moreover, for all $m \geq 1$, $G_m = (tK[[t]],+)$ as a group.

Proof. Similar to the proof of theorem 23. □

References

[1] Christoph Bergbauer and Dirk Kreimer, *Hopf algebras in renormalization theory: locality and Dyson-Schwinger equations from Hochschild cohomology*, IRMA Lect. Math. Theor. Phys., vol. 10, Eur. Math. Soc., Zürich, 2006, arXiv:hep-th/0506190.

[2] D. J. Broadhurst and D. Kreimer, *Towards cohomology of renormalization: bigrading the combinatorial Hopf algebra of rooted trees*, Comm. Math. Phys. **215** (2000), no. 1, 217–236, arXiv:hep-th/0001202.

[3] Frédéric Chapoton, *Algèbres pré-lie et algèbres de Hopf liées à la renormalisation*, C. R. Acad. Sci. Paris Sér. I Math. **332** (2001), no. 8, 681–684.

[4] Frédéric Chapoton and Muriel Livernet, *Pre-Lie algebras and the rooted trees operad*, Internat. Math. Res. Notices **8** (2001), 395–408, arXiv:math/0002069.

[5] C. Chryssomalakos, H. Quevedo, M. Rosenbaum, and J. D. Vergara, *Normal coordinates and primitive elements in the Hopf algebra of renormalization*, Comm. Math. Phys. **255** (2002), no. 3, 465–485, arXiv:hep-th/0105259.

[6] Alain Connes and Dirk Kreimer, *Hopf algebras, Renormalization and Noncommutative geometry*, Comm. Math. Phys **199** (1998), no. 1, 203–242, arXiv:hep-th/9808042.

[7] Héctor Figueroa and José M. Gracia-Bondia, *On the antipode of Kreimer's Hopf algebra*, Modern Phys. Lett. A **16** (2001), no. 22, 1427–1434, arXiv:hep-th/9912170.

[8] Loïc Foissy, *Finite-dimensional comodules over the Hopf algebra of rooted trees*, J. Algebra **255** (2002), no. 1, 85–120, arXiv:math.QA/0105210.

[9] ———, *Faà di Bruno subalgebras of the Hopf algebra of planar trees from combinatorial Dyson-Schwinger equations*, Advances in Mathematics **218** (2008), 136–162, ArXiv:0707.1204.

[10] ———, *Classification of systems of combinatorial Dyson-Schwinger equations in the Hopf algebras of decorated rooted trees*, Advances in Mathematics **224** (2010), 2094–2150.

[11] Robert L. Grossman and Richard G. Larson, *Hopf-algebraic structure of families of trees*, J. Algebra **126** (1989), no. 1, 184–210, arXiv:0711.3877.

[12] ———, *Hopf-algebraic structure of combinatorial objects and differential operators*, Israel J. Math. **72** (1990), no. 1-2, 109–117.

[13] ———, *Differential algebra structures on families of trees*, Adv. in Appl. Math. **35** (2005), no. 1, 97–119, arXiv:math/0409006.

[14] Michael E. Hoffman, *Combinatorics of rooted trees and Hopf algebras*, Trans. Amer. Math. Soc. **355** (2003), no. 9, 3795–3811.

[15] Naihong Hu and Xiuling Wang, *Quantizations of generalized-Witt algebra and of Jacobson-Witt algebra in the modular case*, J. Algebra **312** (2007), no. 2, 902–929, arXiv:math/0602281.

[16] Dirk Kreimer, *Combinatorics of (perturbative) Quantum Field Theory*, Phys. Rep. **4–6** (2002), 387–424, arXiv:hep-th/0010059.

[17] ———, *Dyson-Schwinger equations: from Hopf algebras to number theory*, Universality and renormalization, Fields Inst. Commun., no. 50, Amer. Math. Soc., Providence, RI, 2007, arXiv:hep-th/0609004.

[18] Dirk Kreimer and Karen Yeats, *An étude in non-linear Dyson-Schwinger equations*, Nuclear Phys. B Proc. Suppl. **160** (2006), 116–121, arXiv:hep-th/0605096.

[19] Jean-Louis Loday and Maria Ronco, *Combinatorial Hopf algebras*, arXiv:0810.0435, 2008.

[20] John W. Milnor and John C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264.

[21] Florin Panaite, *Relating the Connes-Kreimer and Grossman-Larson Hopf algebras built on rooted trees*, Lett. Math. Phys. **51** (2000), no. 3, 211–219.