Pre-Lie algebras and systems of Dyson-Schwinger equations

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ABSTRACT. These lecture notes contain a review of the results of [15, 16, 17, 19] about combinatorial Dyson-Schwinger equations and systems. Such an equation or system generates a subalgebra of a Connes-Kreimer Hopf algebra of decorated trees, and we shall say that the equation or the system is Hopf if the associated subalgebra is Hopf. We first give a classication of the Hopf combinatorial Dyson-Schwinger equations. The proof of the existence of the Hopf subalgebra uses pre-Lie structures and is different from the proof of [15, 17].

We consider afterwards systems of Dyson-Schwinger equations. We give a description of Hopf systems, with the help of two families of special systems (quasi-cyclic and fundamental) and four operations on systems (change of variables, dilatation, extension, concatenation). We also give a few result on the dual Lie algebras. Again, the proof of the existence of these Hopf subalgebras uses pre-Lie structures and is different from the proof of [16].

Contents

Introduction

In Quantum Field Theory, the Green's functions of a given theory are developed as a series in the coupling constant, indexed by the set of Feynman graphs of the theory. These series can be seen at the level of the algebra of Feynman graphs. They satisfy then a certain system of *combinatorial Dyson-Schwinger equations*. These equations use a combinatorial operator of insertion, and they allow to inductively compute the homogeneous components of the Green's functions lifted at the level of Feynman graphs [2, 26, 28, 29, 30, 31, 32, 33, 40, 41, 42, 44]. As the Feynman graphs are organised as a Hopf algebra, a natural question is to know if the graded subalgebra generated by the Green's functions is Hopf or not. This problem, and related questions about the nature of the obtained Hopf subalgebras, are the main object of study in [15, 16, 17, 19].

Here is an example coming from Quantum Electrodynamics [44], see the first section of this text for more details. For any Feynman graph γ, the operator B_γ is combinatorially defined by the operation of insertion into γ. The system holds on three series in Feynman graphs, denoted by , and . After a truncation, it is given by the following equations:

$$= B_{\gamma_1}\left(\frac{(1+\quad)^3}{(1-\quad)^2(1-\quad)}\right),$$

$$= B_{\gamma_2}\left(\frac{(1+\quad)^2}{(1-\quad)^2}\right), \qquad = B_{\gamma_3}\left(\frac{(1+\quad)^2}{(1-\quad)(1-\quad)}\right),$$

with $\gamma_1 =$, $\gamma_2 =$, and $\gamma_3 =$.

The insertion operators appearing in this system are 1-cocyles of a certain subspace of a quotient of the Hopf algebra of Feynman graphs, that is to say for all x in this subspace:

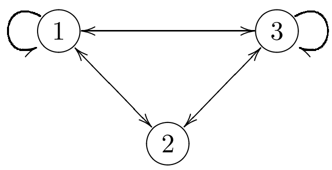
$$\Delta \circ B_\gamma(x) = B_\gamma(x) \otimes 1 + (Id \otimes B_\gamma) \circ \Delta(x).$$

This allows to lift the problem to the level of rooted trees. Replacing insertion by grafting of trees on a root, we obtain a system in the Hopf algebra of rooted trees decorated by $\{1,2,3\}$:

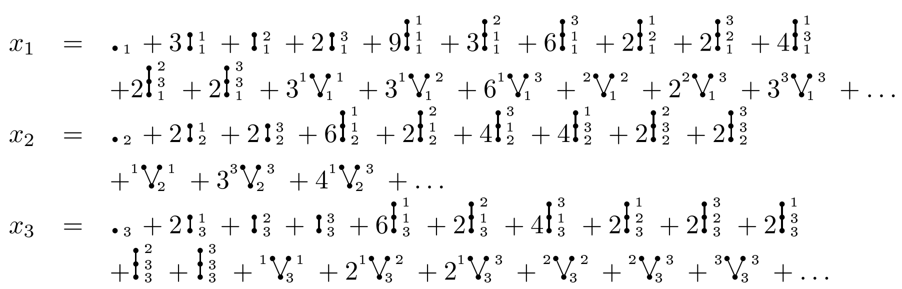
$$x_1 = B_1\left(\frac{(1+x_1)^3}{(1-x_2)(1-x_3)^2}\right),$$

$$x_2 = B_2\left(\frac{(1+x_1)^2}{(1-x_3)^2}\right), \qquad x_3 = B_3\left(\frac{(1+x_1)^2}{(1-x_2)(1-x_3)}\right),$$

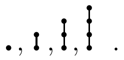
where, for all trees $t_1,\ldots,t_n$, $B_i(t_1\ldots t_n)$ is the tree obtained by grafting $t_1,\ldots,t_n$ on a common root decorated by i. The *graph of dependence* of this system is:



This system has a unique solution $X = (x_1, x_2, x_3)$. Here are the components of degree ≤ 3 of X:



It can be proved that the subalgebra generated by the homogeneous components of x_1, x_2 and x_3 is a Hopf subalgebra. In fact, this system is an example of a *fundamental system* (definition 51). The aim of this text is to present the classification of the systems of combinatorial Dyson-Schwinger equations which give a Hopf subalgebra. We shall limit ourselves to systems with only one 1-cocycle per equation. More general cases are studied in [18]; it turns out that if the corresponding subalgebra is Hopf, then the truncation of the equations to 1-cocycle of degree 1 allows to get back the whole system.

We begin with a single equation $x = B(f(x))$, where f is a formal series in one indeterminate, with $f(0) = 1$. The question is answered in the third and fourth sections. The subalgebra generated by the components of the solution is Hopf, if, and only if, f is constant, or $f = e^{\alpha h}$ for a certain α, or $f = (1 - \alpha\beta h)^{-1/\beta}$ for a certain couple (α, β), with $\beta \neq 0$ (theorem 24). The direct sense is proved using a "leaf-cutting" result (proposition 21), applied on two families of trees, the ladders  ... and the corollas  .... The other sense uses a complementary structure on the dual of the Hopf algebra of trees $\mathcal{H}_{CK}$. By the Cartier-Quillen-Milnor-Moore theorem, it is an enveloping algebra. The associated Lie algebra is based on trees, and is in fact a free *pre-Lie algebra* (definition 6 and theorem 8), that is to say it has a (non-associative) product $\circ$ such that:

$$(x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z).$$

The Lie bracket is given by $[x, y] = x \circ y - y \circ x$. For example, the space of Feynman graphs is a pre-Lie algebra, with a product defined by insertions. In the case of trees, the pre-Lie product is defined by graftings. This pre-Lie algebra is denoted by $\mathfrak{g}_{\mathcal{T}}$. Another especially interesting pre-Lie algebra is the *Faà di Bruno Lie algebra* $\mathfrak{g}_{FdB}$, related to the group of formal diffeomorphisms of the line. As $\mathfrak{g}_{\mathcal{T}}$ is a free pre-Lie algebra (theorem 8), this allows to define morphisms ϕ_λ from $\mathfrak{g}_{\mathcal{T}}$ to $\mathfrak{g}_{FdB}$ (proposition 14). This morphism is computed with the help of an explicit construction of the enveloping algebra of a pre-Lie algebra (theorem 9, applied in propositions 10 and 12). Dually, we obtain a Hopf algebra morphism from the Faà di Bruno Hopf algebra $\mathcal{H}_{FdB}$ to the Connes-Kreimer Hopf algebra, and the image of the generators of $\mathcal{H}_{FdB}$, which are linear spans of trees, satisfy a Dyson-Schwinger equation (proposition 16); as a consequence, this Dyson-Schwinger equation is Hopf. This result is proved in [15, 17] in a different way, with the help of an identity on a family of symmetric polynomials which is not used here.

The case of systems of Dyson-Schwinger equations (briefly, SDSE) is studied in the last four sections. We first generalize the results on a single equation, especially the "leaf-cutting" result and its consequences (proposition 29 and lemma 30). Four operations are introduced on SDSE, *change of variables*, *dilatation*, *extension* and *concatenation*. The latter leads to the notion of connected SDSE, that is to say a SDSE which cannot be obtained by a concatenation of two smaller ones. The main objects of study are now connected systems. Another tool is also introduced, the *graph of dependence*. A graph-theoretical study proves that this graph always contains an oriented cycle (proposition 41). A study of SDSE whose graph is an oriented cycle allows to separate the SDSE into two classes, the *quasi-cyclic* and the *fundamental* case. The quasi-cyclic case is entirely described in theorem 45. The fundamental case is the object of the seventh section. We first introduce the notion of the *level* of a vertex of the graph of dependence. This notion defines a sort of gradation of the graph (proposition 48). A study of vertices, level by level, finally allows to describe all fundamental SDSE. As a conclusion, any SDSE which gives a Hopf subalgebra is obtained from the concatenation of quasi-cyclic or fundamental systems, after the application of a dilatation, a change of variables, and a finite number of extensions.

This text is organised as follows. The first section of the text deals with Feynman graphs. The algebraic structures (product, coproduct, insertions) on Feynman graphs of a given theory are introduced here, and this leads to the first example of a system of Dyson-Schwinger equations, coming from Quantum Electrodynamics. The second section gives the alternative Hopf algebras in quantum field theory, namely the Connes-Kreimer Hopf algebras of decorated rooted trees. Their universal property (theorem 5) allows to define Hopf algebra morphisms from rooted trees to Feynman graphs. The role of the insertion operators on graphs are played for trees by the grafting operators, and Dyson-Schwinger equations are lifted to the level of trees.

The third section adopts the dual point of view. We give the pre-Lie products on $\mathfrak{g}_{\mathcal{T}}$ and $\mathfrak{g}_{FdB}$, and construct the pre-Lie morphism ϕ_λ from $\mathfrak{g}_{\mathcal{T}}$ to $\mathfrak{g}_{FdB}$ with the help of an explicit description of their enveloping algebra. Dually, the image under ϕ_λ^* of the generators of the Faà di Bruno Hopf algebra satisfies a Dyson-Schwinger equation (proposition 16).

Single Dyson-Schwinger equations are reviewed in the fourth section. Proposition 21 gives a combinatorial criterion of "leaf-cutting" to know if the solution of the considered Dyson-Schwinger equation is Hopf. This criterion and proposition 16 for the other direction, imply the main theorem for Dyson-Schwinger equations (theorem 24).

The study of systems of Dyson-Schwinger equations is achieved in the last sections. The fifth section introduces the tool of "leaf-cutting" for systems (lemma 30), and the four operations on Hopf SDSE. The oriented graph of dependence of the equations of a Hopf SDSE is also studied here. The next section introduces quasi-cyclic SDSE, and achieves their description. The second family of SDSE (fundamental ones) is studied in the seventh section. In particular, the notion of level is introduced, and the vertices are separated according to their level being 0, 1, or ≥ 2. The last section gives a few more results and comments on fundamental SDSE, especially on the dual pre-Lie algebras, as well as several examples found in the literature.

Thanks. I would like to thank the organizers of the meeting DSFdB2011, especially for the opportunity of giving a mini-course on the algebraic aspects of Dyson-Schwinger equations. The lecture notes of this mini-course are the framework of the present text. I would also like to thank both referees, for their useful

and relevant comments which help me to greatly improve the quality of this document.

Notations.

(1) Let K be a commutative field of characteristic zero. All the vector spaces, algebras, coalgebras, Lie algebras... of this text will be taken over K.

(2) We use the convention $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{N}^* = \{1, 2, 3, \ldots\}$.

1 Feynman graphs

1.1 Definition

For more precise results and definitions, see [8, 44] and more generally the references listed in the introduction. Let us consider a quantum field theory. In this theory, a certain number of particles interact in different possible ways. The possible configurations of interactions are described by the *Feynman graphs* of the theory. The graphs we shall consider here are described in the following way:

(1) There are several types of edges (one for each particle of the theory).

(2) The vertices can be *external* or *internal*.

  (a) There are at least two internal vertices.

  (b) If a vertex v is external, it is related to a single edge, which is said to be *external*. The other edges are said to be *internal*.

  (c) There are several types of internal vertices (one for each interaction of the theory).

(3) The graph should be connected and 1-*particle irreducible*, that is to say that it remains connected if one deletes any internal edge.

(4) The number of external vertices (or external edges) belongs to a certain set of integers (condition of *global divergence* in Renormalization).

The *number of loops* of a Feynman graph γ is:

$$l(\gamma) = \sharp\{\text{internal edges of } \gamma\} - \sharp\{\text{internal vertices of } \gamma\} + 1.$$

The condition of 1-particle irreducibility implies that $l(\gamma) \geq 1$ for all Feynman graphs γ.

Example. We take in this section the example of Quantum Electrodynamics (QED). In this theory:

(1) There are two types of particles, electrons and photons. So there are two types of edges: electron and photon .

(2) There is one interaction: an electron can capture or eject a photon. So there is one type of internal vertex .

(3) The number of external edges is equal to 2 or 3.

Here are examples of Feynman graphs in QED:

, , , ,

, , , , , , ...

Remark. Feynman graphs are often considered without external vertices. The external edges are then considered as *half-edges*; The internal edges are the union of two half-edges. A Feynman subgraph of γ is then a set of half-edges of γ which forms a Feynman graph.

1.2 Insertion

Let us fix a QFT. For this theory, the external structures of the Feynman graphs correspond to the different types of vertices and edges of the theory. For example, in QED, there are three possible external structures:

(1) Two electron edges, corresponding to the edge .

(2) Two photon edges, corresponding to the edge .

(3) One photon and two electron edges, corresponding to the vertex .

Let γ and γ' be two Feynman graphs. Inserting γ' into γ consists in replacing in γ an internal edge or vertex corresponding to the external structure of γ' by γ'. For example, in QED:

(1) There is one possible insertion of in . The result is .

(2) There are two possible insertions of in . Both of them give .

(3) There are three possible insertions of in itself. The results are , and .

More generally, one can insert a family $\gamma_1, \ldots, \gamma_k$ of Feynman graphs into a Feynman graph γ: one inserts $\gamma_1, \ldots, \gamma_n$ in γ in such a way that the set of internal edges and vertices of the copies of $\gamma_1, \ldots, \gamma_k$ are disjoint. It is not difficult to prove that if Γ is obtained by the insertion of $\gamma_1, \ldots, \gamma_n$ in γ, then:

$$l(\Gamma) = l(\gamma) + l(\gamma_1) + \ldots + l(\gamma_k).$$

Let us describe the "dual" operation. For any Feynman graph Γ, let $\underline{\gamma} = \gamma_1 \ldots \gamma_k$ be a family of disjoint Feynman subgraphs of Γ. The *contraction* of Γ by $\gamma_1, \ldots, \gamma_k$ is the graph obtained from Γ by replacing any γ_i be an edge or a vertex corresponding to its external structure. It is denoted by $\Gamma/\underline{\gamma}$. Moreover:

$$l(\Gamma) = l(\gamma_1) + \ldots + l(\gamma_k) + l(\Gamma/\underline{\gamma}) = l(\underline{\gamma}) + l(\Gamma/\underline{\gamma}).$$

1.3 Algebraic structures on Feynman graphs

See [11, 31, 35, 36, 44]. Let us consider the free commutative algebra generated by the set of Feynman graphs of a given theory. We denote it by $\mathcal{H}_{FG}$, without precising the considered QFT. A basis of this algebra is given by monomials in Feynman graphs, that is to say disjoint unions of Feynman graphs, or equivalently graphs such that every connected component is a Feynman graph. The unit is the empty graph 1. This algebra is given a coassociative coproduct. For any Feynman graph Γ:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\underline{\gamma}} \underline{\gamma} \otimes \Gamma / \underline{\gamma},$$

where the sum is over all the family of disjoint Feynman subgraphs of Γ, not empty nor equal to Γ. With this coproduct, $\mathcal{H}_{FG}$ is a Hopf algebra, graded by the number of loops.

For example, in QED:

$$\begin{array}{lcl} \Delta(\quad) & = & \otimes 1 + 1 \otimes \quad + \quad \otimes \quad , \\ \Delta(\quad) & = & \otimes 1 + 1 \otimes \quad + 2 \quad \otimes \quad . \end{array}$$

Remark. For any Feynman graph Γ, the right factors in the tensor products appearing in $\Delta(\Gamma)$ are 1 or Feynman graphs, wherear the left factors can be products of several Feynman graphs. This is an example of *left combinatorial Hopf algebra* [34]. As a consequence, the space of primitive elements of the dual of $\mathcal{H}_{FG}$ inherits a left pre-Lie product (see definition 6 below); a basis of this pre-Lie algebra is given by the set of Feynman graphs and the pre-Lie product is given by insertion, see [29, 31].

For this coproduct, any Feynman graph with no proper Feynman subgraph is primitive. For example, the following Feynman graphs are primitive in QED:

, , ,

Let us take a primitive Feynman graph γ. The *insertion operator* B_γ sends a monomial $\gamma_1 \ldots \gamma_k$ to the sum of all possible insertions of $\gamma_1, \ldots, \gamma_k$ into γ, up to symmetries coefficients we won't detail here (see [44]). In particular, $B_\gamma(1) = \gamma$. Moreover, B_γ is homogeneous for the number of loops, of degree $l(\gamma)$.

1.4 Dyson-Schwinger equations

See [2, 30, 33, 44]. The Green's functions of the QFT are developped as a series in the coupling constant x (we assume here it is equal to 1), indexed by the set of Feynman graphs of the theory. To any Feynman graph is attached a scalar, by the *Feynman rules* and the procedure of *renormalisation*, [8, 10, 11, 12]. At the level of the Hopf algebra of Feynman graphs, we have then to consider the infinite sum of all Feynman graphs, with a fixed external structure, up to certain symmetry coefficients. Is there an easy way to describe these series?

Let us consider the example of QED. There are three possible external structures, so we have to consider three series, denoted here by , , and . Let us consider a Feynman graph Γ appearing in . It can be obtained by the insertions of certain $\gamma_1, \ldots, \gamma_k$ into a primitive Feynman graph with an external structure of type . So can be written as:

$$= \sum_{\gamma} B_\gamma \left(f_\gamma \left(\quad , \quad , \quad \right) \right),$$

where the sum runs over all the primitive Feynman graphs with a external structure, and f_γ is a formal series in three indeterminates. Let us now determine f_γ. For example, let us take $\gamma =$.

(1) This graph has three vertices , and we can insert $1+$ at any of these vertices.

(2) It has two internal edges , and we can insert $1 + \quad + \quad ^2 + \ldots$ at any of these edges.

(3) It has one internal edge , and we can insert $1 + \quad + \quad ^2 + \ldots$ at this edge.

So:

$$\begin{aligned} f_\gamma(\quad , \quad , \quad) &= (1+ \quad)^3 \left(\sum_{k=0}^{\infty} \quad ^k \right)^2 \left(\sum_{k=0}^{\infty} \quad ^k \right) \\ &= \frac{(1+ \quad)^3}{(1- \quad)^2 (1- \quad)}. \end{aligned}$$

Treating any primitive Feynman graph in this way, one obtains:

$$= \sum_{\gamma} B_\gamma \left(\frac{(1+ \quad)^{1+2l(\gamma)}}{(1- \quad)^{2l(\gamma)} (1- \quad)^{l(\gamma)}} \right). \tag{1.1}$$

Let us then consider a graph appearing in . It can be obtained by an insertion in . As this graph has two vertices and two internal edges , this gives:

$$= B \quad \left(\frac{(1+ \quad)^2}{(1- \quad)^2} \right). \tag{1.2}$$

Similarly, we obtain for the last series:

$$= B \quad \left(\frac{(1+ \quad)^2}{(1- \quad)(1- \quad)} \right). \tag{1.3}$$

The three equations (1.1), (1.2) and (1.3) are the Dyson-Schwinger equations of the QFT. They allow to inductively compute the irreducible components (for the number of loops) of , and . For a more "physical" description, see [44] (we did not pay here attention to signs and we took the coupling constant

x equal to 1).

The question we shall answer here is if the Hopf algebra generated by these homogeneous components is Hopf or not. We restrict ourselves to the case where a single insertion operator, homogeneous of degree 1, appears in any of these equations (this the case for (1.2) and (1.3) only; we should have to truncate (1.1) to apply the obtained result; see [18] for more details). For this, we shall use trees instead of Feynman graphs. The key point is the following:

Proposition 1. *[2, 29] In a suitable subspace of a quotient of $\mathcal{H}_{FG}$, we can assume that the operators appearing in the Dyson-Schwinger equations satisfy the following assertion: for any x,*

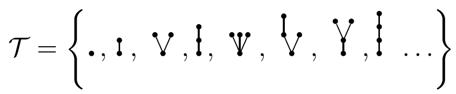
$$\Delta(L(x)) = L(x) \otimes 1 + (Id \otimes L) \circ \Delta(x).$$

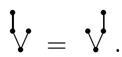
2 Rooted trees

We shall replace Feynman graphs by rooted trees and insertion operators by grafting operators, with the help of the universal property of the Hopf algebra of rooted trees (theorem 5).

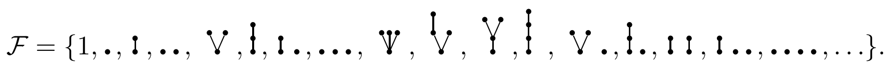
2.1 The Connes-Kreimer Hopf algebra

Let $\mathcal{T}$ be the set of *rooted trees*:



Note that rooted trees are considered unordered; for example, 

The Connes-Kreimer Hopf algebra [10, 13] is the free commutative algebra generated by $\mathcal{T}$. As a consequence, a basis of $\mathcal{H}_{CK}$ is given by the set of rooted forests $\mathcal{F}$:



The product of two forests is their disjoint union. The unit is the empty forest 1.

We give $\mathcal{H}_{CK}$ a coproduct, with the help of *admissible cuts*:

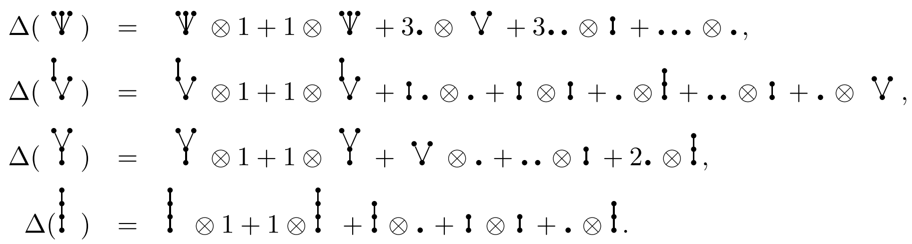
Definition 2. *Let $t \in \mathcal{T}$. An* admissible cut *of t is a non-empty cut such that every downward path in the tree meets at most one cut edge. The set of admissible cuts of t is denoted by $Adm(t)$. If c is an admissible cut of t, one of the trees* *obtained after the application of c contains the root of t: we shall denote it by $R^c(t)$. The product of the other trees will be denoted by $P^c(t)$.*

The coproduct is given for any $t \in \mathcal{T}$ by:

$$\Delta(t) = t \otimes 1 + 1 \otimes t + \sum_{c \in Adm(t)} P^c(t) \otimes R^c(t).$$

The counit ε sends any non-empty forest to 0 and the empty forest 1 to 1.

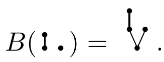
Examples.



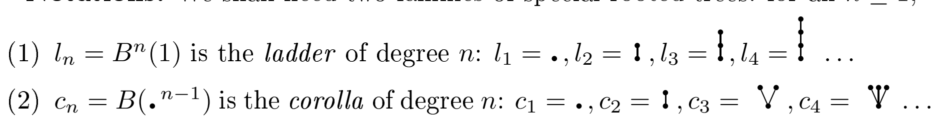
Moreover, this Hopf algebra is graded by the number of vertices of the forests. For any $F \in \mathcal{F}$, we shall denote by $|F|$ its degree, that is to say the number of vertices of F.

The following operator will replace the insertion operators:

Definition 3. *The operator $B : \mathcal{H}_{CK} \longrightarrow \mathcal{H}_{CK}$ is the linear map which sends any rooted forest $F = t_1 \dots t_n$ to the rooted tree obtained by grafting the trees $t_1, \dots, t_n$ on a common root.*

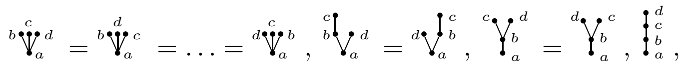
For example, . Clearly, B induces a bijection of degree 1 from $\mathcal{F}$ to $\mathcal{T}$.

Notations. We shall need two families of special rooted trees: for all $n \geq 1$,

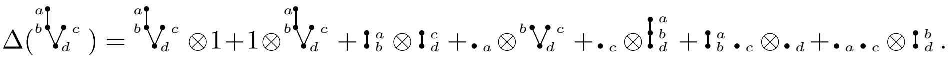


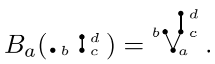
2.2 Decorated rooted trees

In order to treat Dyson-Schwinger systems, we will use decorated rooted trees. We fix a (nonempty) set of decorations I. A *decorated rooted tree* is a pair (t, d), where t is a rooted tree and d is a map from the set of vertices of t to I. The set of rooted trees decorated by I is denoted by $\mathcal{T}^I$. For example, here are the rooted trees decorated by $\mathcal{D}$ with $n \leq 4$ vertices:

 $(a,b,c,d) \in I^4$.

The construction of $\mathcal{H}_{CK}$ is generalized to decorated rooted trees, and we obtain in this way a Hopf algebra $\mathcal{H}_{CK}^I$. A basis of $\mathcal{H}_{CK}^I$ is given by the set of decorated forests, denoted by $\mathcal{F}^I$. Here is an example of the coproduct:



For any $i \in I$, we define the operator $B_i : \mathcal{H}_{CK} \longrightarrow \mathcal{H}_{CK}$, sending a decorated rooted forest F to the decorated tree obtained by grafting the trees of F on a common root decorated by i. For example,  .

Proposition 4. *For all $i \in I$, for all $x \in \mathcal{H}_{CK}^I$:*

$$\Delta \circ B_i(x) = B_i(x) \otimes 1 + (Id \otimes B_i) \circ \Delta(x).$$

Proof. If x is a forest, by a study of the admissible cuts of the trees of x and the admissible cuts of $B_i(x)$. $\square$

Remark. In other words, B_i is a 1-cocycle for a certain cohomology of coalgebras [10], called the *Cartier-Quillen cohomology*, dual to the Hochschild homology for algebras.

Theorem 5 (Universal property). *Let $\mathcal{A}$ be a commutative Hopf algebra and let L_i be a 1-cocycle of $\mathcal{A}$ for all $i \in I$. There exists a unique Hopf algebra morphism $\phi : \mathcal{H}_{CK}^I \longrightarrow \mathcal{A}$ such that $\phi \circ B_i = L_i \circ \phi$ for all $i \in I$.*

Proof. We define $\phi(F)$ for any decorated forest F inductively on the degree of F in the following way:

(1) $\phi(1) = 1$.

(2) If F is not a tree, let us denote $F = t_1 \dots t_k$, with $k \geq 2$ for trees $t_1, \dots, t_k$. We put $\phi(F) = \phi(t_1) \dots \phi(t_k)$.

(3) If F is a tree, there exists a unique $i \in I$ and a unique forest G such that $F = B_i(G)$. We put $\phi(F) = L_i \circ \phi(G)$.

This is well-defined, as A is commutative: in the second point, $\phi(F)$ does not depend on the way to write F as a product of trees (that is to say up to the order of the appearing trees). From the first and second point, it is an algebra morphism. From the third point, $\phi \circ B_i = L_i \circ \phi$ for all $i \in I$. Let us now prove that it is a coalgebra morphism. We put:

$$A = \{x \in \mathcal{H}_{CK}^I \mid (\phi \otimes \phi) \circ \Delta(x) = \Delta \circ \phi(x)\}.$$

As ϕ and Δ are algebra morphisms, A is a subalgebra of $\mathcal{H}_{CK}^I$. Let us take $x \in A$.

For all $i \in I$:

$$\begin{aligned}
(\phi \otimes \phi) \circ \Delta(B_i(x)) &= (\phi \otimes \phi)(B_i(x) \otimes 1 + (Id \otimes B_i) \circ \Delta(x)) \\
&= \phi \circ B_i(x) \otimes 1 + (\phi \otimes \phi \circ B_i) \circ \Delta(x) \\
&= L_i \circ \phi(x) \otimes 1 + (Id \otimes L_i) \circ (\phi \otimes \phi) \circ \Delta(x) \\
&= L_i(\phi(x)) \otimes 1 + (Id \otimes L_i) \circ \Delta(\phi(x)) \\
&= \Delta(L_i(x)).
\end{aligned}$$

So $L_i(x) \in A$, and A is stable under B_i for all i. It is not difficult to show then that A contains any decorated forests, so is equal to $\mathcal{H}_{CK}^I$. Hence, ϕ is a Hopf algebra morphism. It is not difficult to prove that $\varepsilon \circ \phi = \varepsilon_{\mathcal{A}}$. $\square$

Remarks.

(1) The first part of this proof means that $(\mathcal{H}_{CK}, B)$ is an initial object in a certain category, see [37, 43] for applications.

(2) If B_γ is an insertion operator of $\mathcal{H}_{FG}$, homogeneous of degree 1, from theorem 5 there exists a Hopf algebra morphism $\phi_\gamma : \mathcal{H}_{CK} \longrightarrow \mathcal{H}_{FG}$, such that $\phi_\gamma \circ B = B_\gamma \circ \phi_\gamma$. It is not difficult to prove that ϕ_γ is homogeneous of degree 1.

(3) If we consider a Dyson-Schwinger equation $(E) : X = B_\gamma(f(X))$ in $\mathcal{H}_{FG}$, it can be lifted to a Dyson-Schwinger equation $(E') : X = B(f(X))$ in $\mathcal{H}_{CK}$. Moreover, if X is the solution of (E'), then the solution of (E) is $\phi_\gamma(X)$. As a consequence, if the homogeneous components of X generate a Hopf subalgebra of $\mathcal{H}_{CK}$, the homogeneous components of the solution of (E) generate a Hopf subalgebra of $\mathcal{H}_{FG}$. This result is easily extended to Dyson-Schwinger systems.

(4) The construction of the morphism ϕ_γ can easily be extended when we consider several insertion operators, replacing trees by decorated trees, see [27] for a construction of this kind.

2.3 Completion of a graded Hopf algebra

In order to treat Dyson-Schwinger equations, we shall consider series in trees, instead of polynomials in trees, which are elements of $\mathcal{H}_{CK}$. Let us give a general frame to this purpose. Let H be a graded Hopf algebra. We define a *valuation* on H by:

$$val(a) = \max \left\{ n \in \mathbb{N} \mid a \in \bigoplus_{k \geq n} A_k \right\}.$$

In particular, $val(0) = +\infty$. We define a distance on H by $d(a,b) = 2^{-val(a-b)}$. This metric space is not complete. Its completion is denoted by $\overline{H}$. It is equal, as a vector space, to $\prod_{n=0}^{\infty} H_n$.

The product of H, being homogeneous, is continuous, so can be extended as a product from $\overline{H}\otimes\overline{H}$ to $\overline{H}$. The coproduct can also be extended from $\overline{H}$ to $\overline{H\otimes H}$. Note that $\overline{H}$ is not in general a Hopf algebra, as $\overline{H}\otimes\overline{H}\subsetneq\overline{H\otimes H}$ (except if H is finite-dimensional).

For example, the elements of $\overline{\mathcal{H}_{CK}}$ can be uniquely written as $\sum\limits_{F\in\mathcal{F}} a_F F$, where the coefficients a_F are scalars.

3 Pre-Lie algebras

We already mentioned that the space of Feynman graphs is given a pre-Lie algebra structure by insertion. A similar result is here described for rooted trees, and we apply a freeness result (theorem 8) to the Faà di Bruno pre-Lie algebra in order to obtain solutions of Dyson-Schwinger equations. As a consequence, the subalgebras associated to the Dyson-Schwinger equations of proposition 16 are Hopf. This was proved in a different way in [15, 17].

3.1 Definition and examples

Definition 6. *A* (left) pre-Lie algebra *(or* left-symmetric algebra*, or* Vinberg algebra*) is a pair* $(\mathfrak{g},\circ)$*, where* $\mathfrak{g}$ *is a* K*-vector space and* $\circ:\mathfrak{g}\otimes\mathfrak{g}\longrightarrow\mathfrak{g}$*, with the following axiom: for all* $x,y,z\in\mathfrak{g}$*,*

$$(x\circ y)\circ z-x\circ(y\circ z)=(y\circ x)\circ z-y\circ(x\circ z).$$

Remark. A *right pre-Lie algebra* satisfies:

$$(x\circ y)\circ z-x\circ(y\circ z)=(x\circ z)\circ y-x\circ(z\circ y).$$

If $(\mathfrak{g},\circ)$ is right pre-Lie, then $(\mathfrak{g},-\circ^{op})$ is left pre-Lie. In the sequel all the pre-Lie algebras will be left, and we shall write everywhere "pre-Lie algebra" instead of "left pre-Lie algebra".

Proposition 7. *Let* $(\mathfrak{g},\circ)$ *be a pre-Lie algebra. Then* $[x,y]=x\circ y-y\circ x$ *defines a Lie bracket on* $\mathfrak{g}$*.*

Proof. This bracket is obviously skew-symmetric. The Jacobi identity is proved by a direct computation. □

Remarks.

(1) The pre-Lie axiom can be reformulated as $[x,y]\circ z=x\circ(y\circ z)-y\circ(x\circ z)$. In other words, $(\mathfrak{g},\circ)$ is a left-module over $(\mathfrak{g},[-,-])$.

(2) There exists other types of products which induce a Lie bracket by skew-symmetrization: see [21] for other examples.

Examples.

(1) Associative algebras are obviously pre-Lie.

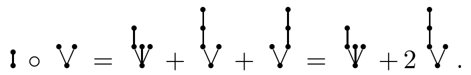
(2) Let $\mathfrak{g}_{FdB} = Vect(e_i \mid i \geq 1)$ and let $\lambda \in K$. One defines a product on $\mathfrak{g}_{FdB}$ by $e_i \circ e_j = (j+\lambda)e_{i+j}$. For all $i, j, k \geq 1$:

$$\begin{array}{rl} & (e_i \circ e_j) \circ e_k - e_i \circ (e_j \circ e_k) \\ = & (j+\lambda)(k+\lambda)e_{i+j+k} - (k+\lambda)(j+k+\lambda)e_{i+j+k} \\ = & -k(k+\lambda)e_{i+j+k}. \end{array}$$

This expression is symmetric in i, j, so $\mathfrak{g}_{FdB}$ is pre-Lie. The associated Lie bracket is given by $[e_i, e_j] = (j-i)e_{i+j}$, so does not depend of λ. This Lie algebra is the *Faà di Bruno Lie algebra.* The graded dual of the enveloping algebra of $\mathfrak{g}_{FdB}$ is known as the Faà di Bruno Hopf algebra or Hopf algebra of formal diffeomorphisms, see [9, 10] for the link with the Hopf algebra of trees.

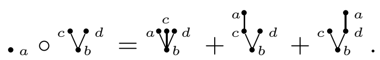
(3) Let $\mathfrak{g}_{\mathcal{T}}$ be the vector space generated by the set $\mathcal{T}$ of rooted trees. We define a product on $\mathfrak{g}_{\mathcal{T}}$ by:

$$t \circ t' = \sum_{s' \text{ vertex of } t'} \text{grafting of } t \text{ over } s'.$$

For example,  . This product is called *natural growth* [3, 13]. It is indeed a pre-Lie product: if t, t', t'' are three rooted trees,

$$\begin{array}{rcl} t \circ (t' \circ t'') - (t \circ t') \circ t'' & = & \displaystyle\sum_{s'' \in t'',\, s' \in t' \cup t''} \text{grafting of } t' \text{ over } s'',\, t \text{ over } s' \\ & & \displaystyle - \sum_{s'' \in t'',\, s' \in t'} \text{grafting of } t' \text{ over } s'',\, t \text{ over } s' \\ & = & \displaystyle\sum_{s', s'' \in t''} \text{grafting of } t \text{ over } s',\, t' \text{ over } s''. \end{array}$$

This is symmetric in t, t', so $\circ$ is pre-Lie. This construction is easily generalized to rooted trees decorated by a set I. The obtained pre-Lie algebra is denoted by $\mathfrak{g}_{\mathcal{T}^I}$. For example, if $a, b, c, d \in I$:



Theorem 8. *[7] $\mathfrak{g}_{\mathcal{T}}$ is, as a pre-Lie algebra, freely generated by $\bullet$, that is to say: if $\mathfrak{g}$ is a pre-Lie algebra and if $x \in \mathfrak{g}$, there exists a unique pre-Lie algebra morphism from $\mathfrak{g}_{\mathcal{T}}$ to $\mathfrak{g}$ sending $\bullet$ to x. More generally, for any set I, the pre-Lie algebra $\mathfrak{g}_{\mathcal{T}^I}$ of rooted trees decorated by I is freely generated by the elements $\bullet_i$, $i \in I$.*

Other examples of pre-Lie algebras are known, see [38] for a list of examples, including vector fields on an affine variety. Generalization of the Faà di Bruno pre-Lie algebras are described in [1].

3.2 Enveloping algebra of a pre-Lie algebra

Let V be a vector space and let $S(V)$ be the symmetric algebra generated by V. It is a cocommutative Hopf algebra, with the coproduct defined by $\Delta(v) = v\otimes 1+1\otimes v$ for all $v \in V$. So, if $v_1, \ldots, v_n \in V$:

$$\Delta(v_1 \ldots v_n) = \sum_{I\subseteq\{1,\ldots,n\}} v_I \otimes v_{\{1,\ldots,n\}-I},$$

where for all $I \subseteq \{1, \ldots, n\}$, v_I is the product of the v_i's, $i \in I$. The underlying coalgebra is denoted by $coS(V)$.

The Poincaré-Birkhoff-Witt theorem implies that the coalgebras $\mathcal{U}(\mathfrak{g})$ and $coS(\mathfrak{g})$ are isomorphic: choosing a basis $(v_i)_{i\in I}$ of $\mathfrak{g}$ indexed by a totally ordered set I, we obtain a coalgebra isomorphism sending the element of the Poincaré-Birkhoff-Witt $v_{i_1}^{a_1} \ldots v_{i_n}^{a_n} \in \mathcal{U}(\mathfrak{g})$, with $i_1 < \ldots < i_n$ in I, to $v_{i_1}^{a_1} \ldots v_{i_n}^{a_n} \in S(\mathfrak{g})$. Except if $\mathfrak{g}$ is abelian, it is not an algebra morphism; moreover, this construction depends of the choice of the basis of $\mathfrak{g}$, especially of the total order on the set of indices I.

When $\mathfrak{g}$ is pre-Lie, one can describe a "canonical" coalgebra isomorphism from $\mathcal{U}(\mathfrak{g})$ to $coS(\mathfrak{g})$. For this, we can give $coS(\mathfrak{g})$ a new product denoted by $\star$, defined by induction on $\mathfrak{g}$ with the help of the pre-Lie product $\mathfrak{g}$. This makes $coS(\mathfrak{g})$ a Hopf algebra, and it is now isomorphic to $\mathcal{U}(\mathfrak{g})$. Here are the formulas defining $\star$:

Theorem 9. *[20, 38] Let $(\mathfrak{g}, \circ)$ a pre-Lie algebra. Let $S_+(\mathfrak{g})$ the augmentation ideal of $S(\mathfrak{g})$. One can extend the product $\circ$ to $S(\mathfrak{g})$ in the following way: if $a, b, c \in S_+(\mathfrak{g})$, $x \in \mathfrak{g}$,*

$$\left\{\begin{array}{rcl} a\circ 1 &=& \varepsilon(a),\\ 1\circ b &=& b,\\ (xa)\circ b &=& x\circ(a\circ b)-(x\circ a)\circ b,\\ a\circ(bc) &=& \sum(a'\circ b)(a''\circ c). \end{array}\right.$$

One then defines a product on $S_+(\mathfrak{g})$ by $a \star b = \sum a'(a'' \circ b)$, with the Sweedler notation $\Delta(a) = \sum a' \otimes a''$. This product is extended to $S(\mathfrak{g})$, making 1 the unit of $\star$. With its usual coproduct, $S(\mathfrak{g})$ is a Hopf algebra, isomorphic to $\mathcal{U}(\mathfrak{g})$ via the isomorphism:

$$\Phi_{\mathfrak{g}} : \left\{\begin{array}{rcl} \mathcal{U}(\mathfrak{g}) &\longrightarrow& (S(\mathfrak{g}), \star)\\ v\in\mathfrak{g} &\longrightarrow& v. \end{array}\right.$$

The proof in [38] is inductive. In particular, the fact that $\circ$ is well-defined (in the second point, the choice of the first letter x in the commutative word xa is arbitrary) uses the pre-Lie axiom. The computations are direct but rather complex.

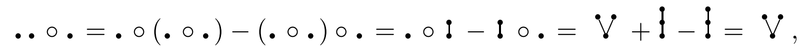
Examples. If $x, y, z, t \in \mathfrak{g}$:

$$\begin{array}{rcl}
x \circ (yz) &=& (x \circ y)z + y(x \circ z) \\
(xy) \circ z &=& x \circ (y \circ z) - (x \circ y) \circ z \\
x \circ (yzt) &=& (x \circ y)zt + y(x \circ z)t + yz(x \circ t) \\
(xy) \circ (zt) &=& (x \circ (y \circ z))t + (y \circ z)(x \circ t) + (x \circ z)(y \circ t) \\
&& +z(x \circ (y \circ t)) - ((x \circ y) \circ z)t - z((x \circ y) \circ t) \\
(xyz) \circ t &=& x \circ (y \circ (z \circ t)) - x \circ ((y \circ z) \circ t) - y \circ ((x \circ z) \circ t) \\
&& +(y \circ (x \circ z)) \circ t - z \circ ((x \circ y) \circ t) + (z \circ (x \circ y)) \circ t.
\end{array}$$

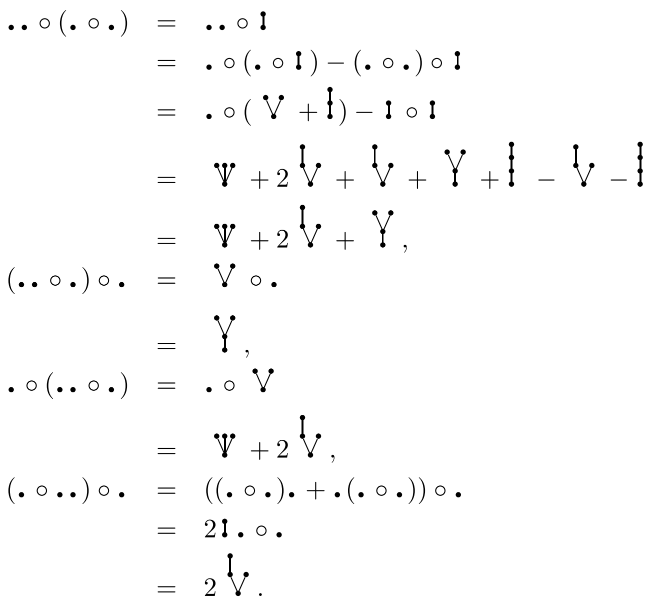
Remarks.

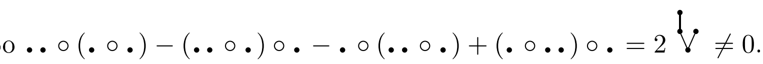
(1) An easy induction proves that for all $n \geq 0$, $\mathfrak{g} \circ S_n(\mathfrak{g}) \subseteq S_n(\mathfrak{g})$. So $(S_n(\mathfrak{g}), \circ)$ is a $\mathfrak{g}$-module for all $n \geq 0$. Moreover, $(S_n(\mathfrak{g}), \circ)$ is isomorphic to $S_n(\mathfrak{g}, \circ)$ as a $\mathfrak{g}$-module (4th point).

(2) $(S_+(\mathfrak{g}), \circ)$ is not pre-Lie. For example, in $\mathfrak{g}_{\mathcal{T}}$:



so:



So 

Remark. It turns out that $S_{\geq n}(\mathfrak{g})$ is a left ideal for $\star$. In particular, $S_{\geq 2}(\mathfrak{g})$ is a left ideal such that $S_+(\mathfrak{g}) = \mathfrak{g} \oplus S_{\geq 2}(\mathfrak{g})$. One deduces that $\mathcal{U}(\mathfrak{g})$ contains a left ideal I such that $\mathcal{U}_+(\mathfrak{g}) = \mathfrak{g} \oplus I$. Dually, we recover the notion of *left-sided combinatorial Hopf algebra* [34].

3.3 Examples

Let us start by $\mathfrak{g}_{\mathcal{T}}$. A basis of $S(\mathfrak{g}_{\mathcal{T}})$ is given by the set of rooted forests $\mathcal{F}$.

Proposition 10. *Let $F = t_1 \ldots t_n, G \in \mathcal{F}$. Then:*

$$F \circ G = \sum_{s_1,\ldots,s_n \in G} \textit{grafting of } t_1 \textit{ over } s_1, \ldots, t_n \textit{ over } s_n.$$

Proof. Inductively on n. Let us start with $n = 1$. We put $G = s_1 \ldots s_m$ and we proceed inductively on m. If $m = 1$, it is the definition of $\circ$ on $\mathfrak{g}_{\mathcal{T}}$. Let us assume the result at rank $m - 1$. We put $G' = s_1 \ldots s_{m-1}$. Then:

$$\begin{aligned}
t_1 \circ G &= t_1 \circ (G' s_m) \\
&= (t_1 \circ G') s_m + G'(t_1 \circ s_m) \\
&= \sum_{s \in G'} (\text{grafting of } t_1 \text{ over } s) s_m + \sum_{s \in s_m} G'(\text{grafting of } t_1 \text{ over } s) \\
&= \sum_{s \in G} \text{grafting of } t_1 \text{ over } s.
\end{aligned}$$

So the result is true at rank 1. Let us assume it at rank $n-1$. We put $F' = t_2 \ldots t_n$. Then:

$$\begin{aligned}
F \circ G &= t_1 \circ (F' \circ G) - (t_1 \circ F') \circ G \\
&= \sum_{s_2,\ldots,s_n \in G} \sum_{s \in F' \cup G} \text{grafting of } t_2 \text{ over } s_2, \ldots, t_n \text{ over } s_n, t_1 \text{ over } s \\
&\quad - \sum_{s_2,\ldots,s_n \in G} \sum_{s \in F'} \text{grafting of } t_2 \text{ over } s_2, \ldots, t_n \text{ over } s_n, t_1 \text{ over } s \\
&= \sum_{s_2,\ldots,s_n \in G} \sum_{s \in G} \text{grafting of } t_2 \text{ over } s_2, \ldots, t_n \text{ over } s_n, t_1 \text{ over } s \\
&= \sum_{s_1,\ldots,s_n \in G} \text{grafting of } t_1 \text{ over } s_1, \ldots, t_n \text{ over } s_n.
\end{aligned}$$

So the result is true for all n. ☐

Corollary 11. *If $F = t_1 \ldots t_m, G \in \mathcal{F}$, then:*

$$F \star G = \sum_{k=0}^{m} \sum_{1 \leq i_1 < \ldots < i_k \leq m} \sum_{s_1,\ldots,s_k \in G} (\textit{grafting of } t_1 \textit{ over } s_1, \ \ldots, \ t_k \textit{ over } s_k) \prod_{i \neq i_1,\ldots,i_k} t_i.$$

The Hopf algebra $S(\mathfrak{g}_{\mathcal{T}})$ is known as the Grossman-Larson Hopf algebra [22, 23, 24]. The first known proof of its existence is direct and does not use the pre-Lie structure.

Let us consider now the Faà di Bruno pre-Lie algebra.

Proposition 12. *In $S(\mathfrak{g}_{FdB})$:*

$$(e_{i_1} \dots e_{i_m}) \circ e_j = (j+\lambda)j(j-\lambda)\dots(j-(m-2)\lambda)e_{i_1+\dots+i_m+j}.$$

Proof. We put $P_m(j) = (j+\lambda)j(j-\lambda)\dots(j-(m-2)\lambda)$. We proceed inductively on m. If $m = 1$, it is the definition of the pre-Lie product of $\mathfrak{g}_{FdB}$. Let us assume the result at rank $m-1$. Then:

$$\begin{aligned}
& (e_{i_1} \dots e_{i_m}) \circ e_j \\
= \; & e_{i_1} \circ ((e_{i_2} \dots e_{i_m}) \circ e_j) - (e_{i_1} \circ (e_{i_2} \dots e_{i_m})) \circ e_j \\
= \; & P_{m-1}(j) e_{i_1} \circ e_{i_2+\dots+i_m+j} - \sum_{k=2}^{m} (i_k+\lambda)(e_{i_2} \dots e_{i_1+i_k} \dots e_{i_m}) \circ e_j \\
= \; & P_{m-1}(j)(i_2+\dots+i_m+j+\lambda) e_{i_1+\dots+i_m+j} - \sum_{k=2}^{m} P_{m-1}(j)(i_k+\lambda) e_{i_1+\dots+i_m+j} \\
= \; & P_{m-1}(j)(i_2+\dots+i_m+j+\lambda-i_2-\dots-i_m-(m-1)\lambda) e_{i_1+\dots+i_m+j} \\
= \; & P_m(j) e_{i_1+\dots+i_m+j}.
\end{aligned}$$

So the result is true for all m. $\square$

Notation. If $\lambda \neq -1$, we put $\alpha = 1+\lambda$ and $\beta = \frac{-\lambda}{1+\lambda}$. Then, for all $i_1, \dots, i_m$:

$$(e_{i_1} \dots e_{i_m}) \circ e_j = \alpha^m (j+\beta(j-1))(j+\beta(j)) \dots (j+\beta(j+m-2)) e_{i_1+\dots+i_m+j}. \quad (3.1)$$

This formula is still true if $\lambda = -1$ and $j = 1$, with $\alpha = 0$, for any value of β. Indeed, if $\lambda = -1$ (so $\alpha = 0$) and $j = 1$, then $(e_{i_1} \dots e_{i_m}) \circ e_1 = 0$.

3.4 From rooted trees to Faà di Bruno

From theorem 8, there exists a unique morphism of pre-Lie algebras $\phi_\lambda : \mathfrak{g}_{\mathcal{T}} \longrightarrow \mathfrak{g}_{FdB}$, sending $\bullet$ to e_1.

Definition 13. *Let $\beta \in K$.*

(1) *For any $n \geq 1$, we put $[n]_\beta = 1 + (n-1)\beta$.*

(2) *For any $n \geq 0$, we put $[n]_\beta! = [1]_\beta \dots [n]_\beta$, with the convention $[0]_\beta! = 1$.*

(3) *Let $t \in \mathcal{T}$ and let x be a vertex of t. The* fertility *of x is the number of children of x.*

(4) *Let $t \in \mathcal{T}$. We put $[t]_\beta! = \displaystyle\prod_{s \text{ vertex of } t} [\text{fertility of } s]_\beta!$.*

Remarks.

(1) If $\beta = 1$, then $[n]_\beta = n$ for all $n \geq 1$.

(2) With these notations, (3.1) becomes, for $j = 1$:

$$(e_{i_1} \dots e_{i_m}) \circ e_1 = \alpha^m [m]_\beta! e_{i_1+\dots+i_m+1}.$$

Proposition 14. *For all $t \in \mathcal{T}$, $\phi_\lambda(t) = \alpha^{|t|-1}[t]_\beta! e_{|t|}$. Moreover, ϕ_λ is surjective if, and only if, $\lambda \neq -1$.*

Proof. We extend ϕ_λ as a Hopf algebra morphism from $(S(\mathfrak{g}_\mathcal{T}), \star)$ to $(S(\mathfrak{g}_{FdB}), \star)$. Then $\phi_\lambda(a \circ b) = \phi_\lambda(a) \circ \phi_\lambda(b)$ for all $a, b \in S(\mathfrak{g}_\mathcal{T})$. We prove the result by induction on the degree of t. It is obvious if $|t| = 1$, as then $t = \bullet$ and $\phi_\lambda(\bullet) = e_1$. Let us assume the result for all trees of degrees strictly smaller than t. Let $t_1, \ldots, t_m$ be the trees obtained by deleting the root of t. Then, from proposition 10 and (3.1):

$$
\begin{aligned}
\phi_\lambda(t) &= \phi_\lambda((t_1 \ldots t_m) \circ \bullet) \\
&= (\phi_\lambda(t_1) \ldots \phi_\lambda(t_m)) \circ e_1 \\
&= \alpha^{|t_1|+\ldots+|t_m|-m}[t_1]_\beta! \ldots [t_m]_\beta! (e_{|t_1|} \ldots e_{|t_m|}) \circ e_1 \\
&= \alpha^{|t_1|+\ldots+|t_m|-m}\alpha^m [m]_\beta! [t_1]_\beta! \ldots [t_m]_\beta! e_{|t_1|+\ldots+|t_m|+1} \\
&= \alpha^{|t|-1}[t]_\beta! e_{|t|}.
\end{aligned}
$$

So the result is true for all trees.

If $\lambda = -1$, then $Im(\phi_\lambda) = Ke_1$ as $\alpha = 0$. If $\lambda \neq -1$, then $\phi_\lambda(l_n) = \alpha^{n-1}e_n$, so ϕ_λ is surjective. $\square$

3.5 Duality

The aim of this section is to describe a family of injections of the dual of the Faà di Bruno Hopf algebra in the Hopf algebra of rooted trees, with the help of the pre-Lie structures. Noncommutative versions are given in [4, 5, 14]; the case of free Faà di Bruno Hopf algebras is studied in [15].

$\mathfrak{g}_\mathcal{T}$ and $\mathfrak{g}_{FdB}$ are graded pre-Lie algebras, so $S(\mathfrak{g}_\mathcal{T})$ and $S(\mathfrak{g}_{FdB})$ are graded Hopf algebras (for the product $\star$). As $(\mathfrak{g}_\mathcal{T})_0 = (\mathfrak{g}_{FdB})_0 = (0)$, the homogeneous components of $S(\mathfrak{g}_\mathcal{T})$ and $S(\mathfrak{g}_{FdB})$ are finite-dimensional, so the graded dual of $S(\mathfrak{g}_\mathcal{T})$ and $S(\mathfrak{g}_{FdB})$ are also Hopf algebras. The graded dual of $S(\mathfrak{g}_{FdB})$ is denoted by $\mathcal{H}_{FdB}$.

Let us give a more precise description of $S(\mathfrak{g}_\mathcal{T})^*$. A basis of $S(\mathfrak{g}_\mathcal{T})$ is given by rooted forests. We identify $S(\mathfrak{g}_\mathcal{T})$ and $S(\mathfrak{g}_\mathcal{T})^*$ as vector spaces with the help of the pairing defined in the following way:

$$\langle F, G \rangle = s_F \delta_{F,G},$$

where s_F is the number of automorphisms of the rooted forest F, that is to say the number of automorphisms of the graph F which map all roots to roots.

Let F, G, H be three forests. We put $F = t_1^{\alpha_1} \ldots t_n^{\alpha_n}$, $G = t_1^{\beta_1} \ldots t_n^{\beta_n}$ and

$H = t_1^{\gamma_1} \dots t_n^{\gamma_n}$, where $t_1, \dots, t_n$ are different rooted trees. Then:

$$\begin{aligned}
&\langle \Delta(H), F \otimes G \rangle \\
= &\sum_{i_1,\dots,i_n} \binom{\gamma_1}{i_1} \dots \binom{\gamma_n}{i_n} \langle t_1^{i_1} \dots t_n^{i_n}, t_1^{\alpha_1} \dots t_n^{\alpha_n} \rangle \langle t_1^{\gamma_1 - i_1} \dots t_n^{\gamma_n - i_n}, t_1^{\beta_1} \dots t_n^{\beta_n} \rangle \\
= &\sum_{i_1,\dots,i_n} \binom{\gamma_1}{i_1} \dots \binom{\gamma_n}{i_n} s_F s_G \delta_{i_1,\alpha_1} \dots \delta_{i_n,\alpha_n} \delta_{\gamma_1 - i_1, \beta_1} \dots \delta_{\gamma_n - i_n, \beta_n}.
\end{aligned}$$

So this is zero if there exists i such that $\gamma_i \neq \alpha_i + \beta_i$. If $\gamma_i = \alpha_i + \beta_i$ for all i, then:

$$\begin{aligned}
s_H &= \gamma_1! \dots \gamma_n! s_{t_1}^{\gamma_1} \dots s_{t_n}^{\gamma_n} \\
&= \binom{\gamma_1}{\alpha_1} \dots \binom{\gamma_n}{\alpha_n} \alpha_1! \dots \alpha_n! s_{t_1}^{\alpha_1} \dots s_{t_n}^{\alpha_n} \beta_1! \dots \beta_n! s_{t_1}^{\beta_1} \dots s_{t_n}^{\beta_n} \\
&= \binom{\gamma_1}{\alpha_1} \dots \binom{\gamma_n}{\alpha_n} s_F s_G.
\end{aligned}$$

So:

$$\langle \Delta(H), F \otimes G \rangle = \binom{\gamma_1}{\alpha_1} \dots \binom{\gamma_n}{\alpha_n} s_F s_G = s_H.$$

In both cases, $\langle \Delta(H), F \otimes G \rangle = \langle H, FG \rangle$. So the product of $S(\mathfrak{g}_{\mathcal{T}})^*$ is the "usual" product of forests (disjoint union).

Let us now consider the coproduct of $S(\mathfrak{g}_{\mathcal{T}})^*$. From the preceding point, $S(\mathfrak{g}_{\mathcal{T}})^*$ is generated by the set of rooted trees. It is then enough to compute $\Delta(t)$ for any rooted tree t. Moreover, by construction of $\star$, for all $n \geq 1$:

$$S(\mathfrak{g}_{\mathcal{T}}) \star S_n(\mathfrak{g}_{\mathcal{T}}) \subseteq \bigoplus_{p \geq n} S_p(\mathfrak{g}_{\mathcal{T}}).$$

So, if F, G are two forests such that G has at least two trees, then $F \star G$ is a sum of forests with at least two trees. Hence, if t is a rooted tree, $\langle F \otimes G, \Delta(t) \rangle = \langle F \star G, t \rangle = 0$. If t' is a tree, from corollary 11:

$$\langle F \otimes t', \Delta(t) \rangle = \langle F \star t', t \rangle = s_t \sharp \{\text{graftings of } F \text{ over } t' \text{ that yield } t\}.$$

This is equal to $s_F s_{t'} \sharp \{\text{admissible cuts } c \text{ of } t \text{ such that } P^c(t) = F \text{ and } R^c(t) = t'\}$, see [25]. So:

$$\langle F \otimes t', \Delta(t) \rangle = \sum_{c \in Adm(t)} \langle F \otimes t', P^c(t) \otimes R^c(t) \rangle.$$

As a conclusion, we obtain the following formula: for any rooted tree $t \in S(\mathfrak{g}_{\mathcal{T}})^*$,

$$\Delta(t) = 1 \otimes t + t \otimes 1 + \sum_{c \in Adm(t)} P^c(t) \otimes R^c(t).$$

In other words, $S(\mathfrak{g}_{\mathcal{T}})^*$ is the Connes-Kreimer Hopf algebra $\mathcal{H}_{CK}$ [6, 25, 39]. This result is proved similarly for decorated rooted trees.

We now give a description of $\mathcal{H}_{FdB}$. Let $(x_n)_{n\geq 1}$ be the dual basis of $(e_n)_{n\geq 1}$. Then a basis of $\mathcal{H}_{FdB}$ is given by the monomials in the x_i's and the duality is given by:

$$\langle x_1^{i_1}\ldots x_n^{i_n}, e_1^{j_1}\ldots e_n^{j_n}\rangle = i_1!\ldots i_n!\delta_{i_1,j_1}\ldots\delta_{i_n,j_n}.$$

Dualising proposition 12 we obtain, for all $n\geq 1$:

$$\Delta(x_n) = x_n\otimes 1+\sum_{j=1}^{n}\sum_{m=0}^{n-j}\sum_{i_1+\ldots+i_m+j=n}\frac{(j+\lambda)\ldots(j-(m-2)\lambda)}{m!}x_{i_1}\ldots x_{i_m}\otimes x_j.$$

Let us reformulate this formula. We put $X=\sum x_n\in\overline{\mathcal{H}_{FdB}}$. If $\lambda\neq 0$:

$$\begin{aligned}
\Delta(X) &= X\otimes 1+\sum_{j=1}^{\infty}\sum_{m=0}^{\infty}\sum_{i_1,\ldots,i_m}\frac{(j+\lambda)\ldots(j-(m-2)\lambda)}{m!}x_{i_1}\ldots x_{i_m}\otimes x_j\\
&= X\otimes 1+\sum_{j=1}^{\infty}\left(\sum_{m=0}^{\infty}\frac{(j+\lambda)\ldots(j-(m-2)\lambda)}{m!}X^m\right)\otimes x_j\\
&= X\otimes 1+\sum_{j=1}^{\infty}(1+\lambda X)^{1+\frac{j}{\lambda}}\otimes x_j. \qquad (3.2)
\end{aligned}$$

If $\lambda=0$, we obtain:

$$\Delta(X)=X\otimes 1+\sum_{j=1}^{\infty}e^{jX}\otimes x_j.$$

Remark. Let us consider more precisely the case $\lambda=1$. As $\mathcal{H}_{FdB}$ is commutative, we can consider it as the Hopf algebra of coordinates on its group of characters G_{FdB}. As $\mathcal{H}_{FdB}$ is the free commutative algebra generated by the x_i's, any element $\phi\in G_{FdB}$ is entirely determined by its values on the x_i's. In other words, there exists a bijection:

$$\left\{\begin{array}{rcl}
G_{FdB} & \longrightarrow & h+h^2K[[h]]\\
\phi & \longrightarrow & F_\phi = h+\displaystyle\sum_{n=1}^{\infty}\phi(x_n)h^{n+1}.
\end{array}\right.$$

So, taking $Y=1+\sum x_n h^n$, this morphism can be summarized as $F_\phi=h\phi(Y)$. Moreover, the formula on X implies that:

$$\Delta(Y)=Y\otimes 1+\sum_{j=1}^{\infty}Y^{j+1}\otimes h^j x_j=\sum_{j=0}^{\infty}h^jY^{j+1}\otimes x_j,$$

with the convention $x_0=1$. Then, if $\phi,\psi\in G_{FdB}$:

$$F_{\phi\psi}=h(\phi\otimes\psi)\circ\Delta(Y)=\sum_{j=0}^{\infty}h^{j+1}\phi(Y)^{j+1}\psi(x_j)=F_\psi\circ F_\phi.$$

So, up to an isomorphism, G_{FdB} is the (opposite of the) group of formal diffeomorphisms tangent to the identity at 0, with the usual composition of formal series.

Let us now dualise the pre-Lie algebra morphism ϕ_λ. The Hopf algebra $\mathcal{H}_{FdB}$ is generated by the elements x_i, $i \geq 1$, dual to the elements $e_i \in \mathfrak{g}_{FdB}$. It is enough to describe the image of the x_i's. By homogeneity, $\phi_\lambda^*(x_i)$ is a linear span of rooted trees of degree i. Let $t \in \mathcal{T}$, of degree i. Then:

$$\langle t, \phi_\lambda^*(x_i)\rangle = \langle \phi_\lambda(t), x_i\rangle = \alpha^{i-1}[t]_\beta \langle e_i, x_i\rangle = \alpha^{i-1}[t]_\beta.$$

As a consequence:

$$\phi_\lambda^*(x_i) = \alpha^{i-1} \sum_{t\in\mathcal{T},\, |t|=i} \frac{1}{s_t}[t]_\beta! t.$$

If $\lambda \neq -1$, ϕ_λ is surjective, so ϕ_λ^* is injective. We proved:

Proposition 15. *For all $n \geq 1$, we put:*

$$x(n) = \alpha^{n-1} \sum_{t\in\mathcal{T},\, |t|=n} \frac{1}{s_t}[t]_\beta! t.$$

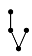
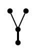
The subalgebra of $\mathcal{H}_{CK}$ generated by these elements is Hopf. If $\lambda \neq -1$ (or equivalently if $\alpha \neq 0$), it is isomorphic to $\mathcal{H}_{FdB}$.

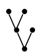
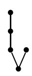
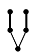
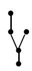
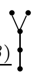
Examples.

$x(1) =$,

$x(2) = \alpha$ ,

$x(3) = \alpha^2 \Big(\frac{(1+\beta)}{2}$  $+$  $\Big)$,

$x(4) = \alpha^3 \Big(\frac{(1+2\beta)(1+\beta)}{6}$  $+ (1+\beta)$  $+ \frac{(1+\beta)}{2}$  $+$  $\Big)$,

$x(5) = \alpha^4 \Big(\frac{(1+3\beta)(1+2\beta)(1+\beta)}{24}$  $+ \frac{(1+2\beta)(1+\beta)}{2}$  $+ \frac{(1+\beta)^2}{2}$  $+ (1+\beta)$ 
$+ \frac{(1+2\beta)(1+\beta)}{6}$  $+ \frac{(1+\beta)}{2}$  $+ (1+\beta)$  $+ \frac{(1+\beta)}{2}$  $+$ $\Big)$.

3.6 From the Faà di Bruno Lie algebra to Dyson-Schwinger equations

Let us use the operator B to inductively describe the $x(n)$'s. We denote by a_t the coefficient of t in $x(|t|)$. Let F be the unique forest such that $t = B(F)$. We put

$F = t_1^{\alpha_1} \dots t_k^{\alpha_k}$, where the t_i's are different rooted trees. Then:

$$\begin{aligned}
a_t &= \alpha^{|t|-1} \frac{[t]_\beta !}{s_t} \\
&= \alpha^{\alpha_1 |t_1| + \dots + \alpha_k |t_k|} \frac{[t_1]_\beta^{\alpha_1} \dots [t_k]_\beta^{\alpha_k} [\alpha_1 + \dots + \alpha_k]_\beta !}{s_{t_1}^{\alpha_1} \dots s_{t_k}^{\alpha_k} \alpha_1 ! \dots \alpha_k !} \\
&= \alpha^{\alpha_1 + \dots + \alpha_k} \frac{[\alpha_1 + \dots + \alpha_k]_\beta !}{(\alpha_1 + \dots + \alpha_k)!} \frac{(\alpha_1 + \dots + \alpha_k)!}{\alpha_1 ! \dots \alpha_k !} a_{t_1}^{\alpha_1} \dots a_{t_k}^{\alpha_k}.
\end{aligned}$$

We put $X = \sum x(i)$. This is *a priori* not an element of $\mathcal{H}_{CK}$ (it is an infinite sum), but it lives in the completion $\overline{\mathcal{H}_{CK}}$ of $\mathcal{H}_{CK}$. The preceding computations imply that X satisfies the following equation:

$$X = B\left(\sum_{n=0}^{\infty} \alpha^n \frac{[n]_\beta !}{n!} X^n \right).$$

This equation is a *combinatorial Dyson-Schwinger equation.* Let us consider the formal series $f = \sum \alpha^n \frac{[n]_\beta !}{n!} h^n$. Denoting its coefficients by a_n, there is the obvious inductive relation:

$$(n+1) a_{n+1} = \alpha (1 + n\beta) a_n.$$

Summing these relations after multiplication by h^n, we obtain:

$$f' = \alpha f + \alpha \beta h f'.$$

An easy induction proves that for all $n \geq 0$:

$$a_n = \frac{[n]_\beta !}{n!} \alpha^n = \begin{cases} \binom{-\frac{1}{\beta}}{n} (-\alpha\beta)^n \text{ if } \beta \neq 0, \\ \frac{\alpha^n}{n!} \text{ if } \beta = 0. \end{cases}$$

Hence, $f(h) = e^{\alpha h}$ if $\beta = 0$ (that is to say if $\lambda = 0$) or $(1 - \alpha\beta h)^{-\frac{1}{\beta}}$ if $\beta \neq 0$.

Proposition 16. *The element $X \in \overline{\mathcal{H}_{CK}}$ defined using the pre-Lie morphism ϕ_λ from $\mathfrak{g}_{\mathcal{T}}$ to $\mathfrak{g}_{FdB}$ satisfies the combinatorial Dyson-Schwinger equation:*

$$X = B(f(X)),$$

where $f = 1$ if $\lambda = -1$, $f = e^h$ if $\lambda = 0$, $f = (1 + \lambda h)^{\frac{1+\lambda}{\lambda}}$ if $\lambda \neq 0, -1$.

Remark. In all cases, $f = \displaystyle\sum_{k=0}^{\infty} \frac{(1+\lambda)(1)(1-\lambda) \dots (1 - \lambda(k-2))}{k!} h^k$.

4 Combinatorial Dyson-Schwinger equations

4.1 Definition

Definition 17. *Let $f \in K[[h]]$. The combinatorial Dyson-Schwinger equation associated to f is:*

$$X = B(f(X)),$$

where $X \in \overline{\mathcal{H}_{CK}}$.

Proposition 18. *The Dyson-Schwinger equation associated to the formal series $f = \sum a_n h^n$ admits a unique solution $X = \sum x(n)$, inductively defined by:*

$$\left\{ \begin{array}{rcl} x(0) & = & 0, \\ x(1) & = & a_0 \bullet, \\ x(n+1) & = & \displaystyle\sum_{k=1}^{n} \sum_{i_1+\cdots+i_k=n} a_k B(x(i_1)\cdots x(i_k)). \end{array} \right.$$

Proof. It is enough to identify the coefficients of each $t \in \mathcal{T}$ in the two sides of the combinatorial Dyson-Schwinger equation associated to f. □

Remark. We can put $X = \sum_{t \in \mathcal{T}} a_t t$. The coefficients a_t are inductively computed by the following formula: if $t = B(t_1^{k_1} \ldots t_n^{k_n})$, where $t_1, \ldots, t_n$ are distinct trees, then:

$$a_t = a_{k_1+\ldots+k_n} \frac{(k_1 + \ldots + k_n)!}{k_1! \ldots k_n!} a_{t_1}^{k_1} \ldots a_{t_n}^{k_n}. \tag{4.1}$$

The induction is initiated by $a_\bullet = a_0$.

Definition 19. *The subalgebra of $\mathcal{H}_{CK}$ generated by the homogeneous components $x(n)$ of the unique solution X of the Dyson-Schwinger equation associated to f will be denoted by $\mathcal{H}_f$.*

We would like to give a necessary and sufficient condition on f for $\mathcal{H}_f$ to be a Hopf subalgebra of $\mathcal{H}_{CK}$. If this is the case, we shall say that the Dyson-Schwinger equation associated to f is *Hopf*.

Remarks.

(1) If $f(0) = 0$, the unique solution of the combinatorial Dyson-Schwinger equation associated to f is 0. As a consequence, $\mathcal{H}_f = K$ is a Hopf subalgebra.

(2) For all $\mu \in K$, if $X = \sum x(n)$ is the solution of the Dyson-Schwinger equation associated to f, the unique solution of the Dyson-Schwinger equation associated to μf is $\sum \mu^n x(n)$. As a consequence, if $\mu \neq 0$, $\mathcal{H}_f = \mathcal{H}_{\mu f}$. We shall then suppose in the sequel that $a_0 = 1$. In this case, $x(1) = \bullet$.

(3) Let $\mu \in K - \{0\}$. the unique solution of the combinatorial Dyson-Schwinger equation associated to $\frac{1}{\mu} f(\mu h)$ is $\frac{1}{\mu} X$. Combining with the preceding remark, the equation associated to $f(\mu h)$ is Hopf if, and only if, the equation associated to f is Hopf (this is the operation of *change of variables*, given for systems in definition 33).

4.2 Pre-Lie structure associated to a Hopf Dyson-Schwinger equation

Lemma 20. *Let V be a subspace of $Vect(\mathcal{T})$ and let us consider the subalgebra A of $\mathcal{H}_{CK}$ generated by V. We consider the following map:*

$$f_\bullet : \left\{ \begin{array}{rcl} \mathcal{H}_{CK} & \longrightarrow & K \\ F \in \mathcal{F} & \longrightarrow & \delta_{F,\bullet}. \end{array} \right.$$

If A is a Hopf subalgebra, then $(f_\bullet \otimes Id) \circ \Delta(V) \subseteq V \oplus K$.

Proof. If A is Hopf, then $\Delta(V) \subseteq A \otimes A$. As $V \subseteq Vect(\mathcal{T})$, $\Delta(V) \subseteq \mathcal{H}_{CK} \otimes (Vect(\mathcal{T}) \oplus K)$. So:

$$\Delta(V) \subseteq (A \otimes A) \cap (\mathcal{H}_{CK} \otimes (Vect(\mathcal{T}) \oplus K)) = A \otimes (V \oplus K).$$

This implies the assertion. □

Remarks.

(1) In the duality between $\mathcal{H}_{CK}$ and $S(\mathfrak{g}_\mathcal{T})$, $f_\bullet = \langle \bullet, - \rangle$.

(2) This result is easily generalized to decorated rooted trees, replacing $\bullet$ by the $\bullet_i$'s, $i \in I$.

4.3 Definition of the structure coefficients

Proposition 21. *Let (E) be a combinatorial Dyson-Schwinger equation. If it is Hopf, then for all $n \geq 1$, there exists a scalar λ_n such that for all $t' \in \mathcal{T}$, of degree n:*

$$\sum_{t \in \mathcal{T}} n(t,t') a_t = \lambda_n a_{t'},$$

where $n(t,t')$ is the number of leaves l of t such that the cut of l gives t'.

Proof. Let us assume that (E) is Hopf. Then $\mathcal{H}_f$ is a Hopf subalgebra of $\mathcal{H}_{CK}$. Let us use lemma 20, with $V = Vect(x(n), n \geq 1)$. So $(f_\bullet \otimes Id) \circ \Delta(x(n+1))$ belongs to $\mathcal{H}_f$, and is a linear span of trees of degree n, so is a multiple of $x(n)$. We then denote:

$$(f_\bullet \otimes Id) \circ \Delta(x(n+1)) = \lambda_n x(n) = \sum_{\substack{t' \in \mathcal{T} \\ |t'|=n}} \lambda_n a_{t'} t'.$$

By definition of the coproduct Δ:

$$(f_\bullet \otimes Id) \circ \Delta(x(n+1)) = \sum_{\substack{t,t' \in \mathcal{T} \\ |t'|=n}} n(t,t') a_t t'.$$

The result is proved by identifying the coefficients in the basis $\mathcal{T}$ of these two expressions of $(f_\bullet \otimes Id) \circ \Delta(x(n+1))$. $\square$

If f is Hopf and if all the $x(n)$'s are non zero, let us consider the dual basis $(e_n)_{n\geq 1}$ of the $x(n)$'s. It is a basis of $\mathfrak{g}_f = Prim(\mathcal{H}_f^*)$. As $\mathcal{H}_f$ is generated by a subspace of $Vect(\mathcal{T})$, $\mathfrak{g}_f$ is naturally a pre-Lie algebra. Let us describe this pre-Lie product.

Proposition 22. *For all $k, l \geq 1$, $e_l \circ e_k = \lambda_k e_{k+l}$.*

Proof. Let us first prove the following result: for all $t', t'' \in \mathcal{T}$,

$$\sum_{t \in \mathcal{T}} n(t', t''; t) a_t = \lambda_{|t''|} a_{t'} a_{t''},$$

where $n(t', t''; t)$ is the number of admissible cuts c of t such that $P^c(t) = t'$ and $R^c(t) = t''$ (that is to say the coefficient of $t' \otimes t''$ in $\Delta(t)$). We proceed by induction on $|t''|$. If $|t''| = 1$, then $t'' = \bullet$ and:

$$\sum_{t \in \mathcal{T}} n(t', t''; t) a_t = a_{B(t')} = a_1 a_{t'} = \lambda_1 a_{t'} a_{t''},$$

as $\lambda_1 = a_1$ and $a_\bullet = 1$. Let us assume the result at all rank $< k$ and let us assume that $|t''| = k$. We put $t'' = B(F)$, with $F = \prod_{s \in \mathcal{T}} s^{p_s}$, $p = \sum_{s \in \mathcal{T}} p_s$.

First step. By definition of λ_k, using (4.1):

$$\begin{aligned}
\lambda_k a_{t''} &= (p_\bullet + 1) a_{B(\bullet F)} + \sum_{s, s' \in \mathcal{T}, p_{s'} \geq 1} (p_s + 1) n(s, s') a_{B\left(\frac{s}{s'} F\right)} \\
&= (p_\bullet + 1) \frac{p+1}{p_\bullet + 1} \frac{a_{p+1}}{a_p} a_{t''} + \sum_{s, s' \in \mathcal{T}, p_{s'} \geq 1} (p_s + 1) n(s, s') \frac{p_{s'}}{p_s + 1} \frac{a_s}{a_{s'}} a_{t''} \\
&= (p+1) \frac{a_{p+1}}{a_p} a_{t''} + \sum_{s' \in \mathcal{T}} p_{s'} \lambda_{|s'|} a_{t''}.
\end{aligned}$$

We obtain $\displaystyle\left((p+1) \frac{a_{p+1}}{a_p} + \sum_{s' \in \mathcal{T}} p_{s'} \lambda_{|s'|} \right) a_{t''} = \lambda_k a_{t''}$.

Second step. Let us now fix $t' \in \mathcal{T}$. Then:

$$\begin{array}{rcl}
\displaystyle\sum_{t\in\mathcal{T}} n(t',t'';t)a_t & = & (p_{t'}+1)a_{B(t'F)} + \displaystyle\sum_{s,s'\in\mathcal{T},p_{s'}\geq 1} (p_s+1)n(t',s';s)a_{B\left(\frac{s}{s'}F\right)} \\
& = & (p_{t'}+1)\dfrac{(p+1)!}{(p_{t'}+1)\prod p_s!} a_{p+1}a_{t'} \displaystyle\prod_s a_s^{p_s} \\
& & + \displaystyle\sum_{s,s'\in\mathcal{T},p_{s'}\geq 1} (p_s+1)n(t',s';s)\frac{p_{s'}}{p_s+1}a_{t''}\frac{a_s}{a_{s'}} \\
& = & (p+1)\dfrac{a_{p+1}}{a_p}a_{t'}a_{t''} + \displaystyle\sum_{s'\in\mathcal{T},p_{s'}\geq 1} p_{s'}\lambda_{|s'|}a_{t'}a_{t''} \\
& = & \lambda_k a_{t''}a_{t'}.
\end{array}$$

We used the induction hypothesis on s' and then the first step.

As a consequence, for all $n \geq 1$:

$$\Delta(x(n)) = \sum_{k=1}^{n-1} \lambda_k x(n-k) \otimes x(k) + \text{terms with forests which are not trees.}$$

Dually, we deduce that $e_{n-k} \circ e_k = \lambda_k e_n$ for all $1 \leq k \leq n$. $\square$

4.4 Main theorem for single equations

Assume that the Dyson-Schwinger equation associated to the formal series $f = 1 + \sum_{n\geq 1} a_n h^n$ is Hopf. If $a_1 \neq 0$, the coefficients λ_n are entirely determined by a_1 and a_2, and this also determines all the a_n's, as it is explained in the following result:

Lemma 23. (1) $\lambda_1 = a_1$.

(2) *For all* $n \geq 2$, $\lambda_n a_1^{n-1} = a_1^{n-2}(a_1^2 + 2a_2(n-1))$.

(3) *For all* $n \geq 2$, $a_n = \dfrac{\lambda_n - a_1(n-1)}{n} a_{n-1}$.

Proof. Recall that the ladders l_n and the corollas c_n are defined in section 2.1. Using proposition 21with $t' = \bullet$, $\lambda_1 a_\bullet = a_{\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}} = a_1$ and $t' = l_n$ gives:

$$\begin{array}{rcl}
\lambda_n a_{l_n} & = & \lambda_n a_1^{n-1} \\
& = & a_{l_{n+1}} + 2a_{B^{n-1}(\bullet\bullet)} + \displaystyle\sum_{i=1}^{n-2} a_{B^i(\bullet B^{n-i}(1))} \\
& = & a_1^n + 2a_2a_1^{n-2} + \displaystyle\sum_{i=1}^{n-2} 2a_1^{n-2}a_2.
\end{array}$$

So $\lambda_n a_1^{n-1} = a_1^n + 2a_2 a_1^{n-2}(n-1)$. We now use proposition 21 with $t' = c_n$, $n \geq 2$:

$$\lambda_n a_{c_n} = n a_{c_{n+1}} + a_{B(\,\mathbf{I}\,\bullet^{n-2})},$$

so $\lambda_n a_{n-1} = n a_n + (n-1) a_1 a_{n-1}$. $\square$

Theorem 24. *Let $f = 1 + a_1 h + \ldots \in K[[h]]$. The following assertions are equivalent:*

1. *The combinatorial Dyson-Schwinger equation associated to f is Hopf.*
2. *There exists $(\alpha, \beta) \in K^2$, such that $f = 1$ if $\alpha = 0$, $f = e^{\alpha h}$ if $\beta = 0$, $f = (1 - \alpha\beta h)^{-\frac{1}{\beta}}$ if $\alpha\beta \neq 0$.*

Proof. $1 \Longrightarrow 2$. If $a_1 = 0$, by lemma 23-3, $a_n = 0$ for all $n \geq 1$, so $f = 1$. We now assume that $a_1 \neq 0$. From lemma 23-2, for all $n \geq 1$, $\lambda_n = a_1 + 2\frac{a_2}{a_1}(n-1)$ and, for all $n \geq 2$:

$$a_n = \frac{a_1}{n}\left(1 + \left(\frac{2a_2}{a_1^2} - 1\right)(n-1)\right) a_{n-1}.$$

We put $\alpha = a_1$ and $\beta = \frac{2a_2}{a_1^2} - 1$. An easy induction proves that for all n, $a_n = \alpha^n [n]_\beta ! / n!$. The result is then proved in section 3.6.

$2 \Longrightarrow 1$. From proposition 16, the result is true if $f = 1$, $f = e^h$ or $f = (1 + \lambda h)^{\frac{1+\lambda}{\lambda}}$, $\lambda \neq 0, -1$. From a preceding remark, as we can replace h by μh for any non-zero μ, the result is already proved for all (α, β) such that $\beta \neq -1$. If $\beta = -1$, we can assume that $\alpha = 1$. Then $f = 1 + h$, so X satisfies $X = B(1 + X)$. Hence, for all $n \geq 1$, $X_n = l_n$ and then:

$$\Delta(l_n) = \sum_{i=0}^{n} l_i \otimes l_{n-i},$$

with the convention $l_0 = 1$. So $X = B(1 + X)$ is Hopf. $\square$

Remarks.

1. If $a_1 \neq 0$, the pre-Lie structure constants λ_k are given by:

$$\lambda_k = \alpha(1 + (1 + \beta)(k - 1)) = \alpha(-\beta + k(1 + \beta)).$$

2. The coproduct of the $x(n)$'s is given by formula (3.2).
3. Apart from $\mathcal{H}_1 = K$, for Hopf equations we find that $\mathcal{H}_f$ is isomorphic to $\mathcal{H}_{FdB}$ whenever $\beta \neq -1$, and otherwise the cocommutative ladders Hopf algebra.

5 Systems of Dyson-Schwinger equations

5.1 Definition

Definition 25. *Let I be a finite, non-empty set, and let $f_i \in K[[h_j, j \in I]]$ be a non-constant formal series for all $i \in I$. The* system of combinatorial Dyson-Schwinger equations *(briefly, the SDSE) associated to $(f_i)_{i\in I}$ is:*

$$\forall i \in I,\ x_i = B_i(f_i(x_j, j \in I)),$$

where $x_i \in \overline{\mathcal{H}_{CK}^I}$ for all $i \in I$.

In order to ease the notation, we shall often assume that $I = \{1, \ldots, N\}$, especially in the proofs, without loss of generality.

Notations.

(1) Let (S) be an SDSE. We shall denote, for all $i \in I$:

$$f_i = \sum_{p_1,\cdots,p_N} a^{(i)}_{(p_1,\cdots,p_N)} h_1^{p_1} \cdots h_N^{p_N}.$$

(2) Let $1 \leq j \leq N$. We put $\varepsilon_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ where the 1 is in position j. We shall denote, for all $i \in I$, $a^{(i)}_j = a^{(i)}_{\varepsilon_j}$; for all $j, k \in I$, $a^{(i)}_{j,k} = a^{(i)}_{\varepsilon_j+\varepsilon_k}$, and so on.

Proposition 26. *Let (S) be an SDSE. Then it admits a unique solution $(x_i)_{i\in I} \in \left(\overline{\mathcal{H}_{CK}^I}\right)^I$.*

Proof. If $(x_1, \cdots, x_N)$ is a solution of S, then x_i is a linear (infinite) span of rooted trees with a root decorated by i. We denote:

$$x_i = \sum a_t t,$$

where the sum is over all trees which root is decorated by i. These coefficients are uniquely determined by the following formulas: if $t \in \mathcal{T}^I$, we put $t = B_i(F)$ and $F = t_1^{p_1} \ldots t_k^{p_k}$, where the t_j's are different trees. Let r_j be the number of roots of F decorated by j for all $j \in I$. Then:

$$a_t = \frac{r_1! \ldots r_N!}{p_1! \ldots p_k!} a^{(i)}_{(r_1,\ldots,r_N)} a_{t_1}^{p_1} \ldots a_{t_k}^{p_k}.$$

So (S) has a unique solution. ☐

Definition 27. *Let (S) be an SDSE and let $X = (x_i)_{i\in I}$ be its unique solution. The subalgebra of $\mathcal{H}_{CK}^I$ generated by the homogeneous components $x_i(k)$'s of the x_i's will be denoted by $\mathcal{H}_{(S)}$. If $\mathcal{H}_{(S)}$ is Hopf, we shall say that the system (S) is* Hopf.

Remarks.

(1) This definition makes sense for systems of equations with a single coupling constant only. If one allows for different coupling constants, the perturbation series are in more than one variable. Algebraically, this corresponds to a refined grading of $\mathcal{H}_{(S)}$, given by counting not just the total number of vertices, but the number of vertices of each decoration type $i \in I$ separately. Taking the homogeneous components then produces larger subalgebras which might be Hopf in more general cases.

(2) We assume that there is no constant f_i. Indeed, if $f_i \in K$, then x_i is a multiple of $\bullet_i$. We shall always avoid this case in all this text. All the same, let us give examples of systems with constant formal series:

Proposition 28. *Let us consider the following system:*

$$(S): \left\{ \begin{array}{rcl} x_1 & = & B_1(1), \\ x_2 & = & B_2\left(\displaystyle\sum_{k=0}^{\infty} a_k x_1^k\right), \end{array} \right.$$

with $a_1 = 1$. It is Hopf, if and only if, the following assertion is satisfied: for all $n \geq 1$,

$$(a_n = 0) \Longrightarrow (a_{n+1} = 0).$$

Moreover, the Hopf algebra $\mathcal{H}_{(S)}$ depends only on $N = \min\{n \mid a_n = 0\} \in \mathbb{N}^* \cup \{\infty\}$, and in particular does not depend of the values of the non-zero a_n's.

Proof. For all $n \geq 1$, we put $d_n = B_2(\bullet_1{}^{n-1})$:

$$d_1 = \bullet_2,\ d_2 = B_2(\bullet_1),\ d_3 = B_2(\bullet_1\bullet_1),\ d_4 = B_2(\bullet_1\bullet_1\bullet_1)\ \ldots$$

Then $x_1(1) = \bullet_1$, $x_1(n) = 0$ if $n \geq 2$, and $x_2(n) = a_{n-1}d_n$ for all $n \geq 1$. So $\mathcal{H}_{(S)}$ is the subalgebra generated by $\bullet_1$ and the d_n's such that $a_{n-1} \neq 0$.
Moreover, for all $n \geq 1$:

$$\Delta(d_n) = d_n \otimes 1 + \sum_{k=0}^{n-1} \binom{n-1}{k} \bullet_1{}^k \otimes d_{n-k}.$$

$\Longrightarrow$. Let us assume that $a_{n+1} \neq 0$. Then $d_{n+2} \in \mathcal{H}_{(S)}$, so $\Delta(d_{n+2}) \in \mathcal{H}_{(S)} \otimes \mathcal{H}_{(S)}$. Taking the terms of $\Delta(d_{n+2})$ in $\mathcal{H}_{(S)}(1) \otimes \mathcal{H}_{(S)}(n+1)$, we obtain that $\bullet_1 \otimes d_{n+1} \in \mathcal{H}_{(S)} \otimes \mathcal{H}_{(S)}$, so $d_{n+1} \in \mathcal{H}_{(S)}$. As a consequence, $a_n \neq 0$.

$\Longleftarrow$. Let us put $N = \min\{n \mid a_n = 0\} \in \mathbb{N}^* \cup \{\infty\}$. Then $\mathcal{H}_{(S)}$ is generated by $\bullet_1$ and the d_n's such that $n - 1 < N$. Clearly, $\mathcal{H}_{(S)}$ is a Hopf subalgebra of $\mathcal{H}_{CK}$. $\square$

5.2 General results

We here generalize the results dealing with single Dyson-Schwinger equations, without detailed proofs.

Proposition 29. *Let (S) be an SDSE. If it is Hopf, then, for all $i,j \in I$, for all $n \geq 1$, there exists a scalar $\lambda_n^{(i,j)}$ such that for all $t' \in \mathcal{T}$, which root is decorated by i:*

$$\sum_{t \in \mathcal{T}} n_j(t,t') a_t = \lambda_{|t'|}^{(i,j)} a_{t'},$$

where $n_j(t,t')$ is the number of leaves l of t decorated by j such that the cut of l gives t'.

Remark. Let (S) be a Hopf SDSE. We assume that $f_i(0) = a_{\bullet_i} = 0$. For any forest F, let $\delta_F : \mathcal{H}_{CK}^I \longrightarrow K$, defined by $\delta_F(G) = \delta_{F,G}$ for any forest G. Then, putting $t = B_i(F)$:

$$a_{t\bullet_i} = (\delta_F \otimes Id) \circ \Delta(x_i(|t|)) \in \mathcal{H}_{(S)}.$$

As $\bullet_i \notin \mathcal{H}_{(S)}$, $a_t = 0$. Hence, $x_i = 0$, so we do not change $\mathcal{H}_{(S)}$ by dropping the index i from I alltogether. From now on, we will assume that $f_i(0) \neq 0$ for all $i \in I$. Applying a change of variables, without loss of generality we restrict to $f_i(0) = 1$ for all $i \in I$.

We generalize lemma 23:

Lemma 30. *Let us assume that (S) is Hopf. Let us fix $i \in I$.*

(1) *For all sequences $i = i_1, \ldots, i_n$ of elements of I such that $a_{i_{p+1}}^{(i_p)} \neq 0$ for all $1 \leq p \leq n-1$:*

$$\lambda_n^{(i,j)} = a_j^{(i_n)} + \sum_{p=1}^{n-1} (1 + \delta_{j,i_{p+1}}) \frac{a_{j,i_{p+1}}^{(i_p)}}{a_{i_{p+1}}^{(i_p)}}.$$

In particular, $\lambda_1^{(i,j)} = a_j^{(i)}$.

(2) *For all $p_1, \cdots, p_N \in \mathbb{N}$:*

$$a_{(p_1,\cdots,p_j+1,\cdots,p_N)}^{(i)} = \frac{1}{p_j+1} \left(\lambda_{p_1+\cdots+p_N+1}^{(i,j)} - \sum_{l \in I} p_l a_j^{(l)} \right) a_{(p_1,\cdots,p_N)}^{(i)}.$$

Proof. The first point is proved using the definition of the coefficients $\lambda_n^{(i,j)}$, with $t' = \begin{array}{l} \bullet_{i_n} \\ \bullet_{i_{n-1}} \\ \vdots \\ \bullet_{i_2} \\ \bullet_{i_1} \end{array}$. The second point uses $t' = B_i(\bullet_1{}^{p_1} \cdots \bullet_N{}^{p_N})$. □

Remarks.

(1) From the second point of lemma 30, if $a_{\underline{m}}^{(i)} = 0$ for a particular $\underline{m} \in \mathbb{N}^I$, then for any $\underline{n} \in \mathbb{N}^I$, $a_{\underline{m}+\underline{n}}^{(i)} = 0$.

(2) If $a_j^{(i)} = 0$, then f_i does not depend on h_j.

(3) We assume that there are no constant f_i, so for all $i \in I$ there exists $j \in I$, such that $a_j^{(i)} \neq 0$. As a consequence, the sequences of elements considered in the first point of lemma 30 exist for any $i \in I$ and any $n \geq 1$.

(4) By lemma 30-1, the coefficients $\lambda_n^{(i,j)}$ are determined by the coefficients of degree 1 and 2 of the f_i's. Moreover, they completely determined the f_i's, according to lemma 30-2.

Lemma 31. *Let (S) be a Hopf SDSE and let $i, i' \in I$ such that $a_{i'}^{(i)} \neq 0$. For all $j \in I$, for all $n \geq 2$:*

$$\lambda_n^{(i,j)} = (1 + \delta_{j,i'}) \frac{a_{j,i'}^{(i)}}{a_{i'}^{(i)}} + \lambda_{n-1}^{(i',j)}.$$

Proof. It is enough to apply proposition 30-1 with $i_2 = i'$. $\square$

Proposition 32. *Let (S) be a Hopf SDSE. Let $i \in I$ such that:*

$$f_i = 1 + \sum_{j \in I} a_j^{(i)} h_j.$$

Then if $a_{i'}^{(i)} \neq 0$, for all j, for all $n \geq 1$, $\lambda_{n+1}^{(i,j)} = \lambda_n^{(i',j)}$. As a consequence, if $a_{i'}^{(i)}, a_{i''}^{(i)} \neq 0$, $f_{i'} = f_{i''}$.

Proof. By hypothesis on f_i, $a_{j,i'}^{(i)} = 0$ for all j, i'. The result comes then immediately from lemma 31. So, if i' and i'' are two direct descendants of i, for all $k \in I$, for all $n \geq 1$, $\lambda_n^{(i',k)} = \lambda_n^{(i'',k)}$. So, $f_{i'} = f_{i''}$. $\square$

5.3 Operations on Hopf SDSE

Proposition 33 (change of variables). *Let (S) be the SDSE associated to $(f_i(h_j,\ j \in I))_{i \in I}$. Let λ_i and μ_i be non-zero scalars for all $i \in I$. The system (S) is Hopf if, and only if, the SDSE system (S') associated to $(\mu_i f_i(\lambda_j h_j,\ j \in J))_{i \in I}$ is Hopf.*

Proof. We assume that $I = \{1, \ldots, N\}$. We consider the following morphism:

$$\phi : \left\{ \begin{array}{rcl} \mathcal{H}^I & \longrightarrow & \mathcal{H}^I \\ F \in \mathcal{F} & \longrightarrow & (\mu_1 \lambda_1)^{n_1(F)} \cdots (\mu_N \lambda_N)^{n_N(F)} F, \end{array} \right.$$

where $n_i(F)$ is the number of vertices of F decorated by i. Then ϕ is a Hopf algebra automorphism and for all i, $\phi \circ B_i = \mu_i \lambda_i B_i \circ \phi$. Moreover, if we put $Y_i = \frac{1}{\lambda_i} \phi(x_i)$ for all i:

$$\begin{array}{rcl} Y_i & = & \dfrac{1}{\lambda_i} \phi \circ B_i(f_i(x_1, \cdots, x_N)) \\ & = & \dfrac{1}{\lambda_i} \mu_i \lambda_i B_i^+(f_i(\phi(x_1), \cdots, \phi(x_N))) \\ & = & \mu_i B_i^+(f_i(\lambda_1 Y_1, \cdots, \lambda_N Y_N)). \end{array}$$

So $(Y_1, \cdots, Y_N)$ is the solution of the system (S'). Moreover, ϕ sends $\mathcal{H}_{(S)}$ onto $\mathcal{H}_{(S')}$. As ϕ is a Hopf algebra automorphism, $\mathcal{H}_{(S)}$ is a Hopf subalgebra of $\mathcal{H}^I$ if, and only if, $\mathcal{H}_{(S')}$ is. $\square$

Proposition 34 (restriction). *Let (S) be the SDSE associated to the family $(f_i(h_j,\ j \in I))_{i\in I}$ and let $I' \subseteq I$, non-empty. Let (S') be the SDSE associated to the family $\left(f_i(h_j, j \in I)_{|h_j=0,\, \forall j\notin I'}\right)_{i\in I'}$. If (S) is Hopf, then (S') also is.*

Proof. Let $\phi : \mathcal{H}^I \longrightarrow \mathcal{H}^{I'}$ be the unique Hopf algebra morphism such that:

$$\phi \circ B_i = \begin{cases} B_i \circ \phi \text{ if } i \in I', \\ 0 \text{ if } i \notin I'. \end{cases}$$

For any forest F, $\phi(F) = 0$ if at least one vertex of F is decorated by an element which is not in I', and F otherwise. Then ϕ sends $\mathcal{H}_{(S)}$ to $\mathcal{H}_{(S')}$. As ϕ is a morphism of Hopf algebras, if $\mathcal{H}_{(S)}$ is a Hopf subalgebra of $\mathcal{H}^I$, $\mathcal{H}_{(S')}$ is a Hopf subalgebra of $\mathcal{H}^{I'}$. $\square$

Proposition 35 (dilatation). *Let (S) be the system associated to $(f_i)_{i\in I}$ and (S') be a system associated to a family $(g_j)_{j\in J}$, such that there exists a partition $J = \bigcup_{i\in I} J_i$, with the following property: for all $i \in I$, for all $p \in J_i$,*

$$g_p = f_i\left(\sum_{q\in J_j} h_q,\ j \in I\right).$$

Then (S) is Hopf, if, and only if, (S') is Hopf. We shall say that (S') is a dilatation *of (S).*

Proof. $\Longleftarrow$. Let us assume that (S) is Hopf. For all $i \in I$, we can then write:

$$\Delta(x_i) = \sum_{n\geq 0} P_n^{(i)}(x_1, \cdots, x_N) \otimes x_i(n),$$

where the $P_n^{(i)}$ are elements of $\overline{\mathcal{H}_{(S)}} = K[[x_1, \ldots, x_N]]$, with the convention $x_i(0) = 1$. Let $\phi : \mathcal{H}^I \longrightarrow \mathcal{H}^J$ be the Hopf algebra morphism such that, for all $1 \leq i \leq N$:

$$\phi \circ B_i = \sum_{j\in J_i} B_j \circ \phi.$$

Then, immediately, for all $1 \leq i \leq N$:

$$\phi(x_i) = \sum_{j\in J_i} x'_j.$$

As a consequence:

$$\sum_{j\in J_i} \Delta(x'_j) = \sum_{j\in J_i} \sum_{n\geq 0} P_n^{(i)}\left(\sum_{k\in J_1} x'_k, \cdots, \sum_{k\in J_N} x'_k\right) \otimes x'_j(n).$$

Conserving the terms of the form $F \otimes t$, where t is a tree with root decorated by j, for all $j \in J_i$:

$$\Delta(x'_j) = \sum_{n \geq 0} P_n^{(i)} \left(\sum_{k \in J_1} x'_k, \cdots, \sum_{k \in J_N} x'_k \right) \otimes x'_j(n).$$

So (S') is Hopf.

$\Longrightarrow$. Let us assume that (S') is Hopf. We choose one representative q_i in each J_i. Taking the restriction of (S') to these elements, we obtain that (S) is Hopf. $\square$

Example. Let $f, g \in K[[h_1, h_2]]$. Let us consider the following SDSE:

$$(S): \left\{ \begin{array}{rcl} x_1 &=& B_1(f(x_1, x_2)), \\ x_2 &=& B_2(g(x_1, x_2)), \end{array} \right.$$

The following SDSE is a dilatation of (S):

$$(S'): \left\{ \begin{array}{rcl} x_1 &=& B_1(f(x_1 + x_2 + x_3, x_4 + x_5)), \\ x_2 &=& B_2(f(x_1 + x_2 + x_3, x_4 + x_5)), \\ x_3 &=& B_3(f(x_1 + x_2 + x_3, x_4 + x_5)), \\ x_4 &=& B_4(g(x_1 + x_2 + x_3, x_4 + x_5)), \\ x_5 &=& B_5(g(x_1 + x_2 + x_3, x_4 + x_5)). \end{array} \right.$$

Remark. If i, i' are in the same J_q, then, by lemma 31, since $g_i = g_{i'}$, for all $n \geq 1$, for all $j \in J$, $\lambda_n^{(i,j)} = \lambda_n^{(i',j)}$. Conversely, if there exists a partition of the set of indices J such that this condition holds, lemma 30 (2) suffices to prove that (S) is a dilatation of another SDSE.

Proposition 36 (extension). *Let (S) be the SDSE associated to $(f_i)_{i \in I}$. Let $0 \notin I$ and let (S') be associated to $(f_i)_{i \in I \cup \{0\}}$, with:*

$$f_0 = 1 + \sum_{i \in I} a_i^{(0)} h_i.$$

We assume that for all $i, j \in I^{(0)} = \left\{ j \in I \,/\, a_j^{(0)} \neq 0 \right\}$, $f_i = f_j$. If (S) is Hopf, then (S') also is. We shall say that (S') is an extension *of (S).*

Proof. As (S) is Hopf, we can put for all $1 \leq i \leq N$:

$$\Delta(x_i) = x_i \otimes 1 + \sum_{k=1}^{+\infty} P_k^{(i)} \otimes x_i(k),$$

where $P_k^{(i)}$ is an element of the completion of $\mathcal{H}_{(S)}$. By the second hypothesis, if $i, j \in I^{(0)}$, $f_i = f_j$, so $P_k^{(i)} = P_k^{(j)}$. We then denote by P_k the common value of

$P_k^{(i)}$ for all $i \in I^{(0)}$. So:

$$\begin{array}{rcl}\Delta(x_0) & = & \bullet_0 \otimes 1 + 1 \otimes \bullet_0 + \displaystyle\sum_{i=1}^{N} a_i^{(0)} \Delta \circ B_0(x_i) \\ & = & x_0 \otimes 1 + \left(1 + \displaystyle\sum_{i=1}^{N} a_i^{(0)} x_i\right) \otimes \bullet_0 + \displaystyle\sum_{i=1}^{N} \sum_{k=1}^{\infty} a_i^{(0)} P_k^{(i)} \otimes B_0(x_i(k)) \\ & = & x_0 \otimes 1 + \left(1 + \displaystyle\sum_{i=1}^{N} a_i^{(0)} x_i\right) \otimes \bullet_0 + \displaystyle\sum_{k=1}^{N} P_j \otimes x_0(k+1).\end{array}$$

This belongs to the completion of $\mathcal{H}_{(S')} \otimes \mathcal{H}_{(S')}$, so (S') is Hopf. $\Box$

Remark. From proposition 32, the condition of equalities of the f_i's for $i \in I^{(0)}$ is necessary.

Example. This construction can be iterated. For example, we consider the following system:

$$(S) : \left\{ \begin{array}{rcl} x_1 & = & B_1(f(x_1, x_2)), \\ x_2 & = & B_2(g(x_1, x_2)). \end{array} \right.$$

Here is an iterated extension of (S):

$$(S') : \left\{ \begin{array}{rcl} x_1 & = & B_1(f(x_1, x_2)), \\ x_2 & = & B_2(g(x_1, x_2)), \\ x_3 & = & B_3(1 + x_1), \\ x_4 & = & B_4(1 + x_1), \\ x_5 & = & B_5(1 - x_2), \\ x_6 & = & B_6(1 + 2x_3 - 4x_4). \end{array} \right.$$

Proposition 37 (concatenation). *Let (S) be the SDSE associated to $(f_i)_{i \in I}$ and let (S') be the SDSE associated to $(g_j)_{j \in J}$, where I and J are two disjoint sets. Then the system (S'') associated to $(f_i)_{i \in I} \cup (g_j)_{j \in J}$ is Hopf if, and only if, (S) and (S') are Hopf. We shall say that (S'') is the* concatenation *of (S) and (S').*

Proof. In this case, $\mathcal{H}_{(S'')} = \mathcal{H}_{(S)} \otimes \mathcal{H}_{(S')} \subseteq \mathcal{H}_{CK}^I \otimes \mathcal{H}_{CK}^J \subseteq \mathcal{H}^{I \sqcup J}$. So if $\mathcal{H}_{(S)}$ and $\mathcal{H}_{(S')}$ are Hopf subalgebras, $\mathcal{H}_{(S'')}$ also is. By restriction, the converse is also true. $\Box$

Example. Let us consider the two following systems:

$$(S) : \left\{ \begin{array}{rcl} x_1 & = & B_1(f_1(x_1, x_2)), \\ x_2 & = & B_2(f_2(x_1, x_2)); \end{array} \right. \qquad (S') : \left\{ \begin{array}{rcl} x_1 & = & B_1(g_1(x_1, x_2, x_3)), \\ x_2 & = & B_2(g_2(x_1, x_2, x_3)), \\ x_3 & = & B_3(g_3(x_1, x_2, x_3)). \end{array} \right.$$

The concatenation of (S) and (S') is:

$$(S''):\left\{\begin{array}{rcl}x_1&=&B_1(f_1(x_1,x_2)),\\x_2&=&B_2(f_2(x_1,x_2)),\\x_3&=&B_3(g_1(x_3,x_4,x_5)),\\x_4&=&B_4(g_2(x_3,x_4,x_5)),\\x_5&=&B_5(g_3(x_3,x_4,x_5)).\end{array}\right.$$

5.4 The graph associated to a Dyson-Schwinger system

Definition 38. *Let (S) be an SDSE.*

(1) *We construct an oriented graph $G_{(S)}$ associated to (S) in the following way:*
   - *The vertices of $G_{(S)}$ are the elements of I.*
   - *There is an edge from i to j if, and only if, $a_j^{(i)} \neq 0$.*

(2) *If $a_i^{(i)} \neq 0$, the vertex i will be said to be* self-dependent*. In other words, if i is self-dependent, there is a loop from i to itself in $G_{(S)}$.*

(3) *If $G_{(S)}$ is connected, we shall say that (S) is* connected*.*

Remarks.

(1) As constant f_i are excluded, each vertex of $G_{(S)}$ hase at least one outgoing edge.

(2) Let us consider the action of the different operations defined earlier on the associated graphs.
   - If (S') is obtained from (S) by a change of variables, then $G_{(S')} = G_{(S)}$.
   - If (S') is obtained from (S) by a dilatation, the set of vertices J of the graph $G_{(S')}$ admits a partition indexed by the vertices of $G_{(S)}$, and there is an edge from $x \in J_i$ to $y \in J_j$ in $G_{(S')}$ if, and only if, there is an edge from i to j in $G_{(S)}$.
   - If (S') is obtained from (S) by an extension, then $G_{(S')}$ is obtained from $G_{(S)}$ by adding a new vertex with no ancestor. The added vertex is called an *extension vertex.*
   - If (S'') is the concatenation of (S) and (S'), then $G_{(S'')}$ is the disjoint union of $G_{(S)}$ and $G_{(S')}$.
   - Conversely, if $G_{(S)}$ is the disjoint union of two subgraphs G' and G'', then (S) is the concatenation of the two subsystems (S') and (S''), formed by the equations indexed by the elements of G' and G'' respectively. As a consequence, taking the connected components of $G_{(S)}$, (S) is the concatenation of a finite number of connected Hopf SDSE.

Notations. Let $i, j \in I$.

(1) We shall write $i \longrightarrow j$ if there is an edge from i to j in $G_{(S)}$, that is to say if $a_j^{(i)} \neq 0$. In this case, we shall say that i is an *direct ancestor* of j or that j is a *direct descendant* of i.

(2) If there is an oriented path from i to j in $G_{(S)}$, we shall say that i is an *ancestor* of j or that j is a *descendant* of i.

5.5 Structure of the graph of a Hopf SDSE

Let us first give two lemmas on the graph of a Hopf SDSE:

Lemma 39. *Let (S) be a Hopf SDSE and let $i \in I$. Let j, k and $l \in I$ such that $a_j^{(i)} \neq 0$, $a_k^{(j)} \neq 0$ and $a_l^{(i)} \neq 0$. Then $a_k^{(i)} \neq 0$ or $a_k^{(l)} \neq 0$.*

Proof. Let us assume that $a_k^{(i)} = 0$. As $a_j^{(i)} \neq 0$, $j \neq k$. As $a_k^{(i)} = 0$, $a_{{}^{j}\vee^{k}_{i}} = a_{j,k}^{(i)} = 0$. Then $\lambda_2^{(i,k)} a_j^{(i)} = \lambda_2^{(i,k)} a_{\substack{j\\i}} = a_{\substack{k\\j\\i}} + a_{{}^{j}\vee^{k}_{i}} = a_j^{(i)} a_k^{(j)} + 0$; hence, $\lambda_2^{(i,k)} = a_k^{(j)}$. Moreover, As $a_l^{(i)} \neq 0$, $l \neq k$. Then $a_l^{(i)} \lambda_2^{(i,k)} = \lambda_2^{(i,k)} a_{\substack{l\\i}} = a_{\substack{k\\l\\i}} + a_{{}^{l}\vee^{k}_{i}} = a_l^{(i)} a_k^{(l)} + 0$, so $\lambda_2^{(i,k)} = a_k^{(l)}$. Hence, $a_k^{(l)} = a_k^{(j)} \neq 0$. $\square$

Remarks.

(1) In other words, if (S) is Hopf, then, in $G_{(S)}$:

$$\begin{tikzcd} i \arrow[r] \arrow[d] & j \arrow[d] \\ l & k \end{tikzcd} \Longrightarrow \begin{tikzcd} i \arrow[r] \arrow[d] & j \arrow[d] \\ l \arrow[r] & k \end{tikzcd} \text{ or } \begin{tikzcd} i \arrow[r] \arrow[d] \arrow[dr] & j \arrow[d] \\ l & k \end{tikzcd}.$$

(2) A first special case is given by $i = k$:

$$\begin{tikzcd} i \arrow[r, leftrightarrow] \arrow[d] & j \\ l & \end{tikzcd} \Longrightarrow \begin{tikzcd} i \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & j \\ l & \end{tikzcd}.$$

(3) A second special case is given by $i = l$, that is to say when i is self-dependent:

$$\begin{tikzcd} i \arrow[loop left] \arrow[r] & j \arrow[d] \\ & k \end{tikzcd} \Longrightarrow \begin{tikzcd} i \arrow[loop left] \arrow[r] \arrow[dr] & j \arrow[d] \\ & k \end{tikzcd}.$$

Hence, any descendant of of a self-dependent vertex is a direct descendant.

Lemma 40. *Let (S) be a Hopf SDSE and let i be a vertex of $G_{(S)}$. We suppose that there exists a vertex j, such that:*

- *j is a descendant of i.*
- *All oriented path from i to j are of length ≥ 3.*

Then $f_i = 1 + \sum\limits_{i \longrightarrow l} a_l^{(i)} h_l$.

Proof. Let L be the minimal length of the oriented paths from i to j. By hypothesis, $L \geq 3$. Then the homogeneous component of degree $L+1$ of x_i contains trees with a leaf decorated by j, and all these trees are decorated ladders. By definition of the coefficients $\lambda_n^{(i,j)}$, if t' is a tree with L vertices and its root decorated by i:

$$\lambda_L^{(i,j)} a_{t'} = \sum_{t \in \mathcal{T}^I} n_j(t, t') a_t.$$

For a good-chosen ladder t', the right-hand side is non-zero, so $\lambda_L^{(i,j)}$ is non-zero. If t' is not a ladder, the right-hand side is 0, so $a_{t'} = 0$. As a conclusion, $x_i(L)$ is a linear span of ladders. Considering its coproduct, for all $p \leq L$, $x_i(p)$ is a linear span of ladders. In particular, $x_i(3)$ is a linear span of ladders. But:

$$x_i(3) = \sum_{l,m} a_l^{(i)} a_m^{(l)} \begin{smallmatrix} \bullet m \\ \bullet l \\ \bullet i \end{smallmatrix} + \sum_{l \leq m} a_{l,m}^{(i)} \, {}^{l}\mathrm{V}^{m}_{i},$$

so $a_{l,m}^{(i)} = 0$ for all l, m. Hence, f_i contains only terms of degree ≤ 1. $\square$

Remark. This lemma can be applied with $i = j$, if i is not a self-dependent vertex.

Let us now study the structure of the graph of a SDSE:

Proposition 41. *Let G be a finite oriented graph, such that any vertex of G has at least one direct descendant. The set of vertices of G is denoted by I. There exists a sequence $G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n = I$ of subgraphs of G such that:*

1. *For any element $i \in G_0$, the descendants of i are all in G_0.*
2. *For any element $i \in G_0$, i has an ancestor in G_0.*
3. *For all $1 \leq k \leq n$, G_k is obtained from G_{k-1} by adding an element i_k, with no ancestor in G_{k-1} and with all its descendants in G_{k-1}.*

Moreover, G_0 contains an oriented cycle. More precisely, any vertex $i \in G_0$ is the descendant of a vertex included in an oriented cycle.

Proof. Let us prove the existence of $G_0, \ldots, G_n$ by induction on the number N of elements of I. If $N = 1$, we take $G_0 = I$. If $N > 1$, and if I has no vertex with no ancestor, we take $G_0 = I$. If I has a vertex i with no ancestor, let us consider the restriction of (S) to $I - \{i\}$. This gives a sequence $G_0 \subseteq \ldots \subseteq G_n = I - \{i\}$. We complete it by putting $G_{n+1} = I$.

Let $i \in G_0$. As any vertex has a direct ancestor, it is possible to define inductively a sequence $(x_l)_{l \geq 0}$ of vertices of G, such that $x_0 = i$ and x_{l+1} is a direct ancestor of x_l for all l. As G is finite, there exists $0 \leq l < m$, such that $x_l = x_m$.

Then $x_l \leftarrow x_{l+1} \leftarrow \cdots \leftarrow x_{m-1} \leftarrow x_m = x_l$ is a closed path of G, included in G_0. If we take such a path of minimal length, it is necessarily an oriented cycle. $\square$

Remark. Although the sequence $(G_i)_{0\leq i\leq n}$ is not unique, it is possible to prove that G_0 is unique; this fact will not be used in the sequel.

We shall classify the Hopf SDSE according to the minimal length L of an oriented cycle included in G_0. If $L = 1$, then the considered SDSE has a self-dependent vertex. We begin with the cases where $L \geq 2$.

6 Quasi-cyclic SDSE

6.1 Structure of the cycles

Proposition 42. *Let (S) be a Hopf SDSE such that $G_{(S)}$ is an oriented cycle of length $N \geq 2$, that is:*

$$G_{(S)} = \begin{tikzcd} 1 \arrow[r] & \ldots \arrow[r] & N \arrow[ll, bend right] \end{tikzcd}.$$

Two cases are possible.

(1) *Up to a change of variables, for all $i \in I$, $f_i = 1 + \sum\limits_{i\rightarrow j} h_j$.*

(2) *$N = 2$ and up to a change of variables, for all $i \in I$, $f_i = \left(1 - \sum\limits_{i\rightarrow j} h_j\right)^{-1}$.*

In this case, $\lambda_n^{(i,j)} = n - \delta_{i,j}$ for all $i, j \in \{1, 2\}$.

Proof. Up to a change of variables, we can assume that $a_{i+1}^{(i)} = 1$ for all $1 \leq i \leq N-1$ and $a_1^{(N)} = 1$. If $N \geq 3$, we can apply lemma 40 and we immediately obtain the first case. Let us study the case $N = 2$. In other words, $G_{(S)} = 1 \longleftrightarrow 2$. We put:

$$f_1(h_2) = \sum_{i=0}^{\infty} a_i h_2^i, \qquad f_2(h_1) = \sum_{i=0}^{\infty} b_i h_1^i,$$

with $a_1 = b_1 = 1$. Then $\lambda_3^{(1,1)} = \lambda_3^{(1,1)} a = 2a^1 = 2b_2$. On the other hand, $\lambda_3^{(1,1)} a_2 = a = 2a_2$, so $2a_2b_2 = 2a_2$: $a_2 = 0$ or $b_2 = 1$. Similarly, $b_2 = 0$ or $a_2 = 1$. So $a_2 = b_2 = 0$ or 1. In the first case, $f_1(h_2) = 1 + h_2$ and $f_2(h_1) = 1 + h_1$. In the second case, let us apply lemma 30-1 with $(i_1, \cdots, i_n) = (1, 2, 1, 2, \cdots)$. If $n = 2k$ is even, we obtain $\lambda_n^{(1,2)} = 2 + 2(k-1) = 2k = n$. If $n = 2k+1$ is odd, $\lambda_n^{(1,2)} = 1 + 2k = n$. So $\lambda_n^{(1,2)} = n$ for all $n \geq 1$. By lemma 30-2, for all $n \geq 1$, $a_{n+1} = a_n$. So for all $n \geq 0$, $a_n = 1$ and $f_1(h_2) = (1 - h_2)^{-1}$. Similarly, $f_2(h_1) = (1 - h_1)^{-1}$. $\square$

The second case is a special case of a *fundamental system*; we postpone its study to section 7. We now concentrate on the first case.

Definition 43. *Let $I = \mathbb{Z}/N\mathbb{Z}$, $N \geq 2$. We consider the SDSE associated to the following formal series:*

$$f_i = 1 + h_{\overline{i+1}}, \text{ for all } i \in I.$$

These SDSE and the ones obtained from them by a dilatation and a change of variables are called N-quasi-cyclic systems.

Example. Here is an example of quasi-cyclic SDSE:

$$\left\{ \begin{array}{rcl} x_1 &=& B_1(1 + x_2 + x_3),\\ x_2 &=& B_2(1 + x_4),\\ x_3 &=& B_3(1 + x_4),\\ x_4 &=& B_4(1 + x_5),\\ x_5 &=& B_5(1 + x_1). \end{array} \right.$$

Remark. If (S) is a N-quasi-cyclic SDSE without dilatation, then $x_{\overline{i}}$ is the sum of all the ladders cyclically decorated, with root decorated by $\overline{i}$. The subalgebra generated by these ladders is clearly Hopf. It is not difficult to prove that $\lambda_n^{(\overline{i},\overline{j})} = \delta_{\overline{i+n},\overline{j}}$.

6.2 Connected Hopf SDSE with a quasi-cycle

Notations.

(1) Let (S) and (S') be two Hopf SDSE. We shall say that (S) *contains* (S') if (S') is a restriction of (S) to a subset of its vertices.

(2) Let G and H be two oriented graphs. We shall say that G *contains* H if the vertices of H are vertices of G, and the edges of H are precisely the edges of G between the vertices of H.

Remark. If (S) contains (S'), then $G_{(S)}$ contains $G_{(S')}$.

Lemma 44. (1) *Let (S) be a Hopf SDSE containing a quasi-cyclic SDSE with set of vertices $I_{\overline{1}} \sqcup \cdots \sqcup I_{\overline{M}}$. Then any vertex of $G_{(S)}$ has direct descendants in at most one $I_{\overline{k}}$. Moreover, if a vertex has at least one direct descendant in a $I_{\overline{k}}$, it is non self-dependent.*

(2) *Let (S) be a Hopf SDSE such that I admits a partition $I = I_{\overline{1}} \sqcup \cdots \sqcup I_{\overline{M}}$ indexed by $\mathbb{Z}/M\mathbb{Z}$, with the following conditions:*

- *For all $1 \leq p \leq M$, the direct descendants of any $i \in I_{\overline{p}}$ are precisely the elements of $I_{\overline{p+1}}$.*
- *For all $i \in I$, $f_i = 1 + \sum\limits_{i \rightarrow j} a_j^{(i)} h_j$.*

Then (S) is quasi-cyclic.

Proof. 1. Let us assume that the vertex 0 of $G_{(S)}$ has direct descendants $x \in I_{\overline{k}}$ and $y \in I_{\overline{l}}$ with $\overline{k} \neq \overline{l}$. Then lemma 39 implies that any direct descendant of x is a direct descendant of 0, so 0 has also a direct descendant in $I_{\overline{k+1}}$. Similarly, 0 has a direct descendant in $I_{\overline{l+1}}$. Iterating this process, 0 has direct descendants in all the $I_{\overline{i}}$'s; it even holds that all the elements of all the $I_{\overline{i}}$'s are direct descendants of 0. Up to a restriction, the situation is the following:

$$\begin{tikzcd}
0 \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrrr] \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & M \arrow[llll, bend left]
\end{tikzcd}$$

with, for all $1 \leq i \leq M$, $f_i(h_{i+1}) = 1 + h_{i+1}$, with the convention $h_{M+1} = h_1$.

We first assume $M \geq 3$. In order to ease the notation, we do not write the index $^{(0)}$ in the sequel of the proof. By lemma 30-1, with $(i_1, i_2) = (0, 1)$ and $(0, 2)$:

$$\lambda_2^{(0,2)} = 1 + \frac{a_{1,2}}{a_1} = 2\frac{a_{2,2}}{a_2}.$$

By lemma 30-1, with $(i_1, i_2, i_3) = (0, 2, 3)$ and $(0, 1, 2)$:

$$\lambda_3^{(0,2)} = \frac{a_{1,2}}{a_1} = 2\frac{a_{2,2}}{a_1}.$$

Finally, $\dfrac{a_{1,2}}{a_1} = 1 + \dfrac{a_{1,2}}{a_1}$, which is absurd. So $M = 2$. By lemma 30-1 with $(i_1, i_2) = (0, 1)$ and $(0, 2)$:

$$\lambda_2^{(0,1)} = \frac{a_{1,2}}{a_2} + 1 = \frac{2a_{1,1}}{a_1}.$$

By lemma 30-1 with $(i_1, i_2, i_3) = (0, 1, 2)$ and $(0, 2, 1)$:

$$\lambda_3^{(0,1)} = \frac{2a_{1,1}}{a_1} + 1 = \frac{a_{1,2}}{a_1}.$$

We obtain:

$$\frac{2a_{1,1}}{a_1} = \frac{a_{1,2}}{a_1} + 1 = \frac{a_{1,2}}{a_1} - 1.$$

This is a contradiction.

Finally, if the vertex 0 has a direct descendant in $I_{\overline{k}}$, it comes that the elements of $I_{\overline{k+1}}$ are descendants of 0 which are not direct. If 0 is self-dependent, this contradicts lemma 39 with $i = l = 0$. So 0 is not self-dependent.

2. Let us choose an element $i_{\overline{p}}$ in each $I_{\overline{p}}$. Up to a change of variables, we can assume that for all $1 \leq p \leq M$:

$$f_{i_{\overline{p}}} = 1 + \sum_{j \in I_{\overline{p+1}}} h_j.$$

Let us choose $j \in I_{\overline{p}}$ and $k \in I_{\overline{p+1}}$; with $(i_1, \ldots, i_{M+1}) = (j, i_{\overline{p+1}}, \ldots, i_{\overline{p+M}})$, we obtain $\lambda_{N+1}^{(j,k)} = a_k^{(i_{\overline{p}})} = 1$. By lemma 30-1 with $(i_1, \ldots, i_{M+1}) = (j, i_{\overline{p+1}}, \ldots, i_{\overline{p+M-1}}, j)$, we obtain $\lambda_{N+1}^{(j,k)} = a_k^{(j)}$, so $a_k^{(j)} = 1$. Hence:

$$f_j = 1 + \sum_{j \rightarrow k} h_k.$$

So (S) is the dilatation of the system associated to the formal series $1 + h_{\overline{j}}$, for $\overline{j} \in \mathbb{Z}/M\mathbb{Z}$. So it is quasi-cyclic. □

Let us state more precisely the structure of connected Hopf SDSE containing a quasi-cycle.

Theorem 45. *Let (S) be a connected Hopf SDSE containing a N-quasi-cyclic SDSE. Then I admits a partition $I = I_{\overline{1}} \sqcup \cdots \sqcup I_{\overline{N}}$, with the following conditions:*

1. *If $i \in I_{\overline{p}}$, its direct descendants are all in $I_{\overline{p+1}}$.*
2. *If i and j have a common direct ancestor, then they have the same direct descendants.*
3. *For all $i \in I$, $f_i = 1 + \sum_{i \longrightarrow j} a_j^{(i)} h_j$.*
4. *If i and j have a common direct ancestor, then $f_i = f_j$.*

Such an SDSE will be called an extended quasi-cyclic SDSE.

Proof. Let S_0 a maximal quasi-cyclic subsystem of SDSE. We denote by $I^{(0)}$ the set of its vertices. By definition 43, it admits a partition $I^{(0)} = I_{\overline{1}}^{(0)} \sqcup \ldots \sqcup I_{\overline{N}}^{(0)}$, and for all $i \in I_{\overline{k}}$:

$$(f_i)_{|h_j = 0 \text{ if } j \notin I^{(0)}} = 1 + \sum_{j \in I^{(0)}_{\overline{k+1}}} a_j^{(i)} h_j.$$

Moreover, if $i \in I_{\overline{k}}^{(0)}$ and $j \in I_{\overline{k+1}}^{(0)}$, $a_j^{(i)} \neq 0$.

Let j be a direct descendant of an element $i \in I^{(0)}$. Let us assume that $j \notin I^{(0)}$. Up to a reindexation, we can suppose that $i \in I_{\overline{1}}^{(0)}$. Applying lemma 39, for all $k \in I_{\overline{3}}^{(0)}$, k is a direct descendant of j. By lemma 44-1, the direct descendants of j which are in $I^{(0)}$ are the elements of $I_{\overline{3}}^{(0)}$, and j is not self-dependent. Similarly, for all $k \in I_{\overline{1}}^{(0)}$, j is a direct descendant of k.

Let us choose a vertex $i_{\overline{k}} \in I_{\overline{k}}^{(0)}$ for all $\overline{k}$. We restrict ourselves to the oriented cycle formed by $i_{\overline{1}}, j, i_{\overline{3}}, \ldots, i_{\overline{N}}$. If $N \geq 3$, by proposition 42, $a_{i_{\overline{3}}, i_{\overline{3}}}^{(j)} = 0 = a_{j,j}^{(i_{\overline{1}})} = 0$.

If $N=2$, we obtain:

$$\lambda_3^{(i_{\overline{1}},i_{\overline{1}})} a_{\substack{i_{\overline{1}}\\ i_{\overline{2}}\\ i_{\overline{1}}}} = a_{{}^{i_{\overline{1}}}\bigvee_{i_{\overline{1}}}^{\substack{i_{\overline{1}}\\ i_{\overline{2}}}}} + a_{{}^{i_{\overline{1}}}\bigvee^{i_{\overline{1}}}_{\substack{i_{\overline{2}}\\ i_{\overline{1}}}}} + a_{\substack{i_{\overline{1}}\\ i_{\overline{1}}\\ i_{\overline{2}}\\ i_{\overline{1}}}} = 0,$$

so $\lambda_3^{(i_{\overline{1}},i_{\overline{1}})}=0$. Restricting to $i_{\overline{1}}$ and j, this implies that case (2) of proposition 42 cannot hold, so $a^{(j)}_{i_{\overline{1}},i_{\overline{1}}} = a^{(i_{\overline{1}})}_{j,j}=0$.

Hence, we have in both cases $a^{(j)}_{k,k}=0$ for all $k \in I^{(0)}_3$ and $a^{(i)}_{j,j}=0$ for all $i \in I^{(0)}_1$. Let us now take two elements k,k' of $I^{(0)}_{\overline{3}}$. Then $\lambda_2^{(j,k)} a_{\substack{k\\ j}} = a_{\substack{k\\ k\\ j}} + 2a_{{}^{k}\bigvee_{j}^{k}} = 0+0$, so $\lambda_2^{(j,k)}=0$. As a consequence, $0=\lambda_2^{(j,k)} a_{\substack{k'\\ j}} = a_{\substack{k'\\ k\\ j}} + a_{{}^{k}\bigvee_{j}^{k'}} = 0 + a^{(j)}_{k,k'}$. Hence:

$$(f_j)_{|h_k=0 \text{ if } k\notin I^{(0)}\cup\{j\}} = 1 + \sum_{k\in I^{(0)}_{\overline{3}}} a^{(j)}_k h_k.$$

Similarly, for all $i \in I^{(0)}_{\overline{1}}$:

$$(f_i)_{|h_k=0 \text{ if } k\notin I^{(0)}\cup\{j\}} = 1 + a^{(i)}_j h_j + \sum_{k\in I^{(0)}_{\overline{2}}} a^{(i)}_k h_k.$$

By lemma 44-2, $I^{(0)}\cup\{j\}$ forms a quasi-cyclic SDSE: this contradicts the maximality of $I^{(0)}$. So all the descendants of $I^{(0)}$ are in $I^{(0)}$.

As a consequence, we shall take $G_0 = I^{(0)}$ in proposition 41. We proceed by induction on n. If $n=0$, (S) is quasi-cyclic and the result is immediate. Let us assume the result at rank $n-1$ and let (S') be the restriction of (S) to all the vertices except the last one, denoted by i. By the induction hypothesis, the set of its vertices admits a partition $I' = I'_{\overline{1}} \cup \cdots \cup I'_{\overline{N}}$, with the required conditions. Let us first prove that all the direct descendants of i are in the same $I'_{\overline{p}}$. Let $j \in I'_{\overline{p}}$ and $k \in I'_{\overline{q}}$ be two direct descendants of i, with $\overline{p} \neq \overline{q}$. Let $j' \in I'_{\overline{p+1}}$ be a direct descendant of j and $k' \in I'_{\overline{q+1}}$ be a direct descendant of k. Lemma 39 implies that i is a direct ancestor of j' and k', as j can't be a direct ancestor of k' and k can't be a direct ancestor of j' because $\overline{p}\neq\overline{q}$. So we can replace j by j' and k by k'. Iterating the process, we can assume that i and j are in the quasi-cycle: this contradicts lemma 44. So the direct descendants of i are all in $I_{\overline{m}}$ for a good m. We then take $I_{\overline{l}} = I'_{\overline{l}}$ if $\overline{l} \neq \overline{m-1}$ and $I_{\overline{m-1}} = I'_{\overline{m-1}} \cup \{i\}$ and this proves the first assertion on $G_{(S)}$.

We now prove the assertion on f_i. We separate the proof into two subcases. Let us first assume $N\geq 3$. There is an oriented path $i \rightarrow i_{\overline{m}} \rightarrow \cdots \rightarrow i_{\overline{m+M-1}}$, with $i_{\overline{i}} \in I'_{\overline{i}}$ for all i. Moreover, there is no shorter oriented path from i to $i_{\overline{m+M-1}}$. As $N \geq 3$, from lemma 40:

$$f_i = 1 + \sum_{i\longrightarrow j} a^{(i)}_j h_j.$$

Let us secondly assume that $N=2$. Let $1,\ldots,p$ be the direct descendants of i

and let 0 be a direct descendant of 1. Then as $1, \ldots, p$ are in the same part of the partition of I', they are not direct descendants of 1. Let us first restrict to $\{i, 1, 0\}$. So $\lambda_3^{(i,0)} a_{\substack{0\\1\\i}} = 0$ as $a_{0,0}^{(1)} = 0$ by the induction hypothesis, $\lambda_3^{(i,0)} = 0$. Moreover, $0 = \lambda_3^{(i,0)} a_{{}^{1}\vee^{1}_{i}} = a_{{}^{1}\vee^{\substack{0\\1}}_{i}}$, so $a_{1,1}^{(i)} = 0$. Similarly, $a_{2,2}^{(i)} = \cdots = a_{p,p}^{(i)} = 0$.

Let us now take $1 \leq j < k \leq p$. Then $\lambda_2^{(i,j)} a_{\substack{j\\i}} = 2a_{{}^{j}\vee^{j}_{i}} = 0$, so $\lambda_2^{(i,j)} = 0$ and $0 = \lambda_2^{(i,j)} a_{\substack{k\\i}} = a_{{}^{j}\vee^{k}_{i}}$, so $a_{j,k}^{(i)} = 0$. As a conclusion, f_i is of the required form.

Proposition 32-3 implies that $f_i = f_{i'}$ if i and i' have a common ancestor, and this implies the second assertion on $G_{(S)}$. $\square$

Remark. In particular, the vertex added to G_i in order to obtain G_{i+1} in proposition 41 is an extension vertex. So (S) is obtained from a quasi-cyclic SDSE by a change of variables, a dilatation, and a finite number of extensions. Hence, it is Hopf.

Example. Here is an example of a quasi-cyclic SDSE:

$$\left\{ \begin{array}{rcl} X_1 & = & B_1(1 + X_2 + X_3) \\ X_2 & = & B_2(1 + X_1) \\ X_3 & = & B_3(1 + X_1) \\ X_4 & = & B_4(1 + aX_1) \\ X_5 & = & B_5(1 + aX_1) \\ X_6 & = & B_6(1 + X_4 + X_5). \end{array} \right.$$

where a is a nonzero scalar. In this case, $N = 2$, $I_{\overline{1}} = \{1, 6\}$ and $I_{\overline{2}} = \{2, 3, 4, 5\}$.

7 Fundamental systems

We now study the case of connected Hopf SDSE containing a self-dependent vertex. We shall use the notion of *level* of a vertex.

7.1 Level of a vertex

Proposition 46. *Let (S) be a Hopf SDSE. Let i be a self-dependent vertex of $G_{(S)}$. Then for all $j \in I$, for all $n \geq 1$:*

$$\lambda_n^{(i,j)} = a_j^{(i)} + (1 + \delta_{i,j})(n - 1)\frac{a_{i,j}^{(i)}}{a_i^{(i)}}.$$

Proof. Apply lemma 30, first point, with $i_1 = \ldots = i_n = i$, as $a_i^{(i)} \neq 0$. $\square$

So for a self-dependent vertex i, the sequences $\left(\lambda_n^{(i,j)}\right)_{n\geq 1}$ are polynomial of degree ≤ 1. We formalize this in the following definition:

Definition 47. *Let (S) be a Hopf SDSE, and let i be a vertex of $G_{(S)}$. It will be said to be of* level $\leq M$ *if for all vertices j, there exist scalar $b_j^{(i)}$, $\tilde{a}_j^{(i)}$, such that for all $n > M$:*

$$\lambda_n^{(i,j)} = b_j^{(i)}(n-1) + \tilde{a}_j^{(i)}.$$

The vertex i will be said to be of level M *if it is of level $\leq M$ and not of level $\leq M-1$.*

Remarks.

(1) $\lambda_1^{(i,j)} = a_j^{(i)}$. So if i is of level 0, $\tilde{a}_j^{(i)} = a_j^{(i)}$.

(2) Self-dependent vertices are of level 0, with $\tilde{a}_j^{(i)} = a_j^{(i)}$ and $b_j^{(i)} = (1+\delta_{i,j})\frac{a_{i,j}^{(i)}}{a_i^{(i)}}$.

(3) In a quasi-cyclic SDSE, $\lambda_n^{(\overline{i},\overline{j})} = \delta_{\overline{i+n},\overline{j}}$, for all $\overline{i},\overline{j} \in I$, as it was observed in section 6.1. So in this case the vertices are not of finite level.

Proposition 48. *Let (S) be a Hopf SDSE, i a vertex of $G_{(S)}$ and i' a direct descendant of $G_{(S)}$.*

(1) *i has level 0 or 1 if, and only if, i' has level 0.*

(2) *Let $M \geq 2$. Then i has level M if, and only if, i' has level $M-1$.*

Moreover, if this holds, then for all $k \in I$, $b_k^{(i)} = b_k^{(i')}$.

Proof. Lemma 31 immediately implies that for all $M \geq 1$, i is of level $\leq M$ if, and only if, i' is of level $\leq M-1$. Moreover, if this holds, then $b_k^{(i)} = b_k^{(i')}$ for all k. The first point is a reformulation of this result for $M = 1$. Let us assume that $M \geq 2$. If i is of level M, then i' is of level $\leq M-1$. If i' is of level $\leq M-2$, then i is of level $\leq M-1$: contradiction. So i' is of level $M-1$. The converse is proved in the same way. $\square$

Corollary 49. *Let (S) be a connected Hopf SDSE. Then if one of the vertices of $G_{(S)}$ is of finite level, then all vertices of $G_{(S)}$ are of finite level. Moreover, the coefficients $b_j^{(i)}$ depend only on j. They will now be denoted by b_j.*

Lemma 50. *Let (S) be a connected Hopf SDSE such that any vertex is of finite level. Let j be a vertex of $G_{(S)}$ such that there exists a vertex i which is not an ancestor of j. Then $b_j = 0$.*

Proof. We apply lemma 30-1. We obtain:

$$\lambda_n^{(i,j)} = a_j^{(i_n)} + \sum_{p=1}^{n-1}(1+\delta_{j,i_{p+1}})\frac{a_{j,i_{p+1}}^{(i_p)}}{a_{i_{p+1}}^{(i_p)}}.$$

Moreover, $i_1, \dots, i_n$ are descendants of i, so j is not a descendant of $i_1, \dots, i_n$ and $a_j^{(i_n)} = a_{j,i_{p+1}}^{(i_p)} = 0$ for all p. So $\lambda_n^{(i,j)} = 0$ for all n. As i is of finite level, we deduce that $\tilde{a}_j^{(i)} = 0$ and $b_j = 0$. $\square$

7.2 Definition of fundamental SDSE

Notations. For any $\beta \in K$, we put:

$$F_\beta(h) = \sum_{k=1}^{\infty} \frac{[n]_\beta!}{n!} x^k = \begin{cases} (1-\beta h)^{-\frac{1}{\beta}} \text{ if } \beta \neq 0, \\ e^h \text{ if } \beta = 0. \end{cases}$$

For all $\beta \neq -1$:

$$F_{\frac{\beta}{1+\beta}}((1+\beta)h) = \sum_{k=0}^{\infty} \frac{(1+\beta)\dots(1+n\beta)}{n!} h^n,$$

so we shall put $F_{\frac{\beta}{1+\beta}}((1+\beta)h) = 1$ if $\beta = -1$.

Definition 51. *Let I be a set with a partition $I = I_0 \cup J_0 \cup K_0 \cup L_0$, such that:*

- *I_0, J_0, K_0, L_0 can be empty.*
- *$I_0 \cup J_0$ is not empty.*
- *If $I_0 = \emptyset$, then J_0 is not reduced to a single element.*

We define a SDSE in the following way:

(1) *For all $i \in I_0$, there exists $\beta_i \in K$, such that:*

$$f_i = F_{\beta_i}(h_i) \prod_{j \in I_0 - \{i\}} F_{\frac{\beta_j}{1+\beta_j}}((1+\beta_j)h_j) \prod_{j \in J_0} F_1(h_j).$$

(2) *For all $i \in J_0$:*

$$f_i = \prod_{j \in I_0} F_{\frac{\beta_j}{1+\beta_j}}((1+\beta_j)h_j) \prod_{j \in J_0 - \{i\}} F_1(h_j).$$

(3) *For all $i \in K_0$:*

$$f_i = \prod_{j \in I_0} F_{\frac{\beta_j}{1+\beta_j}}((1+\beta_j)h_j) \prod_{j \in J_0} F_1(h_j).$$

(4) *For all $i \in L_0$, there exists a family of scalars $\left(a_j^{(i)}\right)_{j \in I_0 \cup J_0 \cup K_0}$, such that $(\exists j \in I_0,\ a_j^{(i)} \neq 1+\beta_j)$ or $(\exists j \in J_0,\ a_j^{(i)} \neq 1)$ or $(\exists j \in K_0,\ a_j^{(i)} \neq 0)$.*

$$f_i = \prod_{j \in I_0} F_{\frac{\beta_j}{a_j^{(i)}}}\left(a_j^{(i)} h_j\right) \prod_{j \in J_0} F_{\frac{1}{a_j^{(i)}}}\left(a_j^{(i)} h_j\right) \prod_{j \in K_0} F_0\left(a_j^{(i)} h_j\right).$$

These SDSE and the ones obtained from them by a dilatation and a change of variables are called fundamental SDSE.

Remarks.

(1) It is not difficult to prove that in such a SDSE, all the vertices are of level 0. Moreover, the coefficients b_j are given by:

j	I_0	J_0	K_0	L_0
b_j	$1+\beta_j$	1	0	0

The following array gives the coefficients $a_j^{(i)} = \tilde{a}_j^{(i)}$:

$j \setminus i$	I_0	J_0	K_0	L_0
I_0	$1+(1-\delta_{i,j})\beta_j$	$1+\beta_j$	$1+\beta_j$	$a_j^{(i)}$
J_0	1	$1-\delta_{i,j}$	1	$a_j^{(i)}$
K_0	0	0	0	$a_j^{(i)}$
L_0	0	0	0	0

(2) The condition in (4) on L_0 is equivalent to the property that for each $i \in L_0$, there is some $j \in I$ with $a_j^{(i)} \neq b_j$.

(3) The elements of I_0 are precisely the self-dependent vertices of a fundamental system.

7.3 Fundamental systems are Hopf

We now give a new proof that fundamental SDSE are Hopf. We shall use for this a pre-Lie algebra attached to the coefficients $\lambda_n^{(i,j)}$.

Let us consider a fundamental SDSE (S), without dilatation. We keep the notations of section 7.2 for coefficients $a_j^{(i)}$ and b_j. The coefficients $\lambda_n^{(i,j)}$ have the form:

$$\lambda_n^{(i,j)} = a_j^{(i)} + b_j(n-1).$$

Proposition 52. *Let $\mathfrak{g}$ be a vector space with basis $(e_n^i)_{i\in I, n\geq 1}$. We define a product on $\mathfrak{g}$ by:*

$$e_m^i \circ e_n^j = \lambda_n^{(j,i)} e_{m+n}^j.$$

Then $\mathfrak{g}$ is a pre-Lie algebra. It is graded, e_n^i being homogeneous of degree n for all $n \geq 1$.

Proof. Let e_m^i, e_n^j and e_p^k be three elements of the basis of $\mathfrak{g}$. Then:

$$\begin{array}{rl}
& e_m^i \circ (e_n^j \circ e_p^k) - (e_m^i \circ e_n^j) \circ e_p^k \\
= & \lambda_p^{(k,j)}(\lambda_{n+p}^{(k,i)} - \lambda_n^{(j,i)}) e_{m+n+p}^j \\
= & (a_j^{(k)} + b_j(p-1))(a_i^{(k)} - a_i^{(j)} + b_i(n+p-1) - b_i(n-1)) e_{m+n+p}^j \\
= & (a_j^{(k)} + b_j p - b_j)(a_i^{(k)} + b_i p - a_i^{(j)}) e_{m+n+p}^j.
\end{array}$$

Three cases are possible.

(1) If $i=j$, this is trivially symmetric in e^i_m and e^j_n.

(2) If i and j are two different elements of $I_0 \cup J_0 \cup K_0$, then $a_i^{(j)} = b_i$ and this expression becomes symmetric in e^i_m and e^j_n.

(3) If $i \in L_0$, then $a_i^{(k)} = a_i^{(j)} = b_i = 0$, so this expression is 0. Similarly, if $j \in L_0$, then $a_j^{(k)} = b_j = 0$, so this expression is 0.

In any case, we obtain the pre-Lie relation for these three elements. So $\mathfrak{g}$ is pre-Lie. $\square$

Theorem 8 implies that there exists a unique pre-Lie algebra morphism:

$$\phi : \left\{ \begin{array}{rcl} \mathfrak{g}_{\mathcal{T}^I} & \longrightarrow & \mathfrak{g} \\ \bullet_i\,,\ i \in I & \longrightarrow & e^i_1. \end{array} \right.$$

Let us study this morphism. We shall use the following notation:

Notation. Let $F \in \mathcal{F}^I$. For all $i \in I$, let $d_i(F)$ be the number of roots of F decorated by i. We put $d(F) = (d_1(F), \ldots, d_N(F))$.

Proposition 53. *For all $t \in \mathcal{T}^I$, there exists a coefficient $a_t \in K$ such that:*

$$\phi(t) = a'_t e^i_{|t|},$$

where i is the decoration of the root of t. Moreover, these coefficients can be inductively computed by:

$$\left\{ \begin{array}{rcl} a'_{\bullet_i} & = & 1, \\ a'_{B_i(t_1 \ldots t_k)} & = & d_1(t_1 \ldots t_k)! \ldots d_N(t_1 \ldots t_k)! a^{(i)}_{d(t_1 \ldots t_k)} a'_{t_1} \ldots a'_{t_k}. \end{array} \right.$$

Proof. Let $t_1, \ldots, t_k \in \mathcal{T}^I$ and $i \in I$. The definition of the pre-Lie product of $\mathfrak{g}_{\mathcal{T}^I}$ in terms of grafting easily gives:

$$B_i(t_1 \ldots t_k) = t_1 \circ B_i(t_2 \ldots t_k) - \sum_{j=2}^{k} B_i(t_2 \ldots (t_1 \circ t_j) \ldots t_k). \tag{7.1}$$

First step. The morphism ϕ is clearly homogeneous. Moreover, for all $i \in I$, $\mathfrak{g}_i = Vect(e^i_n \mid n \geq 1)$ is a right pre-Lie ideal of $\mathfrak{g}$; it is not difficult to prove that the right pre-Lie ideal of $\mathfrak{g}_{\mathcal{T}^I}$ generated by $\bullet_i$ is the subspace generated by rooted trees whose root is decorated by i. Hence, if t is a rooted tree whose root is decorated by i, then $\phi(t)$ is an element of $\mathfrak{g}_i$, homogeneous of degree $|t|$, so is collinear to $e^i_{|t|}$. This proves the existence of the coefficients a_t.

Second step. Let us first prove that there exists a family of coefficients $b^{(i)}_{(p_1,\ldots,p_N)}$ such that for all forest $F = t_1 \ldots t_k \in \mathcal{F}^I$, for all $i \in I$:

$$a'_{B_i(F)} = b^{(i)}_{d(F)} a'_{t_1} \ldots a'_{t_k}.$$

We proceed by induction on k. If $k=0$, then $B_i(F)=\bullet_i$ and $a'_t=1$: we take $b^{(i)}_{(0,\ldots,0)}=1$. If $k=1$, we denote by j the decoration of the root of t_1:

$$\phi(B_i(t_1))=\phi(t_1\circ\bullet_i)=\phi(t_1)\circ e^i_1=a'_{t_1}e^j_{|t_1|}\circ e^i_1=a'_{t_1}\lambda^{(i,j)}_1 e^i_{1+|t_1|}=a'_{t_1}a^{(i)}_j e^i_{|B_i(t_1)|}.$$

We then take $b^{(i)}_{(0,\ldots,0,1,0,\ldots,0)}=a^{(i)}_j$, when the 1 is in position j. Let us assume the result at rank $k-1$. The decoration of the root of t_j is denoted by d_j. For all $k\geq 2$, $t_1\circ t_j$ is a linear span of rooted trees whose root is decorated by d_j, so the induction hypothesis on k gives that:

$$\phi(B_i(t_2\ldots t_1\circ t_j\ldots t_k))=b^{(i)}_{d(t_2\ldots t_k)}a'_{t_1}\ldots a'_{t_k}\lambda^{(d_j,d_1)}_{|t_j|}e^i_{|B_i(t_1\ldots t_k)|}.$$

By (7.1):

$$\begin{aligned}
\phi(B_i(t_1\ldots t_k)) &= b^{(i)}_{d(t_2\ldots t_k)}a'_{t_1}\ldots a'_{t_k}e^{d_1}_{|t_1|}\circ e^i_{|B_i(t_2\ldots t_k)|}\\
&\quad -\sum_{j=2}^k a'_{t_1}\ldots a'_{t_k}b^{(i)}_{d(t_2\ldots t_k)}\lambda^{(d_j,d_1)}_{|t_j|}e^i_{|B_i(t_1\ldots t_k)|}\\
&= \underbrace{b^{(i)}_{d(t_2\ldots t_k)}\left(\lambda^{(i,d_1)}_{|B_i(t_2\ldots t_k)|}-\sum_{j=2}^k\lambda^{(d_j,d_1)}_{|t_j|}\right)}_{B}a'_{t_1}\ldots a'_{t_k}e^i_{|B_i(t_1\ldots t_k)|}.
\end{aligned}$$

Moreover:

$$\begin{aligned}
\lambda^{(i,d_1)}_{|B_i(t_2\ldots t_k)|}-\sum_{j=2}^k\lambda^{(d_j,d_1)}_{|t_j|} &= a^{(i)}_{d_1}+b_{d_1}(|t_2|+\ldots+|t_k|)-\sum_{j=2}^k(a^{(d_j)}_{d_1}+b_j(|t_j|-1)\\
&= a^{(i)}_{d_1}-b_{d_1}(k-1)+\sum_{j=2}^k a^{(d_j)}_{d_1}.
\end{aligned}$$

Hence, B depends only on the decoration of the roots of $t_1,\ldots,t_k$ and on i; note that as the morphism ϕ is well-defined, it does not depend on the choice of t_1. We then put $B=b^{(i)}_{d(t_1\ldots t_k)}$.

Last step. Let us fix $(p_1,\ldots,p_N)\in\mathbb{N}^N$ and $1\leq j\leq N$. We apply the second step to $t=B_i({\bullet_1}^{p_1}\ldots{\bullet_1}^{p_N})$. It immediately gives:

$$a'_t=b^{(i)}_{(p_1,\ldots,p_N)}.$$

Moreover, by (7.1):

$$\begin{aligned}
B_i({\bullet_1}^{p_1}\ldots{\bullet_j}^{p_j+1}\ldots{\bullet_N}^{p_N}) &= \bullet_j\circ B_i({\bullet_1}^{p_1}\ldots{\bullet_1}^{p_N})\\
&\quad -\sum_{k=1}^N p_k B_i(\mathbf{I}^{j}_{k}\,{\bullet_1}^{p_1}\ldots{\bullet_k}^{p_k-1}\ldots{\bullet_N}^{p_N}).
\end{aligned}$$

Applying ϕ:

$$b^{(i)}_{(p_1,\ldots,p_j+1,\ldots,p_N)} = \left(\lambda^{(i,j)}_{p_1+\ldots+p_N+1} - \sum_{k=1}^{N} a_j^{(k)}\right) b^{(i)}_{(p_1,\ldots,p_N)}.$$

Lemma 30-2 easily implies that $b^{(i)}_{(p_1,\ldots,p_N)} = p_1!\ldots p_N! a^{(i)}_{(p_1,\ldots,p_N)}$. □

As in section 3.5, let us use the duality between $S(\mathfrak{g}_{T^I})$ and $\mathcal{H}^I_{CK}$ defined by:

$$\langle F, G\rangle = s_F \delta_{F,G}.$$

Let us dualize the pre-Lie algebra morphism ϕ. It becomes a Hopf algebra morphism $\phi^* : S(\mathfrak{g})^* = S(\mathfrak{g}^*) \longrightarrow \mathcal{H}^I_{CK}$. We denote by $(x^i_n)_{i\in I, n\geq 1}$ the dual basis of the basis $(e^i_n)_{i\in I,n\geq 1}$ of $\mathfrak{g}$. Proposition 53 implies that:

$$\phi(x^i_n) = \sum_{\substack{t\in\mathcal{T}^I, |t|=n \\ \text{the root of } t \text{ is decorated by } i}} a''_t t,$$

where the coefficients a''_t satisfies the following property: if $t \in \mathcal{T}^I$, we put $t = B_i(F)$ and $F = t_1^{p_1}\ldots t_k^{p_k}$, where the t_j's are different trees. Let r_j be the number of roots of F decorated by j for all $j \in I$. Then:

$$a''_t = \frac{r_1!\ldots r_N!}{p_1!\ldots p_k!} a^{(i)}_{(r_1,\ldots,r_N)} a''^{p_1}_{t_1} \ldots a''^{p_k}_{t_k}.$$

By proposition 26, these coefficients are the coefficients a_t's. Hence, the image of ϕ^* is the subalgebra of $\mathcal{H}^I_{CK}$ generated by the homogeneous components of the solution of (S). As ϕ^* is a Hopf algebra morphism, it is a Hopf subalgebra. Finally:

Proposition 54. *Let (S) be a Hopf fundamental SDSE. Then it is Hopf.*

Remark. We also proved that the Hopf algebra $\mathcal{H}_{(S)}$ is dual to the enveloping algebra of the Lie algebra $\mathfrak{g}$, defined by the help of the structure constants $\lambda^{(i,j)}_n$.

7.4 Self-dependent vertices

Theorem 55. *Let (S) be a Hopf SDSE, and let i be a self-dependent vertex of (S). The subsystem formed by i and all its descendants is fundamental, with $K_0 = L_0 = \emptyset$. Moreover, if k is a direct descendant of i and j is not a direct descendant of i, then $a^{(k)}_j = 0$.*

Proof. From lemma 39 with $i = l$, we deduce that any descendant of i is a direct descendant of i. Up to a restriction, we now assume that any vertex of (S) is a direct descendant of i. We won't write the indices $^{(i)}$ in the proof. Up to a change of variables, we assume that $a_j = 1$ for all j. As i has level 0, the coefficients of

$f = f_i$ satisfy an induction of the form (lemma 30-2):

$$\left\{\begin{array}{rcl} a_{(0,\cdots,0)} & = & 1, \\ a_{(p_1,\cdots,p_j+1,\cdots,p_N)} & = & \dfrac{1}{p_j+1}\left(1+\displaystyle\sum_{l=1}^{N}\mu_j^{(l)}p_l\right)a_{(p_1,\cdots,p_N)}, \end{array}\right.$$

with $\mu_j^{(l)} = (1+\delta_{i,j})\,a_{i,j} - a_j^{(l)}$ for all $j, l \in I$.

Let us fix $j \neq k$ in I. For $(p_1, \cdots, p_N) = (0, \cdots, 0)$, as $a_{(0,\cdots,0)} = 1$:

$$\mu_j^{(k)} = \mu_k^{(j)}. \tag{7.2}$$

For $(p_1, \cdots, p_N) = \varepsilon_l$, we obtain:

$$\left(1+\mu_j^{(k)}+\mu_j^{(l)}\right)\left(1+\mu_k^{(l)}\right) = \left(1+\mu_k^{(j)}+\mu_k^{(l)}\right)\left(1+\mu_j^{(l)}\right).$$

So:

$$\mu_j^{(k)}\mu_k^{(l)} = \mu_k^{(j)}\mu_j^{(l)}. \tag{7.3}$$

Let $j, k \in I$. We shall say that $j\,\mathcal{R}\,k$ if $j = k$ or if $\mu_j^{(k)} \neq 0$. Let us show that $\mathcal{R}$ is an equivalence. By (7.2), it is clearly symmetric. Let us assume that $j\,\mathcal{R}\,k$ and $k\,\mathcal{R}\,l$. If $j = k$ or $k = l$ or $j = l$, then $j\,\mathcal{R}\,l$. If j, k, l are distinct, then $\mu_j^{(k)} \neq 0$ and $\mu_k^{(l)} \neq 0$. By (7.3), $\mu_j^{(l)} = \mu_k^{(l)} \neq 0$, so $j\,\mathcal{R}\,l$. We denote by $I_1, \cdots, I_M$ the equivalence classes of $\mathcal{R}$.

Let us assume that $j\,\mathcal{R}\,k$, $j \neq k$. Then $\mu_j^{(k)} \neq 0$, so for all l, $\mu_k^{(l)} = \mu_j^{(l)}$. In particular, $\mu_k^{(j)} = \mu_j^{(j)} = \mu_j^{(k)} = \mu_k^{(k)}$. So, finally, there exists a family of scalars $(\beta_n)_{1\leq n\leq M}$, such that:

- If $j, k \in I_n$, then $\mu_j^{(k)} = \beta_n$.
- If j and k are not in the same I_n, then $\mu_j^{(k)} = \mu_k^{(j)} = 0$.

The coefficients $-a_j^{(k)} + (1+\delta_{i,j})\,a_{i,j}^{(i)}$ are given for all j, k by the array:

$j \setminus k$	I_1	I_2	$\cdots$	I_M
I_1	β_1	0	$\cdots$	0
I_2	0	β_2	$\ddots$	$\vdots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	0
I_M	0	$\cdots$	0	β_M

An easy induction proves:

$$a_{(p_1,\cdots,p_N)} = \frac{1}{p_1!\cdots p_N!}\prod_{n=1}^{M}(1+\beta_n)\cdots\left(1+\beta_n\left(\sum_{l\in I_n}p_l - 1\right)\right).$$

So:

$$f_i = \prod_{p=1}^{M}F_{\beta_p}\left(\sum_{l\in I_p}h_l\right).$$

We assume that $i \in I_1$, without loss of generality. From the expression of $f = f_i$, we deduce that $b_j = (1+\delta_{i,j})a_{i,j}^{(i)} = \beta_1 + 1$ if $j \in I_1$, 1 if $j \in I_2 \cup \cdots \cup I_M$ (see section 7.2). So $a_j^{(k)}$ is given for all j, k by the array:

$j \setminus k$	I_1	I_2	I_3	$\cdots$	I_M
I_1	1	β_1+1	$\cdots$	$\cdots$	β_1+1
I_2	$\vdots$	$1-\beta_2$	1	$\cdots$	1
I_3	$\vdots$	1	$1-\beta_3$	$\ddots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	1
I_M	1	1	$\cdots$	1	$1-\beta_M$

As a consequence, if $j \in I_1$, then for all $1 \leq k \leq N$, $a_k^{(j)} = a_k^{(i)}$ and $\lambda_n^{(j,k)} = \lambda_n^{(i,k)}$ for all $n \geq 1$. Note that if j, j' are in the same I_p, then $\lambda_n^{(j,k)} = \lambda_n^{(j',k)}$ for all $n \geq 1$, for all $k \in I$. So, the Hopf SDSE formed by i and its descendants is the dilatation of a system with the following coefficients $\lambda_n^{(j,k)}$:

$j \setminus k$	1	2	3	$\cdots$	M
1	$(\beta_1+1)(n-1)+1$	n	$\cdots$	$\cdots$	n
2	$(\beta_1+1)n$	$n-\beta_2$	n	$\cdots$	n
3	$\vdots$	n	$n-\beta_3$	$\ddots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	n
M	$(\beta_1+1)n$	n	$\cdots$	n	$n-\beta_M$

with $i = 1$, and $f_1 = \displaystyle\prod_{j=1}^{M} F_{\beta_j}(h_j)$. If $j \neq 1$, for all $(k_1, \cdots, k_M)$:

$$\begin{aligned}
a^{(j)}_{(k_1+1,\cdots,k_M)} &= \left((\beta_1+1)\sum_{l=1}^{M} k_l + \beta_1 + 1 - (\beta_1+1)\sum_{l=2}^{M} k_l - k_1 \right) \frac{a^{(j)}_{(k_1,\cdots,k_M)}}{k_1+1} \\
&= (\beta_1 + 1 + \beta_1 k_1)\frac{a^{(j)}_{(k_1,\cdots,k_M)}}{k_1+1},
\end{aligned}$$

$$\begin{aligned}
a^{(j)}_{(k_1,\cdots,k_j+1,\cdots,k_M)} &= \left(\sum_{l=1}^{M} k_l + 1 - \beta_j - \sum_{l=1}^{M} k_l + \beta_j k_j \right) \frac{a^{(j)}_{(k_1,\cdots,k_M)}}{k_j+1} \\
&= (1 - \beta_j + \beta_j k_j)\frac{a^{(j)}_{(k_1,\cdots,k_M)}}{k_j+1}.
\end{aligned}$$

If $l \neq 1$ and $l \neq j$:

$$a^{(j)}_{(k_1,\cdots,k_l+1,\cdots,k_M)} = \left(\sum_{p=1}^{M} k_p - \sum_{p=1}^{M} k_p + \beta_l k_l \right) \frac{a^{(j)}_{(k_1,\cdots,k_M)}}{k_l+1} = (1+\beta_l k_l)\frac{a^{(j)}_{(k_1,\cdots,k_M)}}{k_l+1}.$$

So, if $j \neq 1$:

$$f_j = F_{\frac{\beta_1}{1+\beta_1}}((1+\beta_1)h_1)F_{\frac{\beta_j}{1-\beta_j}}((1-\beta_j)h_j)\prod_{k\neq 1,j} F_{\beta_k}(h_k).$$

Let us put $I'_0 = \{j \geq 2 \;/\; \beta_j \neq 1\}$ and $J'_0 = \{j \geq 2 \;/\; \beta_j = 1\}$. Then, after the change of variables $h_j \longrightarrow \frac{1}{1-\beta_j}h_j$ for all $j \in I'_0$:

$$\left\{\begin{array}{rcl} f_1 &=& F_{\beta_1}(h_1)\displaystyle\prod_{j\in I'_0} F_{\beta_j}\left(\frac{1}{1-\beta_j}h_j\right)\prod_{j\in J'_0} F_1(h_j),\\ f_j &=& F_{\frac{\beta_1}{1+\beta_1}}((1+\beta_1)h_1)F_{\frac{\beta_j}{1-\beta_j}}(h_j)\displaystyle\prod_{l\in I'_0-\{j\}} F_{\beta_l}\left(\frac{1}{1-\beta_l}h_l\right)\prod_{l\in J'_0} F_1(h_l) \text{ if } j\in I'_0,\\ f_j &=& F_{\frac{\beta_1}{1+\beta_1}}((1+\beta_1)h_1)\displaystyle\prod_{l\in I'_0} F_{\beta_l}\left(\frac{1}{1-\beta_l}h_l\right)\prod_{l\in J'_0-\{j\}} F_1(h_l) \text{ if } j\in J'_0. \end{array}\right.$$

Putting $\gamma_j = \frac{\beta_j}{1-\beta_j}$ for all $j \in I_0$, as $\beta_j = \frac{\gamma_j}{1+\gamma_j}$ and $1-\beta_j = \frac{1}{1+\gamma_j}$:

$$\left\{\begin{array}{rcl} f_1 &=& F_{\beta_1}(h_1)\displaystyle\prod_{j\in I'_0} F_{\frac{\gamma_j}{1+\gamma_j}}((1+\gamma_j)h_j)\prod_{j\in J'_0} F_1(h_j),\\ f_j &=& F_{\frac{\beta_1}{1+\beta_1}}((1+\beta_1)h_1)F_{\gamma_j}(h_j)\displaystyle\prod_{j\in I'_0-\{j\}} F_{\frac{\gamma_j}{1+\gamma_j}}((1+\gamma_j)h_j)\prod_{j\in J'_0} F_1(h_j) \text{ if } j\in I'_0,\\ f_j &=& F_{\frac{\beta_1}{1+\beta_1}}((1+\beta_1)h_1)\displaystyle\prod_{j\in I'_0} F_{\frac{\gamma_j}{1+\gamma_j}}((1+\gamma_j)h_j)\prod_{j\in J'_0-\{j\}} F_1(h_j) \text{ if } j\in J'_0. \end{array}\right.$$

So this a fundamental system, with $I_0 = \{1\} \cup I'_0$, $J_0 = J'_0$, and $K_0 = L_0 = \emptyset$. □

7.5 Hopf SDSE containing a 2-cycle

We first introduce a family of Hopf SDSE with no self-dependent vertices. More precisely, we are looking for the Hopf SDSE (S) such that $G_{(S)}$ is *complete M-partite*, that is to say there exists a partition $I_1 \sqcup \ldots \sqcup I_M$ of the set of vertices of G into nonempty parts, such that if x, y are two vertices of G, there is an edge from x to y if, and only if, x and y are in different I_j's.

Proposition 56. *Let (S) be a Hopf SDSE such that $G_{(S)}$ is a complete M-partite graph. Let $I = I_1 \sqcup \ldots \sqcup I_M$ be the partition of the set of vertices. Then one, and only one, of the following results holds:*

(1) *Up to a change of variables, for all $1 \leq n \leq M$, for all $i \in I_n$, $f_i = \displaystyle\prod_{m\neq n}\left(1-\sum_{j\in I_m} h_j\right)^{-1}$.*

(2) *(S) is 2-quasi-cyclic.*

Proof. First, let us choose two vertices $i \to j$ in $G_{(S)}$. Then $j \to i$ in $G_{(S)}$. Then $a_j^{(j)} = 0$, so $a_{i,j}^{(j)} = 0$; by lemma 30-1 with $(i_1, i_2) = (j, i)$, $a_j^{(i)} = \lambda_2^{(j,j)}$, so $a_j^{(i)}$

depends only on j. So, up to a change of variables, we can suppose that all the $a_j^{(i)}$'s are equal to 0 or 1. We first study two preliminary cases.

First preliminary case. Let us suppose that $G_{(S)}$ is the following graph (which is complete 3-partite):

$$\begin{tikzcd}
1 \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & 2 \arrow[dl, leftrightarrow] \\
& 3 &
\end{tikzcd}$$

So, $a_j^{(i)} = 1$ if $i \neq j$. Moreover, if $\{i,j,k\} = \{1,2,3\}$, by lemma 30-1 with $(i_1,i_2) = (i,k)$:

$$\lambda_2^{(i,j)} = a_{j,k}^{(i)} + 1.$$

Consequently, $\lambda_2^{(i,j)} = \lambda_2^{(i,k)}$. So, applying proposition 42 to the restriction to $\{i,j\}$ implies that two cases are possible:

(1) For all $i \neq j$, for all $n \geq 1$, $\lambda_n^{(i,j)} = n$.

(2) For all $i \neq j$, for all $n \geq 1$, $\lambda_n^{(i,j)} = 1$ if n is odd and 0 if n is even.

In the second case, we deduce that if $\{i,j,k\} = \{1,2,3\}$, $a_{j,k}^{(i)} = -1$. By lemma 30-1 with $(i_1,i_2,i_3) = (1,3,2)$:

$$1 = \lambda_3^{(1,2)} = a_{2,3}^{(1)} = -1.$$

This is a contradiction. So the first case holds. It is then not difficult to prove that if $\{i,j,k\} = \{1,2,3\}$, $f_i(h_j,h_k) = (1-h_j)^{-1}(1-h_k)^{-1}$.

Second preliminary case. We now consider the graph with three vertices

$$1 \longleftrightarrow 2 \longleftrightarrow 3.$$

It is complete 2-partite, with $I_1 = \{1,3\}$ and $I_2 = \{2\}$. By lemma 30-1 with $(i_1,i_2) = (2,3)$ and $(2,1)$:

$$\lambda_2^{(2,1)} = a_{1,3}^{(2)} = \lambda_2^{(2,3)}.$$

Applying proposition 42 to the restriction to $\{1,2\}$ and $\{2,3\}$ shows that two cases are possible:

(1) For all $n \geq 1$, $\lambda_n^{(1,2)} = \lambda_n^{(2,1)} = \lambda_n^{(2,3)} = \lambda_n^{(3,2)} = n$.

(2) For all $n \geq 1$, $\lambda_n^{(1,2)} = \lambda_n^{(2,1)} = \lambda_n^{(2,3)} = \lambda_n^{(3,2)} = 0$ if n is even and 1 if n is odd.

In the first case, proposition 42 implies that $f_1 = f_3 = (1-h_2)^{-1}$. Lemma 30-2 implies that for all $m,n \geq 0$:

$$a_{(m+1,n)}^{(2)} = \frac{m+n+1}{m+1} a_{(m,n)}, \; a_{(m,n+1)}^{(2)} = \frac{m+n+1}{n+1} a_{(m,n)}.$$

Consequently, for all $m,n \geq 0$, $a_{(m,n)}^{(2)} = \frac{(m+n)!}{m!n!}$, so $f_2 = (1-h_1-h_3)^{-1}$. In the second case, proposition 42 implies that $f_1 = f_3 = 1+h_2$. Moreover, $f_2(h_1,0) = 1+h_1$

and $f_2(0,h_3)=1+h_3$, so $a^{(2)}_{1,1}=a^{(2)}_{3,3}=0$. As $\lambda_2^{(2,1)}=a^{(2)}_{1,3}=0$, $f_2=1+h_1+h_3$.

We separate the proof of the general case into two subcases.

General case, first subcase. We assume that $M=2$. We put $I_1=\{i_1,\cdots,i_r\}$ and $I_2=\{j_1,\cdots,j_s\}$. For $i_p\in I_1$, we put:

$$f_{i_p}=\sum_{(q_1,\cdots,q_s)} a^{(i_p)}_{(q_1,\cdots,q_s)} h_{j_1}^{q_1}\cdots h_{j_s}^{q_s}.$$

Restricting to the vertices i_p and j_q, by proposition 42, two cases are possible.

(1) $a^{(i_p)}_{j_q,j_q}=0$. Then, by the second preliminary case, restricting to i_p, j_q and $j_{q'}$, for all $j_{q'}$, $a^{(i_p)}_{j_q,j_{q'}}=a^{(i_p)}_{j'_q,j'_q}=0$. So $f_{i_p}=1+\sum\limits_q h_{j_q}$.

(2) $\lambda_n^{(i_p,j_q)}=n$ for all $n\geq 1$. We obtain:

$$a^{(i_p)}_{(q_1,\cdots,q_m+1,\cdots,q_s)}=\frac{1+q_1+\cdots+q_s}{q_m+1}a^{(i_p)}_{(q_1,\cdots,q_s)}.$$

An easy induction proves that $a^{(i_p)}_{(q_1,\cdots,q_s)}=\frac{(q_1+\cdots+q_s)!}{q_1!\cdots q_s!}$, so:

$$f_{i_p}=\left(1-\sum_q h_{j_q}\right)^{-1}.$$

A similar result holds for the j_q's. So, we prove that for any vertex i of $G_{(S)}$:

$$(a): f_i=1+\sum_{i\longrightarrow j} h_j \text{ or } (b): f_i=\left(1-\sum_{i\longrightarrow j} h_j\right)^{-1}.$$

Moreover, by the second preliminary case, if i and j are related, they satisfy both (a) or both (b). As the graph is connected, every vertex satisfies (a) or every vertex satisfies (b).

General case, second subcase. We now assume that $M\geq 3$. Let us fix $i\in G$ and let j a direct descendant of i. Let us choose a common direct descendant k of i and j: as $M\geq 3$, this exists. By the first preliminary case, after restriction to i,j,k we obtain that $\lambda_n^{(i,j)}=n$ for all $n\geq 1$. We obtain, similarly to the case $M=2$, if $i\in I_p$, $f_i=\prod\limits_{q\neq p}\left(1-\sum\limits_{j\in I_q} h_j\right)^{-1}$. $\Box$

Remark. The system of case (1) is fundamental, with $I=J_0$.

Theorem 57. *Let (S) be a Hopf SDSE, containing a 2-cycle. Then the subsystem formed by the vertices of this 2-cycle and all their descendants is fundamental or is quasi-cyclic.*

Proof. We assume, up to a restriction, that any vertex of (S) is a descendant of a vertex of the 2-cycle. Let $k, l \in G_{(S)}$, such that l is a direct descendant of k. There exists $i, j, i_2, \ldots, i_p$ such that in $G_{(S)}$:

$$j \longleftrightarrow i \longrightarrow i_2 \longrightarrow \ldots \longrightarrow i_p \longrightarrow k \longrightarrow l.$$

Applying repeatedly lemma 39 (case $i = k$), we obtain that there is an edge from i_2 to i, from i_3 to i_2, ..., from l to k. So if there is an edge from k to l in $G_{(S)}$, there is also an edge from l to k. We shall say that $G_{(S)}$ is *symmetric*.

Let us now prove that $G_{(S)}$ is a complete M-partite graph, for a certain $M \geq 2$. Let us consider a maximal complete partite subgraph G' of $G_{(S)}$. This exists, as $G_{(S)}$ contains at least a 2-cycle. Let us assume that $G' \neq G_{(S)}$. As $G_{(S)}$ is connected, there exists a vertex $i \in G_{(S)}$, related to a vertex of G'. Let us put $I' = I'_1 \cup \cdots \cup I'_M$ be the partition of the set of vertices of G'.

First, if i is related to a vertex j of I'_p, it is related to any vertex of I'_p. Indeed, let j' be another vertex of I'_p and let $k \in I'_q$, $q \neq p$. By lemma 39, j' is related to i. As $G_{(S)}$ is symmetric, i is related to j'.

Let us assume that i is not related to at least two I'_p's. Let us take k, l in G', in two different I'_p's, not related to i. By the first step, j, k and l are in different I'_p's, so are related. By lemma 39, k or l is related to i. As $G_{(S)}$ is symmetric, then i is related to k or l: contradiction. So i is not related to at most one I'_p.

As a conclusion:

(1) If i is related to every I'_p's, by the first step i is related to every vertex of G', so $G' \cup \{i\}$ is an complete $M+1$-partite graph, with partition $I'_1 \cup \cdots \cup I'_M \cup \{x\}$.

(2) If i is related to every I'_p's but one, we can suppose up to a reindexation that i is not related to I_M. Then, by the first step, i is related to every vertex of $I'_1 \cup \cdots \cup I'_{M-1}$. So $G' \cup \{x\}$ is an complete M-partite graph, with partition $I'_1 \cup \cdots \cup ('I_M \cup \{x\})$.

Both cases contradict the maximality of G', so $G_{(S)} = G'$ is a complete M-partite graph. From proposition 56, (S) is 2-quasi-cyclic or fundamental, with $I = J_0$. □

7.6 Systems with only vertices of level 0

Theorem 58. *Let (S) be a connected Hopf SDSE with only vertices of degree 0. Then it is fundamental.*

Proof. We use notations of proposition 41. We prove inductively that G_i is a fundamental system for all $i \geq 0$. Let us first consider the case $i = 0$. From theorem 55 and 57, for any vertex $i \in I_0$, i and all its descendants are part of a fundamental system with $K_0 = L_0 = \emptyset$. A simple study of the possible graphs shows that (S_0) (corresponding to G_0) is a concatenation of fundamental systems. If (S_0) is not connected, let us take i and j in two different connected components of G_0. Then i cannot be a descendant of j and j cannot be a descendant of i. By lemma 50, $b_i = b_j = 0$. So the fundamental system corresponding to any connected component of (S_0) satisfies $J_0 = K_0 = L_0 = \emptyset$ and $\beta_i = -1$ for all $i \in I_0$. It is then clear that (S_0) is a fundamental system, with $I = I_0$ and $\beta_i = -1$ for all $i \in I_0$.

Let us assume that the system associated to G_{k-1} is fundamental. The vertex added to G_{k-1} in order to obtain G_k is denoted by 0. For all $i \in I_{k-1}$, $\lambda_n^{(0,i)} = b_i(n-1) + a_i^{(0)}$. Let us take $i, j \in I_{k-1}$, with $i \neq j$. Using lemma 30-1 in two different ways:

$$a_{i,j}^{(0)} = \left(b_j + a_j^{(0)} - a_j^{(i)}\right) a_i^{(0)} = \left(b_i + a_i^{(0)} - a_i^{(j)}\right) a_j^{(0)}.$$

So, for all $i, j \in I_{k-1}$:

$$\left(b_j - a_j^{(i)}\right) a_i^{(0)} = \left(b_i - a_i^{(j)}\right) a_j^{(0)}. \tag{7.4}$$

If the fundamental system formed by G_{k-1} has a dilatation, as $b_j - a_j^{(i)} = b_i - a_i^{(j)} \neq 0$ if i and j are in the same part of the dilatation, we deduce that $a_i^{(0)} = a_j^{(0)}$ and for all $n \geq 1$, $\lambda_n^{(0,i)} = \lambda_n^{(0,j)}$. Hence, up to a restriction, we can assume that there is no dilatation.

Let $i \in L_0$. Let us choose $j \in I_0 \cup J_0 \cup K_0$, such that $a_j^{(i)} \neq b_j$. Then $b_i = a_i^{(j)} = 0$, so (7.4) gives $\left(b_j - a_j^{(i)}\right) a_i^{(0)} = 0$. So $a_i^{(0)} = 0$ for all $i \in L_0$. So the direct descendants of 0 are all in $I_0 \cup J_0 \cup K_0$. Using lemma 30-2 with $i \in I_0 \cup J_0 \cup K_0$:

$$\begin{aligned}
&a^{(0)}_{(p_1,\cdots,p_i+1,\cdots,p_N)} \\
&= \left(a_i^{(0)} + b_i(p_1 + \cdots + p_N) - \sum_{j \in I_0 \cup J_0 \cup K_0 - \{i\}} b_i p_j - a_i^{(i)} p_i\right) \frac{a^{(0)}_{(p_1,\cdots,p_N)}}{p_i + 1} \\
&= \left(a_i^{(0)} + \left(b_i - a_i^{(i)}\right) p_i\right) \frac{a^{(0)}_{(p_1,\cdots,p_N)}}{p_i + 1}.
\end{aligned}$$

So:

$$f_0 = \prod_{i \in I_0} F_{\frac{\beta_i}{a_i^{(0)}}}\left(a_i^{(0)} h_i\right) \prod_{i \in J_0} F_{\frac{1}{a_i^{(0)}}}\left(a_i^{(0)} h_i\right) \prod_{i \in K_0} F_0\left(a_i^{(0)} h_i\right).$$

So the system of equations associated to G_k is fundamental, with $0 \in K_0 \cup L_0$. $\square$

7.7 Vertices of level 1

As a consequence, if (S) is a connected Hopf SDSE, two disjoint cases are possible:

(1) (S) contains a quasi-cyclic subsystem, so is described by theorem 45.

(2) Any vertex of (S) is of finite level, and the subsystem $(S^{(0)})$ formed by the vertices of level 0 is fundamental.

In order to conclude the description of all connected Hopf SDSE, let us study now vertices of level ≥ 1.

Theorem 59. *Let (S) be a connected Hopf SDSE such that any vertex is of finite level. Let (S_0) be the subsytem formed by the vertices of level 0. The set of vertices of level 1 which are not extension vertices can be decomposed into $I_1 \cup J_1$, such that:*

(1) *For all $i \in I_1$, there exists $\nu_i \in K$, a family of scalars $\left(a_j^{(i)}\right)_{j \in I_0 \cup J_0 \cup K_0}$, such that $\nu_i \neq 1$ and, if $\nu_i \neq 0$:*

$$f_i = \frac{1}{\nu_i} \prod_{j \in I_0} F_{\frac{\beta_j}{\nu_i a_j^{(i)}}} \left(\nu_i a_j^{(i)} h_j\right) \prod_{j \in J_0} F_{\frac{1}{\nu_i a_j^{(i)}}} \left(\nu_i a_j^{(i)} h_j\right) \prod_{j \in K_0} F_0 \left(\nu_i a_j^{(i)} h_j\right) + 1 - \frac{1}{\nu_i}.$$

If $\nu_i = 0$:

$$f_i = -\sum_{j \in I_0} \frac{a_j^{(i)}}{\beta_j} \ln(1 - \beta_j h_j) - \sum_{j \in J_0} a_j^{(i)} \ln(1 - h_j) + \sum_{j \in K_0} a_j^{(i)} h_j + 1.$$

(2) *For all $i \in J_1$, there exists $\nu_i \in K - \{0\}$, a family of scalars $\left(a_j^{(i)}\right)_{j \in L_0}$, with the following conditions:*

- *$L_0^{(i)} = \{j \in L_0 \;/\; a_j^{(i)} \neq 0\}$ is not empty.*
- *For all $j, k \in L_0^{(i)}$, $f_j = f_k$. In particular, we put $c_t^{(i)} = a_t^{(j)}$ for any $j \in L_0^{(i)}$, for all $t \in I_0 \cup J_0 \cup K_0$.*

Then:

$$\begin{aligned} f_i &= \frac{1}{\nu_i} \prod_{j \in I_0} F_{\frac{\beta_j}{c_j^{(i)} - 1 - \beta_j}} \left(\left(c_j^{(i)} - 1 - \beta_j\right) h_j\right) \prod_{j \in J_0} F_{\frac{1}{c_j^{(i)} - 1}} \left(\left(c_j^{(i)} - 1\right) h_j\right) \\ & \prod_{j \in K_0} F_0 \left(c_j^{(i)} h_j\right) + \sum_{j \in L_0^{(i)}} a_j^{(i)} h_j + 1 - \frac{1}{\nu_i}. \end{aligned}$$

Proof. First case. Let us assume that 0 is of level 1. Then all the direct descendants of 0 are of level 0, so are in $I_0 \cup J_0 \cup K_0 \cup L_0$. Moreover, for all $i \in I$, $\lambda_1^{(0,i)} = a_i^{(0)}$ and $\lambda_n^{(0,i)} = b_i(n-1) + \tilde{a}_i^{(0)}$ if $n \geq 2$.

First step. Let us first assume that all the direct descendants of 0 are in L_0. Up to a change of variables, we can assume that for all direct descendants i of 0, $a_i^{(0)} = 1$.Let i be a direct descendant of 0 and let $0, j, i_3, \ldots, i_n$ be a sequence of elements of I as in lemma 30-2. Then $j \in L_0$, $i_3, \ldots, i_n \in I_0 \cup J_0 \cup L_0$, so i is not a direct descendant of $j, i_3, \ldots, i_n$. Hence:

$$\lambda_n^{(0,i)} = (1 + \delta_{i,j}) a_{i,j}^{(0)}.$$

Moreover:

$$\lambda_n^{(0,i)} = \lambda_n^{(0,j)} = (1 + \delta_{i,j}) a_{i,j}^{(0)}.$$

So there exists a scalar γ, such that $\lambda_n^{(0,i)} = \gamma$ for all $n \geq 2$, for all direct descendants i of 0. An easy induction using lemma 30-2 proves that for all $n \geq 1$:

$$a^{(0)}_{\underbrace{i,\ldots,i}_{n}} = \frac{\gamma^{n-1}}{n!}.$$

Let i be a direct descendant of 0 and let k be a direct descendant of i. Then k is not a direct descendant of 0, and lemma 30-1 implies that for all $n \geq 2$:

$$\lambda_n^{(0,jk} = \lambda_{n-1}^{(i,k)} = b_k + a_k^{(i)}(n-2).$$

For $t = B_0(\bullet_i{}^n)$, we obtain:

$$\lambda_{n+1}^{(0,k)} a_t = (b_k + a_k^{(i)}(n-1))\frac{\gamma^{n-1}}{n!} = a_{B_0(\mathbf{I}^k_i\, \bullet_i{}^{n-1})} = n a_k^{(i)} \frac{\gamma^{n-1}}{n!}.$$

If $\gamma \neq 0$, we obtain that $b_k = a_k^{(i)}$ for all direct descendants k of i, which contradicts the fact that $i \in L_0$. So $\gamma = 0$. This implies that for all direct descendants i, j of 0, $a_{i,j}^{(0)} = 0$, so $f_0 = 1 + \sum\limits_{i \longrightarrow i} a_i^{(0)} h_i$ and 0 is an extension vertex. We shall assume in the sequel that at least one of the direct descendant of 0 is not in L_0.

Second step. Let us take $i, j \in I$, with $i \neq j$. Using lemma 30-1 in two different ways:

$$a_{i,j}^{(0)} = \left(b_j + \tilde{a}_j^{(0)} - a_j^{(i)}\right) a_i^{(0)} = \left(b_i + \tilde{a}_i^{(0)} - a_i^{(j)}\right) a_j^{(0)}. \tag{7.5}$$

Let us take $i, j \in L_0$. Then $a_j^{(i)} = a_i^{(j)} = b_i = b_j = 0$, so (7.5) gives:

$$\tilde{a}_j^{(0)} a_i^{(0)} = \tilde{a}_i^{(0)} a_j^{(0)}.$$

So $\left(\tilde{a}_i^{(0)}\right)_{i \in L_0}$ and $\left(a_i^{(0)}\right)_{i \in L_0}$ are collinear. We deduce that there exists a scalar $\nu \in K$, such that for all $i \in L_0$, $\tilde{a}_i^{(0)} = \nu a_i^{(0)}$. Let us now take $i, j \in I_0 \cup J_0 \cup K_0$, with $i \neq j$. Then $b_i = a_i^{(j)}$ and $b_j = a_j^{(i)}$, so (7.5) gives:

$$\tilde{a}_j^{(0)} a_i^{(0)} = \tilde{a}_i^{(0)} a_j^{(0)}.$$

So $\left(\tilde{a}_i^{(0)}\right)_{i \in I_0 \cup J_0 \cup K_0}$ and $\left(a_i^{(0)}\right)_{i \in I_0 \cup J_0 \cup K_0}$ are collinear. We deduce that there exists a scalar $\nu' \in K$, such that for all $i \in I_0 \cup J_0 \cup K_0$, $\tilde{a}_i^{(0)} = \nu' a_i^{(0)}$. Let us now take $i \in I_0 \cup J_0 \cup K_0$ and $j \in L_0$. Then $b_j = a_j^{(i)} = 0$, so $\nu a_j^{(0)} a_i^{(0)} = \left(b_i + \nu' a_i^{(0)} - a_i^{(j)}\right) a_j^{(0)}$. In other words:

$$\forall i \in I_0 \cup J_0 \cup K_0,\ \forall j \in L_0,\ (\nu - \nu') a_i^{(0)} a_j^{(0)} = (b_i - a_i^{(j)}) a_j^{(0)}. \tag{7.6}$$

Third step. Let us assume that there is a dilatation on (S_0). If this dilatation holds only on vertices of K_0 or L_0, this system can all the same be considered as a fundamental system with no dilatation. Similarly, if the dilatation holds on a vertex of i of I_0 such that $\beta_i = 0$, then as $F_0(h_1 + h_2) = F_0(h_1) F_0(h_2)$ (as

$F_0 = exp$), this dilated system can be seen as a fundamental system with no dilatation. It remains to consider the case of dilatations holding on a vertex of I_0 with $\beta_i \neq 0$ or on a vertex of J_0. Let us take i, j in the same part of the dilatation. (7.5) becomes:

$$\left(b_j + \nu' a_j^{(0)} - a_j^{(i)}\right) a_i^{(0)} = \left(b_i + \nu' a_i^{(0)} - a_i^{(j)}\right) a_j^{(0)}.$$

As i, j are in the same part of the dilatation, necessarily $b_i = b_j$ and $a_j^{(i)} = a_i^{(j)}$. It remains that $(b_j - a_j^{(i)})(a_i^{(0)} - a_j^{(0)}) = 0$. By hypothesis on the dilatation, $b_j \neq a_j^{(i)}$, so $a_i^{(0)} = a_j^{(0)}$. Consequently, $\tilde{a}_i^{(0)} = \nu' a_i^{(0)} = \nu' a_j^{(0)} = \tilde{a}_j^{(0)}$. As the level of 0 is 1, we deduce that $\lambda_n^{(0,i)} = \lambda_n^{(0,j)}$ for all $n \geq 1$. Hence, the vertex 0 respects the dilatation; up to a restriction, we can assume there is no dilatation in (S_0).

Fourth step. Let us assume that $L_0^{(0)} = \emptyset$. Then all the direct descendants of 0 are in $I_0 \cup J_0 \cup K_0$. Moreover, if $i \in I_0 \cup J_0 \cup K_0$ and $p_1 + \ldots + p_N > 0$:

$$a^{(0)}_{(p_1,\cdots,p_i+1,\cdots,p_N)} = \left(\nu' a_i^{(0)} + \left(b_i - a_i^{(i)}\right) p_i\right) \frac{a^{(0)}_{(p_1,\cdots,p_N)}}{p_i + 1}.$$

It is then not difficult to show that 0 is in I_1. Note that this case holds if $\nu = \nu'$. Indeed, if $\nu = \nu'$, let $j \in L_0$. For a good choice of i, $b_i - a_i^{(j)} \neq 0$ in (7.6), so $a_j^{(0)} = 0$: then $L_0^{(0)} = \emptyset$, and the result is proved in the third step.

Fifth step. Let us assume that $L_0^{(0)} \neq \emptyset$. By the preceding step, $\nu \neq \nu'$. Let us take $j \in L_0^{(0)}$. By (7.6), for all $i \in I_0 \cup J_0 \cup K_0$, $a_i^{(j)} = b_i - (\nu - \nu') a_i^{(0)}$ does not depend on j. As a consequence, $f_j = f_k$ for all $j, k \in L_0^{(0)}$. Let us use lemma 30-2. For all $i \in I_0 \cup J_0 \cup K_0$, if $(p_1, \cdots, p_N) \neq (0, \cdots, 0)$:

$$a^{(0)}_{(p_1,\cdots,p_i+1,\cdots,p_N)} = \left(\nu' a_i^{(0)} + \left(b_i - a_i^{(i)}\right) p_i + (\nu - \nu') a_i^{(0)} \sum_{j \in L_0^{(0)}} p_j\right) \frac{a^{(0)}_{(p_1,\cdots,p_N)}}{p_i + 1}.$$

For all $i \in L_0^{(0)}$, if $(p_1, \cdots, p_N) \neq (0, \cdots, 0)$:

$$a^{(0)}_{(p_1,\cdots,p_i+1,\cdots,p_N)} = \nu a_i^{(0)} \frac{a^{(0)}_{(p_1,\cdots,p_N)}}{p_i + 1}.$$

Let us fix $i \in I_0 \cup J_0 \cup K_0$ and $j \in L_0^{(0)}$. Then:

$$\begin{aligned}
a^{(0)}_{i,i} &= \frac{1}{2}\left(\nu' a_i^{(0)} + b_i - a_i^{(i)}\right) a_i^{(0)}, \\
a^{(0)}_{i,i,j} &= \frac{1}{2}\nu a_i^{(0)} a_j^{(0)} \left(\nu' a_i^{(0)} + b_i - a_i^{(i)}\right), \\
a^{(0)}_{i,j} &= \nu a_i^{(0)} a_j^{(0)}, \\
a^{(0)}_{i,i,j} &= \frac{1}{2}\nu a_i^{(0)} a_j^{(0)} \left(\nu' a_i^{(0)} + b_i - a_i^{(i)} + (\nu - \nu') a_i^{(0)}\right).
\end{aligned}$$

Identifying the two expressions of $a^{(0)}_{i,i,j}$, as $\nu \neq \nu'$ and $a^{(0)}_j \neq 0$, we obtain $\nu \left(a^{(0)}_i\right)^2 = 0$. Let us choose $i \in I_0 \cup J_0 \cup K_0$ such that $a^{(0)}_i \neq b_i$. Then $a^{(0)}_i \neq 0$ by (7.6) and thus $\nu = 0$, $\nu' \neq 0$. We then easily obtain that $0 \in J_1$. $\square$

Remarks.

(1) For all $i \in I_1 \cup J_1$, $b_i = 0$ by lemma 50 and by proposition 48, i cannot be the descendant of a vertex of level 0. The coefficients $a^{(i)}_j$ and $\tilde{a}^{(i)}_j$ are given by the following arrays:

$$a^{(i)}_j : \begin{array}{c|c|c} j \setminus i & I_1 & J_1 \\ \hline I_0 & a^{(i)}_j & (c^{(i)}_j - 1 - \beta_j)/\nu_i \\ \hline J_0 & a^{(i)}_j & (c^{(i)}_j - 1)/\nu_i \\ \hline K_0 & a^{(i)}_j & c^{(i)}_j/\nu_i \\ \hline L_0 & 0 & a^{(i)}_j \end{array} \qquad \tilde{a}^{(i)}_j : \begin{array}{c|c|c} j \setminus i & I_1 & J_1 \\ \hline I_0 & \nu_i a^{(i)}_j & c^{(i)}_j - 1 - \beta_j \\ \hline J_0 & \nu_i a^{(i)}_j & c^{(i)}_j - 1 \\ \hline K_0 & \nu_i a^{(i)}_j & c^{(i)}_j \\ \hline L_0 & 0 & 0 \end{array}$$

(2) It is possible to prove that the SDSE of theorem 59 are Hopf, as this was done for fundamental SDSE in section 7.3.

7.8 Vertices of level ≥ 2

Proposition 60. *Let (S) be a Hopf SDSE and let i be a vertex of (S) of level ≥ 2. Then i is an extension vertex.*

Proof. We denote by M the level of i. Then all the descendants of i are of level $\leq M-1$, so i is not a descendant of itself.

Let M be the level of i and let us assume that $M \geq 3$. Let j be a direct descendant of i, k be a direct descendant of j, l be a direct descendant of k. Then j has level $M-1$, k has level $M-2$, l has level $M-3$. Hence, all the paths from i to l have a length ≥ 3. The result is then deduced from lemma 40.

Let us now assume that i is of level 2. The direct descendants of i are of level 1, and the direct descendants of the direct descendants of i are of level 0. Hence, if $i \rightarrow j \rightarrow k$ in $G_{(S)}$, i, j and k are distinct. Up to a change of variables, we assume that if $i \rightarrow j \rightarrow k$ in $G_{(S)}$, then $a^{(i)}_j = a^{(j)}_k = 1$.

First step. Let us assume that there exists a direct descendant j of i, such that $a^{(i)}_{j,j} \neq 0$. Let us fix a direct descendant k of j. Then k has level 0, so k is not an ancestor of j; by lemma 50, $b_j = 0$. As the level of i is 2, there exists a scalar b such that if $n \geq 3$, $\lambda^{(i,j)}_n = b$. The level of j is 1, so there exists scalars c, d such that:

$$\lambda^{(j,k)}_n = \begin{cases} 1 \text{ if } n = 1, \\ c(n-1) + d \text{ if } n \geq 2. \end{cases}$$

Considering the levels, k is not a direct descendant of i, so $a^{(i)}_{j,k} = 0$. By lemma 31, for all $n \geq 2$, $\lambda^{(i,k)}_n = \lambda^{(j,k)}_{n-1}$. Moreover:

- By lemma 30-1 with $(i_1, i_2, i_3) = (i, j, k)$, $b = \lambda^{(i,j)}_3 = 2a^{(i)}_{j,j}$. So $b \neq 0$.
- By lemma 30-2, $a^{(i)}_{j,j,j} = \frac{b}{3} a^{(i)}_{j,j}$, as $a^{(j)}_j = 0$. So $a^{(i)}_{j,j,j} \neq 0$.
- $(c+d)a^{(i)}_{j,j} = \lambda^{(j,k)}_2 a = \lambda^{(i,k)}_3 a = a = 2a^{(i)}_{j,j}$. As $a^{(i)}_{j,j} \neq 0$, $c + d = 2$.
- $(2c+d)a^{(i)}_{j,j,j} = \lambda^{(j,k)}_3 a = \lambda^{(i,k)}_4 a = a = 3a^{(i)}_{j,j,j}$. As $a^{(i)}_{j,j,j} \neq 0$, $2c + d = 3$.

As a conclusion, $c = d = 1$. Hence, for any direct descendant of j, $\lambda^{(j,k)}_n = n$ for all $n \geq 1$. Lemma 30-2 implies that $f_j(0, \ldots, 0, h_k, 0, \ldots, 0) = (1 - h_k)^{-1}$, so for all $n \geq 0$, $a_{B_j(\bullet_k{}^{n-1})} = 1$.

Let now $l \in I$ which is not a direct descendant of j and let k be a direct descendant of j. For all $n \geq 1$:

$$\lambda^{(j,l)}_n = \lambda^{(j,l)}_n a_{B_j(\bullet_k{}^{n-1})} = a_{B_j(\bullet_k{}^{n-2}\,\mathbf{:}^{\,l}_{\,k})} = (n-1)a^{(k)}_l.$$

We proved that for any vertex l of $G_{(S)}$, for all $n \geq 1$:

$$\lambda^{(j,l)}_n = \begin{cases} n \text{ if } l \text{ is a direct descendant of } j, \\ a^{(k)}_l (n-1) \text{ if } l \text{ is not a direct descendant of } j, \end{cases}$$

where k is any direct descendant of j. This proves that j has level 0, so i has level 1: contradiction. So for all direct descendants j of i, $a^{(i)}_{j,j} = 0$.

Second step. Let j and j' be two different direct descendants of i. Let us use lemma 30-1 with $(i_1, i_2) = (i, j)$ and (i, j'). This gives:

$$\lambda^{(i,j)}_2 = 2a^{(i)}_{j,j} = a^{(i)}_{j,j'} = 0.$$

So all the terms of f_i of degree 2 are equal to 0. Finally:

$$f_i = 1 + \sum_{i \to j} a^{(i)}_j h_j,$$

so i is an extension vertex. □

8 Comments and examples of fundamental systems

8.1 Graph of a fundamental system

Figure 8.1 illustrates the structure of the graph of a fundamental system (with no dilatation). An arrow between two boxes means that there is an arrow from any vertex of the incoming box to any vertex of the outgoing box. A dotted edge

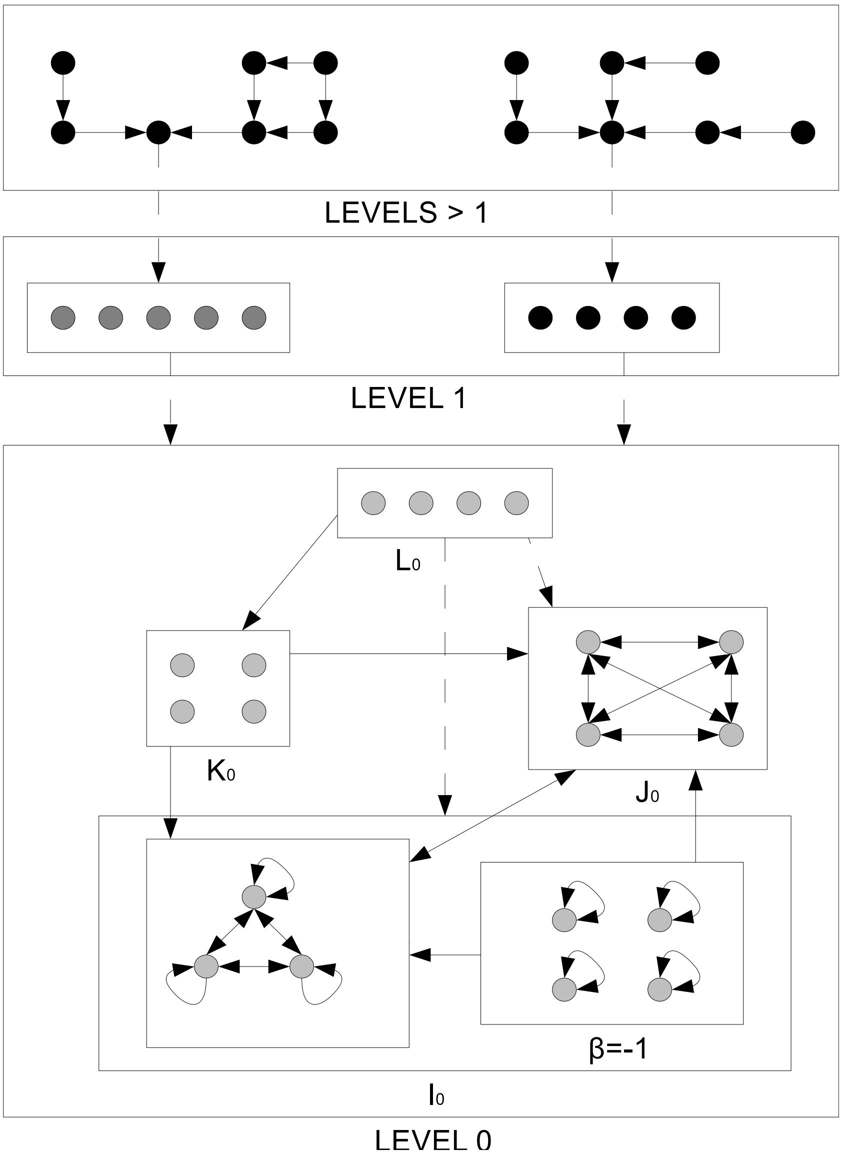


Figure 1. Structure of the graph of a fundamental SDSE.

between two boxes means that there may be an edge from a vertex of the incoming edge to a vertex of the outgoing box. The black vertices are extension vertices.

The subgraph of the vertices of I_0 is separated into two parts. One (the vertices with $\beta_i \neq -1$) is a complete graph with self-dependent vertices; the second one (the vertices with $\beta_i = -1$) is made of isolated self-dependent vertices. The subgraph of the vertices of J_0 is a complete graph with only non self-dependent vertices; the other boxes are made of isolated non self-dependent vertices.

8.2 Examples of fundamental systems

Here are examples of Dyson-Schwinger equations or systems found in the literature:

(1) The following equation is found in [26, 28, 33, 40]:

$$x = B\left(\frac{1}{1-x}\right),$$

where B is a 1-cocyle of a certain graded Hopf algebra. This generates Hopf subalgebra, by theorem 24.

(2) The following equation is found in [2, 29, 40]:

$$x = \sum_{n\geq 1} B_n((1+x)^{n+1}),$$

where for all $n \geq 1$, B_n is a 1-cocyle of certain graded Hopf algebra, homogeneous of degree n. If we truncate all the B_n with $n \neq n_0$, we obtain the equation:

$$x = B_{n_0}((1+x)^{n_0+1}),$$

which gives a Hopf subalgebra. It is possible to prove, working in the Hopf algebra of rooted trees decorated by $\mathbb{N}^*$, that the initial equation gives a Hopf subalgebra [18].

(3) The following system is the truncation of a system appearing in [29, 32]: $N \geq 2$, and for all $1 \leq n \leq N$,

$$x_n = B_n\left(\frac{(1+x_2)^n}{(1+x_1)^n}\right).$$

This is obtained from a fundamental system, with $I_0 = \{1,2\}$, $L_0 = \{3,\ldots,N\}$, $\beta_1 = 1$, $\beta_2 = -1/2$, $a_1^{(n)} = n$ and $a_2^{(n)} = n/2$ if $n \geq 3$, by a change of variables $h_1 \longrightarrow -h_1$ and $h_2 \longrightarrow 2h_2$.

(4) The following system appears in [44] and in the first section of this text:

$$\begin{cases} x_1 &=& B_1\left(\dfrac{(1+x_1)^3}{(1-x_2)(1-x_3)^2}\right), \\ x_2 &=& B_2\left(\dfrac{(1+x_1)^2}{(1-x_3)^2}\right), \\ x_3 &=& B_3\left(\dfrac{(1+x_1)^2}{(1-x_2)(1-x_3)}\right). \end{cases}$$

This is obtained from a fundamental system,with $I_0 = \{1,3\}$, $J_0 = \{2\}$, $\beta_1 = -1/3$, $\beta_3 = 1$, by a change of variables $h_1 \longrightarrow 3h_1$.

8.3 Dual pre-Lie algebras

Let us give a few results on the dual pre-Lie algebras. Let (S) be an extended fundamental SDSE. The pre-Lie algebra of primitive elements of the dual $\mathcal{H}^*_{(S)}$ has a basis $(e^i_n)_{i\in I, n\geq 1}$. As observed in 7.3, the pre-Lie product is given by:

$$e^i_m \circ e^j_n = \lambda^{(j,i)}_n e^j_{m+n}.$$

As a consequence, $\mathfrak{g}_i = Vect(e^i_k, k \geq 1)$ is a pre-Lie subalgebra. Three cases are possible:

(1) $i \in I_0$, with $\beta_i = -1$. Then $e^i_k \circ e^i_l = e^i_{k+l}$: $\mathfrak{g}_i$ is an associative, commutative algebra, isomorphic to the augmentation ideal of $K[X]$.

(2) $i \in K_0 \cup L_0 \cup I_1 \cup J_1$ or is an extension vertex. Then $e^i_k \circ e^i_l = 0$: $\mathfrak{g}_i$ is a trivial pre-Lie algebra.

(3) $i \in I_0$ with $\beta_i \neq -1$, or $i \in J_0$. Then $b_j \neq 0$, and $\mathfrak{g}_i$ is a Faà di Bruno pre-Lie algebra with parameter given by:

$$\lambda_i = \frac{a^{(i)}_i}{b_i} - 1 = \begin{cases} \frac{-\beta_i}{1+\beta_i} \text{ if } i \in I_0, \\ -1 \text{ if } i \in J_0. \end{cases}$$

Note that in both cases (1) and (2), the Lie algebra $\mathfrak{g}_i$ is abelian.

Let us describe the Lie algebra $\mathfrak{g}_{(S)}$ in two simple cases; see [19] for more general results.

Proposition 61. *Let (S) be a fundamental SDSE with no dilatation, such that $L_0 = \emptyset$. Two cases are possible:*

(1) *If $J_0 = \emptyset$ and for all $i \in I_0$, $\beta_i = -1$, then the Lie algebra $\mathfrak{g}_{(S)}$ is abelian.*

(2) *If $J_0 \neq \emptyset$ or if there exists $i \in I_0$, such that $\beta_i \neq -1$, then the Lie algebra $\mathfrak{g}_{(S)}$ can be decomposed in a semi-direct product:*

$$\mathfrak{g}_{(S)} = (M_1 \oplus \ldots \oplus M_k) \rtimes \mathfrak{g}_0,$$

where:

- *$\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}_{(S)}$, isomorphic to the Faà di Bruno Lie algebra, with basis $(f^0_n)_{n\geq 1}$ such that for all $m,n \geq 1$:*

$$[f^0_m, f^0_n] = (n-m) f^0_{n+m}.$$

- *For all $1 \leq i \leq k$, M_i is an abelian Lie subalgebra of $\mathfrak{g}_{(S)}$, with basis $(f^i_n)_{n\geq 1}$.*
- *For all $1 \leq i \leq k$, M_i is a left $\mathfrak{g}_0$-module in the following way:*

$$f^0_m . f^i_n = n f^i_{m+n}.$$

Proof. We use here the notations of section 7.2.

1. In this case, for all $i \in I$, $b_i = 0$, so the pre-Lie product is given by:

$$e_m^i \circ e_n^j = a_i^{(j)} e_{m+n}^j.$$

Moreover, the following array gives the coefficients $a_j^{(i)}$:

$j \setminus i$	I_0	K_0
I_0	$\delta_{i,j}$	0
K_0	0	0

Hence, the pre-Lie product is commutative. Consequently, the associated Lie bracket is abelian.

2. In this case, there exists $0 \in I$, such that $b_0 \neq 0$. Then, for all $i, j \in I$, $a_j^{(i)} = b_j + \delta_{i,j} c_j$. For all $n \geq 1$, we put:

$$\begin{cases} f_n^0 = \dfrac{1}{b_0} e_n^0, \\ f_n^i = e_n^i - \dfrac{b_1}{b_0} e_n^0 \text{ if } i \neq 0. \end{cases}$$

The family $(f_n^i)_{i \in I, n \geq 1}$ is a basis of $\mathfrak{g}_{(S)}$. For all $i \in I$, we put:

$$c_i = \begin{cases} -\beta_i \text{ if } i \in I_0, \\ -1 \text{ if } i \in J_0, \\ 0 \text{ if } i \in K_0. \end{cases}$$

Direct computations show that if $i, j \neq 0$:

$$\begin{array}{rcl} f_m^0 \circ f_n^0 & = & \left(n + \dfrac{c_0}{b_0}\right) f_{m+n}^0, \\ f_m^i \circ f_n^0 & = & -\dfrac{b_i}{b_0} c_0 f_{m+n}^0, \\ f_m^0 \circ f_n^j & = & n f_{m+n}^j - \dfrac{b_j}{b_0} c_0 f_{m+n}^0, \\ f_m^i \circ f_n^j & = & \delta_{i,j} x_i e_{m+n}^j - \dfrac{b_i b_j c_0}{b_0^2} e_{m+n}^0. \end{array}$$

So $Vect(f_n^0 \mid n \geq 1)$ is a Faà di Bruno pre-Lie algebra, with parameter:

$$\frac{c_0}{b_0} = \begin{cases} \dfrac{-\beta_0}{1 + \beta_0} \text{ if } 0 \in I_0, \\ -1 \text{ if } 0 \in J_0. \end{cases}$$

Moreover, we obtain, if $i, j \neq I$:

$$[f_m^0, f_n^0] = (n - m) f_{m+n}^0, \qquad [f_m^0, f_n^j] = n f_{m+n}^j, \qquad [f_m^i, f_n^j] = 0,$$

which is precisely the announced result. $\square$

Proposition 62. *Let (S) be a quasi-cyclic SDSE. The pre-Lie $\mathfrak{g}_{(S)}$ admits a* *basis* $(e^i_n)_{i\in I, n\geq 1}$ *such that:*

$$e^i_m \circ e^j_n = \begin{cases} e^j_{m+n} \text{ if there exists a path from } j \text{ to } i \text{ in } G_{(S)} \text{ of length } n, \\ 0 \text{ if not.} \end{cases}$$

This pre-Lie product is associative.

Proof. Up to a change of variables, for all $i \in I$, we have:

$$x_i = 1 + \sum_{i\to j} B_i(x_j).$$

Hence, for all $i \in I$, for all $n \geq 1$:

$$x_i(n) = \sum_{i\to i_2\to\ldots\to i_n} \begin{array}{c} {\scriptstyle i_n} \\ {\scriptstyle i_{n-1}} \\ \vdots \\ {\scriptstyle i_2} \\ {\scriptstyle i} \end{array}.$$

So:

$$\Delta(x_i(n)) = \sum_{i\to i_2\to\ldots\to i_n} \sum_{k=0}^{n} \begin{array}{c} {\scriptstyle i_n} \\ {\scriptstyle i_{n-1}} \\ \vdots \\ {\scriptstyle i_{k+2}} \\ {\scriptstyle i_{k+1}} \end{array} \otimes \begin{array}{c} {\scriptstyle i_k} \\ {\scriptstyle i_{k-1}} \\ \vdots \\ {\scriptstyle i_2} \\ {\scriptstyle i} \end{array} = \sum_{k=0}^{n} \sum_{i_k} x_{i_k}(n-k) \otimes x_i(k),$$

where the last sum is over all i_k such that there exists a path of length k from i to i_k in $G_{(S)}$. Let $(e^i_n)_{i\in I,n\geq 1}$ be the dual basis of the basis $(x^i(n))_{i\in I,n\geq 1}$; this is a basis of $\mathfrak{g}_{(S)}$ and the formula for the coproduct of $x_i(n)$ implies the formula for the pre-Lie product of two elements of this basis. Moreover, for all $i, j, k \in I$, for all $m, n, p \geq 1$:

- $(e^i_m \circ e^j_n) \circ e^k_p = e^k_{m+n+p}$ if there exists a path from k to j of length p and a path from j to i of length n, and 0 otherwise.
- $e^i_m \circ (e^j_n \circ e^k_p) = e^k_{m+n+p}$ if there exists a path from k to j of length p and a path from k to i of length $n + p$, and 0 otherwise.

As (S) is quasi-cyclic, there exists a partition $I = I_{\overline{1}} \sqcup \ldots \sqcup I_{\overline{N}}$ such that there is an edge from i to j in $G_{(S)}$ if, and only if, there exists $\overline{a} \in \mathbb{Z}/N\mathbb{Z}$ such that $i \in I_{\overline{a}}$ and $j \in I_{\overline{a+1}}$. Let $\overline{a}, \overline{b}, \overline{c} \in \mathbb{Z}/N\mathbb{Z}$ such that $i \in I_{\overline{a}}$, $j \in I_{\overline{b}}$ and $k \in I_{\overline{c}}$. Then:

- $(e^i_m \circ e^j_n) \circ e^k_p = e^k_{m+n+p}$ if $\overline{c+p} = \overline{b}$ and $\overline{b+n} = \overline{a}$, and 0 otherwise.
- $e^i_m \circ (e^j_n \circ e^k_p) = e^k_{m+n+p}$ if $\overline{c+p} = \overline{b}$ and $\overline{c+n+p} = \overline{a}$, and 0 otherwise.

Consequently, $(e^i_m \circ e^j_n) \circ e^k_p = e^i_m \circ (e^j_n \circ e^k_p)$ and the product $\circ$ is associative. □

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