Plane posets, special posets, and permutations

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ABSTRACT. We study the self-dual Hopf algebra $\mathcal{H}_{\mathcal{SP}}$ of special posets introduced by Malvenuto and Reutenauer and the Hopf algebra morphism from $\mathcal{H}_{\mathcal{SP}}$ to to the Hopf algebra of free quasi-symmetric functions **FQSym** given by linear extensions. In particular, we construct two Hopf subalgebras both isomorphic to **FQSym**; the first one is based on plane posets, the second one on heap-ordered forests. An explicit isomorphism between these two Hopf subalgebras is also defined, with the help of two combinatorial transformations on special posets. The restriction of the Hopf pairing of $\mathcal{H}_{\mathcal{SP}}$ to these Hopf subalgebras and others is also studied, as well as certain isometries between them. These problems are solved using duplicial and dendriform structures.

Keywords. Special posets, permutations, self-dual Hopf algebras, duplicial algebras, dendriform algebras.

AMS CLASSIFICATION. 06A11, 05A05, 16W30, 17A30.

Contents

Introduction

The Hopf algebra of double posets is introduced in [17]. Recall that a *double poset* is a finite set with two partial orders; the set of isoclasses of double posets is given a structure of monoid, with a product called *composition* (definition 4). The algebra of this monoid is given a coassociative coproduct, with the help of the notion of *ideal* of a double poset. We then obtain a graded, connected Hopf algebra, non commutative and non cocommutative. This Hopf algebra $\mathcal{H}_{\mathcal{DP}}$ is self-dual: it has a nondegenerate Hopf pairing $\langle -, - \rangle$, such that the pairing of two double posets is given by the number of *pictures* between these double posets (definition 6); see [7] for more details on the nondegeneracy of this pairing.

Other algebraic structures are constructed on $\mathcal{H}_{\mathcal{DP}}$ in [7]. In particular, a second product is defined on $\mathcal{H}_{\mathcal{DP}}$, making it a free 2-*As* Hopf algebra [13]. As a consequence, this object is closely related to operads and the theory of combinatorial Hopf algebras [14]. In particular, it contains the free 2-*As* algebra on one generator: this is the Hopf subalgebra $\mathcal{H}_{\mathcal{WNP}}$ of *WN posets*, see definition 3. Another interesting Hopf subalgebra $\mathcal{H}_{\mathcal{PP}}$ is given by *plane posets*, that is to say double poset with a particular condition of (in)compatibility between the two orders (definition 2).

We investigate in the present text the algebraic properties of the family of *special posets*, that is to say double posets such that the second order is total [17]. They generate a Hopf subalgebra of $\mathcal{H}_{\mathcal{DP}}$ denoted by $\mathcal{H}_{\mathcal{SP}}$. For example, as explained in [7], the two partial orders of a plane poset allow to define a third, total order, so plane posets can also be considered as special posets: this defines an injective morphism of Hopf algebras from $\mathcal{H}_{\mathcal{PP}}$ to $\mathcal{H}_{\mathcal{SP}}$. Its image is denoted by $\mathcal{H}_{\mathcal{SPP}}$. Another interesting Hopf subalgebra of $\mathcal{H}_{\mathcal{SP}}$ is generated by the set of *ordered forests*; it is the Hopf algebra $\mathcal{H}_{\mathcal{OF}}$ used in [8, 3]. A special poset is *heap-ordered* if its second order (recall it is total) is a linear extension of the first one; these objects define another Hopf subalgebra $\mathcal{H}_{\mathcal{HOP}}$ of $\mathcal{H}_{\mathcal{SP}}$. Taking the intersections, we finally obtain a commutative diagram of six Hopf algebras:

$$\begin{tikzcd}
\mathcal{H}_{\mathcal{SPP}} \arrow[r, hook] & \mathcal{H}_{\mathcal{HOP}} \arrow[r, hook] & \mathcal{H}_{\mathcal{SP}} \\
\mathcal{H}_{\mathcal{SPF}} \arrow[r, hook] \arrow[u, hook] & \mathcal{H}_{\mathcal{HOF}} \arrow[r, hook] \arrow[u, hook] & \mathcal{H}_{\mathcal{OF}} \arrow[u, hook]
\end{tikzcd}$$

The Hopf algebra $\mathcal{H}_{\mathcal{HOF}}$ of *heap-ordered forests* is used in [3]. Moreover, $\mathcal{H}_{\mathcal{SPF}}$ is generated by the set of *plane forests*, considered as special posets; it is isomorphic to the non commutative Connes-Kreimer Hopf algebra of plane forests $\mathcal{H}_{\mathcal{SPF}}$ [4, 5, 10].

A Hopf algebra morphism Θ, from $\mathcal{H}_{\mathcal{SP}}$ to the Malvenuto-Reutenauer Hopf algebra of permutations **FQSym** [16], also known as the Hopf algebra of free quasi-symmetric functions [2], is defined in [17]. This construction uses the linear extensions of the first order of a special poset. The morphism Θ is surjective and respects the Hopf pairings defined on $\mathcal{H}_{\mathcal{SP}}$ and **FQSym**. Moreover, its restrictions to $\mathcal{H}_{\mathcal{SPP}}$ and $\mathcal{H}_{\mathcal{HOF}}$ are isometric Hopf algebra isomorphisms (corollary 22). In the particular case of $\mathcal{H}_{\mathcal{SPP}}$, this is proved using, first a bijection from the set of special plane posets of order n to the n-th symmetric group $\mathfrak{S}_n$ for all $n \geq 0$, then intervals in $\mathfrak{S}_n$ for the weak Bruhat order, see proposition 21. As a consequence, we obtain a commutative diagram:

$$\begin{tikzcd}
\mathcal{H}_{\mathcal{SPP}} \arrow[dr, hook] \arrow[drr, hook, two heads] & & \\
& \mathcal{H}_{\mathcal{SP}} \arrow[r, two heads, "\Theta"] & \mathbf{FQSym} \\
\mathcal{H}_{\mathcal{HOF}} \arrow[ur, hook] \arrow[urr, hook, two heads] & &
\end{tikzcd}$$

We then complete this diagram with a Hopf algebra morphism $\Upsilon : \mathcal{H}_{\mathcal{SP}} \longrightarrow \mathcal{H}_{\mathcal{HOF}}$, combinatorially defined (theorem 25), such that its restriction to $\mathcal{H}_{\mathcal{SPP}}$ gives the following commutative diagram:

$$\begin{tikzcd}
\mathcal{H}_{\mathcal{SPP}} \arrow[dd, "\Upsilon"'] \arrow[dr, hook] \arrow[drr, hook, two heads] & & \\
& \mathcal{H}_{\mathcal{SP}} \arrow[r, two heads, "\Theta"] & \mathbf{FQSym} \\
\mathcal{H}_{\mathcal{HOF}} \arrow[ur, hook] \arrow[urr, hook, two heads] & &
\end{tikzcd}$$

The definition of Υ uses two transformations of special posets, summarized by $\mathbf{I}^{i}_{j} \longrightarrow \bullet_i \bullet_j - \mathbf{I}^{j}_{i}$ and ${}_i\Lambda^{k}{}_j \longrightarrow \mathbf{I}^{k}_{i} \bullet_j - {}^{j}\mathrm{V}^{k}_{i} + \mathbf{I}^{k}_{j\,i}$.

In order to prove the cofreeness of $\mathcal{H}_{\mathcal{SPF}}$, $\mathcal{H}_{\mathcal{SP}}$, $\mathcal{H}_{\mathcal{HOP}}$, $\mathcal{H}_{\mathcal{SPP}}$, $\mathcal{H}_{\mathcal{OF}}$ and $\mathcal{H}_{\mathcal{SWNP}}$, we introduce a new product $\nwarrow$ on $\mathcal{H}_{\mathcal{SP}}$ making it a *duplicial algebra* [12], and two non associative coproducts $\Delta_{\prec}$ and $\Delta_{\succ}$, making it a *dendriform coalgebra* [11, 15], see paragraph 5.1. These two complementary structures are compatible, and $\mathcal{H}_{\mathcal{SP}}$ is a *Dup-Dend bialgebra* [8]. By the theorem of rigidity for Dup-Dend bialgebras, all these objects are isomorphic to non-commutative Connes-Kreimer Hopf algebras of decorated plane forests [4, 5, 10] (note that this result was obvious for $\mathcal{H}_{\mathcal{SPF}}$), so are free and cofree. Moreover, it is possible to define a Dup-Dend structure on **FQSym** in such a way that the Hopf algebra morphism Θ becomes a morphism of Dup-Dend bialgebras. Dendriform structures are also used to show that the restriction of the pairing of $\mathcal{H}_{\mathcal{DP}}$ on $\mathcal{H}_{\mathcal{SPF}}$ is nondegenerate, with the help of *bidendriform bialgebras* [6]: in fact, the pairing of $\mathcal{H}_{\mathcal{SP}}$ restricted to $\mathcal{H}_{\mathcal{SPF}}$ respects a certain bidendriform structure.

In the seventh section, we construct an isometric Hopf algebra morphism between $\mathcal{H}_{\mathcal{PP}}$ and $\mathcal{H}_{\mathcal{SPP}}$. These two Hopf algebras are clearly isomorphic, with a very easily-defined isomorphism, which is not an isometry. We prove that these two objects are isometric as Hopf algebras up to two conditions on the base field: it should be not of characteristic two and should contain a root of -1. This is done using the freeness and cofreeness of $\mathcal{H}_{\mathcal{PP}}$ and a lemma on symmetric, invertible matrices with integer coefficients.

This text is organised as follows. The first section recalls the concepts and notations on the Hopf algebra of double posets $\mathcal{H}_{\mathcal{DP}}$. The second section introduces special posets, heap-ordered posets, special plane posets and the other families of double posets here studied. The bijection between the set of special plane posets of order n and $\mathfrak{S}_n$ is defined in the third section. The properties of the morphism Θ from $\mathcal{H}_{\mathcal{SP}}$ to **FQSym** are investigated in the next section. In particular, it is proved that its restrictions to $\mathcal{H}_{\mathcal{SPP}}$ or $\mathcal{H}_{\mathcal{HOF}}$ are isomorphisms, and the induced isomorphism from $\mathcal{H}_{\mathcal{SPP}}$ to $\mathcal{H}_{\mathcal{HOF}}$ is combinatorially defined. The fifth and sixth sections introduce duplicial, dendriform and bidendriform structures and gives applications of these algebraic objects on our families of posets. The problem of finding an isometry from $\mathcal{H}_{\mathcal{SPP}}$ to $\mathcal{H}_{\mathcal{PP}}$ is studied in the seventh section; all the obtained results are sumed up in the conclusion.

Notations.

1. K is a commutative field. Any algebra, coalgebra, Hopf algebra... of the present text will be taken over K.

2. If $\mathcal{H} = (\mathcal{H}, m, 1, \Delta, \varepsilon, S)$ is a Hopf algebra, we shall denote by $\mathcal{H}^+$ its augmentation ideal, that is to say $Ker(\varepsilon)$. This ideal $\mathcal{H}^+$ has a coassociative, non counitary coproduct $\tilde{\Delta}$, defined by $\tilde{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$ for all $x \in \mathcal{H}^+$.

3. For all $n \geq 1$, $\mathfrak{S}_n$ is the n-th symmetric group. Any element σ of $\mathfrak{S}_n$ will be represented by the word $(\sigma(1) \ldots \sigma(n))$. By convention, $\mathfrak{S}_0$ is a group reduced to its unit, denoted by the empty word 1.

1 Reminders on double posets

1.1 Several families of double posets

Definition 1 [17]. A *double poset* is a triple $(P, \leq_1, \leq_2)$, where P is a finite set and $\leq_1$, $\leq_2$ are two partial orders on P. The set of isoclasses of double posets will be denoted by $\mathcal{DP}$. The set of isoclasses of double posets of cardinality n will be denoted by $\mathcal{DP}(n)$ for all $n \in \mathbb{N}$.

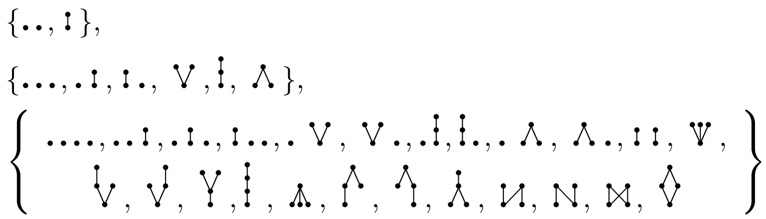
Remark. Let $P \in \mathcal{DP}$. Then any part $Q \subseteq P$ inherits also two partial orders by restriction, so is also a double poset: we shall speak in this way of *double subposets*.

Definition 2 A *plane poset* is a double poset $(P, \leq_h, \leq_r)$ such that for all $x, y \in P$ with $x \neq y$, x and y are comparable for $\leq_h$ if, and only if, x and y are not comparable for $\leq_r$. The set of isoclasses of plane posets will be denoted by $\mathcal{PP}$. For all $n \in \mathbb{N}$, the set of isoclasses of plane posets of cardinality n will be denoted by $\mathcal{PP}(n)$.

If $(P, \leq_h, \leq_r)$ is a plane poset, we shall represent the Hasse graph of $(P, \leq_h)$ such that if $x <_r y$ in P, then y is more on the right than x in the graph.

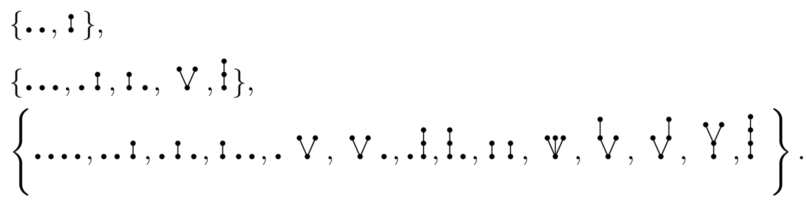
Examples. The empty double poset is denoted by 1.

$\mathcal{PP}(0) = \{1\}$,
$\mathcal{PP}(1) = \{\cdot\}$,
$\mathcal{PP}(2) =$
$\mathcal{PP}(3) =$
$\mathcal{PP}(4) =$



Remark. Let F be a plane forest. We defined in [4] two partial orders on F, which makes it a plane poset. Equivalently, a plane poset is a plane forest if, and only if its Hasse graph is a forest. The set of plane forests will be denoted by $\mathcal{PF}$; for all $n \geq 0$, the set of plane forests with n vertices will be denoted by $\mathcal{PF}(n)$. For example:

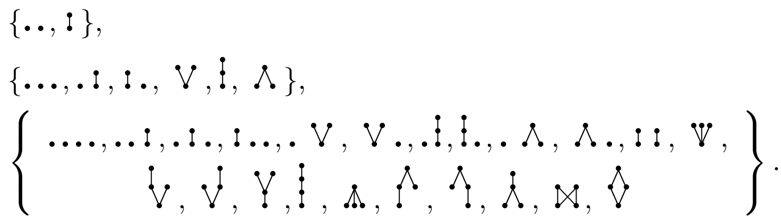
$\mathcal{PF}(0) = \{1\}$,
$\mathcal{PF}(1) = \{\cdot\}$,
$\mathcal{PF}(2) =$
$\mathcal{PF}(3) =$
$\mathcal{PF}(4) =$



Definition 3 Let P be a double poset. We shall say that P is *WN* ("without N") if it is plane and does not contain any double subposet isomorphic to  nor . The set of isoclasses of WN posets will be denoted by $\mathcal{WNP}$. For all $n \in \mathbb{N}$, the set of isoclasses of WN posets of cardinality n will be denoted by $\mathcal{WNP}(n)$.

Examples.

$\mathcal{WNP}(0) = \{1\}$,
$\mathcal{WNP}(1) = \{\cdot\}$,
$\mathcal{WNP}(2) =$
$\mathcal{WNP}(3) =$
$\mathcal{WNP}(4) =$



Remark. $\mathcal{PF} \subsetneq \mathcal{WNP} \subsetneq \mathcal{PP}$.

1.2 Products and coproducts of double posets

Definition 4 Let P and Q be two elements of $\mathcal{DP}$. We define $PQ \in \mathcal{DP}$ by:

- PQ is the disjoint union of P and Q as a set.
- P and Q are double subposets of PQ.
- For all $x \in P$, $y \in Q$, $x \leq_2 y$ in PQ and x and y are not comparable for $\leq_1$ in PQ.

Remarks.

1. This product is called *composition* in [17] and denoted by $\rightsquigarrow$ in [7].
2. The Hasse graph of PQ is the concatenation of the Hasse graphs of P and Q.

This associative product is linearly extended to the vector space $\mathcal{H}_{\mathcal{DP}}$ generated by the set of double posets. Moreover, the subspaces $\mathcal{H}_{\mathcal{PP}}$, $\mathcal{H}_{\mathcal{WNP}}$ and $\mathcal{H}_{\mathcal{PF}}$ respectively generated by the sets $\mathcal{PP}$, $\mathcal{WNP}$ and $\mathcal{PF}$ are stable under this product.

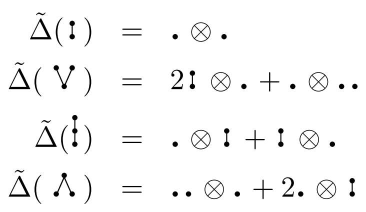
Definition 5 [17].

1. Let $P = (P, \leq_1, \leq_2)$ be a double poset and let $I \subseteq P$. We shall say that I is a 1-*ideal* of P if:
$$\forall x \in I,\ \forall y \in P,\ (x \leq_1 y) \Longrightarrow (y \in I).$$
We shall write shortly "ideal" instead of "1-ideal" in the sequel.

2. The associative algebra $\mathcal{H}_{\mathcal{DP}}$ is given a Hopf algebra structure with the following coproduct: for any double poset P,
$$\Delta(P) = \sum_{I \text{ ideal of } P} (P \setminus I) \otimes I.$$
This Hopf algebra is graded by the cardinality of the double posets.

As any double subposet of a, respectively, plane poset, WN poset, plane forest, is also a, respectively, plane poset, WN poset, plane forest, $\mathcal{H}_{\mathcal{PP}}$, $\mathcal{H}_{\mathcal{WNP}}$ and $\mathcal{H}_{\mathcal{PF}}$ are Hopf subalgebras of $\mathcal{H}_{\mathcal{DP}}$. The latter is the (co-opposite of the) non-commutative Connes-Kreimer Hopf algebra of plane trees [4, 5, 10].

Examples.



1.3 Hopf pairing on double posets

Definition 6 [17]

1. For two double posets P, Q, $S(P, Q)$ is the set of bijections $\sigma : P \longrightarrow Q$ such that, for all $i, j \in P$:

   - $(i \leq_1 j \text{ in } P) \Longrightarrow (\sigma(i) \leq_2 \sigma(j) \text{ in } Q)$.
   - $(\sigma(i) \leq_1 \sigma(j) \text{ in } Q) \Longrightarrow (i \leq_2 j \text{ in } P)$.

   These bijections are called *pictures*.

2. We define a pairing on $\mathcal{H}_{\mathcal{DP}}$ by $\langle P, Q\rangle = Card(S(P, Q))$ for $P, Q \in \mathcal{DP}$. This pairing is a symmetric Hopf pairing.

It is proved in [7] that this pairing is nondegenerate if, and only if, the characteristic of K is zero. Moreover, the restriction of this pairing to $\mathcal{H}_{\mathcal{PP}}$, $\mathcal{H}_{\mathcal{PF}}$ or $\mathcal{H}_{\mathcal{WNP}}$ is nondegenerate, whatever the field K is.

2 Several families of posets

2.1 Special posets

Definition 7 [17]. A double poset $P = (P, \leq_1, \leq_2)$ is *special* if the order $\leq_2$ is total. The set of special double posets will be denoted by $\mathcal{SP}$. The set of special double posets of cardinality n will be denoted by $\mathcal{SP}(n)$.

This notion is equivalent to the notion of labelled posets. If $(P, \leq_1, \leq_2)$ is a special poset of order n, there is a unique isomorphism from $(P, \leq_2)$ to $(\{1, \ldots, n\}, \leq)$, and we shall often identify them.

Examples. We shall graphically represent a special poset $(P, \leq_1, \leq_2)$ by the Hasse graph of $(P, \leq_1)$, with indices on the vertices giving the total order $\leq_2$.

1. Here are $\mathcal{SP}(n)$ for $n \leq 3$:

$$\begin{array}{rcl}
\mathcal{SP}(1) &=& \{\bullet_1\},\\
\mathcal{SP}(2) &=& \{\bullet_1\bullet_2, \mathrm{I}\substack{2\\1}, \mathrm{I}\substack{1\\2}\},\\
\mathcal{SP}(3) &=& \left\{\begin{array}{c}
\bullet_1\bullet_2\bullet_3, \bullet_1\mathrm{I}\substack{3\\2}, \bullet_1\mathrm{I}\substack{2\\3}, \bullet_2\mathrm{I}\substack{3\\1}, \bullet_2\mathrm{I}\substack{1\\3}, \bullet_3\mathrm{I}\substack{2\\1}, \bullet_3\mathrm{I}\substack{1\\2},\\
{}^{2}\mathrm{V}^{3}_{1}, {}^{1}\mathrm{V}^{3}_{2}, {}^{1}\mathrm{V}^{2}_{3}, {}_{2}\Lambda^{1}_{3}, {}_{1}\Lambda^{2}_{3}, {}_{1}\Lambda^{3}_{2}, \mathrm{I}\substack{3\\2\\1}, \mathrm{I}\substack{2\\3\\1}, \mathrm{I}\substack{3\\1\\2}, \mathrm{I}\substack{1\\3\\2}, \mathrm{I}\substack{2\\1\\3}, \mathrm{I}\substack{1\\2\\3}
\end{array}\right\}.
\end{array}$$

2. See [3]. *Ordered forests* are special double posets. The set of ordered forests will be denoted by $\mathcal{OF}$. The set of ordered forests of cardinality n will be denoted by $\mathcal{OF}(n)$. For example:

$$\begin{array}{rcl}
\mathcal{OF}(1) &=& \{\bullet_1\},\\
\mathcal{OF}(2) &=& \{\bullet_1\bullet_2, \mathrm{I}\substack{2\\1}, \mathrm{I}\substack{1\\2}\},\\
\mathcal{OF}(3) &=& \left\{\begin{array}{c}
\bullet_1\bullet_2\bullet_3, \bullet_1\mathrm{I}\substack{3\\2}, \bullet_1\mathrm{I}\substack{2\\3}, \bullet_2\mathrm{I}\substack{3\\1}, \bullet_2\mathrm{I}\substack{1\\3}, \bullet_3\mathrm{I}\substack{2\\1}, \bullet_3\mathrm{I}\substack{1\\2},\\
{}^{2}\mathrm{V}^{3}_{1}, {}^{1}\mathrm{V}^{3}_{2}, {}^{1}\mathrm{V}^{2}_{3}, \mathrm{I}\substack{3\\2\\1}, \mathrm{I}\substack{2\\3\\1}, \mathrm{I}\substack{3\\1\\2}, \mathrm{I}\substack{1\\3\\2}, \mathrm{I}\substack{2\\1\\3}, \mathrm{I}\substack{1\\2\\3}
\end{array}\right\}.
\end{array}$$

3. Let $P = (P, \leq_h, \leq_r)$ be a plane poset. From proposition 11 in [7], the relation $\leq$ defined by $x \leq y$ if, and only if, $x \leq_h y$ or $x \leq_r y$, is a total order on P, called the *induced total order* on P. So $(P, \leq_h, \leq)$ is also a special double poset: we can consider plane posets as special posets. The set of plane posets, seen as special double posets, will be denoted by $\mathcal{SPP}$. The set of plane posets of cardinality n, seen as special double posets, will be denoted by $\mathcal{SPP}(n)$. For example:

$$\begin{array}{rcl}
\mathcal{SPP}(1) &=& \{\bullet_1\},\\
\mathcal{SPP}(2) &=& \{\bullet_1\bullet_2, \mathbf{I}^2_1\},\\
\mathcal{SPP}(3) &=& \left\{ \bullet_1\bullet_2\bullet_3, \bullet_1\mathbf{I}^3_2,, \mathbf{I}^2_1\bullet_3, {}^2\mathbf{V}^3_1, {}_1\Lambda^3_2, \mathbf{I}^{3}_{\substack{2\\1}} \right\}.
\end{array}$$

4. We define the set $\mathcal{SPF}$ of plane forests, seen as special posets, and the set $\mathcal{SWNP}$ of WN posets, seen as special posets. Note that $\mathcal{SPF} = \mathcal{OF} \cap \mathcal{SPP}$. For example:

$$\begin{array}{rcl}
\mathcal{SPF}(1) &=& \{\bullet_1\},\\
\mathcal{SPF}(2) &=& \{\bullet_1\bullet_2, \mathbf{I}^2_1\},\\
\mathcal{SPF}(3) &=& \left\{ \bullet_1\bullet_2\bullet_3, \bullet_1\mathbf{I}^3_2,, \mathbf{I}^2_1\bullet_3, {}^2\mathbf{V}^3_1, \mathbf{I}^{3}_{\substack{2\\1}} \right\}
\end{array}$$

If P and Q are special double posets, then PQ is also special. So the space $\mathcal{H}_{\mathcal{SP}}$ generated by special double posets is a subalgebra of $(\mathcal{H}_{\mathcal{DP}}, \rightsquigarrow)$. Moreover, if P is a special double poset, then any subposet of P is also special. As a consequence, $\mathcal{H}_{\mathcal{SP}}$ is a Hopf subalgebra of $\mathcal{H}_{\mathcal{DP}}$; This Hopf algebra also appears in [1]. Similarly, the spaces $\mathcal{H}_{\mathcal{OF}}$, $\mathcal{H}_{\mathcal{SPP}}$, $\mathcal{H}_{\mathcal{SWNP}}$ and $\mathcal{H}_{\mathcal{SPF}}$ generated by $\mathcal{OF}$, $\mathcal{SPP}$, $\mathcal{SWNP}$ and $\mathcal{SPF}$ are Hopf subalgebras of $\mathcal{H}_{\mathcal{DP}}$.

Remark. It is clear that $\mathcal{H}_{\mathcal{PP}}$ and $\mathcal{H}_{\mathcal{SPP}}$ are isomorphic Hopf algebras, via the isomorphism sending the plane poset $(P, \leq_h, \leq_r)$ to the special poset $(P, \leq_h, \leq)$. The same argument works for $\mathcal{H}_{\mathcal{WNP}}$ and $\mathcal{H}_{\mathcal{SWNP}}$, for $\mathcal{H}_{\mathcal{PF}}$ and $\mathcal{H}_{\mathcal{SPF}}$.

2.2 Heap-ordered posets

Definition 8 Let $P = (P, \leq_1, \leq_2)$ be a special double poset. It is *heap-ordered* if for all $x, y \in P$, $x \leq_1 y$ implies that $x \leq_2 y$. The set of heap-ordered posets will be denoted by $\mathcal{HOP}$. The set of heap-ordered posets of cardinality n will be denoted by $\mathcal{HOP}(n)$. We put $\mathcal{HOF} = \mathcal{HOP} \cap \mathcal{OF}$ and $\mathcal{HOF}(n) = \mathcal{HOP}(n) \cap \mathcal{OF}(n)$ for all n.

Examples. Here are the sets $\mathcal{HOP}(n)$ and $\mathcal{HOF}(n)$ for $n \leq 3$:

$$\begin{array}{rcl}
\mathcal{HOP}(1) &=& \{\bullet_1\},\\
\mathcal{HOP}(2) &=& \{\bullet_1\bullet_2, \mathrm{I}^{2}_{1}\},\\
\mathcal{HOP}(3) &=& \left\{ \bullet_1\bullet_2\bullet_3, \bullet_1\mathrm{I}^{3}_{2}, \bullet_2\mathrm{I}^{3}_{1}, \bullet_3\mathrm{I}^{2}_{1}, {}^{2}\mathrm{V}_{1}{}^{3}, {}_{1}\Lambda^{3}{}_{2}, \substack{\bullet_3\\ \bullet_2\\ \bullet_1} \right\},
\end{array}$$

$$\begin{array}{rcl}
\mathcal{HOF}(1) &=& \{\bullet_1\},\\
\mathcal{HOF}(2) &=& \{\bullet_1\bullet_2, \mathrm{I}^{2}_{1}\},\\
\mathcal{HOF}(3) &=& \left\{ \bullet_1\bullet_2\bullet_3, \bullet_1\mathrm{I}^{3}_{2}, \bullet_2\mathrm{I}^{3}_{1}, \bullet_3\mathrm{I}^{2}_{1}, {}^{2}\mathrm{V}_{1}{}^{3}, \substack{\bullet_3\\ \bullet_2\\ \bullet_1} \right\}.
\end{array}$$

Note that $\mathcal{SPP} \subsetneq \mathcal{HOP}$ and $\mathcal{SPF} \subsetneq \mathcal{HOF}$, as $\bullet_2\mathrm{I}^{3}_{1}$ is not a plane poset. It is well-known that $|\mathcal{HOF}(n)| = n!$ for all $n \geq 0$.

If P and Q are two heap-ordered posets, then PQ also is. As a consequence, the spaces $\mathcal{H}_{\mathcal{HOP}}$, $\mathcal{H}_{\mathcal{HOF}}$ and $\mathcal{H}_{\mathcal{SPF}}$ generated by $\mathcal{HOP}$, $\mathcal{HOF}$ and $\mathcal{SPF}$ are Hopf subalgebras of $\mathcal{H}_{\mathcal{DP}}$. Moreover, plane posets are heap-ordered, so $\mathcal{H}_{\mathcal{SPP}} \subseteq \mathcal{H}_{\mathcal{HOP}}$. We obtain a commutative diagram of canonical injections:

$$\begin{tikzcd}
\mathcal{H}_{\mathcal{SPP}} \arrow[r, hook] & \mathcal{H}_{\mathcal{HOP}} \arrow[r, hook] & \mathcal{H}_{\mathcal{SP}} \\
\mathcal{H}_{\mathcal{SPF}} \arrow[r, hook] \arrow[u, hook] & \mathcal{H}_{\mathcal{HOF}} \arrow[r, hook] \arrow[u, hook] & \mathcal{H}_{\mathcal{OF}} \arrow[u, hook]
\end{tikzcd}$$

Proposition 9 *1. Let $P \in \mathcal{SP}$. Then P is heap-ordered if, and only if, it does not contain any double subposet isomorphic to I^{1}_{2}.*

2. Let $P \in \mathcal{SP}$. Then $P \in \mathcal{SPP}$ if, and only if, it does not contain any double subposet isomorphic to $\mathrm{I}^{3}_{1}\bullet_2$ nor I^{1}_{2}.

Proof. The first point is immediate.

2. $\Longrightarrow$. If $P \in \mathcal{SPP}$, then any subposet of P belongs to $\mathcal{SPP}$. The conclusion comes from the fact that $\mathrm{I}^{3}_{1}\bullet_2$ and I^{1}_{2} are not special plane posets.

2. $\Longleftarrow$. By the first point, $P = (P, \leq_1, \leq_2)$ is heap-ordered. We define a relation $\leq_r$ on P by:

$$x \leq_r y \text{ if } (x = y) \text{ or } \left((x <_2 y) \text{ and not } (x <_1 y)\right).$$

By definition, $x \leq_2 y$ if, and only if, $x \leq_1 y$ or $x \leq_r y$. Moreover, if x and y are comparable for both $\leq_1$ and $\leq_r$, then $x = y$ by definition of $\leq_r$. It remains to prove that $\leq_r$ is a partial order on P. If $x <_r y$ and $y <_r z$, then $x <_2 y <_2 z$, so $x <_2 z$, so $x <_1 z$ or $x <_r z$. If $x <_1 z$, then the subposet $\{x, y, z\}$ of P is equal to $\mathrm{I}^{3}_{1}\bullet_2$, as x, y and y, z are not comparable for $\leq_1$: contradiction. So $x <_r z$. $\Box$

2.3 Pairing on $\mathcal{H}_{\mathcal{SP}}$

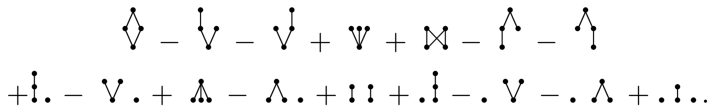
We restrict the pairing of $\mathcal{H}_{\mathcal{DP}}$ to $\mathcal{H}_{\mathcal{SP}}$. The matrix of the restriction of this pairing restricted to $\mathcal{H}_{\mathcal{SP}}(2)$ is:

	$\bullet_1\bullet_2$	$\mathbf{l}^2_1$	$\mathbf{l}^1_2$
$\bullet_1\bullet_2$	2	1	1
$\mathbf{l}^2_1$	1	1	0
$\mathbf{l}^1_2$	1	0	1

Remarks.

1. As a consequence, $\bullet_1\bullet_2 - \mathbf{l}^2_1 - \mathbf{l}^1_2$ in in the kernel of the pairing. Hence, $\langle -,-\rangle_{|\mathcal{H}_{\mathcal{SP}}}$, $\langle -,-\rangle_{|\mathcal{H}_{\mathcal{HOP}}}$ and $\langle -,-\rangle_{|\mathcal{H}_{\mathcal{OF}}}$ are degenerate. The kernels of these restrictions of the pairing are described in corollary 19.

2. A direct (but quite long) computation shows that the following element is in the kernel of $\langle -,-\rangle_{|\mathcal{H}_{\mathcal{SWNP}}}$:



   (We write here the double posets appearing in this element as plane poset, they have to be considered as special posets). So $\langle -,-\rangle_{|\mathcal{H}_{\mathcal{SWNP}}}$ is degenerate.

3. We shall see that $\langle -,-\rangle_{|\mathcal{H}_{\mathcal{HOF}}}$, $\langle -,-\rangle_{|\mathcal{H}_{\mathcal{SPP}}}$ and $\langle -,-\rangle_{|\mathcal{H}_{\mathcal{SPF}}}$ are nondegenerate, see corollaries 23, 26 and 37.

3 Links with permutations

3.1 Plane poset associated to a permutation

Proposition 10 *Let $\sigma \in \mathfrak{S}_n$. We define two relations $\leq_h$ and $\leq_r$ on $\{1,\cdots,n\}$ by:*

- *($i \leq_h j$) if ($i \leq j$ and $\sigma(i) \leq \sigma(j)$).*
- *($i \leq_r j$) if ($i \leq j$ and $\sigma(i) \geq \sigma(j)$).*

Then $(\{1,\cdots,n\},\leq_h,\leq_r)$ is a plane poset. The induced total order on $\{1,\cdots,n\}$ is the usual total order.

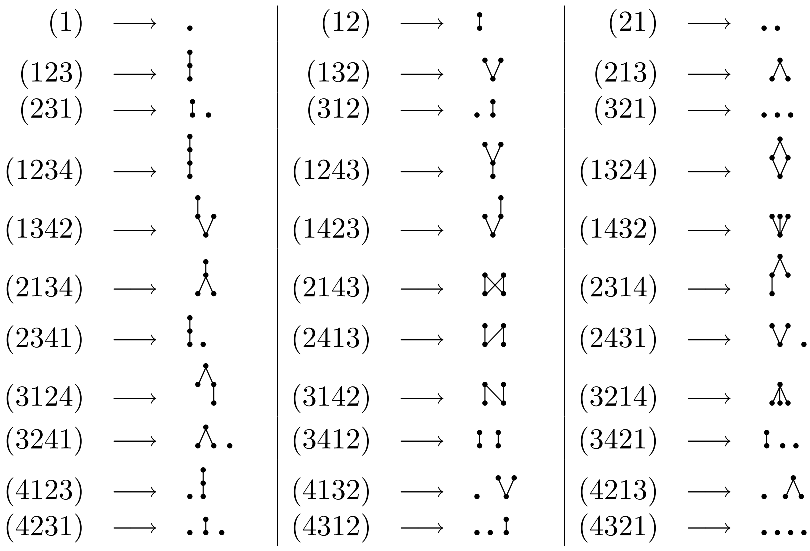
Proof. It is clear that $\leq_h$ and $\leq_r$ are two partial orders on $\{1,\cdots,n\}$. It is immediate for any i,j, i and j are comparable for $\leq_h$ or $\leq_r$. Moreover, if i and j are comparable for both $\leq_h$ and $\leq_r$, then $\sigma(i)=\sigma(j)$, so $i=j$. For all i,j, $i \leq_h j$ or $i \leq_r j$ if, and only if, $i \leq j$. $\square$

Definition 11 Let $n \in \mathbb{N}$. We define a map:

$$\Phi_n : \left\{ \begin{array}{rcl} \mathfrak{S}_n & \longrightarrow & \mathcal{PP}(n) \\ \sigma & \longrightarrow & (\{1, \cdots, n\}, \leq_h, \leq_r), \end{array} \right.$$

where $\leq_h$ and $\leq_r$ are defined in proposition 10.

Examples.



We shall prove in the next section that Φ_n is bijective for all $n \geq 1$.

3.2 Permutation associated to a plane poset

We now construct the inverse bijection. For any $P \in \mathcal{PP}$, we put:

$$\kappa(P) = \max(\{y \in P \,/\, \forall x \in P,\ x \leq y \Rightarrow x \leq_h y\}).$$

Note that $\kappa(P)$ is well-defined: the smallest element of P for its total order belongs to the set $\{y \in P \,/\, \forall x \in P,\ x \leq y \Rightarrow x \leq_h y\}$.

Let $P \in \mathcal{PP}(n)$. Up to a unique increasing bijection, we can suppose that $P = \{1, \cdots, n\}$ as a totally ordered set: we shall take this convention in this paragraph. We define an element σ of $\mathfrak{S}_n$ by:

$$\left\{ \begin{array}{rcl} \sigma^{-1}(n) & = & \kappa(P) \\ \sigma^{-1}(n-1) & = & \kappa\left(P - \{\sigma^{-1}(n)\}\right), \\ \vdots & & \vdots \\ \sigma^{-1}(1) & = & \kappa\left(P - \{\sigma^{-1}(n), \cdots, \sigma^{-1}(2)\}\right). \end{array} \right.$$

This defines a map:

$$\Psi_n : \left\{ \begin{array}{rcl} \mathcal{PP}(n) & \longrightarrow & \mathfrak{S}_n \\ (P, \leq_h, \leq_r) & \longrightarrow & \sigma. \end{array} \right.$$

Lemma 12 $\Psi_n \circ \Phi_n = Id_{\mathfrak{S}_n}$.

Proof. Let $\sigma \in \mathfrak{S}_n$. We put $P = \Phi_n(\sigma)$ and $\tau = \Psi_n(P)$. Then:

$$\{y \in P / \forall x \in P, x \leq y \Rightarrow x \leq_h y\} = \{j \in \{1, \cdots, n\} / \forall 1 \leq i \leq n, i \leq j \Rightarrow \sigma(i) \leq \sigma(j)\}.$$

So $\tau^{-1}(n) = \kappa(P) = \sigma^{-1}(n)$. Iterating this process, we obtain $\sigma^{-1} = \tau^{-1}$, so $\sigma = \tau$. $\Box$

Lemma 13 *Let $P \in \mathcal{PP}(n)$. We put $\Psi_n(P) = \sigma$. If $i \leq_h j$ in P, then $\sigma(i) \leq \sigma(j)$.*

Proof. If $i = j$, this is obvious. Let us assume that $i <_h j$. We put $k = \sigma(i)$ and $l = \sigma(j)$. Then $k \neq l$. Let us assume that $k > l$. We then put:

$$P' = P \backslash \{\sigma^{-1}(n), \ldots, \sigma^{-1}(k+1)\} = \{i_1, \cdots, i_p, i, i_{p+1}, \cdots, i_{p+q}, j, i_{p+q+1}, \cdots, i_{p+q+r}\},$$

with $i_1 < \cdots < i_p < i < i_{p+1} < \cdots < i_{p+q} < j < i_{p+q+1} < \cdots < i_{p+q+r}$. Indeed, as $l < k < k+1$, both $\sigma^{-1}(k) = i$ and $\sigma^{-1}(l) = j$ belongs to this set. As $\kappa(P') = i$, $i_1, \cdots, i_p <_h i$. If $i \leq_h i_{p+1}$, then $\kappa(P') \geq i_{p+1} > i$: contradiction. So $i <_r i_{p+1}$.

Let us prove by induction on s that $i_{p+s} \leq_h j$ for $1 \leq s \leq q$. If $i_{p+1} \leq_r j$, then i and j would be comparable for $\leq_r$, so would not be comparable for $\leq_h$: contradiction. So $i_{p+1} \leq_h j$. Let us suppose that $i_{p+s-1} \leq_h j$, $1 < s \leq q$. As $i_{p+s} < j$, $i_{p+s} <_h j$ or $i_{p+s} <_r j$. Let us assume that $i_{p+s} <_r j$. As $\kappa(P') = i < i_{p+s}$, there exists $x \in P'$, $x <_r i_{p+s}$. By the induction hypothesis, $x \notin \{i_{p+1}, \cdots, i_{p+s}\}$. As $i <_h j$, $x \neq i$, so $x \in \{i_1, \cdots, i_p\}$. But for such an x, $x <_h i <_h j$, so $x <_h j$: contradiction. So $i_{p+s} <_h j$.

Finally, we obtain that $i_1, \cdots, i_p, i, i_{p+1}, \cdots, i_{p+q}, j \leq_h j$, so $i = \kappa(P') \geq j$: contradiction, $i < j$. So $k < l$. $\Box$

Lemma 14 $\Phi_n \circ \Psi_n = Id_{\mathcal{PP}_n}$.

Proof. Let $P \in \mathcal{PP}_n$. We put $\sigma = \Psi_n(P)$ and $Q = \Phi_n(\sigma)$. As totally ordered sets, $P = Q = \{1, \cdots, n\}$. As they are both plane posets, it is enough to prove that $(P, \leq_h) = (Q, \leq_h)$. Let us suppose that $i \leq_h j$ in P. Then $i \leq j$ and $\sigma(i) \leq \sigma(j)$ by lemma 13. So $i \leq_h j$ in Q. Let us suppose that $i \leq_h j$ in Q. So $i \leq j$ and $\sigma(i) \leq \sigma(j)$. We put $k = \sigma(i)$ and $l = \sigma(j)$. As $k < l$:

$$i \in P' = P - \{\sigma^{-1}(n), \cdots, \sigma^{-1}(l+1)\}.$$

By definition of $\kappa(P') = j$, $i \leq_h j$ in P as $i \leq j$. $\Box$

Proposition 15 *Ψ_n is a bijection, of inverse Φ_n. As a consequence, $card(\mathcal{PP}(n)) = n!$ for all $n \in \mathbb{N}$.*

Here are examples of properties of the bijection Ψ_n:

Proposition 16 *Let $P = (P, \leq_h, \leq_r) \in \mathcal{PP}(n)$.*

1. $(n \cdots 1) \circ \Psi_n(P) = \Psi_n((P, \leq_r, \leq_h))$.
2. $\Psi_n(P)^{-1} = \Psi_n((P, \leq_h, \geq_r))$.

Proof. 1. We put $\Psi_n(P) = \sigma = (a_1 \cdots a_n)$. Then $(n \cdots 1) \circ \sigma = (n - a_1 + 1 \cdots n - a_n + 1)$. We put $Q = \Phi_n((n \cdots 1) \circ \sigma)$. For all $i, j \in \{1, \cdots n\}$:

$$\begin{aligned} i \leq_h j \text{ in } Q &\iff i \leq j \text{ and } n - a_i + 1 \leq n - a_j + 1 \\ &\iff i \leq j \text{ and } a_i \geq a_j \\ &\iff i \leq_r j \text{ in } P \end{aligned}$$

Similarly, $i \leq_r j$ in Q if, and only if, $i \leq_h j$ in P. So $Q = (P, \leq_r, \leq_h)$.

2. We put $R = \Phi_n(\sigma^{-1})$. Let $i, j \in \{1, \cdots, n\}$.

$$\begin{aligned} \sigma(i) \leq_h \sigma(j) \text{ in } R &\iff \sigma(i) \leq \sigma(j) \text{ and } i \leq j \\ &\iff i \leq_h j \text{ in } P, \end{aligned}$$

$$\begin{aligned} \sigma(i) \leq_r \sigma(j) \text{ in } R &\iff \sigma(i) \leq \sigma(j) \text{ and } i \geq j \\ &\iff i \geq_r j \text{ in } P. \end{aligned}$$

So $\sigma : (P, \leq_h, \geq_r) \longrightarrow R$ is an isomorphism of plane posets. $\Box$

Remark. In other terms, $(n \cdots 1) \circ \Psi_n(P) = \Psi_n \circ \iota(P)$, where the involution ι is defined in [7] by $\iota((P, \leq_h, \leq_r)) = (P, \leq_r, \leq_h)$.

4 A morphism to FQSym

Note that $\mathcal{H}_{\mathcal{PP}}$, $\mathcal{H}_{\mathcal{SPP}}$ and **FQSym** are both free and cofree, with the same formal series. From a result of [9], $\mathcal{H}_{\mathcal{PP}}$, hence $\mathcal{H}_{\mathcal{SPP}}$, is isomorphic to **FQSym**. Our aim in this section is to define and study an explicit isomorphism between $\mathcal{H}_{\mathcal{SPP}}$ and **FQSym**.

4.1 Reminders on FQSym

Let us first recall the construction of **FQSym** [16, 2]. As a vector space, a basis of **FQSym** is given by the disjoint union of the symmetric groups $\mathfrak{S}_n$, for all $n \geq 0$. By convention, the unique element of $\mathfrak{S}_0$ is denoted by 1. The product of **FQSym** is given, for $\sigma \in \mathfrak{S}_k$, $\tau \in \mathfrak{S}_l$, by:

$$\sigma\tau = \sum_{\epsilon \in Sh(k,l)} (\sigma \otimes \tau) \circ \epsilon,$$

where $Sh(k, l)$ is the set of (k, l)-shuffles, that is to say permutations $\epsilon \in \mathfrak{S}_{k+l}$ such that $\epsilon^{-1}(1) < \ldots < \epsilon^{-1}(k)$ and $\epsilon^{-1}(k + 1) < \ldots < \epsilon^{-1}(k + l)$. In other words, the product of σ and τ is given by shifting the letters of the word representing τ by k, and then summing all the possible shufflings of this word and of the word representing σ. For example:

$$\begin{aligned}(132)(21) &= (13254)+(13524)+(15324)+(51324)+(13542)\\ &\quad +(15342)+(51342)+(15432)+(51432)+(54132).\end{aligned}$$

Let $\sigma \in \mathfrak{S}_n$. For all $0 \leq k \leq n$, there exists a unique triple $\left(\sigma_1^{(k)}, \sigma_2^{(k)}, \zeta_k\right) \in \mathfrak{S}_k \times \mathfrak{S}_{n-k} \times Sh(k, n-k)$ such that $\sigma = \zeta_k^{-1} \circ \left(\sigma_1^{(k)} \otimes \sigma_2^{(k)}\right)$. The coproduct of **FQSym** is then defined by:

$$\Delta(\sigma) = \sum_{k=0}^{n} \sigma_1^{(k)} \otimes \sigma_2^{(k)}.$$

For example:

$$\begin{aligned}\Delta((41325)) &= 1 \otimes (41325) + (1) \otimes (1324) + (21) \otimes (213)\\ &\quad +(312) \otimes (12) + (4132) \otimes (1) + (41325) \otimes 1.\end{aligned}$$

Note that $\sigma_1^{(k)}$ and $\sigma_2^{(k)}$ are obtained by cutting the word representing σ between the k-th and the $k+1$-th letter, and then *standardizing* the two obtained words, that is to say applying to their letters the unique increasing bijection to $\{1, \ldots, k\}$ or $\{1, \ldots, n-k\}$. Moreover, **FQSym** has a nondegenerate, homogeneous, Hopf pairing defined by $\langle \sigma, \tau \rangle = \delta_{\sigma, \tau^{-1}}$ for all permutations σ and τ.

4.2 Linear extensions

Definition 17 Let $P = (P, \leq_1, \leq_2)$ a special poset. Let $x_1 <_2 \ldots <_2 x_n$ be the elements of P. A *linear extension* of P is a permutation $\sigma \in \mathfrak{S}_n$ such that, for all $i, j \in \{1, \ldots, n\}$:

$$(x_i \leq_1 x_j) \Longrightarrow (\sigma^{-1}(i) < \sigma^{-1}(j)).$$

The set of linear extension of P will be denoted by S_P.

Remarks.

1. Let P be a special poset. It is heap-ordered if, and only if, $Id_n \in S_P$.

2. Let P be a special poset of cardinality n. We identify P and $\bullet^n$ with $\{1, \ldots, n\}$ as totally ordered sets. Then the set of linear extensions of P is $S(\bullet^n, P)$ (see definition 6).

The following theorem is proved in :

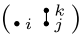
Theorem 18 *[17] The following map is a surjective morphism of Hopf algebras:*

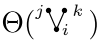
$$\Theta : \left\{ \begin{array}{rcl} \mathcal{H}_{\mathcal{SP}} & \longrightarrow & \mathbf{FQSym} \\ P \in \mathcal{SP} & \longrightarrow & \displaystyle\sum_{\sigma \in S_P} \sigma. \end{array} \right.$$

Moreover, for any $x, y \in \mathcal{H}_{\mathcal{SP}}$, $\langle x, y \rangle = \langle \Theta(x), \Theta(y) \rangle_{\mathbf{FQSym}}$.

Examples. If $\{i, j, k\} = \{1, 2, 3\}$:

$$\Theta(\bullet_i \bullet_j \bullet_k) = (ijk) + (ikj) + (jik) + (jki) + (kij) + (kji)$$

$\Theta($  $) = (ijk) + (jik) + (jki)$

$\Theta($  $) = (ijk) + (ikj)$

$\Theta($  $) = (ijk)$

Corollary 19 *The kernel of the pairing on $\mathcal{H}_{\mathcal{SP}}$ is $Ker(\Theta)$. The kernel of the pairing restricted to $\mathcal{H}_{\mathcal{HOP}}$ and $\mathcal{H}_{\mathcal{OF}}$ is respectively $Ker(\Theta) \cap \mathcal{H}_{\mathcal{HOP}}$ and $Ker(\Theta) \cap \mathcal{H}_{\mathcal{OF}}$.*

Proof. For any $x \in \mathcal{H}_{\mathcal{SP}}$, as Θ is surjective:

$$\begin{array}{rcl} x \in \mathcal{H}_{\mathcal{SP}}^{\perp} & \Longleftrightarrow & \forall y \in \mathcal{H}_{\mathcal{SP}},\ \langle x, y \rangle = 0 \\ & \Longleftrightarrow & \forall y \in \mathcal{H}_{\mathcal{SP}},\ \langle \Theta(x), \Theta(y) \rangle_{\mathbf{FQSym}} = 0 \\ & \Longleftrightarrow & \forall y' \in \mathbf{FQSym},\ \langle \Theta(x), y' \rangle = 0 \\ & \Longleftrightarrow & \Theta(x) \in \mathbf{FQSym}^{\perp} \\ & \Longleftrightarrow & \Theta(x) = 0. \end{array}$$

So $\mathcal{H}_{\mathcal{SP}}^{\perp} = Ker(\Theta)$. The proof is similar for $\mathcal{H}_{\mathcal{HOP}}$ and $\mathcal{H}_{\mathcal{OF}}$. $\square$

4.3 Restriction to $\mathcal{H}_{\mathcal{SPP}}$

Notation. Let $\sigma \in \mathfrak{S}_n$. We put $Desc(\sigma) = \{(i, j) \mid 1 \leq i < j \leq n,\ \sigma(i) > \sigma(j)\}$.

Lemma 20 *1. Let P be a heap-ordered poset. Then:*

$$\{(i, j) \mid 1 \leq i < j \leq n,\ i, j \text{ are not comparable for } \leq_1\} = \bigcup_{\sigma \in S_P} Desc(\sigma^{-1}).$$

2. Let P, Q be two heap-ordered posets. If $S_P = S_Q$, then $P = Q$.

Proof. 1. $\supseteq$. Let $(i, j) \in Desc(\sigma^{-1})$, with $\sigma \in S_P$. Then $i < j$ and $\sigma^{-1}(i) > \sigma^{-1}(j)$. By definition of S_P, we do not have $i <_1 j$ in P. As P is heap-ordered and $i <_2 j$ in P, we do not have $i >_1 j$ in P. So i and j are not comparable for $<_1$.

1. $\subseteq$. Let $1 \leq i < j \leq n$, such that i and j are not comparable for $<_1$. We proceed by induction on n. It is obvious if $n = 1$. As P is heap-ordered, n is a

maximal element for $<_1$. Let P' be the heap-ordered poset $P \setminus \{n\}$. Similarly, 1 is a minimal element for $<_1$. Let P'' be the heap-ordered poset $P \setminus \{1\}$.

If $j < n$, then i and j are not comparable for $<_1$ in P'. By the induction hypothesis, there exists $\sigma' = (i_1, \ldots, i_{n-1}) \in S_{P'}$ such that $(i,j) \in Desc(\sigma'^{-1})$. Then $\sigma = (i_1, \ldots i_{n-1}, n) \in S_P$ and $(i,j) \in Desc(\sigma^{-1})$.

If $1 < i$, then $i-1$ and $j-1$ are not comparable for $<_1$ in P''. By the induction hypothesis, there exists $\sigma'' = (i_1, \ldots, i_{n-1}) \in S_{P''}$ such that $(i-1, j-1) \in Desc(\sigma''^{-1})$. Then $\sigma = (1, i_1+1, \ldots i_{n-1}+1) \in S_P$ and $(i,j) \in Desc(\sigma^{-1})$.

If $i = 1$ and $j = n$, two cases are possible.

- If there exists no element $k \in P$, such that $k <_1 n$, then $P = P'' \bullet_n$. Then $\sigma = (n, 1, \ldots, n-1) \in S_P$, and $(1,n) \in Desc(\sigma^{-1})$.
- Let us suppose that there exists $k \in P$, such that $k <_1 n$. We choose a k such that there is no element l such that $k <_1 l <_1 n$. As 1 and n are not comparable for $<_1$, we cannot have $1 <_1 k$. As P is heap-ordered, 1 and k are not comparable for $<_1$ in P'. By the induction hypothesis, there exists $\sigma' \in I_{P'}$, such that $(1,k) \in Desc(\sigma^{-1})$. So σ' has the form $(\ldots, k, \ldots, 1, \ldots)$. Then $\sigma = (\ldots k, n, \ldots, 1, \ldots) \in I_P$ and $(1,n) \in Desc(\sigma^{-1})$.

2. Let $1 \leq i, j \leq n$. As P, Q are heap-ordered, by the first point:

$$\begin{aligned} i <_1 j \text{ in } P &\Longleftrightarrow i < j \text{ and } i,j \text{ are comparable for } <_1 \text{ in } P \\ &\Longleftrightarrow i < j \text{ and } \forall \sigma \in S_P,\ (i,j) \notin Desc(\sigma^{-1}) \\ &\Longleftrightarrow i < j \text{ and } \forall \sigma \in S_Q,\ (i,j) \notin Desc(\sigma^{-1}) \\ &\Longleftrightarrow i < j \text{ and } i,j \text{ are comparable for } <_1 \text{ in } Q \\ &\Longleftrightarrow i <_1 j \text{ in } Q. \end{aligned}$$

So $P = Q$. □

Proposition 21 *Let $n \in \mathbb{N}$. We partially order $\mathfrak{S}_n$ by the weak Bruhat order [20].*

1. *If $P \in \mathcal{SPP}(n)$, then* $\Theta(P) = \displaystyle\sum_{\sigma \in \mathfrak{S}_n,\, \sigma \leq \Phi_n(P)^{-1}} \sigma.$
2. *Let $P \in \mathcal{SP}(n)$. There exists $\tau \in \mathfrak{S}_n$, such that $S_P = \{\sigma \in \mathfrak{S}_n \mid \sigma \leq \tau\}$ if, and only if, $P \in \mathcal{SPP}$.*

Proof. 1. We put $\tau = \Phi_n(P)$. The aim is to prove that for all $\sigma \in \mathfrak{S}_n$, $\sigma \in S_P$ if, and only if, $\sigma \leq \tau$.

Let us assume that $\sigma \in S_P$. We put:

$$I = \{(i,j) \mid i <_r j,\ \sigma^{-1}(i) < \sigma^{-1}(j)\}.$$

Let us prove that $\sigma \leq \tau$ by induction on $|I|$. If $|I| = 0$, by definition of the elements of S_P, for all $i < j$:

$$i <_h j \Longleftrightarrow \sigma^{-1}(i) < \sigma^{-1}(j) \Longleftrightarrow \tau^{-1}(i) < \tau^{-1}(j).$$

So $\sigma = \tau$. Let us assume now that $|I| \geq 1$. Let us choose $(i,k) \in I$, such that $E = \sigma^{-1}(k) - \sigma^{-1}(i)$ is minimal. If $E \geq 2$, let j such that $\sigma^{-1}(i) < \sigma^{-1}(j) < \sigma^{-1}(k)$. Three cases are possible.

1. If $i < j < k$, by minimality of E, $i <_h j$ et $j <_h k$, so $i <_h k$. This contradicts $i <_r k$.

2. If $j < i < k$, by minimality of E, $j <_h k$. As $\sigma \in S_P$, $j <_r i$. As $i <_r k$, we obtain $j <_r k$. This contradicts $j <_h k$.

3. If $i < k < j$, by minimality of E, $i <_h j$. As $\sigma \in S_P$, $k <_r j$. As $i <_r k$, $i <_r j$. This contradicts $i <_h j$.

In all cases, this gives a contradiction. So $E = 1$, that is to say $\sigma^{-1}(i) = \sigma^{-1}(k) - 1$. The permutation σ' obtained from σ by permuting i and k in the word representing σ is greater than σ for the weak Bruhat order by definition of this order; moreover, it is not difficult to show that it is also an element of S_P (as $(i,k) \in I$), with a strictly smaller $|I|$. By the induction hypothesis, $\sigma \leq \sigma' \leq \tau$.

Let us assume that $\sigma \leq \tau$ and let us prove that $\sigma \in S_P$. Then τ is obtained from σ by a certain number k of elementary transformations (that is to say the permutations of two adjacent letters ij with $i < j$ in the word representing σ). We proceed by induction on k. If $k = 0$, then $\sigma = \tau$. If $k \geq 1$ there exists $\sigma' \in \mathfrak{S}_n$, obtained from σ by one elementary transformation, such that τ is obtained from σ' by $k-1$ elementary transformations. By the induction hypothesis, $\sigma' \in S_P$. We put $\sigma = (\ldots a_i a_{i+1} \ldots)$, $\sigma' = (\ldots a_{i+1} a_i \ldots)$, with $a_i < a_{i+1}$. Let us prove that $\sigma \in S_P$. Let $k <_h l$.

- If $k, l \neq a_i, a_{i+1}$, as $\sigma' \in S_P$, $\sigma^{-1}(k) = \sigma'^{-1}(k) < \sigma'^{-1}(l) = \sigma^{-1}(l)$.

- If $k = a_i$, as $\sigma' \in S_P$, $l \neq a_{i+1}$. So $\sigma^{-1}(l) = \sigma'^{-1}(l) > \sigma'^{-1}(k) = \sigma^{-1}(k) + 1$, and $\sigma^{-1}(k) < \sigma^{-1}(l)$.

- If $k = a_{i+1}$, then $l \neq a_i$ as $k < l$. So $\sigma^{-1}(l)\sigma'^{-1}(l) > \sigma'^{-1}(k) + 1 = \sigma^{-1}(k)$.

- If $l = a_i$, then $k \neq a_{i+1}$ as $k < l$. Then $\sigma^{-1}(k) = \sigma'^{-1}(k) < \sigma'^{-1}(l) - 1 = \sigma^{-1}(l)$.

- If $l = a_{i+1}$, as $\sigma \in S_P$, $k \neq a_i$. Then $\sigma^{-1}(k) = \sigma'^{-1}(k) < \sigma'^{-1}(l) = \sigma^{-1}(l) - 1$, and $\sigma^{-1}(k) < \sigma^{-1}(l)$.

Indeed, $\sigma \in S_P$.

2. $\Longleftarrow$. Comes from the first point, with $\tau = \Phi_n(P)^{-1}$.

2. $\Longrightarrow$. Let us assume that $S_P = \{\sigma \in \mathfrak{S}_n \mid \sigma \leq \tau\}$ for a particular τ. As $Id_n = (1 \ldots n) \leq \tau$, $(1 \ldots n) \in S_P$, so P is heap-ordered. Let $Q = \Psi_n(\tau^{-1}) \in \mathcal{SPP}(n)$. Then:

$$S_Q = \{\sigma \in \mathfrak{S}_n \mid \sigma \leq \tau\} = S_P.$$

By lemma 20, $P = Q$. $\square$

Examples. Here is the Hasse graph of $\mathfrak{S}_3$, partially ordered by the weak Bruhat order:

$$\begin{tikzcd}[column sep=small]
 & (321) \arrow[dl, dash] \arrow[dr, dash] & \\
(231) \arrow[d, dash] & & (312) \arrow[d, dash] \\
(213) \arrow[dr, dash] & & (132) \arrow[dl, dash] \\
 & (123) &
\end{tikzcd}$$

So:

$$\begin{array}{rcl}
\Theta(\cdot\cdot\cdot) & = & (312)+(231)+(312)+(213)+(132)+(123)\\
\Theta(\cdot\,\updownarrow) & = & (231)+(213)+(123)\\
\Theta(\updownarrow\,\cdot) & = & (312)+(132)+(123)\\
\Theta(\wedge) & = & (213)+(123)\\
\Theta(\vee) & = & (132)+(123)\\
\Theta(\vdots) & = & (123).
\end{array}$$

As $\Phi_n : \mathcal{SPP}(n) \longrightarrow \mathfrak{S}_n$ is a bijection:

Corollary 22 *The restriction $\Theta_{|\mathcal{H}_{\mathcal{SPP}}} : \mathcal{H}_{\mathcal{SPP}} \longrightarrow$ **FQSym** is an isomorphism.*

Corollary 23 *The restriction of the pairing to $\mathcal{H}_{\mathcal{SPP}}$ is nondegenerate.*

Proof. As the isomorphism $\Theta_{|\mathcal{H}_{\mathcal{SPP}}}$ is an isometry and the pairing of **FQSym** is nondegenerate. $\square$

4.4 Restriction to $\mathcal{H}_{\mathcal{HOF}}$

Notation. Let $P = (P, \leq_1, \leq_2)$ be a special poset. If $i, j \in P$, we denote by $[i,j]_1$ the set of elements k of P such that $i \leq_1 k \leq_1 j$. we denote by $R_P = \{(i,j) \in P^2 \mid [i,j]_1 = \{i,j\},\ i \neq j\}$. This set is in fact the set of edges of the Hasse graph of $(P, \leq_1)$, so allows to reconstruct the double poset P.

Proposition 24 *Let P be a special poset with n elements.*

1. *Let $i, j \in P$, such that $(j,i) \in R_P$. We define:*
   - *$P_1 \in \mathcal{SP}(n)$ such that $R_{P_1} = R_P \setminus \{(j,i)\}$;*

- $P_2 \in \mathcal{SP}(n)$ *such that* $R_{P_2} = (R_P \setminus \{(j,i)\}) \cup \{(i,j)\}$, *after the elimination of redundant elements.*

Then $\Theta(P) = \Theta(P_1) - \Theta(P_2)$.

2. *Let* $i, j, k \in P$, *all distinct, such that* (i,k) *and* $(j,k) \in R_P$. *We define:*

- $P_3 \in \mathcal{SP}(n)$, *such that* $R_{P_3} = R_P \setminus \{(j,k)\}$;
- $P_4 \in \mathcal{SP}(n)$, *such that* $R_{P_4} = (R_P \setminus \{(j,k)\}) \cup \{(i,j)\}$, *after the elimination of redundant elements;*
- $P_5 \in \mathcal{SP}(n)$, *such that* $R_{P_5} = (R_P \setminus \{(j,k),(i,k)\}) \cup \{(i,j),(j,k)\}$, *after the elimination of redundant elements.*

Then $\Theta(P) = \Theta(P_3) - \Theta(P_4) + \Theta(P_5)$.

Proof. 1. We denote by S the set of permutations $\sigma \in \mathfrak{S}_n$ such that, for all $(x,y) \in R_P \setminus \{(i,j)\}$, $\sigma^{-1}(x) < \sigma^{-1}(y)$. Then:

$$\Theta(P_1) = \sum_{\sigma \in S} \sigma, \qquad \Theta(P) = \sum_{\sigma \in S,\, \sigma^{-1}(j) < \sigma^{-1}(i)} \sigma, \qquad \Theta(P_2) = \sum_{\sigma \in S,\, \sigma^{-1}(j) > \sigma^{-1}(i)} \sigma.$$

As a consequence, $\Theta(P) + \Theta(P_2) = \Theta(P_1)$.

2. Note that i and j are not comparable for $\leq_1$ (otherwise, for example if $i <_1 j$, then $i <_1 j <_1 k$, and this contradicts the definition of R_P). We denote by S' the set of permutations $\sigma \in \mathfrak{S}_n$, such that for all $(x,y) \in R_P \setminus \{(i,k),(j,k)\}$, $\sigma^{-1}(x) < \sigma^{-1}(y)$. Then:

$$\Theta(P) = \sum_{\sigma \in S',\, \sigma^{-1}(i), \sigma^{-1}(j) < \sigma^{-1}(k)} \sigma, \qquad \Theta(P_3) = \sum_{\sigma \in S',\, \sigma^{-1}(i) < \sigma^{-1}(k)} \sigma,$$

$$\Theta(P_4) = \sum_{\sigma \in S',\, \sigma^{-1}(i) < \sigma^{-1}(j), \sigma^{-1}(k)} \sigma, \qquad \Theta(P_5) = \sum_{\sigma \in S',\, \sigma^{-1}(i) < \sigma^{-1}(j) < \sigma^{-1}(k)} \sigma.$$

We put:

$$S_1 = \sum_{\sigma \in S',\, \sigma^{-1}(i) < \sigma^{-1}(j) < \sigma^{-1}(k)} \sigma, \qquad S_2 = \sum_{\sigma \in S',\, \sigma^{-1}(j) < \sigma^{-1}(i) < \sigma^{-1}(k)} \sigma,$$

$$S_3 = \sum_{\sigma \in S',\, \sigma^{-1}(i) < \sigma^{-1}(k) < \sigma^{-1}(j)} \sigma.$$

Then $\Theta(P) = S_1 + S_2$, $\Theta(P_3) = S_1 + S_2 + S_3$, $\Theta(P_4) = S_1 + S_3$ and $\Theta(P_5) = S_1$. Hence, $\Theta(P) + \Theta(P_4) = \Theta(P_3) + \Theta(P_5)$. $\square$

Remark. In other words, in the first case, one replaces a double subposet $\mathbf{I}^{i}_{j}$ of P by $\bullet_i \bullet_j - \mathbf{I}^{j}_{i}$. In the second case, one replaces a double subposet ${}_i\Lambda^k{}_j$ by $\mathbf{I}^{k}_{i} \bullet_j - {}^{j}\mathrm{V}^{k}_{i} + \begin{smallmatrix}k\\j\\i\end{smallmatrix}$.

Theorem 25 *Let $P \in \mathcal{SP}$. Applying repeatedly the two transformations of proposition 24, with $i < j$ in the first case, and $i < j < k$ in the second case, we can associate to P a linear span of heap-ordered forests denoted by $\Upsilon(P)$. Then $\Upsilon(P)$ does not depend of the way the transformations are performed. Moreover, Υ defines a Hopf algebra morphism from $\mathcal{H}_{\mathcal{SP}}$ to $\mathcal{H}_{\mathcal{HOF}}$, such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{H}_{\mathcal{SP}} \arrow[r, "\Theta"] \arrow[d, "\Upsilon"'] & \mathbf{FQSym} \\ \mathcal{H}_{\mathcal{HOF}} \arrow[ur, "\Theta"'] & \end{tikzcd}$$

The restriction $\Theta_{|\mathcal{H}_{\mathcal{HOF}}}$ is an isomorphism, and $\Upsilon_{|\mathcal{H}_{\mathcal{HOF}}} = Id_{\mathcal{H}_{\mathcal{HOF}}}$. Moreover, $\langle \Upsilon(x), \Upsilon(y)\rangle = \langle x, y\rangle$ for all $x, y \in \mathcal{H}_{\mathcal{SP}}$ (that is to say Υ respects the pairings).

Proof. It is clear that, using repeatedly the first transformation, we associate to P a linear span of heap-ordered posets. Then, using repeatedly the second transformation, we associate to this element of $\mathcal{H}_{\mathcal{HOP}}$ a linear span of heap-ordered forest. So $\Upsilon(P)$ exists. Moreover, using proposition 24, $\Theta(\Upsilon(P)) = \Theta(P)$. As $\Theta : \mathcal{H}_{\mathcal{SP}} \longrightarrow \mathbf{FQSym}$ is surjective (as, for example, $\Theta_{|\mathcal{H}_{\mathcal{SPP}}}$ is an isomorphism), $\Theta_{|\mathcal{H}_{\mathcal{HOF}}}$ is surjective. As $Card(\mathcal{HOF}(n)) = Card(\mathfrak{S}_n) = n!$ for all $n \in \mathbb{N}$, $\Theta_{|\mathcal{H}_{\mathcal{HOF}}}$ is bijective.

Hence, for all $P \in \mathcal{H}_{\mathcal{DP}}$, there exists a unique element $Q \in \mathcal{H}_{\mathcal{HOF}}$ such that $\Theta(Q) = \Theta(P)$; this Q is for example $\Upsilon(P)$, so $\Upsilon(P)$ is uniquely defined and does not depend of the way the transformations are performed. Moreover, $\Upsilon = \left(\Theta_{|\mathcal{H}_{\mathcal{HOF}}}\right)^{-1} \circ \Theta$ is a Hopf algebra morphism. As Θ respects the pairings, so does $\Upsilon = \left(\Theta_{|\mathcal{H}_{\mathcal{HOF}}}\right)^{-1} \circ \Theta$. $\Box$

Corollary 26 *1. $\Upsilon_{|\mathcal{H}_{\mathcal{SPP}}} : \mathcal{H}_{\mathcal{SPP}} \longrightarrow \mathcal{H}_{\mathcal{HOF}}$ is an isomorphism of graded Hopf algebras, and respects the pairings.*

2. $\langle -, -\rangle_{|\mathcal{H}_{\mathcal{HOF}}}$ is nondegenerate.

Proof. By restriction in the commutative diagram of theorem 25, we obtain the following commutative diagram:

$$\begin{tikzcd} \mathcal{H}_{\mathcal{SPP}} \arrow[r, "\Theta"] \arrow[d, "\Upsilon"'] & \mathbf{FQSym} \\ \mathcal{H}_{\mathcal{HOF}} \arrow[ur, "\Theta"'] & \end{tikzcd}$$

As the two restrictions of Θ are isomorphisms of graded Hopf algebras and respect the pairing, so is $\Upsilon_{|\mathcal{H}_{\mathcal{SPP}}} = (\Theta_{|\mathcal{H}_{\mathcal{SPP}}})^{-1} \circ \Theta_{|\mathbf{FQSym}}$. As $\Upsilon_{|\mathcal{H}_{\mathcal{SPP}}}$ is an isometry and the pairing on $\mathcal{H}_{\mathcal{SPP}}$ is nondegenerate, the pairing on $\mathcal{H}_{\mathcal{HOF}}$ is nondegenerate. $\Box$

5 More algebraic structures on $\mathcal{H}_{SP}$

5.1 Recalls on *Dup-Dend* bialgebras

Recall that a duplicial algebra [12] is a triple $(A, ., \nwarrow)$, where A is a vector space and $., \nwarrow$ two products on A, with the following axioms: for all $x, y, z \in A$:

$$\left\{\begin{array}{rcl}(xy)z &=& x(yz),\\ (x\nwarrow y)\nwarrow z &=& x\nwarrow (y\nwarrow z),\\ (xy)\nwarrow z &=& x(y\nwarrow z).\end{array}\right. \tag{1}$$

A dendriform coalgebra (dual notion of dendriform algebra, [11, 15]) is a triple $(A, \Delta_\prec, \Delta_\succ)$, where $\Delta_\prec$ and $\Delta_\succ$ are two coproducts on A, with the following axioms: for all $x \in A$,

$$\left\{\begin{array}{rcl}(\Delta_\prec \otimes Id)\circ \Delta_\prec(x) &=& (Id\otimes \tilde{\Delta})\circ \Delta_\prec(x),\\ (\Delta_\succ \otimes Id)\circ \Delta_\prec(x) &=& (Id\otimes \Delta_\prec)\circ \Delta_\succ(x),\\ (\tilde{\Delta} \otimes Id)\circ \Delta_\succ(x) &=& (Id\otimes \Delta_\succ)\circ \Delta_\succ(x).\end{array}\right. \tag{2}$$

Note that these axioms imply that $\tilde{\Delta} = \Delta_\prec + \Delta_\succ$ is coassociative. We shall use the following Sweedler notations: for any $a \in A$,

$$\tilde{\Delta}(a) = a' \otimes a'', \qquad \Delta_\prec(a) = a'_\prec \otimes a''_\prec, \qquad \Delta_\succ(a) = a'_\succ \otimes a''_\succ.$$

A *Dup-Dend* bialgebra [8] is a family $(A, ., \nwarrow, \Delta_\prec, \Delta_\succ)$, where A is a vector space, $., \nwarrow : A \otimes A \longrightarrow A$ and $\Delta_\prec, \Delta_\succ : A \longrightarrow A \otimes A$, with the following properties:

- $(A, ., \nwarrow)$ is a duplicial algebra (axioms 1).
- $(A, \Delta_\prec, \Delta_\succ)$ is a dendriform coalgebra (axioms 2).
- For all $x, y \in A$:

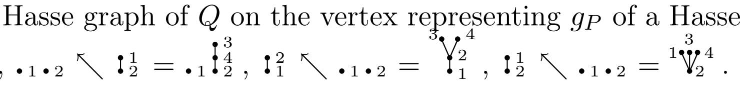
$$\left\{\begin{array}{rcl}\Delta_\prec(xy) &=& y\otimes x + y'_\prec \otimes xy''_\prec + xy'_\prec \otimes y''_\prec + x'y\otimes x'' + x'y'_\prec \otimes x''y''_\prec,\\ \Delta_\succ(xy) &=& x\otimes y + xy'_\succ \otimes y''_\succ + y'_\succ \otimes xy''_\succ + x'\otimes x''y + x'y'_\succ \otimes x''y''_\succ;\\ \Delta_\prec(x\nwarrow y) &=& x\nwarrow y'_\prec \otimes y''_\prec + x'_\prec \nwarrow y\otimes x''_\prec + x'_\prec \nwarrow y'_\prec \otimes x''_\prec y''_\prec,\\ \Delta_\succ(x\nwarrow y) &=& x\otimes y + x\nwarrow y'_\succ \otimes y''_\succ + x'_\succ \otimes x''_\succ \nwarrow y\\ && +x'_\prec \otimes x''_\prec y + x'_\prec \nwarrow y'_\succ \otimes x''_\prec y''_\succ.\end{array}\right. \tag{3}$$

5.2 Another product on $\mathcal{H}_{\mathcal{SP}}$

Definition 27

1. Let $P = (P, \leq_1, \leq_2)$ be a nonempty special poset. The maximal element of $(P, \leq_2)$ will be denoted by g_P.
2. Let P and Q be two nonempty special poset. We define $P \nwarrow Q$ by:

- $P \nwarrow Q = P \sqcup Q$ as a set, and P, Q are special subposets of $P \nwarrow Q$.
- For all $x \in P$, $y \in Q$, $x \leq_2 y$.
- For all $x \in P$, $y \in Q$, $x \leq_1 y$ if, and only if, $x \leq_1 g_P$.

Remark. Let P and Q be two nonempty special posets. A Hasse graph of $P \nwarrow Q$ is obtained by grafting a Hasse graph of Q on the vertex representing g_P of a Hasse graph of P. For example, 

Lemma 28 *$(\mathcal{H}^+_{\mathcal{SP}}, ., \nwarrow)$ is a duplicial algebra.*

Proof. Let P, Q, R be three nonempty special posets. The special posets $(P \nwarrow Q) \nwarrow R$ and $P \nwarrow (Q \nwarrow R)$ are both characterized by:

- $S = P \sqcup Q \sqcup R$ as a set, and P, Q, R are special subposets of S.
- For all $x \in P$, $y \in Q$, $z \in R$, $x \leq_2 y \leq_2 z$.
- For all $x \in P$, $y \in Q$, $z \in R$, $x \leq_1 y$ if, and only if, $x \leq g_P$; $x \leq_1 z$ if, and only if, $x \leq_1 g_P$; $y \leq_1 z$ if, and only if, $y \leq_1 g_Q$.

The last point comes from the fact that $g_{R \nwarrow S} = g_S$ for any non-empty special posets R and S. So they are equal.

The special posets $(PQ) \nwarrow R$ and $P(Q \nwarrow R)$ are both characterized by:

- $S = P \sqcup Q \sqcup R$ as a set, and P, Q, R are special suposets of S.
- For all $x \in P$, $y \in Q$, $z \in R$, $x \leq_2 y \leq_2 z$.
- For all $x \in P$, $y \in Q$, $z \in R$, x and y are not comparable for $\leq_1$; x and z are not comparable for $\leq_1$; $y \leq_1 z$ if, and only if, $y \leq_1 g_Q$.

So $\mathcal{H}^+_{\mathcal{SP}}$ is a duplicial algebra. $\square$

Proposition 29 *Let P, Q be two nonempty special posets. Then $P \nwarrow Q \in \mathcal{HOP}$ (respectively $\mathcal{OF}$, $\mathcal{SPP}$, $\mathcal{HOF}$, $\mathcal{SPF}$, $\mathcal{SWNP}$) if, and only if, $P, Q \in \mathcal{HOP}$ (respectively $\mathcal{OF}$, $\mathcal{SPP}$, $\mathcal{HOF}$, $\mathcal{SPF}$, $\mathcal{SWNP}$).*

Proof. We put $R = P \nwarrow Q$.

$\Longleftarrow$. In all the cases, this comes from the fact that P and Q are double subposets of $P \nwarrow Q$.

$\mathcal{HOP}$. $\Longrightarrow$. Recall that $R \in \mathcal{HOP}$ if, and only if, R does not contain a double subposet isomorphic to $\mathfrak{l}^1_2$ (proposition 9). Let us assume that $P \nwarrow Q$ is not a heap-ordered poset. Then it contains two distinct elements a, b, such that $a \leq_1 b$ and $b \leq_2 a$. If $a \in P$, then, by definition of $\leq_2$ on R, $b \in P$, so P is not a heap-ordered poset. If $a \in Q$, as $b \leq_1 a$, by definition of $\leq_1$ on R, $b \in Q$, so Q is not a heap-ordered poset.

$\mathcal{OF}$. $\Longrightarrow$. Recall that R is an ordered forest if, and only if, $(R, \leq_1)$ does not contain a double subposet isomorphic to Λ (see lemma 13 in [7]). Let us assume that R is not an ordered forest. Then it contains three different elements a, b, c, with $a \leq_2 b \leq_2 c$, such that one of the following assertions holds:

1. $b, c \leq_1 a$ and b, c are not comparable for $\leq_1$: $(\{a, b, c\}, \leq_1) = {}_b\Lambda^a{}_c$.
2. $a, c \leq_1 b$ and a, c are not comparable for $\leq_1$: $(\{a, b, c\}, \leq_1) = {}_a\Lambda^b{}_c$.
3. $a, b \leq_1 c$ and a, b are not comparable for $\leq_1$: $(\{a, b, c\}, \leq_1) = {}_a\Lambda^c{}_b$.

In the three cases, if the maximal element of $\{a, b, c\}$ for $\leq_1$ is in P, then, by definition of $\leq_1$ on R, $a, b, c \in P$, so P is not an ordered forest. Let us assume that this element is in Q. In the first case, then, by definition of $\leq_2$ on R, $b, c \in Q$, so Q is not an ordered forest. In the second case, we deduce similarly that $c \in Q$. If $a \in P$, then $a \leq_1 g_P$ in P as $a \leq_1 b$ in R, so $a \leq_1 c$ in R: contradiction, so $a \in Q$. As a consequence, Q is not an ordered forest. In the last case, then:

- If $a \in P$, $b \in Q$, then $a \leq_1 g_P$ in P as $a \leq_1 c$ in R, so $a \leq_1 b$ in R: contradiction, this case is impossible.
- Similarly, $a \in Q$, $b \in P$ is impossible.

So $a, b \in P$ or $a, b \in Q$. In the first subcase, $a, b \leq_1 g_P$ in P as $a, b \leq_1 c$ in R, so $\{a, b, g_P\}$ is a subposet of $(P, \leq_1)$ isomorphic to Λ: P is not an ordered forest. In the second subcase, Q contains a, b, c, so is not an ordered forest.

$\mathcal{SPP}$. $\Longrightarrow$. Let us recall that R is a plane poset if, and only if, it is heap-ordered and does not contain a double subposet isomorphic to $\mathtt{I}^3_1 \bullet_2$ (proposition 9). Let us assume that R is not a plane poset. If it is not heap-ordered, by the first point P or Q is not heap-ordered, so is not a plane poset. Let us assume that there exists three different elements a, b, c of R, such that $a \leq_2 b \leq_2 c$, $a \leq_1 c$, a, b and b, c are not comparable for $\leq_1$. By definition of $\leq_2$ on R, if $c \in P$, then $a, b \in P$, so $P \notin \mathcal{SPP}$. If $c \in Q$ and $a \in Q$, then $b \in Q$ as $a \leq_2 b$, so $Q \notin \mathcal{SPP}$. If $c \in Q$ and $a \in P$, then $a \leq_1 g_P$ in P. As a and b are not comparable for $\leq_1$ in R, $b \in P$. As b, c are not comparable for $\leq_1$ in R, b and g_P are not comparable for $\leq_1$ in P. Let us consider $\{a, b, g_P\} \subseteq P$. By definition of g_P, $a \leq_2 b \leq_2 g_P$, so $\{a, b, g_P\} = \mathtt{I}^3_1 \bullet_2$, so P is not plane.

$\mathcal{HOF}$. $\Longrightarrow$. Comes from $\mathcal{HOF} = \mathcal{OF} \cap \mathcal{HOP}$.

$\mathcal{SPF}$. $\Longrightarrow$. Comes from $\mathcal{SPF} = \mathcal{OF} \cap \mathcal{SPP}$.

$\mathcal{SWNP}$. $\Longrightarrow$. Let us assume that $P \nwarrow Q$ is not a WN poset. If it is not plane, then by the third point, P or Q is not plane, so is not WN. Let us assume that $P \nwarrow Q$ is plane (so P and Q are plane). Then $P \nwarrow Q$ contains a subposet $\{a, b, c, d\}$ isomorphic to  or . We assume that $a <_2 b <_2 c <_2 d$ in $P \nwarrow Q$. If $d \in P$, then by definition of $P \nwarrow Q$, $\{a, b, c, d\} \subseteq P$, so P is not WN. Similarly, if $a \in Q$, Q is not WN. We now assume that $a \in P$ and $d \in Q$.

- If $\{a, b, c, d\} =$ : as a and d are not comparable for $\leq_1$ in $P \nwarrow Q$, we do not have $a \leq_1 g_P$ in P. As P is plane, it is heap-ordered, so a and g_P are not comparable for $\leq_1$ in P. As $a <_1 c$ in $P \nwarrow Q$, necessarily $c \in P$. As $b <_2 c$ in $P \nwarrow Q$, $b \in P$. Moreover, as $b <_1 d$, $b <_1 g_P$. As c and d are not comparable for $\leq_1$ in $P \nwarrow Q$, c and g_P are not comparable for $\leq_1$ in P. So $\{a, b, c, g_P\} =$ .

- If $\{a, b, c, d\} =$ : as $a <_1 d$ in $P \nwarrow Q$, $a <_1 g_P$ in P. As a and b are not comparable for $\leq_1$ in $P \nwarrow Q$, necessarily $b \in P$. As $b <_1 d$, $b <_1 g_P$. As c and b are not comparable for $\leq_1$ in $P \nwarrow Q$, $c \in P$. As c and d are not comparable for $\leq_1$, c and g_P are not comparable for $\leq_1$ in P. So $\{a, b, c, g_P\} =$ .

In both cases, P is not WN. $\square$

Remarks.

1. As a consequence, the augmentation ideals $\mathcal{H}^+_{\mathcal{SP}}$, $\mathcal{H}^+_{\mathcal{HOP}}$, $\mathcal{H}^+_{\mathcal{SPP}}$, $\mathcal{H}^+_{\mathcal{OF}}$, $\mathcal{H}^+_{\mathcal{HOF}}$, $\mathcal{H}^+_{\mathcal{SWNP}}$ and $\mathcal{H}^+_{\mathcal{SPF}}$ are duplicial algebras.
2. It is proved in [8] that $\mathcal{H}^+_{\mathcal{SPF}}$ is the free duplicial algebra generated by $\bullet$: it is enough to observe that for any plane forest F, g_F is the leaf of F at most on the right, so $\nwarrow$, when restricted to plane forests, is precisely the product $\nwarrow$ defined in [8].

5.3 Dendriform coproducts on $\mathcal{H}_{\mathcal{SP}}$

For any nonempty special poset P, we put:

$$\Delta_{\prec}(P) = \sum_{\substack{I \text{ non trivial ideal of } P \\ g_P \notin I}} P \setminus I \otimes I, \qquad \Delta_{\succ}(P) = \sum_{\substack{I \text{ non trivial ideal of } P \\ g_P \in I}} P \setminus I \otimes I.$$

Note that $\Delta_{\prec} + \Delta_{\succ} = \tilde{\Delta}$. Moreover, $\mathcal{H}^+_{\mathcal{SP}}$, $\mathcal{H}^+_{\mathcal{HOP}}$, $\mathcal{H}^+_{\mathcal{SPP}}$, $\mathcal{H}^+_{\mathcal{OF}}$, $\mathcal{H}^+_{\mathcal{HOF}}$, $\mathcal{H}^+_{\mathcal{SWNP}}$ and $\mathcal{H}^+_{\mathcal{SPF}}$ are stable under the coproducts $\Delta_{\prec}$ and $\Delta_{\succ}$.

Proposition 30 *$\mathcal{H}^+_{\mathcal{SP}}$ is a Dup-Dend bialgebra.*

Proof. The proof is similar to the proof of proposition 20 in [8]. Nevertheless, in order to help the reader, we give here a complete proof. Let us first prove that $(\mathcal{H}^+_{\mathcal{SP}}, \Delta_{\prec}, \Delta_{\succ})$ is a dendriform coalgebra. It is enough to prove (2) if $x = P$ is a nonempty special poset. We put, as $\tilde{\Delta}$ is coassociative, $(\tilde{\Delta}\otimes Id)\circ\tilde{\Delta}(P)=(Id\otimes\tilde{\Delta})\circ\tilde{\Delta}(P)=\sum P^{(1)}\otimes P^{(2)}\otimes P^{(3)}$, where $P^{(1)},P^{(2)},P^{(3)}$ are subposets of P. Then:

$$\left\{\begin{array}{rcccl}(\Delta_\prec\otimes Id)\circ\Delta_\prec(P)&=&(Id\otimes\tilde{\Delta})\circ\Delta_\prec(P)&=&\displaystyle\sum_{g_P\in P^{(1)}}P^{(1)}\otimes P^{(2)}\otimes P^{(3)},\\(\Delta_\succ\otimes Id)\circ\Delta_\prec(P)&=&(Id\otimes\Delta_\prec)\circ\Delta_\succ(P)&=&\displaystyle\sum_{g_P\in P^{(2)}}P^{(1)}\otimes P^{(2)}\otimes P^{(3)},\\(\tilde{\Delta}\otimes Id)\circ\Delta_\succ(P)&=&(Id\otimes\Delta_\succ)\circ\Delta_\succ(P)&=&\displaystyle\sum_{g_P\in P^{(3)}}P^{(1)}\otimes P^{(2)}\otimes P^{(3)}.\end{array}\right.$$

So $\mathcal{H}_{\mathcal{SP}}^+$ is a dendriform coalgebra.

Let us now prove axioms (3). It is enough prove these formulas if $x=P$, $y=Q$ are non-empty plane forests. Let I be a non trivial ideal of PQ or $P\nwarrow Q$. We put $I'=I\cap P$ and $I''=I\cap Q$. As I is non trivial, I' and I'' are not simultaneously empty and not simutaneously total.

Let us first compute $\Delta_\prec(PQ)$. We have to consider non trivial ideals I of PQ, such that $g_{PQ}\notin I$. As $g_{PQ}=g_Q$, $I''\neq Q$. So five case are possible.

- $I'=P$, $I''=\emptyset$: this gives the term $Q\otimes P$.
- $I'=P$, $I''\neq\emptyset,Q$: this gives the term $Q'_\prec\otimes PQ''_\prec$.
- $I'=\emptyset$, $I''\neq\emptyset,Q$: this gives the term $PQ'_\prec\otimes PQ''_\prec$.
- $I'\neq\emptyset,P$, $I''=\emptyset$: this gives the term $P'Q\otimes P''$.
- $I'\neq\emptyset,P$, $I''\neq\emptyset,Q$: this gives the term $P'Q'_\prec\otimes P''Q''_\prec$.

Let us compute $\Delta_\succ(PQ)$. We have to consider non trivial ideals I of PQ, such that $g_{PQ}\in I$. As $g_{PQ}=g_Q$, $I''\neq\emptyset$. So five cases are possible:

- $I'=\emptyset$, $I''=Q$: this gives the term $P\otimes Q$.
- $I'=\emptyset$, $I''\neq\emptyset,Q$: this gives the term $PQ'_\succ\otimes Q''_\succ$.
- $I'=P$, $I''\neq\emptyset,Q$; this gives the term $Q'_\succ\otimes PQ''_\succ$.
- $I'\neq\emptyset,P$, $I''=Q$: this gives the term $P'\otimes P''Q$.
- $I'\neq\emptyset,P$, $I''\neq\emptyset,Q$: this gives the term $P'Q'_\succ\otimes P''Q''_\succ$.

We now compute $\Delta_\prec(P\nwarrow Q)$. We have to consider non trivial ideals I of $P\nwarrow Q$, such that $g_{P\nwarrow Q}\notin I$. As $g_{P\nwarrow Q}=g_Q$, $I''\neq Q$. Moreover, if $g_P\in I$, then, as I is an ideal, $Q\subseteq I$ so $I''=Q$: impossible. So $g_P\notin I'$. So three cases are possible.

- $I'=\emptyset$, $I''\neq\emptyset,Q$; this gives the term $P\nwarrow Q'_\prec\otimes PQ''_\prec$.
- $I'\neq\emptyset,P$, $I''=\emptyset$: this gives the term $P'_\prec\nwarrow Q\otimes P''_\nwarrow$.

- $I' \neq \emptyset, P$, $I'' \neq \emptyset, Q$: this gives the term $P'_{\prec} \nwarrow Q'_{\prec} \otimes P''_{\prec} Q''_{\prec}$.

Finally, let us compute $\Delta_{\succ}(P \nwarrow Q)$. We have to consider non trivial ideals I of $P \nwarrow Q$, such that $g_{P \nwarrow Q} \in I$. As $g_{P \nwarrow Q} = g_Q$, $I'' \neq \emptyset$. Moreover, if $g_P \in I'$, as I is an ideal, $I'' = Q$. As I' and I'' are not simultaneously total, this implies that $I' \neq P$. So five cases are possible:

- $I' = \emptyset$, $I'' = Q$: this gives the term $P \otimes Q$.
- $I' = \emptyset$, $I'' \neq \emptyset, Q$: this gives the term $P \nwarrow Q'_{\succ} \otimes Q''_{\succ}$.
- $I' \neq \emptyset, P$, $g_P \in I'$: this gives the term $P'_{\succ} \otimes P''_{\succ} \nwarrow Q$.
- $I' \neq \emptyset, P$, $g_P \notin I'$, $I'' = Q$: this gives the term $P'_{\prec} \otimes P''_{\prec} Q$.
- $I' \neq \emptyset, P$, $g_P \notin I'$, $I'' \neq \emptyset, Q$: this gives the term $P'_{\prec} \nwarrow Q'_{\succ} \otimes P''_{\prec} Q''_{\succ}$.

So $\mathcal{H}^+_{\mathcal{SP}}$ is a *Dup-Dend* bialgebra. $\Box$

Remarks.

1. As a consequence, the augmentation ideals $\mathcal{H}^+_{\mathcal{SP}}$, $\mathcal{H}^+_{\mathcal{HOP}}$, $\mathcal{H}^+_{\mathcal{SPP}}$, $\mathcal{H}^+_{\mathcal{OF}}$, $\mathcal{H}^+_{\mathcal{HOF}}$, $\mathcal{H}^+_{\mathcal{SWNP}}$ and $\mathcal{H}^+_{\mathcal{SPF}}$ are *Dup-Dend* bialgebras.
2. The rigidity theorem of [8] implies that $\mathcal{H}_{\mathcal{SP}}$, $\mathcal{H}_{\mathcal{HOP}}$, $\mathcal{H}_{\mathcal{SPP}}$, $\mathcal{H}_{\mathcal{OF}}$, $\mathcal{H}_{\mathcal{HOF}}$, $\mathcal{H}_{\mathcal{SWNP}}$ and $\mathcal{H}_{\mathcal{SPF}}$ are isomorphic to non commutative Connes-Kreimer Hopf algebras of decorated plane trees, with particular graded sets of decorations. The cardinal of the components of these graded sets can be computed by manipulations of formal series. For example:

n	1	2	3	4	5	6	7	8
$\lvert\mathcal{D}_{\mathcal{SP}}(n)\rvert$	1	1	10	148	3 336	112 376	5 591 196	406 621 996
$\lvert\mathcal{D}_{\mathcal{OF}}(n)\rvert$	1	1	7	66	786	11 278	189 391	3 648 711
$\lvert\mathcal{D}_{\mathcal{HOF}}(n)\rvert = \lvert\mathcal{D}_{\mathcal{SPF}}(n)\rvert$	1	0	1	6	39	284	2 305	20 682
$\lvert\mathcal{D}_{\mathcal{SWNP}}(n)\rvert$	1	0	1	4	17	76	353	1 688
$\lvert\mathcal{D}_{\mathcal{SPF}}(n)\rvert$	1	0	0	0	0	0	0	0

We obtain sequences A122705 for $\mathcal{D}_{\mathcal{OF}}$ and A122827 for $\mathcal{D}_{\mathcal{HOF}}$ in [19].

5.4 Application to FQSym

Let $\sigma \in \mathfrak{S}_n$ be a permutation ($n \geq 1$). We put:

$$\Delta_{\prec}(\sigma) = \sum_{k=\sigma^{-1}(n)}^{n-1} \sigma_1^{(k)} \otimes \sigma_2^{(k)}, \quad \Delta_{\succ}(\sigma) = \sum_{k=1}^{\sigma^{-1}(n)-1} \sigma_1^{(k)} \otimes \sigma_2^{(k)}.$$

Remark that $\Delta_{\prec} + \Delta_{\succ} = \tilde{\Delta}$.

Examples.

$\Delta_{\prec}((12543)) = (123)\otimes(21)+(1243)\otimes(1), \quad \Delta_{\succ}((12543) = (1)\otimes(1432)+(12)\otimes(321).$

Let σ, τ be two permutations of respective degrees k and l, with $k, l \geq 1$. We put:

$$\sigma \nwarrow \tau = \sum_{\substack{\zeta \in Sh(k,l) \\ \zeta(k+1) \geq \zeta(\sigma^{-1}(k))}} (\sigma \otimes \tau) \circ \zeta^{-1}.$$

In other terms, $\sigma \nwarrow \tau$ is the sum of the shufflings of the word representing σ and the word representing τ shifted by k, such that the letters of τ are all after the greatest letter of σ. In particular, if $\sigma^{-1}(k) = k$, then $\sigma \nwarrow \tau = \sigma \otimes \tau$.

Examples.

$$\begin{array}{rcl}
(123) \nwarrow (12) & = & (12345), \\
(132) \nwarrow (12) & = & (13245) + (13425) + (13452), \\
(312) \nwarrow (12) & = & (31245) + (31425) + (34125) + (34152) + (34512).
\end{array}$$

Proposition 31 *These products and coproducts make* $\mathbf{FQSym}^+$ *a Dup-Dend bialgebra. Moreover,* $\Theta : \mathcal{H}_{\mathcal{SP}} \longrightarrow \mathbf{FQSym}$ *is a morphism of Dup-Dend bialgebras.*

Proof. We first prove the compatibility of Θ with $\nwarrow$. Let P and Q be two non-empty special posets, of respective degrees k and l. We first show:

$$S_{P \nwarrow Q} = \bigsqcup_{\sigma \in S_P,\, \tau \in S_Q} \bigsqcup_{\substack{\zeta \in Sh(k,l) \\ \zeta(k+1) \geq \zeta(\sigma^{-1}(k))}} \{(\sigma \otimes \tau) \circ \zeta^{-1}\}.$$

$\subseteq$. Let $\chi \in S_{P \nwarrow Q}$. There exists a unique $(\sigma, \tau, \zeta) \in \Sigma_k \times \Sigma_l \times Sh(k,l)$, such that $\chi = (\sigma \otimes \tau) \circ \zeta^{-1}$. Let us prove that $\sigma \in S_P$. If $i >_1 j$ in P, then $i >_1 j$ in $P \nwarrow Q$, so:

$$\begin{array}{rcl}
\chi^{-1}(i) & \geq & \chi^{-1}(j), \\
\zeta \circ (\sigma^{-1} \otimes \tau^{-1})(i) & \geq & \zeta \circ (\sigma^{-1} \otimes \tau^{-1})(j), \\
\zeta \circ \sigma^{-1}(i) & \geq & \zeta \circ \sigma^{-1}(j), \\
\sigma^{-1}(i) & \geq & \sigma^{-1}(j),
\end{array}$$

as ζ is increasing on $\{1, \ldots, k\}$. So $\sigma \in S_P$. Similarly, $\tau \in S_Q$. Moreover, the element $\tau(1) + k$ belongs to Q in $P \nwarrow Q$, so $\tau(1) + k >_1 k$ in $P \nwarrow Q$. As a consequence:

$$\begin{array}{rcl}
\chi^{-1}(\tau(1) + k) & \geq & \chi^{-1}(k), \\
\zeta \circ (\sigma^{-1} \otimes \tau^{-1})(\tau(1) + k) & \geq & \zeta \circ (\sigma^{-1} \otimes \tau^{-1})(k), \\
\zeta(k+1) & \geq & \zeta \circ \sigma^{-1}(k).
\end{array}$$

$\supseteq$. Let $\sigma \in S_P$, $\tau \in S_Q$ and $\zeta \in Sh(k,l)$, such that $\zeta(k+1) \geq \zeta(\sigma^{-1}(k))$. We put $\chi = (\sigma \otimes \tau) \circ \zeta^{-1}$. Let i, j be two elements of $P \nwarrow Q$, such that $i >_1 j$. Three cases can occur:

- i, j are elements of P. Then $\sigma^{-1}(i) \geq \sigma^{-1}(j)$, so $(\sigma^{-1} \otimes \tau^{-1})(i) \geq (\sigma^{-1} \otimes \tau^{-1})(j)$, and finally $\sigma^{-1}(i) = \zeta \circ (\sigma^{-1} \otimes \tau^{-1})(i) \geq \zeta \circ (\sigma^{-1} \otimes \tau^{-1})(j) = \sigma^{-1}(j)$.

- i, j are elements of Q. The same proof holds.

- i is an element of Q and j is an element of P. Then $i >_1 k$ in $P \nwarrow Q$. By definition of $P \nwarrow Q$, $k >_1 j$ in P, so by the first point $\sigma^{-1}(k) \geq \sigma^{-1}(j)$.

  Moreover, $i+1 \geq k+1$, so $\sigma^{-1}(i) \geq \zeta(k+1)$ as ζ is increasing on $\{k+1, \ldots, k+l\}$. Then:

$$\sigma^{-1}(i) \geq \zeta(k+1) \geq \zeta(\sigma^{-1}(k)) = \sigma^{-1}(k) \geq \sigma^{-1}(j).$$

Finally, for any non-empty special posets P and Q of respective degrees k and l:

$$\Theta(\mathcal{P} \nwarrow Q) = \sum_{\sigma \in S_P,\, \tau \in S_Q} \sum_{\substack{\zeta \in Sh(k,l) \\ \zeta(k+1) \geq \zeta(\sigma^{-1}(k))}} (\sigma \otimes \tau) \circ \zeta^{-1} = \sum_{\sigma \in S_P,\, \tau \in S_Q} \sigma \nwarrow \tau = \Theta(P) \nwarrow \Theta(Q).$$

We now prove the compatibility of Θ and the two coproducts $\Delta_{\prec}$ and $\Delta_{\succ}$. Let $P \in \mathcal{SP}(n)$. As Θ is a morphism of Hopf algebras, there exists a bijection:

$$\left\{ \begin{array}{rcl} S_P \times \{1, \ldots, n-1\} & \longmapsto & \displaystyle\bigsqcup_{I \text{ non trivial ideal of } P} S_{P \setminus I} \times S_I \\ (\sigma, k) & \longmapsto & \left(\sigma_1^{(k)}, \sigma_2^{(k)}\right), \end{array} \right.$$

where this couple belongs to the term of the union indexed by $I = \{\sigma(k+1), \ldots, \sigma(n)\}$. So, if $(\sigma, k) \in S_F \times \{1, \ldots, n-1\}$, $k \geq \sigma^{-1}(n)$ if, and only if, $n = g_P$ is not an element of I. So:

$$(\Theta \otimes \Theta) \circ \Delta_{\prec}(F) = \sum_{g_P \notin I} P \setminus I \otimes I \sum_{\substack{\sigma \in S_{P \setminus I} \\ \tau \in S_I}} \sigma \otimes \tau = \sum_{\sigma \in S_P} \sum_{k=\sigma^{-1}(n)}^{n-1} \sigma_1^{(k)} \otimes \sigma_2^{(k)} = \sum_{\sigma \in S_P} \Delta_{\prec}(\sigma) = \tilde{\Delta} \circ \Theta(F).$$

Similarly, $(\Theta \otimes \Theta) \circ \Delta_{\succ} = \Delta_{\succ} \circ \Theta$.

As $\Theta_{|\mathcal{H}_{\mathcal{HOF}}} \longrightarrow$ **FQSym** is an isomorphism and $\mathcal{H}_{\mathcal{HOF}}^{+}$ is a *Dup-Dend* bialgebra, **FQSym**$^+$ is also a *Dup-Dend* bialgebra. $\Box$

Remarks.

1. It is of course possible to prove directly that **FQSym**$^+$ a *Dup-Dend* bialgebra.

2. A similar structure of *Dup-Dend* bialgebra structure exists on the Hopf algebra of parking functions**PQSym** [18], replacing, for a parking function σ, $\sigma^{-1}(n)$ by the maximal integer i such that $\sigma(i)$ is maximal.

6 Dendriform structures on $\mathcal{H}_{\mathcal{SPF}}$

The aim of this section is to prove that the restriction of the pairing to $\mathcal{H}_{\mathcal{SPF}}$ is nondegenerate (corollary 37). We first recall the classical result:

Lemma 32 *The restriction of $\langle -,- \rangle$ to $K[\bullet]$ is nondegenerate if, and only if, the characteristic of K is zero.*

Proof. As the homogeneous components of $K[\bullet]$ are one-dimensional, this restriction is non degenerate if, and only if, $\langle \bullet^n, \bullet^n \rangle$ is a non-zero element of K for all $n \in \mathbb{N}$. Moreover, it is not difficult to show that $\langle \bullet^n, \bullet^n \rangle = n!$. $\Box$

6.1 Dendriform coproducts

Notation. Let P be a plane poset, seen as a special poset. The smallest element for the total order of P will be denoted by s_P.

Proposition 33 *For any non-empty plane poset P, we put:*

$$\Delta'_{\prec}(P) = \sum_{\substack{I \text{ non trivial ideal of } P\\ s_P \notin I}} P \setminus I \otimes I, \qquad \Delta'_{\succ}(P) = \sum_{\substack{I \text{ non trivial ideal of } P\\ s_P \in I}} P \setminus I \otimes I.$$

Then $(\mathcal{H}^+_{\mathcal{SPP}}, \Delta'_{\prec}, \Delta'_{\succ})$ is a dendriform coalgebra. Moreover, for all $x, y \in \mathcal{H}^+_{\mathcal{SPP}}$:

$$\begin{aligned} \Delta'_{\prec}(xy) &= x \otimes y + x'_{\prec} y \otimes y''_{\prec} + x'_{\prec} \otimes x''_{\prec} y + xy' \otimes y'' + x'_{\prec} y' \otimes x''_{\prec} y'', \quad (4)\\ \Delta'_{\succ}(xy) &= y \otimes x + x'_{\succ} y \otimes x''_{\succ} + x'_{\succ} \otimes x''_{\succ} y + y' \otimes xy'' + x'_{\succ} y' \otimes x''_{\succ} y''. \quad (5)\end{aligned}$$

Proof. Let us first prove the (2) for all $x \in \mathcal{H}^+_{\mathcal{SPP}}$. It is enough to prove this if $x = P$ is a nonempty special poset. We put, as $\tilde{\Delta}$ is coassociative, $(\tilde{\Delta} \otimes Id) \circ \tilde{\Delta}(P) = (Id \otimes \tilde{\Delta}) \circ \tilde{\Delta}(P) = \sum P^{(1)} \otimes P^{(2)} \otimes P^{(3)}$, where $P^{(1)}, P^{(2)}, P^{(3)}$ are subposets of P. Then:

$$\left\{ \begin{aligned} (\Delta'_{\prec} \otimes Id) \circ \Delta'_{\prec}(P) &= (Id \otimes \tilde{\Delta}) \circ \Delta'_{\prec}(P) &= \sum_{s_P \in P^{(1)}} P^{(1)} \otimes P^{(2)} \otimes P^{(3)},\\ (\Delta'_{\succ} \otimes Id) \circ \Delta'_{\prec}(P) &= (Id \otimes \Delta'_{\prec}) \circ \Delta'_{\succ}(P) &= \sum_{s_P \in P^{(2)}} P^{(1)} \otimes P^{(2)} \otimes P^{(3)},\\ (\tilde{\Delta} \otimes Id) \circ \Delta'_{\succ}(P) &= (Id \otimes \Delta'_{\succ}) \circ \Delta'_{\succ}(P) &= \sum_{s_P \in P^{(3)}} P^{(1)} \otimes P^{(2)} \otimes P^{(3)}. \end{aligned} \right.$$

So $\mathcal{H}^+_{\mathcal{SP}}$ is a dendriform coalgebra.

It is enough prove formulas (4) and (5) if $x = P$, $y = Q$ are non-empty plane forests. Let I be a non trivial ideal of PQ. We put $I' = I \cap P$ and $I'' = I \cap Q$. As I is non trivial, I' and I'' are not simultaneously empty and not simutaneously total.

Let us first compute $\Delta'_{\prec}(PQ)$. We have to consider non trivial ideals I of PQ, such that $s_{PQ} \notin I$. As $s_{PQ} = s_P$, $I' \neq P$. So five case are possible.

- $I' = \emptyset$, $I'' = Q$: this gives the term $P \otimes Q$.
- $I' = \emptyset$, $I'' \neq \emptyset, Q$: this gives the term $PQ' \otimes Q''$.
- $I' \neq \emptyset, P$, $I'' = \emptyset$: this gives the term $P'_\prec Q \otimes P''_\prec$.
- $I' \neq \emptyset, P$, $I'' = Q$: this gives the term $P'_\prec \otimes P''_\prec Q$.
- $I' \neq \emptyset, P$, $I'' \neq \emptyset, Q$: this gives the term $P'_\prec Q' \otimes P''_\prec Q''$.

The proof of formula (5) is similar. $\Box$

Remarks.

1. In other words, $(\mathcal{H}^+_{\mathcal{SPP}}, .^{op}, (\Delta'_\succ)^{op}, (\Delta'_\prec)^{op})$ is a codendriform bialgebra in the sense of [4].
2. $\mathcal{H}^+_{\mathcal{SPF}}$ is clearly stable under both coproducts $\Delta'_\prec$ et $\Delta'_\succ$, so $(\mathcal{H}^+_{\mathcal{SPF}}, .^{op}, (\Delta'_\succ)^{op}, (\Delta'_\prec)^{op})$ is a codendriform subcoalgebra of $\mathcal{H}^+_{\mathcal{SPP}}$.

6.2 Dendriform products on $\mathcal{H}_{\mathcal{SPF}}$

From [5], $\mathcal{H}^+_{\mathcal{SPF}}$ is the free dendriform algebra generated by $\centerdot$. Moreover, for all non-empty plane forest F, $\centerdot \prec F = B^+(F)$, the rooted tree obtained by grafting the roots of F on a common root. It is also proved that $(\mathcal{H}^+_{\mathcal{SPF}}, \prec, \succ, \tilde{\Delta}^{op})$ is a dendriform Hopf algebra [14], so, for all $x, y \in \mathcal{H}^+_{\mathcal{SPF}}$:

$$\begin{aligned}
\tilde{\Delta}(x \prec y) &= x \otimes y + x \prec y' \otimes y'' + x' \otimes x''y + x' \prec y \otimes x'' + x' \prec y' \otimes x''y'', &(6)\\
\tilde{\Delta}(x \succ y) &= y \otimes x + x \succ y' \otimes y'' + y' \otimes xy'' + x' \succ y \otimes x'' + x' \succ y' \otimes x''y''. &(7)
\end{aligned}$$

Proposition 34 *For all $x, y \in \mathcal{H}^+_{\mathcal{SPF}}$:*

$$\begin{aligned}
\Delta'_\prec(x \prec y) &= x \otimes y + x \prec y' \otimes y'' + x'_\prec \otimes x''_\prec y + x'_\prec \prec y \otimes x''_\prec + x'_\prec \prec y' \otimes x''_\prec y'', &(8)\\
\Delta'_\succ(x \prec y) &= x'_\succ \otimes x''_\succ y + x'_\succ \prec y \otimes x''_\succ + x'_\succ \prec y' \otimes x''_\succ y'', &(9)\\
\Delta'_\prec(x \succ y) &= x'_\prec \succ y \otimes x''_\prec + x \succ y' \otimes y'' + x'_\prec \succ y' \otimes x''_\prec y'', &(10)\\
\Delta'_\succ(x \succ y) &= y \otimes x + y' \otimes xy'' + x'_\succ \succ y \otimes x''_\succ + x'_\succ \succ y' \otimes x''_\succ y''. &(11)
\end{aligned}$$

Proof. For fixed x, y, note that $(8)+(10) = (4)$, $(9)+(11) = (5)$, $(8)+(9) = (6)$, and $(10) + (11) = (7)$. As a consequence, for fixed x, y, (8), (9), (10) and (11) are equivalent.

We now prove (8)-(11) for x, y two non empty plane forest, by induction on the degree n of x. If $n = 1$, then $x = \centerdot$. Then:

$$\Delta'_\prec(x \prec y) = \centerdot \otimes y + B^+(y') \otimes y'' = x \otimes y + x \prec y' \otimes y''.$$

So (8) (hence, (9)-(11)) holds for $x = \centerdot$, as $\Delta'_\prec(x) = 0$. Let us assume the result at all rank $< n$. Two subcases occur.

- The plane forest x is a tree. Then there exists x_1 of degree $n-1$, such that $x = B^+(x_1) = x \prec x_1$. So $x \prec y = (\bullet \prec x_1) \prec y = \bullet \prec (x_1 y)$. So:

$$\begin{array}{rcl}
\Delta'_\prec(x \prec y) & = & \Delta'_\prec(\bullet \prec (x_1 y)) \\
& = & \bullet \otimes (x_1 y) + B^+((x_1 y)') \otimes (x_1 y)'' \\
& = & \bullet \otimes (x_1 y) + \bullet \prec x_1 \otimes y + \bullet \prec y \otimes x_1 + \bullet \prec (x_1' y) \otimes x_1'' + \bullet \prec x_1' \otimes x_1'' y \\
& & + \bullet \prec (x_1 y') \otimes y'' + \bullet \prec y' \otimes x_1 y'' + \bullet \prec (x_1' y') \otimes x_1'' y'' \\
& = & (\bullet \prec x_1 \otimes y) + (\bullet \prec (x_1 y') \otimes y'') + (\bullet \otimes (x_1 y) + \bullet \prec x_1' \otimes x_1'' y) \\
& & +(\bullet \prec y \otimes x_1 + \bullet \prec (x_1' y) \otimes x_1'') + (\bullet \prec y' \otimes x_1 y'' + \bullet \prec (x_1' y') \otimes x_1'' y'') \\
& = & x \otimes y + x \prec y' \otimes y'' + x'_\prec \otimes x_1'' y + x'_\prec \prec y \otimes x''_\prec + x'_\prec \prec y' \otimes x''_\prec y''.
\end{array}$$

- The plane forest x is not a tree. Then you can write $x = x_1 x_2$, such that the induction hypothesis holds for x_1 and x_2. Hence:

$$x \prec y = (x_1 \prec x_2) \prec y + (x_1 \succ x_2) \prec y = x_1 \prec (x_2 y) + x_1 \succ (x_2 \prec y).$$

  Applying (8) and (10) for x_1 (induction hypothesis), then (4) for x_2, then arranging the terms, you obtain (8) for x.

So the induction hypothesis holds for x in both cases. $\Box$

Remark. In other words, $(\mathcal{H}^+_{\mathcal{SPF}}, \succ^{op}, \prec^{op}, (\Delta'_\succ)^{op}, (\Delta'_\prec)^{op})$ is a bidendriform bialgebra in the sense of [6]. By the bidendriform rigidity theorem, it is a free dendriform algebra, and a cofree dendriform coalgebra. As a direct consequence:

Lemma 35 *As a dendriform algebra, $\mathcal{H}^+_{\mathcal{SPF}}$ is freely generated by $\bullet$. Moreover, the space $Prim_{tot}(\mathcal{H}^+_{\mathcal{SPF}}) = Ker(\Delta'_\prec) \cap Ker(\Delta'_\succ)$ is one-dimensional, generated by $\bullet$.*

Lemma 36 *For all $x, y, z \in \mathcal{H}^+_{\mathcal{SPF}}$:*

$$\langle x \prec y, z \rangle = \langle x \otimes y, \Delta'_\prec(y) \rangle \text{ and } \langle x \succ y, z \rangle = \langle x \otimes y, \Delta'_\succ(y) \rangle.$$

Proof. As $\langle -, - \rangle$ is a Hopf pairing, it is enough to prove one of these two formulas. Moreover, it is enough to prove it for x, y, z three non empty plane forests. We prove the first one, by induction on the degree n of x. If $n = 1$, then $x = \bullet$ and $x \prec y = B^+(y)$. Let $\sigma \in S(B^+(y), z)$. As 1 is the root of $B^+(y)$, for all j, $1 \leq_h j$ in $B^+(y)$. As $\sigma \in S(B^+(y), z)$, $\sigma(1) \leq \sigma(i)$ for all i, so $\sigma(1) = 1$. Let us denote by z_1 the plane forest obtained by deleting the vertex 1 of z; then $S(B^+(y), z)$ is in bijection by $S(y, z_1)$. Moreover, by definition of $\Delta'_\prec$:

$$\Delta'_\prec(z) = \bullet \otimes z_1 + \text{terms } z' \otimes z'',\ z' \text{ homogeneous of degree} \geq 2.$$

So, by homogeneity of the pairing:

$$\langle x \otimes y, \Delta'_\prec(z) \rangle = \langle \bullet, \bullet \rangle \langle y, z_1 \rangle + 0 = |S(y, z_1)| = |S(B^+(y), z)| = \langle x \prec y, z \rangle.$$

Let us assume the result at all rank $< n$. Two subcases occur.

- The plane forest x is a tree. Let us put $x = B^+(x_1) = \bullet \prec x_1$. Using the result at rank 1:

$$\begin{array}{rcl}\langle x \prec y, z\rangle & = & \langle \bullet \prec (x_1 y), z\rangle \\ & = & \langle \bullet \otimes x_1 y, \Delta'_{\prec}(z)\rangle \\ & = & \langle \bullet \otimes x_1 \otimes y, (Id \otimes \tilde{\Delta}) \circ \Delta'_{\prec}(z)\rangle \\ & = & \langle \bullet \otimes x_1 \otimes y, (\Delta'_{\prec} \otimes Id) \circ \Delta'_{\prec}(z)\rangle \\ & = & \langle \bullet \prec x_1, \Delta'_{\prec}(z)\rangle.\end{array}$$

- The plane forest x is not a tree. Then you can write $x = x_1 x_2$, such that the induction hypothesis holds for x_1 and x_2. Hence:

$$\begin{array}{rcl}\langle (x_1 x_2) \prec y, z\rangle & = & \langle x_1 \prec (x_2 y), z\rangle + \langle x_1 \succ (x_2 \prec y), z\rangle \\ & = & \langle x_1 \otimes x_2 \otimes y, (Id \otimes \tilde{\Delta}) \circ \Delta'_{\prec}(z)\rangle + \langle x_1 \otimes x_2 \otimes y, (Id \otimes \Delta'_{\prec}) \circ \Delta'_{\succ}(z)\rangle \\ & = & \langle x_1 \otimes x_2 \otimes y, (\Delta'_{\prec} \otimes Id) \circ \Delta'_{\prec}(z)\rangle + \langle x_1 \otimes x_2 \otimes y, (\Delta'_{\succ} \otimes Id) \circ \Delta'_{\prec}(z)\rangle \\ & = & \langle x_1 \prec x_2 \otimes y, \Delta'_{\prec}(z)\rangle + \langle x_1 \succ x_2 \otimes y, \Delta'_{\prec}(z)\rangle \\ & = & \langle x_1 x_2 \otimes y, \Delta'_{\prec}(z)\rangle.\end{array}$$

So the induction hypothesis holds for x in both cases. $\square$

Corollary 37 *The restriction of the pairing $\langle -, - \rangle$ to $\mathcal{H}_{\mathcal{SPF}}$ is nondegenerate.*

Proof. Let us assume it is degenerate. By lemma 36, its kernel I is a non trivial dendriform biideal of $\mathcal{H}^+_{\mathcal{SPF}}$. Any non-zero element of I of minimal degree is then in $Prim_{tot}(\mathcal{H}^+_{\mathcal{SPF}})$, as I is a dendriform coideal. By lemma 35, we obtain that $\bullet \in I$: absurd, as $\langle \bullet, \bullet \rangle = 1 \neq 0$. $\square$

7 Isometries between $\mathcal{H}_{\mathcal{PP}}$ and $\mathcal{H}_{\mathcal{SPP}}$

All the pairs of isomorphic Hopf algebras $\mathcal{H}_{\mathcal{PP}}$ and $\mathcal{H}_{\mathcal{SPP}}$, $\mathcal{H}_{\mathcal{WNP}}$ and $\mathcal{H}_{\mathcal{SWNP}}$, $\mathcal{H}_{\mathcal{PF}}$ and $\mathcal{H}_{\mathcal{SPF}}$ have Hopf pairings. The isomorphism between these Hopf algebras are not isometries: for example, $\langle \mathbf{I}, \mathbf{I} \rangle = 0$ whereas $\langle \mathbf{I}^{2}_{1}, \mathbf{I}^{2}_{1} \rangle = 1$. Our aim in this section is to answer the question if there is an isometric Hopf isomorphism between them. The answer is immediately no for $\mathcal{H}_{\mathcal{WNP}}$ and $\mathcal{H}_{\mathcal{SWNP}}$, as the first one is nondegenerate whereas the second is degenerate.

7.1 Isometric Hopf isomorphisms between free Hopf algebras

Proposition 38 *Let us assume that the characteristic of the base field is not 2. Let H and H' be two graded, connected Hopf algebras, both with a homogeneous, nondegenerate Hopf pairing, and both free. The following conditions are equivalent:*

1. *There exists a homogeneous, isometric Hopf algebra isomorphism between H and H'.*

2. For all $n \geq 0$, the spaces H_n and H'_n are isometric.

Proof. $1 \Longrightarrow 2$. Obvious.

$2 \Longrightarrow 1$. Let us fix for all $n \in \mathbb{N}^*$ a complement V_n of $(H^{+2})_n$ in H_n, where H^+ is the augmentation ideal of H. As H is free, the direct sum V of the V_n's freely generates H. Moreover, any subspaces of V generates a free subalgebra of H. In particular, the subalgebra $H_{\langle n \rangle}$ of H generated by $V_1 \oplus \ldots \oplus V_n$ is free. Moreover, it contains $H_0 \oplus \ldots \oplus H_n$, so for all $v \in V_0 \oplus \ldots \oplus V_n$, $\Delta(v) \in H_{\langle n \rangle} \otimes H_{\langle n \rangle}$. So $H_{\langle n \rangle}$ is a Hopf subalgebra of H. Finally, it is the algebra generated by $H_0 \oplus \ldots \oplus H_n$, so does not depend of the choice of V. We similarly define $H'_{\langle n \rangle}$ for all n.

We are going to construct for all $n \geq 0$ a Hopf algebra isomorphism $\phi_n : H_{\langle n \rangle} \longrightarrow H'_{\langle n \rangle}$ such that:

1. ϕ_n is homogeneous of degree 0.
2. For all $x, y \in H_{\langle n \rangle}$, $\langle \phi_n(x), \phi_n(y) \rangle = \langle x, y \rangle$.
3. ϕ_n restricted to $H_{\langle n-1 \rangle}$ is ϕ_{n-1} if $n \geq 1$.
4. For all $i \leq n$, $H'_i = (H'^{+2})_i \oplus \phi_n(V_i)$.

As $H_{\langle 0 \rangle} = H'_{\langle 0 \rangle} = K$, we define ϕ_0 by $\phi_0(1) = 1$. Let us assume that ϕ_{n-1} is defined. Then $H_n = (H^{+2})_n \oplus V_n = (H_{\langle n-1 \rangle})_n \oplus V_n$. By the induction hypothesis, ϕ_{n-1} induces an isometry between $(H_{\langle n-1 \rangle})_n$ and $(H'_{\langle n-1 \rangle})_n = (H'^{+2})_n$. As H_n and H'_n are nondegenerate and isometric, by the Witt extension theorem, it can be extended into an isometry $\tilde{\phi}_{n-1} : H_n \longrightarrow H'_n$. As $H_{\langle n \rangle}$ is freely generated by $V_0 \oplus \ldots \oplus V_n$, we can define an algebra morphism $\phi_n : H_{\langle n \rangle} \longrightarrow H'_{\langle n \rangle}$ by $\phi_n(v) = \phi_{n-1}(v)$ if $v \in V_i$, $i \leq n-1$ and $\phi_n(v) = \tilde{\phi}_{n-1}(v)$ if $v \in V_n$. This algebra morphism immediately satisfies the points 3 and 4 of the induction, by construction of $\tilde{\phi}_{n-1}$, and also extends $\tilde{\phi}_{n-1}$. Moreover, by the fourth point, $\phi_n(V_1 \oplus \ldots \oplus V_n)$ freely generated $H'_{\langle n \rangle}$, so ϕ_n is an algebra isomorphism from $H_{\langle n \rangle}$ to $H'_{\langle n \rangle}$.

Let us prove that ϕ_n is a Hopf algebra isomorphism. Let $x \in H_k$, $k \leq n$. For all $y \in H_i$, $z \in H_j$, $i + j = k$, as ϕ_n extends both ϕ_{n-1} and $\tilde{\phi}_{n-1}$, its restriction in all degree $\leq n$ is an isometry, so:

$$\begin{aligned}
\langle \Delta \circ \phi_n(x), \phi_n(y) \otimes \phi_n(z) \rangle &= \langle \phi_n(x), \phi_n(y)\phi_n(z) \rangle \\
&= \langle \phi_n(x), \phi_n(yz) \rangle \\
&= \langle x, yz \rangle \\
&= \langle \Delta(x), y \otimes z \rangle \\
&= \langle (\phi_n \otimes \phi_n) \circ \Delta(x), \phi_n(y) \otimes \phi_n(z) \rangle.
\end{aligned}$$

As ϕ_n is surjective in degree $\leq n$, and by homogeneity of the pairing of H', we deduce that $(\phi_n \otimes \phi_n) \circ \Delta(x) - \Delta \circ \phi_n(x) \in (H' \otimes H')^{\perp} = (0)$, as the pairing of H' is nondegenerate. As $H_1 \oplus \ldots \oplus H_n$ generates $H_{\langle n \rangle}$, ϕ_n is a Hopf algebra morphism.

Finally, let us prove the second point of the induction. By homogeneity of the pairings of H and H', it is enough to prove it for x, y homogeneous of the same degree k. We proceed by induction on k. If $k \leq n$, we already noticed that ϕ_n is an isometry in degree k. Let us assume that the result is true at all rank $< k$, with $k > n$. As $(H_{\langle n \rangle})_k = ((H_{\langle n \rangle})^{+2})_k$, we can assume that $x = x_1 x_2$, with x_1, x_2 homogeneous of degree $< k$. Then, using the induction hypothesis on x_1 and x_2:

$$
\begin{array}{rcl}
\langle \phi_n(x), \phi_n(y) \rangle & = & \langle \phi_n(x_1)\phi_n(x_2), \phi_n(y) \rangle \\
& = & \langle \phi_n(x_1) \otimes \phi_n(x_2), \Delta \circ \phi_n(y) \rangle \\
& = & \langle \phi_n(x_1) \otimes \phi_n(x_2), (\phi_n \otimes \phi_n) \circ \Delta(y) \rangle \\
& = & \langle x_1 \otimes x_2, \Delta(y) \rangle \\
& = & \langle x, y \rangle.
\end{array}
$$

Conclusion. We define $\phi : H \longrightarrow H'$ by $\phi(x) = \phi_n(x)$ for all $x \in H_{\langle n \rangle}$. By the third point of the induction, this does not depend of the choice of n. Then ϕ is clearly an isometric, homogeneous Hopf algebra isomorphism. $\Box$

We can improve this result, in the following sense:

Proposition 39 *Let us assume that the characteristic of the base field is not* 2*. Let H and H' be two graded, connected Hopf algebras, both with a homogeneous, nondegenerate Hopf pairing, and both free. Let V and V' be subspaces of respectively H and H', W and W' graded subspaces of respectively V and V' generating Hopf subalgebras h and h' of H and H'. We assume that h is a non isotropic subspace of H. The following conditions are equivalent:*

1. *There exists a homogeneous, isometric Hopf algebra isomorphism ϕ between H and H', such that $\phi(h) = h'$.*
2. *For all $n \geq 0$, the spaces H_n and H'_n are isometric and the spaces h_n and h'_n are isometric.*

Proof. $1 \Longrightarrow 2$. Obvious.

$2 \Longrightarrow 1$. For all $n \geq 1$, let us choose a complement U_n of W_n in V_n.

By proposition 38, there exists an isometric, homogeneous Hopf algebra isomorphism $\psi : h \longrightarrow h'$. Let us construct inductively a Hopf algebra isomorphism $\phi_n : H_{\langle n \rangle} \longrightarrow H'_{\langle n \rangle}$, isometric, such that:

1. ϕ_n is homogeneous of degree 0.
2. For all $x, y \in H_{\langle n \rangle}$, $\langle \phi_n(x), \phi_n(y) \rangle = \langle x, y \rangle$.
3. ϕ_n restricted to $H_{\langle n-1 \rangle}$ is ϕ_{n-1} if $n \geq 1$.
4. $\phi_n(x) = \psi(x)$ for all $x \in h_{\langle n \rangle}$.
5. For all $i \leq n$, $H'_i = (H'^{+2})_i \oplus \psi(W_i) \oplus \phi_n(U_i)$.

As $H_{\langle 0 \rangle} = H'_{\langle 0 \rangle} = K$, we define ϕ_0 by $\phi_0(1) = 1$. Let us assume that ϕ_{n-1} is defined. Then $H_n = (H^{+2})_n \oplus W_n \oplus U_n = (H_{\langle n-1 \rangle})_n \oplus W_n \oplus U_n$. By the induction hypothesis, ϕ_{n-1} and ψ induces an isometry between $(H_{\langle n-1 \rangle})_n \oplus W_n$ and $(H'_{\langle n-1 \rangle})_n \oplus W'_n = (H'^{+2})_n \oplus W'_n$. As H_n and H'_n are nondegenerate and isometric, by the extension theorem of Witt, it can be extended into an isometry $\tilde{\phi}_{n-1} : H_n \longrightarrow H'_n$. As $H_{\langle n \rangle n}$ is freely generated by $V_0 \oplus \ldots \oplus V_n$, we can define an algebra morphism $\phi_n : H_{\langle n \rangle} \longrightarrow H'_{\langle n \rangle}$ by $\phi_n(v) = \phi_{n-1}(v)$ if $v \in V_i$, $i \leq n-1$ and $\phi_n(v) = \tilde{\phi}_{n-1}(v)$ if $v \in V_n$. This morphisms clearly satisfy the fourth point of the definition. The end of the proof is similar to the proof of proposition 38. $\square$

We apply these propositions with $H = \mathcal{H}_{\mathcal{PP}}$, $H' = \mathcal{H}_{\mathcal{SPP}}$, V beign the subspace generated by plane posets and V' being the subspace generated by special plane posets. If we take W the subspace of V generated by $\mathcal{WNP}$ and W' the subspace generated by $\mathcal{SWNP}$, we obtain the following results:

Lemma 40 *1. The following assertions are equivalent:*

(a) There exists a homogeneous, isometric Hopf algebra isomorphism between $\mathcal{H}_{\mathcal{PP}}$ and $\mathcal{H}_{\mathcal{SPP}}$.

(b) For all $n \geq 1$, $(\mathcal{H}_{PP})_n$ and $(\mathcal{H}_{\mathcal{SPP}})_n$ are isometric.

2. The following assertions are equivalent:

(a) There exists a homogeneous, isometric Hopf algebra isomorphism ϕ between $\mathcal{H}_{\mathcal{WNP}}$ and $\mathcal{H}_{\mathcal{SWNP}}$.

(b) For all $n \geq 1$, $(\mathcal{H}_{\mathcal{WNP}})_n$ and $(\mathcal{H}_{\mathcal{SWNP}})_n$ are isometric.

3. The following assertions are equivalent:

(a) There exists a homogeneous, isometric Hopf algebra isomorphism ϕ between $\mathcal{H}_{\mathcal{PP}}$ and $\mathcal{H}_{\mathcal{SPP}}$, such that $\phi(\mathcal{H}_{\mathcal{WNP}}) = \mathcal{H}_{\mathcal{SWNP}}$.

(b) For all $n \geq 1$, $(\mathcal{H}_{\mathcal{PP}})_n$ and $(\mathcal{H}_{\mathcal{SPP}})_n$, $(\mathcal{H}_{\mathcal{WNP}})_n$ and $(\mathcal{H}_{\mathcal{SWNP}})_n$ are isometric.

7.2 A lemma on symmetric, invertible integer matrices

Definition 41 Let $A, B \in M_n(\mathbb{Z})$. We shall say that A and B are *congruent* if there exists $P \in GL_n(\mathbb{Z})$ such that $A = PBP^T$.

Thinking of elementary operations on rows or columns as products by certain invertible matrices, it is clear that A and B are congruent if B is obtained from A by one of the following operations:

- Adding to the i-th row of A λ times the j-th row of A and adding to the i-th column of A λ times the j-th column of A (where $\lambda \in \mathbb{Z}$, $i \neq j$).
- Multiplying the i-th row and the i-th column of A by -1.

- Permuting the i-th and j-th rows of A and the i-th and j-th columns of A.

Moreover, as the determinant of any element of $GL_n(\mathbb{Z})$ is ± 1, if A and B are congruent, $det(A) = det(B)$.

Proposition 42 *Let $A \in GL_n(\mathbb{Z})$, symmetric. Then A is congruent to a matrix B, diagonal by blocks, with diagonal blocks equal to* (1), (-1) *or* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proof. We proceed by induction on n. If $n = 1$, as $det(A) = \pm 1$, then $A = (1)$ or (-1). Let us assume the result at all rank $< n$.

First step. Let us show that A is congruent to a symmetric matrix B, with only one non-zero coefficient on its first column. We proceed by induction on $k = |A_{1,1}| + \ldots + |A_{n,1}|$. If $k = 1$, we take $A = B$. Let us assume the result at all rank $< k$. If there is only one non-zero coefficient $A_{i,1}$, the result is obvious. Let us assume that $A_{i,1}$ and $A_{j,1}$ are non-zero, with $i \neq j$. Up to a change of notations, we assume that $|A_{i,1}| \leq |A_{j,1}|$. Two subcases can occur.

First subcase. $j \neq 1$. We put $A_{j,1} = qA_{i,1} + r$, with $|r| < |A_{i,1}|$. Adding $-qR_i$ to R_j and $-qC_i$ to C_j, we obtain a matrix A' congruent to A, such that $A'_{k,1} = A_{k,1}$ if $k \neq j$ and $|A'_{j,1}| = |r| < |A_{j,1}|$. So the induction hypothesis holds for A'. As the congruence is an equivalence, A is congruent to a matrix B with only one non-zero coefficient $B_{i,1}$.

Second subcase. $j = 1$. If $|A_{i,1}| = |A_{j,1}|$, we can permute i and j and we recover the first subcase. Let us assume that $|A_{i,1}| < |A_{j,1}|$. Let ϵ be the sign of $A_{i,1}A_{1,1}$. Adding $-\epsilon R_i$ to R_1 and $-\epsilon C_i$ to C_1, we obtain a matrix A' congruent to A, such that $A'_{k,1} = A_{k,1}$ if $k \neq 1$ and $|A'_{1,1}| = ||A_{1,1}| - 2|A_{i,1}||$. Moreover, as $|A_{i,1}| < |A_{1,1}|$, $-|A_{1,1}| < |A_{1,1}| - 2|A_{i,1}| < |A_{1,1}|$, so $|A'_{1,1}| < |A_{1,1}|$. So the induction hypothesis holds for A'. As the congruence is an equivalence, A is congruent to a matrix B with only one non-zero coefficient $B_{i,1}$.

Second step. Replacing A by B, we can now assume that only one $A_{i,1}$ is non-zero. Developing the determinant of A by the first column, we obtain that $A_{i,1}$ divides $det(A) = \pm 1$, so $A_{i,1} = \pm 1$. If $i \neq 1$, permuting R_i and R_2, C_i and C_2, we can assume that $i = 2$; moreover, if $A_{2,1} = -1$, multiplying the second row and the second column of A by -1, we can assume that $A_{2,1} = 1$. Finally, we obtain two subcases.

First subcase. The matrix A has the following form:

$$\begin{pmatrix} \pm 1 & 0 & \ldots & 0 \\ 0 & * & \ldots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \ldots & * \end{pmatrix}.$$

We conclude by applying the induction hypothesis on the $(n-1) \times (n-1)$ remaining block, which is in $GL_{n-1}(\mathbb{Z})$ and symmetric.

Second subcase. The matrix A has the following form:

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & d & * & \dots & * \\ 0 & * & * & \dots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & * & * & \dots & * \end{pmatrix}.$$

Adding $-A_{3,2}R_1$ to R_3 and $-A_{2,3}C_1$ to C_3 (recall that A is symmetric),..., $-A_{n,2}R_1$ to R_n and $-A_{2,n}C_1$ to C_n, we obtain a matrix B congruent to A of the form:

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & d & 0 & \dots & 0 \\ 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & * & \dots & * \end{pmatrix}.$$

Let us put $d = 2q + r$, with $r = 0$ or 1. Adding $-qR_1$ to R_2 and $-qC_1$ to C_2, A is congruent to a matrix of the form:

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & r & 0 & \dots & 0 \\ 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & * & \dots & * \end{pmatrix}.$$

If $r = 0$, we conclude by applying the induction hypothesis on the $(n-2) \times (n-2)$ remaining block, which is in $GL_{n-2}(\mathbb{Z})$ and symmetric. If $r = 1$, observe that:

$$\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So A is congruent to a matrix of the form:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & * & \dots & * \end{pmatrix}.$$

we conclude by applying the induction hypothesis on the $(n-2) \times (n-2)$ remaining block. □

Remark. The form of proposition 42 is not unique. For example, $P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ are elements of $GL_3(\mathbb{Z})$ and:

$$P \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} P^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; Q \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} Q^T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using these observations, it is possible to prove the following result (which will not be used in te sequel):

Theorem 43 *Let $A \in GL_n(\mathbb{Z})$, symmetric. Then A is congruent to a matrix B, diagonal by blocks, with diagonal blocks all equal to (± 1) or all equal to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Moreover, B is unique, up to the orders of the blocks.*

7.3 Existence of an isometry between $\mathcal{H}_{\mathcal{PP}}$ and $\mathcal{H}_{\mathcal{SPP}}$

We can now prove the following theorem:

Theorem 44 *The following conditions are equivalent:*

1. *There exists a homogeneous, isometric Hopf algebra isomorphism between $\mathcal{H}_{\mathcal{PP}}$ and $\mathcal{H}_{\mathcal{SPP}}$.*

2. *There exists a homogeneous, isometric Hopf algebra isomorphism ϕ between $\mathcal{H}_{\mathcal{WNP}}$ and $\mathcal{H}_{\mathcal{SWNP}}$.*

3. *There exists a homogeneous, isometric Hopf algebra isomorphism ϕ between $\mathcal{H}_{\mathcal{PP}}$ and $\mathcal{H}_{\mathcal{SPP}}$, such that $\phi(\mathcal{H}_{\mathcal{WNP}}) = \mathcal{H}_{\mathcal{SWNP}}$.*

4. *The characteristic of the base field K is not 2 and there exists $i \in K$ such that $i^2 = -1$.*

Proof. By lemma 40, the question is essentially to know whether $(\mathcal{H}_{PP})_n$ and $(\mathcal{H}_{\mathcal{SPP}})_n$, $(\mathcal{H}_{\mathcal{WNP}})_n$ and $(\mathcal{H}_{\mathcal{SWNP}})_n$ are isometric. More precisely, we are going to prove that the following conditions are equivalent:

1. For all $n \geq 1$, $(\mathcal{H}_{\mathcal{PP}})_n$ and $(\mathcal{H}_{\mathcal{SPP}})_n$ are isometric.

2. For all $n \geq 1$, $(\mathcal{H}_{\mathcal{WNP}})_n$ and $(\mathcal{H}_{\mathcal{SWNP}})_n$ are isometric.

3. The characteristic of the base field K is not 2 and there exists $i \in K$ such that $i^2 = -1$.

This will immediately imply theorem 44.

1 or 2 $\Longrightarrow$ 3. We choose $n=2$. In the basis $(\mathbf{I}, \bullet\bullet)$ of $(\mathcal{H}_{\mathcal{PP}})_2 = (\mathcal{H}_{\mathcal{WNP}})_2$, the matrix of the pairing is $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$. In the basis $(\mathbf{I}^2_1, \bullet_1\bullet_2)$ of $(\mathcal{H}_{\mathcal{SPP}})_2 = (\mathcal{H}_{\mathcal{SWNP}})_2$, the matrix of the pairing is $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Considering the determinants of both matrices, we obtain that 1 and -1 differ multiplicatively from a square of K, so -1 is a square of K. For all $x = a\mathbf{I} + b\bullet\bullet \in (\mathcal{H}_{\mathcal{PP}})_2$, $\langle x, x\rangle = 2(ab+b^2)$. As $(\mathcal{H}_{\mathcal{PP}})_2$ is isometric with $(\mathcal{H}_{\mathcal{SPP}})_2$, there exists $x \in (\mathcal{H}_{\mathcal{PP}})_2$, such that $\langle x, x\rangle = 1$. As a consequence, $char(K) \neq 2$.

3 $\Longrightarrow$ 1, 2. Let us consider $V = (\mathcal{H}_{PP})_n$, or $(\mathcal{H}_{\mathcal{SPP}})_n$, or $(\mathcal{H}_{\mathcal{WNP}})_n$, or $(\mathcal{H}_{\mathcal{SWNP}})_n$. In a convenient basis of double posets, the matrix of the pairing on V is a symmetric matrix A with integer coefficients, invertible over K, whenever K is. Its determinant is an integer, let us assume that it is not ± 1. Then it has a prime divisor p. Choosing a field K of characteristic p, this determinant is 0 in K, so A is not invertible over K: contradiction. So $A \in GL_n(\mathbb{Z})$. We apply proposition 42 to A. There exists $P \in GL_n(\mathbb{Z})$, such that $B = P^T AP$ is diagonal by blocks, with diagonal blocks equal to (1), (-1) or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Changing the basis of V according to P (which is invertible over K, as it is invertible over $\mathbb{Z}$), we obtain a basis $\mathcal{B}$ of V such that the matrix of the pairing in the basis $\mathcal{B}$ is B. Now, observe that:

$$(i)(-1)(i) = (1), \qquad \begin{pmatrix} \frac{i}{2} & -i \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So V has an orthogonal basis.

As a conclusion, $(\mathcal{H}_{\mathcal{PP}})_n$ and $(\mathcal{H}_{\mathcal{SPP}})_n$ have the same dimension and have both an orthogonal basis, so they are isometric. Similarly, $(\mathcal{H}_{\mathcal{WNP}})_n$ and $(\mathcal{H}_{\mathcal{SWNP}})_n$ are isometric. $\square$

Example. Let i be one of the two square roots of -1 in K. We define an isometry from $(\mathcal{H}_{\mathcal{PP}})_{\langle 2\rangle}$ to $(\mathcal{H}_{\mathcal{SPP}})_{\langle 2\rangle}$ by:

$$\begin{cases} \phi(\bullet) &= \bullet_1, \\ \phi(\mathbf{I}) &= i\mathbf{I}^2_1 + \dfrac{1+i}{2}\bullet_1\bullet_2. \end{cases}$$

Using direct computations, it is possible to extend ϕ from $(\mathcal{H}_{\mathcal{PP}})_{\langle 3\rangle}$ to $(\mathcal{H}_{\mathcal{SPP}})_{\langle 3\rangle}$ sending $(\mathcal{H}_{\mathcal{WNP}})_{\langle 3\rangle}$ to $(\mathcal{H}_{\mathcal{SWNP}})_{\langle 3\rangle}$ in four families of isometries parametrised by an element $x \in K$ by:

1.

$$
\begin{cases}
\phi_1(\ell_3) = \ell_{[1,2,3]} + (ix-i)\,\ell_{[1,2]}\bullet_3 + (-1-ix)\bullet_1\ell_{[2,3]} + \dfrac{1+i}{2}\bullet_1\bullet_2\bullet_3, \\[2ex]
\phi_1(\mathsf{V}) = (-1-i+3x)\,\ell_{[1,2,3]} - i\,\mathsf{V}_1^{2,3} + \dfrac{3ix^2-2ix}{2}\,\ell_{[1,2]}\bullet_3 \\
\qquad + \dfrac{-3ix^2+(-3+i)x+2+i}{2}\bullet_1\ell_{[2,3]} + x\bullet_1\bullet_2\bullet_3, \\[2ex]
\phi_1(\Lambda) = (-3x+2+2i)\,\ell_{[1,2,3]} - i\,{}_1\Lambda^3_2 + \dfrac{3ix^2-2ix}{2}\,\ell_{[1,2]}\bullet_3 \\
\qquad + \dfrac{3ix^2+(6-4i)x-4-2i}{2}\bullet_1\ell_{[2,3]} + (-x+1+i)\bullet_1\bullet_2\bullet_3.
\end{cases}
$$

2.

$$
\begin{cases}
\phi_2(\ell_3) = \ell_{[1,2,3]} + (ix-i)\,\ell_{[1,2]}\bullet_3 + (-1-ix)\bullet_1\ell_{[2,3]} + \dfrac{1+i}{2}\bullet_1\bullet_2\bullet_3, \\[2ex]
\phi_2(\mathsf{V}) = (-1-i+3x)\,\ell_{[1,2,3]} - i\,\mathsf{V}_1^{2,3} + \dfrac{3ix^2-2ix}{2}\,\ell_{[1,2]}\bullet_3 \\
\qquad + \dfrac{-3ix^2+(-3+i)x+2+i}{2}\bullet_1\ell_{[2,3]} + x\bullet_1\bullet_2\bullet_3, \\[2ex]
\phi_2(\Lambda) = (-3x+2)\,\ell_{[1,2,3]} + 2i\,\mathsf{V}_1^{2,3} + i\,{}_1\Lambda^3_2 + \dfrac{-3ix^2+4ix-6i}{2}\,\ell_{[1,2]}\bullet_3 \\
\qquad + \dfrac{3ix^2+(6-4i)x-4-2i}{2}\bullet_1\ell_{[2,3]} + (-x+1+i)\bullet_1\bullet_2\bullet_3.
\end{cases}
$$

3. If the characteristic of the base field is not 2, nor 3:

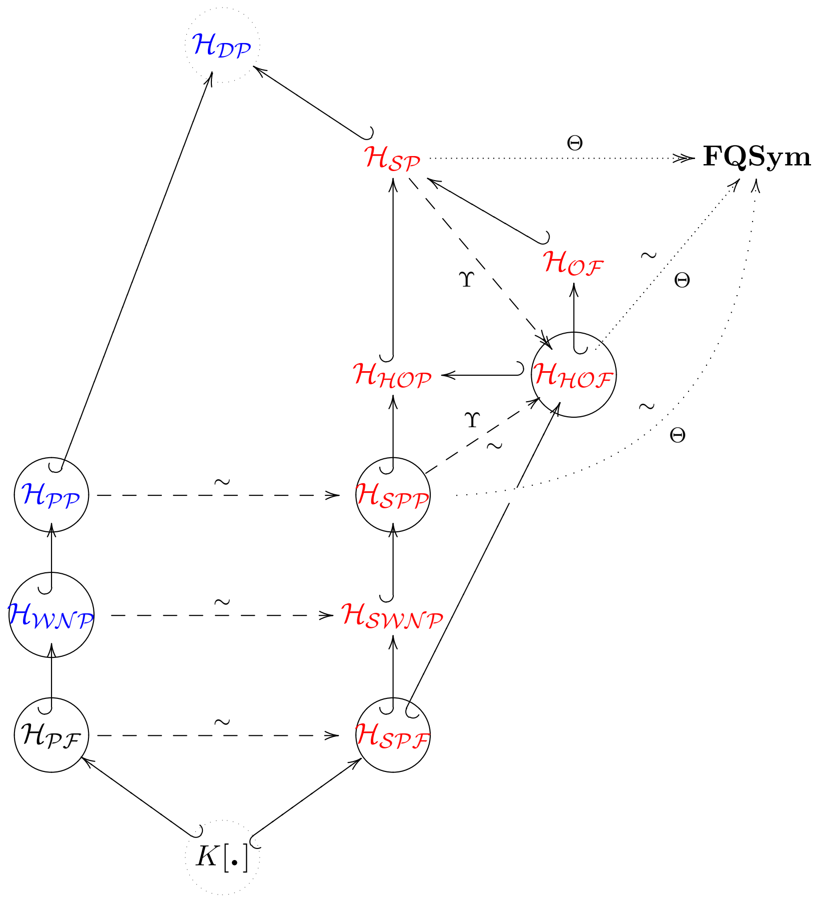
$$
\begin{cases}
\phi_3(\ell_3) = -\ell_{[1,2,3]} + \dfrac{-3ix-i}{3}\,\ell_{[1,2]}\bullet_3 + \dfrac{3ix-2i+3}{3}\bullet_1\ell_{[2,3]} + \dfrac{3i-1}{6}\bullet_1\bullet_2\bullet_3, \\[2ex]
\phi_3(\mathsf{V}) = (-1-i+3x)\,\ell_{[1,2,3]} - i\,\mathsf{V}_1^{2,3} + \dfrac{3ix^2-2ix}{2}\,\ell_{[1,2]}\bullet_3 \\
\qquad + \dfrac{-3ix^2+(-3+i)x+2+i}{2}\bullet_1\ell_{[2,3]} + x\bullet_1\bullet_2\bullet_3, \\[2ex]
\phi_3(\Lambda) = (-3x+2i)\,\ell_{[1,2,3]} - i\,{}_1\Lambda^3_2 + \dfrac{-9ix^2-2i}{6}\,\ell_{[1,2]}\bullet_3 \\
\qquad + \dfrac{9ix^2+18x-10i}{6}\bullet_1\ell_{[2,3]} + \dfrac{-3x+3i+1}{3}\bullet_1\bullet_2\bullet_3.
\end{cases}
$$

4. If the characteristic of the base field is not 2, nor 3:

$$\begin{cases} \phi_4(\mathtt{I\!I}) &= -\mathtt{I\!I}^{3}_{2,1} + \dfrac{-3ix-i}{3}\,\mathtt{I}^{2}_{1}\bullet_3 + \dfrac{3ix-2i+3}{3}\bullet_1\mathtt{I}^{3}_{2} + \dfrac{3i-1}{6}\bullet_1\bullet_2\bullet_3, \\ \phi_4(\mathtt{V}) &= (-1-i+3x)\mathtt{I\!I}^{3}_{2,1} - i\,{}^{2}\mathtt{V}_{1}{}^{3} + \dfrac{3ix^2-2ix}{2}\,\mathtt{I}^{2}_{1}\bullet_3 \\ & \quad + \dfrac{-3ix^2+(-3+i)x+2+i}{2}\bullet_1\mathtt{I}^{3}_{2} + x\bullet_1\bullet_2\bullet_3, \\ \phi_4(\Lambda) &= -3x\,\mathtt{I\!I}^{3}_{2,1} + 2i\,{}^{2}\mathtt{V}_{1}{}^{3} + i\,{}_{1}\Lambda^{3}{}_{2} + \dfrac{-9ix^2-14i}{6}\,\mathtt{I}^{2}_{1}\bullet_3 \\ & \quad + \dfrac{9ix^2+18x-10i}{6}\bullet_1\mathtt{I}^{3}_{2} + \dfrac{-3x+3i+1}{3}\bullet_1\bullet_2\bullet_3. \end{cases}$$

8 Conclusion

We finally obtain the following commuting diagram:



In blue, algebras stable under $\not\downarrow$ and ι (see definitions in [7]). In red, algebras stable under $\nwarrow$, $\Delta_{\prec}$ and $\Delta_{\succ}$. The algebras such that the restriction of the pairing $\langle -,- \rangle$ is nondegenerate are circled. If the circle is dotted, the result is true if, and only if, the characteristic of the base field is zero. The three horizontal dotted lines correspond to the isomorphisms sending $(P, \leq_h, \leq_r)$ to $(P, \leq_h, \leq)$. Moreover, it is not difficult to show that the intersection of two Hopf algebras of this diagram is given by the smallest common ancestor in the oriented graph formed by the black edges of this diagram. This lies on the fact the only plane posets $(P, \leq_h, \leq_r)$ which are special (recall that this means that $\leq_r$ is total) are the double posets $\cdot^n$, for all $n \geq 0$.

All the arrows of the diagram are isometries, at the exception of the three horizontal dotted lines. There exists isometric Hopf algebra isomorphisms between $\mathcal{H}_{\mathcal{PF}}$ and $\mathcal{H}_{\mathcal{SPF}}$, $\mathcal{H}_{\mathcal{PP}}$ and $\mathcal{H}_{\mathcal{SPP}}$, if, and only if, the characteristic of the base field K is not 2 and -1 is a square of K.

If the characteristic of K is zero, all these Hopf algebras are free, cofree, and self-dual.

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