A DIVISIBILITY RESULT ON COMBINATORICS OF GENERALIZED BRAIDS

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ABSTRACT. For every finite Coxeter group Γ, each positive braid in the corresponding braid group admits a unique decomposition as a finite sequence of elements of Γ, the so-called Garside-normal form. The study of the associated adjacency matrix $\mathrm{Adj}(\Gamma)$ allows to count the number of Garside-normal form of a given length. In this paper we prove that the characteristic polynomial of $\mathrm{Adj}(B_n)$ divides the one of $\mathrm{Adj}(B_{n+1})$. The key point is the use of a Hopf algebra based on signed permutations. A similar result was already known for the type A. We observe that this does not hold for type D. The other Coxeter types (I, E, F and H) are also studied.

INTRODUCTION

Let S be a set. A *Coxeter matrix* on S is a symmetric matrix $M = (m_{s,t})$ whose entries are in $\mathbb{N} \cup \{+\infty\}$ and such that $m_{s,t} = 1$ if, and only if, $s = t$. A Coxeter matrix is usually represented by a labelled *Coxeter graph* Γ whose vertices are the elements of S; there is an edge between s and t labelled by $m_{s,t}$ if, and only if, $m_{s,t} \geqslant 3$. From such a graph Γ, we define a group W_Γ by the presentation:

$$W_\Gamma = \left\langle S \;\middle|\; \begin{array}{cl} s^2 = 1 & \text{for } s \in S \\ \mathrm{prod}(s,t;m_{s,t}) = \mathrm{prod}(t,s;m_{t,s}) & \text{for } s,t \in S \text{ and } m_{s,t} \neq +\infty \end{array} \right\rangle.$$

where $\mathrm{prod}(s,t;m_{s,t})$ is the product $s\,t\,s...$ with $m_{s,t}$ terms. The pair (W_Γ, S) is called a *Coxeter system*, and W_Γ is the *Coxeter group* of type Γ. Note that two elements s and t of S commute in W_Γ if, and only if, s and t are not connected in Γ. Denoting by $\Gamma_1, ..., \Gamma_k$ the connected components of Γ, we obtain that W_Γ is the direct product $W_{\Gamma_1} \times ... \times W_{\Gamma_k}$. The Coxeter group W_Γ is said to be *irreducible* if the Coxeter graph Γ is connected. We say that a Coxeter graph is *spherical* if the corresponding group W_Γ is finite. There are four infinite families of connected spherical Coxeter graphs: A_n ($n \geqslant 1$), B_n ($n \geqslant 2$), D_n ($n \geqslant 4$), $I_2(p)$ ($p \geqslant 5$), and six exceptional graphs E_6, E_7, E_8, F_4, H_3 and H_4. For $\Gamma = A_n$, the group W_Γ is the symmetric group $\mathfrak{S}_{n+1}$.

For a Coxeter graph Γ, we define the *braid group* $B(W_\Gamma)$ by the presentation:

$$B(W_\Gamma) = \langle S \mid \mathrm{prod}(s,t;m_{s,t}) = \mathrm{prod}(t,s;m_{t,s}) \text{ for } s,t \in S \text{ and } m_{s,t} \neq +\infty \rangle.$$

and the positive braid monoid to be the monoid presented by:

$$B^+(W_\Gamma) = \langle S \mid \mathrm{prod}(s,t;m_{s,t}) = \mathrm{prod}(t,s;m_{t,s}) \text{ for } s,t \in S \text{ and } m_{s,t} \neq +\infty \rangle^+.$$

The groups $B(W_\Gamma)$ are known as Artin-Tits groups; they have been introduced in [4, 2] and in [10] for spherical type. The embedding of the monoid $B^+(W_\Gamma)$ in

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the corresponding group $B(W_\Gamma)$ was established by L. Paris in [14]. For $\Gamma = A_n$, the braid group $B(W_{A_n})$ is the Artin braid group B_n and $B^+(W_{A_n})$ is the monoid of positive Artin braids B_n^+.

We now suppose that Γ is a spherical Coxeter graph. The Garside normal form allows us to express each braid β of $B^+(W_\Gamma)$ as a unique finite sequence of elements of W_Γ. This defines an injection Gar form $B^+(W_\Gamma)$ to $W_\Gamma^{(\mathbb{N})}$. The Garside length of a braid $\beta \in B^+(W_\Gamma)$ is the length of the finite sequence $\mathrm{Gar}(\beta)$. If, for all $\ell \in \mathbb{N}$, we denote by $B^\ell(W_\Gamma)$ the set of braids whose Garside length is ℓ, the map Gar defines a bijection between $B^\ell(W_\Gamma)$ and $\mathrm{Gar}(B^+(W_\Gamma)) \cap W_\Gamma^\ell$.

A sequence $(s,t) \in W_\Gamma^2$ is said *normal* if (s,t) belongs to $B^2(W_\Gamma)$. From a local characterization of the Garside normal form, for $\ell \geqslant 2$ the sequence $(w_1, ..., w_\ell)$ of W_Γ^ℓ belongs to $\mathrm{Gar}(B^+(W_\Gamma))$ if, and only if, (w_i, w_{i+1}) is normal for all $i = 1, ..., \ell - 1$. Roughly speaking, in order to recognize the elements of $\mathrm{Gar}(B^+(W_\Gamma))$ among thus of $W_\Gamma^{(\mathbb{N})}$ it is enough to recognize the elements of $B^2(W_\Gamma)$ among thus of W_Γ^2.

We define a square matrix $\mathrm{Adj}_\Gamma = (a_{u,v})$, indexed by the elements of W_Γ, by:

$$a_{u,v} = \begin{cases} 1 & \text{if } (u,v) \text{ is normal,} \\ 0 & \text{otherwise.} \end{cases}$$

For $\ell \geqslant 1$, the number of positive braids whose Garside length is ℓ is then:

$$\mathrm{card}(B^\ell(W_\Gamma)) = {}^tX\, \mathrm{Adj}_\Gamma^{\ell-1}\, X, \qquad \text{where } X_u = \begin{cases} 0 & \text{if } u = 1_{W_\Gamma}, \\ 1 & \text{otherwise.} \end{cases}$$

Thus the eigenvalues of Adj_Γ give informations on the growth of $\mathrm{card}(B^\ell(W_\Gamma))$ relatively to ℓ.

Assume that Γ is a connected spherical type graph of one of the infinite family A_n, B_n or D_n. We define χ_n^A, χ_n^B and χ_n^D to be the characteristic polynomials of $\mathrm{Adj}_{A_n}, \mathrm{Adj}_{B_n}$ and Adj_{D_n} respectively. In [3], P. Dehornoy conjectures that χ_n^A is a divisor of χ_{n+1}^A. This conjecture was proved by F. Hivert, J.C. Novelli and J.Y. Thibon in [9]. To prove that χ_n^A divides χ_{n+1}^A, they see Adj_{A_n} as the matrix of an endomorphism Φ_n^A of the Malvenuto-Reutenauer Hopf algebra **FQSym** [11, 6]. We recall that **FQSym** is a connected graded Hopf algebra whose a basis in degree n is indexed by the element of $\mathfrak{S}_n \simeq W_{A_{n-1}}$. The authors of [9] then construct a surjective derivation ∂ of degree -1 satisfying $\partial \circ \Phi_n^A = \Phi_{n-1}^A \circ \partial$, and eventually prove the divisibility result. A combinatorial description of Adj_{A_n} can be found in [3] and in [7], with a more algorithmic approach.

The aim of this paper is to prove that the polynomial χ_n^B divides the polynomial χ_{n+1}^B. The first step is to construct a Hopf algebra **BFQSym** from W_{B_n} which plays the same role for the type B as **FQSym** for the type A; this is a special case of a general construction for families of wreath products, see [13]. We then interpret Adj_{B_n} as the matrix of an endomorphism Φ_n^B of the Hopf algebra **BFQSym**. The next step is to construct a derivation ∂ on **BFQSym** satisfying the relation $\partial \circ \Phi_n^B = \Phi_{n-1}^B \circ \partial$ and establish the divisibility result. Unfortunately there is no such a result for the Coxeter type D_n: the polynomial χ_4^D is not a divisor of χ_5^D and of χ_6^D neither.

The paper is divided as follows. The first section is an introduction to Coxeter groups and braid monoids of type B. The adjacency matrix Adj_{B_n} is introduced here. Section 2 is devoted to the Hopf algebra **BFQSym**. In Section 3, we prove the divisibility result using a derivation on the Hopf algebra **BFQSym**. Conclusions and characteristic polynomials of type D, I, E, F and H are in the last section.

1. Coxeter groups and braid monoids of type B.

The following notational convention will be useful in the sequel: if $p \leqslant q$ in $\mathbb{Z}$, we denote by $[p,q]$ the subset $\{p, ..., q\}$ of $\mathbb{Z}$.

1.1. Signed permutation groups.

Definition 1.1. A *signed permutation* of rank n is a permutation σ of $[-n,n]$ satisfying $\sigma(-i) = -\sigma(i)$ for all $i \in [-n,n]$. We denote by $\mathfrak{S}_n^{\pm}$ the group of signed permutations.

In the literature, the group of signed permutations $\mathfrak{S}_n^{\pm}$ is also known as the hyperoctahedral group of rank n. We note that, by very definition, all signed permutations send 0 to itself. Also by definition, a signed permutation is entirely defined by its values on $[1,n]$. In the sequel, a signed permutation σ of rank n will consequently be written as $(\sigma(1), ..., \sigma(n))$. This notation is often called the *window* notation of the permutation σ.

Definition 1.2. For σ a signed permutation of $\mathfrak{S}_n^{\pm}$, the *word* of σ, denoted by $w(\sigma)$ is the word $\sigma(1) \dots \sigma(n)$ on the alphabet $[-n,n] \setminus \{0\}$.

Example 1.3. Signed permutations of rank 2 are:

$$\mathfrak{S}_2^{\pm} = \{(1,2), (-1,2), (1,-2), (-1,-2), (2,1), (-2,1), (2,-1), (-2,-1)\}.$$

One remarks that for any signed permutation σ of $\mathfrak{S}_n^{\pm}$, the map $|\sigma|$ defined on $[1,n]$ by $|\sigma|(i) = |\sigma(i)|$ is a permutation of $\mathfrak{S}_n$.

Among the signed permutations, we isolate a generating family s_i's which eventually equips $\mathfrak{S}_n^{\pm}$ with a Coxeter structure.

Definition 1.4. Let $n \geqslant 1$. We define a permutation $s_i^{(n)}$ of $\mathfrak{S}_n^{\pm}$ by $s_0^{(n)} = (-1, 2, ..., n)$ and $s_i^{(n)} = (1, ..., i+1, i, ..., n)$ for $i \in [1,n]$.

From the natural injection of $\mathfrak{S}_n^{\pm}$ to $\mathfrak{S}_{n+1}^{\pm}$ we can write s_i instead of $s_i^{(n)}$ without ambiguity. The following proposition is a direct consequence of the previous definition.

Proposition 1.5. *For all $n \geqslant 1$, the permutations $S_n = \{s_0, ..., s_n\}$ are subject to the relations:*

- *$R_1(S_n)$: $s_i^2 = 1$ for all $i \in [0,n]$;*
- *$R_2(S_n)$: $s_0\, s_1\, s_0\, s_1 = s_1\, s_0\, s_1\, s_0$;*
- *$R_3(S_n)$: $s_i\, s_j = s_j\, s_i$ for $i, j \in [0,n]$ with $|i-j| \geqslant 2$;*
- *$R_4(S_n)$: $s_i\, s_j\, s_i = s_j\, s_i\, s_j$ for $1 \leqslant i, j \leqslant n$ with $|i-j| = 1$.*

Each signed permutation σ of $\mathfrak{S}_n^{\pm}$ can be represented as a product of the s_i's. Some of these representations are shorter than the others. The minimal numbers of s_i's required is then a parameter of the signed permutation.

Definition 1.6. Let σ be a signed permutation of $\mathfrak{S}_n^{\pm}$. The *length* of σ denoted by $\ell(\sigma)$ is the minimal integer k such that there exists $x_1, ..., x_k$ in S_n satisfying $\sigma = x_1 \cdot ... \cdot x_k$. An expression of σ in terms of S_n is said to be *reduced* if it has length $\ell(\sigma)$.

Example 1.7. Permutations of $\mathfrak{S}_3^{\pm}$ admit the following decompositions in terms of permutations in s_i's:

$$\begin{array}{rcl@{\qquad\qquad}rcl}
(1,2) & = & \emptyset, & (2,1) & = & s_1, \\
(-1,2) & = & s_0, & (-2,1) & = & s_1 \cdot s_0, \\
(1,-2) & = & s_1 \cdot s_0 \cdot s_1, & (2,-1) & = & s_0 \cdot s_1, \\
(-1,-2) & = & s_0 \cdot s_1 \cdot s_0 \cdot s_1, & (-2,-1) & = & s_0 \cdot s_1 \cdot s_0.
\end{array}$$

Each given expression is reduced. In particular, the length of $(-1,-2)$ is 4, while the length of $(-2,1)$ is 2.

Among all the signed permutations of $\mathfrak{S}_n^{\pm}$, there is a unique one with maximal length called *Coxeter element* of $\mathfrak{S}_n^{\pm}$ and denoted by w_n^B:

$$w_n^B = (-1, ..., -n).$$

A presentation of $\mathfrak{S}_n^{\pm}$ is given by relations R_1, R_2, R_3 and R_4 on S_n. More precisely the group of signed permutations $\mathfrak{S}_n^{\pm}$ is isomorphic to the Coxeter group W_{B_n} with generator set S_n and relations given by the following graph:

$$B_n : \overset{s_0}{\bullet} \underset{4}{\text{———}} \overset{s_1}{\bullet} \underset{3}{\text{———}} \overset{s_2}{\bullet} \underset{3}{\text{———}} \overset{s_3}{\bullet} \cdots\cdots \overset{s_{n-2}}{\bullet} \underset{3}{\text{———}} \overset{s_{n-1}}{\bullet}$$

For more details, the reader can consult [1]. Thanks to this isomorphism, we identify the group $\mathfrak{S}_n^{\pm}$ with W_{B_n} for $n \geqslant 1$.

1.2. **Braid monoids of type B.**

Putting $\Theta_n^B = \{\theta_0, ..., \theta_{n-1}\}$, the braid monoid of type B and of rank n is the monoid BB_n^+ whose presentation is:

$$BB_n^+ = B^+\left(\mathfrak{S}_n^{\pm}\right) = B^+\left(W_{B_n}\right) = \left\langle \Theta_n^B \,|\, R_2\left(\Theta_n^B\right),\ R_3\left(\Theta_n^B\right) \text{ and } R_4\left(\Theta_n^B\right)\right\rangle^+.$$

The group of signed permutations $\mathfrak{S}_n^{\pm}$ is a quotient of BB_n^+ by $\theta_i^2 = 1$. We denote by π the natural surjective homomorphism defined by:

$$\begin{array}{rcl}
\pi : BB_n^+ & \to & \mathfrak{S}_n^{\pm} \\
\theta_i & \mapsto & s_i.
\end{array}$$

Lemma 1.8 (Matsumoto Lemma [12])**.** Let u and v be two reduced expressions of a same signed permutation. We can rewrite u into v using only relations of type R_2, R_3 and R_4; in other words, relations $s_i^2 = 1$ of R_1 can be avoided.

The previous Lemma is a not so direct consequence of the *exchange Lemma*; see [5] for more details.

Definition 1.9. For σ in $\mathfrak{S}_n^{\pm}$ we define $r(\sigma)$ to be the braid $\theta_{i_1} ... \theta_{i_k}$ where $s_{i_1} ... s_{i_k}$ is a reduced expression of σ.

Since relations R_2, R_3 and R_4 are also verified by the θ_i's, the braid $r(\sigma)$ is well defined for every signed permutation σ.

Proposition 1.10. *For $n \geqslant 0$, the map $r : \mathfrak{S}_n^{\pm} \to BB_n^+$ is injective.*

This is a direct consequence of the definition of r.

Definition 1.11. A braid x of BB_n^+ is *simple* if it belongs to $r\left(\mathfrak{S}_n^{\pm}\right)$. We denote by SB_n the set of all simple braids. The element $\Delta_n^B = r(w_n^B)$ is the *Garside braid* of BB_n^+.

In particular, there are $2^n n!$ simple braids in BB_n^+. Simple braids are used to describe the structure of the braid monoid BB_n^+ from the one of the Coxeter group $\mathfrak{S}_n^{\pm} \simeq W_{B_n}$.

Example 1.12. Using Example 1.7, we obtain that the simple braids of BB_2^+ are:

$$SB_2 = \{1, \theta_0, \theta_1, \theta_0\theta_1, \theta_1\theta_0, \theta_1\theta_0\theta_1, \theta_0\theta_1\theta_0, \theta_0\theta_1\theta_0\theta_1\}.$$

The Coxeter element of SB_2 is $w_2^B = (-1, -2)$, whose a decomposition in terms of the s_i's is $w_2^B = s_0\, s_1\, s_0\, s_1$, and so $\Delta_2^B = \theta_0\, \theta_1\, \theta_0\, \theta_1$.

Definition 1.13. Let x and y be two braids of BB_n^+. We say that x left divides y or that y is a right multiple of x if there exists $z \in BB_n^+$ satisfying $x \cdot z = y$.

The Coxeter group $\mathfrak{S}_n^{\pm}$ is equipped with a lattice structure via the relation $\preccurlyeq$ defined by $\sigma \preccurlyeq \tau$ iff $\ell(\tau) = \ell(\sigma) + \ell(\sigma^{-1}\tau)$. Equipped with the left divisibility, the set SB_n is a lattice which is isomorphic to $(\mathfrak{S}_n^{\pm}, \preccurlyeq)$. The maximal element of $\mathfrak{S}_n^{\pm}$ is w_n^B, while the one of SB_n is Δ_n^B. There is also an ordering $\succcurlyeq$ on $\mathfrak{S}_n^{\pm}$ such that SB_n equipped with the right divisibility is a lattice, isomorphic to $(\mathfrak{S}_n^{\pm}, \succcurlyeq)$. In particular, simple elements of BB_n^+ are exactly the left (or the right) divisors of Δ_n^B.

Notation 1.14. For x and y two braids of BB_n^+, we denote by $x \wedge y$ the left great common divisor of x and y.

1.3. **Left Garside normal form.** Let x be a non trivial braid of BB_n^+. The left great common divisor x_1 of x and Δ_n^B is a simple element. Since one of the braids θ_i's (which are simple) left divides x, the braid x_1 is non trivial. We can then write x as $x = x_1 \cdot x'$, with $x' \in BB_n^+$. If the braid x' is trivial, we are done; else, we restart the process, replacing x by x'. As the length of the involved braid strictly decrease, we eventually obtain the trivial braid.

Proposition 1.15. *Let $x \in BB_n^+$ be a non trivial braid. There exists a unique integer $k \geqslant 1$ and unique non trivial simple braids $x_1, ..., x_k$ satisfying:*
(i) $x = x_1 \cdot ... \cdot x_k$;
(ii) $x_i = (x_i \cdot ... \cdot x_k) \wedge \Delta_n^B$ for $i \in [1, k-1]$.
The expression $x_1 \cdot ... \cdot x_k$ is called the left Garside normal form *of the braid x.*

The proof of the previous Proposition is a classic Garside result and can be found in [2]. Note that in Proposition 1.15, we exclude the trivial braid from the decomposition. This must be done in order to have unicity for the integer k. Indeed, one can transform a decomposition $x = x_1 \cdot ... \cdot x_k$ to $x = x_1 \cdot ... \cdot x_k \cdot 1 \cdot ... \cdot 1$ that satisfy conditions (i) and (ii). The price to pay is that the trivial braid must be treated separately.

Definition 1.16. The integer k introduced in the previous proposition is the *Garside length* of the braid x. By convention the *Garside length* of the trivial braid is 0, corresponding to the empty product of simple braids.

Example 1.17. Let $x = \theta_1\theta_1\theta_0\theta_1\theta_0\theta_1$ be a braid of BB_2^+. The maximal prefix of the given expression of x that is a word of a simple braid is θ_1. However, using relation R_2 on the underlined factor of x we obtain:

$$x = \theta_1\underline{\theta_1\theta_0\theta_1\theta_0}\theta_0 = \theta_1\underline{\theta_0\theta_1\theta_0\theta_1}\theta_0.$$

The braid $y = \theta_1\theta_0\theta_1\theta_0$ is then a left divisor of x. As y is equal to the simple braid Δ_2^B, we have $x_1 = y$ and then $x = x_1 \cdot \theta_1\theta_0$. Since $y = \theta_1\,\theta_0$ is simple, we have $x_2 = \theta_1\theta_0$. We finally obtain:

$$x = x_1 \cdot x_0 = \theta_1\theta_0\theta_1\theta_0 \cdot \theta_1\theta_0,$$

establishing that the Garside length of the braid x is 2.

Condition (ii) of Proposition 1.15 is difficult to check in practice. However it can replaced by a local condition, involving only two consecutive terms of the left Garside normal form. More precisely, (ii) is equivalent to:
(ii') the pair (x_i, x_{i+1}) is normal for $i \in [1, k-1]$.

Definition 1.18. A pair $(x, y) \in SB_n^2$ of simple braids is said to be *normal* if x is the left gcd of $x \cdot y$ and the Garside braid Δ_n^B.

Since the number of simple elements is finite, there is a finite number of braids of a given Garside length.

Definition 1.19. For positive integers n and d, we denote by $b_{n,d}$ the number of braids of BB_n^+ whose Garside length is d.

In order to determine $b_{n,d}$, we will switch to the Coxeter context.

1.4. **Combinatorics of normal sequences.** We recall that each simple braid of SB_n can be uniquely expressed as $r(\sigma)$, where σ is a signed permutation. From the definition of normal pair of braids, we obtain a notion of normal pair of signed permutations. We say that a pair (σ, τ) of $\mathfrak{S}_n^{\pm}$ is normal if $(r(\sigma), r(\tau))$ is. Thus Proposition 1.15 can be reformulated as follow:

Proposition 1.20. *For $n \geqslant 2$ and $x \in BB_n^+$ a non trivial braid, there exists a unique integer $k \geqslant 1$ and non trivial signed permutations $\sigma_1, ..., \sigma_k$ of $\mathfrak{S}_n^{\pm}$ satisfying the following relations:*
(i) $x = r(\sigma_1) \cdot ... \cdot r(\sigma_k)$;
(ii) the pair (σ_i, σ_{i+1}) is normal for $i \in [1, k-1]$.

Instead of counting braids of Garside length d, we will count sequences of signed permutations of length d which are normal.

Definition 1.21. A sequence $(\sigma_1, ..., \sigma_k)$ of signed permutations is *normal* if the pair (σ_i, σ_{i+1}) is normal for $i \in [1, k-1]$.

The number $b_{n,d}$ is then the number of length d normal sequences of non trivial signed permutations of $\mathfrak{S}_n^{\pm}$. We now look for a criterion for a pair to be normal in the Coxeter context.

Definition 1.22. The *descent set* of a permutation $\sigma \in \mathfrak{S}_n^{\pm}$ is defined by

$$\mathrm{Des}\,(\sigma) = \{i \in [0, n-1] \mid \ell(\sigma\, s_i) < \ell(\sigma)\}.$$

Example 1.23. Let us compute the descent set of $\sigma = (-2, 1)$. A reduced expression of σ is $s_1\, s_0$ and so σ has length 2. The expression $\sigma\, s_0 = s_1\, s_0\, s_0$ reduces to s_1, which is of length 1. The expression $\sigma\, s_1 = s_1\, s_0\, s_1$ is reduced, and so $\sigma\, s_1$ has length 3. Therefore the descents set of σ is $\mathrm{Des}\,(\sigma) = \{0\}$.

Let us start with two intermediate results.

Lemma 1.24. Let σ be a signed permutation of $\mathfrak{S}_n^{\pm}$, and $i \in [0, n-1]$. The braid $r(\sigma)\theta_i$ is simple if, and only if, $i \notin \mathrm{Des}\,(\sigma)$.

Proof. Let σ be a signed permutation of $\mathfrak{S}_n^{\pm}$ and $x_1 \ldots x_{\ell(\sigma)}$ one of its reduced expressions. If $i \notin \mathrm{Des}\,(\sigma)$ then $\ell(\sigma s_i) > \ell(\sigma)$ holds. Hence $x_1 \ldots x_{\ell(\sigma)}\, s_i$ is a reduced expression of σs_i. It follows $r(\sigma s_i) = r(x_1 \ldots x_{\ell(\sigma)}) r(s_i) = r(\sigma)\theta_i$, and so $r(\sigma)\theta_i$ is simple. Conversely, let us assume that $r(\sigma)\theta_i$ is simple. There exists a signed permutation τ in $\mathfrak{S}_n^{\pm}$ of length $\ell(\sigma) + 1$ satisfying $\pi(r(\sigma)\theta_i) = \tau$. As $\pi(r(\sigma)\theta_i)$ is equal to σs_i, we must have $\ell(\sigma s_i) = \ell(\sigma) + 1$ and so $i \notin \mathrm{Des}\,(\sigma)$. ☐

Lemma 1.25. For τ a signed permutation of $\mathfrak{S}_n^{\pm}$ and $i \in [0, n-1]$, the braids θ_i is a left divisor of $r(\tau)$ if, and only if, $i \in \mathrm{Des}\left(\tau^{-1}\right)$.

Proof. The braids θ_i and $r(\tau)$ are simple. Thanks to the lattice isomorphism between SB_n equipped with the left divisibility and $(\mathfrak{S}_n^{\pm}, \preccurlyeq)$, the braid θ_i is a left divisor of $r(\tau)$ if and only $s_i \preccurlyeq \tau$ holds, and so, by definition of $\preccurlyeq$ if, and only if, $\ell(\tau) = \ell(s_i) + \ell(s_i\tau)$, which is equivalent to $\ell(s_i\tau) < \ell(\tau)$. As the length of a permutation is the length of its inverse, we have $\ell(s_i\tau) < \ell(\tau) \Leftrightarrow \ell(\tau^{-1}s_i) < \ell(\tau^{-1})$ which is equivalent to $i \in \mathrm{Des}\left(\tau^{-1}\right)$. ☐

Proposition 1.26. *A pair (σ, τ) of signed permutations of $\mathfrak{S}_n^{\pm}$ is normal if, and only if, the inclusion $\mathrm{Des}\left(\tau^{-1}\right) \subseteq \mathrm{Des}\,(\sigma)$ holds.*

Proof. Let σ and τ be two signed permutations of $\mathfrak{S}_n^{\pm}$. Assume that (σ, τ) is not normal. Then, there exists a simple braid z which is a left divisor of $r(\sigma)r(\tau)$ and greater than $r(\sigma)$, i.e., $r(\sigma)$ left divides z. Hence, there exists $i \in [0, n]$, such that $r(\sigma)\theta_i$ is simple, and θ_i left divides $r(\tau)$. Denoting by x the simple braid $r(\sigma)\theta_i$ and by y the positive braid $\theta_i{}^{-1}r(\tau)$, we obtain $r(\sigma)r(\tau) = x\,y$.

By Lemma 1.24, the integer i does not belong to $\mathrm{Des}\,(\sigma)$, but in $\mathrm{Des}\left(\tau^{-1}\right)$. To summarize, we have proved that the pair (σ, τ) is not normal if there exists $i \in [0, n]$ such that $i \notin \mathrm{Des}\,(\sigma)$ and $i \in \mathrm{Des}\left(\tau^{-1}\right)$. The converse implication is immediate. Therefore (σ, τ) is normal if, and only if, for all $i \in [0, n]$, we have either $i \in \mathrm{Des}\,(\sigma)$ or $i \notin \mathrm{Des}\left(\tau^{-1}\right)$. Since i is or is not in $\mathrm{Des}\left(\tau^{-1}\right)$, we obtain that the pair (σ, τ) is normal if, and only if, $\mathrm{Des}\left(\tau^{-1}\right) \subseteq \mathrm{Des}\,(\sigma)$ holds, as expected. ☐

The descent set of a signed permutation σ can be defined directly from the window notation of σ.

Proposition 1.27 (Proposition 8.1.2 of [1])**.** *For $n \geqslant 1$, $\sigma \in \mathfrak{S}_n^{\pm}$ and $i \in [0, n-1]$ we have $i \in \mathrm{Des}\,(\sigma)$ if, and only if, $\sigma(i) > \sigma(i+1)$.*

We denote by $\mathbb{Q}\mathfrak{S}_n^{\pm}$ the $\mathbb{Q}$-vector space generated by $\mathfrak{S}_n^{\pm}$. Permutations of $\mathfrak{S}_n^{\pm}$ are then vectors of $\mathbb{Q}\mathfrak{S}_n^{\pm}$. In this way, the expressions 2σ and $\sigma + \tau$ take sense for σ and τ in $\mathbb{Q}\mathfrak{S}_n^{\pm}$.

Definition 1.28. For $n \geqslant 1$, we define a square matrix $\mathrm{Adj}_{B_n} = (a_{\sigma,\tau})$ indexed by the elements of $\mathfrak{S}_n^{\pm}$ by:

$$a_{\sigma,\tau} = \begin{cases} 1 & \text{if } \mathrm{Des}\left(\tau^{-1}\right) \subseteq \mathrm{Des}\,(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.29. There are 8 signed permutations in $\mathfrak{S}_2^{\pm}$. In the above table, we give them with informations about their inverses and descending sets:

σ	σ^{-1}	$\mathrm{Des}\,(\sigma)$	$\mathrm{Des}\,(\sigma^{-1})$
$(1,2)$	$(1,2)$	$\emptyset$	$\emptyset$
$(1,-2)$	$(1,-2)$	$\{1\}$	$\{1\}$
$(-1,2)$	$(-1,2)$	$\{0\}$	$\{0\}$
$(-1,-2)$	$(-1,-2)$	$\{0,1\}$	$\{0,1\}$
$(2,1)$	$(2,1)$	$\{1\}$	$\{1\}$
$(2,-1)$	$(-2,1)$	$\{1\}$	$\{0\}$
$(-2,1)$	$(2,-1)$	$\{0\}$	$\{1\}$
$(-2,-1)$	$(-2,-1)$	$\{0\}$	$\{0\}$

With the same enumeration of $\mathfrak{S}_2^{\pm}$, we obtain:

$$\mathrm{Adj}_{B_2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Lemma 1.30. A pair (σ,τ) of signed permutation of $\mathfrak{S}_n^{\pm}$ is normal if, and only if, the scalar ${}^t\sigma\ \mathrm{Adj}_{B_n}\ \tau$ is equal to 1.

Proof. For a pair of signed permutations (σ,τ), the scalar ${}^t\sigma\ \mathrm{Adj}_{B_n}\ \tau$ corresponds to the coefficient $a_{\sigma,\tau}$ of the matrix Adj_{B_n}. We conclude by definition of Adj_{B_n} and Proposition 1.26. □

Proposition 1.31. *Let σ and τ be permutations of $\mathfrak{S}_n^{\pm}\setminus\{1\}$. For all $d\geqslant 1$, the number $b_{n,d}(\sigma,\tau)$ of normal sequences $(x_1,...,x_d)$ with $\pi(x_1)=\sigma$ and $\pi(x_d)=\tau$ is:*

$$b_{n,d}(\sigma,\tau) = {}^t\sigma\ \mathrm{Adj}_{B_n}^{d-1}\ \tau.$$

Proof. By induction on d. For $d=1$, such a normal sequence exists if, and only if, the permutation σ is equal to τ. Hence $b_{n,1}(\sigma,\tau)$ is δ_σ^τ, which is equal to ${}^t\sigma\cdot\tau$.

Assume now $d\geqslant 2$. A sequence $s=(x_1,x_2,...,x_{d-1},x_d)$ is normal if, and only if, the sequence $s'=(x_1,x_2,...,x_{d-1})$ and the pair (x_{d-1},x_d) are normal. Denoting by κ the permutation $\pi(x_{d-1})$, we obtain:

$$b_{n,d}(\sigma,\tau) = \sum_{\substack{\kappa\in\mathfrak{S}_n^{\pm}\\ (\kappa,\tau)\text{ normal}}} b_{n,d-1}(\sigma,\kappa).$$

As, by Lemma 1.30, the integer ${}^t\kappa\,\mathrm{Adj}_{B_n}\,\tau$ is equal to 1 if, and only if, (κ,τ) is normal and to 0 otherwise, we obtain:

$$b_{n,d}(\sigma,\tau) = \sum_{\kappa\in\mathfrak{S}_n^{\pm}} b_{n,d-1}(\sigma,\kappa)\cdot{}^t\kappa\,\mathrm{Adj}_{B_n}\,\tau.$$

Using induction hypothesis, we get:

$$\begin{aligned} b_{n,d}(\sigma,\tau) &= \sum_{\kappa\in\mathfrak{S}_n^{\pm}} {}^t\sigma(\mathrm{Adj}_{B_n})^{d-2}\kappa\cdot{}^t\kappa\,\mathrm{Adj}_{B_n}\tau, \\ &= {}^t\sigma\,\mathrm{Adj}_{B_n}^{d-2}\cdot\mathrm{Adj}_{B_n}\tau = {}^t\sigma\,\mathrm{Adj}_{B_n}^{d-1}\tau, \end{aligned}$$

as expected. □

Corollary 1.32. For $n \geqslant 1$ and $d \geqslant 1$ we have:

$$b_{n,d} = {}^tX\,\mathrm{Adj}_{B_n}^{d-1}X,$$

where X is the vector $\sum_{\sigma\in\mathfrak{S}_n^{\pm}\setminus\{1\}}\sigma$.

Proof. Let $n \geqslant 1$ and $d \geqslant 1$ be two integers. By Proposition 1.20, the integer $b_{n,d}$ is the number of normal sequences with no trivial entry. As the pair $(1,\sigma)$ is never normal for $\sigma\in\mathfrak{S}_n^{\pm}$, a sequence $(x_1,...,x_d)$ is not normal whenever $x_i = 1$ for any i in $[1,d-1]$. Hence, $b_{n,d}$ is the number of normal sequences $(x_1,...,x_d)$ with $x_1\neq 1$ and $x_d\neq 1$:

$$b_{n,d} = \sum_{\sigma,\tau\in\mathfrak{S}_n^{\pm}\setminus\{1\}} b_{n,d}(\sigma,\tau).$$

which is equal, by Proposition 1.31, to:

$$b_{n,d} = \sum_{\sigma,\tau\in\mathfrak{S}_n^{\pm}\setminus\{1\}} {}^t\sigma\ \mathrm{Adj}_{B_n}^{d-1}\tau = {}^tX\ \mathrm{Adj}_{B_n}^{d-1}X,$$

as expected. □

Example 1.33. In BB_2^+, the only braid of Garside length 0 is the trivial one, i.e., $b_{2,0} = 1$. Except the trivial one, all simple braids have length 1, and so $b_{2,1} = 7$, corresponding to ${}^tXX$. Considering the matrix Adj_{B_n} we obtain the following values of $b_{n,d}$:

d	$b_{2,d}$	$b_{3,d}$	$b_{4,d}$
0	1	47	383
1	7	771	35841
2	25	10413	2686591
3	79	134581	193501825
4	241	1721467	13837222655
5	727	21966231	988224026497

The generating series $F_{B_n}(t) = \sum_{d=0}^{+\infty} b_{n,d}t^d$ is given by ${}^tX\left(\mathrm{I} - t\,\mathrm{Adj}_{B_2}\right)^{-1}X$:

$$\begin{aligned} F_{B_2}(t) &= \frac{7-3t}{(3t-1)(t-1)}, \\ F_{B_3}(t) &= \frac{-60t^4+149t^3-163t^2+169t-47}{(t-1)(3t-1)(20t^3-43t^2+16t-1)}. \end{aligned}$$

Developing $F_{B_2}(t)$, we obtain $b_{2,d} = 3^{d+1}-2$.

The eigenvalues of the matrix Adj_{B_n} give informations on the growth of the function $d\mapsto b_{n,d}$. The first point is to determine if the eigenvalues of $\mathrm{Adj}_{B_{n-1}}$ are also eigenvalues of Adj_{B_n}, i.e., to determine if the characteristic polynomial of the matrix $\mathrm{Adj}_{B_{n-1}}$ divides the one of Adj_{B_n}. In [3], P. Dehornoy conjectured that this divisibility result holds for classical braids (Coxeter type A). The conjecture was proved by F. Hivert, J.-C. Novelli and J.-Y. Thibon in [9]. If we denote by χ_n^B the characteristic polynomial of the matrix Adj_{B_n}, we obtain:

$$\begin{aligned}
&\chi_1^B(x) = (x-1)^2,\\
&\chi_2^B(x) = \chi_1^B(x)\ x^4\ (x-1)\ (x-3),\\
&\chi_3^B(x) = \chi_2^B(x)\ x^{37}\ (x^3 - 16x^2 + 43x - 20),\\
&\chi_4^B(x) = \chi_3^B(x)\ x^{329}\ (x-1)^3\ (x^4 - 85x^3 + 1003x^2 - 2291x + 1260),\\
&\chi_5^B(x) = \chi_4^B(x)\ x^{3449}\ (x^7 - 574x^6 + 39344x^5 - 576174x^4 +\\
&\qquad\qquad 3027663x^3 - 5949972x^2 + 4281984x - 1088640).
\end{aligned}$$

As the reader can see, the polynomial χ_i^B divides χ_{i+1}^B for $i \in \{1,2,3,4\}$. The aim of the paper is to prove the following theorem:

Theorem 1.1. *For all $n \in \mathbb{N}$, the characteristic polynomial of the matrix* Adj_{B_n} *divides the characteristic polynomial of the matrix* $\mathrm{Adj}_{B_{n+1}}$.

For this, we interpret the matrix Adj_{B_n} as the matrix of an endomorphism Φ_n of $\mathbb{Q}\mathfrak{S}_n^{\pm}$. In order to prove the main theorem we equip the vector space $\mathbb{Q}\mathfrak{S}_n^{\pm}$ with a structure of Hopf algebra.

2. The Hopf algebra **BFQSym**.

We describe in this section an analogous of the Hopf algebra **FQSym** for the signed permutation group $\mathfrak{S}_n^{\pm}$. We denote by $\mathbb{Q}\mathfrak{S}^{\pm}$ the $\mathbb{Q}$-vector space $\bigoplus_{n=1}^{+\infty} \mathbb{Q}\mathfrak{S}_n^{\pm}$.

2.1. **Signed permutation words.** We have shown in Section 1.1 that a signed permutation can be uniquely determined by its window notation. In order to have a simple definition for the notions attached to the construction of the Hopf algebra **BFQSym**, we describe a one-to-one construction between signed permutations and some specific words associated to the window notation.

Definition 2.1. For $n \geqslant 1$, we define $\mathrm{W}_n^{\pm}$ to be the set of words $w = w_1 \dots w_n$ on the alphabet $[-n, n]$ satisfying $\{|w_1|, \dots, |w_n|\} = [1, n]$.

If w is an element of $\mathrm{W}_n^{\pm}$, then $(w_1, \dots, w_n)$ is the window notation of some signed permutation of $\mathfrak{S}_n^{\pm}$. For $n \geqslant 1$, we define two maps $w : \mathfrak{S}_n^{\pm} \to \mathrm{W}_n^{\pm}$ and $\rho : \mathrm{W}_n^{\pm} \to \mathfrak{S}_n^{\pm}$ by $w(\sigma) = \sigma(1) \dots \sigma(n)$ and, for $i \in [-n, n]$:

$$\rho(w)(i) = \begin{cases} 0 & \text{if } i = 0,\\ w_i & \text{if } i > 0,\\ -w_{-i} & \text{if } i < 0. \end{cases}$$

Definition 2.2. For $i \in \mathbb{Z} \setminus \{0\}$ and $k \in \mathbb{Z}$, we define the integers $i[k]$ and $i\langle k\rangle$ (whenever $i \neq \pm k$) by:

$$i[k] = \begin{cases} i + k & \text{if } i > 0,\\ i - k & \text{if } i < 0, \end{cases} \qquad i\langle k\rangle = \begin{cases} i + 1 & \text{if } i < -k,\\ i & \text{if } -k < i < k,\\ i - 1 & \text{if } i > k. \end{cases}$$

For $w = w_1 \dots w_\ell$ a word on the letters $[-n, n] \setminus \{0\}$, we define $w_{[k]}$ to be the word $w_{1[k]} \dots w_{\ell[k]}$ and $w\langle k\rangle$ to be the word $w_1\langle k\rangle \dots w_\ell\langle k\rangle$ if $w_j \neq \pm k$ for all j. We also extend these notations to sets of integers.

Example 2.3. If w is the word $1 \cdot -5 \cdot 3 \cdot -2 \cdot 6$, we have $w_{[2]} = 3 \cdot -7 \cdot 5 \cdot -4 \cdot 8$ and $w\langle 4\rangle = 1 \cdot -4 \cdot 3 \cdot -2 \cdot 5$.

2.2. **Shuffle product.**

Definition 2.4. For $k, \ell \geqslant 1$, we denote by $\mathrm{Sh}_{k,\ell}$ all the subsets of $[1, k+\ell]$ of cardinality k. For $X \in \mathrm{Sh}_{k,\ell}$, we write $X = \{x_1 < \dots < x_k\}$ to specify that the x_i's are the elements of X in increasing order.

For example, we have:

$$\mathrm{Sh}_{2,3} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}\}.$$

Definition 2.5. Let $k, \ell \geqslant 1$ be two integers. For two words $u \in \mathrm{W}^{\pm}_k$, $v \in \mathrm{W}^{\pm}_\ell$ and $X = \{x_1 < \dots < x_k\} \in \mathrm{Sh}_{k,\ell}$ we define the *X-shuffle word of u and v* by:

$$u \sqcup\!\sqcup^X v = v_0[k]\, u_1\, v_1[k] \dots v_{k-1}[k]\, u_k\, v_k[k],$$

where $v_0 \dots v_k = v$ and $\ell(v_i) = x_{i+1} - x_i - 1$, with the conventions $x_0 = 0$ and $x_{k+1} = k + \ell$.

One remarks that letters coming from u are in positions belonging to X in the final word.

Example 2.6. Let u be the word $-2 \cdot 1$ and v be the word $3 \cdot -1 \cdot 2$. We then have $k = 2$ and $\ell = 3$. The word $v_{[k]}$ is $5 \cdot -3 \cdot 4$. The $\{2,4\}$-shuffle of u and v is the word $5 \cdot -2 \cdot -3 \cdot 1 \cdot 4$ while the $\{4,5\}$-shuffle of u and v is $5 \cdot -3 \cdot 4 \cdot -2 \cdot 1$; letters in gray are these coming from the word u.

Definition 2.7. For $\sigma \in \mathfrak{S}^{\pm}_k$ and $\tau \in \mathfrak{S}^{\pm}_\ell$ two signed permutations, we define the *shuffle product* of σ and τ to be the signed permutation $\sigma \sqcup\!\sqcup \tau$ of $\mathfrak{S}^{\pm}_{k+\ell}$ defined by:

$$\sigma \sqcup\!\sqcup \tau = \sum_{X \in \mathrm{Sh}_{k,\ell}} \rho\left(w(\sigma) \sqcup\!\sqcup^X w(\tau)\right).$$

Example 2.8. Considering the signed permutations $\sigma = (-2, 1)$ and $\tau = (3, -1, 2)$, we obtain:

$$\begin{aligned} \sigma \sqcup\!\sqcup \tau = \; & (-2, 1, 5, -3, 4) + (-2, 5, 1, -3, 4) + (-2, 5, -3, 1, 4) + (-2, 5, -3, 4, 1) \\ & + (5, -2, 1, -3, 4) + (5, -2, -3, 1, 4) + (5, -2, -3, 4, 1) + (5, -3, -2, 1, 4) \\ & + (5, -3, -2, 4, 1) + (5, -3, 4, -2, 1). \end{aligned}$$

Let $x_1, \dots, x_n$ be n distinct integers. For every sequence $\varepsilon_1, \dots, \varepsilon_n$ of $\{-1, +1\}$, we define $\mathrm{Std}(\varepsilon_1 x_1 \dots \varepsilon_n x_n)$ to be the word $\varepsilon_1 f(x_1) \dots \varepsilon_n f(x_n)$, where f is the unique increasing map from $\{x_1, \dots, x_n\}$ to $[1, n]$. Apart from the ε_i, this notion of standardization of word coincides with the one used on permutations of $\mathfrak{S}^{\pm}_n$.

We define a coproduct on $\mathbb{Q}\mathfrak{S}^{\pm}$ by

$$\forall \sigma \in \mathfrak{S}^{\pm}_n, \quad \Delta(\sigma) = \sum_{k=0}^{n} \rho(\mathrm{Std}(\sigma(1), \dots, \sigma(k))) \otimes \rho(\mathrm{Std}(\sigma(k+1), \dots, \sigma(n))).$$

For example the coproduct of $(4,-2,3,-1)$ is:

$$\begin{aligned}\Delta(4,-2,3,1) =& \emptyset\otimes(4,-2,3,1)+(1)\otimes(-2,3,1)\\ &+(2,-1)\otimes(2,1)+(3,-1,2)\otimes(1)+(4,-2,3,1)\otimes\emptyset.\end{aligned}$$

Equipped with the shuffle product $\sqcup\!\sqcup$ and the coproduct Δ, the vector space $\mathbb{Q}\mathfrak{S}$ is a Hopf algebra denoted **BFQSym**. Details are omitted in this paper and can be found in [13]. Indeed, **BFQSym** corresponds to the Hopf algebra of decorated permutations $\mathbf{FQSym}^D$ with $D=\{-1,1\}$.

2.3. **The dual structure.** Thanks to the non degenerate pairing $\langle\sigma,\tau\rangle=\delta_\sigma^\tau$, we identify **BFQSym** with its dual. The Hopf algebra structure of the dual is given by the product $*$ and the coproduct δ defined by:

$$\langle\sigma*\tau,\kappa\rangle=\langle\sigma\otimes\tau,\Delta(\kappa)\rangle \qquad\text{and}\qquad \langle\delta(\sigma),\tau\otimes\kappa\rangle=\langle\sigma,\tau\sqcup\!\sqcup\kappa\rangle\,.$$

The map ι of $\mathbb{Q}\mathfrak{S}^\pm$ that maps σ to σ^{-1} is a Hopf algebra isomorphism between $(\mathbf{BFQSym},\sqcup\!\sqcup,\Delta)$ and $(\mathbf{BFQSym},*,\delta)$. The following proposition gives a concrete description of $*$.

Proposition 2.9. *Let $\sigma\in\mathfrak{S}_k^\pm$ and $\tau\in\mathfrak{S}_\ell^\pm$ be two permutations. We have:*

$$\sigma*\tau \;=\; \sum_{\substack{u\in W_{k+\ell}^\pm\\ \mathrm{Std}(u_1,\ldots,u_k)=w(\sigma)\\ \mathrm{Std}(u_{k+1},\ldots,u_{k+\ell})=w(\tau)}} \rho(u).$$

Example 2.10. For the signed permutations $\sigma=(2,-1)$ and $\tau=(3,-1,2)$ we have:

$$\begin{aligned}\sigma*\tau=\;&(2,-1,5,-3,4)+(3,-1,5,-2,4)+(4,-1,5,-2,3)+(5,-1,4,-2,3)\\ &+(3,-2,5,-1,4)+(4,-2,5,-1,3)+(5,-2,4,-1,3)+(4,-3,5,-1,2)\\ &+(5,-3,4,-1,2)+(5,-4,3,-1,2).\end{aligned}$$

Definition 2.11. For $n\geqslant 1$, we denote by I_n, J_n, P_n and Q_n the elements of $\mathbb{Q}\mathfrak{S}_n^\pm$ defined by $I_n=(1,\ldots,n)$, $J_n=(-n,\ldots,-1)$, and:

$$P_n=\sum_{\substack{\sigma\in\mathfrak{S}_n^\pm\\ \mathrm{Des}(\sigma^{-1})\subseteq\{0\}}}\sigma, \qquad\qquad Q_n=\sum_{\substack{\sigma\in\mathfrak{S}_n^\pm\\ \mathrm{Des}(\sigma)\subseteq\{0\}}}\sigma.$$

Example 2.12. We have $P_2=(1,2)+(-1,2)+(2,-1)+(-2,-1)$, $Q_2=(1,2)+(-1,2)+(-2,1)+(-2,-1)$ and, for example:

$$\begin{aligned}P_4=\;&(1,2,3,4)+(-1,2,3,4)+(2,-1,3,4)+(-2,-1,3,4)+(2,3,-1,4)\\ &+(-2,3,-1,4)+(2,3,4,-1)+(-2,3,4,-1)+(3,-2,-1,4)\\ &+(-3,-2,-1,4)+(3,-2,4,-1)+(-3,-2,4,-1)+(3,4,-2,-1)\\ &+(-3,4,-2,-1)+(4,-3,-2,-1)+(-4,-3,-2,-1).\end{aligned}$$

In general, P_n and Q_n are linear combinations of 2^n permutations.

Vectors P_n and Q_n are used to describe permutations of $\mathfrak{S}_n^\pm$ whose descent sets are included in a given subset of $[0,n-1]$. The following Lemma exhibits these connections.

Lemma 2.13. Let $k_1, ..., k_{\ell+1} \geqslant 1$ be integers and n be the integer $k_1 + ... + k_{\ell+1}$. Let D be the set $\{k_1, k_1 + k_2, ..., k_1 + ... + k_\ell\}$, we have the following relations:

$$Q_{k_1} * ... * Q_{k_{\ell+1}} = \sum_{\substack{\sigma\in\mathfrak{S}_n^\pm \\ \mathrm{Des}(\sigma)\subseteq\{0\}\cup D}} \sigma, \qquad I_{k_1} * Q_{k_2} * ... * Q_{k_{\ell+1}} = \sum_{\substack{\sigma\in\mathfrak{S}_n^\pm \\ \mathrm{Des}(\sigma)\subseteq D}} \sigma,$$

$$P_{k_1} \sqcup\!\sqcup ... \sqcup\!\sqcup P_{k_{\ell+1}} = \sum_{\substack{\sigma\in\mathfrak{S}_n^\pm \\ \mathrm{Des}(\sigma^{-1})\subseteq\{0\}\cup D}} \sigma, \qquad I_{k_1} \sqcup\!\sqcup P_{k_2} \sqcup\!\sqcup ... \sqcup\!\sqcup P_{k_{\ell+1}} = \sum_{\substack{\sigma\in\mathfrak{S}_n^\pm \\ \mathrm{Des}(\sigma^{-1})\subseteq D}} \sigma.$$

Proof. For $i \in [1, \ell]$ we put $d_i = k_1 + ... + k_i$. By very definition of Q_k, we have:

$$Q_k = \sum_{\substack{\sigma\in\mathfrak{S}_k^\pm \\ \sigma(1)<...<\sigma(k)}} \sigma.$$

Then, by Proposition 2.9, we obtain:

$$Q_{k_1} * ... * Q_{k_{\ell+1}} = \sum_{\substack{\sigma\in\mathfrak{S}_{k+\ell}^\pm \\ \sigma(1)<...<\sigma(d_1) \\ ... \\ \sigma(d_\ell+1)<...<\sigma(n)}} \sigma.$$

Permutations occurring in the previous sum are exactly these having descents in the set $\{0, d_1, ..., d_\ell\}$. Similarly, as I_{k_1} is the only permutation σ of $\mathfrak{S}_{k_1}^\pm$ satisfying $0 < \sigma(1) < ... < \sigma(k_1)$, we have:

$$I_{k_1} * Q_{k_2} * ... * Q_{k_{\ell+1}} = \sum_{\substack{\sigma\in\mathfrak{S}_{k+\ell}^\pm \\ 0<\sigma(1)<...<\sigma(d_1) \\ \sigma(d_1+1)<...<\sigma(d_2) \\ ... \\ \sigma(d_\ell+1)<...<\sigma(n)}} \sigma,$$

which is the sum of permutations of $\mathfrak{S}_n^\pm$ with descent set in $\{d_1, ..., d_\ell\}$.

Applying the isomorphism ι between $(\mathbf{BFQSym}, \sqcup\!\sqcup, \Delta)$ and $(\mathbf{BFQSym}, *, \delta)$ to the previous expression of $Q_{k_1} * ... * Q_{k_{\ell+1}}$, we obtain:

$$\iota(Q_{k_1}) \sqcup\!\sqcup ... \sqcup\!\sqcup \iota(Q_{k_{\ell+1}}) = \sum_{\substack{\sigma\in\mathfrak{S}_n^\pm \\ \mathrm{Des}(\sigma)\in\{0\}\cup D}} \sigma^{-1} = \sum_{\substack{\sigma\in\mathfrak{S}_n^\pm \\ \mathrm{Des}(\sigma^{-1})\in\{0\}\cup D}} \sigma.$$

The expected relation appears, remarking that $\iota(Q_k)$ is equal to P_k. The second relation involving the shuffle product is obtain similarly from $\iota(I_{k_1}) = I_{k_1}$. $\square$

The vector P_n of $\mathbb{Q}\mathfrak{S}_n^\pm$ can also be defined using the shuffle product as suggested by Example 2.12.

Lemma 2.14. For all $n \geqslant 1$, we have $P_n = \sum_{k=0}^n J_k \sqcup\!\sqcup I_{n-k}$.

Proof. By definition of Q_n, we have:

$$Q_n = \sum_{\substack{\sigma\in\mathfrak{S}_n^\pm, \\ \sigma(1)<...<\sigma(n)}} \sigma = \sum_{k=0}^{n} \sum_{\substack{\sigma\in\mathfrak{S}_n^\pm, \\ \sigma(1)<...<\sigma(k)<0 \\ 0<\sigma(k+1)<...<\sigma(n)}} \sigma.$$

In the other hand, by Proposition 2.9, we have:

$$J_k * I_{n-k} = \sum_{\substack{\sigma\in\mathfrak{S}_n^\pm \\ \mathrm{Std}(\sigma(1),\ldots,\sigma(k))=(-k,\ldots,-1) \\ \mathrm{Std}(\sigma(k+1),\ldots,\sigma(n))=(1,\ldots,n-k)}} \sigma = \sum_{\substack{\sigma\in\mathfrak{S}_n^\pm, \\ \sigma(1)<\ldots<\sigma(k)<0 \\ 0<\sigma(k+1)<\ldots<\sigma(n)}} \sigma.$$

We have then established $Q_n = \sum_{k=0}^n J_k * I_{n-k}$. We obtain the expected result applying the isomorphism ι since J_k and I_k are fixed by ι. $\square$

3. The divisibility result.

For $n \in \mathbb{N}$, we define Φ_n to be the endomorphism of $\mathbb{Q}\mathfrak{S}_n^\pm$ whose representative matrix is ${}^t\,\mathrm{Adj}_{B_n}$. We denote by Φ the endomorphism $\bigoplus \Phi_n$ of $\mathbb{Q}\mathfrak{S}^\pm$. By very definition of Adj_{B_n}, for all $\sigma \in \mathfrak{S}_n^\pm$, we have:

$$\Phi(\sigma) = \Phi_n(\sigma) = \sum_{\substack{\tau\in\mathfrak{S}_n^\pm \\ \mathrm{Des}(\tau^{-1})\subseteq\mathrm{Des}(\sigma)}} \tau.$$

For $n \in \mathbb{N}$, we denote by $\mathcal{D}_n$ the set of all subsets of $[0, n-1]$. The descent map from $\mathfrak{S}_n^\pm$ to $\mathcal{D}_n$ can be extended to a unique linear map, also denoted by Des, from $\mathbb{Q}\mathfrak{S}_n^\pm$ to $\mathbb{Q}\mathcal{D}_n$. We denote by $\widetilde{\Phi}_n$ the map from $\mathbb{Q}\mathcal{D}_n$ to $\mathbb{Q}\mathfrak{S}_n^\pm$ defined by:

$$\widetilde{\Phi}_n(I) = \sum_{\substack{\tau\in\mathfrak{S}_n^\pm \\ \mathrm{Des}(\tau^{-1})\subseteq I}} \tau,$$

for any element I of $\mathcal{D}_n$. For all $\sigma \in \mathfrak{S}_n^\pm$, we have $\Phi_n(\sigma) = \widetilde{\Phi}_n(\mathrm{Des}\,(\sigma))$.

A direct consequence of Lemma 2.13 is:

Proposition 3.1. *For every $D = \{d_1 < \ldots < d_\ell\}$ element of $\mathcal{D}_n$, with $0 < d_1$, we have the relations:*

$$\widetilde{\Phi}_n(D) = I_{k_1} \sqcup\!\sqcup P_{k_2} \sqcup\!\sqcup \ldots \sqcup\!\sqcup P_{k_{\ell+1}},$$

$$\widetilde{\Phi}_n(\{0\} \cup D) = P_{k_1} \sqcup\!\sqcup P_{k_2} \sqcup\!\sqcup \ldots \sqcup\!\sqcup P_{k_{\ell+1}},$$

where $k_i = d_i - d_{i-1}$ for $i \in [1, \ell+1]$ and with the conventions $d_0 = 0$, $d_{\ell+1} = n$.

Definition 3.2. An endomorphism Ψ of $\mathbb{Q}\mathfrak{S}^\pm$ is a *surjective derivation* if:
– *(i)* $\Psi(x \sqcup\!\sqcup y) = \Psi(x) \sqcup\!\sqcup y + x \sqcup\!\sqcup \Psi(y)$ holds for all x, y of $\mathbb{Q}\mathfrak{S}^\pm$;
– *(ii)* $\Psi(\mathbb{Q}\mathfrak{S}_n^\pm) = \mathbb{Q}\mathfrak{S}_{n-1}^\pm$ holds for all $n \geqslant 1$.

Proposition 3.3. *If there exists a surjective derivation Ψ of $\mathbb{Q}\mathfrak{S}^\pm$ commuting with Φ, then, for $n \geqslant 1$, the characteristic polynomial of Φ_{n-1} divides the one of Φ_n.*

Proof. Let Ψ be a surjective derivation of $\mathbb{Q}\mathfrak{S}^\pm$ commuting with Φ, and n be an integer greater than 1. Let us denote by Ψ_n the restriction of Ψ to $\mathbb{Q}\mathfrak{S}_n^\pm$. We fix a basis $\mathcal{B} = \mathcal{B}_0 \sqcup \mathcal{B}_1$ of $\mathbb{Q}\mathfrak{S}_n^\pm$, such that $\mathcal{B}_0$ is a basis of $\ker(\Psi_n)$. Restricting the relation $\Psi \circ \Phi = \Phi \circ \Psi$ to $\mathbb{Q}\mathfrak{S}_n^\pm$, we obtain $\Psi_n \circ \Phi_n = \Phi_{n-1} \circ \Psi_n$. For x in $\ker(\Psi_n)$, we have $\Psi_n(\Phi_n(x)) = \Phi_{n-1}(\Psi_n(x)) = \Phi_{n-1}(0) = 0$. Hence, $\ker(\Psi_n)$ is stable under the map Φ_n. In particular, the representative matrix of Φ_n in the basis $\mathcal{B}$ is the upper triangular matrix:

$$M_n = \begin{bmatrix} A_n & B_n \\ 0 & C_n \end{bmatrix}.$$

Denoting by $\chi(.)$ the characteristic polynomial of a matrix or an endomorphism, we obtain:

$$\chi(\Phi_n) = \chi(M_n) = \chi(A_n)\chi(C_n). \tag{1}$$

The matrix of the restriction $\overline{\Phi}_n$ of Φ_n to $\mathbb{Q}\mathfrak{S}_n^{\pm}/\ker(\Psi_n)$ is C_n and so $\chi(\overline{\Phi}_n)$ is equal to $\chi(C_n)$. From the surjectivity of Ψ, we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{Q}\mathfrak{S}_n^{\pm}/\ker(\Psi_n) \arrow[r, "\overline{\Phi}_n"] \arrow[d, hook, "\overline{\Psi}_n"'] & \mathbb{Q}\mathfrak{S}_n^{\pm}/\ker(\Psi_n) \arrow[d, hook, "\overline{\Psi}_n"] \\
\mathbb{Q}\mathfrak{S}_{n-1}^{\pm} \arrow[r, "\Phi_{n-1}"] & \mathbb{Q}\mathfrak{S}_{n-1}^{\pm}
\end{tikzcd}$$

implying that the endomorphism Φ_{n-1} is conjugate to $\overline{\Phi}_n$. Therefore Equation (1) becomes $\chi(\Phi_n) = \chi(A_n)\chi(\Phi_{n-1})$, and so $\chi(\Phi_{n-1})$ divides $\chi(\Phi_n)$. $\square$

As the reader can check, the property (i) of a derivation is not used in the proof, but will be fundamental in order to establish the commutativity with Φ.

It remains to construct a surjective derivation Ψ which commutes with Φ.

3.1. **A derivation of BFQSym.** In order to describe our derivation, we need to introduce some notations.

Definition 3.4. For a and b two distinct integers, we define $\varepsilon(a, b)$ by:

$$\varepsilon(a,b) = \begin{cases} 1 & \text{if } a < b, \\ -1 & \text{if } a > b. \end{cases}$$

For a, b, c three distinct integers, we write $\varepsilon(a, b, c) = \dfrac{1}{2}\left(\varepsilon(a, b) + \varepsilon(b, c)\right) \in \{-1, 0, 1\}$.

Definition 3.5. Let $u = u_1 \dots u_n$ be a word of $\mathrm{W}_n^{\pm}$ and $i \in [1, n]$. We define:

$$\mathrm{sign}_i(u) = \varepsilon(u_{j-1}, u_j, u_{j+1}),$$

where j is the unique integer satisfying $|u_j| = i$, with the conventions $u_0 = 0$ and $u_{n+1} = -\infty$.

Example 3.6. Considering the word $u = -1 \cdot 2 \cdot -4 \cdot -5 \cdot 3 \cdot 6$, augmented to the word $0 \cdot -1 \cdot 2 \cdot 4 \cdot -5 \cdot 3 \cdot 6 \cdot -\infty$, we obtain:

$$\begin{aligned}
&\mathrm{sign}_1(u) = \varepsilon(0, -1, 2) = 0, && \mathrm{sign}_2(u) = \varepsilon(-1, 2, -4) = 0, \\
&\mathrm{sign}_3(u) = \varepsilon(-5, 3, 6) = 1, && \mathrm{sign}_4(u) = \varepsilon(2, -4, -5) = -1, \\
&\mathrm{sign}_5(u) = \varepsilon(-4, -5, 3) = 0, && \mathrm{sign}_6(u) = \varepsilon(3, 6, -\infty) = 0.
\end{aligned}$$

Lemma 3.7. Let $n \geqslant 1$ and $\sigma \in \mathfrak{S}_n^{\pm}$. For $j \in [1, n-1]$, we have:

$$\mathrm{sign}_{|\sigma(j)|}(w(\sigma)) = \begin{cases} 1 & \text{if } \{j-1, j\} \cap \mathrm{Des}\,(\sigma) = \emptyset; \\ -1 & \text{if } \{j-1, j\} \subseteq \mathrm{Des}\,(\sigma); \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the value of $\mathrm{sign}_{|\sigma(n)|}(w(\sigma))$ is -1 if $n-1$ belongs to $\mathrm{Des}\,(\sigma)$, and is 0 otherwise.

Proof. Let σ be a permutation of $\mathfrak{S}_n^\pm$ and j be an integer in $[1, n-1]$. By definition of sign, we have $\mathrm{sign}_{|\sigma(j)|}(w(\sigma)) = 1$ if, and only if, $\sigma(j-1) < \sigma(j) < \sigma(j+1)$, wich is equivalent to $j-1 \notin \mathrm{Des}\,(\sigma)$ and $j \notin \mathrm{Des}\,(\sigma)$. Still by definition of sign, we have $\mathrm{sign}_{|\sigma(j)|}(w(\sigma)) = -1$ if, and only if, $\sigma(j-1) > \sigma(j) > \sigma(j+1)$, i.e., $j-1$ and j belong to $\mathrm{Des}\,(\sigma)$.

Let us now prove the statement for $j = n$. As the relation $\sigma(n) > -\infty$ is always true, the value of $\mathrm{sign}_{|\sigma(j)|}(w(\sigma))$ is -1 for $\sigma(n-1) > \sigma(n)$ and 0 otherwise as expected. □

Example 3.8. The descent set of $\sigma = (-1, 2, -4, -5, 3, 6)$ is $\{0, 2, 3\}$. Hence, the non zero values of $\mathrm{sign}_{|\sigma(j)|}(w(\sigma))$ are obtained for $j = 3$ and $j = 5$, more precisely, we have $\mathrm{sign}_{|\sigma(3)|}(w(\sigma)) = \mathrm{sign}_4(w(\sigma)) = -1$ and $\mathrm{sign}_{|\sigma(5)|}(w(\sigma)) = \mathrm{sign}_3(w(\sigma)) = 1$, corresponding to Example 3.6.

Definition 3.9. For $u \in \mathrm{W}_n^\pm$ and $i \in [1, n]$, we denote by $\mathrm{del}_i(u)$ the word $u_1\langle i\rangle \dots u_{j-1}\langle i\rangle\ u_{j+1}\langle i\rangle \dots u_n\langle i\rangle$ of $\mathrm{W}_{n-1}^\pm$, where j is the unique integer satisfying the relation $|u_j| = i$.

One can remarks that we have $\mathrm{del}_i(u) = \mathrm{Std}(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_n)$.

Example 3.10. Considering $u = -1 \cdot 2 \cdot -4 \cdot -5 \cdot 3 \cdot 6$, we obtain:

$$\begin{aligned}
&\mathrm{del}_1(u) = 1 \cdot -3 \cdot -4 \cdot 2 \cdot 5, && \mathrm{del}_2(u) = -1 \cdot -3 \cdot -4 \cdot 2 \cdot 5,\\
&\mathrm{del}_3(u) = -1 \cdot 2 \cdot -3 \cdot -4 \cdot 5, && \mathrm{del}_4(u) = -1 \cdot 2 \cdot -4 \cdot 3 \cdot 5,\\
&\mathrm{del}_5(u) = -1 \cdot 2 \cdot -4 \cdot 3 \cdot 5, && \mathrm{del}_6(u) = -1 \cdot 2 \cdot -4 \cdot -5 \cdot 3.
\end{aligned}$$

Definition 3.11. Let n and i be two integers such that $i \in [1, n]$. We define a linear map ∂_n^i of $\mathbb{Q}\mathfrak{S}_n^\pm$ to $\mathbb{Q}\mathfrak{S}_{n-1}^\pm$ by:

$$\partial_n^i(\sigma) = \mathrm{sign}_i(w(\sigma))\, \rho(\mathrm{del}_i(w(\sigma))),$$

where $\sigma \in \mathfrak{S}_n^\pm$. Then we define a map ∂_n from $\mathbb{Q}\mathfrak{S}_n^\pm$ to $\mathbb{Q}\mathfrak{S}_{n-1}^\pm$ by:

$$\partial_n(\sigma) = \sum_{k=1}^{n} \partial_n^i(\sigma) \qquad \text{for } \sigma \in \mathfrak{S}_n^\pm,$$

and a map ∂ of $\mathbb{Q}\mathfrak{S}^\pm$ by $\partial = \bigoplus_{n=1}^{+\infty} \partial_n$.

Example 3.12. Considering the permutation $\sigma = (-1, 2, -4, -5, 3, 6)$, we have $\partial_6^1(\sigma) = \partial_6^2(\sigma) = \partial_6^5(\sigma) = \partial_6^6(\sigma) = 0$, while we have:

$$\begin{aligned}
\partial_6^3(\sigma) &= \mathrm{sign}_3(w(\sigma))\rho(\mathrm{del}_3(w(\sigma))) = (-1, 2, -3, -4, 5),\\
\partial_6^4(\sigma) &= \mathrm{sign}_4(w(\sigma))\rho(\mathrm{del}_4(w(\sigma))) = -(-1, 2, -4, 3, 5).
\end{aligned}$$

Finally we obtain $\partial(\sigma) = (-1, 2, -3, -4, 5) - (-1, 2, -4, 3, 5)$.

Example 3.13. The map ∂ sends $\mathbb{Q}\mathfrak{S}_2^\pm$ to $\mathbb{Q}\mathfrak{S}_1^\pm$. The matrix of this map, with the enumeration of $\mathfrak{S}_2^\pm$ of Example 1.29 and the enumeration (1), (-1) of $\mathfrak{S}_1^\pm$, is:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We now prove that ∂ is a surjective derivation of $\mathbb{Q}\mathfrak{S}^\pm$, compatible with the shuffle product.

Lemma 3.14. Let $\sigma \in \mathfrak{S}_k^{\pm}$ and $\tau \in \mathfrak{S}_\ell^{\pm}$ be two signed permutations.
– (i) For all $i \in [1,k]$, we have $\partial^i_{k+\ell}(\sigma \sqcup\!\sqcup \tau) = \partial^i_k(\sigma) \sqcup\!\sqcup \tau$;
– (ii) For all $i \in [k+1, k+\ell]$, we have $\partial^i_{k+\ell}(\sigma \sqcup\!\sqcup \tau) = \sigma \sqcup\!\sqcup \partial^{i-k}_\ell(\tau)$.

Proof. Let σ and τ be two permutations of $\mathfrak{S}_k^{\pm}$ and $\mathfrak{S}_\ell^{\pm}$ and u, v be their respective words. Let i be an integer of $[1,k]$. Then, there exists a unique j such that $u_j = \pm i$. By definition of del_i, we have $\mathrm{del}_i(u) = u_1\langle i\rangle \; ... \; u_{j-1}\langle i\rangle \; u_{j+1}\langle i\rangle \; ... \; u_k\langle i\rangle$. Let:

$$Y = \{y_1 < ... < y_{k-1}\}$$

be an element of $\mathrm{Sh}_{k-1,\ell}$. Writing u'_m for $u_m\langle i\rangle$, there exists k words $v_0, ..., v_{k-1}$ satisfying $v_0 \dots v_{k-1} = v$ whose length is $\ell(v_j) = y_{j+1} - y_j - 1$ for $j \in [1, k-1]$, with the convention $y_k = k+\ell-1$ such that $\mathrm{del}_i(u) \sqcup\!\sqcup^Y v$ is equal to:

$$v_0[k-1]\, u'_1 \dots v_{j-2}[k-1]\, u'_{j-1} \;\cdot\; v_{j-1}[k-1] \;\cdot\; u'_{j+1}\, v_j[k-1] \dots u'_k\, v_{k-1}[k-1]. \tag{2}$$

We now express v_{j-1} as the word $\alpha_1 \dots \alpha_m$, with $m = y_j - y_{j-1} - 1$. For $a \in [0,m]$, we define $k+1$ words $w^a_0, ..., w^a_k$ by:

$$w^a_p = \begin{cases} v_p & \text{for } p \leqslant j-2, \\ \alpha_1 \dots \alpha_a & \text{for } p = j-1, \\ \alpha_{a+1} \dots \alpha_m & \text{for } p = j, \\ v_{p-1} & \text{for } p \geqslant j+1. \end{cases}$$

Then v is equal to $w^a_0 \; ... \; w^a_{j-1}\, w^a_j \dots w^a_k$. We define Y_a, the refinement of Y, by:

$$Y_a = \{y_1 < ... < y_{j-1} < y_{j-1} + a + 1 < y_j + 1 < ... < y_{k-1}\}.$$

Note that Y_a is an element of $\mathrm{Sh}_{k,\ell}$ for all values of $a \in [0,m]$. The shuffle product of u and v relatively to Y_a is:

$$u \sqcup\!\sqcup^{Y_a} v = w^a_0[k]\, u_1 \dots w^a_{j-2}[k]\, u_{j-1} \;\cdot\; w^a_{j-1}[k]\, u_j\, w^a_j[k] \;\cdot\; u_{j+1}\, w^a_{j+1}[k] \dots u_k\, w^a_k[k],$$

Applying del_i to the previous relation gives that $\mathrm{del}_i\left(u \sqcup\!\sqcup^{Y_a} v\right)$ is equal to:

$$w^a_0[k-1]\, u'_1 \dots w^a_{j-2}[k-1]\, u'_{j-1} \;\cdot\; w^a_{j-1}[k-1]\, w^a_j[k-1] \;\cdot\; u'_{j+1}\, w^a_{j+1}[k-1] \dots u'_k\, w^a_k[k-1],$$

which, by definition of the words w^a_p, is exactly the expression of $\mathrm{del}_i(u) \sqcup\!\sqcup^Y v$ given in (2). We then obtain:

$$\sum_{a=0}^{m} \mathrm{sign}_i\left(u \sqcup\!\sqcup^{Y_a} v\right) \mathrm{del}_i\left(u \sqcup\!\sqcup^{Y_a} v\right) = \left(\sum_{a=0}^{m} \mathrm{sign}_i\left(u \sqcup\!\sqcup^{Y_a} v\right)\right) \mathrm{del}_i(u) \sqcup\!\sqcup^Y v.$$

By definition of sign_i and ε together with the conventions $\alpha_0 = u_{j-1}$, $\alpha_{m+1} = u_{j+1}$, and the conventions $u_0 = 0$, $u_{k+1} = -\infty$ used in definition of sign, we obtain:

$$\begin{aligned} \sum_{a=0}^{m} \mathrm{sign}_i\left(u \sqcup\!\sqcup^{Y_a} v\right) &= \sum_{a=0}^{k} \varepsilon(\alpha_a, u_j, \alpha_{a+1}) = \frac{1}{2} \sum_{a=0}^{k} \varepsilon(\alpha_a, u_j) + \varepsilon(u_j, \alpha_{a+1}), \\ &= \frac{1}{2} \sum_{a=0}^{k} \big(\, \varepsilon(\alpha_a, u_j) - \varepsilon(\alpha_{a+1}, u_j)\big) = \varepsilon(\alpha_0, u_j, \alpha_{m+1}), \end{aligned}$$

and the latter is equal to $\varepsilon(u_{j-1}, u_j, u_{j+1}) = \mathrm{sign}_i(u)$. We have then proved:

$$\sum_{a=0}^{m} \mathrm{sign}_i\left(u \sqcup\!\sqcup^{Y_a} v\right) \mathrm{del}_i\left(u \sqcup\!\sqcup^{Y_a} v\right) = \mathrm{sign}_i(u) \mathrm{del}_i(u) \sqcup\!\sqcup^Y v.$$

From the relation $\mathrm{Sh}_{k,\ell} = \{Y_a \,|\, Y \in \mathrm{Sh}_{k-1,\ell} \,\mathrm{and}\, a \in [0, y_{j+1} - y_j - 1]\}$, we get:

$$\begin{aligned}
\partial_k^i(\sigma) \sqcup\!\sqcup\, \tau &= \sum_{Y \in \mathrm{Sh}_{k-1,\ell}} \mathrm{sign}_i(u)\ \mathrm{del}_i(u) \sqcup\!\sqcup^Y v, \\
&= \sum_{Y \in \mathrm{Sh}_{k-1,\ell}} \sum_{a=0}^{y_{j+1}-y_j-1} \mathrm{sign}_i\left(u \sqcup\!\sqcup^{Y_a} v\right) \mathrm{del}_i\left(u \sqcup\!\sqcup^{Y_a} v\right), \\
&= \sum_{X \in \mathrm{Sh}_{k,\ell}} \mathrm{sign}_i\left(u \sqcup\!\sqcup^{X} v\right) \mathrm{del}_i\left(u \sqcup\!\sqcup^{X} v\right), \\
&= \partial_{k+\ell}^i\left(\sigma \sqcup\!\sqcup\, \tau\right).
\end{aligned}$$

We prove (ii) with a similar argument, exchanging the role of u and v. □

Corollary 3.15. The map ∂ is a derivation of $(\mathbf{BFQSym}, \sqcup\!\sqcup)$.

Proof. Let σ and τ be two signed permutations of $\mathfrak{S}_k^{\pm}$ and $\mathfrak{S}_\ell^{\pm}$. By definition of ∂, we have the relation:

$$\partial(\sigma \sqcup\!\sqcup\, \tau) = \sum_{i=1}^{n} \partial_{k+\ell}^i(\sigma \sqcup\!\sqcup\, \tau) = \sum_{i=1}^{k} \partial_{k+\ell}^i(\sigma \sqcup\!\sqcup\, \tau) + \sum_{i=k+1}^{k+\ell} \partial_{k+\ell}^i(\sigma \sqcup\!\sqcup\, \tau).$$

Thus, by Lemma 3.14, we obtain:

$$\partial(\sigma \sqcup\!\sqcup\, \tau) = \sum_{i=1}^{k} \partial_k^i(\sigma) \sqcup\!\sqcup\, \tau + \sum_{i=1}^{\ell} \sigma \sqcup\!\sqcup\, \partial_\ell^i(\tau),$$

and so $\partial(\sigma \sqcup\!\sqcup\, \tau) = \partial(\sigma) \sqcup\!\sqcup\, \tau + \sigma \sqcup\!\sqcup\, \partial(\tau)$. □

From the compatibility of ∂ with the shuffle product $\sqcup\!\sqcup$, we determine the image of P_n under the derivation ∂.

Lemma 3.16. For all $n \geqslant 1$, we have $\partial(I_n) = (n-1)I_{n-1}$, $\partial(J_n) = (n-2)J_{n-1}$ and $\partial(P_n) = (n-2)P_{n-1}$, with the conventions $I_0 = J_0 = P_0 = \emptyset$.

Proof. For $n \geqslant 1$, we have:

$$\partial(I_n) = \sum_{i=1}^{n} \partial_i(I_n) = \sum_{i=1}^{n} \mathrm{sign}_i(I_n) I_{n-1}.$$

By definition of sign, we have $\mathrm{sign}_1(I_n) = ... = \mathrm{sign}_{n-1}(I_n) = 1$ and $\mathrm{sign}_n(I_n) = 0$. These imply $\partial(I_n) = (n-1)I_{n-1}$. Similary, since $\mathrm{sign}_1(J_n) = -1$, $\mathrm{sign}_k(J_n) = 1$ for $k \in [2, n-1]$, $\mathrm{sign}_n(J_n) = 0$ and $\mathrm{del}_i(J_n) = J_{n-1}$, we obtain the relation $\partial(J_n) = (n-2)J_{n-1}$ for $n \geqslant 1$. Let us now prove $\partial(P_n) = (n-2)P_{n-1}$.

By convention, we have $\partial(I_0) = \partial(J_0) = 0$. Using Lemma 2.14 and the compatibility of ∂ and $\sqcup\!\sqcup$ given in Lemma 3.14, we obtain:

$$\begin{aligned}
\partial(P_n) &= \partial\left(\sum_{k=0}^{n} J_k \sqcup\!\sqcup I_{n-k}\right),\\
&= \sum_{k=0}^{n} \partial(J_k) \sqcup\!\sqcup I_{n-k} + \sum_{k=0}^{n} J_k \sqcup\!\sqcup \partial(I_{n-k}),\\
&= \sum_{k=1}^{n} (k-2) J_{k-1} \sqcup\!\sqcup I_{n-k} + \sum_{k=0}^{n} (n-k-1) J_k \sqcup\!\sqcup I_{n-k-1},\\
&= \sum_{k=0}^{n-1} (k-1) J_k \sqcup\!\sqcup I_{n-1-k} + \sum_{k=0}^{n} (n-k-1) J_k \sqcup\!\sqcup I_{n-1-k},\\
&= (n-2) \sum_{k=0}^{n-1} J_k \sqcup\!\sqcup I_{n-1-k} = (n-2) P_{n-1}.
\end{aligned}$$
□

The proof of the surjectivity of ∂_n given in Proposition 3.18 uses a triangular argument that we illustrate on an example:

Example 3.17. Let $\sigma = \sigma_1$ be the permutation $(2, -1, 4, 5, -3)$ of $\mathfrak{S}_5^{\pm}$. We look for the maximal sequence of the form $k \dots 5$ or $-k \dots -5$ in the word $w(\sigma)$. In our example, this sequence is $4, 5$. We define τ_1 to be the permutation $(2, -1, 4, 5, 6, -3)$ obtained form σ_1 by replacing $4, 5$ by $4, 5, 6$. A direct computation gives $\partial_6(\tau_1) = 2\sigma_1 - \sigma_2$ with $\sigma_2 = (2, -1, 3, 4, 5)$, which is the standardization of $(2, -1, 4, 5, 6)$. Hence, we obtain:

$$\sigma_1 = \partial_6\left(\frac{1}{2}\tau_1\right) + \frac{1}{2}\sigma_2. \tag{3}$$

The maximal sequence of the desired form in σ_2 is $3, 4, 5$, which is longer than this of σ_1. We then define τ_2 to be $(2, -1, 3, 4, 5, 6)$ and we compute $\partial_6(\tau_2) = 3\sigma_2$. Hence σ_2 is equal to $\partial_6\left(\frac{1}{3}\tau_2\right)$ and, eventually, substituting this to (3), we obtain:

$$\sigma_1 = \partial_6\left(\frac{1}{2}\tau_1\right) + \partial_6\left(\frac{1}{6}\tau_2\right) = \partial_6\left(\frac{1}{2}(2, -1, 4, 5, 6, -3) + \frac{1}{6}(2, -1, 3, 4, 5, 6)\right).$$

Proposition 3.18. *For all $n \in \mathbb{N}$, the map $\partial_{n+1} : \mathbb{Q}\mathfrak{S}_{n+1}^{\pm} \to \mathbb{Q}\mathfrak{S}_n^{\pm}$ is surjective.*

Proof. Let σ be a permutation of $\mathfrak{S}_n^{\pm}$. We denote by u the word $w(\sigma)$. We have two cases, depending on which from n or $-n$ appears in u.

Case n appears in u: we define $i(\sigma)$ to be the minimal integer such that u can be written as $v \cdot [i \dots n] \cdot w$. We use an induction on $i(\sigma)$. If $i(\sigma)$ is 1 then $\sigma = I_n$. As Lemma 3.16 gives:

$$\partial_{n+1}(I_{n+1}) = nI_n,$$

we obtain $\sigma = \partial_{n+1}(\frac{1}{n}I_{n+1})$. Assume now $i = i(\sigma) > 1$. Let u' be the word $v \cdot [i \dots n+1] \cdot w$. Since each letter of v and w are smaller than $i-1$, the word u' belongs to $\mathrm{W}_{n+1}^{\pm}$. For $j \in \{i, \dots, n\}$ we have $\mathrm{del}_j(u') = u$ and $\mathrm{sign}_j(u') = 1$. As the first letter of w is smaller than $n+1$, we obtain $\mathrm{sign}_{n+1}(u') = 0$, and so:

$$\sum_{j=i}^{n+1} \partial_{n+1}^{j}(\rho(u')) = (n-i+1)\sigma,$$

with $n-i+1 \neq 0$, since $i \leqslant n$. Let j be in $\{1, ..., i-1\}$. Since the letter $\pm j$ appears only in v or in w and $|j| < i$, we have:

$$\mathrm{del}_j(u') = v\langle j\rangle \cdot [i-1 \ ... \ n] \cdot w\langle j\rangle \,.$$

We then obtain that $\partial_{n+1}(\rho(u'))$ is the sum of $(n-i+1)\sigma$ and a linear combination of permutations $\alpha_1, ..., \alpha_k$ of $\mathfrak{S}_n^{\pm}$ satisfying $i(\alpha_j) = i-1 < i = i(\sigma)$. By the induction hypothesis, the α_j's belong to $\mathrm{Im}(\partial_{n+1})$, which implies $\sigma \in \mathrm{Im}(\partial_{n+1})$.

Case n does not appear in u: hence, $-n$ appears in u. For $\ell \geqslant 1$, we denote by K_ℓ the permutation $(-1, ..., -\ell)$ of $\mathfrak{S}_\ell^{\pm}$. A direct computation gives $\partial(K_\ell) = \ell K_{\ell-1}$ for $\ell \geqslant 1$ and $\partial(K_1) = 0$.

We now define $i(\sigma)$ to be the minimal integer such that u can be written as $v \cdot [-i \ ... \ -n] \cdot w$. We use also an induction on $i(\sigma)$. If $i(\sigma) = 1$, then $\sigma = K_n$ and so, $\partial_{n+1}(K_{n+1})$ is $-(n+1)K_n$, implying:

$$\sigma = \partial_{n+1}\left(-\frac{1}{n+1}K_{n+1}\right).$$

Assume now $i = i(\sigma) > 1$. We denote by u' the word $v \cdot [-i \ ... \ -(n+1)] \cdot w$ of $\mathrm{W}_{n+1}^{\pm}$ and by τ the corresponding permutation of $\mathfrak{S}_{n+1}^{\pm}$. For $j < i$, we have:

$$\mathrm{del}_j(u') = v\langle j\rangle \cdot [-(i-1) \ ... \ -n] \cdot w\langle j\rangle \,.$$

Hence $\alpha = \sum_{j=1}^{i-1} \partial_{n+1}^j(\rho(u'))$ is a linear combination of permutations $\alpha_1, ..., \alpha_k \in \mathfrak{S}_n^{\pm}$ satisfying $i(\alpha_j) < i = i(\sigma)$ which, by induction hypothesis, implies $\alpha \in \mathrm{Im}(\partial_{n+1})$. It remains to establish that $\beta = \sum_{j=i}^{n+1} \partial_{n+1}^j(\rho(u'))$ is a multiple of σ. For $j \in \{i, ..., n\}$, we have $\mathrm{del}_j(u') = u$ and $\mathrm{sign}_j(u') = -1$. If w is empty, then $\mathrm{sign}_{n+1}(\rho(u'))$ is equal to -1 and $\mathrm{del}_{n+1}(u') = u$. Therefore, in this case, β is equal to $-(n+2-i)\sigma$ with $n+2-i \neq 0$, since $i \leqslant n$. If w is not empty, then its first letter is greater than $-(n+1)$, implying $\mathrm{sign}_{n+1}(\rho(u')) = 0$. Then, $\beta = -(n+1-i)\sigma$ with $n+1-i \neq 0$, since $i \leqslant n$. In all cases, we obtain that σ belongs to the image of ∂_{n+1}. □

Corollary 3.19. The map ∂ is a surjective derivation of $(\mathbf{BFQSym}, ⧢)$.

This is a direct consequence of Corollary 3.15 and Proposition 3.18.

3.2. **Commutation of ∂ and Φ.** We shall now prove that ∂ and Φ commutes. We start with two intermediate results.

Lemma 3.20. For all $\sigma \in \mathfrak{S}_n^{\pm}$ and $j \in [1, n]$, we have:

$$\mathrm{Des}\left(\mathrm{del}_{|\sigma(j)|}(w(\sigma))\right) = \begin{cases} D_j & \text{if } \sigma(j-1) < \sigma(j+1); \\ D_j \cup \{j-1\} & \text{if } \sigma(j-1) > \sigma(j+1), \end{cases}$$

where $D_j = \mathrm{Des}\,(\sigma) \cap [0, j-2] \cup \{d-1 \mid d \in \mathrm{Des}\,(\sigma) \cap [j+1, n]\}$, and again with the convention $\sigma(0) = 0$.

Proof. Let σ and j be as in the statement. We denote by u the word $w(\sigma)$, i the positive integer $|\sigma(j)|$. We also denote by v the word $\mathrm{del}_i(u)$ and by τ the permutation $\rho(v)$. The word $v = v_1 \dots v_{n-1}$ is then defined by:

$$v_k = \begin{cases} u_k\langle i\rangle & \text{if } k \leqslant j-1, \\ u_{k+1}\langle i\rangle & \text{if } k \geqslant j, \end{cases}$$

where u_k and v_k are the k-th letter of u and v respectively. For $k \in [0, n-1]$, we have $u_k\langle i\rangle > u_{k+1}\langle i\rangle$ if, and only if, $u_k > u_{k+1}$ holds. Hence, each k in $[0, j-2]$ is a descent of τ if, and only if, k is a descent of σ. Similarly, each k in $[j, n-2]$ is a descent of τ if, and only if, $k+1$ is a descent of σ. Considering the set D_j defined in the statement, we have:

$$\mathrm{Des}\,(\tau) \cap ([0, n-2] \setminus \{j-1\}) = D_j.$$

We cannot determine if $j-1$ is a descent of τ from $\mathrm{Des}\,(\sigma)$. We only remark that the integer $j-1$ is a descent of τ if, and only if, $v_{j-1} > v_j$, hence if, and only if, we have $u_{j-1} > u_{j+1}$, as expected. □

Lemma 3.21. Let σ be a permutation of $\mathfrak{S}_n^{\pm}$ and $\{d_1 < \dots < d_\ell\}$ be the set of its non-zero descents. For i in $[1, \ell]$, we have:

$$\mathrm{Des}\left(\sum_{e=d_i+1}^{d_{i+1}} \partial_n^{|\sigma(e)|}(\sigma)\right) = (d_{i+1} - d_i - 2)\mathrm{Des}\,(\sigma)\langle d_{i+1}\rangle\,, \tag{4}$$

with the convention $d_{\ell+1} = n$. Moreover, for $d_1 > 0$, i.e., $0 \notin \mathrm{Des}\,(\sigma)$, we have:

$$\mathrm{Des}\left(\sum_{e=1}^{d_1} \partial_n^{|\sigma(e)|}(\sigma)\right) = \begin{cases} (d_1 - 1)\mathrm{Des}\,(\sigma)\langle d_1\rangle & \text{if } 0 \notin \mathrm{Des}\,(\sigma), \\ (d_1 - 2)\mathrm{Des}\,(\sigma)\langle d_1\rangle & \text{if } 0 \in \mathrm{Des}\,(\sigma). \end{cases} \tag{5}$$

Proof. Let σ be a permutation of $\mathfrak{S}_n^{\pm}$ and $\{d_1 < \dots < d_\ell\}$ the set of its positive descents. Let i be an integer in $[1, \ell]$. As in definition 3.5 we use the convention $\sigma(n+1) = -\infty$. We start proving (4) using three subcases.

Case $d_{i+1} > d_i + 2$. We have:

$$\sigma(d_i) > \sigma(d_i + 1) < \dots < \sigma(d_{i+1} - 1) < \sigma(d_{i+1}) > \sigma(d_{i+1} + 1).$$

By definition of sign, the terms $\partial_n^{|\sigma(d_i+1)|}(\sigma)$ and $\partial_n^{|\sigma(d_{i+1})|}(\sigma)$ are equal to 0. For e an integer of $[d_i + 2, d_{i+1} - 1]$, the value of $\mathrm{sign}_{|\sigma(e)|}(w(\sigma))$ is 1. By Lemma 3.20, since the relation $\sigma(e-1) < \sigma(e+1)$ holds, we have:

$$\begin{aligned} \mathrm{Des}\left(\rho\left(\mathrm{del}_{|\sigma(e)|}(w(\sigma))\right)\right) &= \mathrm{Des}\,(\sigma) \cap [0, e-2] \cup \{d - 1 \mid d \in \mathrm{Des}\,(\sigma) \cap [e+1, n]\} \\ &= \{d_1, .., d_i, d_{i+1} - 1, ..., d_\ell - 1\} \\ &= \mathrm{Des}\,(\sigma)\langle d_{i+1}\rangle\,. \end{aligned}$$

We conclude remarking that the cardinality of $[d_i + 2, d_{i+1} - 1]$ is $d_{i+1} - d_i - 2$.

Case $d_{i+1} = d_i + 2$. We have:

$$\sigma(d_i) > \sigma(d_i + 1) < \sigma(d_{i+1}) > \sigma(d_{i+1} + 1).$$

As for $e \in [d_i + 1, d_{i+1}]$, we have $\mathrm{sign}_{|\sigma(e)|}(w(\sigma)) = 0$, the left hand side of (4) is 0.

Case $d_{i+1} = d_i + 1$. We have $\sigma(d_i) > \sigma(d_{i+1}) > \sigma(d_{i+1} + 1)$. In this case, $\mathrm{sign}_{|\sigma(d_{i+1})|}(w(\sigma))$ is -1. By Lemma 3.20, the descents of $\mathrm{del}_{|\sigma(d_i+1)|}(w(\sigma))$ are:

$$\{d_1, \dots, d_{i-1}, d_{i+1} - 1, \dots, d_k - 1\} \cup \{d_i\} = \mathrm{Des}\,(\sigma)\langle d_{i+1}\rangle$$

since $\sigma(d_i) > \sigma(d_i + 2)$ holds. We conclude by remarking that $d_{i+1} - d_i - 2 = -1$ occurs in this case.

Relation (5) is proved similarly, with a particular attention on 0. □

Theorem 3.1. *The endomorphisms Φ and ∂ commute.*

Proof. Let σ be a permutation of $\mathfrak{S}_n^{\pm}$. Let us denote by $\{d_1 < ... < d_\ell\}$ the set of non-zero descents of σ. For $i \in [1, \ell]$ we denote by k_i the integer $d_i - d_{i-1}$, with the convention $d_0 = 0$ and $d_{\ell+1} = n$. For $k \in \mathbb{N}$, we define X_k and x_k by:

$$X_k = \begin{cases} I_k & \text{for } 0 \notin \mathrm{Des}\,(\sigma), \\ P_k & \text{for } 0 \in \mathrm{Des}\,(\sigma); \end{cases} \quad \text{and} \quad x_k = \begin{cases} k-1 & \text{for } 0 \notin \mathrm{Des}\,(\sigma), \\ k-2 & \text{for } 0 \in \mathrm{Des}\,(\sigma). \end{cases}$$

By Proposition 3.1, we have $\Phi(\sigma) = X_{k_1} \sqcup\!\sqcup P_{k_2} \sqcup\!\sqcup ... \sqcup\!\sqcup P_{k_{\ell+1}}$. Since ∂ is a derivation, by Corollary 3.19, the previous relation gives:

$$\begin{aligned}(\partial \circ \Phi)(\sigma) =& \partial(X_{k_1}) \sqcup\!\sqcup P_{k_2} \sqcup\!\sqcup ... \sqcup\!\sqcup P_{k_{\ell+1}} \\ &+ \sum_{i=2}^{\ell+1} X_{k_1} \sqcup\!\sqcup ... \sqcup\!\sqcup P_{k_{i-1}} \sqcup\!\sqcup \partial(P_{k_i}) \sqcup\!\sqcup P_{k_{i+1}} ... \sqcup\!\sqcup P_{k_{\ell+1}},\end{aligned}$$

and so, using Lemma 3.16, we obtain:

$$\begin{aligned}(\partial \circ \Phi)(\sigma) =& x_{k_1} X_{k_1-1} \sqcup\!\sqcup P_{k_2} \sqcup\!\sqcup ... \sqcup\!\sqcup P_{k_{\ell+1}} \\ &+ \sum_{i=2}^{\ell+1} (k_i - 2) X_{k_1} \sqcup\!\sqcup P_{k_2} \sqcup\!\sqcup ... \sqcup\!\sqcup P_{k_{i-1}} \sqcup\!\sqcup P_{k_i-1} \sqcup\!\sqcup P_{k_{i+1}} \sqcup\!\sqcup ... \sqcup\!\sqcup P_{k_{\ell+1}}.\end{aligned}$$

On the other hand, by Lemma 3.21, we have:

$$\mathrm{Des}\,(\partial(\sigma)) = x_{k_1} \mathrm{Des}\,(\sigma)\langle d_1\rangle + \sum_{i=2}^{\ell+1} (k_i - 2) \mathrm{Des}\,(\sigma)\langle d_i\rangle\,.$$

By Proposition 3.1 we obtain:

$$\widetilde{\Phi}_n(\mathrm{Des}\,(\sigma)\langle d_1\rangle) = X_{k_1-1} \sqcup\!\sqcup P_{k_2} \sqcup\!\sqcup ... \sqcup\!\sqcup P_{k_{\ell+1}},$$

and for i in $[2, n]$ we have:

$$\widetilde{\Phi}_n(\mathrm{Des}\,(\sigma)\langle d_i\rangle) = X_{k_1} \sqcup\!\sqcup P_{k_2} \sqcup\!\sqcup ... \sqcup\!\sqcup P_{k_{i-1}} \sqcup\!\sqcup P_{k_i-1} \sqcup\!\sqcup P_{k_{i+1}} \sqcup\!\sqcup ... \sqcup\!\sqcup P_{k_{\ell+1}}.$$

Since $(\Phi \circ \partial)(\sigma) = (\widetilde{\Phi}_n(\mathrm{Des}\,(\partial(\sigma))))$, we have established $(\Phi \circ \partial)(\sigma) = (\partial \circ \Phi)(\sigma)$. □

We can now prove the main theorem.

Proof of Theorem 1.1. Let n be an integer. By Corollary 3.19, the map ∂ is a surjective derivation of $\mathbb{Q}\mathfrak{S}^{\pm}$, which, by Theorem 3.1, commutes with Φ. Proposition 3.3 guarantees that the characteristic polynomial of Φ_n divides the one of Φ_{n+1}. Since the characteristic polynomial of Φ_n is the one of Adj_{B_n}, we have established the expected divisibility result. □

4. Other types

In this section, we discuss about the becoming of the divisibility result for other infinite Coxeter families, and we describe the combinatorics of normal sequences of braids for some exceptionnal types.

Let Γ be a finite connected Coxeter graph. From a computational point of view, the matrix Adj_Γ is too huge, as its size is exactly the number of elements in W_Γ, whose growth in an exponential in n for the family A_n, B_n and D_n.

The definition of the descent set given in Definition 1.22 has a counterpart in W_Γ for every Coxeter graph Γ (the reader can consult [1] for more details on the subject).

Definition 4.1. For Γ a Coxeter graph we define a square matrix $\mathrm{Adj}'_\Gamma = (a'_{I,J})$ indexed by the subset of vertices of Γ by:

$$a'_{I,J} = \mathrm{card}\{w \in W_\Gamma \mid \mathrm{Des}\left(w^{-1}\right) = I \text{ and } J \subseteq \mathrm{Des}\,(w)\}.$$

For Γ a graph of the family A_n, B_n and D_n, the size of Adj'_Γ is 2^n, which is smaller than $n!, 2^n n!$ and $2^{n-1}n!$ respectively.

For any subset J of Γ, we denote by $b^d_\Gamma(J)$, the numbers of positive braids of $B^+(W_\Gamma)$ whose Garside normal form is $(w_1, ..., w_d)$ with $\mathrm{Des}\,(w_d) \subset J$. An immediate adaptation of Lemma 2.12 of [3] gives:

Lemma 4.2. For Γ a finite connected Coxeter graph, there exists an integer k such that the characteristic polynomial $\chi_\Gamma(x)$ of Adj_Γ is equal to $x^k \chi'_\Gamma(x)$ where $\chi'_\Gamma(x)$ is the one of Adj'_Γ. Moreover, for $d \geqslant 1$ and $J \subset \Gamma$, we have:

$$b^d_\Gamma(J) = {}^tY\,(\mathrm{Adj}'_\Gamma)^{d-1} J \quad \text{where} \quad Y_I = \begin{cases} 0 & \text{if } I = \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

In order to determine the numbers b^d_Γ of braids of $B^+(W_\Gamma)$ whose Garside length is d form Adj'_Γ, we use an inclusion exclusion principle.

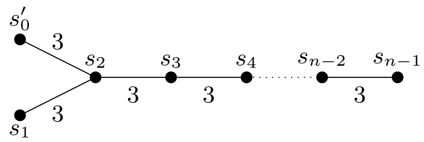
Corollary 4.3. For Γ a finite connected Coxeter graph and $d \geqslant 1$, we have:

$$b^d_\Gamma = {}^tY\,(\mathrm{Adj}'_\Gamma)^{d-1} Z \quad \text{where} \quad Z_I = \begin{cases} 0 & \text{if } I = \emptyset, \\ (-1)^{\mathrm{card}(I)+1} & \text{otherwise,} \end{cases}$$

and Y as in Lemma 4.2.

4.1. **Braids of type** D. For $n \geqslant 4$, the Coxeter graph of type D and rank n is:

Γ_{D_n} :



and the associated Coxeter group is isomorphic to the subgroup of $\mathfrak{S}^{\pm}_{n+1}$ consisting of all signed permutations with an even number of negative entries. Its generators are the signed permutations s_i for $i \in [1, n-1]$, plus the signed permutation $s'_0 = (-2, -1, 3, ..., n)$. We extend the family D_n defined for $n \geqslant 4$ to include $D_1 = A_1$, $D_2 = A_1 \times A_1$ and $D_3 = A_3$. Note that we usually only consider $n \geqslant 4$ in order to have a classification of irreducible Coxeter groups without redundancy.

Denoting by χ_{D_n} the characteristic polynomial of the adjacent matrix Adj_{D_n} of normal sequences of positive braid of type D and rank n, we obtain:

$\chi_{D_1}(x) = (x-1)^2,$

$\chi_{D_2}(x) = (x-1)^4,$

$\chi_{D_3}(x) = x^{19}\ (x-1)^2\ (x-2)\ (x^2 - 6x + 3),$

$\chi_{D_4}(x) = x^{181}\ (x-1)^6\ (x^5 - 44x^4 + 402x^3 - 1084x^2 + 989x - 360),$

$$\begin{aligned}\chi_{D_5}(x) = x^{1906}\ (x-1)^2\ (&x^{12} - 302x^{11} + 17070x^{10} - 328426x^9 + 3077800x^8 \\ &- 16424030x^7 + 4072794x^6 - 113921686x^5 + 154559655x^4 \\ &- 132533636x^3 + 68372600x^2 - 18880000x + 2016000).\end{aligned}$$

As the reader can check, there is no hope to have a divisibility of $\chi_{D_{n+1}}$ by χ_{D_n} except for $n = 1$. The associated generating series are:

$$
\begin{aligned}
F_{D_2}(t) &= \frac{3-t}{(t-1)^2},\\
F_{D_3}(t) &= \frac{-6t^3+15t^2-20t+23}{(t-1)(2t-1)(3t^2-6t-1)},\\
F_{D_4}(t) &= \frac{-360t^5+1709t^4-2246t^3+852t^2+430t+191}{(t-1)(-1+44t-402t^2+1084t^3-989t^4+360t^5)}.
\end{aligned}
$$

which give the following values for the number of D-braids of rank n and of Garside length d:

d	$b_{D_2}(d)$	$b_{D_3}(d)$	$b_{D_4}(d)$
0	1	23	191
1	3	187	9025
2	5	1169	321791
3	7	6697	10737025
4	9	37175	352664255
5	11	203971	11540908225

4.2. **Braids of type I.** For $n \geqslant 2$, the Coxeter graph I_n is:

$$\Gamma_{I_n} : \quad \overset{s}{\bullet}\!\overset{n}{\text{———}}\!\overset{t}{\bullet},$$

which gives the following presentation for the Coxeter group W_{I_n}:

$$W_{I_n} = \left\langle s, t \;\middle|\; \begin{array}{c} s^2 = 1, t^2 = 1 \\ \mathrm{prod}(s,t;n) = \mathrm{prod}(t,s;n) \end{array} \right\rangle.$$

Proposition 4.4. *For $n \geqslant 2$, we have:*

$$\mathrm{Adj}'_{I_n} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ n-1 & b_n & a_n & 0 \\ n-1 & a_n & b_n & 0 \\ n & 1 & 1 & 1 \end{bmatrix},$$

with $a_n = \lfloor \frac{n-1}{2} \rfloor$ and $b_n = \lfloor \frac{n}{2} \rfloor$.

Proof. The elements of W_{I_n} are 1, $w_n = \mathrm{prod}(s,t;n) = \mathrm{prod}(t,s;n)$ and $\mathrm{prod}(s,t;k)$ with $\mathrm{prod}(t,s;k)$ for k in $[1, n-1]$. For k in $[1, n-1]$, we have:

$$\mathrm{prod}(s,t;k)^{-1} = \begin{cases} \mathrm{prod}(t,s;k) & \text{if } k \text{ even,} \\ \mathrm{prod}(s,t;k) & \text{otherwise;} \end{cases}$$

$$\mathrm{Des}\,(\mathrm{prod}(s,t;k)) = \begin{cases} t & \text{if } k \text{ even,} \\ s & \text{otherwise.} \end{cases}$$

From the relation $\mathrm{prod}(s,t;n) = \mathrm{prod}(t,s;n)$ we have $w_n = \mathrm{prod}(s,t;n)^{-1} = \mathrm{prod}(s,t;n)$ and so $\mathrm{Des}\,(w_n) = \{s,t\}$. We organize the elements of $W_{I_n} \setminus \{1, w_n\}$ in 4 sets:

$$
\begin{aligned}
X_1 &= \{\mathrm{prod}(s,t;k) \text{ for } k \text{ even}\}, & X_2 &= \{\mathrm{prod}(s,t;k) \text{ for } k \text{ odd}\},\\
X_3 &= \{\mathrm{prod}(t,s;k) \text{ for } k \text{ even}\}, & X_4 &= \{\mathrm{prod}(t,s;k) \text{ for } k \text{ odd}\}.
\end{aligned}
$$

From the previous study of descents, we obtain:

$\sigma \in$	$\{1\}$	X_1	X_2	X_3	X_4	$\{w_n\}$
$\mathrm{Des}(\sigma)$	$\emptyset$	$\{t\}$	$\{s\}$	$\{s\}$	$\{t\}$	$\{s,t\}$
$\mathrm{Des}(\sigma^{-1})$	$\emptyset$	$\{s\}$	$\{s\}$	$\{t\}$	$\{t\}$	$\{s,t\}$

Denoting by a_n and b_n the integers $\lfloor \frac{n-1}{2} \rfloor$ and $\lfloor \frac{n}{2} \rfloor$ respectively, we obtain that $\mathrm{card}(X_1) = \mathrm{card}(X_3) = a_n$ and $\mathrm{card}(X_2) = \mathrm{card}(X_4) = b_n$. For I, J subsets of $\{s,t\}$ we define $A'_{I,J}$ to be the set $\{\sigma \in W_{I_n} \mid \mathrm{Des}(\sigma^{-1}) = I \text{ and } J \subseteq \mathrm{Des}(w)\}$. For all $K \subset \{s,t\}$ we have $A'_{\{s,t\},K} = \{w_n\}$. We have $A'_{\emptyset,\emptyset} = \{1\}$ and $A'_{\emptyset,K} = \emptyset$ for $K \neq \emptyset$. From the X_i's we get:

$$
\begin{aligned}
&A'_{\{s\},\emptyset} = X_1 \sqcup X_2, \quad A'_{\{s\},\{s\}} = X_2, \quad A'_{\{s\},\{t\}} = X_1, \quad A'_{\{s\},\{s,t\}} = \emptyset,\\
&A'_{\{t\},\emptyset} = X_3 \sqcup X_4, \quad A'_{\{t\},\{s\}} = X_3, \quad A'_{\{t\},\{t\}} = X_4, \quad A'_{\{t\},\{s,t\}} = \emptyset.
\end{aligned}
$$

Using the enumeration $\{\emptyset, \{s\}, \{t\}, \{s,t\}\}$ of subsets of $\{s,t\}$ together with the relation $a_n + b_n = n-1$ we obtain:

$$
\mathrm{Adj}'_{I_n} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_n + b_n & b_n & a_n & 0 \\ a_n + b_n & a_n & b_n & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ n-1 & b_n & a_n & 0 \\ n-1 & a_n & b_n & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \qquad \square
$$

Corollary 4.5. The characteristic polynomial of Adj_{I_n} is:

$$
\chi_{I_n}(x) = \begin{cases} x^{2n-4}(x-1)^3(x-n+1) & \text{if } x \text{ is even,} \\ x^{2n-3}(x-1)^2(x-n+1) & \text{otherwise.} \end{cases}
$$

and the generating series of normal sequence of I_n-braids is:

$$
F_{I_n}(t) = \frac{(n-1)t+1}{((n-1)t-1)(t-1)}.
$$

Proof. From the expression of Adj'_{I_n} given in Proposition 4.4, we obtain:

$$
\begin{aligned}
\chi_{\mathrm{Adj}'_{I_n}}(x) &= (1-x)^2((b_n - x)^2 - a_n^2), \\
&= (1-x)^2(b_n + a_n - x)(b_n - a_n - x), \\
&= (x-1)^2(x-(b_n+a_n))(x-(b_n-a_n)).
\end{aligned}
$$

From the relations:

$$
a_n + b_n = n-1, \qquad\qquad b_n - a_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}
$$

we obtain:

$$
\chi_{\mathrm{Adj}'_{I_n}}(x) = \begin{cases} (x-1)^3(x-n+1) & \text{if } x \text{ is even,} \\ x(x-1)^2(x-n+1) & \text{otherwise.} \end{cases}
$$

Adding the missing powers of x to obtain a degree of $2n$ we obtain the expected value for χ_{I_n}.

For generating series results, Corollary 4.3 gives:

$$
F_{I_n}(t) = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \; (I_4 - t\, \mathrm{Adj}'_{I_n})^{-1} \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}.
$$

By a direct computation (or a use of `Sage` [15] for example) we obtain:

$$F_{I_n}(t) = \frac{(n-1)t+1}{((n-1)t-1)(t-1)}. \qquad \square$$

4.3. **Exceptional Coxeter groups.** Using Adj'_Γ, we can study the combinatorics of normal sequence of braids of type F_4, H_3, H_4, E_6 and E_7. The matrices Adj'_Γ were obtained using `Sage` [15], while the characteristic polynomials and generating series was obtained using the `C` library `flint` [8].

The group W_{F_4} has 1152 elements. The characteristic polynomial of Adj_{F_4} is:

$$\begin{aligned}\chi_{F_4}(x) =& x^{1140}\,(x-1)^3\,(x-4)\,(x^2-25\,x+10)\\ &(x^6-274\,x^5+9194\,x^4-77096\,x^3+250605\,x^2-324870\,x+138600),\end{aligned}$$

and the generating series F_{F_4} is given by:

$$F_{F_4}(t) = \frac{138600\,t^6-187350\,t^5-32055\,t^4+87970\,t^3-15504\,t^2-876\,t-1}{(138600\,t^6-324870\,t^5+250605\,t^4-77096\,t^3+9194\,t^2-274\,t+1)(t-1)}.$$

The group W_{H_3} has 120 elements. The characteristic polynomial of Adj_{H_3} is:

$$\chi_{H_3}(x) = x^{114}\,(x-1)^2\,(x^4-42\,x^3+229\,x^2-244\,x+72),$$

and the generating series F_{H_3} is given by:

$$F_{H_3}(t) = -\frac{72\,t^4-196\,t^3+77\,t^2+76\,t+1}{(72\,t^4-244\,t^3+229\,t^2-42\,t+1)(t-1)}.$$

The group W_{H_4} has 14400 elements. The characteristic polynomial of Adj_{H_4} is:

$$\begin{aligned}\chi_{H_4}(x) = x^{14390}\,(x-1)^2\,(&x^8-3436\,x^7+565470\,x^6-11284400\,x^5+81322353\,x^4\\ &-246756500\,x^3+305430848\,x^2-157717504\,x+27929088),\end{aligned}$$

and the generating series $F_{H_4}(t) = \frac{N_{H_4}(t)}{D_{H_4}(t)(t-1)}$ is given by:

$$\begin{aligned}N_{H_4}(t) = 27929088\,t^8 &- 147220480\,t^7+247258432\,t^6-138197780\,t^5\\ &+465433\,t^4+10247814\,t^3-1205944\,t^2-10962\,t-1,\end{aligned}$$

$$\begin{aligned}D_{H_4}(t) = 27929088\,t^8 &- 157717504\,t^7+305430848\,t^6-246756500\,t^5\\ &+81322353\,t^4-11284400\,t^3+565470\,t^2-3436\,t+1.\end{aligned}$$

The group W_{E_6} has 51840 elements. The characteristic polynomial of Adj_{E_6} is:

$$\begin{aligned}\chi_{E_6}(x) =& x^{51823}\,(x-1)^2\\ &(x^{15}-5454\,x^{14}+3391893\,x^{13}-424089882\,x^{12}+19590731031\,x^{11}\\ &-417118001254\,x^{10}+4673188683575\,x^9-29907005656510\,x^8\\ &+115900067128500\,x^7-282097630883500\,x^6+439789995997000\,x^5\\ &-441496921502000\,x^4+282303310340000\,x^3-110981554480000\,x^2\\ &+24563716800000\,x-2328480000000),\end{aligned}$$

and the generating series $F_{E_6}(t) = \frac{N_{E_6}(t)}{D_{E_6}(t)(t-1)}$ is given by:

$$\begin{aligned} N_{E_6}(t) = & 2328480000000\,t^{15} - 19422916800000\,t^{14} + 59384818480000\,t^{13} \\ & - 64287293380000\,t^{12} - 64835775106000\,t^{11} + 254118878161000\,t^{10} \\ & - 284082015723500\,t^{9} + 148526420487700\,t^{8} - 32460183476310\,t^{7} \\ & - 327255378405\,t^{6} + 1042966224156\,t^{5} - 93297805141\,t^{4} \\ & + 479267710\,t^{3} + 40099205\,t^{2} + 46384\,t + 1, \end{aligned}$$

$$\begin{aligned} D_{E_6}(t) = & 2328480000000\,t^{15} - 24563716800000\,t^{14} + 110981554480000\,t^{13} \\ & - 282303310340000\,t^{12} + 441496921502000\,t^{11} - 439789995997000\,t^{10} \\ & + 282097630883500\,t^{9} - 115900067128500\,t^{8} + 29907005656510\,t^{7} \\ & - 4673188683575\,t^{6} + 417118001254\,t^{5} - 19590731031\,t^{4} \\ & + 424089882\,t^{3} - 3391893\,t^{2} + 5454\,t - 1. \end{aligned}$$

The previous generating series gives the following values for $b_W(d)$, the numbers of W-braids of Garside length d:

d	$b_{F_4}(d)$	$b_{H_3}(d)$	$b_{H_4}(d)$	$b_{E_6}(d)$
0	1	1	1	1
1	1151	119	14399	51839
2	322561	4923	50126401	319483603
3	77804927	179717	164094364799	1567574732717
4	18441371521	6449741	535645654732801	7487770421878165
5	4362177487103	230926603	1748252504973355199	35655729684940971035

The characteristic polynomial and the generating series for braids of type E_7 are available at `http://www.lmpa.univ-littoral.fr/~fromentin/combi.html`.

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