Primitive elements of the Hopf algebra of free quasi-symmetric functions

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ABSTRACT: Using the dendriform and the bidendriform Cartier-Quillen-Milnor-Moore theorem, we construct a basis of the space of primitive elements of the Hopf algebra of free quasi-symmetric functions, indexed by a certain set of trees, and inductively computable.

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Introduction

The Hopf algebra of free quasi-symmetric functions FQSym, also known as the Malvenuto-Reutenauer Hopf algebra, is introduced in [8]. It is a graded self-dual Hopf algebra, with basis the set of all permutations. Certain interesting properties are shown in [2]: in particular, it is shown that it is both free and cofree, and a basis of its space of primitive elements is given, using the self-duality and a monomial basis. Note that computing the primitive elements of degree \( n \) by this method implies to inverse a certain \( n! \times n! \) matrix.

The aim of this paper is to describe another basis of \( \text{Prim}_{\text{coAss}}(\text{FQSym}) \), which can be inductively computed. We use for this the dendriform structure of FQSym. Recall that a dendriform algebra is an associative algebra such that its product can be split into two nonassociative products \( \prec \) and \( \succ \), with good compatibilities ([6, 7, 9]). It is known that FQSym, or more precisely its augmentation ideal, is dendriform. More precisely, it is a dendriform Hopf algebra, in the sense of [9]. This implies, by the dendriform Milnor-Moore theorem, that \( \text{Prim}_{\text{coAss}}(\text{FQSym}) \) is a brace algebra.

We introduce in [3] the notion of bidendriform bialgebra and show that FQSym is bidendriform. The bidendriform Milnor-Moore theorem implies that FQSym is freely generated, as
a dendriform algebra, by the space \( \text{Prim}_{\text{coDend}}(\text{FQSym}) \) of primitive elements in the codendriform sense. Combining this result with the dendriform Milnor-Moore theorem, we show that \( \text{Prim}_{\text{coAss}}(\text{FQSym}) \) is, as a brace algebra, freely generated by \( \text{Prim}_{\text{coDend}}(\text{FQSym}) \). We recall in section 2 a description of free brace algebras. If \( (v_i)_{i \in I} \) is a basis of the vector space \( V \), then the free brace algebra generated by \( V \) has a basis indexed by planar rooted trees decorated by \( I \), and the brace structure is described in this basis by the help of graftings. Hence, for any basis of \( \text{Prim}_{\text{coDend}}(\text{FQSym}) \), it is possible to recover a basis of \( \text{Prim}_{\text{coAss}}(\text{FQSym}) \), indexed by a certain set of planar decorated rooted trees.

Let, for all \( n \in \mathbb{N} \):

\[
\begin{align*}
p_n &= \dim(\text{Prim}_{\text{coAss}}(\text{FQSym})_n), \\
q_n &= \dim(\text{Prim}_{\text{coDend}}(\text{FQSym})_n).
\end{align*}
\]

We then prove in section 3 that for \( n \geq 2 \), \( q_n = (n - 2)p_{n-1} \). We then give \( n - 2 \) applications from \( \text{Prim}_{\text{coAss}}(\text{FQSym})_{n-1} \) to \( \text{Prim}_{\text{coDend}}(\text{FQSym})_n \), which give all elements of \( \text{Prim}_{\text{coDend}}(\text{FQSym})_n \). These applications are given by the insertion of \( n + 1 \) at a given place in elements of the symmetric group \( S_n \), seen as words in letters \( 1, \ldots, n \).

Combining the results of sections 2 and 3, we define inductively in section 4 a new basis of \( \text{Prim}_{\text{coAss}}(\text{FQSym})_n \), indexed by certain planar decorated rooted trees. The trees which are only a root give a basis of \( \text{Prim}_{\text{coDend}}(\text{FQSym})_n \).

**Notations.**

1. \( K \) is a commutative field of any characteristic.
2. If \( V \) is a \( K \)-vector field which is \( \mathbb{N} \)-graded, we shall denote by \( V_k \) the space of homogeneous elements of \( V \) of degree \( k \).

## 1 Bidendriform bialgebras and FQSym

### 1.1 Bidendriform bialgebras

We introduced in [5] the following definition:

**Definition 1** A bidendriform bialgebra is a family \( (A, \prec, \succ, \Delta_\prec, \Delta_\succ) \) such that:

1. \( A \) is a \( K \)-vector space and:

\[
\begin{align*}
\prec &: \left\{ \begin{array}{ccc}
A \otimes A & \rightarrow & A \\
\otimes a b & \rightarrow & a \prec b,
\end{array} \right. \\
\Delta_\prec &: \left\{ \begin{array}{ccc}
A & \rightarrow & A \otimes A \\
a & \rightarrow & \Delta_\prec(a) = a'_\prec \otimes a''_\prec,
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\succ &: \left\{ \begin{array}{ccc}
A \otimes A & \rightarrow & A \\
\otimes a b & \rightarrow & a \succ b,
\end{array} \right. \\
\Delta_\succ &: \left\{ \begin{array}{ccc}
A & \rightarrow & A \otimes A \\
a & \rightarrow & \Delta_\succ(a) = a'_\succ \otimes a''_\succ.
\end{array} \right.
\end{align*}
\]

2. (Dendriform axioms). \( (A, \prec, \succ) \) is a dendriform algebra: for all \( a, b, c \in A \),

\[
\begin{align*}
(a \prec b) \prec c &= a \prec (b \prec c + b \succ c) & (1) \\
(a \succ b) \prec c &= a \succ (b \prec c) & (2) \\
(a \prec b + a \succ b) \succ c &= a \succ (b \succ c) & (3)
\end{align*}
\]

3. (Codendriform axioms). \( (A, \Delta_\prec, \Delta_\succ) \) is a codendriform coalgebra: for all \( a \in A \),

\[
\begin{align*}
(\Delta_\prec \otimes \text{Id}) \circ \Delta_\prec(a) &= (\text{Id} \otimes \Delta_\prec + \text{Id} \otimes \Delta_\succ) \circ \Delta_\prec(a) & (4) \\
(\Delta_\succ \otimes \text{Id}) \circ \Delta_\succ(a) &= (\text{Id} \otimes \Delta_\prec) \circ \Delta_\succ(a) & (5) \\
(\Delta_\prec \otimes \text{Id} + \Delta_\succ \otimes \text{Id}) \circ \Delta_\succ(a) &= (\text{Id} \otimes \Delta_\succ) \circ \Delta_\succ(a). & (6)
\end{align*}
\]
4. (Bidendriform axioms). For all \(a, b \in A\),

\[
\begin{align*}
\Delta_{>}(a \succ b) &= a'b_{>}' \otimes a'' > b''_{>}' + a' \otimes a'' > b + b_{>}' \otimes a > b''_{>} + ab_{>}' \otimes b''_{>} + a \otimes b, \\
\Delta_{>}(a < b) &= a'b_{>}' \otimes a'' < b''_{>}' + a' \otimes a'' < b + b_{>}' \otimes a < b''_{>}, \\
\Delta_{<}(a \succ b) &= a'b_{<}' \otimes a'' > b''_{<}' + ab_{<}' \otimes b''_{<} + a \otimes b_{<}' < a > b''_{<}, \\
\Delta_{<}(a < b) &= a'b_{<}' \otimes a'' < b''_{<}' + a \otimes b_{<}' < a > b''_{<} + b_{<}' \otimes a < b''_{<} + b \otimes a.
\end{align*}
\]

Remarks.

1. If \(A\) is a bidendriform bialgebra, then \(K \oplus A\) is naturally a Hopf algebra, by extending \(< + >\) and \(\Delta_{<} + \Delta_{>}\) on \(K \oplus A\).

2. If \(A\) is a bidendriform bialgebra, it is also a dendriform hopf algebra in the sense of \([9, 10]\), with coassociative coproduct given by \(\tilde{\Delta} = \Delta_{<} + \Delta_{>}\). The compatibilities of dendriform Hopf algebras are given by \((7) + (9)\) and \((8) + (10)\).

If \(A\) is a bidendriform algebra, we define:

\[
Prim_{codend}(A) = Ker(\Delta_{<}) \cap Ker(\Delta_{>}).
\]

The following result is proved in \([5]\) (theorem 35 and corollary 17):

**Theorem 2 (Bidendriform Milnor-Moore theorem)** Let \(A\) be a \(\mathbb{N}\)-graded bidendriform bialgebra, such that \(A_0 = (0)\). Then \(A\) is freely generated as a dendriform algebra by \(Prim_{coDend}(A)\). Moreover, consider the following formal series:

\[
\begin{align*}
R(X) &= \sum_{n=1}^{+\infty} \dim(A)X^n, \\
Q(X) &= \sum_{n=1}^{+\infty} \dim(Prim_{coDend}(A)_n)X^n.
\end{align*}
\]

Then:

\[
Q(X) = \frac{R(X)}{(R(X) + 1)^2}.
\]

1.2 An example: the Hopf algebra \(\text{FQSym}\)

(See \([1, 2, 8]\)). The algebra \(\text{FQSym}\) is the vector space generated by the elements \((\mathbf{F}_u)_{u \in S}\), where \(S\) is the disjoint union of the symmetric groups \(S_n\) \((n \in \mathbb{N})\). Its product and its coproduct are given in the following way: for all \(u \in S_n, v \in S_m\), by putting \(u = (u_1 \ldots u_n)\),

\[
\begin{align*}
\Delta(\mathbf{F}_u) &= \sum_{i=0}^{n} \mathbf{F}_{st(u_1 \ldots u_i)} \otimes \mathbf{F}_{st(u_{i+1} \ldots u_n)}, \\
\mathbf{F}_u \mathbf{F}_v &= \sum_{\zeta \in sh(n, m)} \mathbf{F}_{(u \times v), \zeta}^{-1},
\end{align*}
\]

where \(sh(n, m)\) is the set of \((n, m)\)-shuffles, and \(st\) is the standardisation. Its unit is \(1 = \mathbf{F}_\emptyset\), where \(\emptyset\) is the unique element of \(S_0\). Moreover, \(\text{FQSym}\) is a \(\mathbb{N}\)-graded Hopf algebra, by putting \(|\mathbf{F}_u| = n\) if \(u \in S_n\).

Examples.

\[
\begin{align*}
\mathbf{F}_{(12)} \mathbf{F}_{(123)} &= \mathbf{F}_{(12345)} + \mathbf{F}_{(13245)} + \mathbf{F}_{(13425)} + \mathbf{F}_{(13452)} + \mathbf{F}_{(31245)} + \mathbf{F}_{(31425)} + \mathbf{F}_{(31452)} + \mathbf{F}_{(34125)} + \mathbf{F}_{(34152)} + \mathbf{F}_{(34512)}, \\
\Delta(\mathbf{F}_{(12543)}) &= 1 \otimes \mathbf{F}_{(12543)} + \mathbf{F}_{(1)} \otimes \mathbf{F}_{(1432)} + \mathbf{F}_{(12)} \otimes \mathbf{F}_{(321)} + \mathbf{F}_{(123)} \otimes \mathbf{F}_{(21)} + \mathbf{F}_{(1243)} \otimes \mathbf{F}_{(1)} + \mathbf{F}_{(12543)} \otimes 1.
\end{align*}
\]

3
Let \((FQSym)_{+} = Vect(Fu / u \in S_n, n \geq 1)\) be the augmentation ideal of \(FQSym\). We define \(<, >, \Delta_<, \Delta_\succ\) on \((FQSym)_{+}\) in the following way: for all \(u \in S_n, v \in S_m\), by putting \(u = (u_1 \ldots u_n)\),

\[
F_u \prec F_v = \sum_{\zeta \in sh(n, m)} F_{(u \times v)} \zeta^{-1},
\]

\[
F_u \succ F_v = \sum_{\zeta \in sh(n, m)} F_{(u \times v)} \zeta^{-1},
\]

\[
\Delta_{<}(F_u) = \sum_{i=1}^{u-1(n)} F_{st(u_1 \ldots u_i)} \otimes F_{st(u_{i+1} \ldots u_n)},
\]

\[
\Delta_{\succ}(F_u) = \sum_{i=1}^{u-1(n)-1} F_{st(u_1 \ldots u_i)} \otimes F_{st(u_{i+1} \ldots u_n)}.
\]

**Examples.**

\[
F_{(1 2)} \prec F_{(1 2 3)} = F_{(1 3 4 5 2)} + F_{(3 1 4 5 2)} + F_{(3 4 1 5 2)} + F_{(3 4 5 1 2)},
\]

\[
F_{(1 2)} \succ F_{(1 2 3)} = F_{(1 2 4 3 2)} + F_{(1 3 2 4 5)} + F_{(1 3 4 2 5)} + F_{(3 1 2 4 5)} + F_{(3 1 4 2 5)} + F_{(3 4 1 2 5)}.
\]

\[
\Delta_{<}(F_{(1 2 5 4 3)}) = F_{(1 2 3)} \otimes F_{(2 1)} + F_{(1 2 4 3)} \otimes F_{(1)},
\]

\[
\Delta_{\succ}(F_{(1 2 5 4 3)}) = F_{(1)} \otimes F_{(1 4 3 2)} + F_{(1 2)} \otimes F_{(3 2 1)}.
\]

The following result is proved in [5] (theorem 38):

**Theorem 3** \(((FQSym)_{+}, \prec, \succ, \Delta_{<}, \Delta_{\succ})\) is a connected bidendriform bialgebra.

Moreover, \((FQSym)_{+}\) is \(\mathbb{N}\)-graded, by putting the elements of \(S_n\) homogeneous of degree \(n\). By theorem 2, with \(q_n = dim(Prim_{coDend}(FQSym)_n)\), we obtain:

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>(q_n)</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>39</td>
<td>284</td>
<td>2 305</td>
<td>20 682</td>
<td>203 651</td>
<td>2 186 744</td>
<td>25 463 925</td>
<td>319 989 030</td>
</tr>
</tbody>
</table>

2 Recovering \(Prim_{coAss}(FQSym)\) from \(Prim_{coDend}(FQSym)\)

**2.1 Dendriform Milnor-Moore theorem and variations**

Recall that a brace algebra is a \(K\)-vector space \(A\) together with an \(n\)-multilinear operation for all \(n \geq 2\):

\[
< \ldots >: \left\{ \begin{array}{c}
A^\otimes n \longrightarrow A \\
a_1 \otimes \ldots \otimes a_n \longrightarrow < a_1, \ldots, a_n >
\end{array} \right.,
\]

satisfying certain relations (see [9, 10] for more details). For example:

\[
< a_1, < a_2, a_3 >= < a_1, a_2, a_3 > + << a_1, a_2 >, a_3 > + < a_2, a_1, a_3 > .
\]

The following theorem is proved in [9, 10] (more precisely, the first point of this theorem is proposition 2-8 and theorem 3-4 of [9] and the second point is theorem 4-6 of [10]):

**Theorem 4 (Dendriform Milnor-Moore)** Let \(A\) be a dendriform Hopf algebra. We denote \(Prim_{coAss}(A) = Ker(\Delta)\),
1. Prim\textsubscript{coAss}(\mathcal{A}) is a brace algebra, with brackets given by:
\[
\langle p_1, \ldots, p_n \rangle = \sum_{i=0}^{n-1} (-1)^{n-1-i} (p_1 \prec (p_2 \prec (\ldots \prec p_i \ldots) \succ p_n \prec (\ldots (p_{i+1} \succ p_{i+2}) \succ \ldots) \succ p_{n-1}).
\]

2. If \( \mathcal{A} \) is freely generated as a dendriform algebra by a subvector space \( V \subseteq \text{Prim}_{\text{coAss}}(\mathcal{A}) \), then \( \text{Prim}_{\text{coAss}}(\mathcal{A}) \) is freely generated as a brace algebra by \( V \).

Let us precise the relation between \( \text{Prim}_{\text{coAss}}(\mathcal{A}) \) and \( \text{Prim}_{\text{coDend}}(\mathcal{A}) \) if \( \mathcal{A} \) is a bidendriform bialgebra. Combining the dendriform and the bidendriform Milnor-Moore theorems:

**Theorem 5** Let \( \mathcal{A} \) be a \( \mathbb{N} \)-graded bidendriform bialgebra, with \( A_0 = (0) \). Then \( \text{Prim}_{\text{coAss}}(\mathcal{A}) \) is, as a brace algebra, freely generated by \( \text{Prim}_{\text{coDend}}(\mathcal{A}) \).

**Proof.** By the bidendriform Milnor-Moore theorem, \( \mathcal{A} \) is freely generated as a dendriform algebra by \( \text{Prim}_{\text{coDend}}(\mathcal{A}) \). By the dendriform Milnor-Moore theorem (second point), \( \text{Prim}_{\text{coAss}}(\mathcal{A}) \) is freely generated as a brace algebra by \( \text{Prim}_{\text{coDend}}(\mathcal{A}) \). \( \square \)

**Proposition 6** If \( \mathcal{A} \) is \( \mathbb{N} \)-graded dendriform Hopf algebra, such that \( A_0 = (0) \), then \( \mathcal{A} \) is generated as a dendriform algebra by \( \text{Prim}_{\text{coAss}}(\mathcal{A}) \). Moreover, consider the following formal series:
\[
R(X) = \sum_{n=1}^{\infty} \dim(A_n) X^n, \quad P(X) = \sum_{n=1}^{+\infty} \dim(\text{Prim}_{\text{coDend}}((A)_n)) X^n.
\]

Then:
\[
P(X) = \frac{R(X)}{1 + R(X)}.
\]

**Proof.**

**First step.** Let \( p_1, \ldots, p_n \in \text{Prim}_{\text{coAss}}(\mathcal{A}) \). By induction on \( n \) we define:
\[
\omega(p_1, \ldots, p_n) = \left\{ \begin{array}{ll}
p_1 & \text{if } n = 1, \\
p_n \prec \omega(p_1, \ldots, p_{n-1}) & \text{if } n \geq 2.
\end{array} \right.
\]

An easy induction on \( n \) allows to show the following result, using (8)+(10):
\[
\check{\Delta}(\omega(p_1, \ldots, p_n)) = \sum_{i=1}^{n-1} \omega(p_1, \ldots, p_i) \otimes \omega(p_{i+1}, \ldots, p_n).
\]

We denote by \( \check{\Delta}^n : \mathcal{A} \rightarrow \mathcal{A}^{n+1} \) the iterated coproducts of \( \mathcal{A} \). It comes by induction:
\[
\check{\Delta}^n(\omega(p_1, \ldots, p_n)) = \left\{ \begin{array}{ll}
0 & \text{if } m \geq n,

p_1 \otimes \ldots \otimes p_n & \text{if } m = n - 1.
\end{array} \right.
\]

**Second step.** We consider the tensor (non counitary) coalgebra \( C = T(\text{Prim}_{\text{coAss}}(\mathcal{A})) \):
\[
C = \bigoplus_{n=1}^{\infty} \text{Prim}_{\text{coAss}}(\mathcal{A})^\otimes n.
\]

It is a coalgebra for the deconcatenation coproduct. As \( \text{Prim}_{\text{coAss}}(\mathcal{A}) \) is \( \mathbb{N} \)-graded, \( C \) is a graded coalgebra with formal series:
\[
S(X) = \frac{1}{1 - P(X)} - 1 = \frac{P(X)}{1 - P(X)}.
\]
By the first step, the following application is a morphism of graded coalgebras:

\[ \Psi : \begin{cases} C & \longrightarrow A \\ p_1 \otimes \ldots \otimes p_n & \longrightarrow \omega(p_1, \ldots, p_n). \end{cases} \]

**Third step.** Suppose that \( \text{Ker}(\Psi) \) is non zero. As it is a coideal of \( C \), it contains primitive elements of \( C \), that is to say elements of \( \text{Prim}_{\text{coAss}}(A) \). As \( \Psi \) is obviously monic on \( \text{Prim}_{\text{coAss}}(A) \), this is not possible. So \( \text{Ker}(\Psi) = (0) \) and \( \Psi \) is monic.

Let \( a \in A \). As \( A_0 = (0) \), for a certain \( N(a) \in \mathbb{N}^* \), \( \bar{\Delta}^{N(a)}(a) = 0 \). We prove that \( a \in \text{Im}(\Psi) \) by induction on \( N(a) \). If \( N(a) = 1 \), then \( a \in \text{Prim}_{\text{coAss}}(A) \) and the result is obvious. Suppose that the result is true for all \( b \in A \) such that \( N(b) < N(a) \). As \( \bar{\Delta}^{N(a)}(a) = 0 \), necessarily \( \bar{\Delta}^{N(a)-1}(a) \in \text{Prim}_{\text{coAss}}(A)^{\otimes N(a)} \). We put:

\[ \bar{\Delta}^{N(a)-1}(a) = a_1 \otimes \ldots \otimes a_n, \quad b = a - \omega(a_1, \ldots, a_n). \]

By the first step, \( \bar{\Delta}^{N(a)-1}(b) = 0 \), so \( N(b) < N(a) \). By induction hypothesis, \( b \in \text{Im}(\Psi) \). As \( \omega(a_1, \ldots, a_n) \in \text{Im}(\Psi) \), \( a \in \text{Im}(\Psi) \).

**Last step.** As \( \Psi \) in an isomorphism of graded coalgebras, \( S(X) = R(X) \). Hence:

\[
\begin{align*}
R(X) &= \frac{P(X)}{1 - P(X)}, \\
R(X) - R(X)P(X) &= P(X), \\
P(X) &= \frac{R(X)}{1 + R(X)}. \quad \square
\end{align*}
\]

### 2.2 Free brace algebras

Using a description of the free dendriform algebra generated by a set \( D \) with planar decorated forests, we gave a description of the free brace algebra \( \text{Brace}(D) \) in [4]. A basis of this brace algebra is given by the set \( T^D \) of planar rooted trees decorated by \( D \). For example:

\[
\begin{align*}
\text{Brace}(D)_1 &= \text{Vect}(\ast_a, a \in D), \\
\text{Brace}(D)_2 &= \text{Vect}(\ast_b^a, a, b \in D), \\
\text{Brace}(D)_3 &= \text{Vect}(\ast^a_b \ast_{c}^b, a, b, c \in D), \\
\text{Brace}(D)_4 &= \text{Vect}(\ast^a_b \ast^b_a, \ast^a_t, \ast^c_t, \ast^b_t, a, b, c, d \in D), \ldots
\end{align*}
\]

The brace bracket satisfies, for all \( t_1, \ldots, t_{n-1} \in T^D, \ d \in D \):

\[ < t_1, \ldots, t_{n-1}, \ast_d > = B_d(t_{n-1} \ldots t_1), \]

where \( B_d(t_{n-1} \ldots t_1) \) is the tree obtained by grafting the trees \( t_{n-1}, \ldots, t_1 \) (in this order) on a common root decorated by \( d \). For example, if \( a, b, c, d \in D \),

\[ < \ast_a, \ast_b^c, \ast_d > = \ast^a_c \ast^c_a \ast_d. \]

In consequence, if \( A \) is a connected bidendriform bialgebra and \( (q_d)_{d \in D} \) a basis of \( \text{Prim}_{\text{coDend}}(A) \), a basis of \( \text{Prim}_{\text{coAss}}(A) \) is given by \( (p_t)_{t \in T^D} \) defined inductively by:

\[
\begin{align*}
p \ast_d &= q_d, \\
p B_d^+(t_1 \ldots t_n) &= < p_{t_n}, \ldots, p_{t_1}, q_d >.
\end{align*}
\]
3 Recovering $\text{Prim}_{\text{coDend}}(\text{FQSym})$ from $\text{Prim}_{\text{coAss}}(\text{FQSym})$

For all $n \in \mathbb{N}^*$, we put:

$$\begin{align*}
\left\{ \begin{array}{l}
p_n = \dim(\text{Prim}_{\text{coAss}}(\text{FQSym})_n), \\
q_n = \dim(\text{Prim}_{\text{coDend}}(\text{FQSym})_n).
\end{array} \right.
\end{align*}$$

Proposition 7 For all $n \geq 2$, $q_n = (n - 2)p_{n-1}$.

Proof. We put:

$$R(X) = \sum_{n=1}^{\infty} n!X^n, \quad P(X) = \sum_{n=1}^{\infty} p_n X^n, \quad Q(X) = \sum_{n=1}^{\infty} q_n X^n.$$  

By theorem 2 and proposition 6:

$$P(X) = \frac{R(X)}{1 + R(X)}, \quad Q(X) = \frac{R(X)}{(1 + R(X))^2}.$$  

Hence:

$$P'(X) = \frac{R'(X)}{(1 + R(X))^2}.$$  

Moreover:

$$R'(X) = \sum_{n=1}^{\infty} nn!X^{n-1} = \sum_{n=1}^{\infty} (n+1)!X^{n-1} - \sum_{n=1}^{\infty} n!X^{n-1} = \frac{R(X) - X}{X^2} - \frac{R(X)}{X} = \frac{R(X) - (R(X) + 1)}{X^2}.$$  

We deduce:

$$X^2 P'(X) = \frac{R(X) - X(R(X) + 1)}{(1 + R(X))^2} = Q(X) - \frac{X}{1 + R(X)} = Q(X) - X + XP(X).$$  

So:

$$X^2 P'(X) + XP(X) = \sum_{n=1}^{\infty} (n-1)p_n X^{n+1} = Q(X) - X = \sum_{n=2}^{\infty} q_n X^n.$$  

In conclusion, for all $n \geq 2$, $q_n = (n - 2)p_{n-2}$. □

Definition 8 Let $i \in \mathbb{N}^*$. We define $\Phi_i : \text{FQSym} \rightarrow \text{FQSym}$ in the following way: for all $n \in \mathbb{N}$, for all $\sigma = (\sigma_1, \ldots, \sigma_n) \in S_n$,

$$\Phi_i(F_\sigma) = \begin{cases} 0 & \text{if } i \geq n \\ F_{(\sigma_1, \ldots, \sigma_i, n+1, \sigma_{i+1}, \ldots, \sigma_n)} & \text{if } i < n. \end{cases}$$
Theorem 9 Let \( n \geq 2 \). The following application is bijective:

\[
\Phi : \begin{cases} 
(\text{Prim}_{\text{coAss}}(\text{FQSym})_{n-1})^{n-2} & \rightarrow \text{Prim}_{\text{coDend}}(\text{FQSym})_n \\
(p_1, \ldots, p_{n-2}) & \rightarrow \Phi_1(p_1) + \ldots + \Phi_{n-2}(p_{n-2})
\end{cases}
\]

**Proof.**

*First step.* \( \Phi \) takes its values in  \( \text{Prim}_{\text{coDend}}(\text{FQSym}) \). Let \( p \in \text{Prim}_{\text{coAss}}(\text{FQSym}) \) and \( 1 \leq i \leq n - 2 \). For all \( k \in \mathbb{N} \), let \( \pi_k \) be the projection on \( \text{FQSym}_k \). By definition of \( \Delta_\prec \) and \( \Delta_\succ \), we have immediately, for all \( \sigma \in S_{n-1} \):

\[
\Delta_\prec(\Phi_i(F_\sigma)) = \left( \sum_{j=i+1}^{n-2} \pi_j \otimes \pi_{n-1-j} \right) \circ \tilde{\Delta}(F_\sigma),
\]

\[
\Delta_\succ(\Phi_i(F_\sigma)) = \left( \sum_{j=1}^{i} \pi_j \otimes \pi_{n-1-j} \right) \circ \tilde{\Delta}(F_\sigma),
\]

By linearity, we obtain:

\[
\Delta_\prec(p) = \left( \sum_{j=i+1}^{n-2} \pi_j \otimes \pi_{n-1-j} \right) \circ \tilde{\Delta}(p) = 0,
\]

\[
\Delta_\succ(p) = \left( \sum_{j=1}^{i} \pi_j \otimes \pi_{n-1-j} \right) \circ \tilde{\Delta}(p) = 0.
\]

This proves the first step.

*Second step.* \( \Phi \) is monic. Let \( (p_1, \ldots, p_{n-2}) \in \text{Ker}(\Phi) \). Let be \( 1 \leq i \leq n - 2 \). We define:

\[
\varpi_i : \begin{cases} 
\text{FQSym}_n & \rightarrow \text{FQSym}_n \\
F_\sigma & \rightarrow \begin{cases} 
0 & \text{if } \sigma^{-1}(n) \neq i + 1, \\
F_\sigma & \text{if } \sigma^{-1}(n) = i + 1.
\end{cases}
\end{cases}
\]

Then, in an obvious way, \( \varpi_i(\Phi(p_1, \ldots, p_{n-2})) = \Phi_i(p_i) = 0 \). As \( \Phi_i \) is obviously monic on \( \text{FQSym}_{n-1} \) (because \( i \leq n - 2 \)), \( p_i = 0 \). So \( \Phi \) is monic.

*Last step.* As \( \text{dim} \left( (\text{Prim}_{\text{coAss}}(\text{FQSym})_{n-1})^{n-2} \right) \neq \text{dim} \left( \text{Prim}_{\text{coDend}}(\text{FQSym})_n \right) \), by proposition 7, \( \Phi \) is bijective. \( \square \)

4 An inductive basis of \( \text{Prim}_{\text{coAss}}(\text{FQSym}) \)

We now combine results of section 2 and 3 to obtain an basis of \( \text{Prim}_{\text{coAss}}(\text{FQSym}) \). We first define inductively some set of partially planar decorated trees \( T(n) \) in the following way:

1. \( T(0) \) is the set of non decorated planar trees. The weight of an element of \( T(0) \) is the number of its vertices.

2. Suppose that \( T(n) \) is defined. Then \( T(n+1) \) is the set of planar trees defined by:

   (a) The elements of \( T(n+1) \) are partially decorated planar trees.

   (b) The vertices of the elements of \( T(n+1) \) can eventually be decorated by a pair \( (t, k) \), with \( t \in T(n) \) and \( k \) an integer in {1, \ldots, \text{weight}(t) - 1}.  

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(c) The weight of an element of \( \mathcal{T}(n) \) is the sum of the number of its vertices and of the weights of the trees of \( \mathcal{T}(n) \) that appear in its decorations.

Inductively, for all \( n \in \mathbb{N} \), \( \mathcal{T}(n) \subseteq \mathcal{T}(n+1) \). We put:

\[
\mathcal{T} = \bigcup_{n \in \mathbb{N}} \mathcal{T}(n).
\]

**Examples.**

1. Elements of \( \mathcal{T} \) of weight 1: \( \ldots \)
2. Elements of \( \mathcal{T} \) of weight 2: \( 1 \).
3. Elements of \( \mathcal{T} \) of weight 3: \( \vee, 1, \cdot, \tau \), with \( T = (1,1) \).
4. Elements of \( \mathcal{T} \) of weight 4:
   
   (a) \( \vee, \vee, \vee, \vee \),
   
   (b) \( \cdot, \tau \), with \( T = (\vee,1) \), or \( T = (\vee,2) \), or \( T = (\hat{1},1) \), or \( T = (\hat{1},2) \), or \( T = (\cdot,\tau;1) \) with \( T' = (1,1) \), or \( T = (\cdot,\tau;2) \) with \( T' = (1,1) \),
   
   (c) \( 1^T \), with \( T = (1,1) \), \( 1_T \), with \( T = (1,1) \).

We can then define a basis \( (p_t)_{t \in \mathcal{T}} \) of \( \text{Prim}_{\text{coAss}}(\mathbf{FQSym}) \) inductively in the following way:

1. \( p_\tau = F_{(1)} \).
2. If \( t = \cdot, \tau \), with \( T = (t',i) \), then \( p_t = \Phi_i(p_{t'}) \).
3. If \( t \) is not a single root, let \( t_1, \ldots, t_{n-1} \) be the children of its roots, from left to right, and \( t_n \) its root. Then \( p_t = <p_{t_{n-1}}, \ldots, p_{t_1}, p_{t_n}> \).

By the preceding results:

**Theorem 10** \( (p_t)_{t \in \mathcal{T}} \) is a basis of \( \text{Prim}_{\text{coAss}}(\mathbf{FQSym}) \). A basis of \( \text{Prim}_{\text{coDend}}(\mathbf{FQSym}) \) is given by the \( p_t \)'s, where \( t \) is a single root.

**Examples.**

1. \( p_\tau = F_{(1)} \).
2. \( p_1 = -F_{(21)} + F_{(12)} \).
3. (a) \( p_\tau = -F_{(231)} + F_{(132)} \), with \( T = (1,1) \).
   
   (b) \( p_{\vee} = F_{(231)} - F_{(132)} - F_{(312)} + F_{(213)} \).
   
   (c) \( p_1 = F_{(321)} - F_{(231)} - F_{(213)} + F_{(123)} \).
4. (a) \( p_\tau = -F_{(2431)} + F_{(1432)} \), with \( T = (\cdot,\tau;1) \), where \( T' = (1,1) \).
   
   (b) \( p_\tau = -F_{(2341)} + F_{(1342)} \), with \( T = (\cdot,\tau;2) \), where \( T' = (1,1) \).
   
   (c) \( p_\tau = F_{(2431)} - F_{(1432)} - F_{(3412)} + F_{(2413)} \), with \( T = (\vee,1) \).
   
   (d) \( p_\tau = F_{(2341)} - F_{(1342)} - F_{(3142)} + F_{(2143)} \), with \( T = (\vee,2) \).
   
   (e) \( p_\tau = F_{(3421)} - F_{(2431)} - F_{(2413)} + F_{(1423)} \), with \( T = (1,1) \).
   
   (f) \( p_\tau = F_{(3241)} - F_{(2341)} - F_{(2143)} + F_{(1243)} \), with \( T = (1,2) \).
(g) \( p \gamma = -F_{(2341)} + F_{(1342)} + F_{(3142)} + F_{(3412)} - F_{(2413)} - F_{(2143)} + F_{(3214)}. \)

(h) \( p' \gamma = -F_{(2341)} - F_{(4231)} + F_{(2341)} + F_{(3241)} + F_{(4132)} + F_{(4312)} - F_{(1342)} - F_{(3142)} - F_{(3214)} + F_{(2314)}. \)

(i) \( p' \gamma = -F_{(2341)} + F_{(2431)} + F_{(2143)} + F_{(3241)} - F_{(1243)} - F_{(2314)} - F_{(3124)}. \)

(j) \( p \gamma = -F_{(3421)} + F_{(2431)} + F_{(4231)} - F_{(3241)} + F_{(3214)} - F_{(1324)} - F_{(3124)} + F_{(2134)}. \)

(k) \( \gamma \gamma = -F_{(3421)} - F_{(3214)} + F_{(3241)} - F_{(2314)} - F_{(2134)} + F_{(1234)}. \)

(l) \( p' \gamma = F_{(2341)} + F_{(2431)} + F_{(4231)} - 2F_{(1342)} - F_{(1432)} - F_{(4132)} - F_{(3412)} + F_{(1243)} + F_{(2143)} + F_{(2413)}, \text{ with } T = (1, 1). \)

(m) \( p' \gamma = F_{(3421)} - F_{(2131)} - F_{(2314)} + F_{(1324)}, \text{ with } T = (1, 1). \)

References


