A combinatorial non-commutative Hopf algebra of graphs

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Abstract

A non-commutative, planar, Hopf algebra of rooted trees was defined independently by one of the authors in [14] and by R. Holtkamp in [19]. In this paper we propose such a non-commutative Hopf algebra for graphs. In order to define a non-commutative product we use a quantum field theoretical (QFT) idea, namely the one of introducing discrete scales on each edge of the graph (which, within the QFT framework, corresponds to energy scales of the associated propagators). Finally, we analyze the associated quadri-coalgebra and codendrifrom structures.

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1   Introduction

The Hopf algebra of rooted forests first appeared in the work of A. Dür [12] (and its group of characters, known as the *Butcher group*, appeared even earlier in the work of J. Butcher in numerical analysis [6]). It has been rediscovered by D. Kreimer in the context of quantum field theory [21], see also [5]. A noncommutative version, using ordered forests of planar trees, has been discovered independently by L. Foissy [14] and R. Holtkamp [19]. Remarkably enough, this Hopf algebra is self-dual. Commutative Hopf algebras of graphs have been introduced and studied by A. Connes and D. Kreimer [8, 9, 10], as a powerful algebraic tool unveiling the combinatorial structure of renormalization.

Inspired by constructive quantum field theory [28], we propose in this article a noncommutative version of a Hopf algebra of graphs, by putting a total order on the set of edges. This can be visualized by putting pairwise distinct decorations on each edge, where the decorations take values in the positive integers (or even in any totally ordered infinite set). We prove that the vector space freely generated by these *totally assigned graphs* (TAGs) is a Hopf algebra. The product is given by the disjoint union of graphs with the ordinal sum order on the edges (see Formula (3)), and the coproduct is given by Formula (6), involving subgraphs and contracted graphs. In the last section of this paper we analyze in detail the associated quadri-coalgebra and codendrifrom structures.

It is interesting to notice that what we call here TAGs have already been analyzed, from a completely different perspective (the travelling salesman problem), by O. Boruvka, already in 1926 (see [4]). He proved that the shortest spanning tree of such a graph is unique[1]. Moreover, the same problem was solved through several simpler explicit constructions by the celebrated Kruskal algorithm [22].

Let us mention here that, throughout this paper, we do not deal with graphs which are necessarily 1-particle-irreducible (*i. e.* bridge-less). Moreover we do not consider in this paper *external edges*, as it is done in quantum field theory.

2   Why discrete scales?

As already announced above, the idea of decorating the edges of a graph with discrete scales comes from quantum field theory, or more precisely from the multi-scale analysis technique used in perturbative and in constructive renormalization (see Vincent Rivasseau's book [28]). In quantum field theory each edge of a graph is associated to a *propagator* $C = 1/H$ (which, in elementary particle physics represents a particle). Introducing discrete scales comes to a

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[1]Note however that within this travelling salesman context the decoration associated to a self-loop (known as a *tadpole edge* in quantum field theoretical language) is zero, while in our context one can have strictly positive integers associated to such self-loops.

"slicing" of the propagator

$$C \;\; = \;\; \int_0^\infty e^{-\alpha H} d\alpha \; , \; \sum_{i=0}^\infty C^i \tag{1}$$

$$C^a \;\; = \;\; \int_{M^{-2a}}^{M^{-2(a-1)}} e^{-\alpha H} d\alpha \; , \; C^0 = \int_1^\infty e^{-\alpha H} d\alpha. \tag{2}$$

When some discrete integer a is associated to a given edge, this means that the propagator assigned to this edge lies within a given energy scale. One thus introduces more information (replacing graphs by "assigned graphs") which yields in turn some refinement of the analysis, as we will explain here.

When integrating over the energy scales of the internal propagators in a Feynman graph in quantum field theory, one obtains the *Feynman integral* associated to the respective graphs. Usually, these integrals are divergent. This is when *renormalization* comes in, subtracting (when possible) the divergent parts of these Feynman integrals, in a self-consistent way (see again Vincent Rivasseau's book [28] or any other textbook on renormalization). Nevertheless, these divergences only appear for high energies (the so-called *ultraviolet regime*)[2], which corresponds, within the multi-scale formalism, to the case when all the integer scales associated to the internal edges are higher then the edges associated to the external edges (see again Vincent Rivasseau's book [28] for details).

When dealing with this divergence subtraction (the subtraction of the so-called "counterterms"), an important "technical" complication is given by the issue of "overlapping divergences", which is given by overlapping subgraphs which lead, independently, to divergences. This problem is solved in an elegant way within the multi-scale analysis, where all subgraphs leading to divergences are either disjoint or nested.

Let us also emphasize that the multi-scale renormalization technique splits the counterterms into two categories: "useful" and "useless" counterterms (the useful ones being the ones corresponding to subgraphs where all the integer scales associated to the internal edges are higher then the edges associated to the external edges). This refining is not possible without the scale decoration of edges; furthermore, it also solves another major problem of renormalization, the so-called "renormalon problem" (which is an issue when one wants to sum over the contribution of each term in perturbation theory).

This versatile technique of multi-scale analysis was successfully applied for scalar quantum field theory renormalization (see again [28]), the condensed matter case [2],[13],[29], renormalization of scalar quantum field theory on the non-commutative Moyal space (see [16], [17], [18], [30] and [36]) and recently to the renormalization of quantum gravity tensor models [3],[7].

The combinatorics of the multi-scale renormalization was encoded in a Hopf algebraic framework in [20]. As already announced above, the Hopf algebraic setting of [20] is commutative,

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[2]Divergences for low energies (the *infrared regime*) can also appear in quantum field theory, but one can deal with this type of divergences in a different way. This lies outside the purpose of this section.

and the assigned graphs designed there can have equal scale integers for several edges of the same graph.

3   Non-commutative graph algebra structure

In this section we define the space of totally assigned graphs (TAG) and a non-commutative algebra structure on this space.

Definition 3.1 *A **totally ordered scale assignment** μ for a graph Γ is a total order on the set $E(\Gamma)$ of edges of Γ.*

It will be convenient to visualize the total order μ by choosing a compatible labelling, i.e. an injective increasing map from $\big(E(\Gamma), \mu\big)$ into $\mathbb{N}^* = \{1, 2, 3, \ldots\}$. There is of course an infinite number of possible labellings. The unique such map with values in $\{1, \ldots, |E(\Gamma)|\}$ will be called the *standard labelling* associated with μ.

Example 3.2 *An example of a totally ordered scale assignment with nonstandard labelling is given in Fig. 1.*
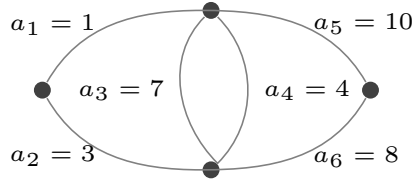

$$a_1 = 1 \qquad a_5 = 10$$
$$a_3 = 7 \qquad a_4 = 4$$
$$a_2 = 3 \qquad a_6 = 8$$

Figure 1: A graph with a totally ordered scale assignment.

Definition 3.3 *A **totally assigned graph** (TAG) is a pair (Γ, μ) formed by a graph Γ (not necessarily connected), together with a totally ordered scale assignment μ.*

Consider now a field $\mathbb{K}$ of characteristic 0, and let $\mathcal{H}$ be the $\mathbb{K}$- vector space freely spanned by TAGs. The **product** m on $\mathcal{H}$ is given by:

$$m\big((\Gamma_1, \mu), (\Gamma_2, \nu)\big) = (\Gamma_1, \mu) \cdot (\Gamma_2, \nu) := (\Gamma_1 \sqcup \Gamma_2, \mu \sqcup \nu), \tag{3}$$

where $\Gamma_1 \sqcup \Gamma_2$ is the disjoint union of the two graphs, and where $\mu \sqcup \nu$ is the *ordinal sum order*, i.e. the unique total order on $E(\Gamma_1) \sqcup E(\Gamma_2)$ which coincides with μ (resp. ν) on Γ_1 (resp. Γ_2), and such that $e_1 < e_2$ for any $e_1 \in \Gamma_1$ and $e_2 \in \Gamma_2$. Although the disjoint union of graphs is commutative, the product is not because the total orders $\mu \sqcup \nu$ and $\nu \sqcup \mu$ are different (see also Remark 3.7 below). Associativity is however obvious. The empty TAG is the empty graph, denoted by $1_{\mathcal{H}}$.
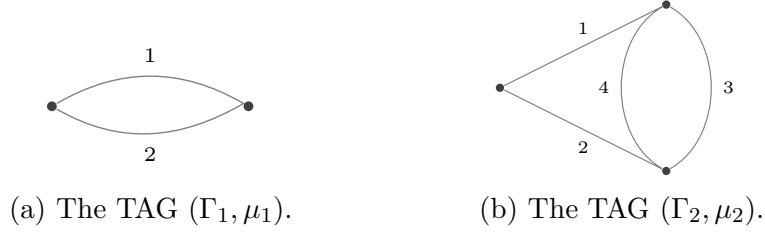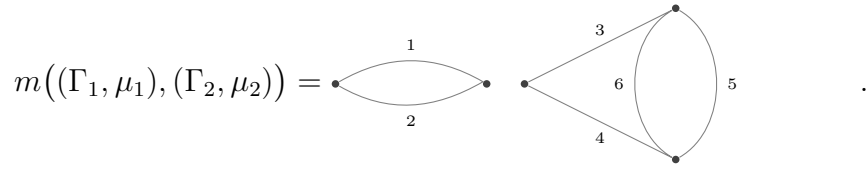
(a) The TAG (Γ_1, μ_1).

(b) The TAG (Γ_2, μ_2).

Figure 2: Two examples of TAGs.

Example 3.4 *Let (Γ_1, μ_1) and (Γ_2, μ_2) be the two graphs in Fig. 2. One has*

$$m\big((\Gamma_1, \mu_1), (\Gamma_2, \mu_2)\big) = \quad\text{}\quad .$$

One has:

Proposition 3.5 $(\mathcal{H}, m, 1_{\mathcal{H}})$ *is a free associative unitary algebra.*

Proof. The only thing which is non-trivial to prove is the freeness of the algebra. Let $G = (\Gamma, \mu)$ be a non-empty TAG. The set $E(\Gamma)$ of its edges is endowed with the total order μ. Say that G is *decomposable* if it can be split into two non-empty components $G_1 = (\Gamma_1, \mu_1)$ and $G_2 = (\Gamma_2, \mu_2)$ such that:

1. μ_i is the restriction of the total order μ to $E(\Gamma_i)$, $i = 1, 2$.

2. For any $e_1 \in E(\Gamma_1)$ and $e_2 \in E(\Gamma_2)$ we have $e_1 < e_2$.

3. The two components are disconnected

In that case we obviously have $G = G_1 \cdot G_2$ for the product just defined above. Otherwise the TAG G is called *indecomposable*. Iterating this decomposition process, we clearly can obtain any TAG G as a finite product of indecomposable TAGs:

$$G_1 \cdots G_k$$

with $G_j = (\Gamma_j, \mu_j)$, $j = 1, \ldots, k$, and where μ_j is the total order μ restricted to $E(\Gamma_j)$. The set $E(\Gamma_k)$ is the smallest terminal segment of $E(\Gamma)$ such that no edge in it hook to other edges in $E(\Gamma)$. Hence the last component G_k is uniquely defined, and the whole decomposition as well by iterating this argument. Hence, the set of TAGs endowed with the product defined above is the free monoid generated by the indecomposable TAGs; this concludes the proof. $\square$

Remark 3.6 *An indecomposable TAG is not necessarily connected (see Fig. 3).*
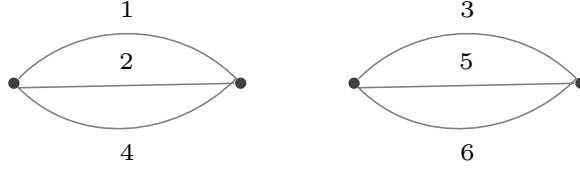
Figure 3: A non connected indecomposable TAG.

Remark 3.7 *The standard labelling of the product* $(\Gamma_1, \mu_1) \cdot (\Gamma_2, \mu_2)$ *is obtained by keeping the standard labelling for* $E(\Gamma_1)$ *and shifting the standard labelling of* $E(\Gamma_2)$ *by* $|E(\Gamma_1)|$.

Let us end this section by the following example illustrating the non-commutativity of our product:

Example 3.8 *One has*

$$m\left(\underset{2}{\overset{1}{\diamondsuit}} , \; \underset{}{\overset{1}{\rule{2cm}{0.4pt}}} \right) = \underset{2}{\overset{1}{\diamondsuit}} \; \underset{}{\overset{3}{\rule{2cm}{0.4pt}}} \tag{4}$$

and

$$m\left(\overset{1}{\rule{2cm}{0.4pt}} , \; \underset{2}{\overset{1}{\diamondsuit}} \right) = \overset{1}{\rule{2cm}{0.4pt}} \; \underset{3}{\overset{2}{\diamondsuit}} = \underset{3}{\overset{2}{\diamondsuit}} \; \overset{1}{\rule{2cm}{0.4pt}} . \tag{5}$$

4   Hopf algebra structure

Let us first give the following definitions:

Definition 4.1 *A subgraph* γ *of a graph* Γ *is the graph formed by a given subset of edges* e *of the set of edges of the graph* Γ *together with the vertices that the edges of* e *hook to in* Γ.

Let us notice that a subgraph is not necessary connected nor spanning.

Definition 4.2 *A* **totally assigned subgraph** (γ, ν) *of a given TAG* (Γ, μ) *is a subgraph* γ *of* Γ *in the sense of Definition 4.1, together with the total order* ν *on* $E(\gamma)$ *induced by* μ. *The* **shrinking** $(\Gamma, \mu)/(\gamma, \nu)$ *of a given TAG* (Γ, μ) *by a totally assigned subgraph* (γ, ν) *is defined as follows: the cograph* Γ/γ *is obtained as usual, by shrinking each connected component of* γ *on a point, and the totally ordered scale assignment* μ/ν *of the cograph* Γ/γ *is given by restricting the total order* μ *on the edges of the cograph, i.e. the edges of* Γ *which are not internal to* γ. *The TAG* $(\Gamma/\gamma, \mu/\nu)$ *is called a totally assigned cograph.*

Let us now define the coproduct $\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ as

$$\Delta\big((\Gamma, \mu)\big) = \sum_{\emptyset \subseteq (\gamma, \nu) \subseteq (\Gamma, \mu)} (\gamma, \nu) \otimes (\Gamma/\gamma, \mu/\nu) \tag{6}$$
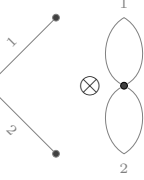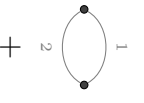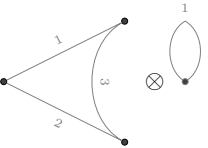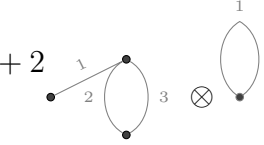
for any TAG (Γ, μ).

Example 4.3 *1) Let (Γ_1, μ_1) be the TAG in Fig. 2a.*
One has the coproduct:

$$\Delta(G, \mu) = (G, \mu) \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes (G, \mu) + 2 \underbrace{\hspace{2cm}}_{1} \otimes \bigcirc$$

2) Let (G, μ) be the TAG given in Figure 2b.

$$\Delta((G, \mu)) = (G, \mu) \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes (G, \mu) + 2 \underbrace{\hspace{1cm}}_{1} \otimes$$

$$+ 2 \underbrace{\hspace{1cm}}_{1} \otimes + 5 \otimes$$

$$+ \otimes + 2 \otimes$$
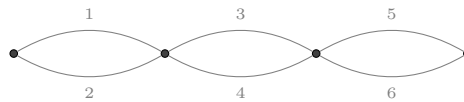
$$+ 2 \otimes$$

Example 4.4 *Let (G, μ) be the TAG given in Fig. 4.*

Figure 4: Another TAG.

$$\Delta((G,\mu)) = (G,\mu) \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes (G,\mu) + \;\cdots$$

$$+ \;\cdots\; + 3 \;\cdots\; + \;\cdots\; + \;\cdots\; + \;\cdots\; + 2 \;\cdots\; + 2 \;\cdots\; + \;\cdots$$

where we have denoted by ... for the similar graph obtained from choosing the same type of subgraph on the LHS on the coproduct; note that this can lead, in some cases, to distinct subgraphs on the RHS (sometimes just because of the distinct labels).

Lemma 4.5 *Let* (Γ, μ) *be a TAG in* $\mathcal{H}$*. Let* (γ, ν) *and* (δ, ν') *be two totally assigned subgraphs such that* $(\delta, \nu') \subseteq (\gamma, \nu) \subseteq (\Gamma, \mu)$*. Then, one has*

$$(\Gamma/\gamma, \mu/\nu) = \big((\Gamma/\delta)/(\gamma/\delta), (\mu/\nu')/(\nu/\nu')\big). \tag{7}$$

Proof. Since, $\delta \subseteq \gamma \subseteq \Gamma$, then one has $\gamma/\delta \subseteq \Gamma/\delta$. One has $\Gamma/\gamma = (\Gamma/\delta)/(\gamma/\delta)$. Moreover, the total order ν (resp. ν') is induced by restriction of μ (resp. ν or μ) to the set of edges of δ. Then $\mu/\nu = (\mu/\nu')/(\nu/\nu')$, which concludes the proof.

$\square$

Proposition 4.6 *The coproduct defined in* (6) *is coassociative.*

Proof. Let $(\Gamma, \mu) \in \mathcal{H}$. Then, one has:

$$
\begin{aligned}
(\Delta \otimes Id) \circ \Delta\big((\Gamma, \mu)\big) &= (\Delta \otimes Id) \left(\sum_{(\gamma,\nu) \subset (\Gamma,\mu)} (\gamma, \nu) \otimes (\Gamma/\gamma, \mu/\nu) \right) \\
&= \sum_{(\gamma,\nu) \subset (\Gamma,\mu)} \left(\sum_{(\gamma',\nu') \subset (\gamma,\nu)} (\gamma', \nu') \otimes (\gamma/\gamma', \nu/\nu') \right) \otimes (\Gamma/\gamma, \mu/\nu) \\
&= \sum_{\substack{(\gamma,\nu) \subset (\Gamma,\mu) \\ (\gamma',\nu') \subset (\gamma,\nu)}} (\gamma', \nu') \otimes (\gamma/\gamma', \nu/\nu') \otimes (\Gamma/\gamma, \mu/\nu).
\end{aligned}
\tag{8}
$$

and

$$
\begin{aligned}
(Id \otimes \Delta) \circ \Delta((\Gamma, \mu)) &= (Id \otimes \Delta) \left(\sum_{(\gamma,\nu)) \subset (\Gamma,\mu)} (\gamma, \nu) \otimes (\Gamma/\gamma, \mu/\nu) \right) \\
&= \sum_{(\gamma,\nu) \subset (\Gamma,\mu)} (\gamma, \nu) \otimes \left(\sum_{(\gamma',\nu') \subset (\Gamma/\gamma,\mu/\nu)} (\gamma', \nu') \otimes \big((\Gamma/\gamma)/\gamma', (\mu/\nu)/\nu'\big) \right).
\end{aligned}
\tag{9}
$$

There is a one-to-one correspondence between the assigned subgraphs $(\gamma', \nu') \subseteq (\Gamma/\gamma, \mu/\nu)$ and the assigned subgraphs $(\gamma_1, \nu_1) \subseteq (\Gamma, \mu)$ such that $(\gamma, \nu) \subseteq (\gamma_1, \nu_1)$. Indeed, starting from an assigned subgraph $(\gamma_1, \nu_1) \subseteq (\Gamma, \mu)$ such that $(\gamma, \nu) \subseteq (\gamma_1, \nu_1)$, we find an assigned subgraph $(\gamma', \nu') \subseteq (\Gamma/\gamma, \mu/\nu)$ by restricting the total order ν_1 to the edges of γ_1 which are not internal to γ, and the inverse operation consists in extending the total order ν' to all edges of γ_1 in the unique way compatible with the total order μ on $E(\Gamma)$.

Applying Lemma 4.5, one has $\big((\Gamma/\gamma)/\gamma', (\mu/\nu)/\nu'\big) = (\Gamma/\gamma_1, \mu/\nu_1)$. Equation (9) can then be rewritten as follows:

$$
\begin{aligned}
(Id \otimes \Delta) &\circ \Delta\big((\Gamma, \mu)\big) \\
&= \sum_{\substack{(\gamma_1,\nu_1) \subset (\Gamma,\mu) \\ (\gamma,\nu) \subset (\gamma_1,\nu_1)}} (\gamma, \nu) \otimes (\gamma_1/\gamma, \nu_1/\nu) \otimes (\Gamma/\gamma_1, \mu/\nu_1).
\end{aligned}
\tag{10}
$$

Using equations (8) and (10), one concludes the proof. $\qquad\square$

Furthermore, we define the counit $\epsilon : \mathcal{H} \longrightarrow \mathbb{K}$ by

$$
\epsilon\big((\Gamma, \mu)\big) = \begin{cases} 1 \text{ if } (\Gamma, \mu) = 1_{\mathcal{H}}; \\ 0 \text{ otherwise.} \end{cases}
\tag{11}
$$

Theorem 4.7 *The triple $(\mathcal{H}, \Delta, \epsilon)$ is a coassociative coalgebra with counit.*
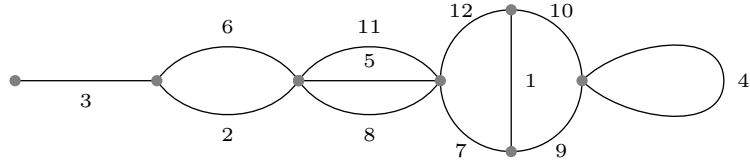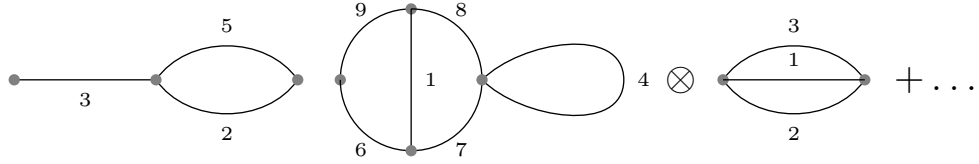
Figure 5: A standard labeled TAG.

Figure 6: One of the terms obtained by acting with the coproduct on the standard labeled TAG of Fig 5.

Proof. Let us show that ϵ is a counit of the coalgebra. For any TAG (Γ, μ), one has

$$(\epsilon \otimes Id) \circ \Delta\big((\Gamma, \mu)\big) = \epsilon\big((\Gamma, \mu)\big)1 + \epsilon(1_{\mathcal{H}})(\Gamma, \mu) + \sum_{(\gamma, \nu) \subsetneq (\Gamma, \mu)} \epsilon(\gamma, \nu)(\Gamma/\gamma, \mu/\nu) = (\Gamma, \mu).$$

Analogously, one has: $(Id \otimes \epsilon) \circ \Delta\big((\Gamma, \mu)\big) = (\Gamma, \mu)$. One thus concludes that the maps Id, $(\epsilon \otimes Id) \circ \Delta$ and $(Id \otimes \epsilon) \circ \Delta$ coincide on TAGs, thus proving that ϵ is a counit of Δ. Using now Proposition 4.6, one concludes the proof. □

Example 4.8 *Let us check the coassociativity of our coproduct on the example of the standard labeled TAG of Fig. 5. When acting with the coproduct on this standard labeled TAG, one gets the term of Fig. 6, which obviously adds up to the rest of the coproduct terms. Another type of term is the one of Fig. 7 (which again adds up to the rest of the coproduct terms). Acting now on these terms with $(\Delta \otimes Id)$ and respectively with $(Id \otimes \Delta)$ leads to the same term represented in Fig. 8 with standard labelling. Let us also emphasize that this term cannot be obtained from other terms of Δ because of the diagrammatic difference of the various disconnected components of the graphs.*
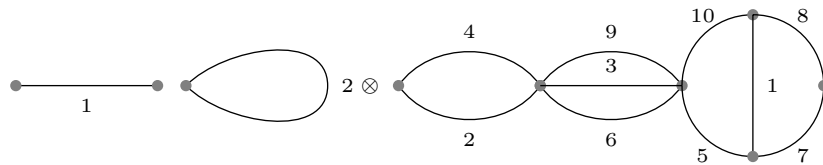

Figure 7: Another terms obtained by acting with the coproduct on the standard labeled TAG of Fig 5.
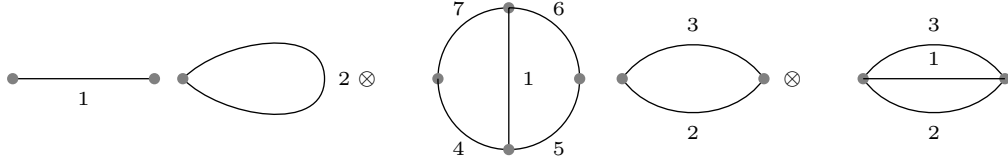
Figure 8: The resulting in the LHS and RHS of the coassociativity identity.

One has:

Proposition 4.9 *Let (Γ_1, μ_1) and (Γ_2, μ_2) be two TAGs in $\mathcal{H}$. One has*

$$\Delta\Big(m\big((\Gamma_1, \mu_1), (\Gamma_2, \mu_2)\big)\Big) = m^{\otimes 2} \circ \tau_{23}\big(\Delta(\Gamma_1, \mu_1), \Delta(\Gamma_2, \mu_2)\big) \tag{12}$$

where τ_{23} is the flip of the two middle factors in $\mathcal{H}^{\otimes 4}$.

Proof. One has

$$\Delta\Big(m\big((\Gamma_1, \mu_1), (\Gamma_2, \mu_2)\big)\Big) = \Delta(\Gamma_1 \sqcup \Gamma_2, \mu_1 \sqcup \mu_2) \tag{13}$$

$$= \sum_{\emptyset \subseteq (g,\nu) \subseteq (\Gamma_1 \sqcup \Gamma_2, \mu_1 \sqcup \mu_2)} (\gamma, \nu) \otimes \big((\Gamma_1 \sqcup \Gamma_2)/\gamma, (\mu_1 \sqcup \mu_2)/\nu\big)$$

$$= \sum_{\substack{(\gamma_1,\nu_1) \subseteq (\Gamma_1,\mu_1) \\ (\gamma_2,\nu_2) \subseteq (\Gamma_2,\mu_2)}} (\gamma_1 \sqcup \gamma_2, \nu_1 \sqcup \nu_2) \otimes (\Gamma_1/\gamma_1 \sqcup \Gamma_2/\gamma_2, \mu_1/\nu_1 \sqcup \mu_2/\nu_2)$$

$$= m^{\otimes 2} \circ \tau_{23}(\Delta(\Gamma_1, \mu_1), \Delta(\Gamma_2, \mu_2)). \tag{14}$$

$\square$

Example 4.10 *Let (Γ_1, μ_1) and (Γ_2, μ_2) be the graph in Fig. 2(a). One has:*

$$\Delta(\Gamma_1 \sqcup \Gamma_2, \mu_1 \sqcup \mu_2) = \quad (\Gamma_1 \sqcup \Gamma_2, \mu_1 \sqcup \mu_2) \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes (\Gamma_1 \sqcup \Gamma_2, \mu_1 \sqcup \mu_2)$$

One has:

Theorem 4.11 *$(\mathcal{H}, m, 1_{\mathcal{H}}, \Delta, \epsilon)$ is a bialgebra.*

Proof. Using Proposition 4.9, it follows that Δ is a morphism of algebras. One thus concludes the proof. $\qquad\square$

For all $n \in \mathbb{N}$, one calls $\mathcal{H}(n)$ the vector space generated by the TAGs with n edges. Then one has $\mathcal{H} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}(n)$. Moreover, one has:

1. For all $m, n \in \mathbb{N}$, $\mathcal{H}(m)\mathcal{H}(n) \subseteq \mathcal{H}(m+n)$.

2. For all $n \in \mathbb{N}$, $\Delta\big(\mathcal{H}(n)\big) \subseteq \sum_{i+j=n} \mathcal{H}(i) \otimes \mathcal{H}(j)$.

One thus concludes that $\mathcal{H}$ is a *graded bialgebra*. Note that $\mathcal{H}$ is *connected*.
We can now state the main result of this section:

Theorem 4.12 *The bialgebra $(\mathcal{H}, m, 1_{\mathcal{H}}, \Delta, \epsilon)$ is a Hopf algebra.*

Proof. The bialgebra $(\mathcal{H}, m, 1_{\mathcal{H}}, \Delta, \epsilon)$ is connected and graded. The conclusion follows. The antipode $S : \mathcal{H} \longrightarrow \mathcal{H}$ verifies $S(1_{\mathcal{H}}) = 1_{\mathcal{H}}$, and is given on a non-empty TAG (Γ, μ) by one of the two following recursive formulas:

$$S(\Gamma, \mu) = -(\Gamma, \mu) - \sum_{\emptyset \subsetneq (\gamma,\nu) \subsetneq (\Gamma,\mu)} S(\gamma, \nu) \cdot (\Gamma/\gamma, \mu/\nu) \tag{15}$$

$$= -(\Gamma, \mu) - \sum_{\emptyset \subsetneq (\gamma,\nu) \subsetneq (\Gamma,\mu)} (\gamma, \nu) \cdot S(\Gamma/\gamma, \mu/\nu). \tag{16}$$

$\qquad\square$

Note that if one considers now the Hopf algebra $\mathcal{H}^c$ of graphs (without any edge scale decoration), one has:

Proposition 4.13 *The map π from $\mathcal{H}$ to $\mathcal{H}^c$ defined on the TAGs by $\pi\big((\Gamma, \mu)\big) = \Gamma$ is a Hopf algebra morphism.*

Proof. This statement directly follows from the definitions. $\qquad\square$

5 Quadri-coalgebra structure

A quadri-algebra ([1]) is a family $(A, \nwarrow, \swarrow, \searrow, \nearrow)$, such that A is a vector space, and $\nwarrow$, $\swarrow$, $\searrow$, $\nearrow$ are four products on A, satisfying nine axioms. Putting $\leftarrow = \nwarrow + \swarrow$, $\rightarrow = \nearrow + \searrow$, $\uparrow = \nwarrow + \nearrow$, $\downarrow = \swarrow + \searrow$, and $\star = \nwarrow + \swarrow + \searrow + \nearrow = \leftarrow + \rightarrow = \uparrow + \downarrow$, these axioms imply that $(A, \uparrow, \downarrow)$ and $(A, \leftarrow, \rightarrow)$ are dendriform algebras ([23, 25]), and $\star$ is an associative (non unitary) product. Dually, a quadri-coalgebra is a family $(C, \Delta_{\nwarrow}, \Delta_{\swarrow}, \Delta_{\searrow}, \Delta_{\nearrow})$, where C is a vector space, , $\Delta_{\nwarrow}, \Delta_{\swarrow}, \Delta_{\searrow}, \Delta_{\nearrow} : C \longrightarrow C \otimes C$, such that:

$$
\begin{aligned}
(\Delta_{\nwarrow} \otimes Id) \circ \Delta_{\nwarrow} &= (Id \otimes \Delta_{\star}) \circ \Delta_{\nwarrow}, & (\Delta_{\nearrow} \otimes Id) \circ \Delta_{\nwarrow} &= (Id \otimes \Delta_{\leftarrow}) \circ \Delta_{\nearrow}, \\
(\Delta_{\uparrow} \otimes Id) \circ \Delta_{\nearrow} &= (Id \otimes \Delta_{\rightarrow}) \circ \Delta_{\nearrow}, & (\Delta_{\swarrow} \otimes Id) \circ \Delta_{\nwarrow} &= (Id \otimes \Delta_{\uparrow}) \circ \Delta_{\swarrow}, \\
(\Delta_{\searrow} \otimes Id) \circ \Delta_{\nwarrow} &= (Id \otimes \Delta_{\nwarrow}) \circ \Delta_{\searrow}, & (\Delta_{\downarrow} \otimes Id) \circ \Delta_{\nearrow} &= (Id \otimes \Delta_{\nearrow}) \circ \Delta_{\searrow}, \\
(\Delta_{\leftarrow} \otimes Id) \circ \Delta_{\swarrow} &= (Id \otimes \Delta_{\downarrow}) \circ \Delta_{\swarrow}, & (\Delta_{\rightarrow} \otimes Id) \circ \Delta_{\swarrow} &= (Id \otimes \Delta_{\swarrow}) \circ \Delta_{\searrow}, \\
(\Delta_{\star} \otimes Id) \circ \Delta_{\searrow} &= (Id \otimes \Delta_{\searrow}) \circ \Delta_{\searrow}, &&
\end{aligned}
$$

where $\Delta_{\leftarrow} = \Delta_{\nwarrow} + \Delta_{\swarrow}$, $\Delta_{\rightarrow} = \Delta_{\searrow} + \Delta_{\nearrow}$, $\Delta_{\uparrow} = \Delta_{\nwarrow} + \Delta_{\nearrow}$, $\Delta_{\downarrow} = \Delta_{\swarrow} + \Delta_{\searrow}$ and $\Delta_{\star} = \Delta_{\nwarrow} + \Delta_{\swarrow} + \Delta_{\searrow} + \Delta_{\nearrow}$. This implies that $(C, \Delta_{\leftarrow}, \Delta_{\rightarrow})$ and $(C, \Delta_{\uparrow}, \Delta_{\downarrow})$ are dendriform coalgebras; moreover, $\Delta_{\star}$ is coassociative (non counitary): it is called the coassociative coproduct induced by the quadri-coalgebra structure.

Definition 5.1   *1. Let $\mathcal{H}_+$ be the augmentation ideal of $\mathcal{H}$. It is given a coassociative, non counitary coprodut $\Delta_{\star}$ defined for all nonempty TAG (Γ, μ) by:*

$$\Delta_{\star}\big((\Gamma, \mu)\big) = \Delta\big((\Gamma, \mu)\big) - (\Gamma, \mu) \otimes 1 - 1 \otimes (\Gamma, \mu) = \sum_{\emptyset \subsetneq (\gamma, \nu) \subsetneq (\Gamma, \mu)} (\gamma, \nu) \otimes (\Gamma/\gamma, \mu/\nu),$$

2. Let (Γ, μ) be a nonempty TAG. We denote by $a(\Gamma, \mu)$ the smallest edge of (Γ, μ) and $b(\Gamma, \mu)$ the greatest edge of (Γ, μ) for the scale assignment μ.

Proposition 5.2 *We define four coproducts on $\mathcal{H}_+$ by:*

$$\Delta_{\nwarrow}\big((G, \mu)\big) = \sum_{\substack{\emptyset \subsetneq (\gamma, \nu) \subsetneq (\Gamma, \mu), \\ a(G) \in \gamma, b(G) \in \gamma}} (\gamma, \nu) \otimes (\Gamma/\gamma, \mu/\nu),$$

$$\Delta_{\swarrow}\big((G, \mu)\big) = \sum_{\substack{\emptyset \subsetneq (\gamma, \nu) \subsetneq (\Gamma, \mu), \\ a(G) \in \gamma, b(G) \notin \gamma}} (\gamma, \nu) \otimes (\Gamma/\gamma, \mu/\nu),$$

$$\Delta_{\searrow}\big((G, \mu)\big) = \sum_{\substack{\emptyset \subsetneq (\gamma, \nu) \subsetneq (\Gamma, \mu), \\ a(G) \notin \gamma, b(G) \notin \gamma}} (\gamma, \nu) \otimes (\Gamma/\gamma, \mu/\nu),$$

$$\Delta_{\nearrow}\big((G, \mu)\big) = \sum_{\substack{\emptyset \subsetneq (\gamma, \nu) \subsetneq (\Gamma, \mu), \\ a(G) \notin \gamma, b(G) \in \gamma}} (\gamma, \nu) \otimes (\Gamma/\gamma, \mu/\nu).$$

Then $(\mathcal{H}_+, \Delta_{\nwarrow}, \Delta_{\swarrow}, \Delta_{\searrow}, \Delta_{\nearrow})$ is a quadri-coalgebra, and the induced coassociative coproduct is $\Delta_{\star}$. Moreover, for all $x, y \in \mathcal{H}_+$:

$$\begin{aligned}
\Delta_{\nwarrow}(xy) &= xy'_{\uparrow} \otimes y''_{\uparrow} + x'_{\leftarrow} y \otimes x''_{\leftarrow} + x'_{\leftarrow} y'_{\uparrow} \otimes x''_{\leftarrow} y''_{\uparrow} \\
\Delta_{\swarrow}(xy) &= x \otimes y + x'_{\leftarrow} \otimes x''_{\leftarrow} y + xy'_{\downarrow} \otimes y''_{\downarrow} + x'_{\leftarrow} y'_{\downarrow} \otimes x''_{\leftarrow} y''_{\downarrow} \\
\Delta_{\searrow}(xy) &= x'_{\rightarrow} \otimes x''_{\rightarrow} y + y'_{\downarrow} \otimes xy''_{\downarrow} + x'_{\rightarrow} y'_{\downarrow} \otimes x''_{\rightarrow} y''_{\downarrow} \\
\Delta_{\nearrow}(xy) &= y \otimes x + y'_{\uparrow} \otimes xy''_{\uparrow} + x'_{\rightarrow} y \otimes x''_{\rightarrow} + x'_{\rightarrow} y'_{\downarrow} \otimes x''_{\rightarrow} y''_{\downarrow},
\end{aligned}$$

where $\Delta_{\leftarrow}(x) = x'_{\leftarrow} \otimes x''_{\leftarrow}$, $\Delta_{\rightarrow}(x) = x'_{\rightarrow} \otimes x''_{\rightarrow}$, $\Delta_{\uparrow}(y) = y'_{\uparrow} \otimes y''_{\uparrow}$, and $\Delta_{\downarrow}(y) = y'_{\downarrow} \otimes y''_{\downarrow}$.

Proof. Obviously, $\Delta_{\nwarrow} + \Delta_{\swarrow} + \Delta_{\searrow} + \Delta_{\nearrow} = \Delta_{\star}$. Let (G, μ) be a nonempty TAG. We write, using the coassociativity of $\Delta_{\star}$:

$$\begin{aligned}
(\Delta_{\star} \otimes Id) \circ \Delta_{\star}\big((\Gamma, \mu)\big) &= (Id \otimes \Delta_{\star}) \circ \Delta_{\star}\big((\Gamma, \mu)\big) \\
&= \sum_{\emptyset \subsetneq (\gamma', \nu') \subsetneq (\gamma, \nu) \subsetneq (\Gamma, \mu)} (\gamma', \nu') \otimes (\gamma/\gamma', \nu/\nu') \otimes (\Gamma/\gamma, \mu/\nu).
\end{aligned}$$

Then each relation defining quadri-coalgebras corresponds to the terms in this sum such that:

$$\begin{array}{ll}
a \in \gamma', b \in \gamma', & a \in \gamma/\gamma', b \in \gamma', \\
a \in \Gamma/\gamma, b \in \gamma', & a \in \gamma', b \in \gamma/\gamma', \\
a \in \gamma/\gamma', b \in \gamma/\gamma', & a \in \Gamma/\gamma, b \in \gamma/\gamma', \\
a \in \gamma', b \in \Gamma/\gamma, & a \in \gamma/\gamma', b \in \Gamma/\gamma, \\
a \in \Gamma/\gamma, b \in \Gamma/\gamma. &
\end{array}$$

Let x, y be nonempty TAGs. By definition of the product, $a(xy) = a(x)$ and $b(xy) = b(y)$. Let $\emptyset \subsetneq (\gamma, \nu) \subsetneq xy$. We put $(\gamma_x, \nu_x) = x \cap (\gamma, \nu)$ and $(\gamma_y, \nu_y) = y \cap (\gamma, \nu)$. Then $(\gamma, \nu) = (\gamma_x, \nu_x)(\gamma_y, \nu_y)$ and $xy/(\gamma, \nu) = x/(\gamma_x, \nu_x)y/(\gamma_y, \nu_y)$. If $a(x,y) \in \gamma$, $b(xy) \notin \gamma$, then $a(x) \in \gamma_x$ and $a(y) \notin \gamma_y$. Four cases are possible:

- $x = \gamma$. This gives the term $x \otimes y$.

- $\gamma \subsetneq x$. This gives the term $x'_{\leftarrow} \otimes x''_{\leftarrow} y$.

- $x \subsetneq \gamma$. This gives the term $xy'_{\downarrow} \otimes y''_{\downarrow}$.

- $\gamma_x, \gamma_y \neq \emptyset$ and $\gamma_x \subsetneq x$, $\gamma_y \subsetneq y$. This gives the term $x'_{\leftarrow} y'_{\downarrow} \otimes x''_{\leftarrow} y''_{\downarrow}$.

This proves the compatibility between $\Delta_{\swarrow}$ and the product. The three other compatibilities are proved in the same way. $\qquad\square$

Summing, we obtain the following compatibilities:

$$\begin{array}{rcl}
\Delta_{\leftarrow}(xy) & = & x \otimes y + xy' \otimes y'' + x'_{\leftarrow} y \otimes x''_{\leftarrow} + x'_{\leftarrow} \otimes x''_{\leftarrow} y + x'_{\leftarrow} y' \otimes x''_{\leftarrow} y'' \\
\Delta_{\rightarrow}(xy) & = & y \otimes x + y' \otimes xy'' + x'_{\rightarrow} \otimes x''_{\rightarrow} y + x'_{\rightarrow} y \otimes x''_{\rightarrow} + x'_{\rightarrow} y' \otimes x''_{\rightarrow} y'' \\
\Delta_{\uparrow}(xy) & = & y \otimes x + x' y \otimes x'' + xy'_{\uparrow} \otimes y''_{\uparrow} + y'_{\uparrow} \otimes xy''_{\uparrow} + x' y'_{\uparrow} \otimes x'' y''_{\uparrow} \\
\Delta_{\downarrow}(xy) & = & x \otimes y + x' \otimes x'' y + xy'_{\downarrow} \otimes y''_{\downarrow} + y'_{\downarrow} \otimes xy''_{\downarrow} + x' y'_{\downarrow} \otimes x'' y''_{\downarrow}.
\end{array}$$

Hence:

Corollary 5.3 $(\mathcal{H}_+, m^{op}, \Delta_{\leftarrow}, \Delta_{\rightarrow})$ *and* $(\mathcal{H}_+, m, \Delta_{\uparrow}, \Delta_{\downarrow})$ *are condendriform Hopf algebras* *([15]).*

Let us end this paper by the following concluding remarks. A further non-commutative Hopf algebra of TAGs can be defined when considering only graphs of a given quantum field theoretical model and defining the coproduct as the appropriate sum on the class of superficially divergent graphs (see for example [20], where such a Hopf algebra was defined, in a commutative setting).

The non-commutative graph Hopf algebraic structure defined here is a combinatorial Hopf algebra (CHA) using a selection/contraction coproduct rule - this type of CHAs being called CHAs of type I in [33]. Examples of such CHAs are the Connes-Kreimer Hopf algebras of quantum field theory [8, 9, 10], of non-commutative Moyal quantum field theory [34,

35] (see also [32] for a short review), of quantum gravity spin-foam models [26, 31], of random tensor models [27], or the word Hopf algebra WMat [11]. This type of coproduct rule is fundamentally different of the selection/deletion rule used in CHAs such as FQSym, MQSym, the Loday-Ronco Hopf algebra *etc.* - CHAs of type II (see again [33]). It seems however interesting to us to investigate in what circumstances one can find some non-trivial mathematical relations between these two types of CHAs.

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