NATURAL ENDOMORPHISMS OF SHUFFLE ALGEBRAS

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1. Introduction

Shuffles have a long history, starting with the probabilistic study of card shufflings in the first part of the 20th century by Borel, Hadamard, Poincaré and others. Their theory was revived in the 50's, for various reasons. In topology, the combinatorics of (non commutative) shuffle products was the key to the definition of topological products such as the ones existing on cochain algebras and the cohomology groups of topological spaces. Commutative shuffle products were the key to the study of the homology of abelian groups and commutative algebras. In combinatorics and for the theory of iterated integrals, commutative shuffle products played a key role resulting in the global picture of the modern theory of free Lie algebras given in C. Reutenauer's seminal *Free Lie algebras* [33].

The classical approach to shuffle algebras, as featured for example in Reutenauer's book, focussed on Lie theoretical properties, that is on the enveloping algebra structure of tensor algebras: the shuffle product arises naturally in this framework by dualizing the Hopf algebra structure of the tensor algebra and many properties of shuffles can be derived from that particular approach.

However, one can try to follow a different path, namely start directly from the combinatorics of shuffles, following the ideas originally developed by M.-P. Schützenberger [32]. A series of recent works by F. Chapoton, C. Malvenuto, C. Reutenauer, the second author of the present article, and others, provides many new tools to revisit the theory of shuffles. This is the purpose of the present article to put these tools to use.

Concretely, we focus on the adaptation to the study of shuffles of the main combinatorial tool in the theory of free Lie algebras, namely the existence of a universal algebra of endomorphisms for tensor and other cocommutative Hopf algebras: the family of Solomon's descent algebras of type A [33, 27]. We show that there exists similarly a natural endomorphism algebra for commutative shuffle algebras, which is a natural extension of the Malvenuto-Reutenauer Hopf algebra of permutations, or algebra of free quasi-symmetric functions. We study this new algebra for its own, establish freeness properties, study its generators, bases, and also feature its relations to the internal structure of shuffle algebras.

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2. Shuffles

As mentioned in the introduction, shuffle products can be understood in the commutative and noncommutative frameworks. The two uses still coexist (topological shuffles are noncommutative, whereas the use in combinatorics is to refer to shuffle products as the commutative ones of the theory of free Lie algebras). We survey briefly the historical foundations of the theory since it will appear later that the combinatorics of the objects that were first considered to study shuffle products (geometrical simplices and tensors) is closely related to the new algebraic structures to be introduced in the present paper.

According to [17], the algebraic theory of these products was first established in [12] together with the introduction of the notion of half-shuffles, which was to become the classical way to define recursively shuffle products. In view of later developments, the fundamental observation [12, Fla 5.7] is that the topological (cartesian) product of simplices $*$ decomposes into two half-products $\prec, \succ$:

$$x * y = x \prec y + x \succ y.$$

The associativity of the product follows then formally from the distributivity of left ($\prec$) and right ($\succ$) half-shuffles with respect to the $*$ product [12, Thm 5.2].

Whereas the topological product is associative but not commutative (the associativity holds automatically, but the commutativity holds only up to orientation and homotopy), Eilenberg and MacLane were the first to consider also the purely commutative case when dealing with the bar construction (a topological object which combinatorial structure is the one of the tensor algebra), see [12, Sect. 18] and [11]. These ideas were rediscovered independently by M.P. Schützenberger [32, 1-18], who also clarified the set of relations necessary to prove the associativity relation (as was realized later, his proof of the associativity relation does not require the commutativity asumption and, up to a rewriting, coincides in the end essentially with the one given by Eilenberg-MacLane in a topological framework). With our previous notation, the half-shuffles associativity relations read:

$$x \prec (y * z) = (x \prec y) \prec z; \tag{1}$$

$$(x * y) \succ z = x \succ (y \succ z); \tag{2}$$

$$(x \succ y) \prec z = x \succ (y \prec z). \tag{3}$$

In the commutative case, the commutativity property translates into $x \prec y = y \succ x$ and these relations simplify to

$$(a \prec b) \prec c = a \prec (b \prec c + c \prec b), \tag{4}$$

see [12, (18.7)], [32, (S0)], [16].

For simplicity, we stick from now on to the current terminology and call *shuffle algebra* a commutative shuffle algebra, that is an algebra with a non associative "half-product" $\prec$ satisfying the relation (4) and *dendriform algebra* a noncommutative shuffle algebra, that is an associative algebra with two half-products satisfying the associativity relations (1-3) (but not the commutativity relation $x \prec y = y \succ x$; the half-shuffles relations have also been attributed to Rota, see [16]). See also [4, 1, 14, 10, 23] for further general informations on the subject and applications of dendriform structures to various problems in algebra and combinatorics related to the ones we consider in the present article.

Shuffle algebras are sometimes refered to as Zinbiel algebras as a follow up of Cuvier's Jan. 1991 Thesis where Leibniz algebras were first introduced and studied (Zinbiel is the word Leibniz inverted, a successful joke suggested by the topologist J.M. Lemaire). Cuvier proved indeed that the cochain complex computing the homology of Leibniz algebras is the tensor algebra [6, 7] -from which one can deduce by standard procedures that the notions of Leibniz algebras and shuffle algebras are Koszul dual [15]. Since the original name "algèbres de shuffle" is better known and accepted we prefer to stick to the usual terminology.

The classical shuffle bialgebra over an alphabet fits into this picture. Let X be a graded, connected alphabet, that is to say $X = \bigcup\limits_{n \geq 1} X_n$. For all $x \in X_n$, we put $|x| = n$: this is the *weight* of x. Let $T(X)$ be the tensor algebra generated by X over $\mathbf{Q}$. For all $n \in \mathbb{N}$, let $T_n(X)$ be the subspace of $T(X)$ generated by the words $y_1...y_n,\ y_i \in X$ of length n, and $T^n(X)$ be the subspace generated by the words of weight n, the weight of a word being the sum of the weights of its letters: $|y_1...y_n| := |y_1| + ... + |y_n|$. The product in the tensor algebra (the concatenation product) is written $\cdot$:

$$y_1...y_n \cdot z_1...z_p := y_1...y_n z_1...z_p.$$

Definition 1. *The shuffle bialgebra* $Sh(X) = \bigoplus\limits_{n \in \mathbb{N}} Sh^n(X)$ *is the graded connected (i.e.* $Sh^0(X) = \mathbf{Q}$*) commutative Hopf algebra such that*

- *The component of degree* n *of* $Sh(X)$*,* $Sh^n(X)$ *is the linear span of the words of weight* n *over* X *(so that as vector spaces* $Sh^n(X) = T^n(X)$*). We write similarly* $Sh_n(X)$ *for the linear span of the words of length* n*;*
- *The product* $\sqcup\!\sqcup$ *is defined recursively as the sum of the two half-shuffle products* $\prec$, $\succ$*:*

$$y_1...y_n \prec z_1...z_p := y_1 \cdot (y_2...y_n \prec z_1...z_p)$$

with $\sqcup\!\sqcup = \prec + \succ$ *and* $y_1...y_n \succ z_1...z_p := z_1...z_p \prec y_1...y_n$.

- *The coalgebra structure is defined by the deconcatenation coproduct:*

$$\Delta(y_1...y_n) := \sum_{0\leq k\leq n} y_1...y_k \otimes y_{k+1}...y_n.$$

Recall that the notions of connected commutative Hopf algebra and connected commutative bialgebra are equivalent since a graded connected commutative bialgebra always has an antipode. The (graded) dual bialgebra of $Sh(X)$ is the tensor algebra $T(X)$ over X, we refer to [33] for details and proofs.

Equivalently, for all $x_1, \dots, x_{k+l} \in X$:

$$\begin{aligned} x_1 \dots x_k \prec x_{k+1} \dots x_{k+l} &= \sum_{\alpha\in Des_{\subseteq\{k\}},\, \alpha^{-1}(1)=1} x_{\alpha^{-1}(1)} \dots x_{\alpha^{-1}(k+l)}, \\ x_1 \dots x_k \succ x_{k+1} \dots x_{k+l} &= \sum_{\alpha\in Des_{\subseteq\{k\}},\, \alpha^{-1}(1)=k+1} x_{\alpha^{-1}(1)} \dots x_{\alpha^{-1}(k+l)}, \end{aligned}$$

where the α are permutations of $[k+l] = \{1, ..., k+l\}$.

The notation $\alpha \in Des_{\subseteq\{k\}}$ means that α has at most one descent in position k. Recall that a permutation σ of $[n]$ is said to have a descent in position $i < n$ if $\sigma(i) > \sigma(i+1)$. The descent set of σ, $desc(\alpha)$ is the set of all descents of α,

$$desc(\sigma) := \{i < n, \sigma(i) > \sigma(i+1)\}.$$

For $I \subset [n]$, we write $Des_I := \{\sigma, desc(\sigma) = I\}$ and $Des_{\subseteq I} := \{\sigma, desc(\sigma) \subseteq I\}$.

Proposition 2. *As a commutative algebra, $Sh(X)$ is the free algebra* over X for the relations (4).

The result goes back to [32], where the reader can also find a discussion of the role of the unit in shuffle algebras (there is a subtelty to make sense of the half-products with 1, however this problem is easily settled and doesn't need to be discussed here: we will only use the fact that 1 is a unit for $\sqcup\!\sqcup$ and, for half-shuffle products of 1 with words w use Schützenberger's conventions $w \prec 1 = w,\ 1 \prec w = 0$).

Whereas most studies focussed on shuffle algebras over sets, shuffle algebras over graded sets are equally important objects. Two classical examples are provided by the iterated bar construction (a key to the computation of the homology of $K(\Pi, n)$ spaces [11, 12]) and *mould calculus* , which focusses on problems such as the study and classification of differential equations by algebraic means [31, 20]. In the first framework, one constructs the shuffle algebra over a graded commutative algebra (this is actually one of the reasons for Eilenberg and MacLane works on shuffles), in the second case, the shuffle algebra over graded derivations (e.g., in dimension 1, the family of the $x^n\partial_x$ with degree $n-1$).

The remaining part of the present article is devoted to the internal study of $Sh(X)$, where X is a graded alphabet. We insist on the action of natural endomorphisms, mimicking what is known for the shuffle algebra over a non graded set. We also recover as a byproduct Chapoton's rigidity theorem showing that an abstract shuffle bialgebra (an abstract shuffle algebra with a suitable coproduct) can always be realized as the shuffle algebra over a graded set [4].

3. Graded permutations

We have mentioned the foundational connexion between shuffles and the geometry of simplices. This relationship can be encoded purely combinatorially by the existence of a noncommutative shuffle product on the direct sum of the symmetric group algebras, this is the "geometrical ring of the symmetric groups" of [24, p. 180], a construction that relates directly the classical geometrical approach to shuffle products with the combinatorial approach.

The direct sum of the symmetric group algebras **S** carries in fact a much richer structure than a mere dendriform product: Malvenuto and Reutenauer first showed that it carries actually a noncommutative noncocommutative Hopf algebra structure and proved that it generalizes naturally various fundamental algebraic structures in the theory of free Lie algebras such as Solomon's descent algebras or quasi-symmetric functions, two noncommutative generalizations of the ring of symmetric functions [18]. This Hopf algebra or Malvenuto-Reutenauer (MR) Hopf algebra can be furthermore realized as an algebra of generalized quasi-symmetric functions and is often refered to in the litterature as the Hopf algebra of free quasi-symmetric functions [8].

The MR Hopf algebra is closely related to various fundamental notions of noncommutative representation theory such as the descent algebra of type A or the algebra of quasi-symmetric functions. These later notions are known to generalize to other Coxeter groups than the symmetric groups and, up to a certain extent, to wreath-products of symmetric groups with cyclic groups [19, 2]. Colored permutations appear naturally in this framework (the finite set of colors corresponding to the elements of the cyclic groups) [22]. Some of our results generalize further these results from the case of finite cyclic groups to the integers.

These new Hopf algebras are typical examples of "combinatorial quantum groups" (graded Hopf algebras which are neither commutative nor cocommutative but can be naturally interpreted as a "group of symmetries", e.g. through the natural action of permutations on tensors) and have originated many studies. The one we will focus on and generalize is due to the second Author of the present article, who introduced the notion of bidendriform bialgebra and showed that the MR Hopf algebra carries such a structure

-with various consequences such as the proof of the Free Lie conjecture (according to which the primitive elements of the MR Hopf algebra form a free Lie algebra) [14]. These results, as we show now, generalize to graded permutations, which are a natural extension of the notions of permutations and colored permutations when studying shuffle algebras over graded sets.

Recall first that, as for any Hopf algebra, $End(Sh(X))$ carries an associative convolution product $\star$ defined by

$$(f \star g) = ⧢ \circ (f \otimes g) \circ \Delta.$$

We define two other products on $End(Sh(X))$ by

$$f \prec g = \prec \circ (f \otimes g) \circ \Delta$$

and

$$f \succ g = \succ \circ (f \otimes g) \circ \Delta.$$

As $\prec + \succ = ⧢$, $\star = \prec + \succ$.

Lemma 3. *$(End(Sh(X)), \prec, \succ)$ is a dendriform algebra.*

Proof. Let $f, g, h \in End(Sh(X))$. Then:

$$\begin{aligned}(f \prec g) \prec h &= \prec \circ (\prec \otimes Id) \circ (f \otimes g \otimes h) \circ (\Delta \otimes Id) \circ \Delta \\ &= \prec \circ (Id \otimes ⧢) \circ (f \otimes g \otimes h) \circ (Id \otimes \Delta) \circ \Delta \\ &= f \prec (g \star h).\end{aligned}$$

The two other axioms are proved in the same way. □

These products on $End(Sh(X))$ dualize to the tensor algebra, for this dual point of view we refer to [13], which contains various applications of dendriform structures to Hopf algebras of graphs and twisted Hopf algebras (Hopf algebras in the category of species).

The following Lemma, although a direct, straightforward, consequence of the recursive definition of the shuffle product on $Sh(X)$ and of the dendriform structure on $End(Sh(X))$ will prove very useful.

Lemma 4 (Rewriting Lemma)**.** *For any $y_1, ..., y_n$ in X, the word $y_1...y_n$ can be rewritten:*

$$y_1...y_n = y_1 \prec (y_2 \prec (... \prec (y_{n-1} \prec y_n)...)).$$

The proof is left as an exercice. The Lemma can be used to prove the Prop.2, see [32, 1.19].

Corollary 5. *If we write $\pi = \sum_{n \in \mathbb{N}^*} \pi_n$ the projection on $Sh_1(X)$ (where π_n is the projection on $Sh_1(X) \cap Sh^n(X)$) orthogonally to the $Sh_n(X)$, $n \neq 1$, we get:*

$$Id = \exp^{\prec}(\pi) := \sum_{n \in \mathbb{N}} \pi^{\prec n} = \sum_{n \in \mathbb{N}} \pi \prec (\pi \prec (...(\pi \prec \pi)...)),$$

where $\pi^{\prec 0}$ stands for the canonical projection on the scalars, $Sh_0(X)$, and $\pi^{\prec n} := \pi \prec \pi^{\prec n-1}$.

The series $\exp^{\prec}(\pi)$ is the "time-ordered exponential" of physicists; it is often called in analyis and physics the Picard series or Dyson-Chen series of π (see e.g. [3], where the link between these series and the Malvenuto-Reutenauer Hopf algebra is explained). Its structure was investigated recently in a series of articles, focussing mainly on the Magnus problem (find an expression for $\Omega := \log(\exp^{\prec}(\pi))$), see [9, 10]). We will be interested here in different issues, related to the meaning of the π_n and their products with respect to the internal structure of shuffle algebras. Notice that, by their very definition and due to the Rewriting Lemma 4, we have:

Lemma 6. *For any $n_1, ..., n_k \in \mathbb{N}^*$, $\pi_{n_1,...,n_k} := \pi_{n_1} \prec (\pi_{n_2} \prec ...(\pi_{n_{k-1}} \prec \pi_{n_k})...)$ is the canonical projection on the linear span of words $x_1...x_k$ with $|x_i| = n_i$. In particular, $\pi_{n_1,...,n_k} \circ \pi_{n_1,...,n_k} = \pi_{n_1,...,n_k}$, and the $\pi_{n_1,...,n_k}$ form a complete family (i.e. with total sum Id) of orthogonal idempotents in $End(Sh(X))$.*

We are now in the position to define and study the algebra of graded permutations.

Definition 7. *Let us fix $k \in \mathbb{N}$. Let $\sigma \in \mathfrak{S}_k$ and $d : [k] \longrightarrow \mathbb{N}^*$. We define a linear endomorphism of $Sh(X)$ by:*

$$\Phi_{(\sigma,d)} : \left\{ \begin{array}{lll} x_1 \dots x_l & \longrightarrow & x_{\sigma(1)} \dots x_{\sigma(l)} \textit{ if } k = l \textit{ and } |x_{\sigma(i)}| = d(i) \textit{ for all } i, \\ & \longrightarrow & 0 \textit{ if not.} \end{array} \right.$$

For example, $\pi_n = \Phi_{(id_n,n)}$.

Notations.

(1) We put $\mathcal{S} = \coprod_{k\geq 0} \mathfrak{S}_k \times Hom([k], \mathbb{N}^*)$, and $\mathbf{S} = Vect(\mathcal{S})$.

(2) Let $\sigma \in \mathfrak{S}_k$ and $d : [k] \longrightarrow \mathbb{N}^*$. We shall represent (σ, d) by the biword $\begin{pmatrix} \sigma(1) & \dots & \sigma(k) \\ d(1) & \dots & d(k) \end{pmatrix}$.

Lemma 8. *For all $(\sigma, d) \in \mathfrak{S}_k \times Hom([k], \mathbb{N}^*)$ and $(\tau, e) \in \mathfrak{S}_l \times Hom([l], \mathbb{N}^*)$,*

$$\Phi(\sigma, d) \circ \Phi(\tau, e) = \left\{ \begin{array}{l} \Phi(\tau \circ \sigma, d) \textit{ if } k = l \textit{ and } d = e \circ \sigma, \\ 0 \textit{ if not.} \end{array} \right.$$

Proof. Direct computation. □

Remark. If for all $n \geq 1$, X_n is infinite, it is not difficult to show that the linear extension $\Phi : \mathbf{S} \longrightarrow End(Sh(X))$ is injective. Hence, we can define an associative internal product on $\mathbf{S}$ by:

$$(\sigma, d) \circ (\tau, e) = \left\{ \begin{array}{l} (\tau \circ \sigma, d) \text{ if } k = l \text{ and } d = e \circ \tau, \\ 0 \text{ if not.} \end{array} \right.$$

We shall from now on identify $\mathbf{S}$ with a subspace of $End(Sh(X))$ via Φ.

Let us write p_n for the canonical projection on $Sh^n(X)$, so that $Id = \sum_n p_n$. A direct inspection shows that the p_n belong to $\mathbf{S}$:

Lemma 9. *For all $n \geq 0$:*

$$p_n = \sum_{k=1}^{n} \sum_{\substack{p:[k]\longrightarrow \mathbb{N}^* \\ p(1)+\ldots+p(k)=n}} (Id_k, p)$$

$$= \sum_{k=1}^{n} \sum_{d(1)+\ldots+d(k)=n} \begin{pmatrix} 1 & \ldots & k \\ d(1) & \ldots & d(k) \end{pmatrix}.$$

Notations.

1. Let $\sigma \in \mathfrak{S}_k$, $\tau \in \mathfrak{S}_l$. We define $\sigma \otimes \tau \in \mathfrak{S}_{k+l}$ by $(\sigma \otimes \tau)(i) = \sigma(i)$ if $1 \leq i \leq k$ and $(\sigma \otimes \tau)(i) = \tau(i-k)+k$ if $k+1 \leq i \leq k+l$.
2. Let $d : [k] \longrightarrow \mathbb{N}^*$ and $e : [l] \longrightarrow \mathbb{N}^*$. We define $d \otimes e : [k+l] \longrightarrow \mathbb{N}^*$ by $(d \otimes e)(i) = d(i)$ if $1 \leq i \leq k$ and $(d \otimes e)(i) = e(i-k)$ if $k+1 \leq i \leq k+l$.

Lemma 10. $\mathbf{S}$ *is a dendriform subalgebra of $End(Sh(X))$.*

Proof. Let (σ, d) and $(\tau, e) \in \mathbf{S}$. We assume that $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_l$. If $n \neq k+l$, then $((\sigma, d) \prec (\tau, e))(x_1 \ldots x_n) = 0$. If $n = k+l$, then:

$$\begin{aligned}
((\sigma,d) \prec (\tau,e))(x_1 \ldots x_{k+l}) &= (\sigma,d).(x_1 \ldots x_k) \prec (\tau,e).(x_{k+1} \ldots x_{k+l}) \\
&= \begin{cases} x_{\sigma\otimes\tau(1)} \ldots x_{\sigma\otimes\tau(k)} \prec x_{\sigma\otimes\tau(k+1)} \ldots x_{\sigma\otimes\tau(k+l)} \\ \qquad \text{if } |x_{\sigma\otimes\tau(i)}| = (d \otimes e)(i) \text{ for all } i, \\ 0 \text{ if not,} \end{cases} \\
&= \begin{cases} \displaystyle\sum_{\alpha \in Des_{\subseteq\{k\}},\, \alpha^{-1}(1)=1} x_{(\sigma\otimes\tau)\circ\alpha^{-1}(1)} \ldots x_{(\sigma\otimes\tau)\circ\alpha^{-1}(k+l)} \\ \qquad \text{if } |x_{(\sigma\otimes\tau)\circ\alpha^{-1}(i)}| = (d \otimes e)\circ\alpha^{-1}(i) \text{ for all } i, \\ 0 \text{ if not.} \end{cases}
\end{aligned}$$

Hence:

$$(\sigma,d) \prec (\tau,e) = \sum_{\alpha \in Des_{\subseteq\{k\}},\alpha^{-1}(1)=1} ((\sigma\otimes\tau)\circ\alpha^{-1}, (d\otimes e)\circ\alpha^{-1}). \tag{5}$$

Similarly:

$$(\sigma,d) \succ (\tau,e) = \sum_{\alpha \in Des_{\subseteq\{k\}},\alpha^{-1}(1)=k+1} ((\sigma\otimes\tau)\circ\alpha^{-1}, (d\otimes e)\circ\alpha^{-1}). \tag{6}$$

So $\mathbf{S}$ is a dendriform subalgebra of $End(Sh(X))$. □

Lemma 11. *The idempotents $\pi_{n_1,\ldots,n_k}$ belong to* $\mathbf{S}$ *and generate a commutative subalgebra thereof for the composition product.*

Proof. The second part of the proposition being a straigthforward consequence of the idempotency property, let us show that π_n belong to $\mathbf{S}$; since $\mathbf{S}$ is a dendriform subalgebra of $End(Sh(X))$, the Lemma will follow.

Since $\pi_1 = (id_1, 1)$, let us assume that $\pi_i \in \mathbf{S}$ for $i < n$. We get:

$$\pi_n = p_n - \sum_{i_1+\ldots+i_k=n,\ k>1} \pi_{i_1,\ldots,i_k}$$

and the Lemma follows by induction. □

Proposition 12. *For all $n \geq 1$, we have:*

$$\pi_n = \begin{pmatrix} 1 \\ n \end{pmatrix} = \sum_{k=1}^{n} (-1)^{k-1} \sum_{a_1+\ldots+a_k=n} p_{a_1} \prec (p_{a_2} \star \ldots \star p_{a_k}).$$

Proof. Indeed, we have, according to the Rewriting lemma (4): $p = \sum_{n\in\mathbb{N}} \pi^{\prec n}$ or, equivalently:

$$(1-\pi) \prec p = \pi^{\prec 0} = p_0$$

or: $\pi \prec p = \sum_{n\in\mathbb{N}^*} p_n$. Let us set $p^+ := \sum_{n\in\mathbb{N}^*} p_n$ and write z for the convolution inverse of p in $End(Sh(X))$ (recall that p_0 is the identity for the convolution product):

$$z = p^{-1} = \sum_{k\in\mathbb{N}} (-1)^k (p^+)^k,$$

we get (recall that according to our conventions, p_0 is a right unit for $\prec$):

$$\pi = \pi \prec p_0 = \pi \prec (p \star z) = (\pi \prec p) \prec z$$

where the third identity follows from the half-shuffle relations, so that:

$$\pi = (\sum_{n\in\mathbb{N}^*} p_n) \prec z = (\sum_{n\in\mathbb{N}^*} p_n) \prec \sum_{n\in\mathbb{N}} (-1)^n (p^+)^n,$$

from which the Proposition follows. □

Notice that the same argument would prove the following Lemma, useful to study Magnus formulas and Picard/Dyson-Chen series:

Lemma 13. *For any formally invertible series $q = 1 + \sum_{n\in\mathbb{N}^*} q_i$ in a dendriform algebra, we have:*

$$\mu = (\sum_{n\in\mathbb{N}^*} q_n) \prec \sum_{k\in\mathbb{N}} (-1)^k (\sum_{n\in\mathbb{N}^*} q_n)^k,$$

where μ is given by: $q = \exp^{\prec} \mu$.

Remark. From (5) and (6), we obtain that $(\sigma, d) \prec (\tau, e)$ is the sum of the shufflings of the biwords $\begin{pmatrix} \sigma(1) & \ldots & \sigma(k) \\ d(1) & \ldots & d(k) \end{pmatrix}$ and $\begin{pmatrix} \tau(1)+k & \ldots & \tau(l)+k \\ e(1) & \ldots & e(k) \end{pmatrix}$ such that the first biletter is $\begin{pmatrix} \sigma(1) \\ d(1) \end{pmatrix}$, and $(\sigma, d) \succ (\tau, e)$ is the sum of the shufflings of these two biwords such that the first biletter is $\begin{pmatrix} \tau(1)+k \\ e(1) \end{pmatrix}$.

For example:

$$\begin{aligned}
\begin{pmatrix} 1 & 2 \\ a & b \end{pmatrix} \prec \begin{pmatrix} 2 & 1 \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & 2 & 4 & 3 \\ a & b & c & d \end{pmatrix} + \begin{pmatrix} 1 & 4 & 2 & 3 \\ a & c & b & d \end{pmatrix} + \begin{pmatrix} 1 & 4 & 3 & 2 \\ a & c & d & b \end{pmatrix},\\
\begin{pmatrix} 1 & 2 \\ a & b \end{pmatrix} \succ \begin{pmatrix} 2 & 1 \\ c & d \end{pmatrix} &= \begin{pmatrix} 4 & 1 & 2 & 3 \\ c & a & b & d \end{pmatrix} + \begin{pmatrix} 4 & 1 & 3 & 2 \\ c & a & d & b \end{pmatrix} + \begin{pmatrix} 4 & 3 & 1 & 2 \\ c & d & a & b \end{pmatrix}.
\end{aligned}$$

4. Bidendriform structures on graded permutations

Definition 14. (1) *Let $w=(i_1,\ldots,i_k)$ be a word with letters in $\mathbb{N}^*$, all distinct. There exists a unique increasing bijection f from $\{i_1,\ldots,i_k\}$ into $[k]$. The* standardization *of w is $std(w)=(f(i_1),\ldots,f(i_k))$. It is an element of $\mathfrak{S}_k$.*

(2) *Let $\sigma\in\mathfrak{S}_k$ and $d:[k]\longrightarrow\mathbb{N}^*$. We put:*

$$\begin{aligned}
\Delta_\prec((\sigma,d)) &= \sum_{k=\sigma^{-1}(1)}^{n-1} \begin{pmatrix} std(\sigma(1),\ldots,\sigma(k)) \\ d(1),\ldots,d(k) \end{pmatrix} \otimes \begin{pmatrix} std(\sigma(k+1),\ldots,\sigma(n)) \\ d(k+1),\ldots,d(n) \end{pmatrix}\\
\Delta_\succ((\sigma,d)) &= \sum_{k=1}^{\sigma^{-1}(1)-1} \begin{pmatrix} std(\sigma(1),\ldots,\sigma(k)) \\ d(1),\ldots,d(k) \end{pmatrix} \otimes \begin{pmatrix} std(\sigma(k+1),\ldots,\sigma(n)) \\ d(k+1),\ldots,d(n) \end{pmatrix}
\end{aligned}$$

This defines two coproducts on the augmentation ideal $\mathbf{S}_+$ of the dendriform algebra $\mathbf{S}_+$.

Example. Let $a_1,a_2,a_3,a_4\in\mathbb{N}^*$.

$$\begin{aligned}
\Delta_\prec\left(\begin{pmatrix} 3 & 1 & 4 & 2 \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix}\right) &= \begin{pmatrix} 2 & 1 \\ a_1 & a_2 \end{pmatrix} \otimes \begin{pmatrix} 2 & 1 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 3 \\ a_1 & a_2 & a_3 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ a_4 \end{pmatrix},\\
\Delta_\succ\left(\begin{pmatrix} 3 & 1 & 4 & 2 \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix}\right) &= \begin{pmatrix} 1 \\ a_1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 & 2 \\ a_2 & a_3 & a_4 \end{pmatrix}.
\end{aligned}$$

In other words, the coproducts of a biword $\begin{pmatrix} \sigma(1) & \ldots & \sigma(n) \\ d(1) & \ldots & d(n) \end{pmatrix}$ are given by the cuts of the biword into two parts and the standardization of the first lines of the two parts of the biword; in $\Delta_\prec$, the biletter $\begin{pmatrix} 1 \\ d\circ\sigma^{-1}(1) \end{pmatrix}$ is in the left part and in $\Delta_\succ$, it is in the right part.

Notations. For all $x\in\mathbf{S}_+$, we put:

$$\Delta_\prec(x)=x'_\prec\otimes x''_\prec,\ \Delta_\succ(x)=x'_\succ\otimes x''_\succ,\ \tilde\Delta(x)=\Delta_\prec(x)+\Delta_\succ(x)=x'\otimes x''.$$

Proposition 15. *For all $x,y\in\mathbf{S}_+$:*

$$\begin{aligned}
\Delta_\prec(x\prec y) &= x'_\prec\prec y'\otimes x''_\prec\star y''+x\otimes y+x\prec y'\otimes y''+ \\
&\quad x'_\prec\otimes x''_\prec\star y+x'_\prec\prec y\otimes x''_\prec, \qquad (7)\\
\Delta_\succ(x\prec y) &= x'_\succ\prec y'\otimes x''_\prec\star y+x'_\succ\prec y\otimes x''_\succ+x'_\succ\otimes x''_\succ\star y, \qquad (8)
\end{aligned}$$

$$(9)\qquad \Delta_\prec(x\succ y) = x'_\prec \succ y'\otimes x''_\prec \star y'' + x'_\prec \succ y\otimes x''_\prec + x\succ y'\otimes y'',$$

$$(10)\qquad \Delta_\succ(x\succ y) = x'_\succ \succ y'\otimes x''_\succ \star y'' + y\otimes x + y'\otimes x\star y'' + x'_\succ \succ y\otimes x''_\succ.$$

Consequently, $(\mathbf{S},\succ^{op},\prec^{op},\Delta_\succ^{op},\Delta_\prec^{op})$ *is a bidendriform bialgebra.*

Proof. These identities are the axioms for dendriform bialgebras, as introduced in [14], to which we also refer for the structure results on dendriform bialgebras we will use further on. We restrict ourselves to the case where $x=(\sigma,d)$ and $y=(\tau,e)$ are two biwords. Then $\Delta_\prec(x\prec y)$ is obtained by taking all the shufflings of x and y such the first letter of the result is the first letter of x, then cutting these words after the letter $\begin{pmatrix} 1\\ d\circ\sigma^{-1}(1)\end{pmatrix}$. As a consequence, there are biletters of x in the left part of the result. Hence, five case are possible:

(1) There are letters of x and y in both parts: this gives the term $x'_\prec \prec y'\otimes x''_\prec \star y''$.
(2) There are letters of x in both parts, and all the letters of y are in the left part: this gives the term $x'_\prec \prec y\otimes x''_\prec$.
(3) There are letters of x in both parts, and all the letters of y are in the right part: this gives the term $x'_\prec \otimes x''_\prec \star y$.
(4) All the letters of x are in the left part, and there are letters of y in both parts: this gives the term $x\prec y'\otimes y''$.
(5) All the letters of x are in the left part, and all the letters of y are in the right part: this gives the term $x\otimes y$.

Let us now consider $\Delta_\succ(x\prec y)$. It is obtained by taking all the shufflings of x and y such the first letter of the result is the first letter of x, then cutting these words before the letter $\begin{pmatrix} 1\\ d\circ\sigma^{-1}(1)\end{pmatrix}$. As a consequence, there are biletters of x in both parts of the result. Hence, three cases are possible:

(1) There are letters of y in both parts of the result: this gives the term $x'_\succ \prec y'\otimes x''_\succ \star y''$.
(2) All the letters of y are in the left part: this gives the term $x'_\succ \prec y\otimes x''_\succ$.
(3) All the letters of y are in the right part: this gives the term $x'_\succ \otimes x''_\succ \star y$.

We now consider $\Delta_\prec(x\succ y)$. It is obtained by taking all the shufflings of x and y such the first letter of the result is the first letter of y, then cutting these words after the letter $\begin{pmatrix} 1\\ d\circ\sigma^{-1}(1)\end{pmatrix}$. Consequently, there are letters of x and y in the left part. So there are three possibilities:

(1) There are letters of x and y in both parts of the result: this gives the term $x'_\prec \succ y'\otimes x''_\prec \star y''$.
(2) All the letters of y are in the left part: this gives the term $x'_\prec \succ y\otimes x''_\prec$.

(3) All the letters of x are in the left part: this gives the term $x \succ y' \otimes y''$.

We now consider $\Delta_\succ(x \succ y)$. t is obtained by taking all the shufflings of x and y such the first letter of the result is the first letter of y, then cutting these words before the letter $\begin{pmatrix} 1 \\ d \circ \sigma^{-1}(1) \end{pmatrix}$. Consequently, there are letters of y in the left part, and letter of x in the right part. Consequently, four cases are possible.

(1) There are letters of x and y in both parts of the result: this gives the term $x'_\succ \succ y' \otimes x''_\succ \star y''$.
(2) There are letters of y in both parts and all the letters of x are in the right part: this gives the term $y' \otimes x \star y''$.
(3) There are letters of x in both parts and all the letters of y are in the left part: this gives the term $x'_\succ \succ y \otimes x''_\succ$.
(4) All the letters of x are in the right part and all the letters of y are in the left part: this gives the term $y \otimes x$.

We obtain in this way the four compatibilities (7)-(10). □

Remark. We define $\Delta : \mathbf{S} \longrightarrow \mathbf{S} \otimes \mathbf{S}$ by $\Delta(x) = x \otimes 1 + 1 \otimes x + \Delta_\prec(x) + \Delta_\succ(x)$ for all $x \in \mathbf{S}_+$, and $\Delta(1) = 1 \otimes 1$. Then $(\mathbf{S}, \star, \Delta)$ is a Hopf algebra.

By the bidendriform rigidity theorem (according to which a bidendriform bialgebra is a free dendriform algebra, see [14] for details):

Corollary 16. $(\mathbf{S}, \prec, \succ)$ *is, as a dendriform algebra, freely generated by the subspace of dendriform primitive elements* $Prim_{Dend}(\mathbf{S}) := Ker(\Delta_\prec) \cap Ker(\Delta_\succ)$.

The generating series of $\mathbf{S}$ as a graded vector space is:

$$R(x) = \left(\sum_{k=0}^{\infty} k! x^k\right) \circ \left(\sum_{k=1}^{\infty} x^k\right).$$

As a consequence of the bidendriform rigidity theorem, the formal series of bidendriform elements of $\mathbf{S}$ is:

$$P(x) = \frac{R(x) - 1}{R(x)^2}.$$

Here are the first coefficients of $R(x)$ and $P(x)$:

n	1	2	3	4	5	6	7	8	9
r_n	1	3	11	49	261	1 631	11 743	95 901	876 809
p_n	1	1	2	10	70	550	4 730	44 378	45 646

5. The dendriform descent algebra

Recall that the descent algebra $\mathbb{D}$ of a tensor algebra and, more generally, of any graded bialgebra H, is the convolution algebra generated by the projections on the graded components of H [27, 33]. The descent algebra is a graded algebra, and its graded components can be equipped with an internal composition product. These components identify with the classical Solomon descent algebras of type A and form the main building block of noncommutative representation theory. Motivated by the structural properties of the descent algebra, Fisher's thesis (where the dual notion is introduced and studied in various particular cases) [13] and by the Proposition 12, which shows that the canonical projection in shuffle algebras belong to the dendrifrom subalgebra of $End(Sh(X))$ generated by the p_n:

Definition 17. *We define* $\mathbf{Descd} \subseteq \mathbf{S}$ *as the dendriform subalgebra of* $End(Sh(X))$ *and* $\mathbf{S}$ *generated by the graded projections* $p_n : Sh(X) \longrightarrow Sh^n(X)$.

The algebra $\mathbf{Descd}$ is naturally graded (the degree of p_n is n): $\mathbf{Descd} = \bigoplus_{n\in\mathbb{N}} Descd_n$ and has the completion $\hat{\mathbf{Descd}} = \prod_{n\in\mathbb{N}} Decsd_n$. For simplicity, we do not emphasize the distinction between $\mathbf{Descd}$ and its completion when dealing with formal power series such as $p = \sum_n p_n$ and will allow us for example to write abusively $p \in \mathbf{Descd}$.

Recall that for all n:

$$p_n = \sum_{k=1}^{n} \sum_{d(1)+\ldots+d(k)=n} \begin{pmatrix} 1 & \ldots & k \\ d(1) & \ldots & d(k) \end{pmatrix}.$$

For the classical descent algebra $\mathbb{D}$, a key property is the group-like behavior of the graded projections p_n ($\Delta(p_n) = \sum_{i\leq n} p_i \otimes p_{n-i}$). We show now that this property is inherited, although in a more sophisticated way, in $\mathbf{Descd}$.

Proposition 18. *For all* $n \geq 1$, $\Delta_{\succ}(p_n) = 0$ *and* $\Delta_{\prec}(p_n) = \sum_{i=1}^{n-1} p_i \otimes p_{n-i}$.

Proof. Clearly, $\Delta_{\succ}(p_n) = 0$. Moreover:

$$\begin{aligned}
\Delta_{\prec}(p_n) &= \sum_{k=1}^{n} \sum_{d(1)+\ldots+d(k)=n} \sum_{i=1}^{k-1} \begin{pmatrix} 1 & \ldots & i \\ d(1) & \ldots & d(i) \end{pmatrix} \otimes \begin{pmatrix} 1 & \ldots & k-i \\ d(i+1) & \ldots & d(k) \end{pmatrix} \\
&= \sum_{i=1}^{n-1} \left(\sum_{p=1}^{i} \sum_{d'(1)+\ldots+d'(p)=i} \begin{pmatrix} 1 & \ldots & p \\ d'(1) & \ldots & d'(p) \end{pmatrix} \right) \otimes \\
&\quad \left(\sum_{q=1}^{n-i} \sum_{d''(1)+\ldots+d''(q)=n-i} \begin{pmatrix} 1 & \ldots & q \\ d''(1) & \ldots & d''(q) \end{pmatrix} \right) \\
&= \sum_{i=1}^{n-1} p_i \otimes p_{n-i}.
\end{aligned}$$

□

Corollary 19. *The dendriform descent algebra* $\mathbf{Descd}$ *is a sub bidendriform bialgebra of* $\mathbf{S}$. *In particular, it is a free dendriform algebra over its dendriform primitive elements.*

Theorem 20. *The family $(\pi_n)_{n\geq 1}$ is a basis of the space of dendriform primitive elements of* **Descd**, *and these elements freely generate* **Descd** *as a dendriform algebra.*

Proof. Recall the notations $p^+ = Id - 1 = \sum\limits_{n\geq 1} p_n$, $p_0 = 1$. Then $\Delta_\succ(p^+) = 0$ and $\Delta_\prec(p^+) = \tilde{\Delta}(p^+) = p^+ \otimes p^+$, so that $\Delta(Id) = Id \otimes Id$.

We set $t := p^{-1} = \sum_{n=0}^{\infty}(-1)^n (p^+)^n$ and $t^+ = t - 1$ and get: $\Delta(t) = \Delta(p^{-1}) = p^{-1} \otimes p^{-1} = t \otimes t$ or, equivalently, $\tilde{\Delta}(t^+) = t^+ \otimes t^+$.

We also have $\pi = p^+ \prec t$. Since $\Delta_\succ(p^+) = 0$, we get $\Delta_\succ(\pi) = 0$ and

$$\Delta_\prec(\pi) = p^+ \otimes p^+ + p^+ \prec t^+ \otimes p^+ t^+ + p^+ \otimes t^+ + p^+ \prec t^+ \otimes t^+ + p^+ \otimes p^+ t^+$$

$$+ p^+ \prec t^+ \otimes p^+ = p^+ \otimes p^+ + p^+ \prec t^+ \otimes (-t^+ - p^+) + p^+ \otimes t^+ + p^+ \prec t^+ \otimes t^+$$

$$+ p^+ \otimes (-t^+ - p^+) + p^+ \prec t^+ \otimes p^+ = 0$$

where we used the identities $pt = (1+p^+)(1+t^+) = 1$ and $p^+t^+ = -p^+ - t^+$. So π is primitive for both coproducts. Taking its homogeneous component of weight n, we obtain that π_n is dendriform primitive for all n.

Recall now that, by the Corollary 19, **Descd** is freely generated as a dendriform algebra by the space of its dendriform primitive elements, and that, by definition, **Descd** is generated by at most one generator in each weight. So the homogeneous components of the space of dendriform primitive elements for the weight are at most one-dimensional. Finally, $(\pi_n)_{n\geq 1}$ is a basis of $Prim_{dend}(\textbf{Descd})$. $\square$

Corollary 21. *The formal series of* **Descd** *is:*

$$\frac{1 - x - \sqrt{(1-x)(1-5x)}}{2x}.$$

Proof. The formal series of **Descd** is $\left(\frac{1-\sqrt{1-4x}}{2x}\right) \circ \left(\frac{x}{1-x}\right)$. $\square$

Examples.

n	1	2	3	4	5	6	7	8	9
$dim(\textbf{Descd}_n)$	1	3	10	36	137	543	2 218	9 285	39 587

This is sequence A002212 of the On-Line Encyclopedia of Integer Sequences.

Remark. As a consequence, it is not difficult to prove that the following families are bases of $\textbf{Descd}_n$:

(1) $n = 1$: $(\pi_1) = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$.

(2) $n = 2$: $(\pi_2, \pi_1 \prec \pi_1, \pi_1 \succ \pi_1) = \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\right)$.

(3) $n = 3$: $(\pi_3, \pi_1 \prec \pi_2, \pi_1 \succ \pi_2, \pi_2 \prec \pi_1, \pi_2 \succ \pi_1, \pi_1 \prec (\pi_1 \prec \pi_1), \pi_1 \prec (\pi_1 \succ \pi_1), \pi_1 \succ (\pi_1 \prec \pi_1), \pi_1 \succ (\pi_1 \succ \pi_1) - (\pi_1 \succ \pi_1) \succ \pi_1, (\pi_1 \succ \pi_1) \succ \pi_1) =$

$$\left(\begin{array}{c} \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \\ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \\ \begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array}\right).$$

So **Descd** is not stable under the internal product $\circ$: for example, $\begin{pmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \notin$ **Descd**.

6. Abstract shuffle bialgebras and the rigidity theorem

The coproduct acting on the shuffle bialgebra $Sh(X)$ satisfies the relation:

$$\Delta(x \prec y) \;=\; x' \prec y' \otimes x'' \sqcup\!\sqcup y'' + 1 \otimes x \prec y \tag{11}$$

where we used the Sweedler notation $\Delta(x) = x' \otimes x''$ and set $1 \prec 1 = 0$ in the formula (recall that $1 \prec w = w \succ 1 = 0$ and $w \prec 1 = 1 \succ w = w$ whenever w is a non empty word). The relation follows from the recursive definition of the shuffle product of words (or from the observation that for two words w and z, the first letter of w is the first letter of $w \prec z$, so that the expansion of $\Delta(w \prec z)$ always starts with the first letter of w).

These identities lead to the abstract definition of a shuffle bialgebra, a particular case of the notion of dendriform bialgebra [4, 29, 14].

Recall from the discussion at the begining of the article that a (non unital) shuffle algebra is, in general, a vector space V equipped with a bilinear map $\prec$ satisfying the axiom (4): $(a \prec b) \prec c = a \prec (b \prec c + c \prec b)$. A unital shuffle algebra A is then obtained by adding a unit to V: $A = A^+ \oplus \mathbf{C} := V \oplus \mathbf{C}$, with $a \prec 1 := a,\ 1 \prec a := 0$. The half-product $1 \prec 1$ is defined to be 0. The shuffle product is defined on $V = A^+$ by $a \sqcup\!\sqcup b := a \prec b + b \prec a$ or $\sqcup\!\sqcup = \prec + \succ$ with $a \succ b := b \prec a$ and extended to A by requiring 1 to be the unit. It provides A with the structure of a commutative (and associative) algebra with unit. The shuffle algebras over generating sets $Sh(X)$ are the free algebras over X for the relation (4) [32]. A shuffle algebra A is graded and connected if its decomposition into graded components $A = \bigoplus_n A_n$ is such $A_0 = \mathbf{C}$ and that the half-product is compatible with the grading ($A_n \prec A_m \subset A_{n+m}$).

Definition 22. *Let $A = V \oplus \mathbf{C}$ be a unital shuffle algebra and Δ a coassociative counital coproduct on A. The coproduct defines a shuffle bialgebra* *structure on A if and only if the relation (11) is satisfied. A shuffle bialgebra is graded connected if it is a graded connected shuffle algebra and if the coproduct is compatible with the grading ($\Delta(A_n) \subset \bigoplus_{i+j=n} A_i \otimes A_j$).*

As expected, shuffle bialgebras over finite sets $Sh(X)$ provide examples for this abstract definition of graded connected shuffle bialgebras.

Lemma 23. *The set of linear endomorphisms of a shuffle bialgebra $End(A)$ is equipped with the structure of a dendriform algebra by the products:*

$$f \prec g(a) := f(a') \prec g(a'')$$

$$f \succ g(a) := f(a') \succ g(a'') = g(a'') \prec f(a').$$

The proof follows from the same arguments as for the Lemma 3.

From now on, A will denote an arbitrary graded connected shuffle bialgebra.

Corollary 24. *There is a unique map ϕ of dendriform algebras from* **Descd** *to $End(A)$ such that $\phi(p_n) := a_n$, where we write a_n for the canonical projection from A to A_n.*

Indeed, as the π_n to which they are related by triangular equations (π_n and p_n are equal up to dendriform products of lower degrees elements), the p_n form a free family of dendriform generators of **Descd**. The Corollary follows by the universal properties of free algebras.

Lemma 25. *Let $\tau = \sum_{n\geq 1} \tau_n := \phi(\pi)$. For any $x, y \in A^+$, we have $\tau(x \prec y) = 0$. In other terms, τ acts trivially on $A^+ \prec A^+$.*

Indeed, let $a^+ = \phi(p^+)$. Since $\pi = p^+ \prec (Id)^{-1}$, and since the convolution inverse of Id in $End(A)$ is the antipode, we get:

$$\tau(x \prec y) = a^+(x' \prec y') \prec S(y'') \sqcup\!\sqcup S(x'').$$

Since $1 \prec u = 0$ for an arbitrary $u \in A^+$, we get:

$$\tau(x \prec y) = (a^+(x') \prec y') \prec S(y'') \sqcup\!\sqcup S(x'')$$

$$= a^+(x') \prec (y' \sqcup\!\sqcup S(y'')) \sqcup\!\sqcup S(x'') = \tau(x) a_0(y) = 0,$$

from which the Lemma follows.

Corollary 26. *The operator τ is an idempotent: $\tau^2 = \tau$.*

Indeed, since $p = \exp^{\prec}(\pi)$, $Id_A =: a = \exp^{\prec}(\tau)$ and:

$$\tau = \tau \circ a = \tau(\exp^{\prec}(\tau)) = \tau \circ \tau$$

since the image of τ in contained in A^+, and therefore $\tau \circ (\tau \prec (\tau ...(\tau \prec \tau)...) = 0$ for iterated products $(\tau \prec (\tau ...(\tau \prec \tau)...)$ of an arbitrary length.

Proposition 27. *The idempotent map τ is a projection onto the primitive elements of A.*

From the identity $\tau = a^+ \prec S$ we get that for a primitive element x of A, $\tau(x) = x \prec 1 = x$. Now, for $y \in A^+$,

$$\Delta(\tau(y)) = \Delta(a^+(y') \prec S(y'')) = \Delta(y' \prec S(y''))$$
$$= (y^1 \prec S(y_4)) \otimes (y_2 \sqcup\!\sqcup S(y_3)) + 1 \otimes y' \prec S(y'')$$

where we used the coassociativity of the coproduct, the property of the antipode $\Delta(S(y)) = S(y'')S(y')$, and an extended Sweedler notation to write somehow abusively $y_1 \otimes ... \otimes y_n$ for the iterated coproduct of order n of y (so that e.g. $(\Delta \otimes Id) \circ \Delta(y) = y_1 \otimes y_2 \otimes y_3$, and so on).

From the coassociativity of the coproduct, we get $y_2 \sqcup\!\sqcup S(y_3) = 1$ (the convolution product of the antipode with the identity is the null map on A^+ and the identity on the scalars). Finally, by cancellation of the non scalar terms in $y_2 \sqcup\!\sqcup S(y_3)$, we get:

$$\Delta(\tau(y)) = y' \prec S(y'') \otimes 1 + 1 \otimes y' \prec S(y'') = \tau(y) \otimes 1 + 1 \otimes \tau(y),$$

from which the Proposition follows.

In 2000, F. Chapoton introduced the breaking new idea that the classical Cartier-Milnor-Moore theorem holds in fact for generalized bialgebras such as dendriform bialgebras, provided an analogue of the Poincaré-Birkhoff-Witt theorem holds [4]. His main Theorem ([4, Thm 1]) implies a "rigidity theorem" in the commutative case: the underlying algebras are then free shuffle algebras. Chapoton's proof follows by adapting the classical proof of the Cartier-Milnor-Moore theorem [21] to dendriform bialgebras. M. Ronco contributed by various remarks to the final preprint version of [4] and proposed soon after another proof by adapting the combinatorial proof [25, 26, 27] of the Cartier-Milnor-Moore theorem [30].

As far as classical bialgebras are concerned, it is a well-known fact that the Leray theorem (which asserts that a graded connected commutative bialgebra over a field of characteristic zero is a free commutative algebra) is much simpler to prove than the Cartier-Milnor-Moore theorem, see e.g. [28] for a modern general proof and further references on the subject. This observation also holds for dendriform and shuffle algebras: the various proofs of the classical Leray theorem can be adapted to shuffle bialgebras to get simple and direct proofs of Chapoton's rigidity theorem. We deduce here a proof of the Theorem from the *Descd* approach.

Theorem 28. *A graded connected shuffle bialgebra A is isomorphic, as a shuffle bialgebra, to the free shuffle algebra over the vector space of its primitive elements $Prim(A)$.*

From the Proposition (27) we know that τ projects to $P := Prim(A)$. From the identity $Id = \exp^{\prec}(\tau)$, we deduce that the image of τ, P, generates A as a shuffle algebra and that an arbitrary element in A can be written as a linear combination of iterated half-shuffle products $p_1 \prec (p_2 \prec ...(p_{n-1} \prec p_n)...)$, with $p_i \in P$

Let us choose a graded basis $\mathcal{B}$ of P (by graded we mean that any $b \in \mathcal{B}$ belongs to a graded component P_n of P). To prove the theorem, it is enough to prove that an arbitrary linear combination of iterated half-shuffle products $\sum_{n\leq N}\sum_{p_1,...,p_n}\lambda_{p_1,...,p_n}p_1 \prec (p_2 \prec ...(p_{n-1} \prec p_n)...)$ with the p_i in $\mathcal{B}$ vanishes if and only if all the coefficients $\lambda_{p_1,...,p_n}$ are null.

Let us prove this property by induction on N. We assume therefore that the $p_1 \prec (p_2 \prec ...(p_{n-1} \prec p_n)...)$, where $n < N-1$ and the p_i run over $\mathcal{B}$ are linearly independent. Assume now that $X = \sum_{n\leq N}\sum_{p_1,...,p_n}\lambda_{p_1,...,p_n}p_1 \prec (p_2 \prec ...(p_{n-1} \prec p_n)...) = 0$ and that $\exists (p_1,...,p_N),\ \lambda_{p_1,...,p_N} \neq 0$. Then, according to eqn (11), $0 = \Delta(X) = \lambda_{p_1,...,p_N}p_1 \otimes p_2 \prec (...(p_{n-1} \prec p_n)...) + Z$, where Z is a linear combination of elements that, by induction, are linearly independent of $p_1 \otimes p_2 \prec (...(p_{n-1} \prec p_n)...)$. The Theorem follows.

Corollary 29. *In particular, the dendriform algebra of graded permutations* $\mathbf{S}$ *acts naturally on an arbitrary graded connected shuffle bialgebra* A.

This follows from the existence of an isomorphism $A \cong Sh(Prim(A))$.

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