The infinitesimal Hopf algebra and the operads of planar forests

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ABSTRACT. We introduce two operads which own the set of planar forests as a basis. With its usual product and two other products defined by different types of graftings, the algebra of planar rooted trees \( H \) becomes an algebra over these operads. The compatibility with the infinitesimal coproduct of \( H \) and these structures is studied. As an application, an inductive way of computing the dual basis of \( H \) for its infinitesimal pairing is given. Moreover, three Cartier-Quillen-Milnor-Moore theorems are given for the operads of planar forests and a rigidity theorem for one of them.

KEYWORDS. Infinitesimal Hopf algebra, Planar rooted trees, Operads.

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Introduction

The Connes-Kreimer Hopf algebra of rooted trees, introduced in [1, 6, 7, 8], is a commutative, non cocommutative Hopf algebra, its coproduct being given by admissible cuts of trees. A non commutative version, the Hopf algebra of planar rooted trees, is introduced in [3, 5]. We furthermore introduce in [4] an infinitesimal version of this object, replacing admissible cuts by left admissible cuts: this last object is here denoted by \( \mathcal{H} \). Similarly with the Hopf case, \( \mathcal{H} \) is a self-dual object and it owns a non-degenerate, symmetric Hopf pairing, denoted by \(<-,->\). This pairing is related to a partial order on the set of planar rooted forests, making it isomorphic to the Tamari poset. As a consequence, \( \mathcal{H} \) is given a dual basis denoted by \((f_F)_{F \in \mathcal{P}}\), indexed by the set \( \mathcal{F} \) of planar forest. In particular, the sub-family \((f_t)_{t \in \mathcal{T}}\) indexed by the set of planar rooted trees \( \mathcal{T} \) is a basis of the space of primitive elements of \( \mathcal{H} \).

The aim of this text is to introduce two structures of operad on the space of planar forests. We introduce two (non-symmetric) operads \( \mathcal{P}_\triangleleft \) and \( \mathcal{P}_\rightarrow \) defined in the following way:

1. \( \mathcal{P}_\triangleleft \) is generated by \( m \) and \( \triangleleft \in \mathcal{P}_\triangleleft (2) \), with relations:
\[
\begin{align*}
m \circ (\triangleleft, I) &= \triangleleft \circ (I, m), \\
m \circ (m, I) &= m \circ (I, m), \\
\triangleleft \circ (m, I) &= \triangleleft \circ (I, \triangleleft).
\end{align*}
\]

2. \( \mathcal{P}_\rightarrow \) is generated by \( m \) and \( \rightarrow \in \mathcal{P}_\rightarrow (2) \), with relations:
\[
\begin{align*}
m \circ (\rightarrow, I) &= \rightarrow \circ (I, m), \\
m \circ (m, I) &= m \circ (I, m), \\
\rightarrow \circ (\rightarrow, I) &= \rightarrow \circ (I, \rightarrow).
\end{align*}
\]

We then introduce two products on \( \mathcal{H} \) or on its augmentation ideal \( \mathcal{M} \), denoted by \( \triangleleft \) and \( \rightarrow \). The product \( F \triangleleft G \) consists of grafting \( F \) on the left leave of \( G \) and the product \( F \rightarrow G \) consists of grafting \( F \) on the left root of \( G \). Together with its usual product \( m \), \( \mathcal{M} \) becomes both a \( \mathcal{P}_\triangleleft \)- and a \( \mathcal{P}_\rightarrow \)-algebra. More precisely, \( \mathcal{M} \) is the free \( \mathcal{P}_\triangleleft \)- and \( \mathcal{P}_\rightarrow \)-algebra generated by a single element \( \triangledown \). As a consequence, \( \mathcal{P}_\triangleleft \) and \( \mathcal{P}_\rightarrow \) inherits a combinatorial representation using planar forests, with composition iteratively described using the products \( \triangleleft \) and \( \rightarrow \).

We then give several applications of these operadic structures. For example, the antipode of \( \mathcal{H} \) is described in term of the operad \( \mathcal{P}_\triangleleft \). We show how to compute elements \( f_t \)'s, with \( t \in \mathcal{T} \), using the action of \( \mathcal{P}_\triangleleft \), and the elements \( f_F \)'s, \( F \in \mathcal{F} \) from the preceding ones using the action of \( \mathcal{P}_\rightarrow \). Combining all these results, it is possible to compute by induction the basis \((f_F)_{F \in \mathcal{F}}\).

We finally study the compatibilities of products \( m, \triangleleft, \rightarrow \), the coproduct \( \tilde{\Delta} \rightarrow \) dual of \( \rightarrow \). This leads to the definition of two types of \( \mathcal{P}_\rightarrow \)-bialgebras, and one type of \( \mathcal{P}_\triangleleft \)-bialgebras. Each type then define a suboperad of \( \mathcal{P}_\rightarrow \) or \( \mathcal{P}_\triangleleft \) corresponding to primitive elements of \( \mathcal{M} \), which are explicitely described:

1. The first one is a free operad, generated by the element \( 1 \in \mathcal{P}_\rightarrow (2) \). As a consequence, the space of primitive elements of \( \mathcal{H} \) admits a basis \((p_t)_{t \in \mathcal{T}}\) indexed by the set of planar binary trees. The link with the basis \((f_t)_{t \in \mathcal{T}}\) is given with the help of the Tamari order.

2. The second one admits a combinatorial representation in terms of planar rooted trees. It is generated by the corollas \( c_n \in \mathcal{P}_\rightarrow (n), n \geq 2 \), with the following relations: for all \( k, l \geq 2 \),
\[
c_k \circ (c_l, I, \ldots, I) = c_l \circ (I, \ldots, I, c_k).
\]
3. The third one admits a combinatorial representation in terms of planar rooted trees, and is freely generated by \( I \in P \setminus (2) \).

We also give the definition of a double \( P \cdot \cdot \cdot \)-bialgebra, combining the two types of \( P \cdot \cdot \cdot \)-bialgebras already introduced. We then prove a rigidity theorem: any double \( P \cdot \cdot \cdot \)-bialgebra connected as a coalgebra is isomorphic to a decorated version of \( M \).

This text is organised as follows: the first section gives several recalls on the infinitesimal Hopf algebra of planar rooted trees and its pairing. The two products \( \setminus \) and \( \nearrow \) are introduced in section 2, as well as the combinatorial representation of the two associated operads. The applications to the computation of \( (fF)_{F \in F} \) is given in section 3. Section 4 is devoted to the study of the suboperads of primitive elements and the last section deals with the rigidity theorem for double \( P \cdot \cdot \cdot \)-bialgebras.

**Notations.**

1. We shall denote by \( K \) a commutative field, of any characteristic. Every vector space, algebra, coalgebra, etc, will be taken over \( K \).

2. Let \( (A, \Delta, \varepsilon) \) be a counitary coalgebra. Let \( 1 \in A \), non zero, such that \( \Delta(1) = 1 \otimes 1 \). We then define the non counitary coproduct:

\[
\tilde{\Delta}: \begin{cases}
(\text{Ker}(\varepsilon)) & \rightarrow (\text{Ker}(\varepsilon)) \otimes (\text{Ker}(\varepsilon)) \\
\tilde{\Delta}(a) & = \Delta(a) - a \otimes 1 - 1 \otimes a.
\end{cases}
\]

We shall use the Sweedler notations \( \Delta(a) = a^{(1)} \otimes a^{(2)} \) and \( \tilde{\Delta}(a) = a' \otimes a'' \).

1 Planar rooted forests and their infinitesimal Hopf algebra

We here recall some results and notations of [4].

1.1 Planar trees and forests

1. The set of planar trees is denoted by \( T \), and the set of planar forests is denoted by \( F \). The weight of a planar forest is the number of its vertices. For all \( n \in \mathbb{N} \), we denote by \( F(n) \) the set of planar forests of weight \( n \).

**Examples.** Planar rooted trees of weight \( \leq 5 \):

\[
., 1, \text{V}, \text{J}, \text{Y}, \text{I}, \text{V}, \text{J}, \text{Y}, \text{I}, \text{V}, \text{J}, \text{Y}, \text{I}, \text{V}, \text{J}, \text{Y}, \text{I}, \text{V}, \text{J}, \text{Y}, \text{I}.
\]

Planar rooted forests of weight \( \leq 4 \):

\[
1, . . . , . . . , 1, . . . , \text{V}, \text{I}, . . . , 1, . . . , \text{V}, . . . , \text{V}, \text{J}, \text{I}, . . . , \text{V}, \text{I}, \text{V}, \text{J}, \text{Y}.
\]

2. The algebra \( H \) is the free associative, unitary algebra generated by \( T \). As a consequence, a linear basis of \( H \) is given by \( F \), and its product is given by the concatenation of planar forests.

3. We shall also need two partial orders and a total order on the set \( \text{Vert}(F) \) of vertices of \( F \in F \), defined in [3, 4]. We put \( F = t_1 \ldots t_n \), and let \( s, s' \) be two vertices of \( F \).

(a) We shall say that \( s \geq_{\text{high}} s' \) if there exists a path from \( s' \) to \( s \) in \( F \), the edges of \( F \) being oriented from the roots to the leaves. Note that \( \geq_{\text{high}} \) is a partial order, whose Hasse graph is the forest \( F \).
(b) If $s$ and $s'$ are not comparable for $\geq_{\text{high}}$, we shall say that $s \geq_{\text{left}} s'$ if one of these assertions is satisfied:

i. $s$ is a vertex of $t_i$ and $s'$ is a vertex of $t_j$, with $i < j$.

ii. $s$ and $s'$ are vertices of the same $t_i$, and $s \geq_{\text{left}} s'$ in the forest obtained from $t_i$ by deleting its root.

This defines the partial order $\geq_{\text{left}}$ for all forests $F$, by induction on the the weight.

(c) We shall say that $s \geq_{\text{h,l}} s'$ if $s \geq_{\text{high}} s'$ or $s \geq_{\text{left}} s'$. This defines a total order on the vertices of $F$.

1.2 Infinitesimal Hopf algebra of planar forests

1. Let $F \in \mathcal{F}$. An admissible cut is a non empty cut of certain edges and trees of $F$, such that each path in a non-cut tree of $F$ meets at most one cut edge. The set of admissible cuts of $F$ will be denoted by $\text{Adm}(F)$. If $c$ is an admissible cut of $F$, the forest of the vertices which are over the cuts of $c$ will be denoted by $P^c(t)$ (branch of the cut $c$), and the remaining forest will be denoted by $R^c(t)$ (trunk of the cut). An admissible cut of $F$ will be said to be left-admissible if, for all vertices $x$ and $y$ of $F$, $x \in P^c(F)$ and $x \leq_{\text{left}} y$ imply that $y \in P^c(F)$. The set of left-admissible cuts of $F$ will be denoted by $\text{Adm}^l(F)$.

2. $\mathcal{H}$ is given a coproduct by the following formula: for all $F \in \mathcal{F}$,

$$\Delta(F) = \sum_{c \in \text{Adm}^l(F)} P^c(F) \otimes R^c(F) + F \otimes 1 + 1 \otimes F.$$ 

Then $(\mathcal{H}, \Delta)$ is an infinitesimal bialgebra, that is to say: for all $x, y \in \mathcal{H}$,

$$\Delta(xy) = (x \otimes 1)\Delta(y) + \Delta(x)(1 \otimes y) - x \otimes y.$$ 

Examples.

$$\Delta(,) = . \otimes 1 + 1 \otimes .,$$

$$\Delta(\ldots) = \ldots \otimes 1 + 1 \otimes \ldots + \ldots \otimes .,$$

$$\Delta(1) = 1 \otimes 1 + 1 \otimes 1 + 1 \otimes .,$$

$$\Delta(1,) = 1 \otimes 1 + 1 \otimes 1 \ldots + 1 \otimes \ldots + 1 \otimes .,$$

$$\Delta(V) = V \otimes 1 + 1 \otimes V + \ldots \otimes + \ldots \otimes V,$$

$$\Delta(\ldots) = \ldots \otimes 1 + 1 \otimes \ldots + \ldots \otimes V + \ldots \otimes V + \ldots \otimes \ldots$$

$$\Delta(\ldots) = \ldots \otimes 1 + 1 \otimes \ldots + \ldots \otimes \ldots + \ldots \otimes 1 + 1 \otimes .$$

$$\Delta(V) = V \otimes 1 + 1 \otimes V + \ldots \otimes + \ldots \otimes V + \ldots \otimes V + \ldots \otimes \ldots$$

$$\Delta(\ldots) = \ldots \otimes 1 + 1 \otimes \ldots + \ldots \otimes 1 + 1 \otimes .$$

$$\Delta(11) = 11 \otimes 1 + 1 \otimes 11 + \ldots \otimes 1 + 1 \otimes 1 \otimes .$$
We proved in [4] that $\mathcal{H}$ is an infinitesimal Hopf algebra, that is to say has an antipode $S$. This antipode satisfies $S(1) = 1$, $S(t) \in \text{Prim}(\mathcal{H})$ for all $t \in \mathcal{T}$, and $S(F) = 0$ for all $F \in \mathcal{F} \setminus (\mathcal{T} \cup \{1\})$.

1.3 Pairing on $\mathcal{H}$

1. We define the operator $B^+ : \mathcal{H} \rightarrow \mathcal{H}$, which associates, to a forest $F \in \mathcal{F}$, the tree obtained by grafting the roots of the trees of $F$ on a common root. For example, $B^+(\mathcal{V}) = \mathcal{V}$, and $B^+(\mathcal{W}) = \mathcal{W}$.

2. The application $\gamma$ is defined by:

\[
\gamma : \begin{cases} 
\delta_{11} \ldots \delta_{tn} \in \mathcal{F} \rightarrow \mathcal{H}, \\
\tau_{1} \ldots \tau_{n} \in \mathcal{F} \rightarrow \mathcal{H} 
\end{cases}
\]

3. There exists a unique pairing $\langle - , - \rangle : \mathcal{H} \times \mathcal{H} \rightarrow K$, satisfying:

i. $\langle 1, x \rangle = \varepsilon(x)$ for all $x \in \mathcal{H}$.

ii. $\langle xy, z \rangle = \langle y \otimes x, \Delta(z) \rangle$ for all $x, y, z \in \mathcal{H}$.

iii. $\langle B^+(x), y \rangle = \langle x, \gamma(y) \rangle$ for all $x, y \in \mathcal{H}$.

Moreover:

iv. $\langle - , - \rangle$ is symmetric and non-degenerate.

v. If $x$ and $y$ are homogeneous of different weights, $\langle x, y \rangle = 0$.

vi. $\langle S(x), y \rangle = \langle x, S(y) \rangle$ for all $x, y \in \mathcal{H}$.

This pairing admits a combinatorial interpretation using the partial orders $\geq_{\text{left}}$ and $\geq_{\text{high}}$ and is related to the Tamari order on planar binary trees, see [4].

4. We denote by $(f_F)_{F \in \mathcal{F}}$ the dual basis of the basis of forests for the pairing $\langle - , - \rangle$. In other terms, for all $F \in \mathcal{F}$, $f_F$ is defined by $(f_F, G) = \delta_{FG}$, for all forest $G \in \mathcal{F}$. The family $(f_t)_{t \in \mathcal{T}}$ is a basis of the space $\text{Prim}(\mathcal{H})$ of primitive elements of $\mathcal{H}$.

2 The operads of forests and graftings

2.1 A few recalls on non-$\Sigma$-operads

1. We shall work here with non-$\Sigma$-operads [11]. Recall that such an object is a family $\mathbb{P} = (\mathbb{P}(n))_{n \in \mathbb{N}}$ of vector spaces, together with a composition for all $n, k_1, \ldots, k_n \in \mathbb{N}$:

\[
\begin{cases}
\mathbb{P}(n) \otimes \mathbb{P}(k_1) \otimes \cdots \otimes \mathbb{P}(k_n) \rightarrow \mathbb{P}(k_1 + \cdots + k_n) \\
p \otimes p_1 \otimes \cdots \otimes p_n \rightarrow p \circ (p_1, \ldots, p_n).
\end{cases}
\]
The following associativity condition is satisfied: for all $p \in \mathbb{P}(n)$, $p_1 \in \mathbb{P}(k_1)$, \ldots, $p_n \in \mathbb{P}(k_n)$, $p_1, \ldots, p_n, k_1, \ldots, k_n \in \mathbb{P}$,

\[
(p \circ (p_1, \ldots, p_n)) \circ (p_{1.1}, \ldots, p_{1.k_1}, \ldots, p_{n.1}, \ldots, p_{n.k_n}) = p \circ (p_1 \circ (p_{1.1}, \ldots, p_{1.k_1}), \ldots, p_n \circ (p_{n.1}, \ldots, p_{n.k_n})).
\]

Moreover, there exists a unit element $I \in \mathbb{P}(1)$, satisfying: for all $p \in \mathbb{P}(n)$,

\[
\begin{align*}
\{ p \circ (I, \ldots, I) &= p, \\
I \circ p &= p.
\end{align*}
\]

An operad is a non-$\Sigma$-operad $\mathbb{P}$ with a right action of the symmetric group $S_n$ on $\mathbb{P}(n)$ for all $n$, satisfying a certain compatibility with the composition.

2. Let $\mathbb{P}$ be a non-$\Sigma$-operad. A $\mathbb{P}$-algebra is a vector space $A$, together with an action of $\mathbb{P}$:

\[
\begin{align*}
\{ \mathbb{P}(n) \otimes A^\otimes & \rightarrow A, \\
p \otimes a_1 \otimes \ldots \otimes a_n & \rightarrow p.(a_1, \ldots, a_n),
\end{align*}
\]

satisfying the following compatibility: for all $p \in \mathbb{P}(n)$, $p_1 \in \mathbb{P}(k_1)$, \ldots, $p_n \in \mathbb{P}(k_n)$, for all $a_1, \ldots, a_{n,k} \in A$,

\[
(p \circ (p_1, \ldots, p_n)).(a_1,1, \ldots, a_{1,k_1}, \ldots, a_{n,1}, \ldots, a_{n,k_n}) = p.(p_1.(a_1,1, \ldots, a_{1,k_1}), \ldots, p_n.(a_{n,1}, \ldots, a_{n,k_n})).
\]

Moreover, $I.a = a$ for all $a \in A$.

In particular, if $V$ is a vector space, the free $\mathbb{P}$-algebra generated by $V$ is:

\[
F_{\mathbb{P}}(V) = \bigoplus_{n \in \mathbb{N}} \mathbb{P}(n) \otimes V^\otimes n,
\]

with the action of $\mathbb{P}$ given by:

\[
p.(p_1 \otimes a_{1,1} \otimes \ldots \otimes a_{1,k_1}, \ldots, p_n \otimes a_{n,1} \otimes \ldots \otimes a_{n,k_n}) = (p \circ (p_1, \ldots, p_n)) \otimes a_{1,1} \otimes \ldots \otimes a_{1,k_1} \otimes \ldots \otimes a_{n,1} \otimes \ldots \otimes a_{n,k_n}.
\]

3. Let $T_b$ be the set of planar binary trees:

\[
T_b = \left\{ 1, Y, Y, Y, Y, Y, Y, Y, Y, Y, \ldots \right\}.
\]

For all $n \in \mathbb{N}$, $T_b(n)$ is the vector space generated by the elements of $T_b$ with $n$ leaves:

\[
T_b(0) = (0),
T_b(1) = Vect(1),
T_b(2) = Vect(Y),
T_b(3) = Vect(Y, Y),
T_b(4) = Vect(Y, Y, Y, Y, Y, Y, Y, Y).
\]
The family of vector spaces $T_b$ is given a structure of non-$\Sigma$-operad by graftings on the leaves. More precisely, if $t, t_1, \ldots, t_n \in T_b$, $t$ with $n$ leaves, then $t \circ (t_1, \ldots, t_n)$ is the binary tree obtained by grafting $t_1$ on the first leaf of $t$, $t_2$ on the second leaf of $t$, and so on (note that the leaves of $t$ are ordered from left to right). The unit is $1$.

It is known that $T_b$ is the free non-$\Sigma$-operad generated by $\Upsilon \in T_b(2)$. Similarly, given elements $m_1, \ldots, m_k$ in $P(2)$, it is possible to describe the free non-$\Sigma$-operad $P$ generated by these elements in terms of planar binary trees whose internal vertices are decorated by $m_1, \ldots, m_k$.

### 2.2 Presentations of the operads of forests

**Definition 1**

1. $P\setminus$ is the non-$\Sigma$-operad generated by $m$ and $\setminus \in P\setminus(2)$, with relations:

   \[
   \left\{ \begin{array}{l}
   m \circ (\setminus, I) = \setminus \circ (I, m), \\
m \circ (m, I) = m \circ (I, m), \\
\setminus \circ (m, I) = \setminus \circ (I, \setminus). 
\end{array} \right.
   \]

2. $P\nearrow$ is the non-$\Sigma$-operad generated by $m$ and $\nearrow \in P\nearrow(2)$, with relations:

   \[
   \left\{ \begin{array}{l}
m \circ (\nearrow, I) = \nearrow \circ (I, m), \\
m \circ (m, I) = m \circ (I, m), \\
\nearrow \circ (\nearrow, I) = \nearrow \circ (I, \nearrow). 
\end{array} \right.
   \]

**Remark.** We shall prove in [2] that these quadratic operads are Koszul.

### 2.3 Grafting on the root

Let $F, G \in F - \{1\}$. We put $G = t_1 \ldots t_n$ and $t_1 = B^+(G_1)$. We define:

\[F \setminus G = B^+(FG_1)G_2 \ldots G_n.\]

In other terms, $F$ is grafted on the root of the first tree of $G$, on the left. In particular, $F \setminus \cdot = B^+(F)$.

**Examples.**

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
\ldots & \setminus & 1 & \ldots & \Upsilon & \ldots & \Upsilon & \ldots & \Upsilon & \ldots & \Upsilon \\
1.1 & \setminus & 1 & \Upsilon & \ldots & 1 & \Upsilon & \ldots & 1 & \Upsilon & \ldots \\
1.1 & \setminus & \Upsilon & 1 & \ldots & \Upsilon & 1 & \ldots & \Upsilon & 1 & \ldots \\
1 & \setminus & \Upsilon & \Upsilon & 1 & \Upsilon & \Upsilon & 1 & \Upsilon & \Upsilon & 1 \\
1 & \setminus & \Upsilon & \Upsilon & \Upsilon & 1 & \Upsilon & \Upsilon & \Upsilon & 1 & \Upsilon \\
1 & \setminus & \Upsilon & \Upsilon & \Upsilon & \Upsilon & 1 & \Upsilon & \Upsilon & \Upsilon & \Upsilon \\
\end{array}
\]

Obviously, $\setminus$ can be inductively defined in the following way: for $F, G, H \in F - \{1\}$,

\[
\left\{ \begin{array}{l}
F \setminus \cdot = B^+(F), \\
F \setminus (GH) = (F \setminus G)H \\
F \setminus B^+(G) = B^+(FG). 
\end{array} \right.
\]

We denote by $\mathcal{M}$ the augmentation ideal of $\mathcal{H}$, that is to say the vector space generated by the elements of $F - \{1\}$. We extend $\setminus: \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ by linearity.
Proposition 2 For all \( x, y, z \in \mathcal{M} \):

\[
\begin{align*}
x \searrow (yz) &= (x \searrow y)z, \\
x \searrow (y \searrow z) &= (xy) \searrow z.
\end{align*}
\] (1) (2)

Proof. We can restrict ourselves to \( x, y, z \in \mathbf{F} - \{1\} \). Then (1) is immediate. In order to prove (2), we put \( z = B^+(z_1)z_2, z_1, z_2 \in \mathbf{F} \). Then:

\[
x \searrow (y \searrow z) = x \searrow (B^+(yz_1)z_2) = B^+(xyz_1)z_2 = (xy) \searrow (B^+(z_1)z_2) = (xy) \searrow z,
\]

which proves (2).

\[\square\]

Corollary 3 \( \mathcal{M} \) is given a graded \( \mathbb{F} \searrow \)-algebra structure by its products \( m \) and by \( \searrow \).

Proof. Immediate, by proposition 2.

\[\square\]

2.4 Grafting on the left leave

Let \( F, G \in \mathbf{F} \). Suppose that \( G \neq 1 \). Then \( F \nearrow G \) is the planar forest obtained by grafting \( F \) on the leave of \( G \) which is at most on the left. For \( G = 1 \), we put \( F \nearrow 1 = F \). In particular, \( F \nearrow \cdot = B^+(F) \).

Examples.

\[
\begin{align*}
... \nearrow 1 &= \nearrow \nearrow \nearrow \nearrow 1 \nearrow ... = \nearrow \nearrow \nearrow 1 \nearrow ... = \nearrow \nearrow \nearrow 1 \nearrow \nearrow ... = \nearrow, \\
1 \nearrow 1 &= \nearrow \nearrow 1 \nearrow 1 \nearrow 1 \nearrow = \nearrow \nearrow 1 \nearrow 1 \nearrow = \nearrow \nearrow 1 \nearrow 1 \nearrow = \nearrow \nearrow 1 \nearrow 1 \nearrow = \nearrow, \\
1 \nearrow 1 &= \nearrow \nearrow 1 \nearrow 1 \nearrow 1 \nearrow = \nearrow \nearrow 1 \nearrow 1 \nearrow = \nearrow \nearrow 1 \nearrow 1 \nearrow = \nearrow \nearrow 1 \nearrow 1 \nearrow = \nearrow, \\
V \nearrow 1 &= \nearrow \nearrow 1 \nearrow V \nearrow V \nearrow = \nearrow \nearrow 1 \nearrow V \nearrow V \nearrow = \nearrow \nearrow 1 \nearrow V \nearrow V \nearrow = \nearrow \nearrow 1 \nearrow V \nearrow V \nearrow = \nearrow, \\
1 \nearrow 1 &= \nearrow \nearrow 1 \nearrow 1 \nearrow 1 \nearrow = \nearrow \nearrow 1 \nearrow 1 \nearrow = \nearrow \nearrow 1 \nearrow 1 \nearrow = \nearrow \nearrow 1 \nearrow 1 \nearrow = \nearrow.
\end{align*}
\]

In an obvious way, \( \nearrow \) can be inductively defined in the following way: for \( F, G, H \in \mathbf{F} \),

\[
\begin{align*}
F \nearrow 1 &= F, \\
F \nearrow (GH) &= (F \nearrow G)H \text{ if } G \neq 1, \\
F \nearrow B^+(G) &= B^+(F \nearrow G).
\end{align*}
\]

We extend \( \nearrow : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \) by linearity.

Proposition 4 1. For all \( x, z \in \mathcal{H}, y \in \mathcal{M} \):

\[
x \nearrow (yz) = (x \nearrow y)z.
\] (3)

2. For all \( x, y, z \in \mathcal{H} \):

\[
x \nearrow (y \nearrow z) = (x \nearrow y) \nearrow z.
\]

So \( (\mathcal{H}, \nearrow) \) is an associative algebra, with unitary element \( 1 \).
Proof. Note that (3) is immediate for \(x, y, z \in \mathbb{F}\), with \(y \neq 1\). This implies the first point. In order to prove the second point, we consider:

\[
Z = \{z \in \mathcal{H} / \forall x, y \in \mathcal{H}, x \not\nearrow (y \nearrow z) = (x \nearrow y) \nearrow z\}.
\]

Let us first prove that \(1 \in Z\): for all \(x, y \in \mathcal{H}\),

\[
x \nearrow (y \nearrow 1) = x \nearrow y = (x \nearrow y) \nearrow 1.
\]

Let \(z_1, z_2 \in Z\). Let us show that \(z_1z_2 \in Z\). By linearity, we can separate the proof into two cases:

1. \(z_1 = 1\). Then it is obvious.
2. \(\varepsilon(z_1) = 0\). Let \(x, y \in \mathcal{H}\). By the first point:

\[
x \nearrow (y \nearrow (z_1z_2)) = x \nearrow ((y \nearrow z_1)z_2) = (x \nearrow (y \nearrow z_1))z_2 = ((x \nearrow y) \nearrow z_1)z_2 = (x \nearrow y) \nearrow (z_1z_2).
\]

So \(Z\) is a subalgebra of \(\mathcal{H}\). Let us show that it is stable by \(B^+\). Let \(z \in Z, x, y \in \mathcal{H}\). Then:

\[
x \nearrow (y \nearrow B^+(z)) = x \nearrow B^+(y \nearrow z) = B^+(x \nearrow (y \nearrow z)) = B^+((x \nearrow y) \nearrow z) = (x \nearrow y) \nearrow B^+(z).
\]

So \(Z\) is a subalgebra of \(\mathcal{H}\), stable by \(B^+\). Hence, \(Z = \mathcal{H}\). \[\square\]

Remarks.

1. (3) is equivalent to: for any \(x, y, z \in \mathcal{H}\),

\[
x \nearrow (yz) - \varepsilon(y)x \nearrow z = (x \nearrow y)z - \varepsilon(y)xz.
\]

2. Let \(F \in \mathbb{F} - \{1\}\). There exists a unique family \((\cdot F_1, \ldots, F_n)\) of elements of \(\mathbb{F}\) such that:

\[
F = (\cdot F_1) \nearrow \ldots \nearrow (\cdot F_n).
\]

For example, \(\vee 1. = (\cdot) \nearrow (\cdot) \nearrow (\cdot)\). As a consequence, \((\mathcal{H}, \nearrow)\) is freely generated by \(\cdot \mathbb{F}\) as an associative algebra.

Corollary 5 \(M\) is given a graded \(\mathbb{P} \nearrow\)-algebra structure by its product \(m\) and by \(\nearrow\).

Proof. Immediate, by proposition 4. \(\square\)

2.5 Dimensions of \(\mathbb{P} \searrow\) and \(\mathbb{P} \nearrow\)

We now compute the dimensions of \(\mathbb{P} \searrow(n)\) and \(\mathbb{P} \nearrow(n)\) for all \(n\) and deduce that \(M\) is the free \(\mathbb{P} \searrow\)- and \(\mathbb{P} \nearrow\)-algebra generated by \(\cdot \cdot\).

Notation. We denote by \(r_n\) the number of planar rooted forests and we put \(R(X) = \sum_{n=1}^{+\infty} r_nX^n\). It is well-known (see [3, 13]) that \(R(X) = \frac{1 - 2X - \sqrt{1 - 4X}}{2X}\).
Proposition 6 For \( \rightarrow \in \{\searrow, \nearrow\} \) and all \( n \in \mathbb{N}^* \), in the \( \mathbb{P}_\rightarrow \)-algebra \( \mathcal{M} \):

\[
\mathbb{P}_\rightarrow (n, \ldots, \ldots) = \text{Vect}(\text{planar forests of weight } n).
\]

As a consequence, \( \mathcal{M} \) is generated as a \( \mathbb{P}_\rightarrow \)-algebra by \( \ldots \).

Proof. \( \subseteq \). Immediate, as \( \mathcal{M} \) is a graded \( \mathbb{P}_\rightarrow \)-algebra.

\( \supseteq \). Induction on \( n \). For \( n = 1 \), \( 1. \) For each internal vertex \( s \) and all \( n \in \mathbb{N}^* \), \( \mathbb{P}_\rightarrow (n) \), two cases are possible.

1. \( F = F_1F_2 \), weight\( (F_1) = n_i < n \). By the induction hypothesis, there exists \( p_1, p_2 \in \mathbb{P}_\rightarrow \), such that \( F_1 = p_1.(\ldots, \ldots) \) and \( F_2 = p_2.(\ldots, \ldots) \). Then \( (m \circ (p_1, p_2)).(\ldots, \ldots) = m.(F_1, F_2) = F_1F_2 \).

2. \( F \in \mathbf{T} \). Let us put \( F = B^+(G) \). Then there exists \( p \in \mathbb{P}_\rightarrow \), such that \( p.(\ldots, \ldots) = G \).

Hence, in both cases, \( F \in \mathbb{P}_\rightarrow (n, \ldots, \ldots) \).

Corollary 7 For all \( \rightarrow \in \{\searrow, \nearrow\} \), \( n \in \mathbb{N}^* \), \( \text{dim}(\mathbb{P}_\rightarrow (n)) \geq r_n \).

Proof. Because we proved the surjectivity of the following application:

\[
\text{ev}_\rightarrow: \{ \mathbb{P}_\rightarrow (n) \to \text{Vect}(\text{planar forests of weight } n) \}
\]

For \( p \to p.(\ldots, \ldots) \).

Lemma 8 For all \( \rightarrow \in \{\searrow, \nearrow\} \), \( n \in \mathbb{N}^* \), \( \text{dim}(\mathbb{P}_\rightarrow (n)) \leq r_n \).

Proof. We prove it for \( \rightarrow = \nearrow \). Let us fix \( n \in \mathbb{N}^* \). Then \( \mathbb{P}_\nearrow (n) \) is linearly generated by planar binary trees whose internal vertices are decorated by \( m \) and \( \nearrow \). The following relations hold:

\[
\begin{array}{c}
\text{\nearrow} = \text{\nearrow} \\
\text{\nearrow}^{(m)} = \text{\nearrow}^{(m)} \\
\text{\nearrow}^{(m)} = \text{\nearrow}^{(m)} \\
\end{array}
\]

In the sequel of the proof, we shall say that such a tree is admissible if it satisfies the following conditions:

1. For each internal vertex \( s \) decorated by \( m \), the left child of \( s \) is a leave.
2. For each internal vertex \( s \) decorated by \( \nearrow \), the left child of \( s \) is a leave or is decorated by \( m \).

For example, here are the admissible trees with 1, 2 or 3 leaves:

\[
\begin{array}{c}
\begin{array}{c}
\text{\nearrow}^{(m)} \end{array} \text{, } \begin{array}{c}
\text{\nearrow}^{(m)} \end{array} \text{, } \begin{array}{c}
\text{\nearrow}^{(m)} \end{array} \text{, } \begin{array}{c}
\text{\nearrow}^{(m)} \end{array} \\
\end{array}
\]

The preceding relations imply that \( \mathbb{P}_\nearrow (n) \) is linearly generated by admissible trees with \( n \) leaves. So \( \text{dim}(\mathbb{P}_\nearrow (n)) \) is smaller than \( a_n \), the number of admissible trees with \( n \) leaves. For \( n \geq 2 \), we denote by \( b_n \) the number of admissible trees with \( n \) leaves whose root is decorated by

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Let \( m \), and by \( c_n \) the number of admissible trees with \( n \) leaves whose root is decorated by \( \nearrow \). We also put \( b_1 = 1 \) and \( c_1 = 0 \). Finally, we define:

\[
A(X) = \sum_{n \geq 1} a_n X^n, \quad B(X) = \sum_{n \geq 1} b_n X^n, \quad C(X) = \sum_{n \geq 1} c_n X^n.
\]

Immediately, \( A(X) = B(X) + C(X) \). Every admissible tree with \( n \geq 2 \) leaves whose root is decorated by \( m \) is of the form \( m \circ \uparrow \), where \( t \) is an admissible tree with \( n - 1 \) leaves. Hence, \( B(X) = XA(X) + X \). Moreover, every admissible tree with \( n \geq 2 \) leaves whose root is decorated by \( \nearrow \) is of the form \( \nearrow \circ (t_1, t_2) \), where \( t_1 \) is an admissible tree with \( k \) leaves whose eventual root is decorated by \( m \) and \( t_2 \) an admissible tree with \( n - k \) leaves \((1 \leq k \leq n - 1)\). Hence, for all \( n \geq 2 \), \( c_n = \sum_{k=1}^{n-1} b_k a_{n-k} \), so \( C(X) = B(X)A(X) \). As a conclusion:

\[
\begin{align*}
A(X) &= B(X) + C(X), \\
B(X) &= XA(X) + X, \\
C(X) &= B(X)A(X).
\end{align*}
\]

So \( A(X) = XA(X) + X + B(X)A(X) = XA(X) + X + XA(X)^2 + XA(X) \), and:

\[
XA(X)^2 + (2X - 1)A(X) + X = 0.
\]

As \( a_1 = 1 \):

\[
A(X) = \frac{1 - 2X - \sqrt{1 - 4X}}{2X} = R(X).
\]

So, for all \( n \geq 1 \), \( \dim(\mathbb{P}_{\nearrow}(n)) \leq a_n = r_n \). The proof is similar for \( \mathbb{P}_{\searrow} \). \( \square \)

As immediate consequences:

**Theorem 9** For \( \gamma \in \{\searrow, \nearrow\} \), \( n \in \mathbb{N}^* \), \( \dim(\mathbb{P}_{\gamma}(n)) = r_n \). Moreover, the following application is bijective:

\[
ev\gamma : \mathbb{P}_{\gamma}(n) \longrightarrow \text{Vect(planar forests of weight } n) \subseteq \mathcal{M}
\]

\[
\begin{array}{c}
p \\
\mapsto p.(\ldots, \ldots)
\end{array}
\]

**Corollary 10**

1. \((\mathcal{M}, m, \searrow)\) is the free \( \mathbb{P}_\searrow \)-algebra generated by \( \ldots \).

2. \((\mathcal{M}, m, \nearrow)\) is the free \( \mathbb{P}_\nearrow \)-algebra generated by \( \ldots \).

### 2.6 A combinatorial description of the composition

Let \( \gamma \in \{\searrow, \nearrow\} \). We identify \( \mathbb{P}_{\gamma} \) and the vector space of non-empty planar forests via theorem 9. In other terms, we identify \( F \in \mathcal{F}(n) \) and \( ev_{\gamma}^{-1}(F) \in \mathbb{P}_{\gamma}(n) \).

**Notations.**

1. In order to distinguish the compositions in \( \mathbb{P}_\searrow \) and \( \mathbb{P}_\nearrow \), we now denote:

   a. \( F \circ_{\searrow} (F_1, \ldots, F_n) \) the composition of \( \mathbb{P}_\searrow \),

   b. \( F \circ_{\nearrow} (F_1, \ldots, F_n) \) the composition of \( \mathbb{P}_\nearrow \).

2. In order to distinguish the action of the operads \( \mathbb{P}_\searrow \) and \( \mathbb{P}_\nearrow \) on \( \mathcal{M} \), we now denote:

   a. \( F \bullet_{\searrow} (x_1, \ldots, x_n) \) the action of \( \mathbb{P}_\searrow \) on \( \mathcal{M} \),

   b. \( F \bullet_{\nearrow} (x_1, \ldots, x_n) \) the action of \( \mathbb{P}_\nearrow \) on \( \mathcal{M} \).
We shall prove the following result:

**Theorem 11**  
1. The composition of $\mathbb{P}\setminus$ in the basis of planar forests can be inductively defined in this way:

\[
\begin{align*}
\mathcal{A}_q(H) &= H, \\
B^+(F)\mathcal{A}_q(H_1, \ldots, H_{n+1}) &= (F\mathcal{A}_q(H_1, \ldots, H_n)) \setminus H_{n+1}, \\
FG\mathcal{A}_q(H_1, \ldots, H_{n+1:n+2}) &= F\mathcal{A}_q(H_1, \ldots, H_{n+1})G\mathcal{A}_q(H_{n+1+1}, \ldots, H_{n+1:n+2}).
\end{align*}
\]

2. The composition of $\mathbb{P}/$ in the basis of planar forests can be inductively defined in this way:

\[
\begin{align*}
\mathcal{B}^\varnothing(H) &= H, \\
B^+(F)\mathcal{B}^\varnothing(H_1, \ldots, H_{n+1}) &= (F\mathcal{B}^\varnothing(H_1, \ldots, H_n)) / H_{n+1}, \\
FG\mathcal{B}^\varnothing(H_1, \ldots, H_{n+1:n+2}) &= F\mathcal{B}^\varnothing(H_1, \ldots, H_{n+1})G\mathcal{B}^\varnothing(H_{n+1+1}, \ldots, H_{n+1:n+2}).
\end{align*}
\]

**Examples.** Let $F_1, F_2, F_3 \in F - \{1\}$.

\[
\begin{align*}
\mathcal{B}^\varnothing(F_1, F_2) &= F_1F_2, & \mathcal{B}(F_1, F_2) &= F_1F_2, \\
\mathcal{B}^\varnothing(F_1, F_2) &= F_1 / F_2, & \mathcal{B}(F_1, F_2) &= F_1 \setminus F_2, \\
\mathcal{B}^\varnothing(F_1, F_2, F_3) &= F_1F_2F_3, & \mathcal{B}(F_1, F_2, F_3) &= F_1F_2F_3, \\
\mathcal{B}^\varnothing(F_1, F_2, F_3) &= F_1(F_2 / F_3), & \mathcal{B}(F_1, F_2, F_3) &= F_1(F_2 \setminus F_3), \\
\mathcal{B}^\varnothing(F_1, F_2, F_3) &= (F_1 / F_2)F_3, & \mathcal{B}(F_1, F_2, F_3) &= (F_1 \setminus F_2)F_3, \\
\mathcal{B}^\varnothing(F_1, F_2, F_3) &= (F_1 / F_2) / F_3, & \mathcal{B}(F_1, F_2, F_3) &= (F_1 \setminus F_2) \setminus F_3.
\end{align*}
\]

**Proposition 12** Let $\mathcal{B}^\varnothing \in \{\setminus, /\}$.

1. $\mathcal{B}^\varnothing$ is the unit element of $\mathbb{P}\setminus$.

2. $\mathcal{B}^\varnothing = m$ in $\mathbb{P}\setminus(2)$. Consequently, in $\mathbb{P}\setminus$, $\mathcal{B}^\varnothing (F, G) = FG$ for all $F, G \in F - \{1\}$.

3. Let $F, G \in F$. In $\mathbb{P}\setminus$, $1 = \mathcal{B}^\varnothing$. Consequently, $1 \mathcal{B}^\varnothing (F, G) = F \mathcal{B}^\varnothing G$ for all $F, G \in F - \{1\}$.

**Proof.**

1. Indeed, $ev_{\setminus}(\cdot) = \mathcal{B}^\varnothing = ev_{\setminus}(I)$. Hence, $\cdot = I$.

2. By definition, $ev_{\setminus}(\cdot) = \mathcal{B}^\varnothing (m)$. So $\mathcal{B} = m$ in $\mathbb{P}\setminus(2)$. Moreover, for all $F, G \in F - \{1\}$:

\[
ev_{\setminus}(FG) = FG = m \mathcal{B}^\varnothing (F, G) = m \mathcal{B}^\varnothing (F \mathcal{B}^\varnothing (\cdot, \ldots, \cdot), G \mathcal{B}^\varnothing (\cdot, \ldots, \cdot)) = \left( m \mathcal{B}^\varnothing (F, G) \right) \mathcal{B}^\varnothing (\cdot, \ldots, \cdot) = ev_{\setminus}(m \mathcal{B}^\varnothing (F, G)).
\]

So $FG = m \mathcal{B}^\varnothing (F, G) = \mathcal{B}^\varnothing (F, G)$. 

12
3. Indeed, \( ev_\rightarrow (1) = \frac{\partial}{\partial x}. = ev_\rightarrow (\frac{\partial}{\partial y}) \). So \( 1 = \frac{\partial}{\partial x} \) in \( P_\rightarrow \{2\} \). Moreover:

\[
ev_\rightarrow (F \rightarrow G) = F \rightarrow G
\]

\[
= \frac{\partial}{\partial x} (F, G)
\]

\[
= \frac{\partial}{\partial y} (F, (\ldots, \ldots), G (\ldots, \ldots))
\]

\[
= (\frac{\partial}{\partial y} (F, G)) (\ldots, \ldots)
\]

\[
= ev_\rightarrow (\frac{\partial}{\partial y} (F, G)).
\]

So, \( F \rightarrow G = \frac{\partial}{\partial y} (F, G) = 1 \phi (F, G). \)

\[\square\]

**Proposition 13** 1. Let \( F, G \in F \), different from 1, of respective weights \( n_1 \) and \( n_2 \). Let \( H_{1,1}, \ldots, H_{1,n_1} \) and \( H_{2,1}, \ldots, H_{2,n_2} \in F - \{1\} \). Let \( \rightarrow \in \{\langle, \rangle\} \). Then, in \( P_\rightarrow \): 

\[
(FG) \phi (H_{1,1}, \ldots, H_{1,n_1}, H_{2,1}, \ldots, H_{2,n_2}) = F \phi (H_{1,1}, \ldots, H_{1,n_1}) G \phi (H_{2,1}, \ldots, H_{2,n_2}).
\]

2. Let \( F \in F \), of weight \( n \geq 1 \). Let \( H_1, \ldots, H_{n+1} \in F \). In \( P_\rightarrow \):

\[
B^+(F) \phi (H_1, \ldots, H_{n+1}) = (F \phi (H_1, \ldots, H_n)) \phi (H_{n+1}).
\]

**Proof.**

1. Indeed, in \( P_\rightarrow \):

\[
(FG) \phi (H_{1,1}, \ldots, H_{1,n_1}, H_{2,1}, \ldots, H_{2,n_2})
\]

\[
= (m \phi (F, G)) \phi (H_{1,1}, \ldots, H_{1,n_1}, H_{2,1}, \ldots, H_{2,n_2})
\]

\[
= m \phi (F \phi (H_{1,1}, \ldots, H_{1,n_1}), G \phi (H_{2,1}, \ldots, H_{2,n_2}))
\]

\[
= F \phi (H_{1,1}, \ldots, H_{1,n_1}) G \phi (H_{2,1}, \ldots, H_{2,n_2}).
\]

2. In \( P_\rightarrow \):

\[
B^+(F) \phi (H_1, \ldots, H_{n+1}) = (F \rightarrow, \phi (H_1, \ldots, H_{n+1})
\]

\[
= (1 \phi (F, \cdot)) \phi (H_1, \ldots, H_{n+1})
\]

\[
= 1 \phi (F \phi (H_1, \ldots, H_n), \phi (H_{n+1}))
\]

\[
= 1 \phi (F \phi (H_1, \ldots, H_n), H_{n+1})
\]

\[
= (F \phi (H_1, \ldots, H_n)) \phi (H_{n+1}).
\]

\[\square\]

**Combining propositions 12 and 13, we obtain theorem 11.**

3 **Applications to the infinitesimal Hopf algebra \( H \)**

3.1 **Antipode of \( H \)**

We here give a description of the antipode of \( H \) in terms of the action \( \langle, \rangle \) of the operad \( P_\rightarrow \).
Notations. For all \( n \in \mathbb{N}^* \), we denote \( l_n = (B^+)^n(1) \in F(n) \). For example:
\[
\begin{align*}
l_1 &= \ldots, \\
l_2 &= 1, \\
l_3 &= 1, \\
l_4 &= 1, \\
l_5 &= 1 \ldots.
\end{align*}
\]

**Lemma 14** Let \( t \in T \). There exists a unique \( k \in \mathbb{N}^* \), and a unique family \( (t_2, \ldots, t_k) \in T^{k-1} \) such that:
\[
t = l_k \bullet (\ldots, t_2, \ldots, t_k).
\]

**Proof.** Induction on the weight \( n \) of \( t \). If \( n = 1 \), then \( t = q \), so \( k = 1 \) and the family is empty.

We suppose the result at all rank \( < n \). We put \( t = B^+ (s_1, \ldots, s_m) \). Necessarily, \( t_k = B^+ (s_2, \ldots, s_m) \) and \( l_{n-1} \bullet (\ldots, t_2, \ldots, t_{k-1}) = s_1 \). We conclude with the induction hypothesis on \( s_1 \). \( \square \)

Example.
\[
\begin{align*}
\mathcal{Y} &= l_4 \bullet (\ldots, 1, 1, \mathcal{Y}).
\end{align*}
\]

**Definition 15** For all \( n \in \mathbb{N}^* \), we put \( p_n = \sum_{k=1}^{n} \sum_{\forall i, a_i > 0} (-1)^k a_1 \ldots a_k \).

**Examples.**
\[
\begin{align*}
p_1 &= \ldots, \\
p_2 &= -1 + \ldots, \\
p_3 &= -1 + 1 + 1 - \ldots, \\
p_4 &= -1 + 1 + 1 + 1 - \ldots - \ldots - \ldots + 1 + \ldots.
\end{align*}
\]

Remark that \( p_n \) is in fact the antipode of \( l_n \) in \( \mathcal{H} \). It is also the antipode of \( l_n \) in the non commutative Connes-Kreimer Hopf algebra of planar trees [3].

**Corollary 16** Let \( t \in T \), written under the form \( t = l_k \bullet (t_1, \ldots, t_k) \), with \( t_1 = \ldots \). Then:
\[
S(t) = p_k \bullet (t_1, \ldots, t_k).
\]

**Proof.** Corollary of proposition 15 of [4], observing that left cuts are cuts on edges from the root of \( t_i \) to the root of \( t_{i+1} \) in \( t \), for \( i = 1, \ldots, n-1 \). \( \square \)

3.2 Inverse of the application \( \gamma \)

**Proposition 17** The restriction \( \gamma : \text{Prim}(\mathcal{H}) \to \mathcal{H} \) is bijective.

**Proof.** By proposition 21 of [4]:
\[
\gamma|_{\text{Prim}(\mathcal{H})} : \begin{cases}
\text{Prim}(\mathcal{H}) & \to \mathcal{H} \\
 f_{B^+(F)} (F \in F) & \to f_F.
\end{cases}
\]

So this restriction is clearly bijective. \( \square \)

We shall denote \( \gamma^{-1}_{|\text{Prim}(\mathcal{H})} : \mathcal{H} \to \text{Prim}(\mathcal{H}) \) the inverse of this restriction. Then, for all \( F \in F \), \( \gamma^{-1}_{|\text{Prim}(\mathcal{H})}(f_F) = f_{B^+(F)} \). Our aim is to express \( \gamma^{-1}_{|\text{Prim}(\mathcal{H})} \) in the basis of forests.
We define inductively a sequence \((q_n)_{n \in \mathbb{N}}\) of elements of \(P_{\neq}\):

\[
\begin{align*}
q_1 &= \cdots \in P_{\neq}(1), \\
q_2 &= \cdots - 1 \in P_{\neq}(2), \\
q_{n+1} &= (\cdots - 1) \underline{\otimes} (q_n, \cdots) \in P_{\neq}(n+1) \text{ for } n \geq 1.
\end{align*}
\]

For all \(F \in F, \cdots \otimes (F, \cdot) = F\) and \(1 \otimes (F, \cdot) = B^+(F)\). So, \(q_n\) can also be defined in the following way:

\[
\begin{align*}
q_1 &= \cdots \in P_{\neq}(1), \\
q_{n+1} &= q_n \cdot B^+(q_n) \in P_{\neq}(n+1) \text{ for } n \geq 1.
\end{align*}
\]

**Examples.**

\[
\begin{align*}
q_3 &= \cdots - 1 - V + 1, \\
q_4 &= \cdots - 1 - \cdots - V + 1 - V + V - Y - \{\}, \\
q_5 &= \cdots - 1 - \cdots - V \cdots + 1 - Y + 1 - V + Y - Y - \{\}.
\end{align*}
\]

\[
\Delta(F \setminus t) = (F \setminus t) \otimes 1 + 1 \otimes (F \setminus t) + F' \otimes F'' \setminus t + F' \otimes t'' + F \otimes t.
\]

**Lemma 18** Let \(F \in F - \{1\}\), and \(t \in T\). Then, in \(H\):

\[
\Delta(F \setminus t) = (F \setminus t) \otimes 1 + 1 \otimes (F \setminus t) + F' \otimes F'' \setminus t + F' \otimes t'' + F \otimes t.
\]

**Proof.** The non-empty and non-total left-admissible cuts of the tree \(F \setminus t\) are:

- The cut on the edges relating \(F\) to \(t\). For this cut \(c\), \(P^c(F \setminus t) = F\) and \(R^c(F \setminus t) = t\).
- Cuts acting only on edges of \(F\) or on edges relating \(F\) to \(t\), at the exception of the preceding case. For such a cut, there exists a unique non-empty, non-total left-admissible cut \(c'\) of \(F\), such that \(P^c(F \setminus t) = P^{c'}(F)\) and \(R^c(F \setminus t) = R^{c'}(F) \setminus t\).
- Cuts acting on edges of \(t\). Then necessarily \(F \subseteq P^c(F \setminus t)\). For such a cut, there exists a unique non-empty, non-total left-admissible cut \(c'\) of \(t\), such that \(P^c(F \setminus t) = FP^{c'}(t)\) and \(R^c(F \setminus t) = R^{c'}(t)\).

Summing these cuts, we obtain the announced compatibility. 

\[\square\]

**Proposition 19** Let \(F = t_1 \cdots t_n \in F\). Then:

\[
\gamma_{\overline{\ell}(\text{Prim}(H))}^{-1}(F) = q_{n+1} \neg \otimes (\cdots, t_1, \ldots, t_n).
\]

**Proof.** First step. Let us show the following property: for all \(x \in \text{Prim}(H), t \in T\), \(q_2 \neg \otimes (x, t)\) is primitive. By lemma 18, using the linearity \(F\):

\[
\begin{align*}
\Delta(x \setminus t) &= (x \setminus t) \otimes 1 + 1 \otimes (x \setminus t) + x \otimes t + xt' \otimes t'', \\
\Delta(xt) &= xt \otimes 1 + 1 \otimes xt + x \otimes t + xt' \otimes t'', \\
\Delta(q_2 \neg \otimes (x, t)) &= \Delta(xt - x \setminus t) \\
&= (xt - x \setminus t) \otimes 1 + 1 \otimes (xt - x \setminus t).
\end{align*}
\]

Second step. Let us show that for all \(x \in \text{Prim}(H), t_1, \ldots, t_n \in T\), \(q_{n+1} \neg \otimes (x, t_1, \ldots, t_n) \in \text{Prim}(H)\) by induction on \(n\). This is obvious for \(n = 0\), as \(q_1 \neg \otimes (x) = x\). Suppose the result at rank \(n - 1\). Then:

\[
q_{n+1} \neg \otimes (x, t_1, \ldots, t_n) = (q_2 \neg \otimes (q_n, I)) \neg \otimes (x, t_1, \ldots, t_n)
\]

\[
= q_2 \neg \otimes (q_n \neg \otimes (x, t_1, \ldots, t_{n-1}), t_n) \in \text{Prim}(H),
\]

\[
\in_{\text{Prim}(H)}
\]
We now suppose that the result is true at all rank \( n \). Proceed by induction on the weight \( n \).

Third step. Let us show that for all \( x, y \in \mathcal{M} \), \( \gamma(q_2 \cdot (x, y)) = \gamma(x)y \). We can limit ourselves to \( x, y \in \mathcal{F} - \{1\} \). Then \( q_2 \cdot (x, y) = xy - x \setminus y \). Moreover, by definition of \( \setminus \), \( x \setminus y \) is a forest whose first tree is not equal to \( . \) Hence, \( \gamma(q_2 \cdot (x, y)) = \gamma(xy) - 0 = \gamma(x)y \).

Last step. Let us show by induction on \( n \) that \( \gamma(q_{n+1} \cdot (\cdot, t_1, \ldots, t_n)) = t_1 \ldots t_n \). Let us suppose the result at rank \( n-1 \). By the third step:
\[
\begin{align*}
\gamma(q_{n+1} \cdot (\cdot, t_1, \ldots, t_n)) &= \gamma(q_2 \cdot (q_n \cdot (\cdot, t_1, \ldots, t_{n-1}), t_n)) \\
&= \gamma(q_n \cdot (\cdot, t_1, \ldots, t_{n-1})))t_n \\
&= t_1 \ldots t_n.
\end{align*}
\]

Consequently, \( x = q_{n+1} \cdot (\cdot, t_1, \ldots, t_n) \in \text{Prim}(\mathcal{H}) \), and satisfies \( \gamma(x) = t_1 \ldots t_n \), which proves proposition 19.

**Examples.** Let \( t_1, t_2, t_3 \in T \).
\[
\begin{align*}
\gamma_{\text{Prim}(\mathcal{H})}^{-1}(t_1) &= \cdot t_1 - \cdot \setminus t_1, \\
\gamma_{\text{Prim}(\mathcal{H})}^{-1}(t_1t_2) &= \cdot t_1t_2 - (\cdot \setminus t_1)t_2 - (\cdot t_1) \setminus t_2 + (\cdot \setminus t_1) \setminus t_2, \\
\gamma_{\text{Prim}(\mathcal{H})}^{-1}(t_1t_2t_3) &= \cdot t_1t_2t_3 - (\cdot \setminus t_1)t_2t_3 - (\cdot t_1) \setminus t_2t_3 + (\cdot \setminus t_1) \setminus t_2t_3 - (\cdot t_1t_2) \setminus t_3 \\
&+ (\cdot \setminus t_1t_2) \setminus t_3 + ((\cdot t_1) \setminus t_2) \setminus t_3 - ((\cdot \setminus t_1) \setminus t_2) \setminus t_3.
\end{align*}
\]

### 3.3 Elements of the dual basis

**Lemma 20** For all \( x, y \in \mathcal{H} \), \( \Delta(x \setminus y) = x \setminus y^{(1)} \otimes y^{(2)} + x^{(1)} \otimes y^{(2)} \setminus y - x \otimes y \). In other terms, \( (\mathcal{H}, \setminus, \Delta) \) is an infinitesimal Hopf algebra.

**Proof.** We restrict to \( x = F \in \mathcal{F} - \{1\}, y = G \in \mathcal{F} - \{1\} \). The non-empty and non-total left-admissible cuts of the tree \( F \setminus G \) are:

- The cut on the edges relating \( F \) to \( G \). For this cut \( c \), \( P^c(F \setminus G) = F \) and \( R^c(F \setminus G) = G \).
- Cuts acting only on edges of \( F \) or on edges relating \( F \) to \( G \), at the exception of the preceding case. For such a cut, there exists a unique non-empty, non-total left-admissible cut \( c' \) of \( F \), such that \( P^c(F \setminus G) = P^{c'}(F) \) and \( R^c(F \setminus G) = R^{c'}(F) \setminus G \).
- Cuts acting on edges of \( G \). Then necessarily \( F \subseteq P^c(F \setminus G) \). For such a cut, there exists a unique non-empty, non-total left-admissible cut \( c' \) of \( t \), such that \( P^c(F \setminus G) = F \setminus P^{c'}(G) \) and \( R^c(F \setminus G) = R^{c'}(G) \).

Summing these cuts, we obtain, denoting \( \Delta(F) = F \otimes 1 + 1 \otimes F + F' \otimes F'' \) and \( \Delta(G) = G \otimes 1 + 1 \otimes G + G' \otimes G'' \):
\[
\tilde{\Delta}(F \setminus G) = (F \setminus G) \otimes 1 + 1 \otimes (F \setminus G) + F \otimes G + F' \otimes F'' \setminus G + F \setminus G' \otimes G'' \\
= (F \setminus 1) \setminus \Delta(G) + \Delta(F) \setminus (1 \otimes G) - F \otimes G.
\]

So \( (\mathcal{H}, \setminus, \Delta) \) is an infinitesimal bialgebra. As it is graded and connected, it has an antipode.

**Proposition 21** Let \( F = t_1 \ldots t_n \in \mathcal{F} \). Then \( f_F = f_{t_n} \setminus \ldots \setminus f_{t_1} \).

**Proof.** First step. We show the following result: for all \( F \in \mathcal{F}, t \in T, f_F \setminus f_t = f_{tF} \). We proceed by induction on the weight \( n \) of \( F \). If \( n = 0 \), then \( F = 1 \) and the result is obvious. We now suppose that the result is true at all rank \( < n \). Let be \( G \in \mathcal{F} \), and let us prove that \( \langle f_F \setminus f_t, G \rangle = \delta_{tF,G} \). Three cases are possible.
1. \( G = 1 \). Then \( \langle f_F / f_t, G \rangle = \langle f_F / f_t, 1 \rangle = \varepsilon(f_F / f_t) = 0 = \delta_{f_F,G} \).

2. \( G = G_1G_2, G_i \neq 1 \). Then, by lemma 20:

\[
\langle f_F / f_t, G \rangle = \langle \Delta(f_F / f_t), G_2 \otimes G_1 \rangle
= \sum_{F_1F_2=F, \text{weight}(F_1)<n} \langle f_{F_2} \otimes f_{F_1} / f_t, G_2 \otimes G_1 \rangle
+ \langle f_F / f_t \otimes 1 + f_F / 1 \otimes f_t, G_2 \otimes G_1 \rangle - \langle f_F \otimes f_t, G_2 \otimes G_1 \rangle
= \sum_{F_1F_2=F, \text{weight}(F_1)<n} \langle f_{F_2} \otimes f_{F_1} / f_t, G_2 \otimes G_1 \rangle + \langle 1 \otimes f_F / f_t, G_2 \otimes G_1 \rangle
= \sum_{F_1F_2=F, \text{weight}(F_1)<n} \delta_{F_2,G_2} \delta_{f_{F_1},G_1}
= \delta_{f_F,G}.
\]

3. \( G = B^+(G_1) \). Note that \( f_F / f_t \) is a linear span of forests \( H_1 / H_2 \), with \( H_1, H_2 \neq 1 \). By definition of \( / \), the first tree of such a forest is not \( . \). Hence, \( \gamma(f_F / f_t) = 0 \) and:

\[\langle f_F \otimes f_t, G \rangle = \langle \gamma(f_F \otimes f_t), G_1 \rangle = 0 = \delta_{f_F,G},\]

as \( tF \notin T \) because \( F \neq 1 \).

**Second step.** We now prove proposition 21 by induction on \( n \). It is obvious for \( n = 1 \). Suppose the result at rank \( n - 1 \). By the first step:

\[ f_{t_1...t_n} / f_{t_1} = (f_{t_n} / \ldots / f_{t_2}) / f_{t_1} = f_{t_n} / \ldots / f_{t_2} / f_{t_1}, \]

using the induction hypothesis for the second equality.

**Remarks.**

1. As an immediate corollary, because \( / \) is associative, for all forests \( F_1, \ldots, F_k \in F \), \( f_{F_1...F_k} = f_{F_k} / \ldots / f_{F_1} \).

2. In term of operads, proposition 21 can be rewritten in the following way:

**Corollary 22** Let \( F_1, \ldots, F_n \in F \). Then \( f_{F_1...F_n} = l_n, \delta^* (f_{F_n}, \ldots, f_{F_1}) \).

**Remark.** Hence, the dual basis \( (\langle f_F \rangle)_{F \in F} \) can be inductively computed, using proposition 21 of [4], together with propositions 19 and 21 of the present text:

\[
\begin{align*}
\langle f_1 \rangle & = 1, \\
\langle f_{t_1...t_n} \rangle & = \langle f_{t_n} / \ldots / f_{t_1} \rangle, \\
\langle f_{B^+(t_1...t_n)} \rangle & = \gamma_{\langle \text{Prim}(H) \rangle}^{-1}(f_{t_1...t_n}).
\end{align*}
\]
For example:

\[
\begin{array}{|c|c|}
\hline
f_1 &= 1 \\
f_\cdot &= 1 \\
f_{\ldots} &= 1 \\
f_{1\ldots} &= -\frac{1}{2} + 1 \\
f_{1\ldots1} &= \frac{3}{2} + Y \\
f_{1\ldots1\ldots} &= -\frac{3}{2} + 1 \\
\hline
f_{\cdot1} &= -\frac{1}{2} + Y \\
f_{\cdot\cdot} &= -\frac{1}{2} + \cdot \\
f_{\cdot\cdot\cdot} &= \frac{3}{2} + \cdot \\
\hline
f_1 &= \frac{1}{2} - \cdot - \cdot + \cdots \\
f_{\cdot1} &= -\frac{1}{2} + Y + \cdot \\
f_{\cdot\cdot} &= -\frac{1}{2} + \cdot + \cdot \\
\hline
f_1 &= -\frac{1}{2} + 1 + \cdot + 1 + \ldots - 1 + \ldots + \cdot \\
\end{array}
\]

4 Primitive suboperads

4.1 Compatibilities between products and coproducts

We define another coproduct \( \Delta_f \) on \( \mathcal{H} \) in the following way: for all \( x, y, z \in \mathcal{H} \),

\[
\langle \Delta_f(x), y \otimes z \rangle = \langle x, z f y \rangle.
\]

**Lemma 23** For all forest \( F \in \mathbf{F} \), \( \Delta_f(F) = \sum_{F_1,F_2 \in \mathbf{F}} F_1 \otimes F_2. \)

**Proof.** Let \( F, G, H \in \mathbf{F} \). Then:

\[
\langle \Delta_f(F), f_G \otimes f_H \rangle = \langle F, f_H f f G \rangle = \langle F, f_{GH} \rangle = \delta_{F,GH} \sum_{F_1,F_2 \in \mathbf{F}} \langle F_1 \otimes F_2, f_G \otimes f_H \rangle.
\]

As \( (f_F)_{F \in \mathbf{F}} \) is a basis of \( \mathcal{H} \) and \( \langle -, - \rangle \) is non degenerate, this proves the result. \( \square \)

**Remark.** As a consequence, the elements of \( \mathbf{T} \) are primitive for this coproduct.

We now have defined three products, namely \( m, /, \) and \( \backslash \), and two coproducts, namely \( \tilde{\Delta} \) and \( \Delta_f \), on \( \mathcal{M} \), obtained from \( \Delta \) and \( \Delta_f \) by substracting their primitive parts. The following properties sum up the different compatibilities.
Proposition 24 For all \( x, y \in M \):

\[
\tilde{\Delta}(xy) = (x \otimes 1)\tilde{\Delta}(y) + \tilde{\Delta}(x)(1 \otimes y) + x \otimes y,
\]

(4)

\[
\tilde{\Delta}(x \nearrow y) = (x \otimes 1)\tilde{\Delta}(y) + \tilde{\Delta}(x)(1 \otimes y) + x \otimes y,
\]

(5)

\[
\tilde{\Delta}_\nearrow(xy) = (x \otimes 1)\tilde{\Delta}_\nearrow(y) + \tilde{\Delta}_\nearrow(x)(1 \otimes y) + x \otimes y,
\]

(6)

\[
\tilde{\Delta}_\nearrow(x \nearrow y) = (x \otimes 1)\tilde{\Delta}_\nearrow(y),
\]

(7)

\[
\tilde{\Delta}_\nearrow(x \nwarrow y) = (x \otimes 1)\tilde{\Delta}_\nearrow(y).
\]

(8)

Proof. It remains to consider the compatibility between \( \nearrow \) or \( \nwarrow \) and \( \tilde{\Delta}_\nearrow \). Let \( F, G \in \mathcal{F} - \{1\} \).

We put \( G = t_1 \ldots t_n \), where the \( t_i \)'s are trees. Then \( F \nearrow G = (F \nearrow t_1)t_2 \ldots t_n \), and \( F \nearrow t_1 \) is a tree. Hence:

\[
\tilde{\Delta}_\nearrow(F \nearrow G) = \sum_{i=1}^{n-1} (F \nearrow t_1)t_2 \ldots t_i \otimes t_{i+1} \ldots t_n
\]

\[
= \sum_{i=1}^{n-1} F \nearrow (t_1t_2 \ldots t_i) \otimes t_{i+1} \ldots t_n
\]

\[
= (F \otimes 1) \nearrow \tilde{\Delta}_\nearrow(G).
\]

The proof is similar for \( F \nwarrow G \). So all these compatibilities are satisfied. \( \square \)

Remark. There is no similar compatibility between \( \Delta \) and \( \nwarrow \). In particular, lemma 19 is not available for \( t \notin \mathcal{T} \).

This justifies the following definitions:

Definition 25

1. A \( \mathbb{P} \nearrow \)-bialgebra of type 1 is a family \( (A, m, \nearrow, \tilde{\Delta}) \), such that:
   (a) \( (A, m, \nearrow) \) is a \( \mathbb{P} \nearrow \)-algebra.
   (b) \( (A, \tilde{\Delta}) \) is a coassociative, non counitary coalgebra.
   (c) Compatibilities (4) and (5) are satisfied.

2. A \( \mathbb{P} \nearrow \)-bialgebra of type 2 is a family \( (A, m, \nearrow, \tilde{\Delta}_\nearrow) \), such that:
   (a) \( (A, m, \nearrow) \) is a \( \mathbb{P} \nearrow \)-algebra.
   (b) \( (A, \tilde{\Delta}_\nearrow) \) is a coassociative, non counitary coalgebra.
   (c) Compatibilities (6) and (7) are satisfied.

3. A \( \mathbb{P} \nwarrow \)-bialgebra is a family \( (A, m, \nwarrow, \tilde{\Delta}_\nearrow) \), such that:
   (a) \( (A, m, \nwarrow) \) is a \( \mathbb{P} \nwarrow \)-algebra.
   (b) \( (A, \tilde{\Delta}_\nearrow) \) is a coassociative, non counitary coalgebra.
   (c) Compatibilities (6) and (8) are satisfied.

Example. The augmentation ideal \( M \) of the infinitesimal Hopf algebra of trees \( \mathcal{H} \) is both a \( \mathbb{P} \nearrow \)-infinitesimal bialgebra of type 1 and 2, and also a \( \mathbb{P} \nwarrow \)-infinitesimal bialgebra.

If \( A \) is a bialgebra of such a type, we denote by \( \text{Prim}(A) \) the kernel of the coproduct. We deduce the definition of the following suboperads:
Definition 26 Let $n \in \mathbb{N}$. We put:

$$\text{PRIM}^{(1)}(n) = \left\{ p \in \mathbb{P} \cap (n) \mid \begin{array}{l}
\text{For all } A, \mathbb{P}_{\rightarrow}-\text{infinitesimal bialgebra of type 1, } \\
\text{and for } a_1, \ldots, a_n \in \text{Prim}(A), \\
p(a_1, \ldots, a_n) \in \text{Prim}(A).
\end{array} \right\},$$

$$\text{PRIM}^{(2)}(n) = \left\{ p \in \mathbb{P} \cap (n) \mid \begin{array}{l}
\text{For all } A, \mathbb{P}_{\rightarrow}-\text{infinitesimal bialgebra of type 2, } \\
\text{and for } a_1, \ldots, a_n \in \text{Prim}_{\rightarrow}(A), \\
p(a_1, \ldots, a_n) \in \text{Prim}_{\rightarrow}(A).
\end{array} \right\},$$

$$\text{PRIM}_{\neg}(n) = \left\{ p \in \mathbb{P} \cap (n) \mid \begin{array}{l}
\text{For all } A, \mathbb{P}_{\neg}-\text{infinitesimal bialgebra, } \\
\text{and for } a_1, \ldots, a_n \in \text{Prim}_{\neg}(A), \\
p(a_1, \ldots, a_n) \in \text{Prim}_{\neg}(A).
\end{array} \right\}.$$

We identify $\mathbb{P} \cap (n)$ and $\mathbb{P} \cap (n)$ with the homogeneous component of weight $n$ of $\mathcal{M}$. We put $\text{Prim}(\mathcal{M}) = \text{Ker}(\tilde{\Delta})$ and $\text{Prim}_{\rightarrow}(\mathcal{M}) = \text{Ker}(\tilde{\Delta}_{\rightarrow})$. We obtain:

Proposition 27

1. For all $n \in \mathbb{N}$:

$$\text{PRIM}^{(1)}(n) = \{ p \in \mathbb{P} \cap (n) \mid p \cdot (\ldots, \ldots) \in \text{Prim}(\mathcal{M}) \} = \mathbb{P} \cap (n) \cap \text{Prim}(\mathcal{M}).$$

2. For all $n \in \mathbb{N}$:

$$\text{PRIM}^{(2)}(n) = \{ p \in \mathbb{P} \cap (n) \mid p \cdot (\ldots, \ldots) \in \text{Prim}_{\rightarrow}(\mathcal{M}) \} = \mathbb{P} \cap (n) \cap \text{Prim}_{\rightarrow}(\mathcal{M}).$$

3. For all $n \in \mathbb{N}$:

$$\text{PRIM}_{\neg}(n) = \{ p \in \mathbb{P} \cap (n) \mid p \cdot (\ldots, \ldots) \in \text{Prim}_{\neg}(\mathcal{M}) \} = \mathbb{P} \cap (n) \cap \text{Prim}_{\neg}(\mathcal{M}).$$

Proof. As $\mathcal{M}$ is a $\mathbb{P}_{\rightarrow}$-infinitesimal bialgebra, by definition:

$$\text{PRIM}^{(1)}(n) \subseteq \{ p \in \mathbb{P} \cap (n) \mid p \cdot (\ldots, \ldots) \in \text{Prim}(\mathcal{M}) \}.$$

Moreover, $\{ p \in \mathbb{P} \cap (n) \mid p \cdot (\ldots, \ldots) \in \text{Prim}(\mathcal{M}) \} = \mathbb{P} \cap (n) \cap \text{Prim}(\mathcal{M})$, as, for all $p \in \mathbb{P} \cap (n)$, $p \cdot (\ldots, \ldots) = p \in \mathcal{M}$.

We now show that $\{ p \in \mathbb{P} \cap (n) \mid p \cdot (\ldots, \ldots) \in \text{Prim}(\mathcal{M}) \} \subseteq \text{PRIM}^{(1)}(n)$. We take $p \in \mathbb{P} \cap (n)$, such that $p \cdot (\ldots, \ldots) \in \text{Prim}(\mathcal{M})$. Let $\mathcal{D} = \{1, \ldots, n\}$ and let $A$ be the free $\mathbb{P}_{\rightarrow}$-algebra generated by $\mathcal{D}$ (with a unit). It can be described as the associative algebra $\mathcal{H}^\mathcal{D}$ generated by the set of planar rooted trees decorated by $\mathcal{D}$, and can be given a structure of $\mathbb{P}_{\rightarrow}$-infinitesimal bialgebra. As $\mathcal{M}$ is freely generated by $\cdot$ as a $\mathbb{P}_{\rightarrow}$-algebra, there exists a unique morphism of $\mathbb{P}_{\rightarrow}$-algebras from $\mathcal{M}$ to $\mathcal{M}^\mathcal{D}$, augmentation ideal of $\mathcal{H}^\mathcal{D}$:

$$\xi: \left\{ \begin{array}{l}
\mathcal{M} \rightarrow \mathcal{M}^\mathcal{D} \\
\cdot \rightarrow \cdot + \ldots + \cdot
\end{array} \right\}.$$

As $\cdot \in \text{Prim}(\mathcal{M})$ and $\cdot + \ldots + \cdot \in \text{Prim}(A)$, $\xi$ is a $\mathbb{P}_{\rightarrow}$-infinitesimal bialgebra morphism from $\mathcal{M}$ to $\mathcal{M}^\mathcal{D}$. So, $\xi(p \cdot (\ldots, \ldots)) \in \text{Prim}(A)$.

Let $F \in A$ be a forest, and $s_1 \geq_h \ldots \geq_h s_k$ its vertices. For all $i \in \{1, \ldots, k\}$, we put $d_i$ the decoration of $s_i$. The decoration word associated to $F$ is the word $d_1 \ldots d_n$. It belongs to $\mathcal{M}(\mathcal{D})$, the free monoid generated by the elements of $\mathcal{D}$. For all $w \in \mathcal{M}(\mathcal{D})$, Let $A_w$ be the subspace of $A$ generated by forests whose decoration word is $w$. This defines a $\mathcal{M}(\mathcal{D})$-gradation of $A$, as a $\mathbb{P}_{\rightarrow}$-infinitesimal bialgebra of type 1.
Consider the projection \( \pi_{1,\ldots,n} \) onto \( A_{1,\ldots,n} \). We get:

\[
\pi_{1,\ldots,n} \circ \xi(p \preceq (\ldots, \ldots)) \in \text{Prim}(A),
\]

\[
= \pi_{1,\ldots,n}(p \preceq (\xi(\cdot), \ldots, \xi(\cdot)))
\]

\[
= \pi_{1,\ldots,n}(p \preceq (1 + \ldots + n, 1 + \ldots + n))
\]

\[
= p \preceq (1, \ldots, n).
\]

So \( p \preceq (1, \ldots, n) \in \text{Prim}(A) \).

Let \( B \) be a \( \mathbb{P} \)-infinitesimal bialgebra and let \( a_1, \ldots, a_n \in \text{Prim}(B) \). As \( \mathcal{M}^D \) is freely generated by the \( \ast \)'s, there exists a unique morphism of \( \mathbb{P} \)-algebras:

\[
\chi: \left\{ \begin{array}{c}
A \\
\ast \\
\end{array} \right\} \rightarrow B
\]

As the \( \ast \) and the \( a_i \)'s are primitive, \( \chi \) is a \( \mathbb{P} \)-infinitesimal bialgebra morphism. So:

\[
\xi(p \preceq (\ast, \ldots, \ast)) = p.(\xi(\ast), \ldots, \xi(\ast)) = p.(a_1, \ldots, a_n) \in \chi(\text{prim}(\mathcal{M}^D)) \subseteq \text{Prim}(A).
\]

Hence, \( p \in \text{PRIM}^{(1)}(n) \). The proof is similar for \( \text{PRIM}^{(2)} \) and \( \text{PRIM}_{<} \). \( \square \)

4.2 Suboperad \( \text{PRIM}^{(1)} \)

Lemma 28 We define inductively the following elements of \( \mathbb{P} \): 

\[
\left\{ \begin{array}{l}
q_1 = \ast , \\
q_{n+1} = (\ldots - 1) \preceq (q_n, \ast) = q_n - B^+(q_n), \text{ for } n \geq 1.
\end{array} \right.
\]

Then, for all \( n \geq 1 \), \( q_n \) belongs to \( \text{PRIM}^{(1)} \). Moreover, for all \( x_1, \ldots, x_n \in \text{Prim}(\mathcal{M}) \):

\[
\gamma(q_n \preceq (x_1, \ldots, x_n)) = \gamma(x_1)x_2 \ldots x_n.
\]

Remark. These \( q_n \)'s are the same as the \( q_n \)'s defined in section 3.2.

Proof. Let us remark that \( f_1 = \cdots = -1 \in \text{Prim}(\mathcal{M}) \). By proposition 27, \( -1 \in \text{PRIM}^{(1)}(n) \).

As \( \text{PRIM}^{(1)} \) is a suboperad of \( \mathbb{P} \), it follows that all the \( q_n \)'s belongs to \( \text{PRIM}^{(1)}(n) \).

Let \( x_1, \ldots, x_n \in \text{Prim}(\mathcal{M}) \). Let us show that \( \gamma(q_n \preceq (x_1, \ldots, x_n)) = \gamma(x_1)x_2 \ldots x_n \) by induction on \( n \). If \( n = 1 \), this is immediate. For \( n = 2 \), \( q_2 \preceq (x_1x_2) = x_1x_2 - x_1 / x_2 \). Moreover, \( x_1 / x_2 \) is a linear span of forests whose first tree is not \( \ast \). So \( \gamma(q_2 \preceq (x_1, x_2)) = \gamma(x_1x_2) = \gamma(x_1)x_2 \).

Suppose now the result true at rank \( n - 1 \). Then:

\[
q_n \preceq (x_1, \ldots, x_n) = q_2 \preceq (\underbrace{q_{n-1} \preceq (x_1, \ldots, x_{n-1})}_{\in \text{Prim}(\mathcal{M})}, x_n),
\]

\[
\gamma(q_n \preceq (x_1, \ldots, x_n)) = \gamma \left( q_2 \preceq \left( q_{n-1} \preceq (x_1, \ldots, x_{n-1}), x_n \right) \right)
\]

\[
= \gamma \left( q_{n-1} \preceq (x_1, \ldots, x_{n-1}) x_n \right)
\]

\[
= \gamma(x_1)x_2 \ldots x_n.
\]

\( \square \)

Theorem 29 The non-\( \Sigma \)-operad \( \text{PRIM}^{(1)} \) is freely generated by \( 1 - \cdots \).
Recall that \( T \) is the free operad generated by an element in \( \Sigma \). Moreover, if we denote by \( \text{Prim}(\mathcal{M}) \) the suboperad of \( \text{Prim} \) generated by the \( (1) \) ‐ operad freely generated by \( q \), then, immediately, \( \text{Prim}(\mathcal{M}) \) is a non‐\( \Sigma \)‐operad freely generated by \( q \). There is a non‐\( \Sigma \)‐operad epimorphism:

\[
\Psi : \begin{cases} 
\mathbb{P}_{q_2} & \rightarrow \text{Prim}(\mathcal{M}) \\
q_2 & \rightarrow q_2.
\end{cases}
\]

The dimension of \( \mathbb{P}_{q_2}(n) \) is the number of planar binary rooted trees with \( n \) leaves, that is to say the Catalan number \( c_n = \frac{(2n - 2)!}{(n - 1)!n!} \). On the other side, the dimension of \( \text{Prim}(\mathcal{M})(n) \) is the number of planar rooted trees with \( n \) vertices, that is to say \( c_n \). So \( \Psi \) is an isomorphism. □

In other terms, in the language of [9]:

**Theorem 30** The triple of operads \((\text{Ass}, \mathbb{P}^\Sigma, \text{FREE}_2)\), where \( \mathbb{P}^\Sigma \) is the symmetrisation of \( \mathbb{P} \) and \( \text{FREE}_2 \) is the free operad generated by an element in \( \text{FREE}_2(2) \), is a good triple of operads.

**Remark.** Note that if \( A \) is a \( \mathbb{P} \) ‐ bialgebra of type 1, then \( (A, m, \Delta) \) is a non unitary infinitesimal bialgebra. Hence, if \( (K \oplus A, m, \Delta) \) has an antipode \( S \), then \( -S \) is an eulerian idempotent for \( A \).

### 4.3 Another basis of \( \text{Prim}(\mathcal{H}) \)

Recall that \( T_b \) is freely generated (as a non‐\( \Sigma \)‐operad) by \( \bar{Y} \). In particular, if \( t_1, t_2 \in T_b \), we denote:

\[
t_1 \lor t_2 = \bar{Y} \circ (t_1, t_2).
\]

Every element \( t \in T_b - \{1\} \) can be uniquely written as \( t = t^l \lor t^r \).

There exists a morphism of operads:

\[
\Theta : \begin{cases} 
T_b & \rightarrow \mathbb{P} \\
\bar{Y} & \rightarrow \ldots - 1.
\end{cases}
\]
By theorem 29, $\Theta$ is injective and its image is $\text{PRIM}^{(1)}$. So, we obtain a basis of $\text{PRIM}^{(1)}$ indexed by $T_b$, given by $p_t = \Theta(t)$. It is also a basis of $\text{Prim}(\mathcal{M})$, which can be inductively computed by:

$$\left\{ \begin{array}{c}
p_1 = \cdots,
p_{t_1 \lor t_2} = \left(\cdots - 1\right) r^a (p_{t_1}, p_{t_2}) = p_{t_1} p_{t_2} - p_{t_1} / p_{t_2}.
\end{array} \right.$$ 

4.4 From the basis $(f_t)_{t \in T}$ to the basis $(p_t)_{t \in T_b}$

We define inductively the application $\kappa: T_b \to T$ in the following way:

$$\kappa: \begin{cases}
T_b &\to T \\
t &\to . \\
t_1 \lor t_2 &\to \kappa(t_2) \setminus \kappa(t_1).
\end{cases}$$

Examples.

\[
\begin{array}{cccc}
\bigvee &\to &. &\bigcup \\
\bigvee &\to &\bigvee &\bigcup \\
\bigvee &\to &\bigvee &\bigcup \\
\bigvee &\to &\bigvee &\bigcup \\
\bigvee &\to &\bigvee &\bigcup \\
\bigvee &\to &\bigvee &\bigcup
\end{array}
\]

It is easy to show that $\kappa$ is bijective, with inverse given by:

$$\kappa^{-1}: \begin{cases}
T &\to T_b \\
. &\to t \\
B^+(s_1 \ldots s_m) &\to \kappa^{-1}(B^+(s_2 \ldots s_m)) \lor \kappa^{-1}(s_1).
\end{cases}$$

Let us recall the partial order $\leq$, defined in [4], on the set $\mathbf{F}$ of planar forests, making it isomorphic to the Tamari poset.

**Definition 31** Let $F \in \mathbf{F}$.

1. An admissible transformation on $F$ is a local transformation of $F$ of one of the following types (the part of $F$ which is not in the frame remains unchanged):

\[
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
S
\end{array}
\end{array}
\end{array}
\end{array}
\to
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
S
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

First kind:

\[
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
S
\end{array}
\end{array}
\end{array}
\end{array}
\to
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
S
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Second kind:

2. Let $F$ and $G \in \mathbf{F}$. We shall say that $F \leq G$ if there exists a finite sequence $F_0, \ldots, F_k$ of elements of $\mathbf{F}$ such that:

\[(a) \text{ For all } i \in \{0, \ldots, k-1\}, F_{i+1} \text{ is obtained from } F_i \text{ by an admissible transformation.}\]

\[(b) \text{ } F_0 = F.\]

\[(c) \text{ } F_k = G.\]

The aim of this section is to prove the following result:
**Theorem 32** Let $t \in T_b$. Then $p_t = \sum_{s \in T} f_s$. 

**Proof.** By induction on the number $n$ of leaves of $t$. If $n = 1$, then $t = 1$ and $p_t = \cdot = f_\cdot$. Suppose the result at all ranks $\leq n - 1$. As $p_t$ is primitive, we can put:

$$p_t = \sum_{s \in T} a_s f_s.$$ 

Write $t = t_1 \lor t_2$. By the induction hypothesis:

$$p_{t_1} = \sum_{s_1 \in T} f_{s_1} \text{ and } p_{t_2} = \sum_{s_2 \in T} f_{s_2}.$$ 

As $t = t_1 \lor t_2$, $p_t = (\cdot - 1) \ast (p_{t_1}, p_{t_2}) = p_{t_1}p_{t_2} - p_{t_1} \lor p_{t_2}$. So, for all $s \in T$, as $s$ is primitive for $\Delta$, \[ \begin{align*}
    a_s &= \langle p_t, s \rangle \\
    &= \langle p_{t_1}p_{t_2} - p_{t_1} \lor p_{t_2}, s \rangle \\
    &= \langle p_{t_2} \otimes p_{t_1}, \Delta(s) - \Delta(s) \rangle \\
    &= \langle p_{t_2} \otimes p_{t_1}, \Delta(s) \rangle \\
    &= \sum_{s_1 \in T} \sum_{s_2 \in T} \langle f_{s_2} \otimes f_{s_1}, \Delta(s) \rangle.
\end{align*} \]

So $a_s$ is the number of left-admissible cuts $c$ of $s$, such that $P^c(s) \leq \kappa(t_2)$ and $R^c(s) \leq \kappa(t_1)$. 

Suppose that $a_s \neq 0$. Then, there exists a left-admissible cut $c$ of $s$, such that $P^c(s) \leq \kappa(t_2)$ and $R^c(s) \leq \kappa(t_1)$. As $s$ is a tree, $s \leq \kappa(t_2) \setminus \kappa(t_1) = \kappa(t)$. Moreover, by considering the degree of $P^c(s)$, this cut $c$ is unique, so $a_s = 1$. Reciproquely, if $s \leq \kappa(t)$, if $c$ is the unique left admissible cut such that weight($P^c(s)$) = weight($t_2$), then $P^c(s) \leq \kappa(t_2)$ and $R^c(s) \leq \kappa(t_1)$. So $a_s \neq 0$. Hence, $(s \leq \kappa(t)) \implies (a_s \neq 0) \implies (a_s = 1) \implies (s \leq \kappa(t))$. This proves theorem 32. \(\square\)

Let $\mu$ be the Möbius function of the poset $F$ ([12, 13]). By the Möbius inversion formula:

**Corollary 33** Let $s \in T$. Then $f_s = \sum_{t \in T_b, \kappa(t) \leq s} \mu(\kappa(t), s)p_t$.

### 4.5 Suboperad $\operatorname{PRIM}^{(2)}$

For all $n \in \mathbb{N}$, we put $c_{n+1} = B^+(\cdot^n)$. In other terms, $c_{n+1}$ is the corolla tree with $n + 1$ vertices, or equivalently with $n$ leaves.

**Examples.** $c_1 = \cdot$, $c_2 = \emptyset$, $c_3 = \vee$, $c_4 = \bigvee$, $c_5 = \bigvee \cdot \ldots$

**Lemma 34** The set $T$ is a basis of the operad $\operatorname{PRIM}^{(2)}$. As an operad, $\operatorname{PRIM}^{(2)}$ is generated by the $c_n$’s, $n \geq 2$. Moreover, for all $k, l \geq 2$,

$$c_k \ast \left( c_1, \underbrace{\cdot, \ldots, \cdot}_{k-1 \text{ times}} \right) = c_l \ast \left( \underbrace{\cdot, \ldots, \cdot}_{l-1 \text{ times}}, c_k \right).$$
Proof. The operad $\text{PRIM}^{(2)}_\wedge$ is identified with $\text{Prim}_\wedge(M)$ by proposition 27. So $\text{Prim}_\wedge(M)$ is equal to $\text{Vect}(T)$. Let $P$ be the suboperad of $\text{PRIM}^{(2)}_\wedge$ generated by the corollas. Let $t \in T$, of weight $n$. Let us prove that $t \in P$ by induction on $n$. If $n = 1$, then $t = \cdot \in P$. If $n \geq 2$, we can suppose that $t = B^+(t_1 \ldots t_k)$, with $t_1, \ldots, t_k \in P$. Then, by theorem 11:

$$c_{k+1} \circ (t_1, \ldots, t_k, \cdot) = (\cdot, \ldots, \cdot) = (t_1 \ldots t_k) \circ \cdot = B^+(t_1 \ldots t_k) = t.$$  

So $t \in P$, hence $P = \text{PRIM}^{(2)}_\wedge$.

Let $k, l \geq 2$. Then, by theorem 11:

$$c_k \circ (c_l) = (\cdot, \ldots, \cdot) = (c_l \circ (c_l, \ldots, \cdot)) \circ \cdot = (c_l) \circ c_k = B^+(c_l \circ (c_l, \ldots, \cdot)) = B^+(B^+(\cdot, \ldots, \cdot), k-2).$$

On the other hand:

$$c_l \circ (c_l) = (\cdot, \ldots, \cdot) = (\cdot, \ldots, \cdot) = (\cdot, \ldots, \cdot) = (\cdot, \ldots, \cdot) = B^+(B^+(\cdot, \ldots, \cdot), k-2).$$

So $c_k \circ c_l = c_l \circ c_k$. □

Definition 35 The operad $T$ is the non-$\Sigma$-operad generated by elements $c_n \in T(n)$, for $n \geq 2$, and the following relations: for all $k, l \geq 2$,

$$c_k \circ (c_l, I, \ldots, I) = c_l \circ (I, \ldots, I, c_k).$$

In other terms, a $T$-algebra $A$ has a family of $n$-multilinear products $[,] : A^\otimes n \to A$ for all $n \geq 2$, with the associativity condition:

$$[[a_1, \ldots, a_l], a_{l+1}, \ldots, a_{l+k}] = [a_1, \ldots, a_{l-1}, [a_l, \ldots, a_{l+k}]].$$

In particular, $[,]$ is associative.

Theorem 36 The operads $T$ and $\text{PRIM}^{(2)}_\wedge$ are isomorphic.

Proof. By lemma 34, there is an epimorphism of operads:

$$\begin{cases}
T & \to \text{PRIM}^{(2)}_\wedge \\
c_n & \to c_n.
\end{cases}$$

In order to prove this is an isomorphism, it is enough to prove that $\dim(T(n)) \leq \dim(\text{PRIM}^{(2)}_\wedge(n))$ for all $n \geq 2$. By lemma 34, $\dim(\text{PRIM}^{(2)}_\wedge(n))$ is the $n$-th Catalan number. Because of the defining relations, $T(n)$ is generated as a vector space by elements of the form $c_l \circ (I, b_2, \ldots, b_l)$, with $b_i \in T(n_i)$, such that $n_1 + \ldots + n_l = n - 1$. Hence, we define inductively the following subsets the free non-$\Sigma$-operad generated by the $c_n$’s, $n \geq 2$:

$$X(n) = \begin{cases}
\{I\} & \text{if } n = 1, \\
\bigcup_{i+j=n-1} \bigcup_{l=2}^n c_l \circ (I, X(i_2), \ldots, X(i_l)) & \text{if } n \geq 2.
\end{cases}$$
Then the images of the elements of \(X(n)\) linearly generate \(T(n)\), so \(\text{dim}(T(n)) \leq \text{card}(X(n))\) for all \(n\). We now put \(a_n = \text{card}(X(n))\) and prove that \(a_n\) is the \(n\)-th Catalan number. We denote by \(A(h)\) their generating formal series. Then:

\[
\begin{cases}
a_1 = 1, \\
a_n = \sum_{i=2}^{n} \sum_{i_2 + \ldots + i_l = n-1} a_{i_1} \ldots a_{i_l} \text{ if } n \geq 2.
\end{cases}
\]

In terms of generating series:

\[
A(h) - a_1 h = h \frac{A(x)}{1 - A(x)}.
\]

So \(A(h)^2 - A(h) + h = 0\). As \(A(h) = 1\):

\[
A(h) = \frac{1 - \sqrt{1 - 4h}}{2}.
\]

So \(a_n\) is the \(n\)-th Catalan number for all \(n \geq 1\). \(\square\)

In other terms:

**Theorem 37** The triple of operads \((\text{Ass}, P^\Sigma, T^\Sigma)\) is a good triple of operads.

**Remark.** Note that if \(A\) is a \(P^\rightarrow\)-bialgebra of type 2, then \((A, m, \Delta^\rightarrow)\) is a non unitary infinitesimal bialgebra. Hence, if \((K \oplus A, m, \Delta^\rightarrow)\) has an antipode \(S^\rightarrow\), then \(-S^\rightarrow\) is an eulerian idempotent for \(A\).

### 4.6 Suboperad \(\text{PRIM}_\downarrow\)

**Lemma 38** The set \(T\) is a basis of the operad \(\text{PRIM}_\downarrow\). As an operad, \(\text{PRIM}_\downarrow\) is generated by \(1\).

**Proof.** Let \(P\) be the suboperad of \(\text{PRIM}_\downarrow\) generated by \(1\). Let \(t \in T\), of weight \(n\). Let us prove that \(t \in P\) by induction on \(n\). If \(n = 1\) or 2, this is obvious. If \(n \geq 2\), suppose that \(t = B^+(t_1 \ldots t_k)\). By the induction hypothesis, \(t_1\) and \(B^+(t_2 \ldots t_k)\) belong to \(P\). Then:

\[
t = t_1 \downarrow B^+(t_2 \ldots t_k) = 1 \ast_\downarrow (t_1, B^+(t_2 \ldots t_k)).
\]

So \(t \in P\). \(\square\)

**Theorem 39** The non-\(\Sigma\)-operad \(\text{PRIM}_\downarrow\) is freely generated by \(1\).

**Proof.** Similar as the proof of theorem 29. \(\square\)

In other terms:

**Theorem 40** The triple of operads \((\text{Ass}, P^\Sigma, F_2)\), where \(F_2\) is the free operad generated by an element in \(F_2(2)\), is a good triple of operads.

**Remark.** Note that if \(A\) is a \(P\downarrow\)-bialgebra, then \((A, m, \Delta)\) is a non unitary infinitesimal bialgebra. Hence, if \((K \oplus A, m, \Delta)\) has an antipode \(S\), then \(-S\) is an eulerian idempotent for \(A\).
5 A rigidity theorem for $\mathbb{P}_\rightarrow$-algebras

5.1 Double $\mathbb{P}_\rightarrow$-infinitesimal bialgebras

Definition 41 A double $\mathbb{P}_\rightarrow$-infinitesimal bialgebra is a family $(A, m, \nearrow, \hat{\Delta}, \hat{\Delta}_\rightarrow)$ where $m: A \otimes A \rightarrow A$, $\hat{\Delta}, \hat{\Delta}_\rightarrow: A \rightarrow A \otimes A$, with the following compatibilities:

1. $(A, m, \nearrow)$ is a (non unitary) $\mathbb{P}_\rightarrow$-algebra.
2. For all $x \in A$:
   \[
   \begin{align*}
   (\hat{\Delta} \otimes \text{Id}) \circ \hat{\Delta}(x) &= (\text{Id} \otimes \hat{\Delta}) \circ \hat{\Delta}(x), \\
   (\hat{\Delta} \otimes \text{Id}) \circ \hat{\Delta}_{\rightarrow}(x) &= (\text{Id} \otimes \hat{\Delta}_{\rightarrow}) \circ \hat{\Delta}_{\rightarrow}(x), \\
   (\hat{\Delta} \otimes \text{Id}) \circ \hat{\Delta}_{\rightarrow}(x) &= (\text{Id} \otimes \hat{\Delta}_{\rightarrow}) \circ \hat{\Delta}(x).
   \end{align*}
   \]
   In other terms, $(A, \hat{\Delta}^\text{cop}, \hat{\Delta}^\text{cop})$ is a $\mathbb{P}_\leftarrow$-coalgebra.
3. $(A, m, \nearrow, \hat{\Delta})$ is a $\mathbb{P}_\rightarrow$-bialgebra of type 1.
4. $(A, m, \nearrow, \hat{\Delta}_\rightarrow)$ is a $\mathbb{P}_\rightarrow$-bialgebra of type 2.

Remark. If $(A, m, \nearrow, \hat{\Delta}, \hat{\Delta}_\rightarrow)$ is a graded double $\mathbb{P}_\rightarrow$-infinitesimal bialgebra, with finite-dimensional homogeneous components, then its graded dual $(A^*, \hat{\Delta}^*\text{cop}, \hat{\Delta}_{\rightarrow}\text{cop}, m^*, \hat{\Delta}^*_{\rightarrow}\text{cop})$ also is.

Theorem 42 $(M, m, \nearrow, \hat{\Delta}, \hat{\Delta}_{\rightarrow})$ is a double $\mathbb{P}_\rightarrow$-infinitesimal bialgebra.

Proof. We already now that $(M, m, \nearrow)$ is a $\mathbb{P}_\rightarrow$-algebra. Moreover, $(M, \hat{\Delta}^\text{cop}, \hat{\Delta}^\text{cop})$ is isomorphic to $(M^*, m^*, \nearrow)$ via the pairing $(\cdot, \cdot)$, so it is a $\mathbb{P}_\leftarrow$-coalgebra. It is already known that $(M, m, \Delta)$ and $(M, \nearrow, \hat{\Delta})$ are infinitesimal bialgebras. As $(M, \nearrow, \hat{\Delta})$ is isomorphic to $(M^\text{cop}, m^\text{cop}, \Delta^\text{cop})$ via the pairing $(\cdot, \cdot)$, it is also an infinitesimal bialgebra. So all the needed compatibilities are satisfied.

Remarks.

1. Via the pairing $(\cdot, \cdot)$, $M$ is isomorphic to its graded dual as an double $\mathbb{P}_\rightarrow$-infinitesimal bialgebra. As a consequence, as $M$ is the free $\mathbb{P}_\rightarrow$-algebra generated by $\cdot$, then $M^\text{cop}$ is also the cofree $\mathbb{P}_\rightarrow$-coalgebra cogenerated by $\cdot$.
2. All these results can be easily extended to infinitesimal Hopf algebras of decorated planar rooted trees, in other terms to every free $\mathbb{P}_\rightarrow$-algebras.

Lemma 43 In the double infinitesimal $\mathbb{P}_\rightarrow$-algebra $M$, $\text{Ker}(\hat{\Delta}) \cap \text{Ker}(\hat{\Delta}_\rightarrow) = \text{Vect}(\cdot)$.

Proof. $\supseteq$. Obvious.
\[\subseteq. \text{Let } x \in \text{Ker}(\hat{\Delta}) \cap \text{Ker}(\hat{\Delta}_\rightarrow). \text{ Then } \hat{\Delta}_\rightarrow(x) = 0, \text{ so } x \text{ is a linear span of trees. We can write:}
   \[x = \sum_{t \in T} a_t t.\]
   Consider the terms in $M \otimes \cdot$ of $\hat{\Delta}(x)$. We get $\sum_{t \in T - \{\cdot\}} a_t B^-(t) \otimes \cdot = 0$, where $B^-(t)$ is the forest obtained by deleting the root of $t$. So, if $t \neq \cdot$, then $a_t = 0$. So $x \in \text{vect}(\cdot)$. $\square$

Remark. This lemma can be extended to any free $\mathbb{P}_\rightarrow$-algebra: if $V$ is a vector space, then the free $\mathbb{P}_\rightarrow$-algebra $F_{\mathbb{P}_\rightarrow}(V)$ generated by $V$ is given a structure of double $\mathbb{P}_\rightarrow$-infinitesimal bialgebra by $\hat{\Delta}(v) = \hat{\Delta}_\rightarrow(v) = 0$ for all $v \in V$. In this case, $\text{Ker}(\hat{\Delta}) \cap \text{Ker}(\hat{\Delta}_\rightarrow) = V$ for $F_{\mathbb{P}_\rightarrow}(V)$.
5.2 Connected double $\mathbb{P}$-infinitesimal bialgebras

**Notations.** Let $A$ be a double $\mathbb{P}$-infinitesimal bialgebra. The iterated coproducts will be denoted in the following way: for all $n \in \mathbb{N}$,

$$
\tilde{\Delta}^n : \begin{cases} 
A & \rightarrow A^{\otimes (n+1)} \\
\rightarrow & \\
a & \rightarrow a^{(1)} \otimes \ldots \otimes a^{(n+1)}, 
\end{cases}
$$

$$
\tilde{\Delta}_n : \begin{cases} 
A & \rightarrow A^{\otimes (n+1)} \\
\rightarrow & \\
a & \rightarrow a^{(1)} \otimes \ldots \otimes a^{(n+1)}.
\end{cases}
$$

**Definition 44** Let $A$ be a double $\mathbb{P}$-infinitesimal bialgebra. It will be said connected if, for any $a \in A$, every iterated coproduct $A \rightarrow A^{\otimes (n+1)}$ vanishes on $a$ for a great enough $n$.

**Theorem 45** Let $A$ be a connected double $\mathbb{P}$-infinitesimal bialgebra. Then $A$ is isomorphic to the free $\mathbb{P}$-algebra generated by $\text{Prim}(A) = \text{Ker}(\tilde{\Delta}) \cap \text{Ker}(\tilde{\Delta}_\mathbb{P})$ as a double $\mathbb{P}$-infinitesimal bialgebra.

**Proof. First step.** We shall use the results on infinitesimal Hopf algebras of [4]. We show that $A = \text{Prim}(A) + A.A + A \setminus A$. As $(A, \tilde{\Delta})$ is a connected non unitary infinitesimal bialgebra, it (or more precisely its unitarisation) has an antipode $S_\mathbb{P}$, defined by:

$$
S_\mathbb{P} : \begin{cases} 
A & \rightarrow A \\
a & \rightarrow \sum_{i=0}^{\infty} (-1)^{i+1} a^{(1)} \ldots a^{(i+1)}. 
\end{cases}
$$

As $(A, \tilde{\Delta})$ is connected, this makes sense. Moreover, $-S_\mathbb{P}$ is the projector on $\text{Ker}(\tilde{\Delta})$ in the direct sum $A = \text{Ker}(\tilde{\Delta}) \oplus A \setminus A$.

In the same order of idea, as $(A, m, \tilde{\Delta}_\mathbb{P})$ is a connected infinitesimal bialgebra, we can define its antipode $S'_\mathbb{P}$ by:

$$
S'_\mathbb{P} : \begin{cases} 
A & \rightarrow A \\
a & \rightarrow \sum_{i=0}^{\infty} (-1)^{i} a^{(1)} \ldots a^{(i+1)}, 
\end{cases}
$$

and $-S'_\mathbb{P}$ is the projector on $\text{Ker}(\tilde{\Delta}_\mathbb{P})$ in the direct sum $A = \text{Ker}(\tilde{\Delta}_\mathbb{P}) \oplus A.A$.

Let $a \in A$, $b \in \text{Ker}(\tilde{\Delta})$. Then $\tilde{\Delta}_\mathbb{P}(a \setminus b) = (a \otimes 1) \tilde{\Delta}_\mathbb{P}(b) = 0$. So $A \setminus A \setminus \text{Ker}(\tilde{\Delta}_\mathbb{P})$ is a subset of $\text{Ker}(\tilde{\Delta}_\mathbb{P})$. Moreover, if $\tilde{\Delta}_\mathbb{P}(a) = 0$, then $(1A \otimes 1) \circ \tilde{\Delta}(a) = (1A \otimes 1) \circ \tilde{\Delta}_\mathbb{P}(a) = 0$. So $\tilde{\Delta}(a) \in A \otimes \text{Ker}(\tilde{\Delta}_\mathbb{P})$. As a consequence, if $n \geq 1$:

$$
\tilde{\Delta}^n(a) = (\tilde{\Delta}^{n-1} \otimes 1A) \circ \tilde{\Delta}(a) \in A^{\otimes n} \otimes \text{Ker}(\tilde{\Delta}_\mathbb{P}).
$$

Hence, for all $n \in \mathbb{N}$, $\tilde{\Delta}^n(\text{Ker}(\tilde{\Delta}_\mathbb{P})) \in A^{\otimes n} \otimes \text{Ker}(\tilde{\Delta}_\mathbb{P})$. Finally, we deduce that $S'_\mathbb{P}(\text{Ker}(\tilde{\Delta}_\mathbb{P})) \subseteq \text{Ker}(\tilde{\Delta}_\mathbb{P})$.

Let $a \in A$. Then $S'_\mathbb{P}(a) \in \text{Ker}(\tilde{\Delta}_\mathbb{P})$ and $S'_\mathbb{P} \circ S'_\mathbb{P}(a) \in \text{Ker}(\tilde{\Delta}) \cap \text{Ker}(\tilde{\Delta}_\mathbb{P}) = \text{Prim}(A)$ by the preceding point. Moreover:

$$
S'_\mathbb{P}(a) = -a + A.A,
$$

$$
S'_\mathbb{P} \circ S'_\mathbb{P}(a) = -S'_\mathbb{P}(a) + A \setminus A,
$$

$$
S'_\mathbb{P} \circ S'_\mathbb{P}(a) = a + A.A + A \setminus A.
$$

This proves the first step.

**Second step.** As $A$ is connected, it classically inherits a filtration of $\mathbb{P}$-algebra given by the kernels of the iterated coproducts. We denote by $\text{deg}_p$ the associated degree. In particular, for
all $x \in A$, $\deg_p(x) \leq 1$ if, and only if, $x \in \text{Prim}(A)$. Let $B$ be the $\mathbb{P}_\to$-subalgebra of $A$ generated by $\text{Prim}(A)$. Let $a \in A$, let us show that $a \in B$ by induction on $n = \deg_p(a)$. If $n \leq 1$, then $a \in \text{Prim}(A) \subseteq B$. Suppose the result true at all ranks $\leq n - 1$. Then, by the first step, we can write:

$$a = b + \sum_i a_i b_i + \sum_j c_j d_j,$$

with $b \in \text{Prim}(A)$, $a_i, b_i, c_j, d_j \in A$. Because of the filtration, we can suppose that $\deg_p(a_i), \deg_p(b_i), \deg_p(c_j), \deg_p(d_j) < n$. By the induction hypothesis, they belong to $B$, so $a \in B$.

**Last step.** So, there is an epimorphism of $\mathbb{P}_\to$-algebras:

$$\phi : \{ F_{\mathbb{P}_\to}(\text{Prim}(A)) \rightarrow A \\
\quad a \in \text{Prim}(A) \rightarrow a,$$

where $F_{\mathbb{P}_\to}(\text{Prim}(A))$ is the free $\mathbb{P}_\to$-algebra generated by $\text{Prim}(A)$. As the elements of $\text{Prim}(A)$ are primitive both in $A$ and $F_{\mathbb{P}_\to}(\text{Prim}(A))$, this is a morphism of double $\mathbb{P}_\to$-infinitesimal bialgebras. Suppose that it is not monic. Take then $x \in \text{Ker}(\phi)$, non-zero, of minimal degree. Then it is primitive, so belongs to $\text{Prim}(A)$ (lemma 43). Hence, $\phi(a) = a = 0$: this is a contradiction. So $\phi$ is a bijection.

In other terms:

**Corollary 46** The triple of operads $\left( (\mathbb{P}_\Sigma)_\text{op}, \mathbb{P}_\Sigma, \text{VECT} \right)$ is a good triple. Here, $\text{VECT}$ denotes the operad of vector spaces:

$$\text{VECT}(k) = \left\{ \begin{array}{ll}KI & \text{if } k = 1, \\
0 & \text{if } k \neq 1, \end{array} \right.$$ 

where $I$ is the unit of $\text{VECT}$.

We also showed that $S_\to \circ S_\to$ is the projection on $\text{Prim}(A)$ in the direct sum $A = \text{Prim}(A) \oplus (A.A + A \cdot A)$. So $S_\to \circ S_\to$ is the eulerian idempotent.

**References**


