Deformation of the Hopf algebra of plane posets

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ABSTRACT. We describe and study a four parameters deformation of the two products and the coproduct of the Hopf algebra of plane posets. We obtain a family of braided Hopf algebras, which are generically self-dual. We also prove that in a particular case (when the second parameter goes to zero and the first and third parameters are equal), this deformation is isomorphic, as a self-dual braided Hopf algebra, to a deformation of the Hopf algebra of free quasi-symmetric functions **FQSym**.

KEYWORDS. Plane posets; Deformation; Braided Hopf algebras; Self- duality.

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Contents

Introduction

A *double poset* is a finite set with two partial orders. As explained in [12], the space generated by the double posets inherits two products and one coproduct, here denoted by $\rightsquigarrow$, $\natural$ and Δ, making it both a Hopf and an infinitesimal Hopf algebra [10]. Moreover, this Hopf algebra is self dual. When the second order is total, we obtain the notion of special posets, also called labelled posets [13] or shapes [1]. A double poset is *plane* if its two partial orders satisfy a (in)compatibility condition, see Definition 1. The subspace $\mathcal{H}_{\mathcal{PP}}$ generated by plane posets is stable under the two products and the coproduct, and is self-dual as a Hopf algebra [3, 4]: in particular, two Hopf pairings are defined on it, using the notion of *picture* [6, 8, 9, 14]. Moreover, as proved in [4], it is isomorphic to the Hopf algebra of free quasi-symmetric functions **FQSym**, also known as the Malvenuto-Reutenauer Hopf algebra of permutations. An explicit isomorphism Θ is given by the linear extensions of plane posets, see Definition 11.

We define in this text a four parameters deformation of the products and the coproduct of $\mathcal{H}_{\mathcal{PP}}$, together with a deformation of the two pairings and of the morphism from $\mathcal{H}_{\mathcal{PP}}$ to **FQSym**. If $q = (q_1, q_2, q_3, q_4) \in K^4$, the product $m_q(P \otimes Q)$ of two plane posets P and Q is a linear span of plane posets R such that $R = P \sqcup Q$ as a set, P and Q being plane subposets of R. The coefficients are defined with the help of the two partial orders of R, see Theorem 14, and are polynomials in q. In particular:

$$\begin{cases} m_{(1,0,0,0)} & = & \natural, \\ m_{(0,1,0,0)} & = & \natural^{op}, \\ m_{(0,0,1,0)} & = & \rightsquigarrow, \\ m_{(0,0,0,1)} & = & \rightsquigarrow^{op}. \end{cases}$$

We also obtain the product dual to the coproduct Δ (considering the basis of double posets as orthonormal) as $m_{(1,0,1,1)}$, and its opposite given by $m_{(0,1,1,1)}$. Dually, we define a family of coassociative coproducts Δ_q. For any plane poset P, $\Delta(P)$ is a linear span of terms $(P \setminus I) \otimes I$, running over the plane subposets I of P, the coefficients being polynomials in q. In the particular cases where at least one of the components of q is zero, only h-ideals, r-ideals or biideals can appear in this sum (Definition 7 and Proposition 22). We study the compatibility of Δ_q with both products $\rightsquigarrow$ and $\natural$ on $\mathcal{H}_{\mathcal{PP}}$ (Proposition 23). In particular, $(\mathcal{H}_{\mathcal{PP}}, \rightsquigarrow, \Delta_q)$ satisfies the axiom

$$\Delta_q(x \rightsquigarrow y) = \sum \sum q_3^{|x'_q||y''_q|} q_4^{|x''_q||y'_q|} (x'_q \rightsquigarrow y'_q) \otimes (x''_q \rightsquigarrow y''_q).$$

If $q_3 = 1$, it is a braided Hopf algebra, with the braiding given by $c_q(P \otimes Q) = q_4^{|P||Q|} Q \otimes P$; if $q_3 = 1$ and $q_4 = 1$, this is a Hopf algebra, and if $q_3 = 1$ and $q_4 = 0$ this is an infinitesimal Hopf algebra. If $q_4 = 1$, this is the coopposite (or the opposite) of a braided Hopf algebra. Similar results hold if we consider the second product $\natural$, permuting the roles of (q_3, q_4) and (q_1, q_2).

We define a symmetric pairing $\langle -, - \rangle_q$ such that:

$$\langle x \otimes y, \Delta_q(z) \rangle_q = \langle x \rightsquigarrow y, z \rangle_q \text{ for all } x, y, z \in \mathcal{H}_{\mathcal{PP}}.$$

If $q = (1, 0, 1, 1)$, we recover the first "classical" pairing of $\mathcal{H}_{\mathcal{PP}}$. We prove that in the case $q_2 = 0$, this pairing is nondegenerate if, and only if, $q_1 \neq 0$ (Corollary 36). Consequently, this pairing is generically nondegenerate.

The coproduct of **FQSym** is finally deformed, in such a way that the algebra morphism Θ from $\mathcal{H}_{\mathcal{PP}}$ to **FQSym** becomes compatible with Δ_q, if q has the form $q = (q_1, 0, q_1, q_4)$. Deforming the second pairing $\langle -, - \rangle'$ of $\mathcal{H}_{\mathcal{PP}}$ and the usual Hopf pairing of **FQSym**, the map Θ becomes also an isometry (Theorem 40). Consequently, the deformation $\langle -, - \rangle'_q$ is nondegenerate if, and only if, $q_1 q_4 \neq 0$.

This text is organized as follows. The first section contains reminders on the Hopf algebra of plane posets $\mathcal{H}_{\mathcal{PP}}$, its two products, its coproducts and its two Hopf pairings, and on the isomorphism Θ from $\mathcal{H}_{\mathcal{PP}}$ to **FQSym**. The deformation of the products is defined in section 2, and we also consider the compatibility of these products with several bijections on $\mathcal{PP}$ and the stability of certain families of plane posets under these products. We proceed to the dual construction in the next section, where we also study the compatibility with the two (undeformed) products. The deformation of the first pairing is described in section 4. The compatibilities with the bijections on $\mathcal{PP}$ or with the second product $\natural$ are also given, and the nondegeneracy is proved for $q = (q_1, 0, q_3, q_4)$ if $q_1 \neq 0$. The last section is devoted to the deformation of the second pairing and of the morphism to **FQSym**.

1   Backgrounds and notations

1.1   Double and plane posets

Definition 1   *1. [12] A double poset is a triple $(P, \leq_1, \leq_2)$, where P is a finite set and $\leq_1$, $\leq_2$ are two partial orders on P.*

2. A plane poset is a double poset $(P, \leq_h, \leq_r)$ such that for all $x, y \in P$, such that $x \neq y$, x and y are comparable for $\leq_h$ if, and only if, x and y are not comparable for $\leq_r$. The set of (isoclasses of) plane posets will be denoted by $\mathcal{PP}$. For all $n \in \mathbb{N}$, the set of (isoclasses of) plane posets of cardinality n will be denoted by $\mathcal{PP}(n)$.

Examples. Here are the plane posets of cardinal ≤ 4. They are given by the Hasse graph of $\leq_h$; if x and y are two vertices of this graph which are not comparable for $\leq_h$, then $x \leq_r y$ if y is more on the right than x.

$$
\begin{aligned}
\mathcal{PP}(0) &= \{1\}, \\
\mathcal{PP}(1) &= \{\cdot\}, \\
\mathcal{PP}(2) &= \{\cdot\cdot, \mathord{\mathrm{l}}\}, \\
\mathcal{PP}(3) &= \{\cdot\cdot\cdot, \cdot\mathord{\mathrm{l}}, \mathord{\mathrm{l}}\cdot, \mathord{\vee}, \mathord{\mathrm{l}}, \mathord{\wedge}\}, \\
\mathcal{PP}(4) &= \left\{
\begin{array}{c}
\cdot\cdot\cdot\cdot, \cdot\cdot\mathord{\mathrm{l}}, \cdot\mathord{\mathrm{l}}\cdot, \mathord{\mathrm{l}}\cdot\cdot, \cdot\mathord{\vee}, \mathord{\vee}\cdot, \cdot\mathord{\mathrm{l}}\mathord{\mathrm{l}}, \mathord{\mathrm{l}}\cdot, \cdot\wedge, \wedge\cdot, \mathord{\mathrm{l}}\mathord{\mathrm{l}}, \\
\mathord{\vee}, \mathord{\vee}, \mathord{\vee}, \mathord{\mathrm{Y}}, \mathord{\mathrm{l}}, \mathord{\wedge}, \mathord{\wedge}, \mathord{\wedge}, \mathord{\lambda}, \mathord{\mathrm{N}}, \mathord{\mathrm{N}}, \mathord{\mathrm{M}}, \mathord{\Diamond}
\end{array}
\right\}.
\end{aligned}
$$

The following proposition is proved in [3] (proposition 11):

Proposition 2 *Let $P \in \mathcal{PP}$. We define a relation $\leq$ on P by:*

$$(x \leq y) \text{ if } (x \leq_h y \text{ or } x \leq_r y).$$

Then $\leq$ is a total order on P.

As a consequence, substituing $\leq$ to $\leq_r$, plane posets are also *special* posets [12], that is to say double posets such that the second order is total. For any plane poset $P \in \mathcal{PP}(n)$, we shall assume that $P = \{1, \ldots, n\}$ as a totally ordered set.

The following theorem is proved in [4]:

Theorem 3 *Let σ be a permutation in the nth symmetric group $\mathfrak{S}_n$. We define a plane poset P_σ in the following way:*

- $P_\sigma = \{1, \ldots, n\}$ *as a set.*

- If $i, j \in P_\sigma$, $i \leq_h j$ if $i \leq j$ and $\sigma(i) \leq \sigma(j)$.

- If $i, j \in P_\sigma$, $i \leq_r j$ if $i \leq j$ and $\sigma(i) \geq \sigma(j)$.

Note that the total order on $\{1, \ldots, n\}$ induced by this plane poset structure is the usual one. Then for all $n \geq 0$, the following map is a bijection:

$$\Psi_n : \begin{cases} \mathfrak{S}_n & \longrightarrow & \mathcal{PP}(n) \\ \sigma & \longrightarrow & P_\sigma. \end{cases}$$

Examples. We here represent the permutation $\sigma \in \mathfrak{S}_n$ by the word $(\sigma(1) \ldots \sigma(n))$. In particular, the identity element of $\mathfrak{S}_n$ is represented by the word $(1 \ldots n)$.

$$\Psi_2((12)) = \mathord{\updownarrow}, \qquad \Psi_2((21)) = \mathord{\cdot\cdot}, \qquad \Psi_3(123)) = \mathord{\updownarrow}, \qquad \Psi_3((132)) = \mathsf{V},$$
$$\Psi_3(213)) = \mathsf{\Lambda}, \qquad \Psi_3((231)) = \mathord{\updownarrow\cdot}, \qquad \Psi_3(312)) = \mathord{\cdot\updownarrow}, \qquad \Psi_3((321)) = \mathord{\cdot\cdot\cdot}.$$

We define several bijections on $\mathcal{PP}$:

Definition 4 *Let $P = (P, \leq_h, \leq_r) \in \mathcal{PP}$. We put:*

$$\begin{cases} \iota(P) & = & (P, \leq_r, \leq_h), \\ \alpha(P) & = & (P, \geq_h, \leq_r), \\ \beta(P) & = & (P, \leq_h, \geq_r), \\ \gamma(P) & = & (P, \geq_h, \geq_r). \end{cases}$$

Remarks.

1. Graphically:

   - A Hasse graph of $\alpha(P)$ is obtained from a Hasse graph of P by a horizontal symmetry.
   - A Hasse graph of $\beta(P)$ is obtained from a Hasse graph of P by a vertical symmetry.
   - A Hasse graph of $\gamma(P)$ is obtained from a Hasse graph of P by a rotation of angle π.

2. These bijections generate a group of permutations of $\mathcal{PP}$ of cardinality 8. It is described by the following array:

$\circ$	α	β	γ	ι	$\iota \circ \alpha$	$\iota \circ \beta$	$\iota \circ \gamma$
α	Id	γ	β	$\iota \circ \beta$	$\iota \circ \gamma$	ι	$\iota \circ \alpha$
β	γ	Id	α	$\iota \circ \alpha$	ι	$\iota \circ \gamma$	$\iota \circ \beta$
γ	β	α	Id	$\iota \circ \gamma$	$\iota \circ \beta$	$\iota \circ \alpha$	ι
ι	$\iota \circ \alpha$	$\iota \circ \beta$	$\iota \circ \gamma$	Id	α	β	γ
$\iota \circ \alpha$	ι	$\iota \circ \gamma$	$\iota \circ \beta$	β	γ	Id	α
$\iota \circ \beta$	$\iota \circ \gamma$	ι	$\iota \circ \alpha$	α	Id	γ	β
$\iota \circ \gamma$	$\iota \circ \beta$	$\iota \circ \alpha$	ι	γ	β	α	Id

This is a dihedral group D_4.

1.2 Algebraic structures on plane posets

Two products are defined on $\mathcal{PP}$. The first is called *composition* in [12] and denoted by $\rightsquigarrow$ in [3].

Definition 5 *Let $P, Q \in \mathcal{PP}$.*

1. The double poset $P \rightsquigarrow Q$ is defined as follows:

- $P \rightsquigarrow Q = P \sqcup Q$ as a set, and P, Q are plane subposets of $P \rightsquigarrow Q$.

- if $x \in P$ and $y \in Q$, then $x \leq_r y$ in $P \rightsquigarrow Q$.

2. The double poset $P \updownarrow Q$ is defined as follows:

- $P \updownarrow Q = P \sqcup Q$ as a set, and P, Q are plane subposets of $P \updownarrow Q$.

- if $x \in P$ and $y \in Q$, then $x \leq_h y$ in $P \updownarrow Q$.

Examples.

1. The Hasse graph of $P \rightsquigarrow Q$ is the concatenation of the Hasse graphs of P and Q.

2. Here are examples for $\updownarrow$: $\bullet \updownarrow \mathbf{l} = \mathbf{\overset{\bullet}{l}}$, $\mathbf{l} \updownarrow \bullet = \mathbf{\overset{l}{\bullet}}$, $\bullet \updownarrow \bullet \bullet = \mathsf{V}$, $\bullet \bullet \updownarrow \bullet = \mathsf{\Lambda}$.

The vector space generated by $\mathcal{PP}$ is denoted by $\mathcal{H}_{\mathcal{PP}}$. These two products are linearly extended to $\mathcal{H}_{\mathcal{PP}}$; then $(\mathcal{H}_{\mathcal{PP}}, \rightsquigarrow)$ and $(\mathcal{H}_{\mathcal{PP}}, \updownarrow)$ are two associative, unitary algebras, sharing the same unit 1, which is the empty plane poset. Moreover, they are both graded by the cardinality of plane posets. They are free algebras, as implied that the following theorem, proved in [3]:

Theorem 6   1.   (a) Let P be a nonempty plane poset. We shall say that P is h-irreducible if for all $Q, \in \mathcal{PP}$, $P = Q \rightsquigarrow R$ implies that $Q = 1$ or $R = 1$.

   (b) Any plane poset P can be uniquely written as $P = P_1 \rightsquigarrow \ldots \rightsquigarrow P_k$, where $P_1, \ldots, P_k$ are h-irreducible. We shall say that $P_1, \ldots, P_k$ are the h-irreducible components of P.

2.   (a) Let P be a nonempty plane poset. We shall say that P is r-irreducible if for all $Q, \in \mathcal{PP}$, $P = Q \updownarrow R$ implies that $Q = 1$ or $R = 1$.

   (b) Any plane poset P can be uniquely written as $P = P_1 \updownarrow \ldots \updownarrow P_k$, where $P_1, \ldots, P_k$ are r-irreducible. We shall say that $P_1, \ldots, P_k$ are the r-irreducible components of P.

Remark. The Hasse graphs of the h-irreducible components of H are the connected components of the Hasse graph of $(P, \leq_h)$, whereas the Hasse graphs of the r-irreducible components of H are the connected components of the Hasse graph of $(P, \leq_r)$.

Definition 7 Let $P = (P, \leq_h, \leq_r)$ be a plane poset, and let $I \subseteq P$.

1. We shall say that I is a h-ideal of P, if, for all $x, y \in P$:

$$(x \in I, \ x \leq_h y) \Longrightarrow (y \in I).$$

2. We shall say that I is a r-ideal of P, if, for all $x, y \in P$:

$$(x \in I, \ x \leq_r y) \Longrightarrow (y \in I).$$

3. We shall say that I is a biideal of P if it both an h-ideal and a r-ideal. Equivalently, I is a biideal of P if, for all $x, y \in P$:

$$(x \in I, \ x \leq y) \Longrightarrow (y \in I).$$

The following proposition is proved in [3] (proposition 29):

5

Proposition 8 $\mathcal{H}_{\mathcal{PP}}$ *is given a coassociative, counitary coproduct in the following way: for any plane poset* P,

$$\Delta(P) = \sum_{I \ h\text{-ideal of } P} (P \setminus I) \otimes I.$$

Moreover, $(\mathcal{H}_{\mathcal{PP}}, \rightsquigarrow, \Delta)$ *is a Hopf algebra, and* $(\mathcal{H}_{\mathcal{PP}}, \natural, \Delta)$ *is an infinitesimal Hopf algebra [10], both graded by the cardinality of the plane posets. In other words, using Sweedler's notations* $\Delta(x) = \sum x^{(1)} \otimes x^{(2)}$, *for all* $x, y \in \mathcal{H}_{\mathcal{PP}}$:

$$\begin{cases} \Delta(x \rightsquigarrow y) & = & \sum x^{(1)} \rightsquigarrow y^{(1)} \otimes x^{(2)} \rightsquigarrow y^{(2)}, \\ \Delta(x \natural y) & = & \sum x \natural y^{(1)} \otimes y^{(2)} + \sum x^{(1)} \otimes x^{(2)} \natural y - x \otimes y. \end{cases}$$

Remarks. The following compatibilities are satisfied:

1. For all $P, Q \in \mathcal{PP}$:

$$\begin{cases} \iota(P \rightsquigarrow Q) & = & \iota(P) \natural \iota(Q), & \iota(P \natural Q) & = & \iota(P) \rightsquigarrow \iota(Q), \\ \alpha(P \rightsquigarrow Q) & = & \alpha(P) \rightsquigarrow \alpha(Q), & \alpha(P \natural Q) & = & \alpha(Q) \natural \alpha(P), \\ \beta(P \rightsquigarrow Q) & = & \beta(Q) \rightsquigarrow \beta(P), & \beta(P \natural Q) & = & \beta(P) \natural \beta(Q), \\ \gamma(P \rightsquigarrow Q) & = & \gamma(Q) \rightsquigarrow \gamma(P), & \gamma(P \natural Q) & = & \gamma(Q) \natural \gamma(P). \end{cases}$$

2. Moreover, $\Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta^{op}$, $\Delta \circ \beta = (\beta \otimes \beta) \circ \Delta$, and $\Delta \circ \gamma = (\gamma \otimes \gamma) \circ \Delta^{op}$.

1.3 Pairings

We also defined two pairings on $\mathcal{H}_{\mathcal{PP}}$, using the notion of *pictures* [6, 8, 9, 14]:

Definition 9 *Let* P, Q *be two elements of* $\mathcal{PP}$.

1. *We denote by* $S(P, Q)$ *the set of bijections* $\sigma : P \longrightarrow Q$ *such that, for all* $i, j \in P$:

   - $(i \leq_h j$ *in* $P) \Longrightarrow (\sigma(i) \leq_r \sigma(j)$ *in* $Q)$.
   - $(\sigma(i) \leq_h \sigma(j)$ *in* $Q) \Longrightarrow (i \leq_r j$ *in* $P)$.

2. *We denote by* $S'(P, Q)$ *the set of bijections* $\sigma : P \longrightarrow Q$ *such that, for all* $i, j \in P$:

   - $(i \leq_h j$ *in* $P) \Longrightarrow (\sigma(i) \leq \sigma(j)$ *in* $Q)$.
   - $(\sigma(i) \leq_h \sigma(j)$ *in* $Q) \Longrightarrow (i \leq j$ *in* $P)$.

The following theorem is proved in [3, 4, 12]:

Theorem 10 *We define two pairings:*

$$\langle -, - \rangle \ : \ \begin{cases} \mathcal{H}_{\mathcal{PP}} \otimes \mathcal{H}_{\mathcal{PP}} & \longrightarrow & K \\ P \otimes Q & \longrightarrow & \langle P, Q \rangle = Card(S(P, Q)), \end{cases}$$

$$\langle -, - \rangle' \ : \ \begin{cases} \mathcal{H}_{\mathcal{PP}} \otimes \mathcal{H}_{\mathcal{PP}} & \longrightarrow & K \\ P \otimes Q & \longrightarrow & \langle P, Q \rangle' = Card(S'(P, Q)). \end{cases}$$

They are both homogeneous, symmetric, nondegenerate Hopf pairings on the Hopf algebra $\mathcal{H}_{\mathcal{PP}} = (\mathcal{H}_{\mathcal{PP}}, \rightsquigarrow, \Delta)$.

1.4 Morphism to free quasi-symmetric functions

We here briefly recall the construction of the Hopf algebra **FQSym** of free quasi-symmetric functions, also called the Malvenuto-Reutenauer Hopf algebra [2, 11]. As a vector space, a basis of **FQSym** is given by the disjoint union of the symmetric groups $\mathfrak{S}_n$, for all $n \geq 0$. By convention, the unique element of $\mathfrak{S}_0$ is denoted by 1.

The product of **FQSym** is given, for $\sigma \in \mathfrak{S}_k$, $\tau \in \mathfrak{S}_l$, by:

$$\sigma\tau = \sum_{\epsilon \in Sh(k,l)} (\sigma \otimes \tau) \circ \epsilon,$$

where $Sh(k, l)$ is the set of (k, l)-shuffles, that is to say permutations $\epsilon \in \mathfrak{S}_{k+l}$ such that $\epsilon^{-1}(1) < \ldots < \epsilon^{-1}(k)$ and $\epsilon^{-1}(k+1) < \ldots < \epsilon^{-1}(k+l)$. In other words, the product of σ and τ is given by shifting the letters of the word representing τ by k, and then summing all the possible shufflings of this word and of the word representing σ. For example:

$$\begin{aligned}
(123)(21) &= (12354) + (12534) + (15234) + (51234) + (12543) \\
&\quad + (15243) + (51243) + (15423) + (51423) + (54123).
\end{aligned}$$

Let $\sigma \in \Sigma_n$. For all $0 \leq k \leq n$, there exists a unique triple $\left(\sigma_1^{(k)}, \sigma_2^{(k)}, \zeta_k\right) \in \mathfrak{S}_k \times \mathfrak{S}_{n-k} \times Sh(k, n-k)$ such that $\sigma = \zeta_k^{-1} \circ \left(\sigma_1^{(k)} \otimes \sigma_2^{(k)}\right)$. The coproduct of **FQSym** is then defined by:

$$\Delta(\sigma) = \sum_{k=0}^{n} \sigma_1^{(k)} \otimes \sigma_2^{(k)}.$$

For example:

$$\begin{aligned}
\Delta((43125)) &= 1 \otimes (43125) + (1) \otimes (3124) + (21) \otimes (123) \\
&\quad + (321) \otimes (12) + (4312) \otimes (1) + (43125) \otimes 1.
\end{aligned}$$

Note that $\sigma_1^{(k)}$ and $\sigma_2^{(k)}$ are obtained by cutting the word representing σ between the k-th and the $(k + 1)$-th letter, and then *standardizing* the two obtained words, that is to say applying to their letters the unique increasing bijection to $\{1, \ldots, k\}$ or $\{1, \ldots, n - k\}$. Moreover, **FQSym** has a nondegenerate, homogeneous, Hopf pairing defined by $\langle \sigma, \tau \rangle = \delta_{\sigma, \tau^{-1}}$ for all permutations σ and τ.

Definition 11 *[5, 12]*

1. *Let $P = (P, \leq_h, \leq_r)$ a plane poset. Let $x_1 < \ldots < x_n$ be the elements of P, which is totally ordered. A* linear extension *of P is a permutation $\sigma \in \mathfrak{S}_n$ such that, for all $i, j \in \{1, \ldots, n\}$:*

$$(x_i \leq_h x_j) \implies (\sigma^{-1}(i) < \sigma^{-1}(j)).$$

The set of linear extensions of P will be denoted by S_P.

2. *The following map is an isomorphism of Hopf algebras:*

$$\Theta : \begin{cases} \mathcal{H}_{\mathcal{PP}} & \longrightarrow & \textbf{FQSym} \\ P \in \mathcal{PP} & \longrightarrow & \displaystyle\sum_{\sigma \in S_P} \sigma. \end{cases}$$

moreover, for all $x, y \in \mathcal{H}_{\mathcal{PP}}$, $\langle \Theta(x), \Theta(y) \rangle' = \langle x, y \rangle$.

2 Deformation of the products

2.1 Construction

Definition 12 *Let $P \in \mathcal{PP}$ and $X, Y \subseteq P$. We put:*

1. $h_X^Y = \sharp\{(x,y) \in X \times Y \; / \; x \leq_h y \text{ in } P\}.$

2. $r_X^Y = \sharp\{(x,y) \in X \times Y \; / \; x \leq_r y \text{ in } P\}.$

Lemma 13 *Let X and Y be disjoint parts of a plane poset P. Then:*

$$h_X^Y + h_Y^X + r_X^Y + r_Y^X = |X||Y|.$$

Proof. Indeed, $h_X^Y + h_Y^X + r_X^Y + r_Y^X = \sharp\{(x,y) \in X \times Y \mid x < y \text{ or } x > y\} = |X \times Y|.$ $\qquad\square$

Remark. If it more generally possible to prove that for any parts X and Y of a plane poset:

$$h_X^Y + h_Y^X + r_X^Y + r_Y^X = 3|X \cap Y|^2 + |X||Y|.$$

Theorem 14 *Let $q = (q_1, q_2, q_3, q_4) \in K^4$. We consider the following map:*

$$m_q : \left\{ \begin{array}{ccc} \mathcal{H}_{\mathcal{PP}} \otimes \mathcal{H}_{\mathcal{PP}} & \longrightarrow & \mathcal{H}_{\mathcal{PP}} \\[2mm] P \otimes Q & \longrightarrow & \displaystyle\sum_{\substack{(R,I) \in \mathcal{PP}^2 \\ I \subseteq R, \, R \backslash I = P, \, I = Q}} q_1^{h_{R \backslash I}^I} q_2^{h_I^{R \backslash I}} q_3^{r_{R \backslash I}^I} q_4^{r_I^{R \backslash I}} R, \end{array} \right.$$

where $P, Q \in \mathcal{PP}$. Then $(\mathcal{H}_{\mathcal{PP}}, m_q)$ is an associative algebra, and its unit is the empty plane poset 1.

Proof. Let $P, Q, R \in \mathcal{PP}$. We put:

$$\left\{ \begin{array}{ccl} P_1 & = & (m_q \otimes Id) \circ m_q(P \otimes Q \otimes R), \\ P_2 & = & (Id \otimes m_q) \circ m_q(P \otimes Q \otimes R). \end{array} \right.$$

Then:

$$\left\{ \begin{array}{ccl} P_1 & = & \displaystyle\sum_{(R_1, I_1, R_2, I_2) \in E_1} q_1^{h_{R_1 \backslash I_1}^{I_1} + h_{R_2 \backslash I_2}^{I_2}} q_2^{h_{I_1}^{R_1 \backslash I_1} + h_{I_2}^{R_2 \backslash I_2}} q_3^{r_{R_1 \backslash I_1}^{I_1} + r_{R_2 \backslash I_2}^{I_2}} q_4^{r_{I_1}^{R_1 \backslash I_1} + r_{I_2}^{R_2 \backslash I_2}} R_2, \\[5mm] P_2 & = & \displaystyle\sum_{(R_1, I_1, R_2, I_2) \in E_2} q_1^{h_{R_1 \backslash I_1}^{I_1} + h_{R_2 \backslash I_2}^{I_2}} q_2^{h_{I_1}^{R_1 \backslash I_1} + h_{I_2}^{R_2 \backslash I_2}} q_3^{r_{R_1 \backslash I_1}^{I_1} + r_{R_2 \backslash I_2}^{I_2}} q_4^{r_{I_1}^{R_1 \backslash I_1} + r_{I_2}^{R_2 \backslash I_2}} R_2, \end{array} \right.$$

With:

$$\begin{array}{ccl} E_1 & = & \{(R_1, I_1, R_2, I_2) \in \mathcal{PP}^4 \; / \; I_1 \subseteq R_1, \; I_1 = Q, \; R_1 \backslash I_1 = P, \; I_2 \subseteq R_2, \; I_2 = R, \; R_2 \backslash I_2 = R_1\}, \\ E_2 & = & \{(R_1, I_1, R_2, I_2) \in \mathcal{PP}^4 \; / \; I_1 \subseteq R_1, \; I_1 = R, \; R_1 \backslash I_1 = Q, \; I_2 \subseteq R_2, \; I_2 = R_1, \; R_2 \backslash I_2 = P\}. \end{array}$$

We shall also consider:

$$E = \{(R, J_1, J_2, J_3) \in \mathcal{PP}^4 \; / \; R = J_1 \sqcup J_2 \sqcup J_3, \; J_1 = P, \; J_2 = Q, \; J_3 = R\}.$$

First step. Let us consider the following maps:

$$\phi : \left\{ \begin{array}{ccc} E_1 & \longrightarrow & E \\ (R_1, I_1, R_2, I_2) & \longrightarrow & (R_2, R_1 \backslash I_1, I_1, I_2), \end{array} \right.$$

$$\phi' : \left\{ \begin{array}{ccc} E & \longrightarrow & E_1 \\ (R, J_1, J_2, J_3) & \longrightarrow & (J_1 \sqcup J_2, J_2, R, J_3). \end{array} \right.$$

By definition of E and E_1, these maps are well-defined, and an easy computation shows that $\phi \circ \phi' = Id_E$ and $\phi' \circ \phi = Id_{E_1}$, so ϕ is a bijection. Let $(R_1, I_1, R_2, I_2) \in E_1$. We put $\phi(R_1, I_1, R_2, I_2) = (R, J_1, J_2, J_3)$. Then:

$$h_{R_1 \setminus I_1}^{I_1} + h_{R_2 \setminus I_2}^{I_2} = h_{J_1}^{J_2} + h_{J_1 \sqcup J_2}^{J_3} = h_{J_1}^{J_2} + h_{J_1}^{J_3} + h_{J_2}^{J_3}.$$

Similar computations finally give:

$$P_1 = \sum_{(R, J_1, J_2, J_3) \in E} q_1^{h_{J_1}^{J_2} + h_{J_1}^{J_3} + h_{J_2}^{J_3}} q_2^{h_{J_2}^{J_1} + h_{J_3}^{J_1} + h_{J_3}^{J_2}} q_3^{r_{J_1}^{J_2} + r_{J_1}^{J_3} + r_{J_2}^{J_3}} q_4^{r_{J_2}^{J_1} + r_{J_3}^{J_1} + r_{J_3}^{J_2}} R.$$

Second step. Let us consider the following maps:

$$\psi : \left\{ \begin{array}{ccc} E_2 & \longrightarrow & E \\ (R_1, I_1, R_2, I_2) & \longrightarrow & (R_2, R_1 \setminus R_1, R_1 \setminus I_1, I_1), \end{array} \right.$$

$$\psi' : \left\{ \begin{array}{ccc} E & \longrightarrow & E_2 \\ (R, J_1, J_2, J_3) & \longrightarrow & (J_2 \sqcup J_3, J_3, R, J_2 \sqcup J_3). \end{array} \right.$$

By definition of E and E_2, these maps are well-defined, and a simple computation shows that $\psi \circ \psi' = Id_E$ and $\psi' \circ \psi = Id_{E_2}$, so ψ is a bijection. Let $(R_1, I_1, R_2, I_2) \in E_2$. We put $\psi(R_1, I_1, R_2, I_2) = (R, J_1, J_2, J_3)$. Then:

$$h_{R_1 \setminus I_1}^{I_1} + h_{R_2 \setminus I_2}^{I_2} = h_{J_2}^{J_3} + h_{J_1}^{J_2 \sqcup J_3} = h_{J_1}^{J_2} + h_{J_1}^{J_3} + h_{J_2}^{J_3}.$$

Similar computations finally give:

$$P_2 = \sum_{(R, J_1, J_2, J_3) \in E} q_1^{h_{J_1}^{J_2} + h_{J_1}^{J_3} + h_{J_2}^{J_3}} q_2^{h_{J_2}^{J_1} + h_{J_3}^{J_1} + h_{J_3}^{J_2}} q_3^{r_{J_1}^{J_2} + r_{J_1}^{J_3} + r_{J_2}^{J_3}} q_4^{r_{J_2}^{J_1} + r_{J_3}^{J_1} + r_{J_3}^{J_2}} R.$$

So m_q is associative.

Last step. Let $P \in \mathcal{PP}$. Then:

$$P.1 = \sum_{\substack{(R, I) \in \mathcal{PP}^2 \\ I \subseteq R, \, R \setminus I = P, \, I = 1}} q_1^{h_{R_I}^{I}} q_2^{h_{I}^{R \setminus I}} q_3^{r_{R \setminus I}^{I}} q_4^{r_{I}^{R \setminus I}} R = q_1^{h_P^1} q_2^{h_1^P} q_3^{r_P^1} q_4^{r_1^P} P = P.$$

Similarly, for all $Q \in \mathcal{PP}$, $1.Q = Q$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Examples.

$$m_q(\cdot \otimes \cdot) = q_3 q_4 \cdot\cdot + q_1 q_2 \mathord{\text{!}},$$

$$m_q(\cdot \otimes \mathord{\text{!}}) = q_3^2 \cdot\mathord{\text{!}} + q_4^2 \mathord{\text{!}}\cdot + q_2(q_3 + q_4) \, \vee \, + q_1(q_3 + q_4) \, \wedge \, + (q_1^2 + q_1 q_2 + q_2^2) \mathord{\text{!}},$$

$$m_q(\cdot \otimes \cdot\cdot) = q_1^2 \, \vee \, + q_2^2 \, \wedge \, + (q_1 + q_2) q_4 \cdot \mathord{\text{!}} + (q_1 + q_2) q_3 \mathord{\text{!}} \cdot + (q_3^2 + q_3 q_4 + q_4^2) \cdot\cdot\cdot,$$

$$m_q(\mathord{\text{!}} \otimes \cdot) = q_3^2 \cdot\mathord{\text{!}} + q_4^2 \mathord{\text{!}}\cdot + q_1(q_3 + q_4) \, \vee \, + q_2(q_3 + q_4) \, \wedge \, + (q_1^2 + q_1 q_2 + q_2^2) \mathord{\text{!}},$$

$$m_q(\cdot\cdot \otimes \cdot) = q_2^2 \, \vee \, + q_1^2 \, \wedge \, + (q_1 + q_2) q_4 \cdot \mathord{\text{!}} + (q_1 + q_2) q_3 \mathord{\text{!}} \cdot + (q_3^2 + q_3 q_4 + q_4^2) \cdot\cdot\cdot.$$

Proposition 15 *Let $(q_1, q_2, q_3, q_4) \in K^4$. Then:*

$$\left\{ \begin{array}{rcl} m_{(q_1, q_2, q_3, q_4)}^{op} & = & m_{(q_2, q_1, q_4, q_3)}, \\[4pt] m_{(q_1, q_2, q_3, q_4)} \circ (\iota \otimes \iota) & = & \iota \circ m_{(q_3, q_4, q_1, q_2)}, \\[4pt] m_{(q_1, q_2, q_3, q_4)} \circ (\alpha \otimes \alpha) & = & \alpha \circ m_{(q_2, q_1, q_3, q_4)}, \\[4pt] m_{(q_1, q_2, q_3, q_4)} \circ (\beta \otimes \beta) & = & \beta \circ m_{(q_1, q_2, q_4, q_3)}, \\[4pt] m_{(q_1, q_2, q_3, q_4)} \circ (\gamma \otimes \gamma) & = & \gamma \circ m_{(q_2, q_1, q_4, q_3)}. \end{array} \right.$$

Proof. Immediate. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.2 Particular cases

Lemma 16 *Let $P \in \mathcal{PP}$ and $X \subseteq P$.*

1. *X is a h-ideal of P if, and only if, $h_X^{P \backslash X} = 0$.*

2. *X is a r-ideal of P if, and only if, $r_X^{P \backslash X} = 0$.*

3. *The h-irreducible components of P are the h-irreducible components of X and $P \backslash X$ if, and only if, $h_{P \backslash X}^X = h_X^{P \backslash X} = 0$.*

4. *The r-irreducible components of P are the r-irreducible components of X and $P \backslash X$ if, and only if, $r_{P \backslash X}^X = r_X^{P \backslash X} = 0$.*

5. *$P = X \rightsquigarrow (P \backslash X)$ if, and only if, $h_{P \backslash X}^X = h_X^{P \backslash X} = r_{P \backslash X}^X = 0$.*

6. *$P = X \, \natural \, (P \backslash X)$ if, and only if, $r_{P \backslash X}^X = r_X^{P \backslash X} = h_{P \backslash X}^X = 0$.*

Proof. We give the proofs of points 1, 3 and 5. The others are similar.

1. $\Longleftarrow$. Let $x \in X$, $y \in P$, such that $x \leq_h y$. As $h_X^{P \backslash X} = 0$, $y \notin P \backslash X$, so $y \in X$: X is a h-ideal.
$\Longrightarrow$. Then, for all $x \in X$, $y \in P \backslash X$, $x \leq_h y$ is not possible. So $h_X^{P \backslash X} = 0$.

3. $\Longrightarrow$. Let us put $P = P_1 \ldots P_k$, where $P_1, \ldots, P_k$ are the h-irreducible components of P. By hypothesis, X is the disjoint union of certain P_i's, and $P \backslash X$ is the disjoint union of the other P_i's. So $h_{P \backslash X}^X = h_X^{P \backslash X} = 0$.
$\Longleftarrow$. Let I be a h-irreducible component of P. If $I \cap X$ and $I \cap (P \backslash X)$, then for any $x \in I \cap X$ and any $y \in I \cap (P \backslash X)$, x and y are not comparable for $\leq_h$: contradiction. So I is included in I or in $P \backslash X$, so is a h-irreducible component of X or $P \backslash X$.

5. $\Longrightarrow$. If $x \in X$ and $y \in P \backslash X$, then $x <_r y$, so $h_{P \backslash X}^X = h_X^{P \backslash X} = r_{P \backslash X}^X = 0$.
$\Longleftarrow$. If $x \in X$ and $y \in P \backslash X$, by hypothesis we do not have $x <_h y$, $x >_h y$ nor $x >_r y$, so $x <_r y$. Hence, $P = X \rightsquigarrow (P \backslash X)$. $\qquad \square$

Proposition 17 *1. Let us assume that $q_3 = q_4 = 0$. Let $P = P_1 \natural \ldots \natural P_k$ and $P' = P_{k+1} \natural \ldots \natural P_{k+l}$ be two plane posets, decomposed into their r-connected components. Then:*

$$m_q(P \otimes P') = \sum_{\sigma \in Sh(k,l)} \prod_{1 \leq i \leq k < j \leq k+l} Q_{i,j}(\sigma)^{|P_i||P_j|} P_{\sigma(1)} \natural \ldots \natural P_{\sigma(k+l)},$$

where $Q_{i,j}(\sigma) = q_1$ if $\sigma^{-1}(i) < \sigma^{-1}(j)$ and q_2 if $\sigma^{-1}(i) > \sigma^{-1}(j)$.

2. *Let us assume that $q_1 = q_2 = 0$. Let $P = P_1 \rightsquigarrow \ldots \rightsquigarrow P_k$ and $P' = P_{k+1} \rightsquigarrow \ldots \rightsquigarrow P_{k+l}$ be two plane posets, decomposed into their r-connected components. Then:*

$$m_q(P \otimes P') = \sum_{\sigma \in Sh(k,l)} \prod_{1 \leq i \leq k < j \leq k+l} Q'_{i,j}(\sigma)^{|P_i||P_j|} P_{\sigma(1)} \rightsquigarrow \ldots \rightsquigarrow P_{\sigma(k+l)},$$

where $Q'_{i,j}(\sigma) = q_3$ if $\sigma^{-1}(i) < \sigma^{-1}(j)$ and q_4 if $\sigma^{-1}(i) > \sigma^{-1}(j)$.

Proof. Let us prove the first point; the proof of the second point is similar. Let us consider a plane poset R such that the coefficient of R in $m_q(P \otimes P')$ is not zero. So there exists $I \subseteq R$ such that $R \backslash I = P$ and $I = P'$. Moreover, as $q_3 = q_4 = 0$, $r_{R \backslash I}^I = r_I^{R \backslash I} = 0$. By Lemma 16-4, the r-connected components of R are the r-components of $R \backslash I$ and I. As a consequence,

there exists a (k,l)-shuffle σ, such that $R = P_{\sigma(1)} \downdownarrows \ldots \downdownarrows P_{\sigma(k+l)}$. Then $h_{R\backslash I}^I$ is the sum of $|P_i||P_j|$, where $1 \leq i \leq k < j \leq k+l$, such that $\sigma^{-1}(i) < \sigma^{-1}(j)$; $h_I^{R\backslash I}$ is the sum of $|P_i||P_j|$, where $1 \leq i \leq k < j \leq k+l$, such that $\sigma^{-1}(i) > \sigma^{-1}(j)$. This immediately implies the announced result. $\qquad\square$

Remarks.

1. The first point implies that:

   - $m_{(q_1,0,0,0)}(P \otimes P') = q_1^{|P||P'|} P \downdownarrows P'$. In particular, $m_{(1,0,0,0)} = \downdownarrows$.

   - $m_{(0,q_2,0,0)}(P \otimes P') = q_2^{|P||P'|} P' \downdownarrows P$. In particular, $m_{(0,1,0,0)} = \downdownarrows^{op}$.

   - $m_{(0,0,q_3,0)}(P \otimes P') = q_3^{|P||P'|} \rightsquigarrow PP'$. In particular, $m_{(1,0,0,0)} = \rightsquigarrow$.

   - $m_{(0,0,0,q_4)}(P \otimes P') = q_4^{|P||P'|} \rightsquigarrow P'P$. In particular, $m_{(1,0,0,0)} = \rightsquigarrow^{op}$.

2. It is possible to define m_q on the space of double posets. The same arguments prove that it is still associative. However, $m_{(1,0,0,0)}$ is not equal to $\downdownarrows$ on $\mathcal{H}_{\mathcal{PP}}$ and $m_{(0,0,1,0)}$ is not equal to $\rightsquigarrow$; for example, if we denote by $\wp_2$ the double poset with two elements x, y, x and y being not comparable for $\leq_h$ and $\leq_r$:

$$m_{(1,0,0,0)}(\,\cdot \otimes \cdot\,) = \mathfrak{l} + 2\wp_2, \qquad m_{(0,0,1,0)}(\,\cdot \otimes \cdot\,) = \cdot\cdot + 2\wp_2.$$

2.3 Subalgebras and quotients

These two particular families of plane posets are used in [3, 4]:

Definition 18 *Let $P \in \mathcal{PP}$.*

1. *We shall say that P is a* plane forest *if it does not contain $\wedge$ as a plane subposet. The set of plane forests is denoted by $\mathcal{PF}$.*

2. *We shall say that P is* WN *("without N") if it does not contain N nor N. The set of WN posets is denoted by $\mathcal{WNP}$.*

Examples. A plane poset is a plane forest if, and only if, its Hasse graph is a rooted forest.

$$
\begin{aligned}
\mathcal{PF}(0) &= \{1\}, \\
\mathcal{PF}(1) &= \{\cdot\}, \\
\mathcal{PF}(2) &= \{\cdot\cdot, \mathfrak{l}\}, \\
\mathcal{PF}(3) &= \{\cdot\cdot\cdot, \cdot\mathfrak{l}, \mathfrak{l}\cdot, \vee, \mathfrak{l}\}, \\
\mathcal{PF}(4) &= \left\{\cdot\cdot\cdot\cdot, \cdot\cdot\mathfrak{l}, \cdot\mathfrak{l}\cdot, \mathfrak{l}\cdot\cdot, \vee, \vee\cdot, \cdot\mathfrak{l}, \mathfrak{l}\cdot, \mathfrak{l}\mathfrak{l}, \mathsf{W}, \overset{\cdot}{\vee}, \overset{\cdot}{\vee}, \mathsf{Y}, \mathfrak{l}\right\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{WNP}(0) &= \{1\}, \\
\mathcal{WNP}(1) &= \{\cdot\}, \\
\mathcal{WNP}(2) &= \{\cdot\cdot, \mathfrak{l}\}, \\
\mathcal{WNP}(3) &= \{\cdot\cdot\cdot, \cdot\mathfrak{l}, \mathfrak{l}\cdot, \vee, \mathfrak{l}, \wedge\}, \\
\mathcal{WNP}(4) &= \left\{\begin{array}{c} \cdot\cdot\cdot\cdot, \cdot\cdot\mathfrak{l}, \cdot\mathfrak{l}\cdot, \mathfrak{l}\cdot\cdot, \vee, \vee\cdot, \cdot\mathfrak{l}, \mathfrak{l}\cdot, \wedge, \wedge\cdot, \mathfrak{l}\mathfrak{l}, \\ \mathsf{W}, \overset{\cdot}{\vee}, \overset{\cdot}{\vee}, \mathsf{Y}, \mathfrak{l}, \mathsf{A}, \overset{\cdot}{\wedge}, \overset{\cdot}{\wedge}, \lambda, \bowtie, \diamondsuit \end{array}\right\}.
\end{aligned}
$$

Definition 19 *We denote by:*

- $\mathcal{H}_{\mathcal{WNP}}$ the subspace of $\mathcal{H}_{\mathcal{PP}}$ generated by WN plane posets.

- $\mathcal{H}_{\mathcal{PF}}$ the subspace of $\mathcal{H}_{\mathcal{PP}}$ generated by plane forests.

- $I_{\mathcal{WNP}}$ the subspace of $\mathcal{H}_{\mathcal{PP}}$ generated by plane posets which are not WN.

- $I_{\mathcal{PF}}$ the subspace of $\mathcal{H}_{\mathcal{PP}}$ generated by plane posets which are not plane forests.

Note that $\mathcal{H}_{\mathcal{WNP}}$ and $\mathcal{H}_{\mathcal{PF}}$ are naturally identified with $\mathcal{H}_{\mathcal{PP}}/I_{\mathcal{WNP}}$ and $\mathcal{H}_{\mathcal{PP}}/I_{\mathcal{PF}}$.

Proposition 20 *Let $q = (q_1, q_2, q_3, q_4) \in K^4$.*

1. *$\mathcal{H}_{\mathcal{WNP}}$ is a subalgebra of $(\mathcal{H}_{\mathcal{PP}}, m_q)$ if and only if, $q_1 = q_2 = 0$ or $q_3 = q_4 = 0$.*

2. *$\mathcal{H}_{\mathcal{PF}}$ is a subalgebra of $(\mathcal{H}_{\mathcal{PP}}, m_q)$ if and only if, $q_1 = q_2 = 0$.*

3. *$I_{\mathcal{WNP}}$ and $I_{\mathcal{PF}}$ are ideals of $(\mathcal{H}_{\mathcal{PP}}, m_q)$.*

Proof. 1. $\Longleftarrow$. We use the notations of Proposition 17-1. If $q_3 = q_4 = 0$, let us consider two WN posets P and P'. then the P_i's are also WN, so for any $\sigma \in \mathfrak{S}_{k+l}$, $P_{\sigma^{-1}(1)} \natural \ldots \natural P_{\sigma^{-1}(k+l)}$ is WN. As a conclusion, $m_q(P \otimes P') \in \mathcal{H}_{\mathcal{WNP}}$. The proof is similar if $q_1 = q_2 = 0$, using Proposition 17-2.

1. $\Longrightarrow$. Let us consider the coefficients of $\lVert$ and $\mathbb{N}$ in certain products. We obtain:

	$\lVert$	$\mathbb{N}$
$m_q(\wedge \otimes \cdot)$	$q_1 q_3^2$	$q_1 q_4^2$
$m_q(\cdot \otimes \wedge)$	$q_2 q_4^2$	$q_2 q_3^2.$

If $\mathcal{H}_{\mathcal{WNP}}$ is a subalgebra of $(\mathcal{H}_{\mathcal{PP}}, m_q)$, then these four coefficients are zero, so, from the first row, $q_1 = 0$ or $q_3 = q_4 = 0$ and from the second row, $q_2 = 0$ or $q_3 = q_4 = 0$. As a conclusion, $q_1 = q_2 = 0$ or $q_3 = q_4 = 0$.

2. $\Longleftarrow$. We use the notations of Proposition 17-2. If $q_1 = q_2 = 0$, let us consider two plane forests P and P'. Then the P_i's are plane trees, so for any $\sigma \in \mathfrak{S}_{k+l}$, $P_{\sigma^{-1}(1)} \natural \ldots \natural P_{\sigma^{-1}(k+l)}$ is a plane forest. As a conclusion, $m_q(P \otimes P') \in \mathcal{H}_{\mathcal{PF}}$.

2. $\Longrightarrow$. Let us consider the coefficients of $\wedge$ in certain products. We obtain:

	$\wedge$
$m_q(\cdot \otimes \cdot\cdot)$	q_2^2
$m_q(\cdot\cdot \otimes \cdot)$	$q_1^2.$

If $\mathcal{H}_{\mathcal{PF}}$ is a subalgebra of $(\mathcal{H}_{\mathcal{PP}}, m_q)$, then $q_1 = q_2 = 0$.

3. Let P and P' be two plane posets such that P or P' is not WN. Let us consider a plane poset R such that the coefficient of R in $m_q(P \otimes P')$ is not zero. There exists $I \subseteq R$, such that $R \setminus I = P$ and $I = P'$. As P or P' is not WN, I or $R \setminus I$ contains $\lVert$ or $\mathbb{N}$, so R contains $\lVert$ or $\mathbb{N}$: R is not WN. So $m_q(P \otimes P') \subseteq I_{\mathcal{WNP}}$. The proof is similar for $I_{\mathcal{PF}}$, using $\wedge$ instead of $\lVert$ and $\mathbb{N}$. $\qquad\square$

3 Dual coproducts

3.1 Constructions

Dually, we give $\mathcal{H}_{\mathcal{PP}}$ a family of coproducts Δ_q, for $q \in K^4$, defined for all $P \in \mathcal{PP}$ by:

$$\Delta_q(P) = \sum_{I \subseteq P} q_1^{h_{P\setminus I}^I} q_2^{h_I^{P\setminus I}} q_3^{r_{P\setminus I}^I} q_4^{r_I^{P\setminus I}} (P \setminus I) \otimes I.$$

These coproducts are coassociative; their common counit is given by:

$$\varepsilon : \begin{cases} \mathcal{H}_{\mathcal{PP}} & \longrightarrow & K \\ P \in \mathcal{PP} & \longrightarrow & \delta_{1,P}. \end{cases}$$

Examples. We put, for all $P \in \mathcal{PP}$, nonempty, $\tilde{\Delta}_q(P) = \Delta(P) - P \otimes 1 - 1 \otimes P$.

$$\begin{aligned}
\tilde{\Delta}_q(\mathord{\text{\textbardbl}}) &= (q_1 + q_2)\mathord{\text{\raisebox{0.4ex}{.}}} \otimes \mathord{\text{\raisebox{0.4ex}{.}}}, \\
\tilde{\Delta}_q(\mathord{\text{..}}) &= (q_3 + q_4) \otimes \mathord{\text{\raisebox{0.4ex}{.}}}, \\
\tilde{\Delta}(\mathord{\text{\textbardbl}}\mathord{\text{\raisebox{0.8ex}{.}}}) &= (q_1^2 + q_1 q_2 + q_2^2)\mathord{\text{\raisebox{0.4ex}{.}}} \otimes \mathord{\text{\textbardbl}} + (q_1^2 + q_1 q_2 + q_2^2)\mathord{\text{\textbardbl}} \otimes \mathord{\text{\raisebox{0.4ex}{.}}}, \\
\tilde{\Delta}(\,\mathsf{V}\,) &= q_2(q_3 + q_4)\mathord{\text{\raisebox{0.4ex}{.}}} \otimes \mathord{\text{\textbardbl}} + q_1(q_3 + q_4)\mathord{\text{\textbardbl}} \otimes \mathord{\text{\raisebox{0.4ex}{.}}} + q_1^2\mathord{\text{\raisebox{0.4ex}{.}}} \otimes \mathord{\text{..}} + q_2^2\mathord{\text{..}} \otimes \mathord{\text{\raisebox{0.4ex}{.}}}, \\
\tilde{\Delta}(\,\mathsf{\Lambda}\,) &= q_1(q_3 + q_4)\mathord{\text{\raisebox{0.4ex}{.}}} \otimes \mathord{\text{\textbardbl}} + q_2(q_3 + q_4)\mathord{\text{\textbardbl}} \otimes \mathord{\text{\raisebox{0.4ex}{.}}} + q_2^2\mathord{\text{\raisebox{0.4ex}{.}}} \otimes \mathord{\text{..}} + q_1^2\mathord{\text{..}} \otimes \mathord{\text{\raisebox{0.4ex}{.}}}, \\
\tilde{\Delta}(\mathord{\text{\textbardbl}}\mathord{\text{\raisebox{0.4ex}{.}}}) &= q_4^2\mathord{\text{\raisebox{0.4ex}{.}}} \otimes \mathord{\text{\textbardbl}} + q_3^2\mathord{\text{\textbardbl}} \otimes \mathord{\text{\raisebox{0.4ex}{.}}} + (q_1 + q_2)q_3\mathord{\text{\raisebox{0.4ex}{.}}} \otimes \mathord{\text{..}} + (q_1 + q_2)q_4\mathord{\text{..}} \otimes \mathord{\text{\raisebox{0.4ex}{.}}}, \\
\tilde{\Delta}(\mathord{\text{\raisebox{0.4ex}{.}}}\mathord{\text{\textbardbl}}) &= q_3^2\mathord{\text{\raisebox{0.4ex}{.}}} \otimes \mathord{\text{\textbardbl}} + q_4^2\mathord{\text{\textbardbl}} \otimes \mathord{\text{\raisebox{0.4ex}{.}}} + (q_1 + q_2)q_4\mathord{\text{\raisebox{0.4ex}{.}}} \otimes \mathord{\text{..}} + (q_1 + q_2)q_3\mathord{\text{..}} \otimes \mathord{\text{\raisebox{0.4ex}{.}}}, \\
\tilde{\Delta}(\mathord{\text{...}}) &= (q_3^2 + q_3 q_4 + q_4^2)\mathord{\text{\raisebox{0.4ex}{.}}} \otimes \mathord{\text{..}} + (q_3^2 + q_3 q_4 + q_4^2)\mathord{\text{..}} \otimes \mathord{\text{..}}.
\end{aligned}$$

Dualizing Proposition 15:

Proposition 21 *Let* $(q_1, q_2, q_3, q_4) \in K^4$. *Then:*

$$\begin{cases}
\Delta_{(q_1,q_2,q_3,q_4)}^{op} &= \Delta_{(q_2,q_1,q_4,q_3)}, \\
(\iota \otimes \iota) \circ \Delta_{(q_1,q_2,q_3,q_4)} &= \Delta_{(q_3,q_4,q_1,q_2)} \circ \iota, \\
(\alpha \otimes \alpha) \circ \Delta_{(q_1,q_2,q_3,q_4)} &= \Delta_{(q_2,q_1,q_3,q_4)} \circ \alpha, \\
(\beta \otimes \beta) \circ \Delta_{(q_1,q_2,q_3,q_4)} &= \Delta_{(q_1,q_2,q_4,q_3)} \circ \beta, \\
(\gamma \otimes \gamma) \circ \Delta_{(q_1,q_2,q_3,q_4)} &= \Delta_{(q_2,q_1,q_4,q_3)} \circ \gamma.
\end{cases}$$

3.2 Particular cases

Proposition 22 *Let* $P \in \mathcal{PP}$. *Then:*

1. $\Delta_{(q,q,q,q)}(P) = \displaystyle\sum_{I \subseteq P} q^{|P\setminus I||I|}(P \setminus I) \otimes I.$

2. $\begin{cases}
\Delta_{(0,q_2,q_3,q_4)}(P) &= \displaystyle\sum_{I \ h\text{-}ideal \ of \ P} q_2^{h_{P\setminus I}^I} q_3^{r_I^{P\setminus I}} q_4^{r_{P\setminus I}^I} I \otimes (P \setminus I), \\
\Delta_{(q_1,0,q_3,q_4)}(P) &= \displaystyle\sum_{I \ h\text{-}ideal \ of \ P} q_1^{h_{P\setminus I}^I} q_4^{r_I^{P\setminus I}} q_3^{r_{P\setminus I}^I} (P \setminus I) \otimes I.
\end{cases}$

3. $\begin{cases}
\Delta_{(q_1,q_2,0,q_4)}(P) &= \displaystyle\sum_{I \ r\text{-}ideal \ of \ P} q_1^{h_I^{P\setminus I}} q_2^{h_{P\setminus I}^I} q_4^{r_{P\setminus I}^I} I \otimes (P \setminus I), \\
\Delta_{(q_1,q_2,q_3,0)}(P) &= \displaystyle\sum_{I \ r\text{-}ideal \ of \ P} q_2^{h_I^{P\setminus I}} q_1^{h_{P\setminus I}^I} q_3^{r_{P\setminus I}^I} (P \setminus I) \otimes I.
\end{cases}$

$$4. \begin{cases} \Delta_{(0,q_2,0,q_4)}(P) = \displaystyle\sum_{I \text{ biideal of } P} q_2^{h_{P\backslash I}^I} q_4^{r_{P\backslash I}^I} I \otimes (P \backslash I), \\ \Delta_{(q_1,0,q_3,0)}(P) = \displaystyle\sum_{I \text{ biideal of } P} q_1^{h_{P\backslash I}^I} q_3^{r_{P\backslash I}^I} (P \backslash I) \otimes I. \end{cases}$$

5. If $P = P_1 \cdots P_k$, where the P_i's are h-connected,

$$\Delta_{(0,0,q_3,q_4)}(P) = \sum_{I \subseteq \{1,\cdots,k\}} q_3^{\alpha_P(I)} q_4^{\alpha_P(\{1,\ldots,k\}\backslash I)} P_I \otimes P_{\{1,\cdots,k\}\backslash I},$$

with, for all $J = \{j_1, \cdots, j_l\}$, $1 \le j_1 < \cdots < j_l \le k$, $P_J = P_{j_1} \cdots P_{j_l}$, and $\alpha_P(J) = \displaystyle\sum_{i \in J, j \notin J, i < j} |P_i||P_j|$.

6. If $P = P_1 \natural \cdots \natural P_k$, where the P_i's are r-connected,

$$\Delta_{(q_1,q_2,0,0)}(P) = \sum_{I \subseteq \{1,\cdots,k\}} q_1^{\beta_P(I)} q_2^{\beta_P(\{1,\ldots,k\}\backslash I)} P_I^\natural \otimes P_{\{1,\cdots,k\}\backslash I}^\natural,$$

with, for all $J = \{j_1, \cdots, j_l\}$, $1 \le j_1 < \cdots < j_l \le k$, $P_J^\natural = P_{j_1} \natural \cdots \natural P_{j_l}$, and $\beta_P(J) = \displaystyle\sum_{i \in J, j \notin J, i < j} |P_i||P_j|$.

$$7. \begin{cases} \Delta_{(q,0,0,0)}(P) = \displaystyle\sum_{P_1 \natural P_2 = P} q^{|P_1||P_2|} P_1 \otimes P_2, \\ \Delta_{(0,q,0,0)}(P) = \displaystyle\sum_{P_1 \natural P_2 = P} q^{|P_1||P_2|} P_2 \otimes P_1, \\ \Delta_{(0,0,q,0)}(P) = \displaystyle\sum_{P_1 P_2 = P} q^{|P_1||P_2|} P_1 \otimes P_2, \\ \Delta_{(0,0,0,q)}(P) = \displaystyle\sum_{P_1 P_2 = P} q^{|P_1||P_2|} P_2 \otimes P_1. \end{cases}$$

8. $\Delta_{(0,0,0,0)}(P) = P \otimes 1 + 1 \otimes P$ if $P \ne 1$.

Proof. We only prove points 1, 2, 5, 7 and 8; the others are proved in the same way.
1. Immediate, as $h_{P\backslash I}^I + h_I^{P\backslash I} + r_{P\backslash I}^I + r_I^{P\backslash I} = |I||P \backslash I|$ by Lemma 13.

2. By Lemma 16:

$$\Delta_{(q_1,0,q_3,q_4)}(P) = \sum_{\substack{I \subseteq P \\ h_{P\backslash I}^I = 0}} q_1^{h_{P\backslash I}^I} q_3^{r_{P\backslash I}^I} q_4^{r_I^{P\backslash I}} (R \backslash I) \otimes I = \sum_{P \backslash I \ h\text{-ideal of } P} q_1^{h_{P\backslash I}^I} q_3^{r_{P\backslash I}^I} q_4^{r_I^{P\backslash I}} (P \backslash I) \otimes I.$$

As $\Delta_{(0,q_2,q_3,q_4)} = \Delta_{(0,q_2,q_4,q_3)}^{op}$, we obtain also the first assertion.

5. By Lemma 16-3:

$$\Delta_{(0,0,q_3,q_4)}(P) = \sum_{\substack{I \subseteq P \\ h_{P\backslash I}^I = h_I^{P\backslash I} = 0}} q_3^{r_{P\backslash I}^I} q_4^{r_I^{P\backslash I}} (R \backslash I) \otimes I = \sum_{I \subseteq \{1,\cdots,k\}} q_3^{r_{P\backslash P_I}^{P_I}} q_4^{r_{P_I}^{P\backslash P_I}} P_I^\otimes P_{\{1,\cdots,k\}\backslash I},$$

and it is immediate that $r_{P\backslash P_I}^{P_I} = \alpha_P(I)$ and $r_{P_I}^{P\backslash P_I} = \alpha_P(\{1,\ldots,k\} - I)$.

7. By Lemma 16:

$$\Delta_{(0,0,q,0)}(P) = \sum_{\substack{I \subseteq P \\ h_{P \setminus I}^I = h_I^{P \setminus I} = r_I^{P \setminus I} = 0}} q^{r_{P \setminus I}^I}(P \setminus I) \otimes I = \sum_{P_1 P_2 = P} q^{r_{P_1}^{P_2}} P_1 \otimes P_2 = \sum_{P_1 P_2 = P} q^{|P_1||P_2|} P_1 \otimes P_2.$$

8. Let $I \subseteq P$, such that $h_{P \setminus I}^I = h_I^{P \setminus I} = r_{P \setminus I}^I = r_I^{P \setminus I} = 0$. Then:

$$|I||P \setminus I| = h_{P \setminus I}^I + h_I^{P \setminus I} + r_{P \setminus I}^I + r_I^{P \setminus I} = 0,$$

so $I = 1$ or $I = P$. $\qquad \square$

Remark. In particular, the coproduct defined in section 1 is $\Delta_{(1,0,1,1)}$. The coproduct of deconcatenation, dual of m, is $\Delta_{(0,0,1,0)}$ and the coproduct of deconcatenation, dual of $\natural$, is $\Delta_{(1,0,0,0)}$.

3.3 Compatibilities with the products

Proposition 23 *Let $x, y \in \mathcal{H}_{\mathcal{PP}}$. We put $\Delta_q(x) = \sum x_q' \otimes x_q''$ and $\Delta_q(y) = \sum y_q' \otimes y_q''$, with the x_q''s, x_q'''s, y_q''s, y_q'''s homogeneous. Then:*

1. $\Delta_q(x \rightsquigarrow y) = \sum \sum q_3^{|x_q'||y_q''|} q_4^{|x_q''||y_q'|}(x_q' \rightsquigarrow y_q') \otimes (x_q'' \rightsquigarrow y_q'').$

2. $\Delta_q(x \natural y) = \sum \sum q_1^{|x_q'||y_q''|} q_2^{|x_q''||y_q'|}(x_q' \natural y_q') \otimes (x_q'' \natural y_q'').$

Proof. We only prove the first point; the proof of the second point is similar. For the sake of simplicity, we simply denote in this proof the composition of two plane posets P et Q by PQ, as in [12]. Let $P, Q \in \mathcal{PP}$. Then:

$$\begin{aligned}
\Delta_q(PQ) &= \sum_{I \subseteq P, J \subseteq Q} q_1^{h_{(PQ) \setminus (IJ)}^{IJ}} q_2^{h_{IJ}^{(PQ) \setminus (IJ)}} q_3^{r_{(PQ) \setminus (IJ)}^{IJ}} q_4^{r_{IJ}^{(PQ) \setminus (IJ)}} ((PQ) \setminus (IJ)) \otimes (IJ) \\
&= \sum_{I \subseteq P, J \subseteq Q} q_1^{h_{(PQ) \setminus (IJ)}^{IJ}} q_2^{h_{IJ}^{PQ \setminus IJ}} q_3^{r_{PQ \setminus IJ}^{IJ}} q_4^{r_{IJ}^{PQ \setminus IJ}} ((P \setminus I)(Q \setminus J)) \otimes (IJ).
\end{aligned}$$

Moreover, for all $I \subseteq P$, $J \subseteq Q$:

- For all $x \in P \setminus I$, $y \in J$, $x <_r y$ in PQ, so $h_{P \setminus I}^J = 0$. Similarly, $h_{Q \setminus J}^I = 0$. Hence:

$$h_{PQ \setminus IJ}^{IJ} = h_{P \setminus I}^I + h_{Q \setminus J}^J + h_{P \setminus I}^J + h_{Q \setminus J}^I = h_{P \setminus I}^I + h_{Q \setminus J}^J.$$

- In the same way, $h_{IJ}^{PQ \setminus IJ} = h_I^{P \setminus I} + h_J^{Q \setminus J} + h_J^{P \setminus I} + h_I^{Q \setminus J} = h_I^{P \setminus I} + h_J^{Q \setminus J}.$

- For all $x \in P \setminus I$, $y \in J$, $x <_r y$ in PQ, so $r_{P \setminus I}^J = |P \setminus I||J|$. For all $x \in Q \setminus J$, $y \in I$, $x >_r y$, so $r_{Q \setminus J}^I = 0$. Hence:

$$r_{PQ \setminus IJ}^{IJ} = r_{P \setminus I}^I + r_{Q \setminus J}^J + r_{P \setminus I}^J + r_{Q \setminus J}^I = h_{P \setminus I}^I + h_{Q \setminus J}^J + |P \setminus I||J|.$$

- In the same way, $r_{IJ}^{PQ \setminus IJ} = r_I^{P \setminus I} + r_J^{Q \setminus J} + r_J^{P \setminus I} + r_I^{Q \setminus J} = r_I^{P \setminus I} + r_J^{Q \setminus J} + |I||Q \setminus J|.$

So:

$$\Delta_q(PQ) = \sum \sum q_3^{|P_q'||Q_q''|} q_4^{|P_q''||Q_q'|}(P_q' Q_q') \otimes (P_q'' Q_q''),$$

which is the announced formula. $\qquad \square$

Examples.

1. If $q_3 = 1$, then $(\mathcal{H}_{\mathcal{PP}}, \rightsquigarrow, \Delta_q)$ is a braided Hopf algebra. The braiding is given, for all $P, Q \in \mathcal{PP}$, by:
$$c_{q_4}(P \otimes Q) = q_4^{|P||Q|} Q \otimes P.$$
In particular, if $q_4 = 1$, it is a Hopf algebra; if $q_4 = 0$, it is an infinitesimal Hopf algebra.

2. If $q_4 = 0$, the compatibility becomes the following:
$$\Delta_q(x \rightsquigarrow y) = \sum q_3^{|x||y_q''|}(x \rightsquigarrow y_q') \otimes y_q'' + \sum q_3^{|x_q'||y|} x_q' \otimes (x_q'' \rightsquigarrow y) - q_3^{|x||y|} x \otimes y.$$
In particular, if $q_3 = 1$, then it is an infinitesimal Hopf algebra.

3. If $q_2 = q_4 = 0$, then for all $x, y \in \mathcal{H}_{\mathcal{PP}}$:
$$\begin{aligned}
\Delta_q(x \rightsquigarrow y) &= (x \rightsquigarrow y) \otimes 1 + 1 \otimes (x \rightsquigarrow y) + \varepsilon(x)(\Delta_q(y) - y \otimes 1 - 1 \otimes y) \\
&\quad + \varepsilon(y)(\Delta_q(x) - x \otimes 1 - 1 \otimes x) + \varepsilon(x)\varepsilon(y) 1 \otimes 1.
\end{aligned}$$

In other terms, for all $x, y \in \mathcal{H}_{\mathcal{PP}}$, such that $\varepsilon(x) = \varepsilon(y) = 0$, $x \rightsquigarrow y$ is primitive.

4 Self-duality results

4.1 A first pairing on $\mathcal{H}_q$

Notations. If $P, Q \in \mathcal{PP}$, we denote by $Bij(P, Q)$ the set of bijections from P to Q.

Definition 24 *Let P, Q be two double posets and let $\sigma \in Bij(P, Q)$. We put:*
$$\begin{cases}
\phi_1(\sigma) &= \sharp\{(x,y) \in P^2 \mid x <_h y \text{ and } \sigma(x) <_h \sigma(y)\} \\
&\quad + \sharp\{(x,y) \in P^2 \mid x <_h y \text{ and } \sigma(x) >_h \sigma(y)\} \\
&\quad + \sharp\{(x,y) \in P^2 \mid x <_h y \text{ and } \sigma(x) <_r \sigma(y)\} \\
&\quad + \sharp\{(x,y) \in P^2 \mid x <_r y \text{ and } \sigma(x) <_h \sigma(y)\}, \\
\phi_2(\sigma) &= \sharp\{(x,y) \in P^2 \mid x <_h y \text{ and } \sigma(x) <_h \sigma(y)\} \\
&\quad + \sharp\{(x,y) \in P^2 \mid x <_h y \text{ and } \sigma(x) >_h \sigma(y)\} \\
&\quad + \sharp\{(x,y) \in P^2 \mid x <_h y \text{ and } \sigma(x) >_r \sigma(y)\} \\
&\quad + \sharp\{(x,y) \in P^2 \mid x <_r y \text{ and } \sigma(x) >_h \sigma(y)\}, \\
\phi_3(\sigma) &= \sharp\{(x,y) \in P^2 \mid x <_r y \text{ and } \sigma(x) <_r \sigma(y)\}, \\
\phi_4(\sigma) &= \sharp\{(x,y) \in P^2 \mid x <_r y \text{ and } \sigma(x) >_r \sigma(y)\}.
\end{cases}$$

Lemma 25 *Let P, Q be two double posets and let $\sigma \in Bij(P, Q)$. For all $i \in \{1, \ldots, 4\}$, $\phi_i(\sigma^{-1}) = \phi_i(\sigma)$.*

Proof. Indeed, as σ is a bijection:
$$\begin{aligned}
\phi_1(\sigma^{-1}) &= \sharp\{(x,y) \in Q^2 \mid x <_h y \text{ and } \sigma^{-1}(x) <_h \sigma^{-1}(y)\} \\
&\quad + \sharp\{(x,y) \in P^2 \mid x <_h y \text{ and } \sigma^{-1}(x) >_h \sigma^{-1}(y)\} \\
&\quad + \sharp\{(x,y) \in P^2 \mid x <_h y \text{ and } \sigma^{-1}(x) <_r \sigma^{-1}(y)\} \\
&\quad + \sharp\{(x,y) \in P^2 \mid x <_r y \text{ and } \sigma^{-1}(x) <_h \sigma^{-1}(y)\} \\
&= \sharp\{(x,y) \in P^2 \mid x <_h y \text{ and } \sigma(x) <_h \sigma(y)\} \\
&\quad + \sharp\{(x,y) \in P^2 \mid x >_h y \text{ and } \sigma(x) <_h \sigma(y)\} \\
&\quad + \sharp\{(x,y) \in P^2 \mid x <_r y \text{ and } \sigma(x) <_h \sigma(y)\} \\
&\quad + \sharp\{(x,y) \in P^2 \mid x <_h y \text{ and } \sigma(x) <_r \sigma(y)\} \\
&= \phi_1(\sigma).
\end{aligned}$$

The other equalities are proved similarly. $\qquad\square$

Lemma 26 *Let P_1, P_2, Q be double posets. There is a bijection:*

$$\begin{cases} Bij(P_1 \rightsquigarrow P_2, Q) & \longrightarrow & \bigcup_{I \subseteq Q} Bij(P_1, R \setminus I) \times R(P_2, I) \\ \sigma & \longrightarrow & (\sigma_{|P_1}, \sigma_{|P_2}), \text{ with } I = \sigma(P_2). \end{cases}$$

Let $\sigma \in Bij(P_1 \rightsquigarrow P_2, Q)$ and let (σ_1, σ_2) be its image by this bijection. Then:

$$\begin{cases} \phi_1(\sigma) & = & \phi_1(\sigma_1) + \phi_1(\sigma_2) + h_{Q \setminus I}^I, \\ \phi_2(\sigma) & = & \phi_2(\sigma_1) + \phi_2(\sigma_2) + h_I^{Q \setminus I}, \\ \phi_3(\sigma) & = & \phi_3(\sigma_1) + \phi_3(\sigma_2) + r_{Q \setminus I}^I, \\ \phi_4(\sigma) & = & \phi_4(\sigma_1) + \phi_4(\sigma_2) + r_I^{Q \setminus I}. \end{cases}$$

Proof. We put $P = P_1 \rightsquigarrow P_2$. if $x \in P_1$ and $y \in P_2$, then $x <_r y$, so:

$$\begin{aligned} \phi_3(\sigma) & = & \sharp\{(x,y) \in P_1^2 \mid x <_r y \text{ and } \sigma(x) <_r \sigma(y)\} \\ & & + \sharp\{(x,y) \in P_2^2 \mid x <_r y \text{ and } \sigma(x) <_r \sigma(y)\} \\ & & + \sharp\{(x,y) \in P_1 \times P_2 \mid x <_r y \text{ and } \sigma(x) <_r \sigma(y)\} \\ & & + \sharp\{(x,y) \in P_2 \times P_1 \mid x <_r y \text{ and } \sigma(x) <_r \sigma(y)\} \\ & = & \phi_3(\sigma) + \phi_3(\sigma_2) + \sharp\{(x,y) \in P_1 \times P_2 \mid \sigma(x) <_r \sigma(y)\} + 0 \\ & = & \phi_3(\sigma) + \phi_3(\sigma_2) + \sharp\{(x,y) \in (Q \setminus I) \times I \mid x <_r y\} \\ & = & \phi_3(\sigma_1) + \phi_3(\sigma_2) + r_{Q \setminus I}^I. \end{aligned}$$

The other equalities are proved similarly. $\square$

Theorem 27 *Let $q = (q_1, q_2, q_3, q_4) \in K^4$. We define a pairing on $\mathcal{H}_{\mathcal{PP}}$ by:*

$$\langle P, Q \rangle_q = \sum_{\sigma \in Bij(P,Q)} q_1^{\phi_1(\sigma)} q_2^{\phi_2(\sigma)} q_3^{\phi_3(\sigma)} q_3^{\phi_4(\sigma)}.$$

This pairing is symmetric and for all $x, y, z \in \mathcal{H}_{\mathcal{PP}}$, $\langle x \rightsquigarrow y, z \rangle_q = \langle x \otimes y, \Delta_q(z) \rangle_q$.

Proof. Let P, Q be two double posets. Then, by Lemma 25:

$$\begin{aligned} \langle P, Q \rangle_q & = & \sum_{\sigma \in Bij(P,Q)} q_1^{\phi_1(\sigma)} q_2^{\phi_2(\sigma)} q_3^{\phi_3(\sigma)} q_3^{\phi_4(\sigma)} \\ & = & \sum_{\sigma \in Bij(Q,P)} q_1^{\phi_1(\sigma^{-1})} q_2^{\phi_2(\sigma^{-1})} q_3^{\phi_3(\sigma^{-1})} q_3^{\phi_4(\sigma^{-1})} \\ & = & \sum_{\sigma \in Bij(Q,P)} q_1^{\phi_1(\sigma)} q_2^{+\phi_2(\sigma)} q_3^{\phi_3(\sigma)} q_3^{\phi_4(\sigma)} \\ & = & \langle Q, P \rangle_q. \end{aligned}$$

So this pairing is symmetric. Let P_1, P_2, Q be three double posets. By Lemma 26:

$$\begin{aligned} \langle P_1 \rightsquigarrow P_2, Q \rangle_q & = & \sum_{\sigma \in Bij(P_1 \rightsquigarrow P_2, Q)} q_1^{\phi_1(\sigma)} q_2^{\phi_2(\sigma)} q_3^{\phi_3(\sigma)} q_3^{\phi_4(\sigma)} \\ & = & \sum_{I \subseteq Q} \sum_{\sigma_1 \in Bij(P_1, Q \setminus I)} \sum_{\sigma_2 \in Bij(P_2, I)} q_1^{\phi_1(\sigma_1) + \phi_1(\sigma_2) + h_{Q \setminus I}^I} \\ & & \times q_2^{\phi_2(\sigma_1) + \phi_2(\sigma_2) + h_I^{Q \setminus I}} q_3^{\phi_3(\sigma_1) + \phi_3(\sigma_2) + r_{Q \setminus I}^I} q_4^{\phi_4(\sigma_1) + \phi_4(\sigma_2) + r_I^{Q \setminus I}} \\ & = & \sum_{I \subseteq Q} q_1^{h_{R \setminus I}^I} q_2^{h_I^{R \setminus I}} q_3^{r_{R \setminus I}^I} q_4^{r_I^{R \setminus I}} \langle P_1, Q \setminus I \rangle_q \langle P_2, I \rangle_q \\ & = & \langle P_1 \otimes P_2, \Delta_q(Q) \rangle_q. \end{aligned}$$

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So $\langle -, - \rangle_q$ is a Hopf pairing. $\qquad \square$

Remark. More generally, adapting this proof, it is possible to show that for all $x, y, z \in \mathcal{H}_{\mathcal{PP}}$:

$$\langle m_{(0,0,q,0)}(x \otimes y), \Delta(z) \rangle_{(q_1,q_2,q_3,q_4)} = \langle x \otimes y, \Delta_{(qq_1,qq_2,qq_3,qq_4)}(z) \rangle_{(q_1,q_2,q_3,q_4)}.$$

Examples. $\langle \cdot, \cdot \rangle_q = 1$. The pairing of plane posets of degree 2 is given by the following array:

	$\updownarrow$	$\cdot\cdot$
$\updownarrow$	$2q_1 q_2$	$q_1 + q_2$
$\cdot\cdot$	$q_1 + q_2$	$q_3 + q_4$

Proposition 28 *For all double posets P, Q:*

$$\langle P, Q \rangle_{(q_1,0,q_3,q_4)} = \sum_{\sigma \in S(P,Q)} q_1^{\phi_1(\sigma)} q_3^{\phi_3(\sigma)} q_4^{\phi_4(\sigma)}.$$

Consequently, $\langle -, - \rangle_{(1,0,1,1)}$ is the pairing $\langle -, - \rangle$ described in section 1.

Proof. Let $\sigma \in Bij(P,Q)$, such that the contribution of σ in the sum defining $\langle P, Q \rangle_{(q_1,0,q_3,q_4)}$ is non zero. As $q_2 = 0$, necessarily $\phi_2(\sigma) = 0$. Hence, if $x <_h y$ in P, then $\sigma(x) <_r \sigma(y)$ in Q; if $\sigma(x) <_h \sigma(y)$ in Q, then $x <_r y$ in P. So $\sigma \in S(P,Q)$. $\qquad \square$

4.2 Properties of the pairing

Lemma 29 *Let P_1, P_2, Q be double posets. There is a bijection:*

$$\begin{cases} Bij(P_1 \natural P_2, Q) & \longrightarrow & \displaystyle\bigcup_{I \subseteq Q} Bij(P_1, Q \setminus I) \times Bij(P_2, I) \\ \sigma & \longrightarrow & (\sigma_{|P_1}, \sigma_{|P_2}), \text{ with } I = \sigma(P_2). \end{cases}$$

Let $\sigma \in Bij(P_1 P_2, Q)$ and let (σ_1, σ_2) be its image by this bijection. Then:

$$\begin{cases} \phi_1(\sigma) & = & \phi_1(\sigma_1) + \phi_1(\sigma_2) + h_{Q \setminus I}^I + h_I^{Q \setminus I} + r_{Q \setminus I}^I, \\ \phi_2(\sigma) & = & \phi_2(\sigma_1) + \phi_2(\sigma_2) + h_{Q \setminus I}^I + h_I^{Q \setminus I} + r_I^{Q \setminus I}, \\ \psi_2(\sigma) & = & \psi_2(\sigma_1) + \psi_2(\sigma_2) + h_{Q \setminus I}^I + h_I^{Q \setminus I}, \\ \phi_3(\sigma) & = & \phi_3(\sigma_1) + \phi_3(\sigma_2), \\ \phi_4(\sigma) & = & \phi_4(\sigma_1) + \phi_4(\sigma_2). \end{cases}$$

Proof. We put $P = P_1 \natural P_2$. Let $x, y \in P$. If $x <_h y$, then $x, y \in P_1$, or $x, y \in P_2$, or $x \in P_1, y \in P_2$. If $x <_r y$, then $x, y \in P_1$, or $x, y \in P_2$. Moreover, if $x \in P_1$ and $x \in P_2$, then $x <_h y$. So:

$$\begin{aligned} \phi_1(\sigma) & = & \phi_1(\sigma_1) + \phi_1(\sigma_2) \\ & & + \sharp\{(x,y) \in P_1 \times P_2 \mid (x <_h y) \text{ and } \sigma(x) <_h \sigma(y))\} \\ & & + \sharp\{(x,y) \in P_1 \times P_2 \mid (x <_h y) \text{ and } \sigma(x) >_h \sigma(y))\} \\ & & + \sharp\{(x,y) \in P_1 \times P_2 \mid (x <_h y) \text{ and } \sigma(x) <_r \sigma(y))\} \\[2mm] & = & \phi_1(\sigma_1) + \phi_1(\sigma_2) \\ & & + \sharp\{(x,y) \in P_1 \times P_2 \mid \sigma(x) <_h \sigma(y))\} \\ & & + \sharp\{(x,y) \in P_1 \times P_2 \mid \sigma(x) >_h \sigma(y))\} \\ & & + \sharp\{(x,y) \in P_1 \times P_2 \mid \sigma(x) <_r \sigma(y))\} \\[2mm] & = & \phi_1(\sigma_1) + \phi_1(\sigma_2) + h_{Q \setminus I}^I + h_I^{Q \setminus I} + r_{Q \setminus I}^I. \end{aligned}$$

The other equalities are proved similarly. $\qquad \square$

Proposition 30 *For all $x, y, z \in \mathcal{H}_{\mathcal{PP}}$, $\langle x \natural y, z \rangle_q = \langle x \otimes y, \Delta_{(q_1 q_2, q_1 q_2, q_1, q_2)}(z) \rangle_q$.*

Proof. Let P_1, P_2, Q be three double posets. By Lemma 29:

$$
\begin{aligned}
\langle P_1 \natural P_2, Q \rangle_q &= \sum_{\sigma \in Bij(P_1 \natural P_2, Q)} q_1^{\phi_1(\sigma)} q_2^{\phi_2(\sigma)} q_3^{\phi_3(\sigma)} q_3^{\phi_4(\sigma)} \\
&= \sum_{I \subseteq Q} \sum_{\sigma_1 \in Bij(P_1, Q \setminus I)} \sum_{\sigma_2 \in Bij(P_2, I)} q_1^{\phi_1(\sigma_1) + \phi_1(\sigma_2) + h_{Q \setminus I}^I + h_I^{Q \setminus I} + r_{Q \setminus I}^I} \\
&\quad \times q_2^{\phi_2(\sigma_1) + \phi_2(\sigma_2) + h_{Q \setminus I}^I + h_I^{Q \setminus I} + r_I^{Q \setminus I}} q_3^{\phi_3(\sigma_1) + \phi_3(\sigma_2)} q_4^{\phi_4(\sigma_1) + \phi_4(\sigma_2)} \\
&= \sum_{I \subseteq Q} (q_1 q_2)^{h_{R \setminus I}^I} (q_1 q_2)^{h_I^{R \setminus I}} q_1^{r_{R \setminus I}^I} q_2^{r_I^{R \setminus I}} \langle P_1, Q \setminus I \rangle_q \langle P_2, I \rangle_q \\
&= \langle P_1 \otimes P_2, \Delta_{(q_1 q_2, q_1 q_2, q_1, q_2)}(Q) \rangle_q,
\end{aligned}
$$

which is the announced formula. $\qquad\square$

Remarks.

1. In particular, if $q = (1, 0, 1, 1)$, for all $P_1, P_2, Q \in \mathcal{PP}$:

$$
\langle P_1 \natural P_2, Q \rangle = \langle P_1 \otimes P_2, \Delta_{(0,0,1,0)}(Q) \rangle = \sum_{Q = Q_1 Q_2} \langle P_1 \otimes P_2, Q_1 \otimes Q_2 \rangle.
$$

This formula was already proved in [3].

2. More generally, it is possible to show that for all $x, y, z \in \mathcal{H}_{\mathcal{PP}}$:

$$
\langle m_{(q,0,0,0)}(x \otimes y), \Delta(z) \rangle_{(q_1, q_2, q_3, q_4)} = \langle x \otimes y, \Delta_{(q q_1 q_2, q q_1 q_2, q q_1, q q_2)}(z) \rangle_{(q_1, q_2, q_3, q_4)}.
$$

Proposition 31 *For all $x, y \in \mathcal{HPP}$:*

$$
\begin{aligned}
\langle x, \beta(y) \rangle_{(q_1, q_2, q_3, q_4)} &= \langle x, y \rangle_{(q_1, q_2, q_4, q_3)}, & \langle x, \gamma(y) \rangle_{(q_1, q_2, q_3, q_4)} &= \langle x, y \rangle_{(q_2, q_1, q_4, q_3)}, \\
\langle \alpha(x), \alpha(y) \rangle_{(q_1, q_2, q_3, q_4)} &= \langle x, y \rangle_{(q_2, q_1, q_3, q_4)}, & \langle \alpha(x), \beta(y) \rangle_{(q_1, q_2, q_3, q_4)} &= \langle x, y \rangle_{(q_1, q_2, q_4, q_3)}, \\
\langle \alpha(x), \gamma(y) \rangle_{(q_1, q_2, q_3, q_4)} &= \langle x, y \rangle_{(q_2, q_1, q_4, q_3)}, & \langle \beta(x), \beta(y) \rangle_{(q_1, q_2, q_3, q_4)} &= \langle x, y \rangle_{(q_2, q_1, q_3, q_4)}, \\
\langle \gamma(x), \gamma(y) \rangle_{(q_1, q_2, q_3, q_4)} &= \langle x, y \rangle_{(q_1, q_2, q_3, q_4)}.
\end{aligned}
$$

Proof. Let P, Q be two double posets. We put $P' = \alpha(P)$ and $Q' = \alpha(Q)$. Note that $Bij(P, Q) = Bij(P', Q')$. Let $\sigma \in Bij(P, Q)$. We denote it by σ' is we consider it as an element of $Bij(P', Q')$. By definition of P' and Q', it is clear that $\phi_1(\sigma') = \phi_2(\sigma)$, $\phi_2(\sigma') = \phi_2(\sigma)$, $\phi_3(\sigma') = \phi_3(\sigma)$, and $\phi_4(\sigma') = \phi_4(\sigma)$. Hence:

$$
\begin{aligned}
\langle P', Q' \rangle_{q_1, q_2, q_3, q_4} &= \sum_{\sigma' \in Bij(P', Q')} q_1^{\phi_1(\sigma')} q_2^{\phi_2(\sigma')} q_3^{\phi_3(\sigma')} q_4^{\phi_4(\sigma')} \\
&= \sum_{\sigma \in Bij(P, Q)} q_1^{\phi_2(\sigma)} q_2^{\phi_1(\sigma)} q_3^{\phi_3(\sigma)} q_4^{\phi_4(\sigma)} \\
&= \langle P, Q \rangle_{(q_2, q_1, q_3, q_4)}.
\end{aligned}
$$

The other assertions are proved in the same way. $\qquad\square$

Remark. In general, the pairing defined by $x \otimes y \longrightarrow \langle x, \alpha(y) \rangle_{(q_1, q_2, q_3, q_4)}$ is not a pairing $\langle -, - \rangle_q$. However, when $q_1 = q_2$, it is possible to prove that for all $x, y \in \mathcal{H}_{\mathcal{PP}}$, $\langle x, \alpha(y) \rangle_{(q_1, q_1, q_3, q_4)} = \langle x, y \rangle_{(q_1, q_1, q_3, q_4)}$. Similarly, $\langle \beta(x), \gamma(y) \rangle_{(q_1, q_1, q_3, q_4)} = \langle x, y \rangle_{(q_1, q_1, q_4, q_3)}$.

4.3   Comparison of pairings with colinear parameters

Proposition 32 *Let $q \in K$. We define the following map:*

$$v_q : \begin{cases} \mathcal{H}_{\mathcal{PP}} & \longrightarrow & \mathcal{H}_{\mathcal{PP}} \\ P \in \mathcal{PP} & \longrightarrow & q^{h(P)}P, \end{cases}$$

where $h(P) = \sharp\{(x,y) \in P \mid x <_h y\}$. Then v_q is an algebra and coalgebra morphism from $\left(\mathcal{H}_{\mathcal{PP}}, \rightsquigarrow, \Delta_{(qq_1,qq_2,q_3,q_4)}\right)$ to $\left(\mathcal{H}_{\mathcal{PP}}, \rightsquigarrow, \Delta_{(q_1,q_2,q_3,q_4)}\right)$. Moreover, for all $x,y \in \mathcal{H}_{\mathcal{PP}}$:

$$\langle v_q(x), v_q(y) \rangle_{(q_1,q_2,q_3,q_4)} = \langle x, y \rangle_{(qq_1,qq_2,q_3,q_4)}.$$

Proof. Let $P_1, P_2 \in \mathcal{PP}$. Then $h(P_1 \rightsquigarrow P_2) = h(P_1) + h(P_2)$, so $v_q(P_1 \rightsquigarrow P_2) = v_q(P_1) \rightsquigarrow v_q(P_2)$, and v_q is an algebra morphism. Let $P \in \mathcal{PP}$. For any $I \subseteq P$, as $P^2 = I^2 \sqcup (P \setminus I)^2 \sqcup (I \times (P \setminus I)) \sqcup ((P \setminus I) \times I)$:

$$h(P) = h(I) + h(P \setminus I) + h^I_{P \setminus I} + h^{P \setminus I}_I.$$

Consequently:

$$\begin{aligned}
\Delta_{(q_1,q_2,q_3,q_4)} \circ v_q(P) &= \sum_{I \subseteq P} (qq_1)^{h^I_{P \setminus I}} (qq_2)^{h^{P \setminus I}_I} q_3^{r^I_{P \setminus I}} q_4^{r^{P \setminus I}_I} q^{h(P \setminus I)}(P \setminus I) \otimes q^{h(I)}I \\
&= (v_q \otimes v_q) \circ \Delta_{(qq_1,qq_2,q_3,q_4)}(P).
\end{aligned}$$

So v_q is a coalgebra morphism. Let $P, Q \in \mathcal{PP}$ and let $\sigma \in Bij(P,Q)$. We define:

$$\begin{aligned}
a_1 &= \sharp\{(x,y) \in P^2 \mid x <_h y, \sigma(x) <_h \sigma(y)\} \\
a_2 &= \sharp\{(x,y) \in P^2 \mid x <_h y, \sigma(x) >_h \sigma(y)\} \\
a_3 &= \sharp\{(x,y) \in P^2 \mid x <_h y, \sigma(x) <_r \sigma(y)\} \\
a_4 &= \sharp\{(x,y) \in P^2 \mid x <_h y, \sigma(x) >_r \sigma(y)\} \\
a_5 &= \sharp\{(x,y) \in P^2 \mid x <_r y, \sigma(x) <_h \sigma(y)\} \\
a_6 &= \sharp\{(x,y) \in P^2 \mid x <_r y, \sigma(x) >_h \sigma(y)\} \\
a_7 &= \sharp\{(x,y) \in P^2 \mid x <_r y, \sigma(x) <_r \sigma(y)\} \\
a_8 &= \sharp\{(x,y) \in P^2 \mid x <_r y, \sigma(x) >_r \sigma(y)\}.
\end{aligned}$$

In order to sum up the notations, we give a following array:

$\sharp\{(x,y) \in P^2 \mid \ldots$	$x <_h y, \ldots$	$x >_h y, \ldots$	$x <_r y, \ldots$	$x >_r y, \ldots$
$\sigma(x) <_h \sigma(y)\}$	a_1	a_2	a_5	a_6
$\sigma(x) >_h \sigma(y)\}$	a_2	a_1	a_6	a_5
$\sigma(x) <_r \sigma(y)\}$	a_3	a_4	a_7	a_8
$\sigma(x) >_r \sigma(y)\}$	a_4	a_3	a_8	a_7

So $\phi_1(\sigma) = a_1 + a_2 + a_3 + a_5$, $\phi_2(\sigma) = a_1 + a_2 + a_4 + a_6$, $\phi_3(\sigma) = a_7$ and $\phi_4(\sigma) = a_8$; $h(P) = a_1 + a_2 + a_3 + a_4$ and $h(Q) = a_1 + a_2 + a_5 + a_6$. Then:

$$\begin{aligned}
q^{h(P)}q^{h(Q)}q_1^{\phi_1(\sigma)}q_2^{\phi_2(\sigma)}q_3^{\phi_3(\sigma)}q_4^{\phi_4(\sigma)} &= (q^2 q_1 q_2)^{a_1}(q^2 q_1 q_2)^{a_2}(qq_1)^{a_3}(qq_2)^{a_4}(qq_1)^{a_5}(qq_2)^{a_6}q_3^{a_7}q_4^{a_8} \\
&= (qq_1)^{\phi_1(\sigma)}(qq_2)^{\phi_2(\sigma)}q_3^{\phi_3(\sigma)}q_4^{\phi_4(\sigma)}.
\end{aligned}$$

Finally:

$$\begin{aligned}
\langle v_q(P), v_q(Q) \rangle_{(q_1,q_2,q_3,q_4)} &= \sum_{\sigma \in Bij(P,Q)} q^{h(P)}q^{h(Q)}q_1^{\phi_1(\sigma)}q_2^{\phi_2(\sigma)}q_3^{\phi_3(\sigma)}q_4^{\phi_4(\sigma)} \\
&= \sum_{\sigma \in Bij(P,Q)} (qq_1)^{\phi_1(\sigma)}(qq_2)^{\phi_2(\sigma)}q_3^{\phi_3(\sigma)}q_4^{\phi_4(\sigma)} \\
&= \langle P, Q \rangle_{(qq_1,qq_2,q_3,q_4)}.
\end{aligned}$$

So v_q is an isometry. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Remark. It is easy to prove that for $P, Q \in \mathcal{PP}$, $v_q(P \natural Q) = v_q(P) \natural v_q(Q)$.

Similarly, one can prove:

Proposition 33 *Let* $q \in K$. *We define the following map:*

$$v'_q : \begin{cases} \mathcal{H}_{\mathcal{PP}} & \longrightarrow & \mathcal{H}_{\mathcal{PP}} \\ P \in \mathcal{PP} & \longrightarrow & q^{r(P)} P, \end{cases}$$

where $r(P) = \sharp\{(x, y) \in P \mid x <_r y\}$. *Then* v'_q *is an algebra and coalgebra morphism from* $\left(\mathcal{H}_{\mathcal{PP}}, \natural, \Delta_{(q_1, q_2, qq_3, qq_4)}\right)$ *to* $\left(\mathcal{H}_{\mathcal{PP}}, \natural, \Delta_{(q_1, q_2, q_3, q_4)}\right)$. *Moreover, for all* $x, y \in \mathcal{H}_{\mathcal{PP}}$:

$$\langle v'_q(x), v'_q(y)\rangle_{(q_1, q_2, q_3, q_4)} = \langle x, y\rangle_{(q_1, q_2, qq_3, qq_4)}.$$

4.4 Non-degeneracy of the pairing

Lemma 34 *For all* $P \in \mathcal{PP}(n)$, $S(P, \iota(P))$ *is reduced to a single element and:*

$$\langle P, \iota(P)\rangle_{(q_1, 0, q_3, q_4)} = q_1^{\frac{n(n-1)}{2}}.$$

Proof. Clearly, $Id_P : P \longrightarrow P$ belongs to $S(P, \iota(P))$. Let $\sigma : P \longrightarrow P$ be a bijection. Then $\sigma \in S(P, \iota(P))$ if, and only if, for all $i, j \in P$:

- $(i \leq_h j$ in $P) \Longrightarrow (\sigma(i) \leq_h \sigma(j)$ in $P)$.

- $(\sigma(i) \leq_r \sigma(j))$ in $P \Longrightarrow (i \leq_r j$ in $P)$.

It is clear $S(P, \iota(P))$ is a submonoid of $\mathfrak{S}_P$. As the group $\mathfrak{S}_P$ is finite, $S(P, \iota(P))$ is a subgroup of $\mathfrak{S}_P$. So if $\sigma \in S(P, \iota(P))$, then $\sigma^{-1} \in S(P, \iota(P))$. Hence, if $\sigma \in S(P, \iota(P))$:

- $(i \leq_h j$ in $P) \Longleftrightarrow (\sigma(i) \leq_h \sigma(j)$ in $P)$.

- $(i \leq_r j$ in $P) \Longleftrightarrow (\sigma(i) \leq_r \sigma(j)$ in $P)$.

So σ is the unique increasing bijection from P to P, that is to say Id_P. Moreover:

$$\begin{aligned}
\phi_1(Id_P) &= \sharp\{(x, y) \in P^2 \mid x <_h y \text{ and } x <_r y\} + \sharp\{(x, y) \in P^2 \mid x <_h y \text{ and } x >_r y\} \\
&\quad + \sharp\{(x, y) \in P^2 \mid x <_h y \text{ and } x <_h y\} + \sharp\{(x, y) \in P^2 \mid x <_r y \text{ and } x <_r y\} \\
&= 0 + 0 + \sharp\{(x, y) \in P^2 \mid x <_h y\} + \sharp\{(x, y) \in P^2 \mid x <_r y\} \\
&= \sharp\{(x, y) \in P^2 \mid x < y\} \\
&= \frac{n(n-1)}{2}, \\
\phi_3(Id_P) &= \sharp\{(x, y) \in P^2 \mid (x <_r y) \text{ and } (x <_h y)\} \\
&= 0, \\
\phi_4(Id_P) &= \sharp\{(x, y) \in P^2 \mid (x >_r y) \text{ and } (x >_h y)\} \\
&= 0.
\end{aligned}$$

So $\langle P, \iota(P)\rangle_{(q_1, 0, q_2, q_3)} = q_1^{\frac{n(n-1)}{2}}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

For all $n \in \mathbb{N}$, we totally order $\mathfrak{S}_n$ lexicographically. For example, in $\mathfrak{S}_3$:

$$(123) \leq (132) \leq (213) \leq (231) \leq (312) \leq (321).$$

We then define a total order $\ll$ on $\mathcal{PP}(n)$ by $P \ll Q$ if, and only if, $\Psi_n(P) \leq \Psi_n(Q)$ in $\mathfrak{S}_n$. For example, if $n = 3$:

$$\substack{\textstyle\cdot\\[-3pt]\textstyle\cdot\\[-3pt]\textstyle\cdot} \ \ll\ \vee\ \ll\ \wedge\ \ll\ \substack{\textstyle\cdot\\[-2pt]\textstyle\cdot}\,.\ \ll\ .\substack{\textstyle\cdot\\[-2pt]\textstyle\cdot}\ \ll\ \dots.$$

For all $P \in \mathcal{PP}(n)$, we put:

$$m(P) = \min_{\ll}\{Q \in \mathcal{PP}(n) \ / \ S(P,Q) \neq \emptyset\}.$$

Lemma 35 *For all $P \in \mathcal{PP}$, $m(P) = \iota(P)$.*

Proof. By Lemma 34, $m(P) \ll \iota(P)$. Let $Q \in \mathcal{PP}$, such that $S(P,Q) \neq \emptyset$. Let us prove that $\iota(P) \ll Q$. We denote $\sigma = \Psi_n(P)$ and $\tau = \Psi_n(Q)$; we can suppose that $P = \Phi_n(\sigma)$ and $Q = \Phi_n(\tau)$. Moreover, it is not difficult to prove that $\Psi_n(\iota(P)) = (n \cdots 1) \circ \sigma$.

First step. Let us prove that $\tau(1) \geq n - \sigma(1) + 1$. In $P = \Psi_n(\sigma)$, $1 \leq_h j$ if, and only if, $\sigma(1) \leq \sigma(j)$. So there are exactly $n - \sigma(1) + 1$ elements of P which satisfy $1 \leq_h j$ in P. Let $\alpha \in S(P,Q)$ (which is non-empty by hypothesis). Then if $1 \leq_h j$, $\alpha(1) \leq_r \alpha(j)$: there are at least $n - \sigma(1) + 1$ elements of Q which satisfy $\alpha(1) \leq_r j$. Let us put $\alpha^{-1}(1) = i$. Then $\alpha(i) = 1 \leq \alpha(1)$ in Q, that is to say $1 \leq_h \alpha(1)$ or $1 \leq_r \alpha(1)$ in Q. If $1 = \alpha(i) \leq_h \alpha(1)$ in Q, then $i \leq_r 1$ in P, so $i = 1$: in both cases, $1 \leq_r \alpha(1)$ in P. So there are at least $n - \sigma(1) + 1$ elements of Q which satisfy $1 \leq_r j$. In Q, $1 \leq_r j$ if, and only if, $\tau(j) \leq \tau(1)$, so there are exactly $\tau(1)$ such elements. As a consequence, $\tau(1) \geq n - \sigma(1) + 1$. Moreover, if there is equality, necessarily $1 = \alpha(1)$ for all $\alpha \in S(P,Q)$.

Second step. We consider the assertion H_i: if $\tau(1) = n - \sigma(1) + 1, \cdots, \tau(i-1) = n - \sigma(i-1) + 1$, then $\tau(i) \geq n - \sigma(i) + 1$ and, if there is equality, then $\alpha(1) = 1, \cdots, \alpha(i) = i$ for all $\alpha \in S(P,Q)$. We proved H_0 in the first step. Let us prove H_i by induction on i. Let us assume H_{i-1}, $1 \leq i \leq n$, and let us prove H_i. Let $\alpha \in S(P,Q)$ (non empty by hypothesis). By H_{i-1}, as the equality is satisfied, $\alpha(1) = 1, \cdots, \alpha(i-1) = i-1$.

In P, the number N_i of elements j such that $i \leq_h j$ is the cardinality of $\sigma^{-1}(\{\sigma(i), \cdots, n\}) \cap \{i, \cdots, n\}$, so $N_i = n - \sigma(i) + 1 - |\{k \leq i \ / \ \sigma(k) > \sigma(i)\}|$. Using α, there exists at least N_i elements j of Q such that $\alpha(i) \leq_r j$ in Q.

Let us put $\alpha^{-1}(i) = j$. As $\alpha(1) = 1, \cdots, \alpha(i-1) = i-1$, $j \geq i$ and $i \leq \alpha(i)$. If $i \leq_h \alpha(i)$ in Q, then $j \leq_r i$ in P, so $j = i$: we always have $i \leq_r \alpha(i)$ in Q, and at least N_i elements k of Q satisfy $i \leq_r j$ in Q. Hence:

$$\begin{aligned}
N_i &\leq |\tau^{-1}(\{1, \cdots, \tau(i)\}) \cap \{i, \cdots, n\}| \\
n - \sigma(i) + 1 - |\{k \leq i \ / \ \sigma(k) > \sigma(i)\}| &\leq \tau(i) - |\{k \leq i \ / \ \tau(k) < \tau(i)\}| \\
&\leq \tau(i) - |\{k \leq i \ / \ n - \sigma(k) + 1 < n - \sigma(i) + 1\}| \\
&\leq \tau(i) - |\{k \leq i \ / \ \sigma(k) > \sigma(i)\}|,
\end{aligned}$$

so $\tau(i) \geq n - \sigma(i) + 1$. If there is equality, then necessarily $i = \alpha(i)$ for all $\alpha \in S(P,Q)$.

Conclusion. The hypothesis H_i is true for all $0 \leq i \leq n$. So $\tau \geq (n \cdots 1) \circ \sigma$ in $\mathfrak{S}_n$, so $\iota(P) \ll Q$ in $\mathcal{PP}(n)$. As a conclusion, $\iota(P) \ll m(P)$. $\qquad\square$

Corollary 36 *Let us assume that $q_2 = 0$. Then the pairing $\langle -, - \rangle_q$ is non-degenerate if, and only if, $q_1 \neq 0$.*

Proof. Let us fix an integer $n \in \mathbb{N}$. We consider the basis $\mathcal{PP}(n)$ of $\mathcal{H}(n)$, totally ordered by $\ll$. In this basis, the matrix of $\langle -, - \rangle_q$ restricted to $\mathcal{H}(n)$ has the following form, coming from

Lemmas 34 and 35:

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 & q_1^{\frac{n(n-1)}{2}} \\ \vdots & & \iddots & q_1^{\frac{n(n-1)}{2}} & * \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & q_1^{\frac{n(n-1)}{2}} & \iddots & & \vdots \\ q_1^{\frac{n(n-1)}{2}} & * & \cdots & \cdots & * \end{pmatrix},$$

so its determinant is $\pm q_1^{\frac{n!n(n-1)}{2}}$. Hence, $\langle -, - \rangle_q$ is non-degenerate if, and only if, $q_1 \neq 0$. $\square$

Remark. With the help of the isometry α (Proposition 31), it is possible to prove that if $q_1 = 0$, $\langle -, - \rangle_q$ is non-degenerate if, and only if, $q_2 \neq 0$.

Corollary 37 *If q_1, q_2, q_3, q_4 are algebraically independent over $\mathbb{Z}$, then the pairing $\langle -, - \rangle_q$ is non-degenerate.*

Proof. For all n, let us consider the matrix of the pairing restricted to $\mathcal{H}_{\mathcal{PP}}(n)$ in the basis formed by the plane posets of degree n. Its determinant D_n is clearly an element of $\mathbb{Z}[q_1, q_2, q_3, q_4]$. Moreover, $D_n(1, 0, q_3, q_4) \neq 0$ by Corollary 36, so D_n is a non-zero polynomial. As a consequence, if q_1, q_2, q_3, q_4 are algebraically independent over $\mathbb{Z}$, $D_n(q_1, q_2, q_3, q_4) \neq 0$. $\square$

5 Morphism to free quasi-symmetric functions

5.1 A second Hopf pairing on $\mathcal{H}_{(q_1, 0, q_1, q_4)}$

We here assume that $q_2 = 0$ and $q_1 = q_3$. For all double poset P,

$$\Delta_{(q_1, 0, q_1, q_4)}(P) = \sum_{I \ h\text{-ideal of } P} q_1^{h_{R\setminus I}^I + r_{R\setminus I}^I} q_4^{r_I^{P\setminus I}} (P \setminus I) \otimes I.$$

For any h-ideal I of P, $h_I^{P\setminus I} = 0$, so:

$$\begin{aligned} h_{R\setminus I}^I + r_{R\setminus I}^I &= \sharp\{(x, y) \in (P \setminus I) \times I \mid x < y\}, \\ r_I^{P\setminus I} &= r_I^{P\setminus I} + h_I^{P\setminus I} \\ &= \sharp\{(x, y) \in (P \setminus I) \times I \mid x > y\}. \end{aligned}$$

Hence:

$$\Delta_{(q_1, 0, q_1, q_4)}(P) = \sum_{I \ h\text{-ideal of } P} q_1^{\sharp\{(x,y)\in(P\setminus I)\times I | x<y\}} q_4^{\sharp\{(x,y)\in(P\setminus I)\times I | x<y\}} (P \setminus I) \otimes I.$$

Notations. Let $P, Q \in \mathcal{P}(n)$ and let $\sigma \in Bij(P, Q)$. As totally ordered sets, $P = Q = \{1, \ldots, n\}$, so σ can be seen as an element of the symmetric group $\mathfrak{S}_n$. Its *length* $\ell(\sigma)$ is then its length in the Coxeter group $\mathfrak{S}_n$ [7].

Theorem 38 *We define a pairing $\langle -, - \rangle_q' : \mathcal{H}_q \otimes \mathcal{H}_q \longrightarrow K$ by:*

$$\langle P, Q \rangle_q' = \sum_{\sigma \in S'(P,Q)} q_1^{\frac{n(n-1)}{2} - \ell(\sigma)} q_4^{\ell(\sigma)},$$

for all $P, Q \in \mathcal{PP}$, with $n = deg(P)$. Then $\langle -, - \rangle_q'$ is an homogeneous symmetric Hopf pairing on the braided Hopf algebra $\mathcal{H}_q = (\mathcal{H}, \rightsquigarrow, \Delta_q)$.

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Proof. This pairing is clearly homogeneous. For any double posets P and Q, the map from $S'(P, Q)$ to $S'(Q, P)$ sending σ to its inverse is a bijection and conserves the length: this implies that the $\langle P, Q \rangle'_q = \langle Q, P \rangle'_q$.

Let P, Q, R be three double posets. There is a bijection:

$$\Theta : \begin{cases} Bij(P \leadsto Q, R) & \longrightarrow & \displaystyle\bigcup_{I \subseteq R} Bij(P, R \setminus I) \times Bij(Q, I) \\ \sigma & \longrightarrow & (\sigma_{|P}, \sigma_{|Q}), \end{cases}$$

with $I = \sigma(Q)$. Let us first prove that $\sigma \in S'(P \leadsto Q, R)$ if, and only if, I is a h-ideal of R and $(\sigma_1, \sigma_2) \in S'(P, R \setminus I) \times S'(Q, I)$.

$\implies$. Let $x' \in I$ and let $y' \in R$ such that $x' \leq_h y'$. We put $x' = \sigma(x)$ and $y' = \sigma(y)$. As $\sigma \in S'(PQ, R)$, $x \leq y$. As $x \in Q$, $y \in Q$, so $y' \in I$: I is a h-ideal of R. By restriction, $\sigma_{|P} \in S'(P, R \setminus I)$ and $\sigma_{|Q} \in S'(Q, I)$.

$\impliedby$. Let us assume that $x \leq_h y$ in $P \leadsto Q$. Then $x, y \in P$ or $x, y \in Q$. As $\sigma_{|P} \in S'(P, R \setminus I)$ and $\sigma_{|Q} \in S'(Q, I)$, $\sigma(x) \leq \sigma(Q)$ in R. Let us assume that $\sigma(x) \leq_h \sigma(y)$ in R. As I is a h-ideal, there are two possibilities:

- $\sigma(x), \sigma(y) \in R \setminus I$ or $\sigma(x), \sigma(y) \in I$. Hence, $x, y \in P$ or $x, y \in Q$. As $\sigma_{|P} \in S'(P, R \setminus I)$ and $\sigma_{|Q} \in S'(Q, I)$, $x \leq y$ in $\leadsto PQ$.

- $\sigma(x) \in R \setminus I$ and $\sigma(y) \in I$. Hence, $x \in P$ and $y \in Q$, so $x \leq y$ in $P \leadsto Q$.

Finally, we obtain a bijection:

$$\Theta : \begin{cases} S'(P \leadsto Q, R) & \longrightarrow & \displaystyle\bigcup_{I \text{ } h\text{-ideal of } R} S'(P, R \setminus I) \times S'(Q, I) \\ \sigma & \longrightarrow & (\sigma_{|P}, \sigma_{|Q}), \end{cases}$$

Let $\sigma \in S'(P \leadsto Q, R)$ and we put $\Theta(\sigma) = (\sigma_1, \sigma_2)$. Let $R \setminus I = \{i_1, \ldots, i_k\}$ and $I = \{j_1, \ldots, j_l\}$. Let $\zeta \in \mathfrak{S}_{k+l}$ defined by $\zeta^{-1}(1) = i_1 \ldots, \zeta^{-1}(k) = i_k, \zeta^{-1}(k+1) = j_1, \ldots, \zeta^{-1}(k+l) = j_l$. Then ζ is a (k, l)-shuffle and $\sigma = \zeta^{-1} \circ (\sigma_1 \otimes \sigma_2)$, so $\ell(\sigma) = \ell(\zeta) + \ell(\sigma_1) + \ell(\sigma_2)$. Moreover:

$$\ell(\zeta) = \sharp\{(p, q) \mid j_q < i_p\} = \sharp\{(i, j) \in (R \setminus I) \times I \mid j < i\} = r_I^{R \setminus I} + h_I^{R \setminus I} = r_I^{R \setminus I},$$

as I is a h-ideal of R. Finally, $\ell(\sigma) = \ell(\sigma_1) + \ell(\sigma_2) + r_I^{R \setminus I}$. Moreover, if $|R| = n$ and $|I| = k$, by Lemma 13:

$$h_{R \setminus I}^I + r_{R \setminus I}^I + r_I^{R \setminus I} = h_{R \setminus I}^I + r_{R \setminus I}^I + h_I^{R \setminus I} + r_I^{R \setminus I} = k(n - k),$$

so:

$$\begin{aligned} & h_{R \setminus I}^I + r_{R \setminus I}^I + \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} \\ = \quad & k(n-k) + \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} - r_I^{R \setminus I} \\ = \quad & \frac{n(n-1)}{2} - r_I^{R \setminus I}. \end{aligned}$$

Finally:

$$
\begin{aligned}
\langle P \rightsquigarrow Q, R\rangle_q' &= \sum_{\sigma \in S'(P \rightsquigarrow Q, R)} q_1^{\frac{n(n-1)}{2}-\ell(\sigma)} q_4^{\ell(\sigma)} \\
&= \sum_{I \; h\text{-ideal of } R} \sum_{\sigma_1 \in S'(P, R\setminus I), \sigma_2 \in S'(Q, I)} q_1^{\frac{n(n-1)}{2}-\ell(\sigma_1)-\ell(\sigma_2)-r_I^{R\setminus I}} q_4^{\ell(\sigma_1)+\ell(\sigma_2)+r_I^{R\setminus I}} \\
&= \sum_{I \; h\text{-ideal of } R} \sum_{\sigma_1 \in S'(P, R\setminus I), \sigma_2 \in S'(Q, I)} q_1^{h_{R\setminus I}^I + r_{R\setminus I}^I + \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2}} q_4^{\ell(\sigma_1)+\ell(\sigma_2)+r_I^{R\setminus I}} \\
&= \sum_{I \; h\text{-ideal of } R} q_1^{h_{R\setminus I}^I + r_{R\setminus I}^I} q_4^{r_I^{R\setminus I}} \langle P, R \setminus I\rangle_q' \langle Q, I\rangle_q' \\
&= \langle P \otimes Q, \Delta_q(R)\rangle_q'.
\end{aligned}
$$

So this pairing is a Hopf pairing on $\mathcal{H}_q$. $\qquad\square$

Remark. In particular, $\langle -, -\rangle_{(1,0,1,1)}'$ is the pairing $\langle -, -\rangle'$ of the section 1.

Examples. Here are the matrices of the pairing $\langle -, -\rangle_q'$ restricted to $\mathcal{H}_q(n)$, for $n = 1, 2, 3$.

$\cdot$	$\cdot$
$\cdot$	1

	$\mathbf{\updownarrow}$	$\cdot\,\cdot$
$\mathbf{\updownarrow}$	q_1	q_1
$\cdot\,\cdot$	q_1	$q_1 + q_4$

	$\mathbf{\updownarrow}$	$\vee$	$\wedge$	$\mathbf{\updownarrow}\,\cdot$	$\cdot\,\mathbf{\updownarrow}$	$\cdot\,\cdot\,\cdot$
$\mathbf{\updownarrow}$	q_1^3	q_1^3	q_1^3	q_1^3	q_1^3	q_1^3
$\vee$	q_1^3	$q_1^2(q_1+q_4)$	q_1^3	$q_1^2(q_1+q_4)$	q_1^3	$q_1^2(q_1+q_4)$
$\wedge$	q_1^3	q_1^3	$q_1^2(q_1+q_4)$	q_1^3	$q_1^2(q_1+q_4)$	$q_1^2(q_1+q_4)$
$\mathbf{\updownarrow}\,\cdot$	q_1^3	$q_1^2(q_1+q_4)$	q_1^3	$q_1^2(q_1+q_4)$	$q_1(q_1^2+q_1^4)$	$q_1(q_1^2+q_1q_4+q_4^2)$
$\cdot\,\mathbf{\updownarrow}$	q_1^3	q_1^3	$q_1^2(q_1+q_4)$	$q_1(q_1^2+q_1^4)$	$q_1^2(q_1+q_4)$	$q_1(q_1^2+q_1q_4+q_4^2)$
$\cdot\,\cdot\,\cdot$	q_1^3	$q_1^2(q_1+q_4)$	$q_1^2(q_1+q_4)$	$q_1(q_1^2+q_1q_4+q_4^2)$	$q_1(q_1^2+q_1q_4+q_4^2)$	$(q_1+q_4)(q_1^2+q_1q_4+q_4^2)$

Proposition 39 *For all $x, y, z \in \mathcal{H}_q$, $\langle x \natural y, z\rangle_q' = \langle x \otimes y, \Delta_{(q_1,0,q_1,0)}(z)\rangle_q'$.*

Proof. Let $P, Q, R \in \mathcal{PP}$. We consider the following map:

$$
\Theta : \begin{cases} Bij(P \natural Q, R) & \longrightarrow & \bigcup_{I \subseteq R} Bij(P, R \setminus I) \times Bij(Q, I) \\ \sigma & \longrightarrow & (\sigma_{|P}, \sigma_{|Q}), \end{cases}
$$

with $I = \sigma(Q)$. Let us first prove that $\sigma \in S'(PQ, R)$ if, and only if, I is a biideal of R and $(\sigma_1, \sigma_2) \in S'(P, R \setminus I) \times S'(Q, I)$.

$\Longrightarrow$. Obviously, $(\sigma_1, \sigma_2) \in S'(P, R \setminus I) \times S'(Q, I)$. Let $x' \in I$, $y' \in R$, such that $x' \leq y'$. We put $x' = \sigma(x)$ and $y' = \sigma(y)$. If $y \notin Q$, then $y \in P$ and $x \in Q$, so $y <_h x$. As $\sigma \in S'(P \natural Q, R)$, $y' < x'$: contradiction. So $y \in Q$ and $y' \in I$.

$\Longleftarrow$. Let $x, y \in P \natural Q$, such that $x \leq_h y$. Two cases are possible.

- If $x, y \in P$ or $x, y \in Q$, as $(\sigma_1, \sigma_2) \in S'(P, R \setminus I) \times S'(Q, I)$, $\sigma(x) \leq \sigma(y)$ in R.

- If $x \in P$ and $y \in Q$, as $I = \sigma(Q)$ is a biideal, it is not possible to have $y \leq x$, so $x \leq y$.

Let $x, y \in P \natural Q$, such that $\sigma(x) \leq_h \sigma(y)$ in R. As $I = \sigma(Q)$ is a biideal, two cases are possible.

- If $x, y \in P$ or $x, y \in Q$, as $(\sigma_1, \sigma_2) \in S'(P, R \setminus I) \times S'(Q, I)$, $x \leq y$ in $P \natural Q$.

- If $x \in P$ and $y \in Q$, then $x \leq y$ in $P \natural Q$.

Moreover, if $I = \sigma(Q)$ is a biideal of R, then $I = \{k+1, \ldots, k+l\} \subseteq R$, where $k = |P|$ and $l = |Q|$. Then $\sigma = \sigma_1 \otimes \sigma_2$, so $\ell(\sigma) = \ell(\sigma_1) + \ell(\sigma_2)$. So, with $n = |R|$:

$$
\begin{aligned}
\langle P \natural Q, R \rangle'_q &= \sum_{I \text{ biideal of } R} \sum_{\sigma_1 \in S'(P, R \setminus I), \sigma_2 \in S'(Q, I)} q_1^{\frac{n(n-1)}{2} - \ell(\sigma_1) - \ell(\sigma_2)} q_4^{\ell(\sigma_1) + \ell(\sigma_2)} \\
&= \sum_{I \text{ biideal of } R} q_1^{\frac{n(n-1)}{2} - \frac{k(k-1)}{2} - \frac{(n-k)(n-k-1)}{2}} \langle P \otimes Q, (R \setminus I) \otimes I \rangle'_q \\
&= \sum_{I \text{ biideal of } R} q_1^{(n-k)k} \langle P \otimes Q, (R \setminus I) \otimes I \rangle'_q \\
&= \sum_{I \text{ biideal of } R} q_1^{|R \setminus I| \cdot |I|} \langle P \otimes Q, (R \setminus I) \otimes I \rangle'_q \\
&= \sum_{I \text{ biideal of } R} q_1^{h_{R \setminus I}^I + r_{R \setminus I}^I} \langle P \otimes Q, (R \setminus I) \otimes I \rangle'_q \\
&= \langle P \otimes Q, \Delta_{(q_1, 0, q_1, 0)}(R) \rangle'_q,
\end{aligned}
$$

with the observation that, as I is a biideal, $h_I^{R \setminus I} = r_I^{R \setminus I} = 0$, so $|R \setminus I| \cdot |I| = h_{R \setminus I}^I + r_{R \setminus I}^I$. □

Remark. In particular, if $q = (1, 0, 1, 1)$, for all $P, Q, R \in \mathcal{PP}$:

$$
\langle P \natural Q, R \rangle' = \langle P \otimes Q, \Delta_{(1,0,1,0)}(R) \rangle' = \sum_{I \text{ biideal of } R} \langle P \otimes Q, (R \setminus I) \otimes I \rangle'.
$$

This formula was already proved in [4].

5.2 Quantization of the Hopf algebra of free quasi-symmetric functions

Theorem 40 *1. We define a coproduct on* **FQSym** *in the following way: for all $\sigma \in \mathfrak{S}_n$,*

$$
\Delta_q(\sigma) = \sum_{k=0}^{n} q_1^{k(n-k) - \ell(\sigma) + \ell(\sigma_k^{(1)}) + \ell(\sigma_k^{(2)})} q_4^{\ell(\sigma) - \ell(\sigma_k^{(1)}) - \ell(\sigma_k^{(2)})} \sigma_k^{(1)} \otimes \sigma_k^{(2)}.
$$

For all $q \in K$, **FQSym**$_q = ($**FQSym**$, m, \Delta_q)$ *is a graded braided Hopf algebra.*

2. We define a Hopf pairing on **FQSym**$_q$ *by* $\langle \sigma, \tau \rangle' = q_1^{\frac{n(n-1)}{2} - \ell(\sigma)} q_4^{\ell(\sigma)} \delta_{\sigma, \tau^{-1}}$.

3. The following map is an isomorphism of braided Hopf algebras:

$$
\Theta : \begin{cases} \mathcal{H}_q & \longrightarrow \ \mathbf{FQSym}_q \\ P \in \mathcal{PP} & \longrightarrow \ \displaystyle\sum_{\sigma \in S_P} \sigma. \end{cases}
$$

Moreover, for all $x, y \in \mathcal{H}_q$, $\langle \Theta(x), \Theta(y) \rangle'_q = \langle x, y \rangle'_q$.

Remarks.

1. Let $\sigma \in \mathfrak{S}_n$. If $0 \leq k \leq n$, we put $\sigma(\{1, \ldots, k\}) = \{i_1, \ldots, i_k\}$ and $\sigma(\{k+1, \ldots, n\}) = \{j_1, \ldots, j_l\}$. Let ζ be the (k, l)-shuffle defined by $\zeta^{-1}(1) = i_1 \ldots, \zeta^{-1}(k) = i_k, \zeta^{-1}(k+1) = j_1, \ldots, \zeta^{-1}(k+l) = j_l$. Then $\sigma = \zeta^{-1} \circ (\sigma_k^{(1)} \otimes \sigma_k^{(2)})$, and $\ell(\sigma) = \ell(\zeta) + \ell(\sigma_k^{(1)}) + \ell(\sigma_k^{(2)})$. Moreover, $0 \leq \ell(\zeta) \leq k(n-k)$, so:

$$0 \leq \ell(\sigma) - \ell(\sigma_k^{(1)}) - \ell(\sigma_k^{(2)}) \leq k(n-k).$$

As a consequence, the coproduct on $\mathbf{FQSym}_q$ is well-defined, even if $q_1 = 0$ or $q_4 = 0$.

2. It is clear that the pairing $\langle -, - \rangle_q'$ defined on $\mathbf{FQSym}_q$ is non degenerate if, and only if, $q_1 \neq 0$ and $q_2 \neq 0$.

Proof. From [12], we already know that Θ is an algebra isomorphism. Moreover, it is also proved there (Lemma 3.1) that $S'(P, Q) = S(P) \cap S(Q)^{-1}$ for any $P, Q \in PP$, so:

$$
\begin{aligned}
\langle \Theta(P), \Theta(Q) \rangle_q' &= \sum_{\sigma \in S(P), \tau \in S(Q)} \langle \sigma, \tau \rangle_q' \\
&= \sum_{\sigma \in S(P), \tau \in S(Q)} q_1^{\frac{n(n-1)}{2} - \ell(\sigma)} q_4^{\ell(\sigma)} \delta_{\sigma, \tau^{-1}} \\
&= \sum_{\sigma \in S(P) \cap S(Q)^{-1}} q_1^{\frac{n(n-1)}{2} - \ell(\sigma)} q_4^{\ell(\sigma)} \\
&= \sum_{\sigma \in S'(P, Q)} q_1^{\frac{n(n-1)}{2} - \ell(\sigma)} q_4^{\ell(\sigma)} \\
&= \langle P, Q \rangle_q'.
\end{aligned}
$$

So Θ is an isometry.

Consequently, for all $x \in \mathcal{H}_q$, $y, z \in \mathbf{FQSym}_q$:

$$
\begin{aligned}
\langle \Delta \circ \Theta(x), y \otimes z \rangle_q' &= \langle \Theta(x), yz \rangle_q' \\
&= \langle x, \Theta^{-1}(yz) \rangle_q' \\
&= \langle x, \Theta^{-1}(y) \Theta^{-1}(z) \rangle_q' \\
&= \langle \Delta_q(x), \Theta^{-1}(y) \otimes \Theta^{-1}(z) \rangle_q' \\
&= \langle (\Theta \otimes \Theta) \circ \Delta_q(x), y \otimes z \rangle_q'.
\end{aligned}
$$

Let us now assume that $q_1, q_4 \neq 0$. Then the pairing $\langle -, - \rangle_q'$ on $\mathbf{FQSym}_q$ is non-degenerate, so $\Delta \circ \Theta = (\Theta \otimes \Theta) \circ \Delta$. Hence, Θ is an isometric isomorphism of braided Hopf algebras. This proves the three points immediately.

Up to an extension, we can assume that K is infinite. It is not difficult to show that the set A of elements $(q_1, q_4) \in K^2$ such that the three points are satisfied is given by polynomial equations with coefficients in $\mathbb{Z}$, As it contains $(K - \{0\})^2$ from the preceding observations, which is dense in K^2, it is is equal to K^2, so the result also holds for any (q_1, q_4). $\square$

Corollary 41 *The pairing $\langle -, - \rangle_q'$ is non-degenerate if, and only if, $q_1 \neq 0$ and $q_4 \neq 0$.*

Proof. This result is implied via the isomorphism Θ from the fact that the pairing $\langle -, - \rangle_q'$ on $\mathbf{FQSym}_q$ is non-degenerate if, and only if, $q_1 \neq 0$ and $q_4 \neq 0$. $\square$

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