Primitive elements of the Hopf algebra of free quasi-symmetric functions

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ABSTRACT: Using the dendriform and the bidendriform Cartier-Quillen-Milnor-Moore theorem, we construct a basis of the space of primitive elements of the Hopf algebra of free quasi-symmetric functions, indexed by a certain set of trees, and inductively computable.

Contents

Introduction

The Hopf algebra of free quasi-symmetric functions **FQSym**, also known as the Malvenuto-Reutenauer Hopf algebra, is introduced in [7]. It is a graded self-dual Hopf algebra, with the set of all permutations as a basis. Certain interesting properties are shown in [2]: in particular, it is shown that it is both free and cofree, and a basis of the space of its primitive elements is given, using the self-duality and a monomial basis. Note that computing the primitive elements of degree n by this method implies to inverse a certain $n! \times n!$ matrix.

The aim of this paper is to describe another basis of $Prim_{coAss}(\mathbf{FQSym})$, which can be inductively computed. We use for this the dendriform structure of **FQSym**. Recall that a dendriform algebra is an associative algebra such that its product can be split into two non-associative products $\prec$ and $\succ$, with good compatibilities [5, 6, 8]. It is known that **FQSym**, or more precisely its augmentation ideal, is dendriform. More precisely, it is a dendriform Hopf algebra, in the sense of [8]. This implies, by the dendriform Cartier-Quillen-Milnor-Moore theorem, that $Prim_{coAss}(\mathbf{FQSym})$ is a brace algebra.

We introduced in [4] the notion of bidendriform bialgebra and showed that **FQSym** is bidendriform. The bidendriform Cartier-Quillen-Milnor-Moore theorem implies that **FQSym** is freely generated, as a dendriform algebra, by the space $Prim_{coDend}(\mathbf{FQSym})$ of primitive elements in the codendriform sense. Combining this result with the dendriform Cartier-Quillen-Milnor-Moore theorem, we show that $Prim_{coAss}(\mathbf{FQSym})$ is, as a brace algebra, freely generated by $Prim_{coDend}(\mathbf{FQSym})$. We recall in section 2 a description of free brace algebras. If $(v_i)_{i\in I}$ is a basis of the vector space V, then the free brace algebra generated by V has a basis indexed by planar rooted trees decorated by I, and the brace structure is described in this basis by the help of graftings. Hence, for any basis of $Prim_{coDend}(\mathbf{FQSym})$, it is possible to recover a basis of $Prim_{coAss}(\mathbf{FQSym})$, indexed by a certain set of planar rooted trees.

Let, for all $n \in \mathbb{N}$:

$$\left\{ \begin{array}{rcl} p_n & = & dim(Prim_{coAss}(\mathbf{FQSym})_n), \\ q_n & = & dim(Prim_{coDend}(\mathbf{FQSym})_n). \end{array} \right.$$

We then prove in section 3 that for $n \geq 2$, $q_n = (n-2)p_{n-1}$. We then give $n-2$ applications from $Prim_{coAss}(\mathbf{FQSym})_{n-1}$ to $Prim_{coDend}(\mathbf{FQSym})_n$, which give all elements of $Prim_{coDend}(\mathbf{FQSym})_n$. These applications are given by the insertion of $n+1$ at a given place in elements of the symmetric group S_n, seen as words in letters $1, \ldots, n$.

Combining the results of the second and third sections, we define inductively in the fourth section a new basis of $Prim_{coAss}(\mathbf{FQSym})_n$, indexed by certain planar decorated rooted trees. The trees which are only a root give a basis of $Prim_{coDend}(\mathbf{FQSym})_n$.

Notations.

1. K is a commutative field of any characteristic.
2. If V is a K-vector field which is $\mathbb{N}$-graded, we shall denote by V_k the space of homogeneous elements of V of degree k.

1 Bidendriform bialgebras and FQSym

1.1 Bidendriform bialgebras

We introduced in [4] the following definition:

Definition 1 A bidendriform bialgebra is a family $(A, \prec, \succ, \Delta_\prec, \Delta_\succ)$ such that:

1. A is a K-vector space and:

$$\prec: \left\{ \begin{array}{rcl} A \otimes A & \longrightarrow & A \\ a \otimes b & \longrightarrow & a \prec b, \end{array} \right. \qquad \Delta_\prec : \left\{ \begin{array}{rcl} A & \longrightarrow & A \otimes A \\ a & \longrightarrow & \Delta_\prec(a) = a'_\prec \otimes a''_\prec, \end{array} \right.$$

$$\succ: \left\{ \begin{array}{rcl} A \otimes A & \longrightarrow & A \\ a \otimes b & \longrightarrow & a \succ b, \end{array} \right. \qquad \Delta_\succ : \left\{ \begin{array}{rcl} A & \longrightarrow & A \otimes A \\ a & \longrightarrow & \Delta_\succ(a) = a'_\succ \otimes a''_\succ. \end{array} \right.$$

2. (Dendriform axioms). $(A, \prec, \succ)$ is a dendriform algebra: for all $a, b, c \in A$,

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c), \tag{1}$$
$$(a \succ b) \prec c = a \succ (b \prec c), \tag{2}$$
$$5a \prec b + a \succ b) \succ c = a \succ (b \succ c). \tag{3}$$

3. (Codendriform axioms). $(A, \Delta_\prec, \Delta_\succ)$ is a codendriform coalgebra: for all $a \in A$,

$$(\Delta_\prec \otimes Id) \circ \Delta_\prec(a) = (Id \otimes \Delta_\prec + Id \otimes \Delta_\succ) \circ \Delta_\prec(a), \tag{4}$$
$$(\Delta_\succ \otimes Id) \circ \Delta_\prec(a) = (Id \otimes \Delta_\prec) \circ \Delta_\succ(a), \tag{5}$$
$$(\Delta_\prec \otimes Id + \Delta_\succ \otimes Id) \circ \Delta_\succ(a) = (Id \otimes \Delta_\succ) \circ \Delta_\succ(a). \tag{6}$$

4. (Bidendriform axioms). For all $a, b \in A$,

$$\begin{aligned}
\Delta_{\succ}(a \succ b) &= a'b'_{\succ} \otimes a'' \succ b''_{\succ} + a' \otimes a'' \succ b + b'_{\succ} \otimes a \succ b''_{\succ} + ab'_{\succ} \otimes b''_{\succ} + a \otimes b, && (7)\\
\Delta_{\succ}(a \prec b) &= a'b'_{\succ} \otimes a'' \prec b''_{\succ} + a' \otimes a'' \prec b + b'_{\succ} \otimes a \prec b''_{\succ}, && (8)\\
\Delta_{\prec}(a \succ b) &= a'b'_{\prec} \otimes a'' \succ b''_{\prec} + ab'_{\prec} \otimes b''_{\prec} + b'_{\prec} \otimes a \succ b''_{\prec}, && (9)\\
\Delta_{\prec}(a \prec b) &= a'b'_{\prec} \otimes a'' \prec b''_{\prec} + a'b \otimes a'' + b'_{\prec} \otimes a \prec b''_{\prec} + b \otimes a. && (10)
\end{aligned}$$

Remarks.

1. If A is a bidendriform bialgebra, then $K \oplus A$ is naturally a Hopf algebra, extending $\prec + \succ$ and $\Delta_{\prec} + \Delta_{\succ}$ on $K \oplus A$.

2. If A is a bidendriform bialgebra, it is also a dendriform hopf algebra in the sense of [8, 9], with coassociative coproduct given by $\tilde{\Delta} = \Delta_{\prec} + \Delta_{\succ}$. The compatibilities of dendriform Hopf algebras are given by (7) + (9) and (8) + (10).

If A is a bidendriform algebra, we define:

$$Prim_{codend}(A) = Ker(\Delta_{\prec}) \cap Ker(\Delta_{\succ}).$$

The following result is proved in [4] (theorem 35 and corollary 17):

Theorem 2 (Bidendriform Cartier-Quillen-Milnor-Moore theorem) *Let A be a $\mathbb{N}$-graded bidendriform bialgebra, such that $A_0 = (0)$. Then A is freely generated as a dendriform algebra by $Prim_{coDend}(A)$. Moreover, consider the following formal series:*

$$R(X) = \sum_{n=1}^{+\infty} dim(A), \qquad Q(X) = \sum_{n=1}^{+\infty} dim\left(Prim_{coDend}(A_n)\right) X^n.$$

Then $Q(X) = \dfrac{R(X)}{(R(X)+1)^2}$.

1.2 An example: the Hopf algebra FQSym

See [1, 2, 7]. The algebra **FQSym** is the vector space generated by the elements $(\mathbf{F}_u)_{u \in \mathbb{S}}$, where $\mathbb{S}$ is the disjoint union of the symmetric groups S_n ($n \in \mathbb{N}$). Its product and its coproduct are given in the following way: for all $u \in S_n$, $v \in S_m$, putting $u = (u_1 \ldots u_n)$,

$$\begin{aligned}
\Delta(\mathbf{F}_u) &= \sum_{i=0}^{n} \mathbf{F}_{st(u_1 \ldots u_i)} \otimes \mathbf{F}_{st(u_{i+1} \ldots u_n)},\\
\mathbf{F}_u.\mathbf{F}_v &= \sum_{\zeta \in sh(n,m)} \mathbf{F}_{(u \times v).\zeta^{-1}},
\end{aligned}$$

where $sh(n,m)$ is the set of (n,m)-shuffles, and st is the standardisation. Its unit is $1 = \mathbf{F}_{\emptyset}$, where $\emptyset$ is the unique element of S_0. Moreover, **FQSym** is a $\mathbb{N}$-graded Hopf algebra, by putting $|\mathbf{F}_u| = n$ if $u \in S_n$.

Examples.

$$\begin{aligned}
\mathbf{F}_{(1\,2)}\mathbf{F}_{(1\,2\,3)} &= \mathbf{F}_{(1\,2\,3\,4\,5)} + \mathbf{F}_{(1\,3\,2\,4\,5)} + \mathbf{F}_{(1\,3\,4\,2\,5)} + \mathbf{F}_{(1\,3\,4\,5\,2)} + \mathbf{F}_{(3\,1\,2\,4\,5)}\\
&\quad + \mathbf{F}_{(3\,1\,4\,2\,5)} + \mathbf{F}_{(3\,1\,4\,5\,2)} + \mathbf{F}_{(3\,4\,1\,2\,5)} + \mathbf{F}_{(3\,4\,1\,5\,2)} + \mathbf{F}_{(3\,4\,5\,1\,2)};
\end{aligned}$$

$$\begin{aligned}
\Delta\left(\mathbf{F}_{(1\,2\,5\,4\,3)}\right) &= 1 \otimes \mathbf{F}_{(1\,2\,5\,4\,3)} + \mathbf{F}_{(1)} \otimes \mathbf{F}_{(1\,4\,3\,2)} + \mathbf{F}_{(1\,2)} \otimes \mathbf{F}_{(3\,2\,1)}\\
&\quad + \mathbf{F}_{(1\,2\,3)} \otimes \mathbf{F}_{(2\,1)} + \mathbf{F}_{(1\,2\,4\,3)} \otimes \mathbf{F}_{(1)} + \mathbf{F}_{(1\,2\,5\,4\,3)} \otimes 1.
\end{aligned}$$

Let $(\mathbf{FQSym})_+ = Vect(\mathbf{F}_u \ / \ u \in S_n, \ n \geq 1)$ be the augmentation ideal of $\mathbf{FQSym}$. We define $\prec, \succ, \Delta_\prec$ and $\Delta_\succ$ on $(\mathbf{FQSym})_+$ in the following way: for all $u \in S_n$, $v \in S_m$, by putting $u = (u_1 \ldots u_n)$,

$$\begin{aligned}
\mathbf{F}_u \prec \mathbf{F}_v &= \sum_{\substack{\zeta \in sh(n,m) \\ \zeta^{-1}(n+m)=n}} \mathbf{F}_{(u\times v).\zeta^{-1}}, \\
\mathbf{F}_u \succ \mathbf{F}_v &= \sum_{\substack{\zeta \in sh(n,m) \\ \zeta^{-1}(n+m)=n+m}} \mathbf{F}_{(u\times v).\zeta^{-1}}, \\
\Delta_\prec(\mathbf{F}_u) &= \sum_{i=u^{-1}(n)}^{n-1} \mathbf{F}_{st(u_1\ldots u_i)} \otimes \mathbf{F}_{st(u_{i+1}\ldots u_n)}, \\
\Delta_\succ(\mathbf{F}_u) &= \sum_{i=1}^{u^{-1}(n)-1} \mathbf{F}_{st(u_1\ldots u_i)} \otimes \mathbf{F}_{st(u_{i+1}\ldots u_n)}.
\end{aligned}$$

Examples.

$$\begin{aligned}
\mathbf{F}_{(1\,2)} \prec \mathbf{F}_{(1\,2\,3)} &= \mathbf{F}_{(1\,3\,4\,5\,2)} + \mathbf{F}_{(3\,1\,4\,5\,2)} + \mathbf{F}_{(3\,4\,1\,5\,2)} + \mathbf{F}_{(3\,4\,5\,1\,2)}, \\
\mathbf{F}_{(1\,2)} \succ \mathbf{F}_{(1\,2\,3)} &= \mathbf{F}_{(1\,2\,3\,4\,5)} + \mathbf{F}_{(1\,3\,2\,4\,5)} + \mathbf{F}_{(1\,3\,4\,2\,5)} + \mathbf{F}_{(3\,1\,2\,4\,5)} + \mathbf{F}_{(3\,1\,4\,2\,5)} + \mathbf{F}_{(3\,4\,1\,2\,5)},
\end{aligned}$$

$$\begin{aligned}
\Delta_\prec\left(\mathbf{F}_{(1\,2\,5\,4\,3)}\right) &= \mathbf{F}_{(1\,2\,3)} \otimes \mathbf{F}_{(2\,1)} + \mathbf{F}_{(1\,2\,4\,3)} \otimes \mathbf{F}_{(1)}, \\
\Delta_\succ\left(\mathbf{F}_{(1\,2\,5\,4\,3)}\right) &= \mathbf{F}_{(1)} \otimes \mathbf{F}_{(1\,4\,3\,2)} + \mathbf{F}_{(1\,2)} \otimes \mathbf{F}_{(3\,2\,1)}.
\end{aligned}$$

The following result is proved in [4], theorem 38:

Theorem 3 $((\mathbf{FQSym})_+, \prec, \succ, \Delta_\prec, \Delta_\succ)$ *is a connected bidendriform bialgebra.*

Moreover, $(\mathbf{FQSym})_+$ is $\mathbb{N}$-graded, by putting the elements of S_n homogeneous of degree n. By theorem 2, with $q_n = dim(Prim_{coDend}(\mathbf{FQSym})_n)$, we obtain:

n	1	2	3	4	5	6	7	8	9	10	11	12
q_n	1	0	1	6	39	284	2 305	20 682	203 651	2 186 744	25 463 925	319 989 030

2 Recovering $Prim_{coAss}(\mathbf{FQSym})$ from $Prim_{coDend}(\mathbf{FQSym})$

2.1 Dendriform Cartier-Quillen-Milnor-Moore theorem and variations

Recall that a brace algebra is a K-vector space A together with a n-multilinear operation for all $n \geq 2$:

$$\langle \ldots \rangle : \left\{ \begin{array}{rcl} A^{\otimes n} & \longrightarrow & A \\ a_1 \otimes \ldots \otimes a_n & \longrightarrow & \langle a_1, \ldots, a_n \rangle, \end{array} \right.$$

satisfying certain relations; see [8, 9] for more details. For example:

$$\langle a_1, \langle a_2, a_3 \rangle \rangle = \langle a_1, a_2, a_3 \rangle + \langle \langle a_1, a_2 \rangle, a_3 \rangle + \langle a_2, a_1, a_3 \rangle.$$

The following theorem is proved in [8, 9]; more precisely, the first point of this theorem is proposition 2-8 and theorem 3-4 of [8] and the second point is theorem 4-6 of [9]:

Theorem 4 (Dendriform Cartier-Quillen-Milnor-Moore theorem) *Let A be a dendriform Hopf algebra. We denote $Prim_{coAss}(A) = Ker(\tilde{\Delta})$.*

1. *$Prim_{coAss}(A)$ is a brace algebra, with brackets given by:*

$$\langle p_1, \ldots, p_n \rangle$$

$$= \sum_{i=0}^{n-1} (-1)^{n-1-i} \left(p_1 \prec (p_2 \prec (\ldots \prec p_i) \ldots) \succ p_n \prec (\ldots (p_{i+1} \succ p_{i+2}) \succ \ldots) \succ p_{n-1}\right).$$

2. *If A is freely generated as a dendriform algebra by a subvector space $V \subseteq Prim_{coAss}(A)$, then $Prim_{coAss}(A)$ is freely generated as a brace algebra by V.*

Let us precise the relation between $Prim_{coAss}(A)$ and $Prim_{coDend}(A)$ if A is a bidendriform bialgebra. Combining the dendriform and the bidendriform Cartier-Quillen-Milnor-Moore theorems:

Theorem 5 *Let A be a $\mathbb{N}$-graded bidendriform bialgebra, with $A_0 = (0)$. Then $Prim_{coAss}(A)$ is, as a brace algebra, freely generated by $Prim_{coDend}(A)$.*

Proof. By the bidendriform Cartier-Quillen-Milnor-Moore theorem, A is freely generated as a dendriform algebra by $Prim_{coDend}(A)$. By the second point of the dendriform Cartier-Quillen-Milnor-Moore theorem, $Prim_{coAss}(A)$ is freely generated as a brace algebra by the space $Prim_{coDend}(A)$. $\square$

Proposition 6 *If A is $\mathbb{N}$-graded dendriform Hopf algebra, such that $A_0 = (0)$, then A is generated as a dendriform algebra by $Prim_{coAss}(A)$. Moreover, consider the following formal series:*

$$R(X) = \sum_{n=1}^{\infty} dim(A_n) X^n, \qquad P(X) = \sum_{n=1}^{+\infty} dim(Prim_{coDend}((A)_n) X^n.$$

Then $P(X) = \dfrac{R(X)}{1 + R(X)}$.

Proof. *First step.* Let $p_1, \ldots, p_n \in Prim_{coAss}(A)$. We define by induction on n:

$$\omega(p_1, \ldots, p_n) = \left\{ \begin{array}{rl} p_1 & \text{if} \quad n = 1, \\ p_n \prec \omega(p_1, \ldots, p_{n-1}) & \text{if} \quad n \geq 2. \end{array} \right.$$

An easy induction on n allows to show the following result, using (8)+(10):

$$\tilde{\Delta}(\omega(p_1, \ldots, p_n)) = \sum_{i=1}^{n-1} \omega(p_1, \ldots, p_i) \otimes \omega(p_{i+1}, \ldots, p_n).$$

We denote by $\tilde{\Delta}^n : A \longrightarrow A^{n+1}$ the iterated coproducts of A. It comes by induction:

$$\tilde{\Delta}^m(\omega(p_1, \ldots, p_n)) = \left\{ \begin{array}{rl} 0 & \text{if} \quad m \geq n, \\ p_1 \otimes \ldots \otimes p_n & \text{if} \quad m = n-1. \end{array} \right.$$

Second step. We consider the tensor (non counitary) coalgebra $C = T(Prim_{coAss}(A))$:

$$C = \bigoplus_{n=1}^{\infty} Prim_{coAss}(A)^{\otimes n}.$$

It is a coalgebra for the deconcatenation coproduct. As $Prim_{coAss}(A)$ is $\mathbb{N}$-graded, C is a graded coalgebra with formal series:

$$S(X) = \frac{1}{1 - P(X)} - 1 = \frac{P(X)}{1 - P(X)}.$$

By the first step, the following application is a morphism of graded coalgebras:

$$\Psi : \left\{ \begin{array}{rcl} C & \longrightarrow & A \\ p_1 \otimes \ldots \otimes p_n & \longrightarrow & \omega(p_1, \ldots, p_n). \end{array} \right.$$

Third step. Suppose that $Ker(\Psi)$ is non zero. As it is a coideal of C, it contains primitive elements of C, that is to say elements of $Prim_{coAss}(A)$. As Ψ is obviously monic on $Prim_{coAss}(A)$, this is impossible. So $Ker(\Psi) = (0)$ and Ψ is monic.

Let $a \in A$. As $A_0 = (0)$, for a certain $N(a) \in \mathbb{N}^*$, $\tilde{\Delta}^{N(a)}(a) = 0$. We prove that $a \in Im(\Psi)$ by induction on $N(a)$. If $N(a) = 1$, then $a \in Prim_{coAss}(A)$ and the result is obvious. Suppose that the result is true for all $b \in A$ such that $N(b) < N(a)$. As $\tilde{\Delta}^{N(a)}(a) = 0$, necessarily $\tilde{\Delta}^{N(a)-1}(a) \in Prim_{coAss}(A)^{\otimes N(a)}$. We put:

$$\tilde{\Delta}^{N(a)-1}(a) = a_1 \otimes \ldots \otimes a_n, \qquad b = a - \omega(a_1, \ldots, a_n).$$

By the first step, $\tilde{\Delta}^{N(a)-1}(b) = 0$, so $N(b) < N(a)$. By the induction hypothesis, $b \in Im(\Psi)$. As $\omega(a_1, \ldots, a_n) \in Im(\Psi)$, $a \in Im(\Psi)$.

Last step. As Ψ in an isomorphism of graded coalgebras, $S(X) = R(X)$. Hence:

$$R(X) = \frac{P(X)}{1 - P(X)},$$

so $R(X) - R(X)P(X) = P(X)$ and $P(X) = \dfrac{R(X)}{1 + R(X)}$. $\square$

2.2 Free brace algebras

Using a description of the free dendriform algebra generated by a set $\mathcal{D}$ with planar decorated forests, we gave a description of the free brace algebra $Brace(\mathcal{D})$ in [3]. A basis of this brace algebra is given by the set $T^{\mathcal{D}}$ of planar rooted trees decorated by $\mathcal{D}$. For example:

$$\begin{array}{rcl} Brace(\mathcal{D})_1 & = & Vect(\bullet_a,\ a \in \mathcal{D}), \\ Brace(\mathcal{D})_2 & = & Vect(\mathbf{I}^b_a,\ a, b \in \mathcal{D}), \\ Brace(\mathcal{D})_3 & = & Vect({}^c\mathbf{V}^b_a,\ \mathbf{I}^{c}_{b\,a},\ a, b, c \in \mathcal{D}), \\ Brace(\mathcal{D})_4 & = & Vect({}^{d}\mathbf{W}^{c\,b}_a,\ {}^{d}_{c}\mathbf{V}^b_a,\ {}^d\mathbf{V}^{c}_{b\,a},\ {}^{d}\mathbf{Y}^{c}_{b\,a},\ \mathbf{I}^{d}_{c\,b\,a},\ a, b, c, d \in \mathcal{D}), \ldots \end{array}$$

The brace bracket satisfies, for all $t_1, \ldots, t_{n-1} \in T^{\mathcal{D}}$, $d \in \mathcal{D}$:

$$\langle t_1, \ldots, t_{n-1}, \bullet_d \rangle = B_d(t_{n-1} \ldots t_1),$$

where $B_d(t_{n-1} \ldots t_1)$ is the tree obtained by grafting the trees $t_{n-1}, \ldots, t_1$ (in this order) on a common root decorated by d. For example, if $a, b, c, d \in \mathcal{D}$,

$$\langle \bullet_a, \mathbf{I}^c_b, \bullet_d \rangle = {}^{c}_{b}\mathbf{V}^a_d.$$

As a consequence, if A is a connected bidendriform bialgebra and if $(q_d)_{d \in \mathcal{D}}$ is a basis of $Prim_{coDend}(A)$, then a basis of $Prim_{coAss}(A)$ is given by $(p_t)_{t \in T^{\mathcal{D}}}$ defined inductively by:

$$\left\{ \begin{array}{rcl} p_{\bullet_d} & = & q_d, \\ p_{B_d^+(t_1 \ldots t_n)} & = & \langle p_{t_n}, \ldots, p_{t_1}, q_d \rangle. \end{array} \right.$$

3 Recovering $Prim_{coDend}(\mathbf{FQSym})$ from $Prim_{coAss}(\mathbf{FQSym})$

For all $n \in \mathbb{N}^*$, we put:

$$\left\{ \begin{array}{rcl} p_n & = & dim(Prim_{coAss}(\mathbf{FQSym})_n), \\ q_n & = & dim(Prim_{coDend}(\mathbf{FQSym})_n). \end{array} \right.$$

Proposition 7 *For all $n \geq 2$, $q_n = (n-2)p_{n-1}$.*

Proof. We put:

$$R(X) = \sum_{n=1}^{\infty} n! X^n, \qquad P(X) = \sum_{n=1}^{\infty} p_n X^n, \qquad Q(X) = \sum_{n=1}^{\infty} q_n X^n.$$

By theorem 2 and proposition 6:

$$P(X) = \frac{R(X)}{1+R(X)}, \qquad Q(X) = \frac{R(X)}{(1+R(X))^2}.$$

Hence:

$$P'(X) = \frac{R'(X)}{(1+R(X))^2}.$$

Moreover:

$$\begin{array}{rcl} R'(X) & = & \displaystyle\sum_{n=1}^{\infty} nn! X^{n-1} \\ & = & \displaystyle\sum_{n=1}^{\infty} (n+1)! X^{n-1} - \sum_{n=1}^{\infty} n! X^{n-1} \\ & = & \displaystyle\frac{R(X)-X}{X^2} - \frac{R(X)}{X} \\ & = & \displaystyle\frac{R(X) - X(1+R(X))}{X^2}. \end{array}$$

We deduce:

$$X^2 P'(X) = \frac{R(X) - X(1+R(X))}{(1+R(X))^2} = Q(X) - \frac{X}{1+R(X)} = Q(X) - X + XP(X).$$

So:

$$X^2 P'(X) + XP(X) = \sum_{n=1}^{\infty} (n-1) p_n X^{n+1} = Q(X) - X = \sum_{n=2}^{\infty} q_n X^n.$$

As a conclusion, for all $n \geq 2$, $q_n = (n-2)p_{n-2}$. $\square$

Definition 8 Let $i \in \mathbb{N}^*$. We define $\Phi_i : \mathbf{FQSym} \longrightarrow \mathbf{FQSym}$ in the following way: for all $n \in \mathbb{N}$, for all $\sigma = (\sigma_1, \ldots, \sigma_n) \in S_n$,

$$\Phi_i(\mathbf{F}_\sigma) = \left\{ \begin{array}{l} 0 \text{ if } i \geq n, \\ \mathbf{F}_{(\sigma_1, \ldots, \sigma_i, n+1, \sigma_{i+1}, \ldots, \sigma_n)} \text{ if } i < n. \end{array} \right.$$

Theorem 9 *Let $n \geq 2$. The following application is bijective:*

$$\Phi : \left\{ \begin{array}{rcl} (Prim_{coAss}(\mathbf{FQSym})_{n-1})^{n-2} & \longrightarrow & Prim_{coDend}(\mathbf{FQSym})_n \\ (p_1, \ldots, p_{n-2}) & \longrightarrow & \Phi_1(p_1) + \ldots + \Phi_{n-2}(p_{n-2}). \end{array} \right.$$

Proof. *First step.* Let us first prove that Φ takes its values in $Prim_{coDend}(\mathbf{FQSym})$. Let $p \in Prim_{coAss}(\mathbf{FQSym})$ and $1 \leq i \leq n-2$. For all $k \in \mathbb{N}$, let π_k be the projection on $\mathbf{FQSym}_k$. By definition of $\Delta_\prec$ and $\Delta_\succ$, for all $\sigma \in S_{n-1}$:

$$\begin{array}{rcl}\Delta_\prec(\Phi_i(\mathbf{F}_\sigma)) & = & \left(\displaystyle\sum_{j=i+1}^{n-2} \pi_j \otimes \pi_{n-1-j}\right) \circ \tilde{\Delta}(\mathbf{F}_\sigma),\\ \Delta_\succ(\Phi_i(\mathbf{F}_\sigma)) & = & \left(\displaystyle\sum_{j=1}^{i} \pi_j \otimes \pi_{n-1-j}\right) \circ \tilde{\Delta}(\mathbf{F}_\sigma).\end{array}$$

By linearity, we obtain:

$$\begin{array}{rcl}\Delta_\prec(p) & = & \left(\displaystyle\sum_{j=i+1}^{n-2} \pi_j \otimes \pi_{n-1-j}\right) \circ \tilde{\Delta}(p) \;=\; 0,\\ \Delta_\succ(p) & = & \left(\displaystyle\sum_{j=1}^{i} \pi_j \otimes \pi_{n-1-j}\right) \circ \tilde{\Delta}(p) \;=\; 0.\end{array}$$

This proves the first step.

Second step. We now prove that Φ is monic. Let $(p_1, \ldots, p_{n-2}) \in Ker(\Phi)$. Let be $1 \leq i \leq n-2$. We define:

$$\varpi_i : \left\{ \begin{array}{rcl} \mathbf{FQSym}_n & \longrightarrow & \mathbf{FQSym}_n \\ \mathbf{F}_\sigma & \longrightarrow & \left\{ \begin{array}{rl} 0 & \text{if} \quad \sigma^{-1}(n) \neq i+1, \\ \mathbf{F}_\sigma & \text{if} \quad \sigma^{-1}(n) = i+1. \end{array} \right. \end{array} \right.$$

Then, in an obvious way, $\varpi_i(\Phi(p_1, \ldots, p_{n-2})) = \Phi_i(p_i) = 0$. As Φ_i is obviously monic on $\mathbf{FQSym}_{n-1}$ (because $i \leq n-2$), $p_i = 0$. So Φ is monic.

Last step. As $dim\left((Prim_{coAss}(\mathbf{FQSym})_{n-1})^{n-2}\right) = dim\,(Prim_{coDend}(\mathbf{FQSym})_n)$, from proposition 7, Φ is bijective. $\square$

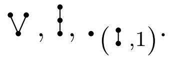
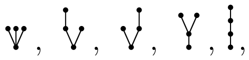
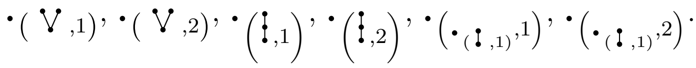
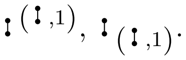
4 An inductive basis of $Prim_{coAss}(\mathbf{FQSym})$

We now combine results of the second and third sections to obtain an basis of the space $Prim_{coAss}(\mathbf{FQSym})$. We first define inductively some set of partially planar decorated trees $\mathbb{T}(n)$ in the following way:

1. $\mathbb{T}(0)$ is the set of non decorated planar trees. The weight of an element of $\mathbb{T}(0)$ is the number of its vertices.
2. Suppose that $\mathbb{T}(n)$ is defined. Then $\mathbb{T}(n+1)$ is the set of planar trees defined by :
   (a) The elements of $\mathbb{T}(n+1)$ are partially decorated planar trees.
   (b) The vertices of the elements of $\mathbb{T}(n+1)$ can eventually be decorated by a pair (t,k), with $t \in \mathbb{T}(n)$ and k an integer in $\{1, \ldots, weight(t)-1\}$.
   (c) The weight of an element of $\mathbb{T}(n)$ is the sum of the number of its vertices and of the weights of the trees of $\mathbb{T}(n)$ that appear in its decorations.

Inductively, for all $n \in \mathbb{N}$, $\mathbb{T}(n) \subseteq \mathbb{T}(n+1)$. We put $\mathbb{T} = \displaystyle\bigcup_{n \in \mathbb{N}} \mathbb{T}(n)$.

Examples.

1. Elements of $\mathbb{T}$ of weight 1: $\bullet$.
2. Elements of $\mathbb{T}$ of weight 2: .
3. Elements of $\mathbb{T}$ of weight 3: 
4. Elements of $\mathbb{T}$ of weight 4:
   (a) 
   (b) 
   (c) 

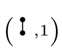
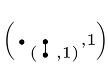
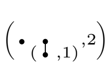
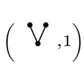
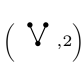
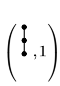
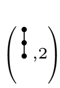
We can then define a basis $(p_t)_{i\in\mathbb{T}}$ of $Prim_{coAss}(\mathbf{FQSym})$ inductively in the following way:

1. $p_\bullet = \mathbf{F}_{(1)}$.
2. If $t = \bullet_{(t',i)}$, then $p_t = \Phi_i(p_{t'})$.
3. If t is not a single root, let $t_1, \ldots, t_{n-1}$ be the children of its roots, from left to right, and t_n its root. Then $p_t = \langle p_{t_{n-1}}, \ldots, p_{t_1}, p_{t_n}\rangle$.

By the preceding results:

Theorem 10 *$(p_t)_{t\in\mathbb{T}}$ is a basis of $Prim_{coAss}(\mathbf{FQSym})$. A basis of $Prim_{coDend}(\mathbf{FQSym})$ is given by the p_t's, where t is a single root.*

Examples.

1. $p_\bullet = \mathbf{F}_{(1)}$.
2. p $= -\mathbf{F}_{(21)} + \mathbf{F}_{(12)}$.
3. (a) p $= -\mathbf{F}_{(231)} + \mathbf{F}_{(132)}$.
   (b) p $= \mathbf{F}_{(231)} - \mathbf{F}_{(132)} - \mathbf{F}_{(312)} + \mathbf{F}_{(213)}$.
   (c) p $= \mathbf{F}_{(321)} - \mathbf{F}_{(231)} - \mathbf{F}_{(213)} + \mathbf{F}_{(123)}$.
4. (a) p $= -\mathbf{F}_{(2431)} + \mathbf{F}_{(1432)}$.
   (b) p $= -\mathbf{F}_{(2341)} + \mathbf{F}_{(1342)}$.
   (c) p $= \mathbf{F}_{(2431)} - \mathbf{F}_{(1432)} - \mathbf{F}_{(3412)} + \mathbf{F}_{(2413)}$.
   (d) p $= \mathbf{F}_{(2341)} - \mathbf{F}_{(1342)} - \mathbf{F}_{(3142)} + \mathbf{F}_{(2143)}$.
   (e) p $= \mathbf{F}_{(3421)} - \mathbf{F}_{(2431)} - \mathbf{F}_{(2413)} + \mathbf{F}_{(1423)}$.
   (f) p $= \mathbf{F}_{(3241)} - \mathbf{F}_{(2341)} - \mathbf{F}_{(2143)} + \mathbf{F}_{(1243)}$.
   (g) p $= -\mathbf{F}_{(2341)} + \mathbf{F}_{(1342)} + \mathbf{F}_{(3142)} + \mathbf{F}_{(3412)} - \mathbf{F}_{(2143)} - \mathbf{F}_{(2413)} - \mathbf{F}_{(4213)} + \mathbf{F}_{(3214)}$.
   (h) p $= -\mathbf{F}_{(2431)} - \mathbf{F}_{(4231)} + \mathbf{F}_{(2341)} + \mathbf{F}_{(3241)} + \mathbf{F}_{(1432)} + \mathbf{F}_{(4132)} + \mathbf{F}_{(4312)} - \mathbf{F}_{(1342)} - \mathbf{F}_{(3142)} - \mathbf{F}_{(3412)} - \mathbf{F}_{(3214)} + \mathbf{F}_{(2314)}$.

(i) $p = -\mathbf{F}_{(3241)} + \mathbf{F}_{(2341)} + \mathbf{F}_{(2143)} + \mathbf{F}_{(2413)} + \mathbf{F}_{(4213)} - \mathbf{F}_{(1243)} - \mathbf{F}_{(1423)} - \mathbf{F}_{(4123)} - \mathbf{F}_{(2314)} - \mathbf{F}_{(3214)} + \mathbf{F}_{(1324)} + \mathbf{F}_{(3124)}$.

(j) $p = -\mathbf{F}_{(3421)} + \mathbf{F}_{(2431)} + \mathbf{F}_{(4231)} - \mathbf{F}_{(3241)} + \mathbf{F}_{(2314)} - \mathbf{F}_{(1324)} - \mathbf{F}_{(3124)} + \mathbf{F}_{(2134)}$.

(k) $p = -\mathbf{F}_{(4321)} + \mathbf{F}_{(3421)} + \mathbf{F}_{(3241)} - \mathbf{F}_{(2341)} + \mathbf{F}_{(3214)} - \mathbf{F}_{(2314)} - \mathbf{F}_{(2134)} + \mathbf{F}_{(1234)}$.

(l) $p_{(\,,1)} = \mathbf{F}_{(2341)} + \mathbf{F}_{(2431)} + \mathbf{F}_{(4231)} - 2\mathbf{F}_{(1342)} - \mathbf{F}_{(1432)} - \mathbf{F}_{(4132)} - \mathbf{F}_{(3142)} - \mathbf{F}_{(3412)} + \mathbf{F}_{(1243)} + \mathbf{F}_{(2143)} + \mathbf{F}_{(2413)}$.

(m) $p_{(\,,1)} = \mathbf{F}_{(3421)} - \mathbf{F}_{(2431)} - \mathbf{F}_{(2314)} + \mathbf{F}_{(1324)}$.

References

[1] Marcelo Aguiar and Frank Sottile, *Structure of the Malvenuto-Reutenauer Hopf algebra of permutations*, Adv. Math. **191** (2005), no. 2, 225–275, math.CO/0203282.

[2] Gérard Duchamp, Florent Hivert, and Jean-Yves Thibon, *Some generalizations of quasi-symmetric functions and noncommutative symmetric functions*, Springer, Berlin, 2000, math.CO/0105065.

[3] Loïc Foissy, *Les algèbres de Hopf des arbres enracinés, II*, Bull. Sci. Math. **126** (2002), 249–288.

[4] ______, *Bidendriform bialgebras, trees, anf quasi-symmetric functions*, J. Pure Appl. Algebra **209** (2007), 429–459.

[5] Jean-Louis Loday, *Dialgebras*, Lecture Notes in Math., no. 1763, Springer, Berlin, 2001, math.QA/0102053.

[6] Jean-Louis Loday and Maria O. Ronco, *Hopf algebra of the planar binary trees*, Adv. Math. **139** (1998), no. 2, 293–309.

[7] Claudia Malvenuto and Christophe Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra*, J. Algebra **177** (1995), no. 3, 967–982.

[8] Maria O. Ronco, *Primitive elements of a free dendriform algebra*, Contemp. Math. **267** (2000), 245–263.

[9] ______, *Eulerian idempotents and Milnor-Moore theorem for certain non-cocommutative Hopf algebras*, J. Algebra **254** (2002), no. 1, 152–172.