Ordered forests, permutations and iterated integrals

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We construct an explicit Hopf algebra isomorphism from the algebra of heap-ordered trees to that of quasi-symmetric functions, generated by formal permutations, which is a lift of the natural projection of the Connes-Kreimer algebra of decorated rooted trees onto the shuffle algebra. This isomorphism gives a universal way of lifting measure-indexed characters of the Connes-Kreimer algebra into measure-indexed characters of the shuffle algebra, already introduced in [28] in the framework of rough path theory as the so-called Fourier normal ordering algorithm.

Keywords: rough paths, Hölder continuity, Hopf algebra of decorated rooted trees, Hopf algebra of free quasi-symmetric functions, shuffle algebra.

Mathematics Subject Classification (2000): 05C05, 16W30, 60F05, 60G15, 60G18, 60H05

Contents

[1]Laboratoire associé au CNRS UMR 7502

Introduction

Let us consider d regular functions $\Gamma_1, \dots, \Gamma_d$ and $s, t \in \mathbb{R}$. To any word $a_1 \dots a_n$ in the letters $\{1, \dots, d\}$, we associate the iterated integral

$$I_\Gamma^{ts}(a_1 \dots a_n) = \int_s^t d\Gamma_{a_1}(x_1) \int_s^{x_1} d\Gamma_{a_2}(x_2) \dots \int_s^{x_n-1} d\Gamma_{a_n}(x_n).$$

This function extends into a function $\overline{I}_\Gamma^{ts}$ on decorated rooted trees. For example,

$$\overline{I}_\Gamma^{ts}({}^{b}\mathbf{V}_{a}^{c}) = \int_s^t d\Gamma_a(x_1) \int_s^{x_1} d\Gamma_b(x_2) \int_s^{x_1} d\Gamma_c(x_3).$$

Such an integral $\overline{I}_\Gamma^{ts}(\mathbb{T})$ can be decomposed as a sum of iterated integrals $I_\Gamma^{ts}(a_1 \dots a_n)$, where $(a_1 \dots a_n)$ ranges in a certain set of words associated to the decorated rooted tree $\mathbb{T}$. For example:

$$\overline{I}_\Gamma^{ts}({}^{b}\mathbf{V}_{a}^{c}) = I_\Gamma^{ts}(abc) + I_\Gamma^{ts}(acb),$$

see section 4.2 of the present text for more details. Seeing the maps I_Γ^{ts} and $\overline{I}_\Gamma^{ts}$ as characters of two Hopf algebras, this construction is equivalent to the definition of a linear map θ^d from the algebra $\mathbf{H}^d$ generated by the set of decorated rooted trees (that is to say the Connes-Kreimer Hopf algebra of rooted trees) to the vector space $\mathbf{Sh}^d$ generated by words (the *shuffle algebra* on d letters, see [1]), and it turns out that this map is a Hopf algebra morphism, surjective and not injective.

Let us consider a word $w = (\ell(1) \dots \ell(n))$ in $\mathbf{Sh}^d$. The first preimage by θ^d of w we may think of is the trunk tree (that is to say the tree with no branching) $\mathcal{T}$ with decorations given from the root to the leaf by $\ell(1), \dots, \ell(n)$. This defines a section of θ^d; however, this section is a coalgebra morphism, but not an algebra morphism, as trunk trees do not satisfy the shuffle relations (see section 3.1 for more explanations). However, fixing a word w of length n, we can construct for any $\sigma \in \Sigma_n$ a preimage $\mathcal{T}^\sigma$ of the permuted word w^σ with the help of Fubini's theorem, as explained in [28], such that certain shuffle relations are satisfied, see section 4.1 and lemma 11 of the present text. We would like in this article to reformulate this construction from an algebraic point of view, more precisely in terms of morphisms of Hopf algebras.

The initial motivation of this work comes from the theory of rough paths. Assume $t \mapsto \Gamma(t) = (\Gamma_1(t), \dots, \Gamma_d(t))$ is a smooth d-dimensional path, and let $V_1, \dots, V_d : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be smooth vector fields. Differential equations of the form

$$dy(t) = \sum_{i=1}^{d} V_i(y(t)) d\Gamma_i(t) \tag{1}$$

are called differential equations driven by Γ (a term coming from control theory, where $\Gamma_1, \dots, \Gamma_d$ are called the controls); the derivatives $d\Gamma_i/dt$ should be seen as forces 'driving' the system. The solution may be formally written as the Chen-Fliess series

$$y(t) = y(0) + \sum_{n \geq 1} \sum_{1 \leq a_1, \dots, a_n \leq d} I_\Gamma^{ts}(a_1 \dots a_n) \left[V_{a_1} \cdots V_{a_n} \cdot \mathrm{Id}\right](y(0)). \tag{2}$$

Such differential equations are solved by the Cauchy-Lipschitz theorem. When Γ is replaced by Brownian motion, one obtains stochastic differential (or equivalently diffusion) equations, which may be solved by using Itô or Stratonovich stochastic calculus. Assume now that Γ is some arbitrary α-Hölder path (see section 6), for some $\alpha \in (0, 1)$. Then the Cauchy-Lipschitz theorem does not hold any more, because one first needs to give a meaning to the iterated integrals of the path Γ.

The theory of rough paths, introduced by T. Lyons [19] and further developed by V. Friz, N. Victoir [11] and M. Gubinelli [14], implies the possibility to solve (1) by a *redefinition of the integration* along Γ, using as an essential ingredient a *rough path* $\mathbf{\Gamma}$ of order $N = \lfloor 1/\alpha \rfloor$ along Γ, see definition 17 of the present text. It is an essential tool, in particular in the context of integration with respect to stochastic processes or of stochastic differential equations, when the driving process is *less* regular than the family of Brownian motion[2]. Fractional Brownian motion with Hurst (or regularity) index $\alpha \in (0, 1/2)$ is probably the most prominent example, and the main application of the above cited article [28] so far is indeed to the case of fractional Brownian motion with $\alpha \leq 1/4$ [29], for which more elementary procedures (such as piecewise linear approximation [7] or the Malliavin calculus [22] for instance) do not work any more. Here we bother only about the algebraic properties of rough paths, i.e. about *formal rough paths* [4, 5], see definition 13; rough paths should also satisfy a Hölder continuity property which we discuss briefly in the last section only.

The axioms of the definition of a formal rough path can be reformulated in terms of Hopf algebras. Recall that if $H = (H, m, \Delta)$ is a Hopf algebra, then for any commutative algebra A, the set $Char_H(A)$ of algebra morphisms from H to A is a group for the convolution product $*$ induced by the coproduct of H. With this formalism, a *formal rough path* may be seen as a a family $(\mathbf{\Gamma}^{ts})_{t,s\in\mathbb{R}}$ of characters of the shuffle algebra $\mathbf{Sh}^d$, such that for any $s, t, u \in \mathbb{R}$, $\mathbf{\Gamma}^{ts} = \mathbf{\Gamma}^{tu} * \mathbf{\Gamma}^{us}$.

As explained earlier, the aim of this text is to give an algebraic frame to the construction of [28], in terms of Hopf algebra morphisms; this will describe in a simple and explicit way *all* formal rough paths over Γ by means of algebraic tools. We use for this two families of combinatorial Hopf algebras. The first one is the Hopf algebra $\mathbf{H}$ introduced in [6] for Renormalization in Quantum Field Theory. It is based on (decorated or not) rooted trees; its product is given by commutative concatenation of rooted trees, giving rooted forests, and its coproduct by admissible cuts of trees, as recalled in section 2.2. We generalise this construction in section 3.1 to ordered rooted forests, that is to say rooted forests whose vertices are totally ordered. The obtained Hopf algebra $\mathbf{H}_o$ is neither commutative nor cocommutative. If the total order of the vertices of the ordered forest $\mathbb{F}$ is compatible with the oriented graph structure of $\mathbb{F}$, we shall say that $\mathbb{F}$ is *heap-ordered*. The set of heap-ordered forests generates a Hopf subalgebra $\mathbf{H}_{ho}$ of $\mathbf{H}_o$. All these constructions are also generalized to decorated rooted forests.

On the other side, working with permutations instead of words, we obtain a Hopf algebra structure on the vector space generated by the elements of all symmetric groups Σ_n. This object, first introduced by C. Malvenuto and Ch. Reutenauer [21], is known as the Hopf algebra of free quasi-symmetric functions **FQSym**, because of its numerous relations with the Hopf algebra of symmetric functions, see [8].

We construct in section 3.1 a Hopf algebra morphism from $\mathbf{H}_o$ to **FQSym** using the notion of forest-order-preserving symmetries, see definition 5. When restricted to $\mathbf{H}_{ho}$, this Hopf algebra morphism Θ becomes an isomorphism. It is then natural to consider the inverse image of $\sigma \in \Sigma_n$ by Θ: this element of $\mathbf{H}_{ho}$ is denoted by $\mathbb{T}^{\sigma^{-1}}$. The product and coproduct of the elements $\mathbb{T}^\sigma$ is decribed in lemma 9. Note that Θ extends the injective Hopf algebra morphism from the non-commutative Connes-Kreimer Hopf algebra of planar rooted trees NCK, into **FQSym**, described in [2] using the Hopf algebra isomorphism from NCK to the Loday-Ronco Hopf algebra of binary trees [10, 17].

There exist canonical projections from a decorated version of **FQSym** to the shuffle algebra $\mathbf{Sh}^d$, and from the Hopf algebra of heap-ordered decorated forests $\mathbf{H}^d_{ho}$ to the Hopf algebra of decorated rooted trees $\mathbf{H}^d$. Completing the last edge, we define a commutative square of Hopf

[2]Recall that Brownian paths are $(\frac{1}{2} - \varepsilon)$-Hölder continuous for every $\varepsilon > 0$.

algebra morphisms:

$$\begin{tikzcd} \mathbf{H}^d_{ho} \arrow[r, "\Theta^d"] \arrow[d, "\pi^d_{ho}"'] & \mathbf{FQSym}^d \arrow[d, "\pi^d_\Sigma"] \\ \mathbf{H}^d \arrow[r, "\theta^d"'] & \mathbf{Sh}^d \end{tikzcd}$$

Considering characters, the group of characters $Char_{\mathbf{H}^d}(A)$ is now seen as a subgroup of $Char_{\mathbf{H}^d_{ho}}(A)$, more precisely, as the subgroup of characters invariant under forest-order-preserving symmetries. The Hopf algebra morphism θ^d induces a group injection from $Char_{\mathbf{Sh}^d}(A)$ to $Char_{\mathbf{H}^d}(A)$, sending ϕ to $\phi \circ \theta^d$. Using characters given by iterated integrals, this allows to compute easily the elements $\mathbb{T}^\sigma$ with the help of Fubini's theorem, see section 4.2.

Section 5 explains how to use this formalism to construct, first characters of the shuffle algebra from a character of a certain algebra of measures graded by the monoid of heap-ordered forests (see definition 15 and lemma 16), then a rough path, using the notion of *measure splitting*, see Definition 14. In particular, constructing a *formal rough path* over Γ is definitely a *very underdetermined* problem, since essentially any choice of function on the set of rooted trees yields by linear and multiplicative extension a family of characters of $\mathbf{H}^d$ and then a formal rough path.

Finally, we briefly present in section 6 in guise of conclusion how the results of this paper may be combined with analytic tools to produce rough paths satisfying the proper Hölder requirements necessary for applications to analysis.

Remark. The base field is $K = \mathbb{R}$ or $\mathbb{C}$. For any set X, we shall denote $Vect(X)$ the K-vector space generated by X.

1 Hopf algebras of words

1.1 The shuffle algebra

Let $d \geq 1$. A *d-word* is a finite sequence of elements taken in $\{1, \ldots, d\}$. The degree of a word is the number of its letters. In particular, there exists only one word of degree 0, the empty word, denoted by 1.

The *shuffle Hopf algebra* $\mathbf{Sh}^d$ is, as a vector space, generated by the set of d-words. The product $ш$ of $\mathbf{Sh}^d$ is given in the following way: if w is a d-word of degree k, w' is a d-word of degree l, then

$$w \,ш\, w' = \sum_{w'' \in Sh(w,w')} w'',$$

where $Sh(w, w')$ is the set of all words obtained by shuffling the letters of w and w'. For example, if $(a_1a_2a_3)$ and (a_4a_5) are two d-words (that is to say $1 \leq a_1, a_2, a_3, a_4, a_5 \leq d$):

$$\begin{aligned} (a_1a_2a_3) \,ш\, (a_4a_5) &= (a_1a_2a_3a_4a_5) + (a_1a_2a_4a_3a_5) + (a_1a_2a_4a_5a_3) \\ &\quad +(a_1a_4a_2a_3a_5) + (a_1a_4a_2a_5a_3) + (a_1a_4a_5a_2a_3) \\ &\quad +(a_4a_1a_2a_3a_5) + (a_4a_1a_2a_5a_3) + (a_4a_1a_5a_2a_3) + (a_4a_5a_1a_2a_3). \end{aligned}$$

This product is commutative; the unit is the empty word 1. The coproduct is defined on any d-word $w = (a_1 \ldots a_n)$ by:

$$\Delta(w) = \sum_{i=0}^{n} (a_1 \ldots a_i) \otimes (a_{i+1} \ldots a_n).$$

For example:

$$\Delta(a_1a_2a_3a_4) = a_1a_2a_3a_4 \otimes 1 + a_1a_2a_3 \otimes a_1 + a_1a_2 \otimes a_3a_4 + a_1 \otimes a_2a_3a_4 + 1 \otimes a_1a_2a_3a_4.$$

The counit sends 1 to 1 and any non-empty word to 0. The antipode S sends the word $(a_1 \dots a_n)$ to $(-1)^n(a_n \dots a_1)$.

We shall consider in the sequel d-words as *trunk trees*, that is to say decorated trees with no branching. For example, we shall identify the d-word (abc) with the trunk tree  . Considering d-words as trunk trees, $\mathbf{Sh}^d$ becomes a vector subspace and a sub-coalgebra (but not a subalgebra) of $\mathbf{H}^d$ whose definition we shall recall in section 2.

1.2 Hopf algebra of permutations

Notations. Let k, l be integers. A (k,l)-*shuffle* is a permutation ζ of $\{1, \dots, k+l\}$, such that $\zeta^{-1}(1) < \dots < \zeta^{-1}(k)$ and $\zeta^{-1}(k+1) < \dots < \zeta^{-1}(k+l)$. The set of (k,l)-shuffles will be denoted by $Sh(k,l)$.

Remarks.

1. We represent a permutation $\sigma \in \Sigma_n$ by the word $(\sigma(1) \dots \sigma(n))$. For example, $Sh(2,1) = \{(123), (132), (312)\}$.

2. For any integers k, l, any permutation $\sigma \in \Sigma_{k+l}$ can be uniquely written as $(\sigma_1 \otimes \sigma_2) \circ \epsilon$, where $\sigma_1 \in \Sigma_k$, $\sigma_2 \in \Sigma_l$, and $\epsilon \in Sh(k,l)$. Similarly, considering the inverses, any permutation $\tau \in \Sigma_{k+l}$ can be uniquely written as $\zeta^{-1} \circ (\tau_1 \otimes \tau_2)$, where $\tau_1 \in \Sigma_k$, $\tau_2 \in \Sigma_l$, and $\zeta \in Sh(k,l)$. Note that, whereas ϵ shuffles the lists $(\sigma(1), \dots, \sigma(k)), (\sigma(k+1), \dots, \sigma(k+l))$, ζ^{-1} renames the numbers of each lists $(\tau(1), \dots, \tau(k)), (\tau(k+1), \dots, \tau(k+l))$ without changing their orderings. For instance, $\{((21) \otimes 3) \circ \epsilon, \epsilon \in Sh(2,1)\} = \{(213), (231), (321)\}$, whereas $\{\zeta^{-1} \circ ((21) \otimes 3), \zeta \in Sh(2,1)\} = \{(213), (312), (321)\}$.

We here briefly recall the construction of the Hopf algebra **FQSym** of free quasi-symmetric functions, also called the Malvenuto-Reutenauer Hopf algebra [8, 21]. As a vector space, a basis of **FQSym** is given by the disjoint union of the symmetric groups Σ_n, for all $n \geq 0$. By convention, the unique element of S_0 is denoted by 1. The product of **FQSym** is given, for $\sigma \in \Sigma_k$, $\tau \in \Sigma_l$, by:

$$\sigma.\tau = \sum_{\epsilon \in Sh(k,l)} (\sigma \otimes \tau) \circ \epsilon.$$

In other words, the product of σ and τ is given by shifting the letters of the word representing τ by k, and then summing all the possible shufflings of this word and of the word representing σ. For example:

$$\begin{array}{rcl}(123)(21) & = & (12354) + (12534) + (15234) + (51234) + (12543) \\ & & +(15243) + (51243) + (15423) + (51423) + (54123).\end{array}$$

Let $\sigma \in \Sigma_n$. For all $0 \leq k \leq n$, there exists a unique triple $\left(\sigma_1^{(k)}, \sigma_2^{(k)}, \zeta_k\right) \in \Sigma_k \times \Sigma_{n-k} \times Sh(k,l)$ such that $\sigma = \zeta_k^{-1} \circ \left(\sigma_1^{(k)} \otimes \sigma_2^{(k)}\right)$. The coproduct of **FQSym** is then defined by:

$$\Delta(\sigma) = \sum_{k=0}^{n} \sigma_1^{(k)} \otimes \sigma_2^{(k)} = \sum_{k=0}^{n} \sum_{\substack{\sigma = \zeta^{-1} \circ (\sigma_1 \otimes \sigma_2) \\ \zeta \in Sh(k,l), \sigma_1 \in \Sigma_k, \sigma_2 \in \Sigma_l}} \sigma_1 \otimes \sigma_2.$$

Note that $\sigma_1^{(k)}$ and $\sigma_2^{(k)}$ are obtained by cutting the word representing σ between the k-th and the $k+1$-th letter, and then *standardizing* the two obtained words, that is to say applying to their letters the unique increasing bijection to $\{1, \ldots, k\}$ or $\{1, \ldots, n-k\}$. For example:

$$\begin{array}{rcl} \Delta((41325)) & = & 1 \otimes (41325) + Std(4) \otimes Std(1325) + Std(41) \otimes Std(325) \\ & & +Std(413) \otimes Std(25) + Std(4132) \otimes Std(5) + (41325) \otimes 1 \\ & = & 1 \otimes (41325) + (1) \otimes (1324) + (21) \otimes (213) \\ & & +(312) \otimes (12) + (4132) \otimes (1) + (41325) \otimes 1. \end{array}$$

Then **FQSym** is a Hopf algebra. It is graded, with $\mathbf{FQSym}(n) = Vect(\Sigma_n)$ for all $n \geq 0$.

It is also possible to give a decorated version of **FQSym**. A d-decorated permutation is a couple (σ, ℓ), where $\sigma \in \Sigma_n$ and ℓ is a map from $\{1, \ldots, n\}$ to $\{1, \ldots, d\}$. A d-decorated permutation is represented by two superposed words $\binom{a_1 \ldots a_n}{b_1 \ldots b_n}$, where $(a_1 \ldots a_n)$ is the word representing σ and for all i, $b_i = \ell(a_i)$. The vector space $\mathbf{FQSym}^d$ generated by the set of d-decorated permutations is a Hopf algebra. For example, if $1 \leq a, b, c \leq d$:

$$\begin{array}{rcl} \binom{213}{bac} \cdot \binom{1}{a} & = & \binom{2134}{baca} + \binom{2143}{baac} + \binom{2413}{baac} + \binom{4213}{abac}, \\ \Delta \binom{4321}{acba} & = & \binom{4321}{acba} \otimes 1 + \binom{321}{acb} \otimes \binom{1}{a} + \binom{21}{ac} \otimes \binom{21}{ba} + \binom{1}{a} \otimes \binom{321}{cba} + 1 \otimes \binom{4321}{acba}. \end{array}$$

In other words, if (σ, ℓ) and (τ, ℓ') are decorated permutations of respective degrees k and l:

$$(\sigma, \ell).(\tau, \ell') = \sum_{\epsilon \in Sh(k,l)} ((\sigma \otimes \tau) \circ \epsilon, \ell \otimes \ell'), \tag{3}$$

where $\ell \otimes \ell'$ is defined by $(\ell \otimes \ell')(i) = \ell(i)$ if $1 \leq i \leq m$ and $(\ell \otimes \ell')(m+j) = \ell'(j)$ if $1 \leq j \leq m'$. If (σ, ℓ) is a decorated permutation of degree n:

$$\Delta((\sigma, \ell)) = \sum_{k=0}^{n} \sum_{\substack{\sigma = \zeta^{-1} \circ (\sigma_1 \otimes \sigma_2) \\ \zeta \in Sh(k,l), \sigma_1 \in \Sigma_k, \sigma_2 \in \Sigma_l}} (\sigma_1 \otimes \sigma_2, (\ell \otimes \ell') \circ \zeta). \tag{4}$$

In some sense, a d-decorated permutation can be seen as a word with a total order on the set of its letters.

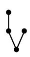
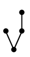
Definition 1 *For any d, let π_Σ^d be the linear map from* $\mathbf{FQSym}^d$ *to* $\mathbf{Sh}^d$*, sending the decorated permutation $\binom{a_1 \ldots a_n}{b_1 \ldots b_n}$ to the word $(b_1 \ldots b_n)$.*

It is clear that π_Σ^d is a surjective morphism of Hopf algebras, so $\mathbf{Sh}^d$ may also be seen as a quotient Hopf algebra of $\mathbf{FQSym}^d$, which accounts for the notation $\bar{S}$ for the antipode of $\mathbf{Sh}^d$.

2 Hopf algebras of forests

We shall here recall the construction of product and the coproduct of the Hopf algebra of rooted trees $\mathbf{H}^d$ and generalize it to the space generated by ordered rooted forests.

2.1 Reminders on rooted trees and forests

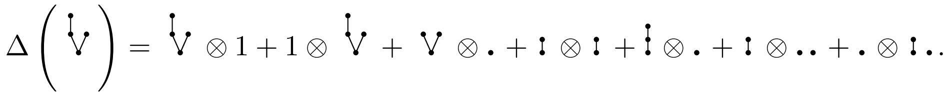
A *rooted tree* is a finite tree with a distinguished vertex called the root [26]. A rooted forest is a finite graph $\mathcal{F}$ such that any connected component of $\mathcal{F}$ is a rooted tree. The set of vertices of the rooted forest $\mathcal{F}$ is denoted by $V(\mathcal{F})$. Note that we work with non-planar trees; for example,  = . Let $\mathcal{F}$ be a rooted forest. The edges of $\mathcal{F}$ are oriented downwards (from the leaves to the roots). If $v, w \in V(\mathcal{F})$, with $v \neq w$, we shall write $v \to w$ if there is an edge in $\mathcal{F}$ from v to w and $v \twoheadrightarrow w$ if there is an oriented path from v to w in $\mathcal{F}$.

Let $\boldsymbol{v}$ be a subset of $V(\mathcal{F})$. We shall say that $\boldsymbol{v}$ is an admissible cut of $\mathcal{F}$, and we shall write $\boldsymbol{v} \models V(\mathcal{F})$, if $\boldsymbol{v}$ is totally disconnected, that is to say that $v \not\twoheadrightarrow w$ for any couple (v,w) of two different elements of $\boldsymbol{v}$. If $\boldsymbol{v} \models V(\mathcal{F})$, we denote by $Lea_{\boldsymbol{v}}\mathcal{F}$ the rooted sub-forest of $\mathcal{F}$ obtained by keeping only the vertices above $\boldsymbol{v}$, that is to say $\{w \in V(\mathcal{F}),\ \exists v \in \boldsymbol{v},\ w \twoheadrightarrow v\} \cup \boldsymbol{v}$. We denote by $Roo_{\boldsymbol{v}}\mathcal{F}$ the rooted sub-forest obtained by keeping the other vertices.

Connes and Kreimer proved in [6] that the vector space $\mathbf{H}$ generated by the set of rooted forests is a Hopf algebra. Its product is given by the disjoint union of rooted forests, and the coproduct is defined for any rooted forest $\mathcal{F}$ by:

$$\Delta(\mathcal{F}) = \sum_{\boldsymbol{v} \models V(\mathcal{F})} Roo_{\boldsymbol{v}}\mathcal{F} \otimes Lea_{\boldsymbol{v}}\mathcal{F}.$$

For example:



The antipode $\bar{S}$ is inductively defined by:

$$\begin{aligned} \bar{S}(1) &= 1, \\ \bar{S}(\mathcal{F}) &= -\mathcal{F} - \sum_{\substack{\boldsymbol{v} \models V(\mathcal{F}) \\ Roo_{\boldsymbol{v}}\mathcal{F} \neq \mathcal{F}, Lea_{\boldsymbol{v}}\mathcal{F} \neq \mathcal{F}}} Roo_{\boldsymbol{v}}\mathcal{F}\ \bar{S}(Lea_{\boldsymbol{v}}\mathcal{F}). \end{aligned}$$

This construction is easily generalised to d-decorated rooted forests. A d-decorated forest is a couple $(\mathcal{F}, \ell)$, where $\mathcal{F}$ is a rooted forest and ℓ a map from $V(\mathcal{F})$ to $\{1,\ldots,d\}$. If $\mathcal{F}$ and $\mathcal{G}$ are two d-decorated forests, then $\mathcal{F}\mathcal{G}$ is naturally d-decorated. For any $\boldsymbol{v} \models V(\mathcal{F})$, $Lea_{\boldsymbol{v}}\mathcal{F}$ and $Roo_{\boldsymbol{v}}\mathcal{F}$ are also d-decorated by restriction, so the vector space $\mathbf{H}^d$ generated by d-decorated rooted forests is a Hopf algebra.

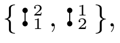
Remark. This is indeed the coproduct (up to a flip) of Connes and Kreimer: for any $\boldsymbol{v} \models V(\mathcal{F})$, there exists a unique admissible cut c such that $Roo_{\boldsymbol{v}}\mathcal{F} = R^c(\mathcal{F})$ and $Lea_{\boldsymbol{v}}\mathcal{F} = P^c(\mathcal{F})$, with the notations of [6].

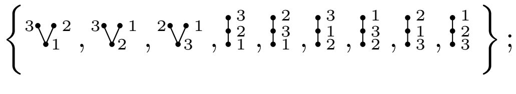
2.2 Hopf algebra of ordered trees

Definition 2 *An ordered (rooted) forest is a rooted forest with a total order on the set of its vertices. The set of ordered forests will be denoted by* $\mathbf{F}_o$*; for all* $n \geq 0$*, the set of ordered forests with* n *vertices will be denoted by* $\mathbf{F}_o(n)$*. An ordered (rooted) tree is a connected ordered forest. The set of ordered trees will be denoted by* $\mathbf{T}_o$*; for all* $n \geq 1$*, the set of ordered trees with* n *vertices will be denoted by* $\mathbf{T}_o(n)$*. The* K*-vector space generated by* $\mathbf{F}_o$ *is denoted by* $\mathbf{H}_o$*. It is a graded space, the homogeneous component of degree* n *being* $Vect(\mathbf{F}_o(n))$ *for all* $n \in \mathbb{N}$*.*

For example:

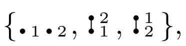
$\mathbf{T}_o(1) = \{\bullet_1\}$,

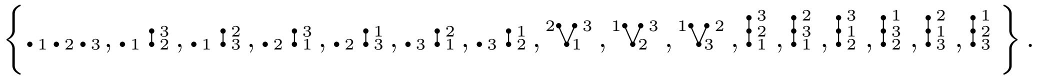
$\mathbf{T}_o(2) =$ 

$\mathbf{T}_o(3) =$ 

$\mathbf{F}_o(0) = \{1\}$,

$\mathbf{F}_o(1) = \{\bullet_1\}$,

$\mathbf{F}_o(2) =$ 

$\mathbf{F}_o(3) =$ 

Remark. The underlying rooted forest of an ordered forest is non-planar, so, for example, ${}^{3}\mathrm{V}_{1}^{2} = {}^{2}\mathrm{V}_{1}^{3}$, and $\mathfrak{l}_{\substack{3\\1}} \bullet_2 = \bullet_2 \mathfrak{l}_{\substack{3\\1}}$.

If $\mathbb{F}$ and $\mathbb{G}$ are two ordered forests, then the rooted forest $\mathbb{F}\mathbb{G}$ is also an ordered forest with, for all $v \in V(\mathbb{F})$, $w \in V(\mathbb{G})$, $v < w$. This defines a non-commutative product on the the set of ordered forests. For example, the product of $\bullet_1$ and $\mathfrak{l}_{\substack{2\\1}}$ gives $\bullet_1 \mathfrak{l}_{\substack{3\\2}}$, whereas the product of $\mathfrak{l}_{\substack{2\\1}}$ and $\bullet_1$ gives $\mathfrak{l}_{\substack{2\\1}} \bullet_3 = \bullet_3 \mathfrak{l}_{\substack{2\\1}}$. This product is linearly extended to $\mathbf{H}_o$, which in this way becomes a graded algebra.

If $\mathbb{F}$ is an ordered forest, then any subforest of $\mathbb{F}$ is also ordered. So we can define a coproduct $\Delta : \mathbf{H}_o \longmapsto \mathbf{H}_o \otimes \mathbf{H}_o$ on $\mathbf{H}_o$ in the following way: for all $\mathbb{F} \in \mathbf{F}_o$,

$$\Delta(\mathbb{F}) = \sum_{\boldsymbol{v} \models V(\mathbb{F})} Roo_{\boldsymbol{v}}\mathbb{F} \otimes Lea_{\boldsymbol{v}}\mathbb{F}.$$

As for the Connes-Kreimer Hopf algebra of rooted trees [6], one can prove that this coproduct is coassociative, so $\mathbf{H}_o$ is a graded Hopf algebra. For example:

$$\Delta\left({}^{\substack{1\\4}}\mathrm{V}_{2}^{3}\right) = {}^{\substack{1\\4}}\mathrm{V}_{2}^{3} \otimes 1 + 1 \otimes {}^{\substack{1\\4}}\mathrm{V}_{2}^{3} + {}^{2}\mathrm{V}_{1}^{3} \otimes \bullet_1 + \mathfrak{l}_{\substack{2\\1}} \otimes \mathfrak{l}_{\substack{1\\2}} + \mathfrak{l}_{\substack{1\\3\\2}} \otimes \bullet_1 + \mathfrak{l}_{\substack{2\\1}} \otimes \bullet_1 \bullet_2 + \bullet_1 \otimes \mathfrak{l}_{\substack{1\\3}} \bullet_2.$$

2.3 Hopf algebra of heap-ordered trees

Definition 3 *[12] An ordered forest is heap-ordered if for all $i, j \in V(\mathbb{F})$, $(i \twoheadrightarrow j) \Longrightarrow (i > j)$. The set of heap-ordered forests will be denoted by $\mathbf{F}_{ho}$; for all $n \geq 0$, the set of heap-ordered forests with n vertices will be denoted by $\mathbf{F}_{ho}(n)$. A heap-ordered tree is a connected heap-ordered forest. The set of heap-ordered trees will be denoted by $\mathbf{T}_{ho}$; for all $n \geq 1$, the set of heap-ordered trees with n vertices will be denoted by $\mathbf{T}_{ho}(n)$.*

For example:

$$\begin{array}{rcl}
\mathbf{T}_{ho}(1) & = & \{\bullet_1\}, \\
\mathbf{T}_{ho}(2) & = & \{\mathfrak{l}_{\substack{2\\1}}\}, \\
\mathbf{T}_{ho}(3) & = & \left\{ {}^{2}\mathrm{V}_{1}^{3}, \mathfrak{l}_{\substack{3\\2\\1}} \right\}; \\
\\
\mathbf{F}_{ho}(0) & = & \{1\}, \\
\mathbf{F}_{ho}(1) & = & \{\bullet_1\}, \\
\mathbf{F}_{ho}(2) & = & \{\bullet_1 \bullet_2, \mathfrak{l}_{\substack{2\\1}}\}, \\
\mathbf{F}_{ho}(3) & = & \left\{ \bullet_1 \bullet_2 \bullet_3, \bullet_1 \mathfrak{l}_{\substack{3\\2}}, \bullet_2 \mathfrak{l}_{\substack{3\\1}}, \bullet_3 \mathfrak{l}_{\substack{2\\1}}, {}^{2}\mathrm{V}_{1}^{3}, \mathfrak{l}_{\substack{3\\2\\1}} \right\}.
\end{array}$$

If $\mathbb{F}$ and $\mathbb{G}$ are two heap-ordered forests, then $\mathbb{F}\mathbb{G}$ is also heap-ordered. If $\mathbb{F}$ is a heap-ordered forest, then any subforest of $\mathbb{F}$ is heap-ordered. So the subspace $\mathbf{H}_{ho}$ of $\mathbf{H}_o$ generated by the heap-ordered forests is a graded Hopf subalgebra of $\mathbf{H}_o$.

For example, $(\bullet_1)(\mathfrak{l}_{\substack{2\\1}}) = \bullet_1 \mathfrak{l}_{\substack{3\\2}}$ and $(\mathfrak{l}_{\substack{2\\1}})(\bullet_1) = \mathfrak{l}_{\substack{2\\1}} \bullet_3$ and this shows that $\mathbf{H}_{ho}$ is not commutative. It is neither cocommutative. Indeed:

$$\Delta({}^{2}\mathrm{V}_{1}^{3}) = {}^{2}\mathrm{V}_{1}^{3} \otimes 1 + 1 \otimes {}^{2}\mathrm{V}_{1}^{3} + 2\mathfrak{l}_{\substack{2\\1}} \otimes \bullet_1 + \bullet_1 \otimes \bullet_1 \bullet_2.$$

So neither $\mathbf{H}_{ho}$ nor its graded dual $\mathbf{H}_{ho}^*$, is isomorphic to the Hopf algebra of heap-ordered trees of [12, 13], which is cocommutative, although these Hopf algebras are all based on the same objects.

It is not difficult to generalize these constructions to decorated versions. A d-decorated ordered forest is a couple $(\mathbb{F},\ell)$, where $\mathbb{F}$ is an ordered forest and ℓ is a map from $V(\mathbb{F})$ to $\{1,\ldots,d\}$. A d-decorated ordered forest will be denoted by $(\mathbb{F},a_1\ldots a_n)$, where a_i is the value of ℓ on the i-th vertex of $\mathbb{F}$.

If $\mathbb{F}$ and $\mathbb{G}$ are two d-decorated ordered forests, then $\mathbb{F}\mathbb{G}$ is naturally d-decorated. For any $\boldsymbol{v}\models V(\mathbb{F})$, $Lea_{\boldsymbol{v}}\mathbb{F}$ and $Roo_{\boldsymbol{v}}\mathbb{F}$ are also d-decorated by restriction, so the vector space $\mathbf{H}_o^d$ generated by d-decorated ordered forests is a Hopf algebra. The subspace $\mathbf{H}_{ho}^d$ of $\mathbf{H}_o^d$ generated by d-decorated heap-ordered forests is a Hopf subalgebra. For example, if $1\leq a_1,a_2,a_3\leq d$:

$$\begin{array}{rcl}
(\bullet_1,a_1).(\mathrm{I}^{2}_{1},a_2a_3) & = & (\bullet_1\,\mathrm{I}^{3}_{2},a_1a_2a_3),\\
\Delta(({}^{2}\mathrm{V}_{1}^{3},a_1a_2a_3)) & = & ({}^{2}\mathrm{V}_{1}^{3},a_1a_2a_3)\otimes 1+1\otimes({}^{2}\mathrm{V}_{1}^{3},a_1a_2a_3)+(\mathrm{I}^{2}_{1},a_1a_2)\otimes(\bullet_1,a_3)\\
& & +(\mathrm{I}^{2}_{1},a_1a_3)\otimes(\bullet_1,a_2)+(\bullet_1,a_1)\otimes(\bullet_1\bullet_1,a_2a_3).
\end{array}$$

Notations. For all $n\geq 1$, we shall denote the trunk tree with n vertices by $\mathcal{T}_n$. This tree has a unique heap-ordering, from the root to the unique leaf. Identifying d-words with d-decorated trunk trees, $\mathbf{Sh}^d$ is now seen as a subspace and a sub-coalgebra of $\mathbf{H}_{ho}^d$. For example, we shall identify:

$$(a_1a_2a_3)=\begin{smallmatrix}\bullet\, a_3\\ \bullet\, a_2\\ \bullet\, a_1\end{smallmatrix}=(\mathcal{T}_3,\ell),$$

where $\ell:\{1,2,3\}\longrightarrow\{1,\ldots,d\}$ sends i to a_i for all $1\leq i\leq 3$.

Definition 4 *For any d, let π_{ho}^d be the linear map from $\mathbf{H}_{ho}^d$ to $\mathbf{H}^d$, sending a d-decorated, ordered forest to the underlying d-decorated forest.*

Note that π_{ho}^d is a surjective morphism of Hopf algebras.

Remark. Let NCK be the non-commutative Connes-Kreimer Hopf algebra of planar forests [10, 17]. Let F be a planar forest. We shall consider F as a heap-ordered forest by ordering its vertices in the "north-west" direction (this is the order defined in [9]): for example, the planar forest $F=\mathrm{I}\ \mathrm{V}$ is identified with the heap-ordered forest $\mathrm{I}^{2}_{1}\,{}^{4}\mathrm{V}_{3}^{5}$. In this way, NCK becomes a subalgebra of $\mathbf{H}_{ho}$. In other terms, this is the order given by the Depth First Search algorithm.

3 From ordered forests to permutations

3.1 Construction of the Hopf algebra morphism

Definition 5 (forest-order-preserving symmetries) *Let $n\geq 0$. For all $\mathbb{F}\in\mathbf{F}_o(n)$, let $S_{\mathbb{F}}$ be the set of permutations $\sigma\in\Sigma_n$ such that for all $1\leq i,j\leq n$, $(i\twoheadrightarrow j)\Longrightarrow(\sigma^{-1}(i)>\sigma^{-1}(j))$. The elements of $S_{\mathbb{F}}$ are called the forest-order-preserving symmetries.*

Proposition 6 *Let us define:*

$$\Theta:\left\{\begin{array}{rcl}\mathbf{H}_o & \longrightarrow & \mathbf{FQSym}\\ \mathbb{F}\in\mathbf{F}_o & \longmapsto & \displaystyle\sum_{\sigma\in S_{\mathbb{F}}}\sigma.\end{array}\right.$$

Then Θ is a Hopf algebra morphism, homogeneous of degree 0.

For example:

$$
\begin{array}{rcl}
\Theta(\bullet_1) & = & (1), \\
\Theta(\bullet_1\bullet_2) & = & (12)+(21), \\
\Theta(\mathbf{I}^{2}_{1}) & = & (12), \\
\Theta(\bullet_1\bullet_2\bullet_3) & = & (123)+(132)+(213)+(231)+(312)+(321), \\
\Theta(\bullet_1\mathbf{I}^{3}_{2}) & = & (123)+(213)+(231), \\
\Theta(\bullet_2\mathbf{I}^{3}_{1}) & = & (123)+(132)+(213), \\
\Theta(\bullet_3\mathbf{I}^{2}_{1}) & = & (123)+(132)+(312), \\
\Theta({}^{2}\mathbf{V}^{3}_{1}) & = & (123)+(132), \\
\Theta(\mathbf{I}^{3}_{2\,1}) & = & (123).
\end{array}
$$

In particular, if $\mathbb{F}$ is the product of the trunk trees with respectively k and l vertices, totally ordered from their roots to their leaves, then $S_{\mathbb{F}} = Sh(k,l)$.

Proof. Obviously, Θ is homogeneous of degree 0. Let $\mathbb{F} \in \mathbf{F}_o(k)$, $\mathbb{G} \in \mathbf{F}_o(l)$. Let $\sigma \in S_{\mathbb{F}\mathbb{G}}$. Then σ can be uniquely written as $\sigma = (\sigma_1 \otimes \sigma_2) \circ \epsilon$, with $\sigma_1 \in \Sigma_k$, $\sigma_2 \in \Sigma_l$, and $\epsilon \in Sh(k,l)$. If $i \twoheadrightarrow j$ in $\mathbb{F}$, then $i \twoheadrightarrow j$ in $\mathbb{F}\mathbb{G}$, so:

$$
\begin{array}{rcl}
\sigma^{-1}(i) & > & \sigma^{-1}(j) \\
\epsilon^{-1} \circ \left(\sigma_1^{-1} \otimes \sigma_2^{-1}\right)(i) & > & \epsilon^{-1} \circ \left(\sigma_1^{-1} \otimes \sigma_2^{-1}\right)(j) \\
\epsilon^{-1}\left(\sigma_1^{-1}(i)\right) & > & \epsilon^{-1}\left(\sigma_1^{-1}(j)\right) \\
\sigma_1^{-1}(i) & > & \sigma_1^{-1}(j),
\end{array}
$$

as ϵ^{-1} is increasing on $\{1, \ldots, k\}$. So $\sigma_1 \in S_{\mathbb{F}}$. If $i \twoheadrightarrow j$ in $\mathbb{G}$, then $k+i \twoheadrightarrow k+j$ in $\mathbb{F}\mathbb{G}$, so:

$$
\begin{array}{rcl}
\sigma^{-1}(k+i) & > & \sigma^{-1}(k+j) \\
\epsilon^{-1} \circ \left(\sigma_1^{-1} \otimes \sigma_2^{-1}\right)(k+i) & > & \epsilon^{-1} \circ \left(\sigma_1^{-1} \otimes \sigma_2^{-1}\right)(k+j) \\
\epsilon^{-1}\left(k+\sigma_2^{-1}(i)\right) & > & \epsilon^{-1}\left(k+\sigma_2^{-1}(j)\right) \\
\sigma_2^{-1}(i) & > & \sigma_2^{-1}(j),
\end{array}
$$

as ϵ^{-1} is increasing on $\{k+1, \ldots, k+l\}$. So $\sigma_2 \in S_{\mathbb{G}}$. Conversely, if $\sigma = (\sigma_1 \otimes \sigma_2) \circ \epsilon$, with $\sigma_1 \in \Sigma_k$, $\sigma_2 \in \Sigma_l$, and $\epsilon \in Sh(k,l)$, the same computations shows that $\sigma \in S_{\mathbb{F}\mathbb{G}}$. So:

$$S_{\mathbb{F}\mathbb{G}} = \bigsqcup_{\epsilon \in Sh(k,l)} (S_{\mathbb{F}} \otimes S_{\mathbb{G}}) \circ \epsilon.$$

So:

$$\Theta(\mathbb{F}\mathbb{G}) = \sum_{\epsilon \in Sh(k,l)} \sum_{\sigma_1 \in S_{\mathbb{F}}} \sum_{\sigma_2 \in S_{\mathbb{G}}} (\sigma_1 \otimes \sigma_2) \circ \epsilon = \Theta(\mathbb{F})\Theta(\mathbb{G}).$$

So Θ is an algebra morphism.

Let $\mathbb{F} \in \mathbf{F}_o(n)$ and let $\boldsymbol{v}$ be an admissible cut of $\mathbb{F}$. The vertices of $Roo_{\boldsymbol{v}}\mathbb{F}$ are $i_1 < \ldots < i_k$ and the vertices of $Lea_{\boldsymbol{v}}\mathbb{F}$ are $j_1 < \ldots < j_l$, with $k+l=n$. Let $\zeta_{\boldsymbol{v}}$ be the inverse of the permutation $(i_1, \ldots, i_k, j_1, \ldots, j_l)$. Note that $\zeta_{\boldsymbol{v}}$ is a (k,l)-shuffle. Let $\sigma_1 \in S_{Roo_{\boldsymbol{v}}\mathbb{F}}$ and $\sigma_2 \in S_{Lea_{\boldsymbol{v}}\mathbb{F}}$. Let us show that $\sigma = \zeta_{\boldsymbol{v}}^{-1} \circ (\sigma_1 \otimes \sigma_2) \in S_{\mathbb{F}}$. If $i \twoheadrightarrow j$ in $\mathbb{F}$, then three cases are possible:

- i and j belong to $Roo_{\boldsymbol{v}}\mathbb{F}$, say $i = i_p$ and $j = i_q$. Then $i \twoheadrightarrow j$ in $Roo_{\boldsymbol{v}}\mathbb{F}$, so $\sigma_1^{-1}(p) > \sigma_1^{-1}(q)$. Then:

$$
\begin{array}{rcl}
\sigma^{-1}(i) & = & \left(\sigma_1^{-1} \otimes \sigma_2^{-1}\right) \circ \zeta_{\boldsymbol{v}}(i) \\
 & = & \left(\sigma_1^{-1} \otimes \sigma_2^{-1}\right)(p) \\
 & = & \sigma_1^{-1}(p).
\end{array}
$$

Similarly, $\sigma^{-1}(j) = \sigma_1^{-1}(q)$. So $\sigma^{-1}(i) > \sigma^{-1}(j)$.

- i and j belong to $Lea_{\boldsymbol{v}}\mathbb{F}$. The proof is similar.
- i belongs to $Lea_{\boldsymbol{v}}\mathbb{F}$ and j belongs to $Roo_{\boldsymbol{v}}\mathbb{F}$. Then $k+1 \leq \zeta_{\boldsymbol{v}}(i) \leq k+l$ and $1 \leq \zeta_{\boldsymbol{v}}(j) \leq k$, so $\sigma^{-1}(j) \leq k < k+1 \leq \sigma^{-1}(i)$.

Conversely, let $\sigma \in S_{\mathbb{F}}$ and $0 \leq k \leq n$. We put $\zeta = \zeta_k$, $\sigma_1 = \sigma_1^{(k)}$ and $\sigma_2 = \sigma_2^{(k)}$, so that $\sigma = \zeta^{-1} \circ (\sigma_1 \otimes \sigma_2)$. Let $\mathbb{G}$ be the sub-forest of $\mathbb{F}$ formed by the vertices $\zeta(1), \ldots, \zeta(k)$ and $\mathbb{H}$ be the sub-forest of $\mathbb{F}$ formed by the vertices $\zeta(k+1), \ldots, \zeta(k+l)$, with $l = n-k$. If i is a vertex of $\mathbb{F}$ and j is a vertex of $\mathbb{H}$ such that $i \twoheadrightarrow j$ in $\mathbb{F}$, then $\sigma^{-1}(i) > \sigma^{-1}(j)$. As $k+1 \leq \zeta(j) \leq k+l$, $k+1 \leq \sigma^{-1}(j) \leq k+l$, so $k+1 \leq \sigma^{-1}(i) \leq k+l$ and $k+1 \leq \zeta(i) \leq k+l$: i is a vertex of $\mathbb{H}$. As a consequence, there exists a (unique) admissible cut $\boldsymbol{v}$ such that $\mathbb{G} = Roo_{\boldsymbol{v}}\mathbb{F}$ and $\mathbb{H} = Lea_{\boldsymbol{v}}\mathbb{F}$. By definition, $\zeta = \zeta_{\boldsymbol{v}}$. It is not difficult to prove that $\sigma_1 \in S_{Roo_{\boldsymbol{v}}\mathbb{F}}$ and $\sigma_2 \in S_{Lea_{\boldsymbol{v}}\mathbb{F}}$. Hence, there is a bijection:

$$\left\{ \begin{array}{rcl} S_{\mathbb{F}} \times \{0, \ldots, n\} & \longrightarrow & \displaystyle\bigsqcup_{\boldsymbol{v} \models V(\mathbb{F})} S_{Roo_{\boldsymbol{v}}\mathbb{F}} \times S_{Lea_{\boldsymbol{v}}\mathbb{F}} \\ (\sigma, k) & \longmapsto & \left(\sigma_1^{(k)}, \sigma_2^{(k)}\right). \end{array} \right.$$

Finally:

$$\Delta \circ \Theta(\mathbb{F}) = \sum_{\sigma \in S_{\mathbb{F}}} \sum_{k=0}^{n} \sigma_1^{(k)} \otimes \sigma_2^{(k)} = \sum_{\boldsymbol{v} \models V(\mathbb{F})} \sum_{\sigma_1 \in S_{Roo_{\boldsymbol{v}}\mathbb{F}}} \sum_{\sigma_2 \in S_{Lea_{\boldsymbol{v}}\mathbb{F}}} \sigma_1 \otimes \sigma_2 = (\Theta \otimes \Theta) \circ \Delta(\mathbb{F}).$$

So Θ is a coalgebra morphism. $\square$

Remarks.

1. In [2, 20, 24], several morphisms of Hopf algebra are defined between NCK, **FQSym**, the Loday-Ronco Hopf algebra of binary trees LR and its dual **YSym**. Composing the isomorphism $\Phi : NCK \longrightarrow LR$ and the transpose $\Lambda^* : LR \longrightarrow$ **FQSym** of the morphism Λ : **FQSym** $\longrightarrow$ **YSym** of [2], we obtain an injection of Hopf algebra $\Upsilon : NCK \longrightarrow$ **FQSym** such that, for all planar forest F with n vertices:

$$\Upsilon(F) = \sum_{\tau^{-1} \leq \upsilon(\sigma)} \tau,$$

   where, with the notations of [2], υ is the map $\gamma \circ \phi$ from the set of planar forests with n vertices into Σ_n, and $\leq$ is the Bruhat order on Σ_n. For example, $\upsilon(\vee) = (132)$ and $\upsilon(\cdot\,\mathfrak{l}) = (312)$, so $\Upsilon(\vee) = (123) + (132)$ and $\Upsilon(\cdot\,\mathfrak{l}) = (123) + (213) + (231)$. Identifying NCK as a subalgebra of $\mathbf{H}_{ho}$, it is then not difficult to show that $\Upsilon = \Theta_{|NCK}$. Replacing binary trees by binary trees with level, it may be possible to prove proposition 6 in the same way as in [2].

2. There is a natural section ω of Θ, sending σ to the trunk tree decorated, from the root to the leaf, by $\sigma(1), \ldots, \sigma(n)$. This section is a coalgebra morphism but not an algebra morphism. For example, $\omega((1))\omega((1)) = \cdot_1 \cdot_2$ whereas $\omega((1)(1)) = \omega((12) + (21)) = \mathfrak{l}_1^2 + \mathfrak{l}_2^1$.

3.2 Restriction to heap-ordered forests

Proposition 7 *The restriction of Θ to $\mathbf{H}_{ho}$ is an isomorphism of graded Hopf algebras.*

Proof. As Θ is homogenous, $\Theta(\mathbf{H}_{ho}(n)) \subseteq$ **FQSym**(n) for all $n \geq 0$. Let us first recall that $dim(\mathbf{H}_{ho}(n)) = n!$. From Lemma 6.5 of [13], the number of heap-ordered trees with $n+1$

vertices is $n!$. This is proved inductively, using the bijection:

$$\left\{\begin{array}{rcl}\mathbf{T}_{ho}(n-1)\times\{1,\ldots,n-1\} & \longmapsto & \mathbf{T}_{ho}(n)\\ (t,i) & \longmapsto & \text{the heap-ordered tree obtained}\\ & & \quad\text{by grafting } n \text{ on the vertex } i \text{ of } t.\end{array}\right.$$

If t is a heap-ordered tree, then its root is its smallest element, so there is a bijection:

$$\left\{\begin{array}{rcl}\mathbf{T}_{ho}(n+1) & \longmapsto & \mathbf{F}_{ho}(n)\\ t & \longmapsto & \text{the heap-ordered forest obtained by deleting the root of } t.\end{array}\right.$$

So $card(\mathbf{F}_{ho}(n)) = n! = dim(\mathbf{H}_{ho}(n))$.

In order to prove that $\Theta_{|\mathbf{H}_{ho}}$ is an isomorphism, it is now enough to prove that $\Theta_{|\mathbf{H}_{ho}(n)}$ is injective. We totally order the elements of Σ_n by the lexicographic order. For any heap-ordered forest $\mathbb{F}$ with n vertices , let $m(\mathbb{F})\in\Sigma_n$ be the greatest element of $S_{\mathbb{F}}$. It is not difficult to prove that $m(\mathbb{F})$ can be inductively computed in the following way:

1. If $\mathbb{F}$ is an ordered tree, let $\mathbb{G}$ be the ordered forest obtained from $\mathbb{F}$ by deleting its root. Then $m(\mathbb{F}) = (1)\otimes m(\mathbb{G})$.

2. If $\mathbb{F}$ is not an ordered tree, let $\mathbb{F}_1,\ldots,\mathbb{F}_m$ be its connected components, ordered by their roots, that is to say $root(\mathbb{F}_1)<\ldots<root(\mathbb{F}_m)$ in $\mathbb{F}$. Let $i_{j,1}<\ldots<i_{j,k_j}$ be the vertices of $\mathbb{F}_j$. Let σ be the following permutation:

$$\left(\begin{array}{ccccccccccc}1 & \ldots & k_m & k_m+1 & \ldots & k_m+k_{m-1} & \ldots & k_m+\ldots+k_2+1 & \ldots & k_m+\ldots+k_1\\ i_{m,1} & \ldots & i_{m,k_m} & i_{m-1,1} & \ldots & i_{m-1,k_{m-1}} & \ldots & i_{1,1} & \ldots & i_{1,k_1}\end{array}\right).$$

   Then $m(\mathbb{F}) = \sigma\circ(m(\mathbb{F}_m)\otimes\ldots\otimes m(\mathbb{F}_1))$.

The induction is initiated by $m(\bullet_1) = (1)$. For example:

$$\begin{array}{rclrcl} m\left(\substack{\bullet_2\\|\\\bullet_1}\right) &=& (12), & m(\bullet_1\bullet_2) &=& (21),\\ m\left(\substack{\bullet_3\\|\\\bullet_2\\|\\\bullet_1}\right) &=& (123), & m\left(\substack{\bullet_2\ \ \bullet_3\\ \vee\\ \bullet_1}\right) &=& (132),\\ m\left(\substack{\bullet_2\\|\\\bullet_1}\bullet_3\right) &=& (312), & m\left(\substack{\bullet_3\\|\\\bullet_1}\bullet_2\right) &=& (213),\\ m\left(\bullet_1\substack{\bullet_3\\|\\\bullet_2}\right) &=& (231), & m(\bullet_1\bullet_2\bullet_3) &=& (321).\end{array}$$

Let us prove that m is injective by induction on n. It is obvious if $n=1$. If $n\geq 2$, let $\mathbb{F},\mathbb{F}'$ be two heap-ordered forests such that $m(\mathbb{F}) = m(\mathbb{F}') = \sigma$. If $\sigma(1) = 1$, then both $\mathbb{F}$ and $\mathbb{F}$ are heap-ordered trees. By the induction hypothesis, the heap-ordered forests obtained by deleting the roots of $\mathbb{F}$ and $\mathbb{F}'$ are equal, so $\mathbb{F} = \mathbb{F}'$. If $\sigma(1)\neq 1$, we put $i = \sigma^{-1}(1)-1$. With the notations of subsection 1.2, by the induction hypothesis, the permutation $\sigma_2^{(i)}$ is $m(\mathbb{G})$, where $\mathbb{G}$ is the connected components of $\mathbb{F}$ and $\mathbb{F}'$ with the smallest root; the permutation $\sigma_1^{(i)}$ is $m(\mathbb{H})$, where $\mathbb{H}$ is the subforest of $\mathbb{F}$ and $\mathbb{F}'$ formed by the vertices which are not in $\mathbb{G}$. Finally, $\sigma\circ\left(\sigma_1^{(i)}\otimes\sigma_2^{(i)}\right)^{-1}$ allows to reconstruct $\mathbb{F}$ and $\mathbb{F}'$ from $\mathbb{G}\mathbb{H}$, so $\mathbb{F} = \mathbb{F}'$.

As a conclusion, m is injective. Consequently, the family $(\Theta(\mathbb{F}))_{\mathbb{F}\in\mathbf{F}_{ho}(n)}$ is a free family of **FQSym**, so $\Theta_{|\mathbf{H}_{ho}}$ is injective, hence, bijective. $\Box$

Definition 8 *Let $\sigma\in\Sigma_n$. The element $\mathbb{T}^\sigma$ is the unique element of $\mathbf{H}_{ho}$ such that $\Theta(\mathbb{T}^\sigma) = \sigma^{-1}$.*

Lemma 9 *1. For any $(\sigma,\tau)\in\Sigma_k\times\Sigma_l$,*

$$\mathbb{T}^\sigma\mathbb{T}^\tau = \sum_{\zeta\in Sh(k,l)}\mathbb{T}^{\zeta^{-1}\circ(\sigma\otimes\tau)}. \tag{5}$$

2. For any $\sigma \in \Sigma_n$,

$$\Delta(\mathbb{T}^{\sigma}) = \sum_{k=0}^{n} \sum_{\substack{\sigma=(\sigma_1\otimes\sigma_2)\circ\epsilon \\ \sigma_1\in\Sigma_k,\, \sigma_2\in\Sigma_{n-k},\, \epsilon\in Sh(k,n-k)}} \mathbb{T}^{\sigma_1}\otimes\mathbb{T}^{\sigma_2}. \tag{6}$$

Proof. 1. Indeed:

$$\Theta(\mathbb{T}^{\sigma}\mathbb{T}^{\tau}) = \Theta(\mathbb{T}^{\sigma})\Theta(\mathbb{T}^{\tau}) = \sigma^{-1}\tau^{-1} = \sum_{\epsilon\in Sh(k,l)} (\sigma^{-1}\otimes\tau^{-1})\circ\epsilon = \sum_{\zeta\in Sh(k,l)} \Theta\left(\mathbb{T}^{\zeta^{-1}\circ(\sigma\otimes\tau)}\right).$$

We conclude with the injectivity of $\Theta_{|\mathbf{H}_{ho}}$.

2. Indeed:

$$\begin{aligned}
(\Theta\otimes\Theta)\circ\Delta(\mathbb{T}^{\sigma}) &= \Delta(\sigma^{-1}) \\
&= \sum_{k=0}^{n} \sum_{\substack{\sigma^{-1}=\zeta^{-1}\circ(\tau_1\otimes\tau_2) \\ \tau_1\in\Sigma_k,\, \tau_2\in\Sigma_{n-k},\, \zeta\in Sh(k,n-k)}} \tau_1\otimes\tau_2 \\
&= \sum_{k=0}^{n} \sum_{\substack{\sigma=(\sigma_1\otimes\sigma_2)\circ\epsilon \\ \sigma_1\in\Sigma_k,\, \sigma_2\in\Sigma_{n-k},\, \epsilon\in Sh(k,n-k)}} \sigma_1^{-1}\otimes\sigma_2^{-1} \\
&= (\Theta\otimes\Theta)\left(\sum_{k=0}^{n} \sum_{\substack{\sigma=(\sigma_1\otimes\sigma_2)\circ\epsilon \\ \sigma_1\in\Sigma_k,\, \sigma_2\in\Sigma_{n-k},\, \epsilon\in Sh(k,n-k)}} \mathbb{T}^{\sigma_1}\otimes\mathbb{T}^{\sigma_2}\right).
\end{aligned}$$

We conclude with the injectivity of $\Theta_{|\mathbf{H}_{ho}}\otimes\Theta_{|\mathbf{H}_{ho}}$. $\square$

For example:

$$\begin{aligned}
\mathbb{T}^{(1)} &= \bullet_1, \\
\mathbb{T}^{(12)} &= \mathrm{I}_{\substack{2\\1}}, \\
\mathbb{T}^{(21)} &= \bullet_1\bullet_2 - \mathrm{I}_{\substack{2\\1}}, \\
\mathbb{T}^{(123)} &= \mathrm{I}_{\substack{3\\2\\1}}, \\
\mathbb{T}^{(132)} &= {}^{2}\mathrm{V}_{1}^{3} - \mathrm{I}_{\substack{3\\2\\1}}, \\
\mathbb{T}^{(213)} &= \bullet_2\,\mathrm{I}_{\substack{3\\1}} - {}^{2}\mathrm{V}_{1}^{3}, \\
\mathbb{T}^{(231)} &= \bullet_3\,\mathrm{I}_{\substack{2\\1}} - {}^{2}\mathrm{V}_{1}^{3}, \\
\mathbb{T}^{(312)} &= \bullet_1\,\mathrm{I}_{\substack{3\\2}} - \mathrm{I}_{\substack{3\\2\\1}} - \bullet_2\,\mathrm{I}_{\substack{3\\1}} + {}^{2}\mathrm{V}_{1}^{3}, \\
\mathbb{T}^{(321)} &= \bullet_1\bullet_2\bullet_3 - \bullet_3\,\mathrm{I}_{\substack{2\\1}} - \bullet_1\,\mathrm{I}_{\substack{3\\2}} + \mathrm{I}_{\substack{3\\2\\1}}.
\end{aligned}$$

So:

$$\begin{aligned}
\mathbb{T}^{(21)}\mathbb{T}^{(1)} &= \mathbb{T}^{(213)} + \mathbb{T}^{(312)} + \mathbb{T}^{(321)}, \\
\Delta\left(\mathbb{T}^{(321)}\right) &= 1\otimes\mathbb{T}^{(321)} + \bullet_1\otimes(\bullet_1\bullet_2 - \mathrm{I}_{\substack{2\\1}}) + (\bullet_1\bullet_2 - \mathrm{I}_{\substack{2\\1}})\otimes\bullet_1 + \mathbb{T}^{(321)}\otimes 1 \\
&= 1\otimes\mathbb{T}^{(321)} + \mathbb{T}^{(1)}\otimes\mathbb{T}^{(21)} + \mathbb{T}^{(21)}\otimes\mathbb{T}^{(1)} + \mathbb{T}^{(321)}\otimes 1.
\end{aligned}$$

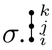
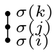
We can also give a decorated version of this result. If ℓ is an map from $\{1,\ldots,m\}$ to $\{1,\ldots,d\}$ and ℓ' is an map from $\{1,\ldots,m'\}$ to $\{1,\ldots,d\}$, let $(\mathbb{T}^{\sigma},\ell)$ be the element of the Hopf algebra obtained by decorating all the forests appearing in $\mathbb{T}^\sigma$ by ℓ; we define similarly $(\mathbb{T}^{\sigma'}, \ell')$. Then it comes directly from lemma 9 that:

$$(\mathbb{T}^\sigma, \ell).(\mathbb{T}^\tau, \ell') = \sum_{\zeta \in Sh(m,m')} \left(\mathbb{T}^{\zeta^{-1}\circ(\sigma\otimes\tau)}, \ell \otimes \ell'\right).$$

Moreover, for any $\sigma \in \Sigma_n$,

$$\Delta(\mathbb{T}^\sigma, \ell) = \sum_{k=0}^{n} \sum_{\substack{\sigma=(\sigma_1\otimes\sigma_2)\circ\epsilon \\ \sigma_1\in\Sigma_k,\, \sigma_2\in\Sigma_{n-k},\, \epsilon\in Sh(k,n-k)}} (\mathbb{T}^{\sigma_1} \otimes \mathbb{T}^{\sigma_2}, \ell \circ \epsilon^{-1}).$$

3.3 Action of the symmetric groups on $\mathbf{H}_o$

The symmetric group Σ_n acts naturally on the set of ordered forests with n vertices by changing the order of the vertices according to σ. For example, $\sigma.$ $=$  if $\{i, j, k\} = \{1, 2, 3\}$. This action is extended by linearity to the homogeneous component $\mathbf{H}_o(n)$ of degree n of $\mathbf{H}_o$.

Let $\mathbb{F}$ be an ordered forest of degree n and let $\tau \in \Sigma_n$. For any $\sigma \in \Sigma_n$:

$$\begin{aligned}
\sigma \in S_{\tau.\mathbb{F}} &\iff \forall i, j \in V(\tau.\mathbb{F}),\ (i \twoheadrightarrow j \text{ in } \tau.\mathbb{F}) \Longrightarrow (\sigma^{-1}(i) > \sigma^{-1}(j)) \\
&\iff \forall i, j \in V(\tau.\mathbb{F}),\ (\tau(i) \twoheadrightarrow \tau(j) \text{ in } \tau.\mathbb{F}) \Longrightarrow (\sigma^{-1}\circ\tau(i) > \sigma^{-1}\circ\tau(j)) \\
&\iff \forall i, j \in V(\mathbb{F}),\ (i \twoheadrightarrow j \text{ in } \mathbb{F}) \Longrightarrow (\sigma^{-1}\circ\tau(i) > \sigma^{-1}\circ\tau(j)) \\
&\iff \tau^{-1}\circ\sigma \in S_{\mathbb{F}}.
\end{aligned}$$

As a consequence, $S_{\tau.\mathbb{F}} = \tau \circ S_{\mathbb{F}}$ so, for any $\mathbb{F} \in \mathbf{H}_o(n)$, for any $\tau \in \Sigma_n$, $\Theta(\tau.\mathbb{F}) = \tau \circ \Theta(\mathbb{F})$.

The subspace $\mathbf{H}_{ho}(n)$ of $\mathbf{H}_o(n)$ is clearly not stable under the action of Σ_n. More precisely, if $\mathbb{F}$ is a heap-ordered forest and $\sigma \in \Sigma_n$, then $\sigma.\mathbb{F}$ is heap-ordered if, and only if, $\sigma^{-1} \in S_{\mathbb{F}}$. In particular, if $\mathbb{F}$ and $\mathbb{G}$ are heap-ordered forests with respectively k and l vertices, then for any (k,l)-shuffle ζ, $\zeta \in S_{\mathbb{F}\mathbb{G}}$, so $\zeta^{-1}.\mathbb{F}\mathbb{G}$ is an element of $\mathbf{H}_{ho}(k+l)$. As a consequence, if $\sigma \in \Sigma_k$, $\tau \in \Sigma_l$ and $\epsilon \in Sh(k,l)$, $\epsilon^{-1}.\mathbb{T}^\sigma\mathbb{T}^\tau \in \mathbf{H}_{ho}$. We compute:

$$\Theta(\epsilon^{-1}.\mathbb{T}^\sigma\mathbb{T}^\tau) = \epsilon^{-1}\circ(\sigma^{-1}\tau^{-1}) = \sum_{\zeta\in Sh(k,l)} \epsilon^{-1}\circ(\sigma^{-1}\otimes\tau^{-1})\circ\zeta = \sum_{\zeta\in Sh(k,l)} \Theta\left(\mathbb{T}^{\zeta^{-1}\circ(\sigma\otimes\tau)\circ\epsilon}\right).$$

From the injectivity of $\Theta_{|\mathbf{H}_{ho}}$, we deduce:

Lemma 10 *For any $(\sigma, \tau, \epsilon) \in \Sigma_k \times \Sigma_l \times Sh(k,l)$,*

$$\epsilon^{-1}.\mathbb{T}^\sigma\mathbb{T}^\tau = \sum_{\zeta\in Sh(k,l)} \mathbb{T}^{\zeta^{-1}\circ(\sigma\otimes\tau)\circ\epsilon}. \tag{7}$$

4 A commuting square of Hopf algebra epimorphisms

4.1 Definition of the square

For any d, there is an isomorphism of Hopf algebras $\Theta^d : \mathbf{H}^d_{ho} \longrightarrow \mathbf{FQSym}^d$. We also defined two natural epimorphisms of Hopf algebras π^d_Σ and π^d_{ho}, see definitions 1 and 4. We obtain the following diagram:

$$\begin{tikzcd}
\mathbf{H}^d_{ho} \arrow[r, "\Theta^d"] \arrow[d, "\pi^d_{ho}"'] & \mathbf{FQSym}^d \arrow[d, "\pi^d_\Sigma"] \\
\mathbf{H}^d & \mathbf{Sh}^d
\end{tikzcd}$$

Let us consider a d-decorated rooted forest $\mathcal{F}$. We give it a heap-order to obtain an element $\mathbb{F}$ of $\mathbf{H}^d_{ho}$, such that $\pi^d_{ho}(\mathbb{F}) = \mathcal{F}$. It is then not difficult to show that $\pi^d_\Sigma \circ \Theta^d(\mathbb{F})$ does not depend of the choice of the heap-order on $\mathcal{F}$, so this defines a map $\theta^d : \mathbf{H}^d \longrightarrow \mathbf{Sh}^d$, making the following diagram commuting:

$$\begin{tikzcd}
\mathbf{H}^d_{ho} \arrow[r, "\Theta^d"] \arrow[d, "\pi^d_{ho}"'] & \mathbf{FQSym}^d \arrow[d, "\pi^d_\Sigma"] \\
\mathbf{H}^d \arrow[r, "\theta^d"'] & \mathbf{Sh}^d
\end{tikzcd}$$

As π^d_{ho} is surjective, θ^d is a morphism of Hopf algebras. For example, if $1 \leq a, b, c \leq d$:

$$\begin{array}{rcl}
\theta^d(\bullet_a) &=& (a),\\
\theta^d(\left|\begin{smallmatrix}b\\a\end{smallmatrix}\right.) &=& (ab),\\
\theta^d(\bullet_a\bullet_b) &=& (ab)+(ba),\\
\theta^d({}^b\mathrm{V}^c_a) &=& (abc)+(acb),\\
\theta^d(\left|\begin{smallmatrix}c\\b\\a\end{smallmatrix}\right.) &=& (abc),\\
\theta^d(\left|\begin{smallmatrix}b\\a\end{smallmatrix}\right.\bullet_c) &=& (abc)+(acb)+(cab),\\
\theta^d(\bullet_a\bullet_b\bullet_c) &=& (abc)+(acb)+(bac)+(bca)+(cab)+(cba).
\end{array}$$

Seeing d-words as d-decorated trunk trees (see subsection 2.1), one can write:

$$\theta^d({}^b\mathrm{V}^c_a) = \left|\begin{smallmatrix}c\\b\\a\end{smallmatrix}\right. + \left|\begin{smallmatrix}b\\c\\a\end{smallmatrix}\right. .$$

In other words, the image of a decorated tree by θ^d is the sum of all trunk trees with same decorations, whose total ordering is compatible with the partial ordering of the initial tree.

Let us now consider a d-word $\ell(1)\ldots\ell(n)$ of degree n , or equivalently a trunk tree $\mathcal{T}$ with decoration ℓ. We put:

$$\mathcal{T}^\sigma = \pi^d_{ho} \circ (\Theta^d)^{-1}(\sigma^{-1}, \ell) = \pi^d_{ho}(\mathbb{T}^\sigma, \ell).$$

In other words, $\mathcal{T}^\sigma$ is obtained from $\mathbb{T}^\sigma$ by decorating the i-th vertex of the forests in $\mathbb{T}^\sigma$ by $\ell(i)$ for all i, and then deleting the orders on the the vertices. For example, if $1 \leq a, b, c \leq d$:

$$\begin{array}{rcl}
\left|\begin{smallmatrix}c\\b\\a\end{smallmatrix}\right.^{(123)} &=& \left|\begin{smallmatrix}c\\b\\a\end{smallmatrix}\right. ,\\
\left|\begin{smallmatrix}c\\b\\a\end{smallmatrix}\right.^{(132)} &=& {}^b\mathrm{V}^c_a - \left|\begin{smallmatrix}c\\b\\a\end{smallmatrix}\right. ,\\
\left|\begin{smallmatrix}c\\b\\a\end{smallmatrix}\right.^{(213)} &=& \bullet_b \left|\begin{smallmatrix}c\\a\end{smallmatrix}\right. - {}^b\mathrm{V}^c_a ,\\
\left|\begin{smallmatrix}c\\b\\a\end{smallmatrix}\right.^{(231)} &=& \bullet_c \left|\begin{smallmatrix}b\\a\end{smallmatrix}\right. - {}^b\mathrm{V}^c_a ,\\
\left|\begin{smallmatrix}c\\b\\a\end{smallmatrix}\right.^{(312)} &=& \bullet_a \left|\begin{smallmatrix}c\\b\end{smallmatrix}\right. - \left|\begin{smallmatrix}c\\b\\a\end{smallmatrix}\right. - \bullet_b \left|\begin{smallmatrix}c\\a\end{smallmatrix}\right. + {}^b\mathrm{V}^c_a ,\\
\left|\begin{smallmatrix}c\\b\\a\end{smallmatrix}\right.^{(321)} &=& \bullet_a\bullet_b\bullet_c - \bullet_c \left|\begin{smallmatrix}b\\a\end{smallmatrix}\right. - \bullet_a \left|\begin{smallmatrix}c\\b\end{smallmatrix}\right. + \left|\begin{smallmatrix}c\\b\\a\end{smallmatrix}\right. .
\end{array}$$

For instance, $\left|\begin{smallmatrix}c\\b\\a\end{smallmatrix}\right.^{(132)}$ is a particular example of $\mathcal{T}^\sigma$, with $\mathcal{T} = \left|\begin{smallmatrix}c\\b\\a\end{smallmatrix}\right.$ and $\sigma = (132)$. The commutative square implies that $\theta^d(\mathcal{T}^\sigma)$ is the d-word $\ell \circ \sigma^{-1}(1)\ldots\ell\circ\sigma^{-1}(n)$.

As the edges of the commutative square are Hopf algebra morphisms, lemmas 9 and 10 imply:

Lemma 11 *1. Let $\mathcal{T}_k = (\mathbb{T}_k, \ell_1)$ and $\mathcal{T}_l = (\mathbb{T}_l, \ell_2)$ be two d-words of respective degrees k and l, seen as d-decorated trunk trees. For any $(\sigma, \tau, \epsilon) \in \Sigma_k \times \Sigma_l \times Sh(k, l)$:*

$$\epsilon^{-1}.\left((\mathbb{T}_k, \ell_1)^\sigma . (\mathbb{T}_l, \ell_2)^\tau\right) = \sum_{\zeta \in Sh(k,l)} (\mathbb{T}_{k+l}, \ell_1 \otimes \ell_2)^{\zeta\circ(\sigma\otimes\tau)\circ\epsilon^{-1}} . \tag{8}$$

Here, the action of ϵ is given by permutations of the decorations.

2. *Let $\mathcal{T} = (\mathbb{T}_n, \ell)$ be a d-word of degree n, seen as a d-decorated trunk tree. For any $\sigma \in \Sigma_n$,*

$$\Delta(\mathcal{T}^\sigma) = \sum_{k=0}^{n} \sum_{\substack{\sigma=(\sigma_1\otimes\sigma_2)\circ\epsilon \\ \sigma_1\in\Sigma_k,\, \sigma_2\in\Sigma_{n-k},\, \epsilon\in Sh(k,n-k)}} \epsilon^{-1}.\left((\mathbb{T}^{\sigma_1}, \ell_1) \otimes (\mathbb{T}^{\sigma_2}, \ell_2)\right). \tag{9}$$

   Here also, the action of ϵ on tensors of decorated forests is given by permutation of the decorations (the ϵ^{-1} in the right member comes from the fact that each decoration follows its vertex when we cut the forests of $\mathcal{T}^\sigma$, so this permutes the letters of ℓ according to ϵ).

For example:

$$\begin{array}{rcl}
\tddeux{a}{b}^{(21)}\tdun{c}^{(1)} &=& \tdtroisdeux{a}{b}{c}^{(213)} + \tdtroisdeux{a}{b}{c}^{(312)} + \tdtroisdeux{a}{b}{c}^{(321)}, \\
\Delta\left(\tdtroisdeux{a}{b}{c}^{(321)}\right) &=& 1 \otimes \tdtroisdeux{a}{b}{c}^{(321)} + \tdun{c} \otimes (\tdun{a}\tdun{b} - \tddeux{a}{b}) + (\tdun{b}\tdun{c} - \tddeux{b}{c}) \otimes \tdun{a} + \tdtroisdeux{a}{b}{c}^{(321)} \otimes 1 \\
&=& 1 \otimes \tdtroisdeux{a}{b}{c}^{(321)} + \tdun{c}^{(1)} \otimes \tddeux{a}{b}^{(21)} + \tddeux{b}{c}^{(21)} \otimes \tdun{a}^{(1)} + \tdtroisdeux{a}{b}{c}^{(321)} \otimes 1 \\
&=& 1 \otimes \tdtroisdeux{a}{b}{c}^{(321)} \otimes 1 + (231).\left(\tdun{a}^{(1)} \otimes \tddeux{b}{c}^{(21)}\right) + (312).\left(\tddeux{a}{b}^{(21)} \otimes \tdun{c}^{(1)}\right) + \tdtroisdeux{a}{b}{c}^{(321)} \otimes 1.
\end{array}$$

4.2 Applications to iterated integrals

Let H be a Hopf algebra and A a commutative algebra. Then the set $Char_H(A)$ of algebra morphisms from H to A is a group for the convolution. More precisely, if $\phi, \psi \in Char_H(A)$, then $\phi * \psi = m_A \circ (\phi \otimes \psi) \circ \Delta_H$, where m_A is the product of A and Δ_H the coproduct of H. The unit of $Char_H(A)$ is $x \mapsto \varepsilon_H(x)1_A$, where ε_H is the counit of H, and the inverse of ϕ is $\phi \circ S_H$, where S_H is the antipode of H.

In particular, a character of the shuffle algebra $\mathbf{Sh}^d$ can be seen as a map ϕ from the set of d-words to A, such that $\phi(1) = 1_A$ and, for any d-words w and w':

$$\phi(w)\phi(w') = \sum_{w''\in Sh(w,w')} \phi(w'').$$

The convolution product of ϕ and ψ is given by:

$$(\phi * \psi)(a_1 \ldots a_k) = \sum_{i=0}^{k} \phi(a_1 \ldots a_i)\psi(a_{i+1} \ldots a_n).$$

The inverse of the character ϕ is $\phi^{-1} = \phi \circ \bar{S}$:

$$\phi^{-1}(a_1 \ldots a_n) = (-1)^n \phi(a_n \ldots a_1).$$

A character of $\mathbf{H}^d$ can be seen as a map from the set of d-decorated rooted trees to A, extended to $\mathbf{H}^d$ by multiplicativity. The convolution product of ϕ and ψ is given by:

$$(\phi * \psi)(\mathbb{F}) = \sum_{\boldsymbol{v}\models V(\mathbb{F})} \phi(Roo_{\boldsymbol{v}}\mathbb{F})\psi(Lea_{\boldsymbol{v}}\mathbb{F}).$$

Seeing $\mathbf{Sh}^d$ as a sub-coalgebra of $\mathbf{H}^d$, this formula for the convolution product also works for d-words seen as trunk trees.

Proposition 12 *The canonical surjection π^d_{ho} from $\mathbf{H}^d_{ho}$ to $\mathbf{H}^d$ induces a canonical injection of the group $Char_{\mathbf{H}^d}(A)$ into the group $Char_{\mathbf{H}^d_{ho}}(A)$: a character of $\mathbf{H}^d_{ho}$ is a character of $\mathbf{H}^d$ if, and only if, it does not depend of the orders on the vertices of the ordered forests. In other words, for any $\psi \in Char_{\mathbf{H}^d_{ho}}(A)$, ψ belongs to $Char_{\mathbf{H}^d}(A)$ if, and only if, it is invariant under forest-order-preserving symmetries, that is to say for any ordered forest $\mathbb{F}$, for any $\sigma \in S_{\mathbb{F}}$, $\psi(\sigma^{-1}.\mathbb{F}) = \psi(\mathbb{F})$.*

As $\theta^d : \mathbf{H}^d \longrightarrow \mathbf{Sh}^d$ is a Hopf algebra morphism, there is a group morphism from $Char_{\mathbf{Sh}^d}(A)$ to $Char_{\mathbf{H}^d}(A)$, sending a character ϕ to $\phi \circ \theta^d$. The character $\bar{\phi} := \phi \circ \theta^d$ of $\mathbf{H}^d$ will be called the *extension* of ϕ.

For example, let us fix d regular functions $\Gamma_1, \ldots, \Gamma_d$ and s, t in $\mathbb{R}$. For any d-word $a_1 \ldots a_n$, we let

$$I^{ts}_{\Gamma}(a_1 \ldots a_n) = \int_s^t d\Gamma_{a_1}(x_1) \int_s^{x_1} d\Gamma_{a_2}(x_2) \ldots \int_s^{x_n - 1} d\Gamma_{a_n}(x_n).$$

It is well-known that I^{ts}_{Γ} is a character of $\mathbf{Sh}^d$. The extension of I^{ts}_{Γ} is defined on any rooted forest $\mathbb{F}$ with decoration ℓ in the following way:

- Choose a heap-order on the vertices of $\mathbb{F}$.
- Put $[s,t]^{\mathbb{F}} = \{(x_1, \ldots, x_n) \in [s,t] \mid \forall i, j,\ (i \twoheadrightarrow j \text{ in } \mathbb{F}) \Longrightarrow (x_i \le x_j)\}$, where $n = |\mathbb{F}|$.

Then, denoting by i^- the ancestor of the vertex i in $\mathbb{F}$:

$$\begin{aligned} \bar{I}^{ts}_{\Gamma}(\mathbb{F}) &= \int_{[s,t]^{\mathbb{F}}} d\Gamma_{\ell(1)}(x_1) \ldots d\Gamma_{\ell(n)}(x_n) \\ &= \int_s^t d\Gamma_{\ell(1)}(x_1) \int_s^{x_{2-}} d\Gamma_{\ell(2)}(x_2) \ldots \int_s^{x_{n-}} d\Gamma_{\ell(n)}(x_n). \end{aligned} \tag{10}$$

For example:

$$\begin{aligned} \bar{I}^{ts}_{\Gamma}({}^{c}\mathbb{V}_{a}{}^{b}) &= I^{ts}_{\Gamma}(\theta^d({}^{c}\mathbb{V}_{a}{}^{b})) \\ &= I^{ts}_{\Gamma}(abc) + I^{ts}_{\Gamma}(acb) \\ &= \int_s^t d\Gamma_a(x_1) \int_s^{x_1} d\Gamma_b(x_2) \int_s^{x_2} d\Gamma_c(x_3) + \int_s^t d\Gamma_a(x_1) \int_s^{x_1} d\Gamma_c(x_3) \int_s^{x_3} d\Gamma_b(x_2) \\ &= \int_s^t d\Gamma_a(x_1) \int_s^{x_1} d\Gamma_b(x_2) \int_s^{x_1} d\Gamma_c(x_3). \end{aligned}$$

Note that $\bar{I}^{ts}_{\Gamma}$ is in fact defined on heap-ordered forests, so it is a character of $\mathbf{H}^d_{ho}$. It is clearly invariant under forest-order-preserving symmetries, so it is a character of $\mathbf{H}^d$.

Note that one may alternatively define characters of $\mathbf{H}^d_{ho}$ in the following way. Let $\mu = \mu(dx_1, \ldots, dx_n)$ be some signed measure on $\mathbb{R}^n$, and $\mathbb{F}$ a heap-ordered forest with vertices $1, \ldots, n$. Then one lets

$$I^{ts}_{\mu}(\mathbb{F}) = \int_{[s,t]^{\mathbb{F}}} \mu(dx_1, \ldots, dx_n). \tag{11}$$

In particular, letting $\mu_{(\Gamma,\ell)} = d\Gamma_{\ell(1)} \otimes \ldots \otimes d\Gamma_{\ell(n)}$, one has $I^{ts}_{\Gamma}(\mu_{(\Gamma,\ell)}) = I^{ts}_{\mu_{(\Gamma,\ell)}}(\mathbb{F})$, but the notation I^{ts}_{μ} can also be used for arbitrary measures that are not tensor measures $d\Gamma_{\ell(1)} \otimes \ldots \otimes d\Gamma_{\ell(n)}$.

As a consequence, if ϕ is a character of $\mathbf{Sh}^d$ extended to $\mathbf{H}^d$, and $\mathcal{T}$ is the trunk tree associated to the d-word $a_1 \ldots a_k$, then for any $\sigma \in \Sigma_k$, as $\theta^d(\mathcal{T}^{\sigma}) = (a_{\sigma^{-1}(1)} \ldots a_{\sigma^{-1}(k)})$:

$$\bar{\phi}(\mathcal{T}^{\sigma}) = \phi \circ \theta^d(\mathcal{T}^{\sigma}) = \phi(a_{\sigma^{-1}(1)} \ldots a_{\sigma^{-1}(k)}).$$

This allows to compute easily $\mathbb{T}^\sigma$ and $\mathcal{T}^\sigma$, as explained below. Indeed, if $\mathcal{T}$ is the trunk tree associated to the d-word $a_1 \dots a_k$:

$$\begin{aligned}\bar{I}^{st}_\Gamma(\mathcal{T}^\sigma) &= I^{st}_\Gamma(a_{\sigma^{-1}(1)} \dots a_{\sigma^{-1}(k)}) \\ &= \int_s^t d\Gamma_{a_{\sigma^{-1}(1)}}(x_{\sigma^{-1}(1)}) \int_s^{x_{\sigma^{-1}(1)}} d\Gamma_{a_{\sigma^{-1}(2)}}(x_{\sigma^{-1}(2)}) \dots \int_s^{x_{\sigma^{-1}(n-1)}} d\Gamma_{a_{\sigma^{-1}(n)}}(x_{\sigma^{-1}(n)}).\end{aligned}$$

Using Fubini's theorem, we can write:

$$\bar{I}^{st}_\Gamma(\mathcal{T}^\sigma) = \int_s^t d\Gamma_{a_1}(x_1) \int_{s_2}^{t_2} d\Gamma_{a_2}(x_2) \dots \int_{s_n}^{t_n} d\Gamma_{a_n}(x_n),$$

where for all i, $s_j \in \{s, x_1, \dots, x_{j-1}\}$ and $t_j \in \{t, x_1, \dots, x_{j-1}\}$. Now decompose $\int_{s_i}^{t_i} d\Gamma_{a_i}(x_i)$ into:

$$\left(\int_s^{t_i} - \int_s^{s_i}\right) d\Gamma_{a_i}(x_i).$$

Then $\bar{I}^{st}_\Gamma(\mathbb{T}^\sigma)$ can be written as a sum of terms of the form:

$$\pm \int_s^{\tau_1} d\Gamma_{a_1}(x_1) \int_s^{\tau_2} d\Gamma_{a_2}(x_2) \dots \int_s^{\tau_n} d\Gamma_{a_n}(x_n),$$

with $\tau_1 = t$ and $\tau_i \in \{t, x_1, \dots, x_{i-1}\}$ for $i = 2 \dots n$. Each of these expressions is of the form $\pm I^{st}_\Gamma(\mathbb{F})$ for a particular d-decorated rooted forest $\mathbb{F}$, and this gives the expression of $\mathcal{T}^\sigma$, as for example:

$$\begin{aligned} I^{ts}_\Gamma\left(\mathbf{I}^{a_3 (231)}_{a_2 a_1}\right) &= \int_s^t d\Gamma_{a_3}(x_3) \int_s^{x_3} d\Gamma_{a_1}(x_1) \int_s^{x_1} d\Gamma_{a_2}(x_2) \\ &= \int_s^t d\Gamma_{a_1}(x_1) \int_s^{x_1} d\Gamma_{a_2}(x_2) \int_{x_1}^t d\Gamma_{a_3}(x_3) \\ &= \int_s^t d\Gamma_{a_1}(x_1) \int_s^{x_1} d\Gamma_{a_2}(x_2) \int_s^t d\Gamma_{a_3}(x_3) \\ &\quad - \int_s^t d\Gamma_{a_1}(x_1) \int_s^{x_1} d\Gamma_{a_2}(x_2) \int_s^{x_1} d\Gamma_{a_3}(x_3) \\ &= I^{ts}_\Gamma(\mathbf{l}^{a_2}_{a_1} \bullet_{a_3}) - I^{ts}_\Gamma({}^{a_2}\mathbf{V}^{a_3}_{a_1}), \end{aligned}$$

so $\mathbf{I}^{a_3 (231)}_{a_2 a_1} = \mathbf{l}^{a_2}_{a_1} \bullet_{a_3} - {}^{a_2}\mathbf{V}^{a_3}_{a_1}$. Choosing three different decorations $a_1 = 1$, $a_2 = 2$ and $a_3 = 3$, we obtain $\mathbb{T}^{(231)} = \mathbf{l}^{2}_{1} \bullet_{3} - {}^{2}\mathbf{V}^{3}_{1}$.

5 Application to rough path theory: the Fourier normal ordering algorithm

We shall now finally apply the previous results to a general construction of formal rough paths. For real applications to analysis, the reader should wait until the next section. Here we let $\Gamma = (\Gamma_1(t), \dots, \Gamma_d(t)) : \mathbb{R} \to \mathbb{R}^d$ be some smooth path, compactly supported in $[0, T]$.

5.1 Definition of a formal rough path

Definition 13 (formal rough path) *A formal rough path over Γ is a functional $J^{ts}_\Gamma(i_1, \dots, i_n)$, $n \le \lfloor 1/\alpha \rfloor$, $i_1, \dots, i_n \in \{1, \dots, d\}$, such that $J^{ts}_\Gamma(i) = \Gamma_i(t) - \Gamma_i(s)$ are the increments of Γ, and the following two properties are satisfied:*

(i) (Chen property)

$$J_\Gamma^{ts}(i_1,\ldots,i_n)=J_\Gamma^{tu}(i_1,\ldots,i_n)+J_\Gamma^{us}(i_1,\ldots,i_n)+\sum_{n_1+n_2=n}J_\Gamma^{tu}(i_1,\ldots,i_{n_1})J_\Gamma^{us}(i_{n_1+1},\ldots,i_n); \tag{12}$$

(ii) (shuffle property)

$$J_\Gamma^{ts}(i_1,\ldots,i_{n_1})J_\Gamma^{ts}(j_1,\ldots,j_{n_2})=\sum_{\boldsymbol{k}\in Sh(\boldsymbol{i},\boldsymbol{j})}J_\Gamma^{ts}(k_1,\ldots,k_{n_1+n_2}), \tag{13}$$

where $Sh(\boldsymbol{i},\boldsymbol{j})$ – the set of shuffles of the words $\boldsymbol{i}$ and $\boldsymbol{j}$ – has been defined in subsection 1.1.

Axioms (i) and (ii) may be rewritten in a Hopf algebraic language: indexing the $J_\Gamma^{ts}(i_1,\ldots,i_n)$ by trunk trees $\mathcal{T}$ with decoration $\ell(j)=i_j$, $j=1,\ldots,n$, they are equivalent to

(i)bis

$$J_\Gamma^{ts}(\mathcal{T})=\sum_{\boldsymbol{v}\models V(\mathcal{T})}J_\Gamma^{tu}(\mathrm{Roo}_{\boldsymbol{v}}(\mathcal{T}))J_\Gamma^{us}(\mathrm{Lea}_{\boldsymbol{v}}(\mathcal{T})),\quad \mathcal{T}\in\mathbf{Sh}^d; \tag{14}$$

in other words, $J_\Gamma^{ts}=J_\Gamma^{tu}*J_\Gamma^{us}$ for the shuffle convolution defined in subsection 4.2;

(ii)bis

$$J_\Gamma^{ts}(\mathcal{T})J_\Gamma^{ts}(\mathcal{T}')=J_\Gamma^{ts}(\mathcal{T}\sqcup\!\sqcup\mathcal{T}'),\quad \mathcal{T},\mathcal{T}'\in\mathbf{Sh}^d. \tag{15}$$

In other words, J_Γ^{ts} is a character of $\mathbf{Sh}^d$.

Such a functional indexed by *trunk trees* extends easily as in subsection 4.2 to a general *tree-indexed formal rough path* by setting $\bar{J}_\Gamma^{ts}(\mathbb{T}):=J_\Gamma^{ts}\circ\theta^d(\mathbb{T})$, where $\theta^d:\mathbf{H}^d\to\mathbf{Sh}^d$ is the canonical projection morphism defined in subsection 4.1. Since θ^d is a Hopf algebra morphism, one gets immediately the generalized properties

(i)ter $\bar{J}_\Gamma^{ts}=\bar{J}_\Gamma^{tu}*\bar{J}_\Gamma^{us}$ for the convolution of $\mathbf{H}^d$;

(ii)ter $\bar{J}_\Gamma^{ts}(\mathcal{T})\bar{J}_\Gamma^{ts}(\mathcal{T}')=\bar{J}_\Gamma^{ts}(\mathcal{T}.\mathcal{T}')$, in other words, $\bar{J}_\Gamma^{ts}$ is a character of $\mathbf{H}^d$.

Properties (i), (ii) and their generalizations are satisfied for the usual integration operators I_Γ^{ts} and their tree extension $\bar{I}_\Gamma^{ts}$, provided Γ is a smooth path so that iterated integrals make sense [15, 18].

Suppose now one wishes to construct a formal rough path over Γ. Assume one has constructed characters of $\mathbf{Sh}^d$, $J_\Gamma^{ts_0}$, $t\in[0,T]$ with s_0 fixed, such that $J_\Gamma^{ts_0}(i)=\Gamma_i(t)-\Gamma_i(s_0)$, then one immediately checks that $J_\Gamma^{ts}:=J_\Gamma^{ts_0}*(J_\Gamma^{ss_0}\circ\bar{S})$ satisfies properties (i)bis and (ii)bis.

Surely enough, examples of such characters are easy to obtain. The obvious first example is $J^{ts}=I^{ts}$ (the canonical iterated integrals). Considering the specific case of rough paths of order 2, a straightforward comuptation shows that

$$J_\Gamma^{ts}(i_1i_2):=I_\Gamma^{ts}+(f_{i_1,i_2}(t)-f_{i_1,i_2}(s)) \tag{16}$$

where $f_{i_1,i_2}=-f_{i_2,i_1}$ are arbitrary (smooth) functions, is also a character of the shuffle algebra. It is more difficult to figure out what are *all* the possibilities for J_Γ^{ts} with the restriction $J_\Gamma^{ts}(i)=\Gamma_i(t)-\Gamma_i(s)$. Thinking in advance about the case of *irregular paths* (see next section), it is even more difficult a priori to guess whether this or that character is regular, especially when I_Γ^{ts} itself is not and eq. (16) does not make sense.

In a certain sense, Fourier normal ordering solves simultaneously the above combinatorial and analytic problems. The Fourier transform is an essential tool. The Fourier transformed path writes $\mathcal{F}\Gamma:\mathbb{R}\to\mathbb{C}^d$, $\xi\mapsto\frac{1}{\sqrt{2\pi}}\int_0^T e^{-\mathrm{i}t\xi}\Gamma(t)dt$, or $\mathcal{F}\Gamma':\xi\mapsto\frac{1}{\sqrt{2\pi}}\int_0^T e^{-\mathrm{i}t\xi}d\Gamma(t)$.

5.2 Skeleton integrals

Definition 14 *(i) (skeleton integral) Let*

$$\mathrm{SkI}^t_\Gamma(a_1\ldots a_n) := (2\pi)^{-n/2}\int_{\mathbb{R}^n}\prod_{j=1}^n \mathcal{F}\Gamma'_{a_j}(\xi_j)d\xi_j \cdot \int^t dx_1 \int^{x_1} dx_2 \ldots \int^{x_{n-1}} dx_n e^{\mathrm{i}(x_1\xi_1+\ldots+x_n\xi_n)}, \tag{17}$$

where, by definition, $\int^x e^{\mathrm{i}y\xi}dy = \frac{e^{\mathrm{i}x\xi}}{\mathrm{i}\xi}$. *It may be checked that* SkI^t_Γ *is a character of* $\mathbf{Sh}^d$*, just as for usual iterated integrals.*

The projection θ^d *yields immediately a generalization of this notion to tree skeleton integrals, compare with eq. (10), if one sees* $\overline{Skl}^t_\Gamma$ *defined on heap-ordered forests:*

$$\overline{\mathrm{SkI}}^t_\Gamma(\mathbb{T}) = \mathrm{SkI}^t_\Gamma \circ \pi^d_\Sigma \circ \Theta^d(\mathbb{T}) = \int^t d\Gamma_{\ell(1)}(x_1)\int_s^{x_2-} d\Gamma_{\ell(2)}(x_2)\ldots\int^{x_n-} d\Gamma_{\ell(n)}(x_n), \quad \mathbb{T}\in\mathbf{H}^d_{ho} \tag{18}$$

An explicit computation yields ([28], Lemma 4.5):

$$\overline{\mathrm{SkI}}^t_\Gamma(\mathbb{T}) = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\prod_{j=1}^n \mathcal{F}\Gamma'_{\ell(j)}(\xi_j)d\xi_j \cdot \frac{e^{\mathrm{i}t(\xi_1+\ldots+\xi_n)}}{\prod_{i=1}^n[\xi_i+\sum_{j\twoheadrightarrow i}\xi_j]}. \tag{19}$$

As for usual iterated integrals, all this extends to non-tensor measures. Thus one may define $\overline{\mathrm{SkI}}^t_\mu(\mathbb{F})$ *if* μ *is a signed measure on* $\mathbb{R}^n$*, and* $\mathbb{F}$ *a heap-ordered forest with* n *vertices.*

(ii) (measure-splitting) Let μ *be some signed measure with compact support, typically,* $\mu = \mu_{(\Gamma,\ell)}(dx_1,\ldots,dx_n) = \otimes_{j=1}^n d\Gamma_{\ell(j)}(x_j)$*. Then*

$$\mu = \sum_{\sigma\in\Sigma_n}\mathcal{P}^\sigma\mu = \sum_{\sigma\in\Sigma_n}\mu^\sigma\circ\sigma^{-1}, \tag{20}$$

where

$$\mathcal{P}^\sigma : \mu \mapsto \mathcal{F}^{-1}\left(\mathbf{1}_{|\xi_{\sigma(1)}|\le\ldots\le|\xi_{\sigma(n)}|}\mathcal{F}\mu(\xi_1,\ldots,\xi_n)\right) \tag{21}$$

is a Fourier projection, and μ^σ *is defined by*

$$\mu^\sigma := \mathcal{P}^{\mathrm{Id}}(\mu\circ\sigma) = (\mathcal{P}^\sigma\mu)\circ\sigma. \tag{22}$$

In particular, the following obvious formulas hold:

$$(\mu\circ\epsilon)^\sigma = \mathcal{P}^{\mathrm{Id}}\left((\mu\circ\epsilon)\circ\sigma\right) = \mu^{\epsilon\circ\sigma}, \quad \epsilon,\sigma\in\Sigma_n; \tag{23}$$

$$\begin{aligned}\mu_1^{\sigma_1}\otimes\mu_2^{\sigma_2} &= \sum_{\epsilon\ \text{shuffle}}\mathcal{P}^\epsilon\left((\mu_1\otimes\mu_2)\circ(\sigma_1\otimes\sigma_2)\right)\\ &= \sum_{\epsilon\ \text{shuffle}}\left((\mu_1\otimes\mu_2)\circ(\sigma_1\otimes\sigma_2)\right)^\epsilon\circ\epsilon^{-1}\\ &= \sum_{\epsilon\ \text{shuffle}}(\mu_1\otimes\mu_2)^{(\sigma_1\otimes\sigma_2)\circ\epsilon}\circ\epsilon^{-1}.\end{aligned} \tag{24}$$

The set of all measures whose Fourier transform is supported in $\{(\xi_1,\ldots,\xi_n);|\xi_1|\le\ldots\le|\xi_n|\}$ *will be denoted by* $\mathcal{P}^{\mathrm{Id}}Meas(\mathbb{R}^n)$*. Thus* $\mu^\sigma\in\mathcal{P}^{\mathrm{Id}}Meas(\mathbb{R}^n)$*.*

To say things shortly, *skeleton integrals* are convenient when using Fourier coordinates, since they avoid awkward boundary terms such as those generated by usual integrals, $\int_0^x e^{\mathrm{i}y\xi}dy = \frac{e^{\mathrm{i}x\xi}}{\mathrm{i}\xi} - \frac{1}{\mathrm{i}\xi}$, which create terms with different homogeneity degree in ξ by iterated integrations. *Measure splitting* gives the *relative scales* of the Fourier coordinates; orders of magnitude of the corresponding integrals may be obtained separately in each sector $|\xi_{\sigma(1)}|\le\ldots\le|\xi_{\sigma(n)}|$. It turns out that these are easiest to get after a permutation of the integrations (applying Fubini's theorem) such that *innermost (or rightmost) integrals bear highest Fourier frequencies.* This is the essence of *Fourier normal ordering.*

5.3 Main result

Now comes the connection to the preceding sections. Let, for $\mathbb{T} \in \mathcal{F}_{ho}(n)$,

$$\mathcal{P}^{\mathbb{T}} Meas(\mathbb{R}^n) = \{\mu; \boldsymbol{\xi} \in \mathrm{supp}(\mathcal{F}\mu) \Rightarrow ((i \twoheadrightarrow j) \Rightarrow (|\xi_i| > |\xi_j|))\}. \tag{25}$$

This generalizes the spaces $\mathcal{P}^{\mathrm{Id}} Meas(\mathbb{R}^n)$ defined above for trunk trees. Assume one finds a way to define regularized tree-index skeleton integrals $\phi^t_\mu(\mathbb{T})$ for every $\mu \in \mathcal{P}^{\mathbb{T}} Meas(\mathbb{R}^n)$, such that

(i) $\phi^t_{\mu_{(\Gamma,\ell)}}(\bullet) = \mathrm{SkI}^t_{\mu_{(\Gamma,\ell)}}(\bullet) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}\Gamma_{\ell(1)}(\xi) e^{\mathrm{i}t\xi} d\xi$ gives back the original path Γ for the tree with one single vertex, and

(ii) if $\mathbb{T} \in \mathbf{H}_{ho}(n), \mathbb{T}' \in \mathbf{H}_{ho}(n')$, and $\mu \in \mathcal{P}^{\mathbb{T}} Meas(\mathbb{R}^n)$, resp. $\mu' \in \mathcal{P}^{\mathbb{T}'} Meas(\mathbb{R}^{n'})$, then

$$\phi^t_\mu(\mathbb{T}) \phi^t_{\mu'}(\mathbb{T}') = \phi^t_{\mu \otimes \mu'}(\mathbb{T}.\mathbb{T}'), \tag{26}$$

which is an extension of the multiplicative property defining a character of $\mathbf{H}^d_{ho}$.

Then one is tempted to think that, interpreting $\phi^t_{\mu^\sigma_{(\Gamma,\ell)}}(\mathbb{T})$ as coming from a rough path $J^t_{\mu_{(\Gamma,\ell)}}$ over Γ by the above defined measure-splitting procedure, the following sequence of postulated equalities, where $\mathbb{T}_n \in \mathbf{H}_{ho}$ is the trunk tree with n vertices,

$$\begin{aligned} J^t_{\mu_{(\Gamma,\ell)}}(\mathbb{T}_n) &= \sum_{\sigma \in \Sigma_n} J^t_{\mu^\sigma_{(\Gamma,\ell)} \circ \sigma}(\mathbb{T}_n) \quad \text{by linearity} \\ &= \sum_{\sigma \in \Sigma_n} J^t_{\mu^\sigma_{(\Gamma,\ell)}}(\mathbb{T}^\sigma) \quad \text{by the results of subsection 4.2} \\ &= \sum_{\sigma \in \Sigma_n} \phi^t_{\mu^\sigma_{(\Gamma,\ell)}}(\mathbb{T}^\sigma) \end{aligned} \tag{27}$$

define a character of $\mathbf{Sh}^d$, and thus allow (leaving aside the regularity properties which must be checked independently) to define a rough path over Γ. This is the content of the following Lemma, which is actually, as we shall see, equivalent to stating that Θ is a Hopf algebra morphism:

Definition 15 *(i) Let $\phi^t_{\mathbb{T}} : \mathcal{P}^{\mathbb{T}} Meas(\mathbb{R}^n) \to \mathbb{R}, \mu \mapsto \phi^t_{\mathbb{T}}(\mu)$, also written $\phi^t_\mu(\mathbb{T})$ ($t \in \mathbb{R}, \mathbb{T} \in \mathbf{H}_{ho}(n)$) be a family of linear forms such that*

(a) $\phi^t_\bullet(\mu_{(\Gamma,i)}) - \phi^s_\bullet(\mu_{(\Gamma,i)}) = \Gamma_t(i) - \Gamma_s(i)$;

*(b) if $(\mathbb{T}_i, \mu_i) \in \mathbf{H}_{ho}(n_i) \times \mathcal{P}^{\mathbb{T}_i} Meas(\mathbb{R}^{n_i})$, $i = 1, 2$, the following $\mathbf{H}_{ho}$-*multiplicative property *holds,*

$$\phi^t_{\mu_1}(\mathbb{T}_1) \phi^t_{\mu_2}(\mathbb{T}_2) = \phi^t_{\mu_1 \otimes \mu_2}(\mathbb{T}_1.\mathbb{T}_2); \tag{28}$$

(c) ϕ^t is invariant under forest-ordering preserving symmetries, that is to say:

$$\phi^t_\mu(\mathbb{F}) = \phi^t_{\mu \circ \sigma}(\sigma^{-1}.\mathbb{F}) \text{ if } \sigma \in S_{\mathbb{F}}. \tag{29}$$

(ii) Let, for $\Gamma = (\Gamma(1), \ldots, \Gamma(d))$, $\chi^t_\Gamma : \mathbf{Sh}^d \to \mathbb{R}$ be the linear form on $\mathbf{Sh}^d$ defined by

$$\chi^t_\Gamma(\mathbb{T}_n, \ell) := \sum_{\sigma \in \Sigma_n} \phi^t_{\mu^\sigma_{(\Gamma,\ell)}}(\mathbb{T}^\sigma). \tag{30}$$

Remark. The family $\Phi^t_{\mathbb{T}}$ does not define a character of $\mathbf{H}_{ho}$. However, consider the vector space:

$$\mathcal{M}eas = \bigoplus_{\mathbb{T} \text{ heap-ordered forest}} \mathcal{P}^{\mathbb{T}} Meas(\mathbb{R}^{|\mathbb{T}|}).$$

It is given an associative product in the following way: if $\mu_1 \in \mathcal{P}^{\mathbb{T}_1} Meas(\mathbb{R}^{|\mathbb{T}_1|})$ and $\mu_2 \in \mathcal{P}^{\mathbb{T}_2} Meas(\mathbb{R}^{|\mathbb{T}_2|})$, then $\mu_1.\mu_2 = \mu_1 \otimes \mu_2 \in \mathcal{P}^{\mathbb{T}_1\mathbb{T}_2} Meas(\mathbb{R}^{|\mathbb{T}_1\mathbb{T}_2|})$. Then $\mathcal{M}eas$ is an associative algebra, graded by the monoid of heap-ordered forests. Then $\Phi^t_{\mathbb{T}}$ now defines a map from $\mathcal{M}eas$ to $\mathbb{R}$, and (28) means that this map is a character.

Moreover, this algebra $\mathcal{M}eas$ is also graded by the number of vertices of the forest $\mathbb{T}$. The action of the symmetric group Σ_n on $\mathbb{R}^n$ by permutation of the coordinates induces an action of Σ_n on the homogeneous component of degree n of $\mathcal{M}eas$: if $\mu \in \mathcal{P}^{\mathbb{T}} Meas(\mathbb{R}^n)$ and $\sigma \in \Sigma_n$, then $\mu^\sigma = \mu \circ \sigma \in \mathcal{P}^{\sigma^{-1}.\mathbb{T}} Meas(\mathbb{R}^n)$. Then (29) means that the map defined on $\mathcal{M}eas$ by the family $\Phi^t_{\mathbb{T}}$ is invariant under this action.

Lemma 16 *1. For every path Γ such that χ^t_Γ is well-defined, χ^t_Γ is a character of $\mathbf{Sh}^d$.*

Consequently, the following formula for $\mathbb{T}_n \in \mathbf{Sh}^d$, $n \geq 1$, with n vertices and decoration ℓ,

$$(J')^{ts}_\Gamma(\ell(1)\ldots\ell(n)) := \chi^t_\Gamma * (\chi^s_\Gamma \circ S)(\mathbb{T}_n) \tag{31}$$

defines a rough path over Γ.

2. $(J')^{ts}_\Gamma(\ell(1)\ldots\ell(n)) = J^{ts}_\Gamma(\ell(1)\ldots\ell(n))$ where

$$J^{ts}_\Gamma(\mathbb{T}_n) := \sum_{\sigma\in\Sigma_n} \big(\phi^t * (\phi^s \circ \bar{S})\big)_{\mu^\sigma_{(\Gamma,\ell)}}(\mathbb{T}^\sigma). \tag{32}$$

The convolution $\phi^t * (\phi^s \circ \bar{S})$ in the last formula is defined by reference to the $\mathbf{H}_{ho}$-coproduct, namely, one sets

$$(\phi^t * (\phi^s \circ \bar{S}))_\nu(\mathbb{T}) = \sum_{\boldsymbol{v}\models V(\mathbb{T})} \phi^t_{\otimes_{v\in V(\mathrm{Roo}_{\boldsymbol{v}}\mathbb{T})}\nu_v}(\mathrm{Roo}_{\boldsymbol{v}}\mathbb{T})\phi^s_{\otimes_{v\in V(\mathrm{Lea}_{\boldsymbol{v}}\mathbb{T})}\nu_v}(\bar{S}(\mathrm{Lea}_{\boldsymbol{v}}\mathbb{T})) \tag{33}$$

for a tensor measure $\nu = \nu_1 \otimes \ldots \otimes \nu_n$, and by multilinear extension

$$\big(\phi^t * (\phi^s \circ \bar{S})\big)_\nu(\mathbb{T}) = (2\pi)^{-n/2} \int \mathcal{F}\nu(\xi_1,\ldots,\xi_n)d\xi_1\ldots d\xi_n \quad \cdot$$
$$\cdot \sum_{\boldsymbol{v}\models V(\mathbb{T})} \phi^t_{\otimes_{v\in V(Roo_{\boldsymbol{v}}(\mathbb{T}))}e^{\mathrm{i}x_v\xi_v}dx_v}(\mathrm{Roo}_{\boldsymbol{v}}\mathbb{T})\phi^s_{\otimes_{v\in V(Lea_{\boldsymbol{v}}(\mathbb{T}))}e^{\mathrm{i}x_v\xi_v}dx_v}(\bar{S}(\mathrm{Lea}_{\boldsymbol{v}}\mathbb{T})), \quad \mathbb{T}\in\mathcal{F}_{ho}(n). \tag{34}$$

for an arbitrary measure $\nu \in Meas(\mathbb{R}^n)$.

Proof.

1. Let $\mathbb{T}_{n_i} \in \mathbf{H}_{ho}$ be the heap-ordered trunk trees with n_i vertices, $i = 1, 2$; define $n := n_1+n_2$. All right-, resp. left shuffles ϵ, ζ below are intended to be shuffles of $(1,\ldots,n_1),(n_1+1,\ldots,n_2)$. Then, with $\ell = \ell_1 \otimes \ell_2$ and writing μ instead of $\mu_{(\Gamma,\ell)}$:

$$\begin{aligned}
\chi^t_\Gamma((\mathbb{T}_{n_1},\ell_1)⧢(\mathbb{T}_{n_2},\ell_2)) &= \sum_\zeta \chi^t_\Gamma((\mathbb{T}_n,\ell\circ\zeta)) \text{ by (30)} \\
&= \sum_{\zeta,\sigma} \phi^t_{\mu^\sigma_{(\Gamma,\ell\circ\zeta)}}(\mathbb{T}^\sigma) \\
&= \sum_{\zeta,\sigma} \phi^t_{(\mu\circ\zeta)^\sigma}(\mathbb{T}^\sigma) \\
&= \sum_{\zeta,\sigma} \phi^t_{\mu^{\zeta\circ\sigma}}(\mathbb{T}^\sigma) \text{ by (23)} \\
&= \sum_{\zeta,\sigma} \phi^t_{\mu^\sigma}(\mathbb{T}^{\zeta^{-1}\circ\sigma}).
\end{aligned} \tag{35}$$

On the other hand, denoting $\mu_i = \mu_{(\Gamma,\ell_i)}$:

$$\begin{aligned}
&\chi^t_\Gamma((\mathbb{T}_{n_1},\ell_1))\chi^t_\Gamma((\mathbb{T}_{n_2},\ell_2)) \\
&= \sum_{\sigma_1,\sigma_2} \phi^t_{\mu_1^{\sigma_1}}(\mathbb{T}^{\sigma_1})\phi^t_{\mu_2^{\sigma_2}}(\mathbb{T}^{\sigma_2}) \\
&= \sum_{\sigma_1,\sigma_2,\epsilon} \phi^t_{\mu^{(\sigma_1\otimes\sigma_2)\circ\epsilon}\circ\epsilon^{-1}}(\mathbb{T}^{\sigma_1}.\mathbb{T}^{\sigma_2}) \text{ by (24)} \\
&= \sum_{\sigma_1,\sigma_2,\epsilon} \phi^t_{\mu^{(\sigma_1\otimes\sigma_2)\circ\epsilon}}(\epsilon^{-1}(\mathbb{T}^{\sigma_1}.\mathbb{T}^{\sigma_2})) \\
&= \sum_{\sigma_1,\sigma_2,\epsilon,\zeta} \phi^t_{\mu^{(\sigma_1\otimes\sigma_2)\circ\epsilon}}\left(\mathbb{T}^{\zeta^{-1}\circ(\sigma_1\otimes\sigma_2)\circ\epsilon}\right) \text{ by lemma 10} \\
&= \sum_{\zeta,\sigma} \phi^t_{\mu^\sigma}(\mathbb{T}^{\zeta^{-1}\circ\sigma}).
\end{aligned} \tag{36}$$

2. Let us check that, for a tensor measure $\mu \in Meas(\mathbb{R}^n)$,

$$\sum_{\sigma\in\Sigma_n} (\phi^t * (\phi^s \circ \bar{S}))_{\mu^\sigma}(\mathbb{T}^\sigma) = (\chi^t * (\chi^s \circ \bar{S}))_\mu(\mathcal{T}_n).$$

Assume $\mathcal{F}\mu(\xi_1,\ldots,\xi_n) = \delta(\xi-\xi^0)$ is a Dirac distribution, where $|\xi^0{}_{\sigma_0(1)}| < \ldots < |\xi^0{}_{\sigma_0(n)}|$ for some $\sigma_0 \in \Sigma_n$. By construction, $\mu^\sigma = 0$ unless $\sigma = \sigma_0$, so

$$\sum_{\sigma\in\Sigma_n} (\phi^t * (\phi^s \circ \bar{S}))_{\mu^\sigma}(\mathbb{T}^\sigma) = (\phi^t * (\phi^s \circ \bar{S}))_{\mu^{\sigma_0}}(\mathbb{T}^{\sigma_0}). \tag{37}$$

We now apply the coproduct formula (9) to $\mathcal{T}^{\sigma_0}$, with $\mathcal{T} = (\mathbb{T}_n, \ell)$, such that $\ell(i) = i$ for all i. We obtain:

$$\Delta(\mathcal{T}^{\sigma_0}) = \sum_{k=0}^n \sum_{\sigma_0=(\sigma_1\otimes\sigma_2)\circ\epsilon} \epsilon^{-1}.((\mathbb{T}^{\sigma_1},\ell_1)\otimes(\mathbb{T}^{\sigma_2},\ell_2)). \tag{38}$$

For fixed $k \in \{0,\ldots,n\}$, let us write $\mu = \mu_1 \otimes \mu_2$, with $\mu_1 \in Meas(\mathbb{R}^k)$ and $\mu_2 \in Meas(\mathbb{R}^{n-k})$. The shuffle ϵ may be made to act on μ^{σ_0} instead of $(\mathbb{T}^{\sigma_1},\ell_1)\otimes(\mathbb{T}^{\sigma_2},\ell_2)$ because of the invariance condition, resulting in the product measure $\mu^{\sigma_0}\circ\epsilon^{-1} = \mu_1^{\sigma_1}\otimes\mu_2^{\sigma_2}$, see eq. (24), so the convolution in (34) writes simply

$$\begin{aligned}
&\sum_{k=0}^n \sum_{\sigma_0=(\sigma_1\otimes\sigma_2)\circ\epsilon} \phi^t_{\mu_1^{\sigma_1}}(\mathbb{T}^{\sigma_1})(\phi^s_{\mu_2^{\sigma_2}}\circ\bar{S})(\mathbb{T}^{\sigma_2}) \\
&= \sum_{k=0}^n \sum_{\sigma_1,\sigma_2} \phi^t_{\mu_1^{\sigma_1}}(\mathbb{T}^{\sigma_1})(\phi^s_{\mu_2^{\sigma_2}}\circ\bar{S})(\mathbb{T}^{\sigma_2}) \\
&= (\chi^t * (\chi^s \circ \bar{S}))_\mu(\mathbb{T}_n),
\end{aligned} \tag{39}$$

where $\mathbb{T}_n$ is the trunk tree with n vertices.

□

Conversely, assume one has some path-dependent shuffle character $\Gamma \rightsquigarrow \chi^t_\Gamma(\mathcal{T})$ for every Γ, which (when extended to a measure-indexed character) is linear; $\chi^t_\Gamma * (\chi^s_\Gamma \circ S)$ is then a rough path over Γ. One may define $\phi^t(\mathbb{T}^\sigma)$, $\sigma \in \Sigma_n$ from eq. (30), and then $\phi^t(\mathbb{T})$ for an arbitrary tree since permutation graphs generate $\mathbf{H}_{ho}$. Eq. (29) is then trivially satisfied, and we claim that eq. (28) also holds. Namely, circulating through eq. (35) and (36), and extending χ^t to a projected measure $\mu = (\mu_1^{\sigma_1} \circ \sigma_1^{-1}) \otimes (\mu_2^{\sigma_2} \circ \sigma_2^{-1})$, one gets

$$\begin{aligned}
&\phi^t_{\mu_1^{\sigma_1}}(\mathbb{T}^{\sigma_1})\phi^t_{\mu_2^{\sigma_2}}(\mathbb{T}^{\sigma_2}) = \chi^t_{\mu_1^{\sigma_1}}((\mathbb{T}_{n_1},\ell_1))\chi_{\mu_2^{\sigma_2}}((\mathbb{T}_{n_2},\ell_2)) = \chi^t_{\mu_1^{\sigma_1}\otimes\mu_2^{\sigma_2}}((\mathbb{T}_{n_1},\ell_1)\boxplus(\mathbb{T}_{n_2},\ell_2)) \\
&= \sum_{\zeta,\sigma} \phi^t_{\mu^\sigma}(\mathbb{T}^{\zeta^{-1}\circ\sigma}) = \sum_{\sigma_1,\sigma_2,\epsilon} \phi^t_{\mu(\sigma_1\otimes\sigma_2)\circ\epsilon}(\epsilon^{-1}(\mathbb{T}^{\sigma_1}.\mathbb{T}^{\sigma_2})) \\
&= \phi^t_{\mu_1^{\sigma_1}\otimes\mu_2^{\sigma_2}}(\mathbb{T}^{\sigma_1}.\mathbb{T}^{\sigma_2}).
\end{aligned} \tag{40}$$

Hence all axioms of Definition 15 hold.

In this sense the Fourier normal ordering algorithm for constructing rough paths yields *all* possible formal rough paths.

6 Analytic epilogue

Assume that Γ is not differentiable, but only α-Hölder for some $0 < \alpha < 1$, i.e. bounded in the $\mathcal{C}^\alpha$-norm,

$$||\gamma||_{\mathcal{C}^\alpha} := \sup_{t\in[0,T]} ||\Gamma(t)|| + \sup_{s,t\in[0,T]} \frac{||\Gamma(t)-\Gamma(s)||}{|t-s|^\alpha}. \tag{41}$$

We let $N = \lfloor 1/\alpha \rfloor$ be the integer part of $1/\alpha$.

Definition 17 (α-Hölder rough path) *Let Γ be an α-Hölder path. Then* $(J^{ts}_\Gamma(i_1,\ldots,i_n))_{1\le n\le N, 1\le i_1,\ldots,i_n\le d}$ *is an α-Hölder rough path above Γ (equivalently, an α-Hölder* lift *of Γ) if J^{ts}_Γ is a formal rough path of order N above Γ, i.e. satisfies properties (i) and (ii) of definition 13, and if J^{ts}_Γ satisfies furthermore the following Hölder property,*
(iii) (Hölder continuity) $J^{ts}_\Gamma(i_1,\ldots,i_n)$ is $n\alpha$-Hölder continuous as a function of two variables, namely, $\sup_{s,t\in\mathbb{R}} \frac{|J^{ts}_\Gamma(i_1,\ldots,i_n)|}{|t-s|^\alpha} < \infty$.

As explained in the Introduction, rough path theory may be seen as a black box taking as input some lift of Γ called *rough path over* Γ, and producing e.g. solutions of differential equations driven by Γ. This means that the very meaning of a differential equation driven by Γ depends on the choice of the lift. One knows that essential properties of solutions (such as the possibility to define global solutions for instance, or to show that the solution has a density as a random variable when Γ is a random process) of such differential equations depend crucially on this choice. In this respect we claim that Fourier normal ordering is an essential tool, in that natural choices of tree data generate rough paths (not simply *formal* ones) with significantly better analytic properties than arbitrary rough paths. The last claim is really work in progress, so we shall content ourselves with presenting three natural choices of tree data generating rough paths. The proof of Hölder bounds rely in general on Besov (wavelet-type) estimates, which require smoothening up characteristic functions used in the measure-splitting lemma such as $\mathbf{1}_{|\xi_1|\le\cdots|\xi_n|}$, and cutting the integral over $\xi_1,\ldots,\xi_n$ into an infinite sum of integrals over dyadic domains where $\log|\xi_j|$ is approximately constant. This does not change at all the algebraic construction but only makes formulas uglier, so we shall skip this detail (see [28]). For paths with a regularly varying Fourier transform this is not required.

Example 1 (zero tree data). Choose $\phi^t_\mu(\mathbf{T}) \equiv 0$ for any non-trivial tree T with at least two vertices. Then $\left(J^{ts}_\Gamma(\mathbf{T}), |V(\mathbf{T})| = 1,\ldots,N\right)$ is a rough path over Γ for any α-Hölder path Γ. Very explicit formulas may be given in this case. Formula (31) from Lemma 16 entails (with obvious notations)

$$J^{ts}_\Gamma(\mathbf{T}_n) = \sum_{k=0}^n (-1)^{n-k} \sum_{\sigma_1,\sigma_2} \phi^t_{\mu_1^{\sigma_1}}(\mathbf{T}_k^{\sigma_1})\phi^s_{\mu_2^{\sigma_2}}(\mathbf{T}_{n-k}^{\sigma_2}) \tag{42}$$

where $\mu_1 := \otimes_{i=1}^{k} d\Gamma_{x_i}(\ell(i))$, $\mu_2 := \otimes_{i=1}^{n-k} d\Gamma_{x_i}(\ell(n-i+1))$, and $\sigma_1 : \{1,\ldots,k\} \to \{1,\ldots,k\}$, $\sigma_2 : \{k+1,\ldots,n\} \to \{k+1,\ldots,n\}$ are permutations. Now, the only permutation graph $\mathbf{T}^{\sigma}$ involving products of trivial trees (i.e. with only one vertex) is $\mathbf{T}_m^{\sigma^0}$, $\sigma^0 = \begin{pmatrix} 1 & \cdots & m \\ m & \cdots & 1 \end{pmatrix}$, which contains the product $\bullet_1 \cdots \bullet_m$. One obtains thus explicit formulas in which branching iterated integrals have disappeared:

$$J_\Gamma^{ts}(\mathbf{T}_n) = \sum_{k=0}^{n} (-1)^{n-k} \int \cdots \int_{|\xi_k|<\ldots<|\xi_1|} \prod_{j=1}^{k} e^{\mathrm{i}t\xi_j} \mathcal{F}\Gamma_{\ell(j)}(\xi_j) d\xi_j$$
$$\int \cdots \int_{|\xi_{k+1}|<\ldots|\xi_n|} \prod_{j=k+1}^{n} e^{\mathrm{i}s\xi_j} \mathcal{F}\Gamma_{\ell(j)}(\xi_j) d\xi_j. \tag{43}$$

If one splits again the domain of integration into $\{|\xi_1| > \ldots > |\xi_k|, |\xi_k| < \ldots < |\xi_n|\} \amalg \{|\xi_1| > \ldots > |\xi_{k+1}|, |\xi_{k+1}| < \ldots < |\xi_n|\}$, one finds finally

$$J_\Gamma^{ts}(\mathbf{T}_n) = \sum_{k=0}^{n} (-1)^{n-k} \int \cdots \int_{|\xi_1|>\ldots>|\xi_k|,|\xi_k|<\ldots|\xi_n|}$$
$$\left[\prod_{j=1}^{k-1} e^{\mathrm{i}t\xi_j} \mathcal{F}\Gamma_{\ell(j)}(\xi_j) d\xi_j \right]$$
$$(e^{\mathrm{i}t\xi_k} - e^{\mathrm{i}s\xi_k}) \mathcal{F}\Gamma_{\ell(k)}(\xi_k) d\xi_k \left[\prod_{j=k+1}^{n} e^{\mathrm{i}s\xi_j} \mathcal{F}\Gamma_{\ell(j)}(\xi_j) d\xi_j \right], \tag{44}$$

valid in this form for paths with a regularly varying Fourier transform, e.g. for fractional Brownian motion, by replacing $\mathcal{F}\Gamma_{\ell(j)}(\xi_j)d\xi_j$ with $c_\alpha \frac{|\xi_j|^{1/2-\alpha}}{i\xi_j} dW_{\xi_j}(\ell(j))$ for some constant c_α (see [29] for the Fourier transform of fBm and [31] for this formula). A very similar formula (using a Volterra kernel representation instead of a Fourier transform) has been obtained in [23] and proved without appealing to Fourier normal ordering.

Example 2 (domain regularization). A less trivial choice consists in *regularizing* tree data. There are of course many possibilities. One of them (called *domain regularization*) has been presented in the original papers [28, 29]. It consists in restricting the range of the Fourier variables $\xi_1, \ldots, \xi_n$ ($n \geq 2$) in tree skeleton integrals where from the start $|\xi_1| < \ldots < |\xi_n|$ to a domain such as

$$\mathbb{R}^T_{reg} := \{(\xi_1,\ldots,\xi_n) \in \mathbb{R}^n \mid |\xi_1| < \ldots < |\xi_n| \text{ and } \forall i, |\xi_i + \sum_{j \twoheadrightarrow i} \xi_j| > C_{reg} \sup_{j \twoheadrightarrow} |\xi_j|\}, \tag{45}$$

so as to avoid the small denominator problem in eq. (17) which is responsible for divergences. When $C_{reg} > N$, $\mathbb{R}^T_{reg}$ is empty, and one is back to Example 1. Otherwise the formulas and the proof of Hölder property use the full strength of the Fourier normal ordering algorithm.

Example 3 (renormalized rough paths). In [30] – a paper written specifically for the case of fBm, but with no doubt generalizable to arbitrary Hölder paths – tree skeleton integrals are shown to be encodable by Feynman diagrams of a special type. Adapting the BPHZ (Bogolioubov et al.) renormalization algorithm [16] to this case (with zero-momentum regularization) yields renormalized, finite skeleton integrals with the appropriate Hölder continuity. Recombining them by the Fourier normal ordering algorithm yields a rough path over fBm. Quite interestingly, as well-known since the seminal papers by Connes and Kreimer [6], the BPHZ algorithm has a nice Hopf algebraic interpretation in terms of 'decorated' trees (Feynman diagrams). So the Connes-Kreimer algebra comes twice into the picture for different reasons: first as a combinatorial tool (for Fourier normal ordering), then as a machinery to remove divergences. We conjecture (work in progress), on the basis of the estimates proved in this construction, that any choice of tree data with the appropriate Hölder continuity which does not increase the Fourier support of the measures yields after recombination a rough path, thus defining a new restricted class of rough paths that would appropriately be called *Fourier normal-ordered rough paths*. Such rough paths are much more amenable to analysis than general rough paths. This would provide a further justification for this article.

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