Commutative and non-commutative bialgebras of quasi-posets and applications to Ehrhart polynomials

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Abstract

To any poset or quasi-poset is attached a lattice polytope, whose Ehrhart polynomial we study from a Hopf-algebraic point of view. We use for this two interacting bialgebras on quasi-posets. The Ehrhart polynomial defines a Hopf algebra morphism with values in $\mathbb{Q}[X]$; we deduce from the interacting bialgebras an algebraic proof of the duality principle, a generalization and a new proof of a result on B-series due to Wright and Zhao, using a monoid of characters on quasi-posets, and a generalization of Faulhaber's formula.

We also give non-commutative versions of these results: polynomials are replaced by packed words. We obtain in particular a non-commutative duality principle.

Keywords. Ehrhart polynomials; Quasi-posets; Characters monoids; Interacting bialgebras

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Contents

Introduction

Let P be a lattice polytope, that is to say that all its vertices are in $\{0,1\}^n$. The Ehrhart polynomial $ehr_P^{cl}(X)$ is such that for all $k \geq 1$, $ehr_P^{cl}(k)$ is the number of points of $\mathbb{Z}^n \cap kP$, where kP is the image of P by the homothety of center 0 and ratio k. For example, if S is the square $[0,1]^n$ and T is the triangle of vertices $(0,0)$, $(1,0)$ and $(1,1)$:

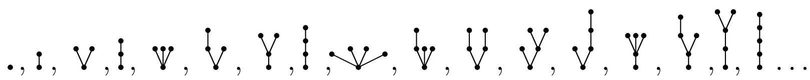
$$ehr_S^{cl}(X) = (X+1)^2, \qquad ehr_T^{cl}(X) = \frac{(X+1)(X+2)}{2}.$$

These polynomial satisfy the reciprocity principle: for all $k \geq 1$, $(-1)^{dim(P)} ehr^{cl}(-k)$ is the number of points of $\mathbb{Z}^n \cap kP'$, where P' is the interior of P. For example:

$$ehr_S^{cl}(-X) = (X-1)^2, \qquad ehr_T^{cl}(-X) = \frac{(X-1)(X-2)}{2}.$$

We refer to [2] for general results on Ehrhart polynomials.

It turns out that these polynomials appear in the theory of B-series (B is for Butcher [4]), as explained in [3, 6]. We now consider rooted trees:



If t is a rooted tree, we orient its edges from the root to the leaves. If i, j are two vertices of t, we shall write $i \xrightarrow{t} j$ if there is an edge from i to j in t.

To any rooted tree t, whose vertices are indexed by $1 \ldots n$, we associate a lattice polytope $pol(t)$ in a following way:

$$pol(t) = \left\{ (x_1, \ldots, x_n) \in [0,1]^n \mid \forall\, 1 \leq i,j \leq n, (i \xrightarrow{t} j) \Longrightarrow (x_i \leq x_j) \right\}$$

For example, if $t =$, indexed as $^2_1$, then $pol(t) = T$.

We can consider the Ehrhart polynomial $ehr_{pol(t)}^{cl}(X)$, which we shall simply denote by $ehr_t^{cl}(X)$: for all $k \geq 1$,

$$ehr_t^{cl}(k) = \sharp \left\{ (x_1, \ldots, x_n) \in \{0, \ldots, k\}^n \mid \forall\, 1 \leq i,j \leq n, (i \xrightarrow{t} j) \Longrightarrow (x_i \leq x_j) \right\}.$$

Note that ehr_t^{cl} does not depend on the indexation of the vertices of t. By the duality principle:

$$(-1)^n ehr_t^{cl}(-k) = \sharp \left\{ (x_1, \ldots, x_n) \in \{1, \ldots, k-1\}^n \mid \forall\, 1 \leq i,j \leq n, (i \xrightarrow{t} j) \Longrightarrow (x_i < x_j) \right\}.$$

A B-series is a formal series indexed by rooted trees, of the form:

$$\sum_t a_t \frac{t}{aut(t)} = a_{\bullet}\bullet + a_{\mathrm{I}}\,\mathrm{I} + a_{\mathrm{V}}\frac{\mathrm{V}}{2} + a_{\vdots}\vdots + \ldots,$$

where $aut(t)$ is the number of automorphisms of t. The following B-series is of special importance in numerical analysis:

$$E = \sum_t \frac{1}{t!}\frac{t}{aut(t)} = \bullet + \frac{1}{2}a_{\mathrm{I}}\,\mathrm{I} + \frac{1}{3}\frac{\mathrm{V}}{2} + \frac{1}{6}\vdots + \ldots,$$

where $t!$ is the tree factorial (see definition 30). This series is the formal solution of an ordinary differential equation describing the flow equation of a vector field. The set of B-series is given a group structure by a substitution operation, which is dually represented by the contraction-extraction coproduct defined in [5]. Its inverse is called the backward error analysis:

$$E^{-1} = \sum_t \lambda_t \frac{t}{aut(t)!}.$$

Wright and Zhao [18] proved that these coefficients λ_t are related to Ehrhart polynomials:

$$\lambda_t = (-1)^{|t|} \frac{dehr_t^{cl}(X)}{dX}_{|X=-1}.$$

We shall in this text study Ehrhart polynomial attached to quasi-posets in a combinatorial Hopf-algebraic way. A quasi-poset P is a pair $(A, \leq_P)$, where A is a finite set and $\leq_P$ is a reflexive and transitive relation on A. The isoclasses of quasi-posets are represented by their Hasse graphs:

$$1, \bullet, \bullet\bullet, \mathrm{I}, \bullet_2, \bullet\bullet\bullet, \bullet\mathrm{I}, \bullet\bullet_2, \mathrm{V}, \Lambda, \vdots, \mathrm{I}^2, \mathrm{I}_2, \bullet_3, \ldots$$

In particular, rooted trees can be seen as quasi-posets. For any quasi-poset $P = (\{1, \ldots, n\}, \leq_P)$, the polytope associated to P is:

$$top(P) = \{(x_1, \ldots, x_n) \in [0,1]^n \mid \forall\, 1 \leq i,j \leq n, (i \leq_P j) \Longrightarrow (x_i \leq x_j)\}.$$

We put $ehr_P(X) = ehr^{cl}_{top(P)}(X-1)$; note the translation by -1, which will give us an object more suitable for our purpose. In other words, for all $k \geq 1$:

$$ehr_P(k) = \sharp\{(x_1, \ldots, x_n) \in \{1, \ldots, k\}^n \mid \forall\, 1 \leq i,j \leq n, (i \leq_P j) \Longrightarrow (x_i \leq x_j)\}.$$

We also define a polynomial $ehr_P^{str}(X)$ such that for all $k \geq 1$:

$$ehr_P^{str}(k) = \sharp\{(x_1, \ldots, x_n) \in \{1, \ldots, k\}^n \mid \forall\, 1 \leq i,j \leq n, (i \leq_P j \text{ and not } j \leq_P i) \Longrightarrow (x_i < x_j)\}.$$

See definition 15 and proposition 16 for more details. These polynomials can be inductively computed, with the help of the minimal elements of P (proposition 20).

We shall consider two products m and $\downarrow$, and two coproducts Δ and δ on the space $\mathcal{H}_{\mathbf{qp}}$ generated by isoclasses of quasi-posets. The coproduct Δ, defined in [9, 10] by the restriction to open and closed sets of the topologies associated to quasi-posets, makes $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$ a graded, connected Hopf algebra and $(\mathcal{H}_{\mathbf{qp}}, \downarrow, \Delta)$ an infinitesimal bialgebra; the coproduct δ, defined in [8] by an extraction-contraction operation, makes $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$ a bialgebra. Moreover, δ is also a right coaction of $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$ over $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$, and $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$ becomes a Hopf algebra in the category of $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$-comodules, which we summarize telling that $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$ and $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$

are two bialgebras in cointeraction (definition 1). For example, the bialgebras $(\mathbb{K}[X], m, \Delta)$ and $(\mathbb{K}[X], m, \delta)$ where m is the usual product of $\mathbb{K}[X]$ and Δ, δ are the coproducts defined by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad\qquad \delta(X) = X \otimes X,$$

are two cointeracting bialgebras.

Ehrhart polynomials $ehr_P(X)$ and $ehr_P^{str}(X)$ can now be seen as maps from $\mathcal{H}_{\mathbf{qp}}$ to $\mathbb{K}[X]$, and both are Hopf algebra morphisms from $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$ to $\mathbb{K}[X]$ with its usual Hopf algebra structure (theorem 17); we shall prove in corollary 44 that ehr^{str} is the unique morphism from $\mathcal{H}_{\mathbf{qp}}$ to $\mathbb{K}[X]$ compatible with both bialgebra structures on $\mathcal{H}_{\mathbf{qp}}$ and $\mathbb{K}[X]$. Using the cointeraction between the two bialgebra structures on $\mathcal{H}_{\mathbf{qp}}$, we show that the monoid $M_{\mathbf{qp}}$ of characters of $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$ acts on the set $E_{\mathcal{H}_{\mathbf{qp}} \to \mathbb{K}[X]}$ of Hopf algebra morphisms from $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$ to $\mathbb{K}[X]$ (proposition 27). Moreover, there exists a particular homogeneous morphism $\phi_0 \in E_{\mathcal{H}_{\mathbf{qp}} \to \mathbb{K}[X]}$ such that for all quasi-poset P:

$$\phi_0(P) = \lambda_P X^{cl(P)} = \frac{\mu_P}{cl(P)!} X^{cl(P)},$$

where μ_P is the number of linear extensions of P and $cl(P)$ is the number of equivalence classes of the equivalence associated to the quasi-order of P (proposition 29). This formula simplifies if P is a rooted tree: in this case,

$$\phi_0(P) = \frac{1}{P!} X^{|P|}.$$

We prove that for any $\phi \in E_{\mathcal{H}_{\mathbf{qp}} \to \mathbb{K}[X]}$, there exists a unique $f \in M_{\mathbf{qp}}$, such that $\phi = \phi_0 \leftarrow f$ (proposition 27). Consequently, this holds for both morphisms ehr and ehr^{str}: the associated characters are denoted by α and α^{str}. This implies that for any quasi-poset P:

$$ehr_P(X) = \sum_{\sim \triangleleft P} \frac{\mu_{P/\sim}}{cl(\sim)!} \alpha_{P|\sim} X^{cl(\sim)}, \qquad\qquad ehr_P^{str}(X) = \sum_{\sim \triangleleft P} \frac{\mu_{P/\sim}}{cl(\sim)!} \alpha^{str}_{P|\sim} X^{cl(\sim)},$$

where the sum is over a certain family of equivalence relations $\sim$, $P|\sim$ is a restriction operation and $P/\sim$ is a contraction operation. Applied to corollas, this gives Faulhaber's formula. We prove that α^{str} is the inverse of the character λ associated to ϕ_0, up to signs (proposition 34), which is a generalization, as well as a Hopf-algebraic proof, of Wright and Zhao's result. We also give an algebraic proof of the duality principle, and we define a Hopf algebra automorphism $\theta : (\mathcal{H}_{\mathbf{qp}}, m, \Delta) \longrightarrow (h_{\mathbf{qp}}, m, \Delta)$ with the help of the cointeraction of the two bialgebra structures on $\mathcal{H}_{\mathbf{qp}}$, satisfying $ehr^{str} \circ \theta = ehr$ (proposition 37).

We propose non-commutative versions of these results in the last section of the paper. Here, (isoclasses of) quasi-posets are replaced by quasi-posets indexed by sets $\{1, \ldots, n\}$, making a Hopf algebra $\mathcal{H}_{\mathbf{QP}}$, in cointeraction with $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$, and $\mathbb{K}[X]$ is replaced by the Hopf algebra of packed words **WQSym** [14]. We define two surjective Hopf algebra morphisms EHR and EHR^{str} from $\mathcal{H}_{\mathbf{QP}}$ to **WQSym** (proposition 39), generalizing ehr and ehr^{str}. The automorphism θ is generalized as a Hopf algebra $\Theta : \mathcal{H}_{\mathbf{QP}} \longrightarrow \mathcal{H}_{\mathbf{QP}}$, such that $EHR^{str} \circ \Theta = EHR$ (proposition 40), and we formulate a non-commutative duality principle (theorem 48), and we obtain a commutative diagram of Hopf algebras:

$$\begin{tikzcd}
\mathcal{H}_{\mathbf{QP}} \arrow[d, hook, "\Theta"'] \arrow[dr, two heads, "EHR"] \arrow[dddrrr, bend left] & & & & \\
\mathcal{H}_{\mathbf{QP}} \arrow[r, two heads, "EHR^{str}"] \arrow[d, hook, "\Psi"'] \arrow[dddrrr, bend left] & \mathbf{WQSym} \arrow[d, hook, "\Phi_{-1}"'] \arrow[dddrrr, "H"] & & & \\
\mathcal{H}_{\mathbf{QP}} \arrow[r, two heads, "EHR"] \arrow[dddrrr, bend left] & \mathbf{WQSym} \arrow[dddrrr, "H"] & & & \\
& & & \mathcal{H}_{\mathbf{qp}} \arrow[d, hook, "\theta"'] \arrow[dr, two heads, "ehr"] & \\
& & & \mathcal{H}_{\mathbf{qp}} \arrow[r, two heads, "ehr^{str}"'] \arrow[d, hook, "\psi"'] & \mathbb{K}[X] \arrow[d, hook, "\phi_{-1}"'] \\
& & & \mathcal{H}_{\mathbf{qp}} \arrow[r, two heads, "ehr"'] & \mathbb{K}[X]
\end{tikzcd}$$

The two triangles reflects the properties of morphisms Θ and θ, whereas the two squares are the duality principles.

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Notations. We denote by $\mathbb{K}$ a commutative field of characteristic zero. All the objects (vector spaces, algebra, and so on) in this text are taken over $\mathbb{K}$.

1 Bialgebras in cointeraction

1.1 Definition

Definition 1 *Let A and B be two bialgebras. We shall say that A and B are in cointeraction if:*

- *B coacts on A, via a map* $\rho : \left\{ \begin{array}{rcl} A & \longrightarrow & A \otimes B \\ a & \longrightarrow & \rho(a) = a_1 \otimes a_0. \end{array} \right.$
- *A is a bialgebra in the category of B-comodules, that is to say:*
  - $\rho(1_A) = 1_A \otimes 1_B$.
  - $m_{2,4}^3 \circ (\rho \otimes \rho) \circ \Delta_A = (\Delta_A \otimes Id) \circ \rho$, *with:*

$$m_{2,4}^3 : \left\{ \begin{array}{rcl} A \otimes B \otimes A \otimes B & \longrightarrow & A \otimes A \otimes B \\ a_1 \otimes b_1 \otimes a_2 \otimes b_2 & \longrightarrow & a_1 \otimes a_2 \otimes b_1 b_2. \end{array} \right.$$

Equivalently, in Sweedler's notations, for all $a \in A$:

$$(a^{(1)})_1 \otimes (a^{(2)})_1 \otimes (a^{(1)})_0 (a^{(2)})_0 = (a_1)^{(1)} \otimes (a_1)^{(2)} \otimes a_0.$$

  - *For all $a, b \in A$, $\rho(ab) = \rho(a)\rho(b)$.*
  - *For all $a \in A$, $(\varepsilon_A \otimes Id) \circ \rho(a) = \varepsilon_A(a) 1_B$.*

Examples of bialgebras in interaction can be found in [5] (for rooted trees) and in [13] (for various families of graphs). Another example is given by the algebra $\mathbb{K}[X]$, with its usual product m, and the two coproducts defined by:

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad \delta(X) = X \otimes X.$$

The bialgebras $(\mathbb{K}[X], m, \Delta)$ and $(\mathbb{K}[X], m, \delta)$ are in cointeractions, via the coaction $\rho = \delta$. Then $(\mathbb{K}[X], m, \delta)$ is a bialgebra. Note that for all $x, y \in \mathbb{K}$, if $P \in \mathbb{K}[X]$, identifying $\mathbb{K}[X] \otimes \mathbb{K}[X]$ and $\mathbb{K}[X, Y]$:

$$\Delta(P)(x, y) = P(x + y), \qquad \delta(P)(x, y) = P(xy).$$

Remark. If A and B are in cointeraction, the coaction of B on A is an algebra morphism.

Proposition 2 *Let A and B be two bialgebras in cointeraction. We assume that A is a Hopf algebra, with antipode S. Then S is a morphism of B-comodules, that is to say:*

$$\rho \circ S = (S \otimes Id) \circ \rho$$

Proof. We work in the space $End_{\mathbb{K}}(A, A \otimes B)$. As $A \otimes B$ is an algebra and A is a coalgebra, it is an algebra for the convolution product $\circledast$:

$$\forall f, g \in End_{\mathbb{K}}(A, A \otimes B),\ f \circledast g = m_{A \otimes B} \circ (f \otimes g) \circ \Delta_A.$$

Its unit is denoted by η:

$$\eta : \left\{ \begin{array}{rcl} A & \longrightarrow & A \otimes B \\ a & \longrightarrow & \varepsilon(a) 1_A \otimes 1_B. \end{array} \right.$$

We consider three elements in this algebra, respectively ρ, $F_1 = (S \otimes Id) \circ \rho$ and $F_2 = \rho \circ S$. Firstly:

$$\begin{aligned} (F_1 \circledast \rho)(a) &= S((a^{(1)})_1)(a^{(2)})_1 \otimes (a^{(1)})_0 (a^{(2)})_0 \\ &= S((a_1)^{(1)})(a_1)^{(2)} \otimes a_0 \\ &= \varepsilon_A(a_1) 1_A \otimes a_0 \\ &= \varepsilon_A(a) 1_A \otimes 1_B \\ &= \eta(a). \end{aligned}$$

Secondly:

$$\begin{aligned} (\rho \circledast F_2)(a) &= (a^{(1)})_1 S(a^{(2)})_1 \otimes (a^{(1)})_0 (S(a^{(2)}))_0 \\ &= \varepsilon_A(a)(1_A)_1 \otimes (1_A)_0 \\ &= \varepsilon_A(a) 1_A \otimes 1_B \\ &= \eta(a). \end{aligned}$$

We obtain that $F_1 \circledast \rho = \rho \circledast F_2 = \eta$, so $F_1 = F_1 \circledast \eta = F_1 \circledast \rho \circledast F_2 = \eta \circledast F_2 = F_2$. $\square$

1.2 Monoids actions

Proposition 3 *Let A and B be two bialgebras in cointeraction, through the coaction ρ. We denote by M_A and M_B the monoids of characters of respectively A and B. Then B acts on A by monoid endomorphisms, via the map:*

$$\leftarrow : \left\{ \begin{array}{rcl} M_A \times M_B & \longrightarrow & M_A \\ (\phi, \psi) & \longrightarrow & \phi \leftarrow \psi = (\phi \otimes \psi) \circ \rho. \end{array} \right.$$

Proof. We denote by $*$ the convolution product of M_B and by $\star$ the convolution product of M_A. As $\rho : A \longrightarrow A \otimes B$ is an algebra morphism, $\leftarrow$ is well-defined. Let $\phi \in M_A$, $\psi_1, \psi_2 \in M_B$.

$$\begin{aligned} (\phi \leftarrow \psi_1) \leftarrow \psi_2 &= (\phi \otimes \psi_1 \otimes \psi_2) \circ (\rho \otimes Id) \circ \rho \\ &= (\phi \otimes \psi_1 \otimes \psi_2) \circ (Id \otimes \Delta_B) \circ \rho \\ &= \phi \leftarrow (\phi_1 * \phi_2). \end{aligned}$$

So $\leftarrow$ is an action. Let $\phi_1,\phi_2\in M_A$, $\psi\in M_B$. For all $a\in A$:

$$\begin{aligned}((\phi_1\star\phi_2)\circ\rho)(a)&=(\phi_1\otimes\phi_2\otimes\psi)\circ(\Delta_A\otimes Id)\circ\rho(a)\\&=(\phi_1\otimes\phi_2\otimes\psi)((a_0)^{(1)}\otimes(a_0)^{(2)}\otimes a_1)\\&=(\phi_1\otimes\phi_2\otimes\psi)((a^{(1)})_0\otimes(a^{(2)})_0\otimes(a^{(1)})_1(a^{(2)})_1)\\&=\phi_1((a^{(1)})_0)\psi((a^{(1)})_1)\phi_2((a^{(2)})_0)\psi((a^{(2)})_1)\\&=(\phi_1\leftarrow\psi)(a^{(1)})(\phi_2\leftarrow\psi)(a^{(2)})\\&=((\phi_1\leftarrow\psi)\star(\phi_2\leftarrow\psi))(a).\end{aligned}$$

So $\leftarrow$ is an action by monoid endomorphisms. $\square$

Example. We take $A=(\mathbb{K}[X],m,\Delta)$, $B=(\mathbb{K}[X],m,\delta)$ and $\rho=\delta$. We consider the map:

$$ev:\left\{\begin{array}{rcl}\mathbb{K}&\longrightarrow&\mathbb{K}[X]^*\\\lambda&\longrightarrow&\left\{\begin{array}{rcl}\mathbb{K}[X]&\longrightarrow&\mathbb{K}\\P(X)&\longrightarrow&ev_\lambda(P)=P(\lambda).\end{array}\right.\end{array}\right.$$

Then ev is a isomorphism from $(\mathbb{K},+)$ to $(M_A,\star)$ and from $(\mathbb{K},.)$ to $(M_B,*)$. Moreover, for all $\lambda,\mu\in\mathbb{K}$:

$$ev_\lambda\leftarrow ev_\mu=ev_{\lambda\mu}.$$

Proposition 4 *Let A and B be two bialgebras in cointeraction, through the coaction ρ.*

1. *Let H be any bialgebra. We denote by M_B the monoid of characters of B and by $E_{A\to H}$ the set of bialgebra morphisms from A to H. Then M_B acts on $E_{A\to H}$ via the map:*

$$\leftarrow:\left\{\begin{array}{rcl}E_{A\to H}\times M_B&\longrightarrow&E_{A\to H}\\(\phi,\lambda)&\longrightarrow&\phi\leftarrow\lambda=(\phi\otimes\lambda)\circ\rho\end{array}\right.$$

2. *Let H_1 and H_2 be two bialgebras and let $\theta:H_1\longrightarrow H_2$ be a bialgebra morphism. For all $\phi\in E_{A\leftarrow H_1}$, for all $\lambda\in M_B$, in $E_{A\leftarrow H_2}$:*

$$\theta\circ(\phi\leftarrow\lambda)=(\theta\circ\phi)\leftarrow\lambda.$$

3. *if $\lambda,\mu\in M_B$, in $E_{A\to A}$:*

$$(Id\leftarrow\lambda)\circ(Id\leftarrow\mu)=Id\leftarrow(\lambda*\mu).$$

In other words, the following map is a monoid endomorphism:

$$\left\{\begin{array}{rcl}(M_B,*)&\longrightarrow&(E_{A\to A},\circ)\\\lambda&\longrightarrow&Id\leftarrow\lambda.\end{array}\right.$$

Proof. 1. For all $\phi\in E_{A\leftarrow B}$, $\lambda\in M_B$, $\phi\leftarrow\lambda:A\longrightarrow H\otimes\mathbb{K}=H$. As ϕ, λ and ρ are algebra morphisms, by composition $\phi\leftarrow\lambda$ is an algebra morphism. Let $a\in A$.

$$\begin{aligned}\Delta_H(\phi\leftarrow\lambda(a))&=\Delta_H(\phi(a_0)\lambda(a_1))\\&=\lambda(a_1)\Delta_H\circ\phi(a_1)\\&=\lambda(a_1)\phi(a_0)^{(1)}\otimes\phi(a_0)^{(2)}\\&=\lambda(a_1)\phi((a_0)^{(1)})\otimes\phi((a_0)^{(2)})\\&=\lambda((a^{(1)})_1(a^{(2)})_1)\phi((a^{(1)})_0)\otimes\phi((a^{(2)})_0)\\&=\lambda((a^{(1)})_1)\lambda((a^{(2)})_1)\phi((a^{(1)})_0)\otimes\phi((a^{(2)})_0)\\&=\phi((a^{(1)})_0)\lambda((a^{(1)})_1)\otimes\phi((a^{(2)})_0)\lambda((a^{(2)})_1)\\&=\phi\leftarrow\lambda(a^{(1)})\otimes\phi\leftarrow\lambda(a^{(2)})\\&=((\phi\leftarrow\lambda)\otimes(\phi\leftarrow\lambda))\circ\Delta_A(a).\end{aligned}$$

So $\phi \leftarrow \lambda \in E_{A\to H}$.

Let $\phi \in E_{A\to H}$, $\lambda, \mu \in M_B$.

$$(\phi \leftarrow \lambda) \leftarrow \mu = (\phi \otimes \lambda \otimes \mu) \circ (\rho \otimes Id) \circ \rho = (\phi \otimes \lambda \otimes \mu) \circ (Id \otimes \Delta_B) \circ \rho = \phi \leftarrow (\lambda * \mu).$$

For all $a \in A$, $\phi \leftarrow \eta \circ \varepsilon(a) = \phi(a_0)\varepsilon(a_1) = \phi(a)$. So $\leftarrow$ is indeed an action of M_B on $E_{A\to H}$.

2. Let $a \in H$.

$$(\theta \circ \phi) \leftarrow \lambda(a) = \theta \circ \phi(a_1)\lambda(a_0) = \theta(\phi(a_1)\lambda(a_0)) = \theta(\phi \leftarrow \lambda(a)) = \theta \circ (\phi \leftarrow \lambda)(a).$$

So $(\theta \circ \phi) \leftarrow \lambda = \theta \circ (\phi \leftarrow \lambda)$.

3. Consequently, if $\lambda, \mu \in M_B$, in $E_{A\to A}$: $(Id \leftarrow \lambda) \circ (Id \leftarrow \lambda) = (Id \leftarrow \lambda) \leftarrow \mu) = Id \leftarrow (\lambda * \mu)$. $\Box$

Example. We take $A = (\mathbb{K}[X], m, \Delta)$, $B = (\mathbb{K}[X], m, \delta)$ and $\rho = \delta$. In $E_{A\longrightarrow A}$, for any $\lambda \in \mathbb{K}$:

$$Id \leftarrow ev_\lambda(X) = ev_\lambda(X)X = \lambda X,$$

so for any $P \in \mathbb{K}[X]$, $(Id \leftarrow ev_\lambda)(P) = P(\lambda X)$.

2 Examples from quasi-posets

2.1 Definition

Definition 5 *1. Let A be a finite set. A quasi-order on A is a transitive, reflexive relation $\leq$ on A. If $\leq$ is a quasi-order on A, we shall say that $(A, \leq)$ is a quasi-poset. If P is a quasi-poset:*

(a) Its isoclass is denoted by $\lfloor P \rfloor$.

(b) $\sim_P$ is defined by:

$$\forall a, b \in A,\ a \sim_P b \text{ if } a \leq b \text{ and } b \leq a.$$

It is an equivalence on A.

(c) $\overline{A} = A/\sim_P$ is given an order by:

$$\forall a, b \in A,\ \overline{a} \leq \overline{b} \text{ if } a \leq b.$$

The poset $(\overline{A}, \leq)$ is denoted by $\overline{P}$.

(d) The cardinality of $\overline{P}$ is denoted by $cl(P)$.

2. Let $n \in \mathbb{N}$.

(a) The set of quasi-posets which underlying set is $[n] = \{1, \ldots, n\}$ is denoted by $\mathbf{QP}(n)$.

(b) The set of posets which underlying set is $[n]$ is denoted by $\mathbf{P}(n)$.

(c) The set of isoclasses of quasi-posets of cardinality n is denoted by $\mathbf{qp}(n)$.

(d) The set of isoclasses of quasi-posets of cardinality n is denoted by $\mathbf{p}(n)$.

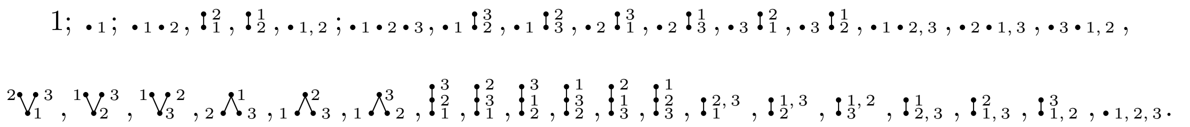
We put:

$$\mathbf{QP} = \bigsqcup_{n\geq 0} \mathbf{QP}(n), \qquad \mathbf{P} = \bigsqcup_{n\geq 0} \mathbf{P}(n), \qquad \mathbf{qp} = \bigsqcup_{n\geq 0} \mathbf{qp}(n), \qquad \mathbf{p} = \bigsqcup_{n\geq 0} \mathbf{p}(n),$$
$$\mathcal{H}_{\mathbf{QP}} = Vect(\mathbf{QP}), \qquad \mathcal{H}_{\mathbf{P}} = Vect(\mathbf{P}), \qquad \mathcal{H}_{\mathbf{qp}} = Vect(\mathbf{qp}) \qquad \mathcal{H}_{\mathbf{p}} = Vect(\mathbf{p}).$$

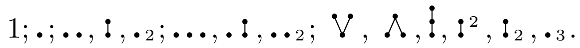
As posets are quasi-posets, there are canonical injections from $\mathcal{H}_{\mathbf{P}}$ into $\mathcal{H}_{\mathbf{QP}}$ and from $\mathcal{H}_{\mathbf{p}}$ into $\mathcal{H}_{\mathbf{qp}}$. Moreover, the map $P \longrightarrow \overline{P}$ induces surjective maps from $\mathcal{H}_{\mathbf{QP}}$ to $\mathcal{H}_{\mathbf{P}}$ and from $\mathcal{H}_{\mathbf{qp}}$ to $\mathcal{H}_{\mathbf{p}}$, both denoted by Ξ. The map $P \longrightarrow \lfloor P \rfloor$ induces maps $\lfloor\rfloor : \mathcal{H}_{\mathbf{QP}} \longrightarrow \mathcal{H}_{\mathbf{qp}}$ and $\lfloor\rfloor : \mathcal{H}_{\mathbf{P}} \longrightarrow \mathcal{H}_{\mathbf{p}}$. The following diagram commutes:

$$\begin{tikzcd}
\mathcal{H}_{\mathbf{QP}} \arrow[rr, two heads, "\lfloor\rfloor"] \arrow[dr, "\Xi"] & & \mathcal{H}_{\mathbf{qp}} \arrow[dr, "\Xi"] & \\
& \mathcal{H}_{\mathbf{P}} \arrow[rr, two heads, "\lfloor\rfloor"] & & \mathcal{H}_{\mathbf{p}} \\
\mathcal{H}_{\mathbf{P}} \arrow[uu, hook] \arrow[ur, "Id"'] \arrow[rr, two heads, "\lfloor\rfloor"] & & \mathcal{H}_{\mathbf{p}} \arrow[uu, hook] \arrow[ur, "Id"'] &
\end{tikzcd} \tag{1}$$

We shall represent any element P of $\mathbf{QP}$ by the Hasse graph of $\overline{P}$, indicating on the vertices the elements of the corresponding equivalence class. For example, the elements of $\mathbf{QP}(n)$, $n \leq 3$, are:



We shall represent any element $P \in \mathbf{qp}$ by the Hasse graph of $\overline{P}$, indicating on the vertices the cardinality of the corresponding equivalence class, if this cardinality is not equal to 1. For example, the elements of $\mathbf{qp}(n)$, $n \leq 3$, are:



2.2 First coproduct

By Alexandroff's theorem [1, 17], finite quasi-posets are in bijection with finite topological spaces. Let us recall the definition of the topology attached to a quasi-poset.

Definition 6 *1. Let $P = (A, \leq)$ be a quasi-poset. An open set of P is a subset O of A such that:*

$$\forall i, j \in A,\ (i \in O \text{ and } i \leq j \Longrightarrow j \in O).$$

The set of open sets of P (the topology associated to P) is denoted by $top(P)$.

2. *Let $P = (A, \leq)$ be a quasi-poset and $B \subseteq A$. We denote by $P_{|B}$ the quasi-poset $(B, \leq_{|B})$.*

3. *Let $P = (A, \leq_P)$ be a quasi-poset. We assume that A is also given a total order $\leq$: for example, A is a subset of $\mathbb{N}$. If the cardinality of A is n, there exists a unique increasing bijection f from $[n]$, with its usual order, to $(A, \leq)$. We denote by $Std(P)$ the quasi-poset in $\mathbf{QP}(n)$ defined by:*

$$\forall i, j \in [n],\ i \leq_{Std(P)} j \Longleftrightarrow f(i) \leq_P f(j).$$

Proposition 7 *1. We define a product m on $\mathcal{H}_{\mathbf{QP}}$ in the following way: if $P \in \mathbf{QP}(k)$, $Q \in \mathbf{QP}(l)$, then $PQ = m(P, Q) \in \mathbf{QP}(k + l)$ and*

$$\begin{aligned}\forall i, j \in [k+l],\ i \leq_{PQ} \Longleftrightarrow & (1 \leq i, j \leq k \text{ and } i \leq_P j) \\ & \text{or } (k+1 \leq i, j \leq k+l \text{ and } i-k \leq_Q j-k).\end{aligned}$$

2. *We define a second product* $\downarrow$ *on* $\mathcal{H}_{\mathbf{QP}}$ *in the following way: if* $P \in \mathbf{QP}(k)$, $Q \in \mathbf{QP}(l)$, *then* $PQ = m(P,Q) \in \mathbf{QP}(k+l)$ *and*

$$\begin{aligned}\forall i,j \in [k+l],\ i \leq_{PQ} \Longleftrightarrow & (1 \leq i,j \leq k \text{ and } i \leq_P j)\\ & \text{or } (k+1 \leq i,j \leq k+l \text{ and } i-k \leq_Q j-k)\\ & \text{or } (1 \leq i \leq k < j \leq k+l).\end{aligned}$$

3. *We define a coproduct* Δ *on* $\mathcal{H}_{\mathbf{QP}}$ *in the following way:*

$$\forall P \in \mathbf{QP}(n),\ \Delta(P) = \sum_{O \in top(P)} Std(P_{|[n]\setminus O}) \otimes Std(P_{|O}).$$

Then $(\mathcal{H}_{\mathbf{QP}}, m, \Delta)$ *is a non-commutative, non-cocommutative Hopf algebra, and* $(\mathcal{H}_{\mathbf{QP}}, \downarrow, \Delta)$ *is an infinitesimal bialgebra.*

Proof. See [9, 10]. $\square$

Examples. If $\{a,b\} = \{1,2\}$ and $\{i,j,k\} = \{1,2,3\}$:

$$\begin{aligned}
\Delta(\bullet_1) &= \bullet_1 \otimes 1 + 1 \otimes \bullet_1,\\
\Delta(\mathbf{I}_a^b) &= \mathbf{I}_a^b \otimes 1 + 1 \otimes \mathbf{I}_a^b + \bullet_a \otimes \bullet_b,\\
\Delta({}^j\mathbf{V}_i^k) &= {}^j\mathbf{V}_i^k \otimes 1 + 1 \otimes {}^j\mathbf{V}_i^k + \mathbf{I}_i^j \otimes \bullet_k + \mathbf{I}_i^k \otimes \bullet_j + \bullet_i \otimes \bullet_j \bullet_k,\\
\Delta({}_j\mathbf{\Lambda}^i{}_k) &= {}_j\mathbf{\Lambda}^i{}_k \otimes 1 + 1 \otimes {}_j\mathbf{\Lambda}^i{}_k + \bullet_j \otimes \mathbf{I}_k^i + \bullet_k \otimes \mathbf{I}_j^i + \bullet_j \bullet_k \otimes \bullet_i,\\
\Delta(\mathbf{I}_{i}^{\begin{smallmatrix}k\\ j\end{smallmatrix}}) &= \mathbf{I}_{i}^{\begin{smallmatrix}k\\ j\end{smallmatrix}} \otimes 1 + 1 \otimes \mathbf{I}_{i}^{\begin{smallmatrix}k\\ j\end{smallmatrix}} \bullet_i \otimes \mathbf{I}_j^k + \mathbf{I}_i^j \otimes \bullet_k.
\end{aligned}$$

Remark. This Hopf algebraic structure is compatible with the morphisms of (1), that is to say:

1. $\mathcal{H}_{\mathbf{P}}$ is a Hopf subalgebra of $\mathcal{H}_{\mathbf{QP}}$.

2. observe that:

   - If (P_1, P_2) and (Q_1, Q_2) are pairs of isomorphic quasi-posets, then P_1Q_1 and P_2Q_2 are isomorphic.
   - If P_1 and P_2 are isomorphic quasi-posets of $\mathbf{QP}(n)$, and if $\phi : [n] \longrightarrow [n]$ is an isomorphism from P_1 to P_2, then the topology associated to P_2 is the image by ϕ of the topology associated to P_1 and for any subset I of P_1, $\phi_{|I}$ is an isomorphism from $(P_1)_{|I}$ to $(P_2)_{|\phi(I)}$.

   Consequently, the surjective map $\lfloor\rfloor : \mathcal{H}_{\mathbf{QP}} \longrightarrow \mathcal{H}_{\mathbf{qp}}$ is compatible with the product and the coproduct: $\mathcal{H}_{\mathbf{qp}}$ inherits a Hopf algebra structure. Its product is the disjoint union of quasi-posets. For any quasi-poset $P = (A, \leq_P)$:

$$\Delta(\lfloor P \rfloor) = \sum_{O \in top(P)} \lfloor P_{|A\setminus O} \rfloor \otimes \lfloor P_{|O} \rfloor.$$

3. $\mathcal{H}_{\mathbf{p}}$ is a Hopf subalgebra of $\mathcal{H}_{\mathbf{qp}}$.

4. All the morphisms in (1) are Hopf algebra morphisms.

Definition 8 *1. We shall say that a finite quasi-poset* $P = (A, \leq_P)$ *is connected if its associated topology is connected.*

2. For any finite quasi-poset P, we denote by $cc(P)$ the number of connected components of its associated topology.

It is well-known that P is connected if, and only if, the Hasse graph of $\overline{P}$ is connected. Any quasi-poset P can be decomposed as the disjoint union of its connected components; in an algebraic setting, $\mathcal{H}_{\mathbf{qp}}$ is generated as a polynomial algebra by the connected quasi-posets. This is not true in $\mathcal{H}_{\mathbf{QP}}$: for example, $\overset{\bullet}{\underset{\bullet}{|}}{}^{3}_{1}\,\bullet_{2}$ is not connected and is indecomposable in $\mathcal{H}_{\mathbf{QP}}$.

2.3 Second coproduct

Definition 9 *Let $P = (A, \leq_P)$ be a quasi-poset and let $\sim$ be an equivalence on A.*

1. We define a second quasi-order $\leq_{P|\sim}$ on A by the relation:

$$\forall x, y \in A,\ x \leq_{P|\sim} y \text{ if } (x \leq_P y \text{ and } x \sim y).$$

2. We define a third quasi-order $\leq_{P/\sim}$ on A as the transitive closure of the relation defined by:

$$\forall x, y \in A,\ xRy \text{ if } (x \leq_P y \text{ or } x \sim y).$$

3. We shall say that $\sim$ is P-compatible and we shall denote $\sim \lhd P$ if the two following conditions are satisfied:

- *The restriction of P to any equivalence class of $\sim$ is connected.*
- *The equivalences $\sim_{P/\sim}$ and $\sim$ are equal. In other words:*

$$\forall x, y \in A,\ (x \leq_{P/\sim} y \text{ and } y \leq_{P/\sim} x) \Longrightarrow x \sim y;$$

note the converse assertion trivially holds.

Remarks.

1. $P| \sim$ is the disjoint union of the restrictions of $\leq_P$ to the equivalence classes of $\sim$.
2. Let $x, y \in P$. Then $x \leq_{P/\sim} y$ if there exist $x_1, x_1', \ldots, x_k, x_k' \in A$ such that:

$$x \leq_P x_1 \sim x_1' \leq_P \ldots \leq_P x_k \sim x_k' \leq_P y.$$

3. If $\sim \lhd P$, then:

   (a) The equivalence classes of $\sim_{P/\sim}$ are the equivalence classes of $\sim$ and are included in a connected component of P. This implies that the connected components of $P/\sim$ are the connected components of P. Consequently:

$$cl(P/\sim) = cl(\sim), \qquad cc(P/\sim) = cc(P), \tag{2}$$

   where $cl(\sim)$ is the number of equivalence classes of $\sim$.

   (b) If $x \sim_P y$ and $x \sim y$, then $x \sim_{P|\sim} y$: the equivalence classes of $\sim_{P|\sim}$ are the equivalence classes of $\sim_P$; the connected components of $P|\sim$ are the equivalence classes of $\sim$. Consequently:

$$cl(P|\sim) = cl(P), \qquad cc(P|\sim) = cl(\sim). \tag{3}$$

Definition 10 *We define a second coproduct δ on $\mathcal{H}_{\mathbf{QP}}$ in the following way: for all $P \in \mathbf{QP}$,*

$$\delta(P) = \sum_{\sim \lhd P} (P/\sim) \otimes (P\,|\sim).$$

Then $(\mathcal{H}_{\mathbf{QP}}, m, \delta)$ is a bialgebra.

Proof. Firstly, let us prove the compatibility of δ with m. Let $P=(A,\leq_P)$ and $Q=(B,\leq_Q)$ be two elements of $\mathbf{QP}$. Let $\sim$ be an equivalence relation on P. We denote by $\sim'$ and $\sim''$ the restriction of $\sim$ to P and Q. Then:

1. If $\sim \lhd PQ$, then as the equivalence classes of $\sim$ are connected, they are included in A or in B. Consequently, if $x\in A$ and $y\in B$, x and y are not equivalent for $\sim$. Moreover, $\sim'\lhd P$ and $\sim''\lhd Q$, and:
$$PQ|\sim=(P|\sim')(Q|\sim''), \qquad\qquad PQ/\sim=(P/\sim')(Q/\sim'').$$
2. Conversely, if $\sim'\lhd P$,$\sim''\lhd Q$ and for all $x\in A$, $y\in B$, x and y not are not $\sim$-equivalent, then $\sim\lhd PQ$.

Hence:
$$\begin{aligned}\delta(PQ)&=\sum_{\sim\lhd PQ}(PQ/\sim)\otimes(PQ|\sim)\\&=\sum_{\sim'\lhd P,\sim''\lhd Q}(P/\sim')(Q/\sim'')\otimes(P|\sim')(Q|\sim'')\\&=\delta(P)\delta(Q).\end{aligned}$$

Let us now prove the coassociativity of δ. Let $P\in\mathbf{QP}$.

First step. We put:
$$A=\{(r,r')\mid r\lhd P,\ r'\lhd P/r\}, \qquad\qquad B=\{(s,s')\mid s\lhd P,\ s'\lhd P_{|}s\}.$$
We consider the maps:
$$F:\left\{\begin{array}{rcl}A&\longrightarrow&B\\(r,r')&\longrightarrow&(r',r),\end{array}\right. \qquad\qquad G:\left\{\begin{array}{rcl}B&\longrightarrow&A\\(s,s')&\longrightarrow&(s',s).\end{array}\right.$$

F is well-defined: we put $(s,s')=(r',r)$. The equivalence classes of s are the equivalence classes of r', so are P-connected. If $x\sim_{P/s}y$, there exist $x_1,x_1',\ldots,x_k,s_k'$ and $y_1,y_1',\ldots,y_l,y_l$ such that:
$$x\leq_P x_1r'x_1'\leq_P\ldots\leq_P x_kr'x_k'\leq_P y, \qquad y\leq_P y_1r'y_1'\leq_P\ldots\leq_P y_lr'y_l'\leq_P x.$$
Hence:
$$x\leq_{P/r}x_1r'x_1'\leq_{P/r}\ldots\leq_{P/r}x_kr'x_k'\leq_{P/r}y, \quad y\leq_{P/r}y_1r'y_1'\leq_{P/r}\ldots\leq_{P/r}y_lr'y_l'\leq_{P/r}x.$$
So $x\sim_{P/r}y$. As $r'\lhd P/r$, $x\sim_P y$: $s\lhd P$.

Let us assume that $xs'y$. Then xry, so, as $r\lhd y$, there exists a path from x to y in the Hasse graph of P, made of vertices all r-equivalent to x and y. If x' and y' are two elements of this path, Then $x'ry'$, so $x'\leq_{G/r}y'$ and finally $x'\leq_{(P/r)/r'}y'$. As $r'\lhd P/r$, $x'r'y'$, so xsy. So the elements of this path are all $P|s$-equivalent: the equivalence classes of s' are $P|s$-connected.

Let us assume that $x\sim_{(P|s)/s'}y$. There exists $x_1,x_1',\ldots,x_k,s_k'$ and $y_1,y_1',\ldots,y_l,y_l$ such that:
$$x\leq_{P|r'}x_1rx_1'\leq_{P|r'}\ldots\leq_{P|r'}x_krx_k'\leq_{P|r'}y, \quad y\leq_{P|r'}y_1ry_1'\leq_{P|r'}\ldots\leq_{P|r'}y_lry_l'\leq_{P|r'}x.$$
Then:
$$x\leq_P x_1rx_1'\leq_P\ldots\leq_P x_krx_k'\leq_P y, \qquad y\leq_P y_1ry_1'\leq_P\ldots\leq_P y_lry_l'\leq_P x,$$
So $x\leq_{P/r}y$ and $y\leq_{P/r}x$. As $r\lhd P$, xry, so $xs'y$: we obtain that $s'\lhd P\mid s$.

G is well-defined: let $(s,s') \in B$ and let us put $G(s,s') = (r,r')$. The equivalence classes of r are $P|s$-connected, so are P-connected. Let us assume that $x \sim_{P/r} y$. There exists $x_1, x'_1, \ldots, x_k, s'_k$ and $y_1, y'_1, \ldots, y_l, y_l$ such that:

$$x \leq_P x_1 s' x'_1 \leq_P \ldots \leq_P x_k s' x'_k \leq_P y, \qquad y \leq_P y_1 s' y'_1 \leq_P \ldots \leq_P y_l s' y'_l \leq_P x.$$

As the equivalence classes of s' are $P|s$-connected, all this elements are in the same connected component of $P|s$, so are s-equivalent:

$$x \leq_{P|s} x_1 s' x'_1 \leq_{P|s} \ldots \leq_{P|s} x_k s' x'_k \leq_{P|s} y, \qquad y \leq_{P|s} y_1 s' y'_1 \leq_{P|s} \ldots \leq_{P|s} y_l s' y'_l \leq_{P|s} x.$$

Hence, $x \sim_{(P|s)/s'} y$, so as $s' \triangleleft P \mid s$, $xs'y$, so xry: $r \triangleleft P$.

The equivalence classes of r' are the equivalence classes of s, so are P-connected and therefore P/r-connected. Let us assume that $x \sim_{(P/r)/r'} y$. Note that if $x's'y'$, then x' and y' are in the same connected component of $P|s$, so $x'sy$. By the definition of $\leq_{P/s'}$ as a transitive closure, using this observation, we obtain:

$$x \leq_P x_1 s x'_1 \leq_P \ldots \leq_P x_k s x'_k \leq_P y, \qquad y \leq_P y_1 s y'_1 \leq_P \ldots \leq_P y_l s y'_l \leq_P x.$$

So $x \sim_{P/s} y$. As $s \triangleleft P$, xsy, so $xr'y$: $r' \triangleleft P/r$.

Clearly, F and G are inverse bijections.

Second step. Let $(r,r') \in A$ and let $F(r,r') = (s,s')$. Note that if xry, then $x/\sim_{P/r} y$, so $x/\sim_{(P/r)/r'} y$, so $xr'y$ as $r' \triangleleft P/r$. Then:

$$\begin{aligned}
\leq_{(P/r)/r'} &= \text{transitive closure of } ((xr'y) \text{ or } (x \leq_{P/r} y))\\
&= \text{transitive closure of } ((xr'y) \text{ or } (x \leq_P y) \text{ or } (x \leq_r y))\\
&= \text{transitive closure of } ((xr'y) \text{ or } (x \leq_P y))\\
&= \text{transitive closure of } ((xsy) \text{ or } (x \leq_P y))\\
&= \leq_{P/s}.
\end{aligned}$$

So $P/s = (P/r)/r'$.

$$\begin{aligned}
\leq_{(P|s)/s'} &= \text{transitive closure of } ((xs'y) \text{ or } (x \leq_{P|s} y))\\
&= \text{transitive closure of } ((xry) \text{ or } (x \leq_{P|r'} y))\\
&= \text{transitive closure of } ((xry) \text{ or } ((x \leq_P y) \text{ and } (xr'y)))\\
&= \text{transitive closure of } (((xry) \text{ or } (x \leq_P y)) \text{ and } ((sry) \text{ or } (xr'y)))\\
&= \text{transitive closure of } ((x \leq_{Pr/r} y) \text{ and } (sr'y))\\
&= \leq_{(P/r)|r'}.
\end{aligned}$$

So $(P|s)/s' = (P/r)|r'$. For all x, y:

$$\begin{aligned}
x \leq_{(P|s)/s'} y &\Longleftrightarrow (x \leq_{P|s} y) \text{ and } (xs'y)\\
&\Longleftrightarrow (x \leq_P y) \text{ and } xsy \text{ and } (xs'y)\\
&\Longleftrightarrow (x \leq_P y) \text{ and } xr'y \text{ and } (xry)\\
&\Longleftrightarrow (x \leq_P y) \text{ and } (xry)\\
&\Longleftrightarrow x \leq_{P|r} y.
\end{aligned}$$

So $(P|s)|s' = P|r$. Finally:

$$\begin{aligned}
(\delta \otimes Id) \circ \delta(P) &= \sum_{(r,r') \in A} (P/r)/r' \otimes (P/r)|r' \otimes P|r\\
&= \sum_{(s,s') \in B} P/s \otimes (P|s)/s' \otimes (P|s)|s'\\
&= (Id \otimes \delta) \circ \delta(P).
\end{aligned}$$

So $\mathcal{H}_{\mathbf{QP}}$ is a bialgebra. $\square$

Examples. If $\{a,b\}=\{1,2\}$ and $\{i,j,k\}=\{1,2,3\}$:

$$\begin{aligned}
\delta(\bullet_1) &= \bullet_1 \otimes \bullet_1,\\
\delta(\mathbf{I}^{b}_{a}) &= \mathbf{I}^{b}_{a} \otimes \bullet_a \bullet_b + \bullet_{a,b} \otimes \mathbf{I}^{b}_{a},\\
\delta({}^{j}\mathbf{V}^{k}_{i}) &= {}^{j}\mathbf{V}^{k}_{i} \otimes \bullet_i \bullet_j \bullet_k + \mathbf{I}^{k}_{i,j} \otimes \mathbf{I}^{j}_{i} \bullet_k + \mathbf{I}^{j}_{i,k} \otimes \mathbf{I}^{k}_{i} \bullet_j + \bullet_{i,j,k} \otimes {}^{j}\mathbf{V}^{k}_{i},\\
\delta({}_{j}\mathbf{\Lambda}^{i}_{k}) &= {}_{j}\mathbf{\Lambda}^{i}_{k} \otimes \bullet_i \bullet_j \bullet_k + \mathbf{I}^{i,j}_{k} \otimes \mathbf{I}^{i}_{j} \bullet_k + \mathbf{I}^{i,k}_{j} \otimes \mathbf{I}^{i}_{k} \bullet_j + \bullet_{i,j,k} \otimes {}_{j}\mathbf{\Lambda}^{i}_{k},\\
\delta(\mathbf{I}^{\substack{k\\j}}_{i}) &= \mathbf{I}^{\substack{k\\j}}_{i} \otimes \bullet_i \bullet_j \bullet_k + \mathbf{I}^{k}_{i,j} \otimes \mathbf{I}^{j}_{i} \bullet_k + \mathbf{I}^{j,k}_{i} \otimes \bullet_i \mathbf{I}^{k}_{j} + \bullet_{i,j,k} \otimes \mathbf{I}^{\substack{k\\j}}_{i}.
\end{aligned}$$

Remarks.

1. δ is the internal coproduct of [8].
2. $(\mathcal{H}_{\mathbf{QP}}, m, \delta)$ is not a Hopf algebra: for all $n \geq 1$, $\delta(\bullet_n) = \bullet_n \otimes \bullet_n$, and $\bullet_n$ has no inverse in $\mathcal{H}_{\mathbf{QP}}$.
3. This coproduct is also compatible with the map $\lfloor\rfloor$, so we obtain a bialgebra structure on $\mathcal{H}_{\mathbf{qp}}$ with the coproduct defined by:
$$\delta(\lfloor P \rfloor) = \sum_{\sim \lhd P} \lfloor P/\sim \rfloor \otimes \lfloor P|\sim \rfloor.$$
4. $\mathcal{H}_{\mathbf{P}}$ and $\mathcal{H}_{\mathbf{p}}$ are not stable under δ, as if P is a poset and $\sim \lhd P$, $P/\sim$ is not necessarily a poset (although $P|\sim$ is). However, there is a way to define a coproduct $\overline{\delta} = (\Xi \otimes Id) \circ \delta$ on $\mathcal{H}_p$:
$$\forall P \in \mathbf{P}(n),\ \overline{\delta}(\lfloor P \rfloor) = \sum_{\sim \lhd P} = \overline{\lfloor P/\sim \rfloor} \otimes \lfloor P|\sim \rfloor.$$
$(\mathcal{H}_p, m, \overline{\delta})$ is a quotient of $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$ through the map Ξ.

2.4 Characters of the second coproduct

We denote by $M_{\mathbf{qp}}$ the monoid of characters of $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$. Its product, as well as the convolution product on the dual $\mathcal{H}^*_{\mathbf{qp}}$ of $\mathcal{H}_{\mathbf{qp}}$ induced by δ, is denoted by $*$.

Proposition 11 *Let $f \in M_{\mathbf{qp}}$. It has an inverse in $M_{\mathbf{qp}}$ if, and only if, for all $n \geq 1$, $f(\bullet_n) \neq 0$.*

Proof. $\Longrightarrow$. If f has an inverse g, then for all $n \geq 1$, as $\delta(\bullet_n) = \bullet_n \otimes \bullet_n$, $\varepsilon(\bullet_n) = 1$ and $f(\bullet_n)g(\bullet_n) = 1$: $f(\bullet_n) \neq 0$.

$\Longleftarrow$. Let us first define elements $g, h \in \mathcal{H}^*_{\mathbf{qp}}$ such that $f * g = h * f = \varepsilon$. Let us define $g(P)$ and $h(P)$ by induction on $cl(P)$. We first put $g(1) = h(1) = 1$. If $cl(P) = 1$, then $P = \bullet_n$ and we put $g(\bullet_n) = h(\bullet_n) = f(\bullet_n)^{-1}$. Let us suppose that $g(Q)$ and $h(Q)$ are defined for any quasi-poset Q such that $cl(Q) < cl(P)$. Let $\sim \lhd P$. By construction, if $x \sim_P y$, then $x \sim_{P/\sim} y$; as $\sim \lhd P$, $x \sim y$. So the number of equivalence classes of $\sim$ is smaller than $cl(P)$, with an equality if, and only if, $\sim = \sim_P$. Note that $\sim_P \lhd P$: indeed, $P/\sim_P = P$. Moreover, $cl(P/\sim)$ is the number of equivalence classes of $\sim$, so we can write:
$$\delta(P) = P \otimes P|\sim_P + \sum P_1 \otimes P_2,$$
where the terms $P_1 \otimes P_2$ are such that $cl(P_1) < cl(P)$. As $P|\sim_P$ is a product of $\bullet_k$, $f(P) \neq 0$. We then put:
$$g(P) = \frac{1}{f(P|\sim_P)} \left(\varepsilon(P) - \sum g(P_1) f(P_2) \right).$$

Then $g * f(P) = \varepsilon(P)$ by construction. We now define $h(P)$ by decreasing induction on the number $cc(P)$ of connected component of P. Note that $1 \leq cc(P) \leq cl(P)$. If $cl(P) = ccl(P)$, then P is a product of $\bullet_k$, so $f(P) \neq 0$ and $\delta(P) = P \otimes P$: we put $h(P) = \dfrac{1}{f(P)}$. Let us assume that $h(Q)$ is defined for all quasi-posets Q such that $cl(Q) = cl(P)$ and $cc(Q) > cc(P)$. We denote by $\sim_0$ the equivalence on P defined by $x \sim_0 y$ if x and y are in the same connected component of P. Note that $\sim_0 \triangleleft P$, $P/\sim_0$ is a product of $\bullet_k$ (so $f(P/\sim_0) \neq 0$) and $P|\sim_0 = P$. If $\sim \triangleleft P$, then if $x \sim y$, then x and y are in the same connected component of P, so $x \sim_0 y$. Hence, the number of equivalence classes of $\sim$, which is also the number of connected components of $P|\sim$, is greater than $cc(P)$, with equality if, and only if, $\sim = \sim_0$; moreover, $cl(P|\sim) = cl(P)$. We can write:

$$\delta(P) = P/\sim_0 \otimes P + \sum P_1' \otimes P_2',$$

where the terms $P_1' \otimes P_2'$ are such that $cl(P_2') = cl(P)$ and $cc(P_2') > cc(P)$. We put:

$$h(P) = \frac{1}{f(P/\sim_0)} \left(\varepsilon(P) - \sum f(P_1')h(P_2') \right).$$

Then $f * h(P) = \varepsilon(P)$ by construction.

Finally:

$$h = \varepsilon * h = (g * f) * h = g * f * h = g * (f * h) = g * \varepsilon = g.$$

So f is invertible in $(\mathcal{H}_{\mathbf{qp}}^*, *)$, with inverse $g = h$.

As $\mathcal{H}_{\mathbf{qp}}$ is the polynomial algebra generated by connected quasi-posets, we can define a character g' on $\mathcal{H}_{\mathbf{qp}}$ by $g'(P) = g(P)$ for any connected quasi-poset P. If P is a connected quasi-poset P, then for any $\sim \triangleleft P$, $P/\sim$ is also connected, so:

$$g' * f(P) = (g' \otimes f) \circ \delta(P) = (g \otimes f) \circ \delta(P) = g * f(P) = \varepsilon(P).$$

As $g' * f$ and ε are both characters and coincide on connected quasi-posets, they are equal: the inverse of f is the character g', so f is invertible in $M_{\mathbf{qp}}$. $\Box$

2.5 Cointeractions

Theorem 12 *We consider the map:*

$$\rho = (Id \otimes \lfloor\rfloor) \circ \delta : \left\{ \begin{array}{rcl} \mathcal{H}_{\mathbf{QP}} & \longrightarrow & \mathcal{H}_{\mathbf{QP}} \otimes \mathcal{H}_{\mathbf{qp}} \\ P \in \mathbf{QP} & \longrightarrow & \displaystyle\sum_{\sim \triangleleft P} (P/\sim) \otimes \lfloor P \mid \sim \rfloor. \end{array} \right.$$

The bialgebras $(\mathcal{H}_{\mathbf{QP}}, m, \Delta)$ and $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$ are in cointeraction via ρ.

Proof. By composition, ρ is an algebra morphism. Let us take $P \in \mathbf{QP}(n)$. We put:

$$A = \{(r, O) \mid r \triangleleft P, O \in top(P/r)\}, \qquad B = \{(O, s, s') \mid O \in top(P), s \triangleleft P_{[n] \setminus O}, s' \triangleleft P_{|}O\}.$$

First step. We define a map $F : A \longrightarrow B$, sending (r, O) to (O, s, s'), by:

- xsy if xry and x, y are in the same connected component of $P_{|[n]\setminus O}$.
- $xs'y$ if xry and x, y are in the same connected component of $P_{|O}$.

Let us prove that F is well-defined. Let us take $x, y \in [n]$, with $x \in O$ and $x \leq_P y$. Then $x \leq_{P/r} y$. as O is an open set of P/r, $y \in O$: O is an open set of P. By definition, the equivalence classes of s' are the intersection of the equivalence classes of r and of the connected components of O. As O is a union of equivalence classes of r, they are $P_{|O}$-connected. If $x \sim_{P_{|O}/s'} y$, then $x \sim_{P/r} y$ and x and y are in the same connected component of O. As $r \triangleleft r$, xry, so $xs'y$: $s' \triangleleft P_{|O}$. Similarly, $s \triangleleft P_{|[n]\setminus O}$.

Second step. We define a map $G : B \longrightarrow A$, sending (O, s, s') to (O, r, r'), by:

$$xry \text{ if } (x, y \notin O \text{ and } xsy) \text{ or } (x, y \in O \text{ and } xs'y).$$

Let us prove that G is well-defined. Let $x, y \in [N]$, with $x \in O$ and $x \leq_{P/r} y$. There exists $x_1, x'_1, \ldots, x_k, x'_k$ such that:

$$x \leq_P x_1 r x'_1 \leq_P \ldots \leq_P x_k r x'_k \leq_P y.$$

As O is an open set of P, $x_1 \in O$; by definition of r, $x'_1 \in O$. Iterating, we obtain that $x_2, x'_2, \ldots, x_k, x'_k, y \in O$. So O is open in P/r.

Let us assume that xry. Then $x, \in O$ or $x, y \notin O$. As $s \triangleleft P_{|[n]\setminus O}$ and $P_{|O}$, there exists a path from x to y in the Hasse graph of P formed by elements s- or $s'-$ equivalent to x and y, so the equivalence classes of r are P-connected.

Let us assume that $x \sim_{P/r} y$. here exists $x_1, x'_1, \ldots, x_k, s'_k$ and $y_1, y'_1, \ldots, y_l, y_l$ such that:

$$x \leq_P x_1 r x'_1 \leq_P \ldots \leq_P x_k r x'_k \leq_P y, \qquad y \leq_P y_1 r y'_1 \leq_P \ldots \leq_P y_l r y'_l \leq_P x.$$

If $x, y \in O$, then all these elements are in O, so $x \sim_{P_{|O}/s'} y$, and then $xs'y$, so xry. If $x, y \notin O$, as O is an open set, none of these elements is in O, so $x \sim_{P_{|[n]\setminus O}/s} y$, so xsy and finally xry: $r \triangleleft P$.

Third step. Let $(r, O) \in A$. We put $F(r, O) = (O, s, s')$ and $G(O, s, s') = (\tilde{r}, O)$. If xry, as O is an open set of P/r, both x and y are in O or both are not in O. Hence, xsy or $xs'y$, so $x\tilde{r}y$.

If $x\tilde{r}y$, then xsy or $xs'y$, so xry: $\tilde{r} = r$ and $G \circ F = Id_A$.

Let $(O, s, s') \in B$. We put $G(O, s, s') = (r, O)$ and $F(r, O) = (O, \tilde{s}, \tilde{s}')$. If xsy, then x and y are in the same connected component of $[n] \setminus O$ as $s \triangleleft P_{|[n]\setminus O}$ and xry, so $x\tilde{s}y$. If $x\tilde{s}y$, then xry, so xsy: we obtain that $\tilde{s} = s$. Similarly, $\tilde{s}' = s'$, which proves that $F \circ G = Id_B$.

We proved that F and G are inverse bijections. Let $(r, O) \in A$ and $(O, s, s') = F(O, r)$.

$$\begin{aligned}
\leq_{(P/r)_{|[n]\setminus O}} &= \text{transitive closure of } (xry \text{ and } x \leq_P y) \text{ restricted to } [n] \setminus O \\
&= \text{transitive closure of } (xry \text{ and } x \leq_{P_{|[n]\setminus O}} y) \\
&= \text{transitive closure of } (xsy \text{ and } x \leq_{P_{|[n]\setminus O}} y) \\
&= \leq_{P_{|[n]\setminus [n]}/s}.
\end{aligned}$$

So $(P/r)_{|[n]\setminus O} = P_{|[n]\setminus O}/s$. Similarly, $(P/r)_{|O} = P_{|O}/s'$.

Let us now consider $P_{|R}$. Its connected components are the equivalence classes of r, that is to say the equivalence classes of s and s'; for any such equivalence class I, $(P_{|R})_{|I} = P_{|I}$. So $P_{|R}$ is the disjoint union of $(P_{|[n]\setminus O})_{|s}$ and $(P_{|O})_{|s'}$, and therefore is isomorphic to $Std(P_{|[n]\setminus O})_{|s})Std((P_{|O})_{|s'})$, but not equal, because of the reindexation induced by the standardization. Hence, $\lfloor P_{|R} \rfloor = \lfloor (P_{|[n]\setminus O})_{|s} \rfloor \lfloor (P_{|O})_{|s'} \rfloor$.

Finally:

$$\begin{aligned}
(\Delta \otimes Id) \circ \rho(P) &= \sum_{(r,O) \in A} (G/r)_{|[n]\setminus O} \otimes (G/r)_{|O} \otimes \lfloor G_{|r} \rfloor \\
&= \sum_{(O,s,s') \in B} (P_{|[n]\setminus O})/s \otimes (P_{|O})/s' \otimes \lfloor (P_{|[n]\setminus O})_{|s} \rfloor \lfloor (P_{|O})_{|s'} \rfloor \\
&= m^3_{2,4} \circ (\rho \otimes \rho) \circ \Delta(P).
\end{aligned}$$

Moreover, $(\varepsilon \otimes Id) \circ \rho(P) = \delta_{P,1} 1 \otimes 1 = \varepsilon(P) 1 \otimes 1$. $\Box$

Remark. As noticed in [8], $(\mathcal{H}_{\mathbf{QP}}, m, \Delta)$ and $(\mathcal{H}_{\mathbf{QP}}, m, \delta)$ are not in cointeraction through δ.

Taking the quotient through $\lfloor \rfloor$:

Corollary 13 *The bialgebras $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$ and $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$ are in cointeraction via δ.*

Using proposition 4 on $\mathcal{H}_{\mathbf{QP}}$:

Corollary 14 *Let $(\lambda_{\lfloor P \rfloor})$ be a family of scalars indexed by the set of connected quasi-posets. We define a character λ on $\mathcal{H}_{\mathbf{qp}}$ by $\lambda_{\lfloor P \rfloor} = \lambda_{\lfloor P_1 \rfloor} \ldots \lambda_{\lfloor P_k \rfloor}$ if $P_1, \ldots, P_k$ are the restrictions of P to its connected components. The following map is a Hopf algebra endomorphism:*

$$\phi_\lambda : \begin{cases} (\mathcal{H}_{\mathbf{QP}}, m, \Delta) & \longrightarrow & (\mathcal{H}_{\mathbf{QP}}, m, \Delta) \\ P \in \mathbf{QP} & \longrightarrow & \displaystyle\sum_{\sim \triangleleft P} \lambda_{\lfloor P|\sim \rfloor} P/\sim . \end{cases}$$

It is bijective if, and only if, for all $n \geq 1$, $\lambda_{\bullet_n} \neq 0$.

Proof. $\phi_\lambda = Id \leftarrow \lambda$, so is an element of $E_{\mathcal{H}_{\mathbf{QP}} \rightarrow \mathcal{H}_{\mathbf{QP}}}$.

$\Longrightarrow$. For all $n \geq 0$, $\phi_\lambda(\bullet_n) = \bullet_n \lambda_{\bullet_n}$. As ϕ_λ is injective, $\lambda_{\bullet_n} \neq 0$.
$\Longleftarrow$. By proposition 11, the character λ is invertible in $M_{\mathbf{qp}}$: let us denote its inverse by μ. Then, by proposition 4:

$$\phi_\lambda \circ \phi_\mu = Id \leftarrow (\lambda * \mu) = Id \leftarrow \varepsilon = Id.$$

Similarly, $\phi_\mu \circ \phi_\lambda = Id$. $\Box$

3 Ehrhart polynomials

Notations. For all $k \geq 0$, we denote by H_k the k-th Hilbert polynomial:

$$H_k(X) = \frac{X(X-1)\ldots(X-k+1)}{k!}.$$

3.1 Definition

Definition 15 *Let $P \in \mathbf{QP}(n)$ and let $k \geq 1$. We put:*

$$\begin{aligned} L_P(k) &= \{f : [n] \longrightarrow [k] \mid \forall i, j \in [n], i \leq_P j \Longrightarrow f(i) \leq f(j)\}, \\ L_P^{str}(k) &= \{f \in L_P(k) \mid \forall i, j \in [n], (i \leq_P j \text{ and } f(i) = f(j)) \Longrightarrow i \sim_P j\}, \\ W_P(k) &= \{w \in L_P(k) \mid w([n]) = [k]\}, \\ W_P^{str}(k) &= \{w \in L_P^{str}(k) \mid w([n]) = [k]\}. \end{aligned}$$

By convention:

$$L_P(0) = L_P^{str}(0) = W_P(0) = W_P^{str}(0) = \begin{cases} \emptyset \text{ if } P \neq 1, \\ \{1\} \text{ if } P = 1. \end{cases}$$

We also put:

$$L_P = \bigcup_{k \geq 0} L_P(k), \qquad L_P^{str} = \bigcup_{k \geq 0} L_P^{str}(k), \qquad W_P = \bigsqcup_{k \geq 0} W_P(k), \qquad W_P^{str} = \bigsqcup_{k \geq 0} W_P^{str}(k).$$

Note that the elements of W_P and W_P^{str} are packed words.

Proposition 16 *Let $P \in \mathbf{QP}$. There exist unique polynomials ehr_P and $ehr_P^{str} \in \mathbb{Q}[X]$, such that for $k \geq 0$:*

$$ehr_P(k) = \sharp L_P(k), \qquad\qquad ehr_P^{str}(k) = \sharp L_P^{str}(k).$$

Proof. This is obvious if $P = 1$, with $ehr_1(X) = ehr_1^{str}(X) = 1$. Let us assume that $P \in \mathbf{QP}(n)$, $n \geq 1$. Note that if $i > n$, $W_P(i) = 0$. For all $k \geq 1$:

$$\sharp L_P(k) = \sum_{i=1}^{k} \sharp W_P(i) \binom{k}{i} = \sum_{i=1}^{k} \sharp W_P(i) H_i(k) = \sum_{i=1}^{n} \sharp W_P(i) H_i(k).$$

So:

$$ehr_P(X) = \sum_{i=1}^{n} \sharp W_P(i) H_i(X).$$

Moreover, if $k = 0$:

$$ehr_P(0) = \sum_{i=1}^{n} \sharp W_P(i) H_i(0) = \sharp L_P(0).$$

In the same way:

$$ehr_P^{str}(X) = \sum_{i=1}^{n} \sharp W_P^{str}(i) H_i(X).$$

These are indeed elements of $\mathbb{Q}[X]$. $\Box$

Remarks.

1. Let $P, Q \in \mathbf{QP}(n)$.
   - If they are isomorphic, then $ehr_P(k) = ehr_Q(k)$ for all $k \geq 1$, so $ehr_P = ehr_Q$.
   - If $w \in L_P$, for all $x, y \in P$ such that $x \sim_P y$, then $w(x) \leq w(y)$ and $w(y) \leq w(x)$, so $w(x) = w(y)$: w goes through the quotient by $\sim_P$. We obtain in this way a bijection from $L_P(k)$ to $L_{\overline{P}}(k)$ for all k, so $ehr_P = ehr_{\overline{P}}$. Similarly, $ehr_P^{str} = ehr_{\overline{P}}^{str}$.

   Hence, we obtain maps, all denoted by ehr and ehr^{str}, such the following diagrams commute:

$$\begin{tikzcd}
\mathcal{H}_{\mathbf{QP}} \arrow[r, two heads, "\sqcup"] \arrow[dr, "ehr"'] & \mathcal{H}_{\mathbf{qp}} \arrow[r, two heads, "\Xi"] \arrow[d, "ehr"] & \mathcal{H}_{\mathbf{p}} \arrow[dl, "ehr"] \\
& \mathbb{K}[X] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{H}_{\mathbf{QP}} \arrow[r, two heads, "\sqcup"] \arrow[dr, "ehr^{str}"'] & \mathcal{H}_{\mathbf{qp}} \arrow[r, two heads, "\Xi"] \arrow[d, "ehr^{str}"] & \mathcal{H}_{\mathbf{p}} \arrow[dl, "ehr^{str}"] \\
& \mathbb{K}[X] &
\end{tikzcd}$$

2. Let $P \in \mathbf{P}(n)$. The classical definition of the Ehrhart polynomial $ehr'(t)$ is the number of of integral points of $tPol(P)$, where $Pol(P)$ is the polytope associated to P. Hence, $ehr'(X) = ehr(X+1)$.

Theorem 17 *The morphisms $ehr, ehr^{str} : \mathcal{H}_{\mathbf{QP}}, \mathcal{H}_{\mathbf{qp}}, \mathcal{H}_{\mathbf{p}} \longrightarrow \mathbb{K}[X]$ are Hopf algebra morphisms.*

Proof. It is enough to prove it for $ehr, ehr^{str} : \mathcal{H}_{\mathbf{p}} \longrightarrow \mathbb{K}[X]$.

First step. Let $P \in \mathbf{P}(n)$. Let us prove that for all $k, l \geq 0$:

$$ehr_P(k+l) = \sum_{O \in Top(P)} ehr_{P_{|[n] \setminus O}}(k) ehr_{P_{|O}}(l), \quad ehr_P^{str}(k+l) = \sum_{O \in Top(P)} ehr_{P_{|[n] \setminus O}}^{str}(k) ehr_{P_{|O}}^{str}(l).$$

Let $f \in L_P(k+l)$. We put $O = f^{-1}(\{k+1,\ldots,k+l\})$. If $x \in O$ and $x \leq_P y$, then $f(x) \leq f(y)$, so $y \in O$: O is an open set of P. By restriction, the following maps are elements of respectively $L_{P_{|[n]\setminus O}}(k)$ and $L_{P_{|O}}(l)$:

$$f_1 : \left\{ \begin{array}{rcl} [n]\setminus O & \longrightarrow & [k] \\ x & \longrightarrow & f(x), \end{array} \right. \qquad f_2 : \left\{ \begin{array}{rcl} O & \longrightarrow & [l] \\ x & \longrightarrow & f(x)-k. \end{array} \right.$$

This defines a map:

$$\upsilon : \left\{ \begin{array}{rcl} L_P(k+l) & \longrightarrow & \displaystyle\bigsqcup_{O \in Top(P)} L_{P_{|[n]\setminus O}}(k) \times L_{P_{|O}}(l) \\ f & \longrightarrow & (f_1, f_2). \end{array} \right.$$

This map is clearly injective; moreover:

$$\nu(L_P^{str}(k+l)) \subseteq \bigsqcup_{O \in Top(P)} L^{str}_{P_{|[n]\setminus O}}(k) \times L^{str}_{P_{|O}}(l).$$

Let us prove that f is surjective. Let $(f_1, f_2) \in L_{P_{|[n]\setminus O}}(k) \otimes L_{P_{|O}}(l)$, with $O \in Top(P)$. We define a map $f : P \longrightarrow [k+l]$ by:

$$f(x) = \begin{cases} f_1(x) \text{ if } x \notin O, \\ f_2(x) + k \text{ if } x \in O. \end{cases}$$

Let $x \leq_P i$. As O is an open set of P, three cases are possible:

- $x, y \notin O$: then $f_1(x) \leq f_1(y)$, so $f(x) \leq f(y)$.
- $x, y \in O$: then $f_2(x) \leq f_2(y)$, so $f(x) \leq f(y)$.
- $x \notin O$, $y \in O$: then $f(x) \leq k < fyj)$.

So $f \in L_P(k+l)$, and $\upsilon(f) = (f_1, f_2)$: υ is surjective, and finally bijective. Moreover, if $f_1 \in L^{str}_{[n]\setminus O}(k)$ and $f_2 \in L^{str}_{P_{|O}}(l)$, then $f = \upsilon^{-1}(f_1, f_2) \in L_P^{str}(k+l)$. Finally:

$$\begin{aligned} f(L_P(k+l)) &= \bigsqcup_{O \in Top(P)} L_{P_{|[n]\setminus O}}(k) \times L_{P_{|O}}(l), \\ f(L_P^{str}(k+l)) &= \bigsqcup_{O \in Top(P)} L^{str}_{P_{|[n]\setminus O}}(k) \times L^{str}_{P_{|O}}(l). \end{aligned}$$

Taking the cardinals, we obtain the announced result.

Second step. Let $P \in \mathbf{P}(m)$, $Q \in \mathbf{P}(n)$, and $f : [m+n] \longrightarrow [k]$. Then $f \in L_{PQ}(k)$ if, and only if, $f_{|[m]} \in L_P(k)$ and $Std(f_{|[m+n]\setminus[m]} \in L_Q(k)$. So $ehr_{PQ}(k) = ehr_P(k)ehr_Q(k)$, and then $ehr_{PQ}(X) = ehr_P(X)ehr_Q(X)$: ehr is an algebra morphism.

Let P be a finite poset, and $k, l \geq 0$. By the first step:

$$\begin{aligned} (ehr \otimes ehr) \circ \Delta(P)(k,l) &= \sum_{O \in Top(P)} ehr_{P_{|[n]\setminus O}}(k) ehr_{P_{|O}}(l) \\ &= ehr_P(k+l) \\ &= \Delta \circ ehr(P)(k,l). \end{aligned}$$

As this is true for all $k, l \geq 1$, $(ehr \otimes ehr) \circ \Delta(P) = \Delta \circ ehr(P)$. Moreover:

$$\varepsilon \circ ehr(P) = ehr_P(0) = \begin{cases} 1 \text{ if } P = 1, \\ 0 \text{ otherwise,} \end{cases}$$

so $\varepsilon \circ ehr = \varepsilon$.

The proof is similar for ehr^{str}. $\square$

3.2 Recursive computation of ehr and ehr^{str}

Lemma 18 *We consider the following maps:*

$$L : \left\{ \begin{array}{rcl} \mathbb{K}[X] & \longrightarrow & \mathbb{K}[X] \\ H_k(X) & \longrightarrow & H_{k+1}(X). \end{array} \right.$$

The map L is injective, and $L(\mathbb{K}[X]) = \mathbb{K}[X]_+$. Moreover, for all $P \in \mathbb{K}[X]$, for all $n \geq 0$:

$$L(P)(n+1) = P(0) + \ldots + P(n).$$

Proof. Let us consider $P = H_k(X)$. For all $n \geq 0$:

$$\begin{aligned} H_k(0) + \ldots + H_k(n) &= \binom{0}{k} + \ldots + \binom{n}{k} \\ &= \binom{k}{k} + \ldots + \binom{n}{k} \\ &= \binom{n+1}{k+1} \\ &= H_{k+1}(n+1) \\ &= L(H_k)(n+1). \end{aligned}$$

By linearity, $L(P)(n+1) = P(0) + \ldots + P(n)$ for all $n \geq 1$. $\Box$

Definition 19 *Let $P \in \mathbf{QP}$. We shall say that P is* discrete *if $\lfloor \overline{P} \rfloor = \mathbf{.}^k$ for a certain $k \geq 0$.*

Proposition 20 *Let $P \in \mathbf{P}(n)$.*

$$\begin{aligned} ehr_P(X) &= L\left(\sum_{\emptyset \neq O \in Top(P)} ehr_{P_{|[n]\setminus O}}(X) \right), \\ ehr_P^{str}(X) &= L\left(\sum_{\emptyset \neq O \in Top(P),\ discrete} ehr^{str}_{P_{|[n]\setminus O}}(X) \right). \end{aligned}$$

Proof. Let $n \geq 1$. As $L_Q(1)$ is reduced to a singleton for all finite poset Q:

$$\begin{aligned} ehr_P(n+1) &= \sum_{O \in Top(P)} ehr_{P_{|[n]\setminus O}}(n) ehr_{P_{|O}}(1) \\ &= \sum_{\emptyset \neq O \in Top(P)} ehr_{P_{|[n]\setminus O}}(n) + ehr_P(n). \end{aligned}$$

We put:

$$Q(X) = \sum_{\emptyset \neq O \in Top(P)} ehr_{P_{|[n]\setminus O}}(X).$$

In particular:

$$Q(0) = \sum_{\emptyset \neq O \in Top(P)} ehr_{P_{|[n]\setminus O}}(0) = ehr_\emptyset(0) + 0 = 1 = ehr_P(1).$$

Then:

$$\begin{aligned}
ehr_P(n+1) &= Q(n) + ehr_P(n) \\
&= Q(n) + Q(n-1) + ehr_P(n-1) \\
&\vdots \\
&= Q(n) + Q(n-1) + \ldots + Q(1) + ehr_P(1) \\
&= Q(n) + \ldots + Q(1) + Q(0) \\
&= L(Q)(n+1).
\end{aligned}$$

So $ehr_P = L(Q)$.

For ehr_P^{str}, observe that $ehr_Q^{str}(1) = 1$ if Q is discrete, and 0 otherwise, which implies:

$$ehr_P^{str}(n+1) = \sum_{\emptyset \neq O \in Top(P),\ \text{discrete}} ehr^{str}_{P_{|[n]\setminus O}}(n) + ehr_P^{str}(n).$$

The end of the proof is similar. $\square$

Examples.

$$\begin{aligned}
ehr_{\bullet}(X) &= H_1(X) = X, \\
ehr_{\substack{\bullet\\ \vert\\ \bullet}}(X) &= H_1(X) + H_2(X) = \frac{X(X+1)}{2}, \\
ehr_{\vee}(X) = ehr_{\wedge}(X) &= H_1(X) + 3H_2(X) + 2H_3(X) = \frac{X(X+1)(2X+1)}{6}, \\
ehr_{\substack{\bullet\\ \vert\\ \bullet\\ \vert\\ \bullet}}(X) &= H_1(X) + 2H_2(X) + H_3(X) = \frac{X(X+1)(X+2)}{6};
\end{aligned}$$

$$\begin{aligned}
ehr^{str}_{\bullet}(X) &= H_1(X) = X, \\
ehr^{str}_{\substack{\bullet\\ \vert\\ \bullet}}(X) &= H_2(X) = \frac{X(X-1)}{2}, \\
ehr^{str}_{\vee}(X) = ehr^{str}_{\wedge}(X) &= H_2(X) + 2H_3(X) = \frac{X(X-1)(2X-1)}{6}, \\
ehr^{str}_{\substack{\bullet\\ \vert\\ \bullet\\ \vert\\ \bullet}}(X) &= H_3(X) = \frac{X(X-1)(X-2)}{6}.
\end{aligned}$$

3.3 Characterization of quasi-posets by packed words

Lemma 21 *Let $P = ([n], \leq_P)$ be a quasi poset and let $I_1, \ldots, I_k$ be distinct minimal classes of the poset $\overline{P}$; let $w' \in W^{str}_{P_{|[n]\setminus(I_1 \sqcup \ldots \sqcup I_k)}}$. The following map belongs to W_P^{str}:*

$$w : \begin{cases} [n] & \longrightarrow & \mathbb{N}^* \\ x \in I_p, 1 \leq p \leq k & \longrightarrow & p \\ x \notin I_1 \sqcup \ldots \sqcup I_k & \longrightarrow & w'(x). \end{cases}$$

Proof. Let us assume that $i \leq_P j$.

- If $i \in I_p$, as I_p is a minimal class of $\overline{P}$, $j \in I_p$ or $j \notin I_1 \sqcup \ldots \sqcup I_k$. In the first case, $w(i) = w(j)$; in the second case, $w(i) \leq k < w(j)$. If moreover $w(i) = w(j)$, then necessarily $j \in I_p$, so $i \sim_P j$.

- If $i \notin I_1 \sqcup \ldots \sqcup I_k$, as $i \leq_P j$, $j \notin I_1 \sqcup \ldots \sqcup I_k$, so $i \leq_{P_{\mid [n] \setminus (I_1 \sqcup \ldots \sqcup I_k)}} j$ and $w'(i) \leq w'(j)$, so $w'(i) \leq w'(j)$. If moreover $w(i) = w(j)$, then $w'(i) = w'(j)$, so $i \sim_{P_{\mid [n] \setminus (I_1 \sqcup \ldots \sqcup I_k)}} j$ and finally $i \sim_P j$.

As a conclusion, $w \in W_P^{str}$. $\square$

Note that this lemma implies that W_P^{str} is non-empty for any non-empty quasi-poset P.

Proposition 22 *Let $P = ([n], \leq_P)$ be a quasi-poset and let $i, j \in [n]$. The following assertions are equivalent:*

1. *$i \leq_P j$.*
2. *$\forall w \in L_P$, $w(i) \leq w(j)$.*
3. *$\forall w \in L_P^{str}$, $w(i) \leq w(j)$.*
4. *$\forall w \in W_P$, $w(i) \leq w(j)$.*
5. *$\forall w \in W_P^{str}$, $w(i) \leq w(j)$.*

Proof. Obviously:

$$\begin{tikzcd} & & 3. \arrow[dr, Rightarrow] & \\ 1. \arrow[r, Rightarrow] & 2. \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & 5. \\ & & 4. \arrow[ur, Rightarrow] & \end{tikzcd}$$

It is enough to prove that 5. $\Longrightarrow$ 1. We proceed by induction on n. If $n = 1$, there is nothing to prove. Let us assume the result at all ranks $< n$. Let $i, j \in [n]$, such that we do not have $i \leq_P j$. Let us prove that there exists $w \in W_P^{str}$, such that $w(i) > w(j)$. There exists a minimal element $k \in [n]$, such that $k \leq_P j$; let I be the class of k in $\overline{P}$. By hypothesis on i, i and k are not equivalent for $\sim_P$, so $i \notin I$. If $j \in I$, let us choose an element $w' \in W^{str}_{P_{\mid [n] \setminus I}}$; if $j \notin I$, then by the induction hypothesis, there exists $w' \in W^{str}_{P_{\mid [n] \setminus I}}$, such that $w'(i) > w'(j)$. By lemma 21, the following map is an element of W_P^{str}:

$$w : \left\{ \begin{array}{rcl} [n] & \longrightarrow & \mathbb{N} \\ x \in I & \longrightarrow & 1 \\ x \notin I & \longrightarrow & w'(x) + 1 \end{array} \right.$$

If $j \in I$, then $w(j) = 1 < w(i)$; if $j \notin I$, $w(i) = w'(i) + 1 > w'(j) + 1 = w(j)$. In both cases, $w(i) > w(j)$. $\square$

3.4 A link with linear extensions

Let $P \in \mathbf{QP}(n)$. Linear extensions, as defined in [9], belong to W_P^{str}: they are the elements $f \in W_P^{str}$ such that

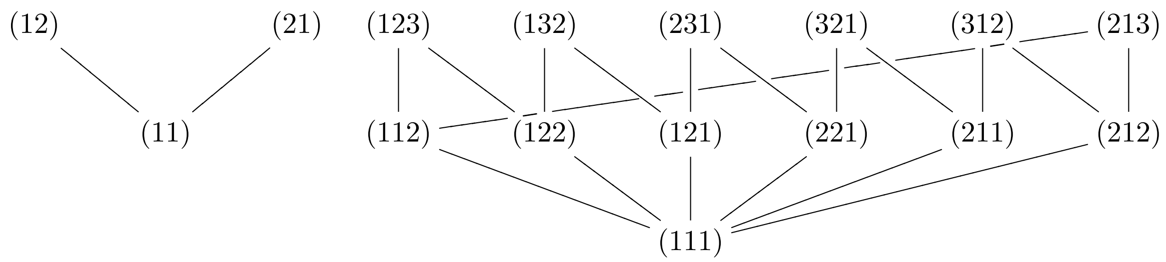
$$\forall i, j \in [n], f(i) = f(j) \Longleftrightarrow i \sim_P j.$$

It may happens that not all elements of W_P^{str} are linear extensions. For example, if $P = {}^{2}\mathtt{V}_{1}^{3}$, $W_P^{str}(3) = \{(123), (132), (122)\}$, and (122) is not a linear extension of P. The set of linear extensions of P will be denoted by E_P.

Definition 23 *Let w and w' be two packed words of the same length n. We shall say that $w \leq w'$ if:*

$$\forall i, j \in [n], \ (w(i) < w(j)) \Longrightarrow (w'(i) < w'(j)).$$

This defines a partial order on packed words of the same length n. For example, here are the Hasse graph of this order for $n = 2$ and $n = 3$:



Proposition 24 *Let $P \in \mathbf{QP}(n)$. Then:*

$$W_P = \bigcup_{w \in E_P} \{w' \mid w' \leq w\}.$$

This union may be not disjoint. Moreover, the maximal elements of W_P for the order of definition 23 are the elements of E_P.

Proof. $\subseteq$. Let $w \in W_P$. For all $1 \leq p \leq \max(w)$, we put $I_p = w^{-1}(p)$. Let f_p be a linear extension of $P_{|I_p}$. We define $f : [n] \longrightarrow \mathbb{N}$ by:

$$f(i) = \max(f_1) + \ldots + \max(f_{p-1}) + f_p(i) \text{ if } i \in I_p.$$

By construction, if $w(i) < w(j)$, then $f(i) < f(j)$: $w \leq f$. Let us prove that $f \in E_P$.

If $i \leq_P j$, then as $w \in W_P$, $w(i) \leq w(j)$. If $w(i) = w(j) = p$, then $i \leq_{P_{|I_p}} j$, so $f_p(i) \leq f_p(j)$, and $f(i) \leq f(j)$. If $w(i) < w(j)$, then $f(i) < f(j)$.

If $f(i) = f(j)$, then $w(i) = w(j) = p$, and $f_p(i) = f_p(j)$. As $f_p \in E_{P_{|P_p}}$, $i \sim_{P_{|P_p}} j$, so $i \sim_P j$.

$\supsetneq$. Let $w \in E_P$ and $w' \leq w$. If $i \leq_P j$, then $w(i) \leq w(j)$ as w is a linear extension of P. As $w' \leq w$, $w'(i) \leq w'(j)$, so $w' \in W_P$.

Let w be a maximal element of W_P. There exists a linear extension w' of P, such that $w \leq w'$. As w is maximal, $w = w'$ is a linear extension of P. Conversely, if w is a linear extension of P and $w \leq w'$ in W_P, then as w is a linear extension of P, $\max(w) = cl(P)$. Moreover, as $w \leq w'$, $\max(w) \leq \max(w')$. As $w' \in W_P$, $\max(w') \leq cl(P)$, which implies that $\max(w) = \max(w') = cl(P)$, and finally $w = w'$: w is a maximal element of W_P. $\Box$

Example. For $P = {}^{2}\mathtt{V}_{1}{}^{3}$:

$$\begin{aligned}
E_P &= \{(123), (132)\};\\
W_P &= \{(123), (122), (112), (111)\} \cup \{(132), (122), (121), (111)\}\\
&= \{(123), (132), (122), (112), (121), (111)\}.
\end{aligned}$$

Note that the two components of W_P are not disjoint.

Remark. A similar result is proved in [9] for T-partitions of a quasi-poset, generalizing Stanley's result [16] for P-partitions of posets; nevertheless, this is different here, as the union is not disjoint.

4 Characters associated to ehr and ehr^{str}

Recall that $(M_{\mathbf{qp}}, *)$ is the monoid of characters of $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$.

4.1 The monoid action on Hopf algebra morphisms

Notation. We denote by π the map from $\mathbb{K}[X]$ to $\mathbb{K}$, sending any polynomial $P(X)$ to the coefficient of X in P. In other words:

$$\pi(P) = \frac{dP}{dX}(0) = \left(\frac{P(X) - P(0)}{X}\right)_{|X=0}.$$

Lemma 25 *Let A be a graded, connected bialgebra and let $\phi, \psi : A \longrightarrow \mathbb{K}[X]$ be bialgebra morphisms. Let G be a set of generators of the algebra A, included in the augmentation ideal of A. If for all $x \in G$, $\pi \circ \phi(x) = \pi \circ \psi(x)$, then $\phi = \psi$.*

Proof. *First step.* Let us prove that $\pi \circ \phi(a) = \pi \circ \psi(a)$ for all $a \in A$. As G generates A, we can assume that $a = x_1 \ldots x_k$, with $k \geq 0$ and $x_1, \ldots, x_k \in G$. If $k = 0$, $a = 1$ and $\pi \circ \phi(1) = \pi \circ \phi(1) = 0$. If $k = 1$, this is the hypothesis of the lemma. If $k \geq 2$, as $G \subseteq Ker(\varepsilon_A)$, $a \in Ker(\varepsilon_A)^2$ and both $\phi(a)$ and $\psi(a)$ belong to $Ker(\varepsilon_{\mathbb{K}[X]})^2 = \langle X^2 \rangle$. So $\pi \circ \phi(a) = \pi \circ \phi(a) = 0$.

Second step. Let us take $a \in A$, homogeneous of degree n. Let us prove that $\phi(a) = \psi(a)$ by induction on n. If $n = 0$, we can assume that $a = 1$ by connectivity, so $\phi(a) = \phi(a) = 1$. Let us assume the result at all ranks $< n$. By the induction hypothesis:

$$\tilde{\Delta}(\phi(a) - \psi(a)) = (\phi \otimes \phi) \circ \tilde{\Delta}(a) - (\psi \otimes \psi) \circ \tilde{\Delta}(a) = (\psi \otimes \psi) \circ \tilde{\Delta}(a) - (\psi \otimes \psi) \circ \tilde{\Delta}(a) = 0.$$

So $\phi(a) - \psi(a) \in Prim(\mathbb{K}[X]) = Vect(X)$ and:

$$\phi(a) - \psi(a) = \pi(\phi(a) - \psi(a))X = 0.$$

So $\phi = \psi$. $\Box$

Proposition 26 *There exists a unique Hopf algebra morphism $\phi_0 : (\mathcal{H}_{\mathbf{qp}}, m, \Delta) \longrightarrow \mathbb{K}[X]$ such that:*

$$\forall x \in \mathcal{H}_{\mathbf{qp}},\ \pi \circ \phi_0(x) = \varepsilon_B(x),$$

where ε_B is the counit of $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$. This morphism is homogeneous for the graduation of $\mathcal{H}_{\mathbf{qp}}$ by the number $cl(P)$ of equivalence classes of $\sim_P$.

Proof. *Unicity.* It is guaranteed by lemma 25, where G is the set of connected quasi-posets.

Existence. We identify $\mathbb{K}[X]$ and its graded dual via the Hopf pairing defined by:

$$\forall k, l \geq 0,\ \langle X^k, X^l \rangle = \delta_{k,l} k!.$$

We consider the dual basis of the basis of posets of $\mathcal{H}_{\mathbf{p}}$, which we denote by $(P^*)_{\mathbf{P} \in \mathbf{p}}$; this is a basis of the Hopf algebra $\mathcal{H}_{\mathbf{p}}^*$. As $\centerdot^*$ is primitive and homogeneous of degree 1 in $\mathcal{H}_{\mathbf{p}}^*$, there exists a homogeneous Hopf algebra morphism $\psi_0' : \mathbb{K}[X] \longrightarrow \mathcal{H}_{\mathbf{p}}^*$, sending X to $\centerdot^*$. Let us consider its transpose $\psi_0 : \mathcal{H}_{\mathbf{p}}^* \longrightarrow \mathbb{K}[X]$; it is homogeneous and sends $\centerdot$ to X. If P is a quasi-poset of cardinality ≥ 2, $\psi_0(P)$ is homogeneous of degree ≥ 2, so $\pi \circ \psi_0(P) = 0$. To summarize:

$$\pi \circ \psi_0(P) = \begin{cases} 1 \text{ if } P = \centerdot, \\ 0 \text{ otherwise.} \end{cases}$$

Consequently, if P is connected, $\pi \circ \psi_0(P) = \varepsilon_B(P)$. We then take $\phi_0 = \psi_0 \circ \Xi$. By composition, it is a Hopf algebra morphism, homogeneous of the graduation of $\mathcal{H}_{\mathbf{qp}}$ by cl, and for any connected quasi-poset P, $\pi \circ \phi_0(P) = \pi \circ \psi_0(\overline{P}) = \varepsilon_B(\overline{P}) = \varepsilon_B(P)$. $\Box$

Proposition 27 *Let $E = E_{\mathcal{H}_{\mathbf{qp}}\to\mathbb{K}[X]}$ be the set of Hopf algebra morphisms from $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$ to $\mathbb{K}[X]$. The monoid $(M_{\mathbf{qp}}, *)$ acts on E via the map:*

$$\leftarrow: \left\{ \begin{array}{rcl} E \times M_B & \longrightarrow & E \\ (\phi, f) & \longrightarrow & \phi \leftarrow f = (\phi \otimes f) \circ \delta. \end{array} \right.$$

For any $\phi \in E$, there exists a unique $f \in M_{\mathbf{qp}}$ such that $\phi = \phi_0 \leftarrow f$. Moreover, for any connected quasi-poset P:

$$f(P) = \pi \circ \phi(P).$$

Proof. *Unicity.* If $\phi = \phi_0 \leftarrow f$, for any connected quasi-poset P:

$$\phi(P) = \sum_{\sim \triangleleft P} \phi_0(P/\sim) f(P|\sim).$$

Note that $P/\sim$ is homogeneous of degree the number of equivalence classes of $\sim$, so $\phi_0(P/\sim)$ is homogeneous of degree 1 if, and only if, $\sim$ has only one equivalence class; in this case, $P|\sim = P$. Hence:

$$\pi \circ \phi(P) = \varepsilon_B(\centerdot) f(P) = f(P).$$

As connected quasi-posets generate $\mathcal{H}_{\mathbf{qp}}$, this entirely determines f.

Existence. As $\mathcal{H}_{\mathbf{qp}}$ is the polynomial algebra generated by connected quasi-posets, there exists a character f such that for all connected poset P, $f(P) = \pi \circ \phi(P)$. Then for all connected poset P, $\pi \circ \phi(P) = \pi \circ (\phi_0 \leftarrow f)(P) = f(P)$. By lemma 25, $\phi = \phi_0 \leftarrow f$. $\square$

4.2 Associated characters

By homogeneity, for any quasi-poset P, there exists a scalar λ_P such that

$$\phi_0(P) = \lambda_P X^{cl(P)}.$$

If P, Q are two quasi-posets:

$$\phi_0(PQ) = \lambda_{PQ} X^{cl(PQ)} = \phi_0(P)\phi_0(Q) = \lambda_P \lambda_Q X^{cl(P)+cl(Q)},$$

So $\lambda_{PQ} = \lambda_P \lambda_Q$: λ defines a character of $\mathcal{H}_{\mathbf{qp}}$. Moreover, as $\phi_0(P) = \phi_0(\overline{P})$ for any $P \in \mathbf{qp}$, $\lambda_P = \lambda_{\overline{P}}$: it is enough to consider posets here.

Lemma 28 *For all $P \in \mathbf{P}(n)$, $n \geq 0$:*

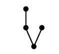
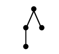
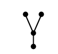
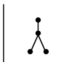
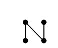
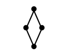
$$\lambda_{\lfloor P \rfloor} = \begin{cases} 1 \text{ if } P = 1, \\ \dfrac{1}{n} \displaystyle\sum_{M \in max(P)} \lambda_{\lfloor P_{|[n]\setminus\{M\}} \rfloor} = \dfrac{1}{n} \displaystyle\sum_{m \in min(P)} \lambda_{\lfloor P_{|[n]\setminus\{m\}} \rfloor} \text{ otherwise.} \end{cases}$$

Proof. Let $P \in \mathbf{P}(n)$, with $n \geq 0$.

$$\begin{aligned} (Id \otimes \pi) \circ \Delta \circ \phi_0(\lfloor P \rfloor) &= \lambda_{\lfloor P \rfloor} (Id \otimes \pi) \circ \Delta(X^n) \\ &= \lambda_{\lfloor P \rfloor} n X^{n-1}; \\ = (Id \otimes \pi) \circ (\phi_0 \otimes \phi_0) \circ \Delta(\lfloor P \rfloor) &= \sum_{O \in Top(P)} \lambda_{\lfloor P_{|O} \rfloor} \lambda_{\lfloor P_{|[n]\setminus O} \rfloor} X^{|[n]\setminus O|} \pi(X^{|O|}) \\ &= \sum_{O \in Top(P),\, |O|=1} \lambda_{\lfloor P_{|O} \rfloor} \lambda_{\lfloor P_{|[n]\setminus O} \rfloor} X^{n-1} \\ &= \sum_{M \in max(P)} \lambda_{\lfloor P_{|[n]\setminus\{M\}} \rfloor} X^{n-1}. \end{aligned}$$

This implies the first equality. The second is proved by considering $(\pi \otimes Id) \circ \Delta \circ \phi_0(\lfloor P \rfloor)$. $\square$

This lemma allows to inductively compute λ_P. This gives:

P															
λ_P	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{5}{24}$	$\frac{1}{6}$	$\frac{1}{12}$

Proposition 29 *Let $P \in \mathbf{P}(n)$. The number of elements of $W_P(n)$ of P is denoted by μ_P: in other words, μ_P is the number of bijections f from $[n]$ to $[n]$ such that for all $x, y \in [n]$,*

$$x \leq_P y \Longrightarrow f(x) \leq f(y),$$

that is to say heap-orderings of P. For any finite poset P, $\lambda_P = \frac{\mu_P}{n!}$.

Proof. Let us fix a non-empty finite poset $P \in \mathbf{P}(n)$. For any poset Q, the set of heap-orderings of Q is $W_Q(|Q|)$. We consider the map:

$$\left\{ \begin{array}{rcl} W_P(n) & \longrightarrow & \displaystyle\bigsqcup_{M \in max(P)} W_{P \setminus \{M\}}(n-1) \\ f & \longrightarrow & f_{|[n-1]} \in W_{P \setminus \{f^{-1}(n)\}}(n-1). \end{array} \right.$$

It is not difficult to prove that this is a bijection. So:

$$\mu_P = \sum_{M \in max(P)} \mu_{P \setminus \{M\}}; \qquad \frac{\mu_P}{n!} = \frac{1}{n} \sum_{M \in max(P)} \frac{\mu_{P \setminus \{M\}}}{|P \setminus \{M\}|!}.$$

An easy induction on $|P|$ then proves that $\lambda_P = \frac{\mu_P}{n!}$ for all P. $\square$

This is simpler for rooted forests:

Definition 30 *Let P be a non-empty finite poset.*

1. *We put:*

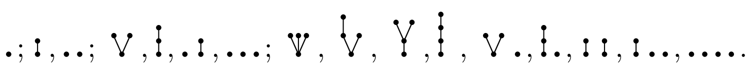
$$P! = \prod_{i \in V(P)} \sharp\{j \in V(P) \mid i \leq_P j\}.$$

By convention, $1! = 1$.

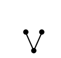
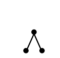
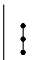
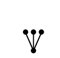
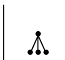
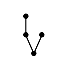
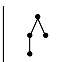
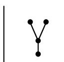
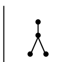
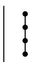
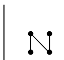
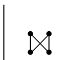
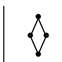
2. *We shall say that P is a rooted forest if P does not contain any subposet isomorphic to*  *.*

Examples.

1. Here are isoclasses of rooted forests of cardinality ≤ 4:

1; 

2. Here are examples of values of $P!$:

P															
$P!$	1	2	3	4	6	4	8	8	12	12	18	24	6	9	16

Proposition 31 *For all finite poset P, $\lambda_P \geq \frac{1}{P!}$, with equality if, and only if, P is a rooted forest.*

Proof. We proceed by induction on $n = |P|$. It is obvious if $n = 0$. Let us assume the result at all ranks $< n$.

$$\begin{aligned}
\lambda_P &= \frac{1}{|P|} \sum_{m \in min(P)} \lambda_{P \setminus \{m\}} \\
&\geq \frac{1}{|P|} \sum_{m \in min(P)} \prod_{i \in V(P), i \neq m} \frac{1}{\sharp\{j \in V(P) \mid j \neq m, i \leq_P j\}} \\
&= \frac{1}{|P|} \sum_{m \in min(P)} \prod_{i \in V(P), i \neq m} \frac{1}{\sharp\{j \in V(P) \mid i \leq_P j\}} \\
&= \frac{1}{P!} \frac{1}{|P|} \sum_{m \in min(P)} \sharp\{j \in V(P) \mid m \leq_P j\}.
\end{aligned}$$

For any $j \in A$, there exists $m \in min(P)$ such that $m \leq_P j$, so:

$$\sum_{m \in min(P)} \sharp\{j \in V(P) \mid m \leq_P j\} \geq |P|.$$

Consequently, $\lambda_P \geq \dfrac{1}{P!}$.

Let us assume that this is an equality. Then:

$$\sum_{m \in min(P)} \sharp\{j \in V(P) \mid m \leq_P j\} = |P|.$$

Consequently, for all $j \in min(P)$, there exists a unique $m \in min(P)$ such that $m \leq_P j$. Moreover, for all $m \in min(P)$, $\lambda_{P \setminus \{m\}} = \frac{1}{P \setminus \{m\}!}$. By the induction hypothesis, $P \setminus \{m\}$ is a rooted forest; this implies that P is also a rooted forest.

Let us assume that P is a rooted forest. For any $j \in V(P)$, there exists a unique $m \in min(P)$ such that $m \leq_P j$, so:

$$\sum_{m \in min(P)} \sharp\{j \in V(P) \mid m \leq_P j\} = |P|.$$

Moreover, for all $m \in min(P)$, $P \setminus \{m\}$ is also a rooted forest. By the induction hypothesis, $\lambda_{P \setminus \{m\}} = \frac{1}{P \setminus \{m\}!}$. Hence, $\lambda_P = \dfrac{1}{P!}$. $\Box$

Let us now apply proposition 27 to ehr and ehr^{str}:

Theorem 32 *For all finite connected quasi-poset P, let us put:*

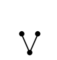
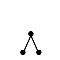
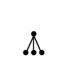
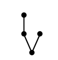
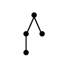
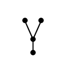
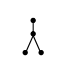
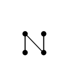
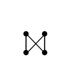
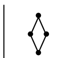
$$\alpha_P = \frac{d}{dX} ehr_P(X)_{|X=0}, \qquad \alpha_P^{str} = \frac{d}{dX} ehr_P^{str}(X)_{|X=0}.$$

These scalars define characters α and α^{str} in $M_{\mathbf{qp}}$. For any quasi-poset P:

$$ehr_P(X) = \sum_{\sim \triangleleft P} \frac{\mu_{P/\sim}}{cl(\sim)!} \alpha_{P|\sim} X^{cl(\sim)}, \qquad ehr_P^{str}(X) = \sum_{\sim \triangleleft P} \frac{\mu_{P/\sim}}{cl(\sim)!} \alpha^{str}_{P|\sim} X^{cl(\sim)},$$

where $cl(\sim)$ is the number of equivalence classes of $\sim$.

Examples. Let us give a few values of α:

P															
α_P	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	0	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$

Lemma 33 *Let $P \in \mathbf{QP}$, not discrete. Then $ehr_P(-1) = 0$.*

Proof. *First step.* Let us prove that $L(H_k(-X)) = -H_{k+1}(-X)$ for all $k \geq 0$. For all $l, n \geq 0$:

$$H_l(-n) = (-1)^l \binom{n+l-1}{l}.$$

For all $k, n \geq 0$:

$$\begin{aligned}
L(H_k(-X))(n+1) &= H_k(0) + \ldots + H_k(-n) \\
&= (-1)^k \sum_{i=0}^{n} \binom{i+k-1}{k} \\
&= (-1)^k \sum_{j=k}^{n+k-1} \binom{j}{k} \\
&= (-1)^k \binom{n+k}{k+1} \\
&= -H_{k+1}(-(n+1)).
\end{aligned}$$

Second step. Let us prove that $L(\langle X+1 \rangle) \subseteq \langle X+1 \rangle$. For all $k \geq 2$, let us put $H_k(-X) = X(X+1)L_k(X)$; $(L_k(X))_{k\geq 2}$ is a basis of $\mathbb{K}[X]$, which implies that $(H_k(-X))_{k\geq 2}$ is a basis of $\langle X(X+1) \rangle$, and that $(X+1) \sqcup (H_k(-X))_{k\geq 2}$ is a basis of $\langle X+1 \rangle$. First:

$$L(X+1) = L(H_1(X) + H_0(X)) = H_2(X) + H_1(X) = \frac{X(X-1)}{2} + X = \frac{X(X+1)}{2} \in \langle X+1 \rangle;$$

if $k \geq 2$, by the first step, $L(H_k(-X)) = -H_{k+1}(-X) \in \langle X+1 \rangle$.

Last step. We can replace P by $\overline{P}$, and we now assume that $P \in \mathbf{P}(n)$. There is nothing to prove if $n = 0, 1$. Let us assume the result at all rank $< n$. Then, by the second step and the induction hypothesis:

$$\begin{aligned}
ehr_{\lfloor P \rfloor}(-1) &= L\left(\sum_{\emptyset \neq O \in Top(P)} ehr_{\lfloor P_{|[n]\setminus O} \rfloor}(X) \right)_{|X=-1} \\
&= L\left(\sum_{\substack{\emptyset \neq O \in Top(P) \\ P_{|[n]\setminus O} \text{ discrete}}} ehr_{\lfloor P_{|[n]\setminus O} \rfloor}(X) \right)_{|X=-1} \\
&= L\left(\sum_{[n] \neq J \subseteq min(P)} ehr_{\lfloor P_{|J} \rfloor}(X) \right)_{|X=-1} \\
&= L\left(\sum_{J \subseteq min(P)} ehr_{\lfloor P_{|J} \rfloor} J(X) \right)_{|X=-1} \\
&= L\left(\sum_{J \subseteq min(P)} X^{|J|} \right)_{|X=-1} \\
&= L(\underbrace{(1+X)^{|min(P)|}}_{\in \langle X+1 \rangle})_{|X=-1} \\
&= 0.
\end{aligned}$$

For the fourth equality, note that P is not discrete, so $min(P) \neq P$. $\square$

Corollary 34 *The character α is invertible in $M_{\mathbf{qp}}$. We denote its inverse by β. For any quasi-poset P:*

$$\beta_P = (-1)^{cl(P)+cc(P)} \frac{\mu_P}{cl(P)!}.$$

Proof. As $\alpha_{\bullet_n} = 1$ for all n, α is invertible by proposition 11. We can restrict ourselves to posets. A connected poset Q is discrete if, and only if, $Q = \bullet$. Let P be a connected poset. If $P = \bullet$, then:

$$\alpha * \beta(\bullet) = \alpha_\bullet * \beta_\bullet = 1.$$

If not, then:

$$\begin{aligned}
0 = -ehr_P(-1) &= \sum_{\sim \lhd P} \frac{\mu_{P/\sim}}{cl(\sim)!} \alpha_{P|\sim} (-1)^{cl(\sim)+1} \\
&= \sum_{\sim \lhd P} \frac{\mu_{P/\sim}}{cl(P/\sim)!} (-1)^{cl(P/\sim)+1} \alpha_{P|\sim} \\
&= \sum_{\sim \lhd P} \beta_{P/\sim} \alpha_{P/\sim} \\
&= \beta * \alpha(P).
\end{aligned}$$

So $\beta * \alpha(P) = 0 = \varepsilon_B(P)$. Hence, β is the inverse in $M_{\mathbf{qp}}$ of α. $\square$

4.3 Duality principle

Proposition 35 *Let $\nu \in \mathbb{K}$, non-zero. There exists a unique Hopf algebra morphism $\phi_\nu : \mathcal{H}_{\mathbf{qp}} \longrightarrow \mathbb{K}[X]$ such that for any quasi-poset P:*

$$\phi_\nu(P)(-\nu) = \varepsilon_B(P).$$

This morphism is given by:

$$\phi_\nu(P) = (-1)^{cl(P)} ehr_P \left(\frac{X}{\nu} \right).$$

Proof. *Unicity.* Let ϕ be such a morphism. There exists a character $\gamma \in M_{\mathbf{qp}}$ such that $\phi = \phi_0 \leftarrow \gamma$. for any quasi-poset P:

$$\phi(P)(-\nu) = \sum_{\sim \lhd P} \lambda_{P/\sim} (-\nu)^{cl(P/\sim)} \gamma_{P|\sim} = \varepsilon_B(P).$$

Let us consider the map $\lambda^{(\nu)} : \mathcal{H}_{\mathbf{qp}} \longrightarrow \mathbb{K}$, which associates to any finite quasi-poset P the scalar $\lambda_P(-\nu)^{cl(P)}$. This is obviously a character of $\mathcal{H}_{\mathbf{qp}}$. As $\lambda^{(\nu)}_{\bullet_n} = (-\nu) \neq 0$ for all n, by lemma 11 $\lambda^{(\nu)}$ is invertible in $M_{\mathbf{qp}}$; moreover, $\lambda^{(\nu)} * \gamma = \varepsilon$, so γ is the (unique) inverse of $\lambda^{(\nu)}$ in $M_{\mathbf{qp}}$.

Existence. For all non-zero scalar η, let us consider the following Hopf algebra isomorphisms:

$$\theta_\eta : \left\{ \begin{array}{rcl} \mathbb{K}[X] & \longrightarrow & \mathbb{K}[X] \\ P(X) & \longrightarrow & P(\eta X), \end{array} \right. \qquad \theta'_\eta : \left\{ \begin{array}{rcl} \mathcal{H}_{\mathbf{qp}} & \longrightarrow & \mathcal{H}_{\mathbf{qp}} \\ P & \longrightarrow & \eta^{cl(P)} P. \end{array} \right.$$

Let $\phi = \theta_{\nu^{-1}} \circ ehr \circ \theta'_{-1}$. By composition, ϕ is a Hopf algebra morphism and for any quasi-poset P:

$$\phi(P) = (-1)^{cl(P)} ehr_P \left(\frac{X}{\nu} \right).$$

Hence, if P is a quasi-poset:

$$\phi(P)(-\lambda) = (-1)^{cl(P)} ehr_P(-1) = \begin{cases} (-1)^{cl(P)}(-1)^{cl(P)} = 1 = \varepsilon_B(P) \text{ if } \overline{P} \text{ is discrete,} \\ 0 = \varepsilon_B(P) \text{ otherwise.} \end{cases}$$

So such a ϕ exists. $\Box$

Remark. Such a morphism does not exist if $\nu = 0$. Indeed, for any non-empty poset P, if $\phi : \mathcal{H}_{\mathbf{qp}} \longrightarrow \mathbb{K}[X]$ is a Hopf algebra morphism, $\phi(\centerdot)(0) = \varepsilon_{\mathbb{K}[X]} \circ \phi(\centerdot) = \varepsilon_A(\centerdot) = 0 \neq \varepsilon_B(\centerdot)$.

Corollary 36 *1. (Duality principle). For any quasi-poset P:*

$$ehr_P^{str}(X) = (-1)^{cl(P)} ehr_P(-X).$$

2. For any quasi-poset P, $\alpha_P^{str} = (-1)^{cl(P)+1}\alpha_P$.

3. α^{str} is invertible in $M_{\mathbf{qp}}$. We denote by β^{str} its inverse. For any quasi-poset P:

$$\beta_P^{str} = \frac{\mu_P}{cl(P)!}.$$

Proof. We can restrict ourselves to posets.

1. It is enough to prove that $ehr^{str} = \phi_{-1}$, that is to say $ehr_P^{str}(1) = 0$ if P is not discrete and 1 otherwise. Let $P \in \mathbf{P}(n)$. There exists a unique map f from $[n]$ to $[1]$. If P is not discrete, $f \notin L_P^{str}(1)$, so $ehr_P^{str}(1) = 0$. If P is discrete, $f \in L_P^{str}(1)$, so $ehr_P^{str}(1) = 1$.

2. and 3. Immediate consequences of the first point. $\Box$

Proposition 37 *The following map is a Hopf algebra automorphism:*

$$\theta : \left\{ \begin{array}{rcl} (\mathcal{H}_{\mathbf{qp}}, m, \Delta) & \longrightarrow & (\mathcal{H}_{\mathbf{qp}}, m, \Delta) \\ P & \longrightarrow & \displaystyle\sum_{\sim \lhd P} P/\sim . \end{array} \right.$$

Its inverse is:

$$\theta^{-1} : \left\{ \begin{array}{rcl} (\mathcal{H}_{\mathbf{qp}}, m, \Delta) & \longrightarrow & (\mathcal{H}_{\mathbf{qp}}, m, \Delta) \\ P & \longrightarrow & (-1)^{cl(P)} \displaystyle\sum_{\sim \lhd P} (-1)^{cl(\sim)} P/\sim . \end{array} \right.$$

Moreover:

$$ehr^{str} \circ \theta = ehr, \qquad\qquad ehr \circ \theta^{-1} = ehr^{str}.$$

Proof. Let ι be the character of $\mathcal{H}_{\mathbf{qp}}$ which sends any quasi-poset to 1. Then $\theta = Id \leftarrow \iota$; moreover, ι is invertible in $M_{\mathbf{qp}}$ by lemma 11, so θ is a Hopf algebra automorphism. For any quasi-poset P:

$$\iota(1) = 1 = ehr_P(1) = \sum_{\sim \lhd P} \frac{\mu_{P/\sim}}{cl(P/\sim)!} \alpha_{P|\sim} = \sum_{\sim \lhd P} \beta_{P/\sim}^{str} \alpha_{P|\sim} = \beta^{str} * \alpha(P),$$

so $\iota = \beta^{str} * \alpha$; hence, its inverse is $\beta * \alpha^{str}$, and for any quasi-poset P:

$$\begin{aligned} \beta * \alpha^{str}(P) &= \sum_{\sim \lhd P} (-1)^{cl(\sim)+cc(P)} \frac{\mu_{P/\sim}}{cl(P/\sim)!} \alpha_{P|\sim}^{str} \\ &= (-1)^{cc(P)} ehr_P^{str}(-1) \\ &= (-1)^{cc(P)+cl(P)} ehr^{str}(1) \\ &= (-1)^{cc(P)+cl(P)}. \end{aligned}$$

Hence:

$$\theta^{-1}(P) = Id \leftarrow (\beta * \alpha^{str})(P) = \sum_{\sim \lhd P} (-1)^{cc(P|\sim)+cl(P|\sim)} P/\sim = \sum_{\sim \lhd P} (-1)^{cl(\sim)+cl(P)} P/\sim .$$

Moreover:

$$\begin{aligned}
ehr^{str} \circ \theta &= (\phi_0 \leftarrow \alpha^{str}) \circ (Id \leftarrow \iota) \\
&= ((\phi_0 \leftarrow \alpha^{str}) \circ Id) \leftarrow \iota \\
&= (\phi_0 \leftarrow \alpha^{str}) \leftarrow \iota \\
&= \phi_0 \leftarrow (\alpha^{str} * \iota) \\
&= \phi_0 \leftarrow (\alpha^{str} * \beta^{str} * \alpha) \\
&= \phi_0 \leftarrow \alpha \\
&= ehr.
\end{aligned}$$

$\square$

Examples.

$$\begin{aligned}
\theta(\bullet) &= \bullet, & \theta^{-1}(\bullet) &= \bullet, \\
\theta(\mathtt{I}) &= \mathtt{I} + \bullet_2, & \theta^{-1}(\mathtt{I}) &= \mathtt{I} - \bullet_2, \\
\theta(\mathtt{V}) &= \mathtt{V} + 2\mathtt{I}_2 + \bullet_3, & \theta^{-1}(\mathtt{V}) &= \mathtt{V} - 2\mathtt{I}_2 + \bullet_3, \\
\theta(\Lambda) &= \Lambda + 2\mathtt{I}^2 + \bullet_3, & \theta^{-1}(\mathtt{V}) &= \Lambda - 2\mathtt{I}^2 + \bullet_3, \\
\theta(\vdots) &= \mathtt{V} + \mathtt{I}_2 + \mathtt{I}^2 + \bullet_3, & \theta^{-1}(\mathtt{V}) &= \mathtt{V} - \mathtt{I}_2 - \mathtt{I}^2 + \bullet_3.
\end{aligned}$$

4.4 A link with Bernoulli numbers

For any $k \in \mathbb{N}$, let c_k be the corolla quasi-poset with k leaves: $c_k = ([k+1], \leq_{c_k})$, with $1 \leq_{c_k} 2, \ldots, k+1$:

$$c_0 = \bullet_1, \qquad c_1 = \mathtt{I}_1^2, \qquad c_2 = {}^2\mathtt{V}_1^3, \qquad c_3 = {}^2\mathtt{\Psi}_1^{3\,4} \ \ldots$$

B proposition 31, $\lambda_{c_k} = \frac{1}{k+1}$. Moreover:

$$\begin{aligned}
L_{c_k} &= \{f : [k+1] \longrightarrow \mathbb{N}^* \mid f(1) \leq f(2), \ldots, f(k+1)\}, \\
L_{c_k}^{str} &= \{f : [k+1] \longrightarrow \mathbb{N}^* \mid f(1) < f(2), \ldots, f(k+1)\},
\end{aligned}$$

so, for all $n \geq 1$:

$$Ehr_{c_k}^{str}(n) = (n-1)^k + \ldots + 1^k = S_k(n),$$

where $S_k(X)$ is the unique polynomial such that for all $n \geq 1$, $S_k(n) = 1^k + \ldots + (n-1)^k$. As a consequence, $\alpha_{c_k}^{str}$ is equal to the k-th Bernoulli number b_k.

Let $\sim \triangleleft c_k$. As the equivalence classes of $\sim$ are connected:

- The equivalence class of the minimal element of c_k contains i leaves, $0 \leq i \leq k$.
- The other equivalence classes are formed by a unique leaf.

Hence:

$$\delta(\lfloor c_k \rfloor) = \sum_{i=0}^{k} \binom{k}{i} \lfloor c'_{i,k-i} \rfloor \otimes \lfloor c_i \rfloor \bullet^{k-i},$$

where $c'_{i,k-i}$ is the quasi poset on $[k+1]$ such that:

$$1 \sim_{c'_{i,k-i}} \ldots \sim_{c'_{i,k-i}} i+1 \leq_{c'_{i,k-i}} i+2, \ldots k+1.$$

Hence, by theorem 32:

$$\begin{aligned}
S_k(X) &= \sum_{i=0}^{k} \binom{k}{i} \lambda_{c'_{i,k-i}} b_i X^{k-i+1} \\
&= \sum_{i=0}^{k} \binom{k}{i} \lambda_{\overline{c'_{i,k-i}}} b_i X^{k-i+1} \\
&= \sum_{i=0}^{k} \binom{k}{i} \lambda_{c_{k-i}} b_i X^{k-i+1} \\
&= \sum_{i=0}^{k} \binom{k}{i} \frac{b_i}{k-i+1} X^{k-i+1}.
\end{aligned}$$

We recover in this way Faulhaber's formula. For all $n \geq 1$, $ehr_{c_k}(n) = n^k + \ldots + 1^k$, and the duality principle gives, for all $n \geq 1$:

$$(-1)^{k+1} S_k(-n) = 1^k + \ldots + n^k = S_k(n) + n^k.$$

5 Noncommutative version

5.1 Reminders on packed words

Let us recall the construction of the Hopf algebra of packed words **WQSym** [14, 15].

Definition 38 *Let $w = x_1 \ldots x_n$ be a word which letters are positive integers.*

1. *We shall say that w is a packed word if there exists an integer k such that $\{x_1, \ldots, x_n\} = [k]$. The set of packed words of length n is denoted by $\mathbf{PW}(n)$; the set of all packed words is denoted by $\mathbf{PW}$.*

2. *There exists a unique increasing bijection $f : \{x_1, \ldots, x_n\} \longrightarrow [k]$ for a well-chosen k. We denote by $Pack(w)$ the packed word $f(x_1) \ldots f(x_k)$. Note that w is packed if, and only if, $w = Pack(w)$.*

3. *Let $I \subseteq \mathbb{N}$. Let $i_1 < \ldots < i_p$ be the indices i such that $x_i \in I$. We denote by w_I the word $x_{i_1} \ldots x_{i_p}$.*

As a vector space, **WQSym** is generated by the set **PW**. The product is given by:

$$\forall u \in \mathbf{PW}(k),\ \forall v \in \mathbf{PW}(l),\ u.v = \sum_{\substack{w = x_1 \ldots x_{k+l} \in \mathbf{PW}(k+l),\\ Pack(x_1 \ldots x_k) = u,\\ Pack(x_{k+1} \ldots x_{k+l}) = v}} w.$$

The unit is the empty word 1. The coproduct is given by:

$$\forall w \in \mathbf{PW},\ \Delta(w) = \sum_{k=0}^{\max(w)} w_{\{1,\ldots,k\}} \otimes Pack(w_{\{k+1,\ldots,\max(w)\}}).$$

For example:

$$\begin{aligned}
(11).(11) &= (1111) + (1122) + (2211),\\
(11).(12) &= (1112) + (1123) + (2212) + (2213) + (3312),\\
(11).(21) &= (1121) + (1132) + (2231) + (3321),\\
(12).(11) &= (1211) + (1222) + (1233) + (1322) + (2311),\\
(12).(12) &= (1212) + (1213) + (1223) + (1234) + (1323) + (1324)\\
&\quad + (1423) + (2312) + (2313) + (2314) + (2413) + (3412),\\
(12).(21) &= (1221) + (1231) + (1232) + (1243) + (1332) + (1342)\\
&\quad + (1432) + (2321) + (2331) + (2341) + (2431) + (3421),
\end{aligned}$$

$$\begin{aligned}
\Delta(111) &= (111) \otimes 1 + (111) \otimes 1,\\
\Delta(212) &= (212) \otimes 1 + (1) \otimes (11) + 1 \otimes (212),\\
\Delta(312) &= (312) \otimes 1 + (1) \otimes (21) + (12) \otimes (1) + 1 \otimes (312).
\end{aligned}$$

5.2 Hopf algebra morphisms in WQSym

Proposition 39 *The two following maps are surjective Hopf algebra morphisms:*

$$EHR : \left\{ \begin{array}{rcl} (\mathcal{H}_{\mathbf{QP}}, m, \Delta) & \longrightarrow & \mathbf{WQSym} \\ P & \longrightarrow & \displaystyle\sum_{w \in W_P} w, \end{array} \right.$$

$$EHR^{str} : \left\{ \begin{array}{rcl} (\mathcal{H}_{\mathbf{QP}}, m, \Delta) & \longrightarrow & \mathbf{WQSym} \\ P & \longrightarrow & \displaystyle\sum_{w \in W_P^{str}} w. \end{array} \right.$$

Proof. Let $P \in \mathbf{QP}(k)$, $Q \in \mathbf{QP}(l)$, and w be a packed word of length $k + l$. Then:

- $w \in W_{PQ}$ if, and only if, $Pack(w_1 \ldots w_k) \in W_P$ and $Pack(w_{k+1} \ldots w_{k+l}) \in W_Q$.
- $w \in W_{PQ}^{str}$ if, and only if, $Pack(w_1 \ldots w_k) \in W_P^{str}$ and $Pack(w_{k+1} \ldots w_{k+l}) \in W_Q^{str}$.

This implies that :

$$EHR(PQ) = EHR(P)EHR(Q), \qquad EHR^{str}(PQ) = EHR^{str}(P)EHR^{str}(Q).$$

Let $P \in \mathbf{QP}(n)$. We consider the two sets:

$$\begin{aligned}
A &= \{(w, k) \mid w \in W_P, 0 \leq k \leq \max(w)\},\\
B &= \{(O, w_1, w_2) \mid O \in Top(P), w_1 \in W_{Pack(P_{|[n]\setminus O})}, w_2 \in W_{Pack(P_{|O})}\}.
\end{aligned}$$

We define a bijection between A and B by $F(w, k) = (O, w_1, w_2)$, where:

- $O = w^{-1}(\{k + 1, \ldots, \max(w)\})$.
- $w_1 = Pack(w_{\{1,\ldots,k\}})$.
- $w_2 = Pack(w_{\{k+1,\ldots,\max(w)\}})$.

Then:

$$\begin{aligned}
\Delta \circ EHR(P) &= \sum_{(w,k) \in A} Pack(w_{\{1,\ldots,k\}}) \otimes Pack(w_{\{k+1,\ldots,\max(w)\}})\\
&= \sum_{(O,w_1,w_2) \in B} w_1 \otimes w_2\\
&= (EHR \otimes EHR) \circ \Delta(P).
\end{aligned}$$

So EHR is a Hopf algebra morphism. In the same way, EHR^{str} is a Hopf algebra morphism.

Let w be a packed word of length n. We define a quasi-poset structure on $[n]$ by $i \leq_P j$ if, and only if, $w_i \leq w_j$. Then $W_P^{str} = \{w\}$, so $EHR^{str}(P) = w$: EHR^{str} is surjective. If $w' \in W_P$, then $\max(w') \leq \max(w)$ with equality if, and only if, $w = w'$. Hence:

$$EHR(P) = w + \text{words } w' \text{ with } \max(w') < \max(w).$$

By a triangular argument, EHR is surjective. $\square$

Examples.

$$\begin{array}{rlrl}
EHR(\bullet_1) &= (1), & EHR^{str}(\bullet_1) &= (1),\\
EHR(\updownarrow^{2}_{1}) &= (12) + (11), & EHR^{str}(\updownarrow^{2}_{1}) &= (12),\\
EHR(\updownarrow^{1}_{2}) &= (21) + (11), & EHR^{str}(\updownarrow^{1}_{2}) &= (21),\\
EHR(\bullet_1 \bullet_2) &= (12) + (21) + (11), & EHR^{str}(\bullet_1 \bullet_2) &= (12) + (21) + (11),\\
EHR(\bullet_{1,2}) &= (11), & EHR^{str}(\bullet_{1,2}) &= (11).
\end{array}$$

Proposition 40 *The following map is a Hopf algebra automorphism:*

$$\Theta : \left\{ \begin{array}{rcl} (\mathcal{H}_{\mathbf{QP}}, m, \Delta) & \longrightarrow & (\mathcal{H}_{\mathbf{QP}}, m, \Delta) \\ P & \longrightarrow & \displaystyle\sum_{\sim \triangleleft P} P/\sim. \end{array} \right.$$

Its inverse is:

$$\Theta^{-1} : \left\{ \begin{array}{rcl} (\mathcal{H}_{\mathbf{QP}}, m, \Delta) & \longrightarrow & (\mathcal{H}_{\mathbf{QP}}, m, \Delta) \\ P & \longrightarrow & \displaystyle\sum_{\sim \triangleleft P} (-1)^{cl(P)+cl(\sim)} P/\sim. \end{array} \right.$$

Moreover, $EHR^{str} \circ \Theta = EHR$ *and* $EHR \circ \Theta^{-1} = EHR^{str}$*.*

Proof. The monoid $M_{\mathbf{qp}}$ acts on the set $E_{\mathcal{H}_{\mathbf{QP}} \longrightarrow \mathcal{H}_{\mathbf{QP}}}$, and $\Theta = Id \leftarrow \iota$, where ι is the character defined in the proof of proposition 37. So Θ is an automorphism, and its inverse is $Id \leftarrow \iota^{*-1}$.

Let us prove that:

$$W_G = \sum_{\sim \triangleleft P} W_{G/\sim}^{str}.$$

Let $w \in W_G$; we define an equivalence $\sim_w$ by $x \sim_w y$ if $w(x) = w(y)$ and x and y are in the same connected component of $w^{-1}(w(x))$. By definition, the equivalence classes of $\sim_w$ are connected. If $x \sim_{P/\sim_w} y$, there exists $x_1, x'_1 \ldots, x_k, x'_k, y_1, y'_1 \ldots, y_l, y'_l$ such that:

$$\begin{array}{c} x \leq_P x_1 \sim_w x'_1 \leq_P \ldots \leq_P x_k \sim_w x'_k \leq_P y, \\ y \leq_P y_1 \sim_w y'_1 \leq_P \ldots \leq_P y_l \sim_w y'_l \leq_P x. \end{array}$$

As $w \in W_P$, $w(x) \leq w(x_1) = w(x'_1) \leq \ldots \leq w(x_k) = w(x'_k) \leq w(y)$; by symmetry, $w(x) = w(x_1) = \ldots = w(x'_k) = w(y) = i$. Moreover, as the equivalence classes of $\sim_w$ are connected, x and y are in the same connected component of $w^{-1}(i)$, so $x \sim_w y$: $\sim_w \triangleleft P$.

If $x \leq_P y$ or $x \sim_w y$, then $w(x) \leq w(q)$. By transitive closure, if $x \leq_{P/\sim_w} y$, then $w(x) \leq w(y)$, so $w \in W_{P/\sim_w}$. Moreover, if $w(i) \neq w(j)$, we do not have $x \sim_w y$, so $w \in W_{P/\sim_w}^{str}$.

Let us assume that $\sim \triangleleft P$ and let $w \in W_{P/\sim}^{str}$. If $x \leq_P y$, then $x \leq_{P/\sim} y$, so $w(x) \leq w(y)$: $W_{P/\sim}^{str} \subseteq W_P$.

Let us assume that $w \in W_{P/\sim}^{str}$, with $\sim \triangleleft P$. If $x \sim y$, then $w(x) = w(y) = i$ and x and y are in the same connected component of $P| \sim$, so are in the the same connected component of $w^{-1}(i)$: $x \sim_w y$. If $x \sim_w y$, then $w(x) = w(y) = i$ and there exists $x_1, x'_1 \ldots, x_k, x'_k$ with $w(x_1) = w(x'_1) = \ldots = w(x_k) = w(x'_k) = i$ such that:

$$x \leq_P x_1 \geq_P x'_1 \leq_P \ldots \geq_P x'_k \leq_P y.$$

As $w \in W^{str}_{P/\sim}$, $x \sim_{P/\sim} x_1$, $x_1 \sim_{P/\sim} x'_1, \ldots, x'_k \sim_{P/\sim} y$. So $x \sim_{P/\sim} y$; as $\sim \lhd P$, $x \sim y$. Finally, $\sim = \sim_w$.

We obtain that:

$$EHR(P) = \sum_{w \in W_P} w = \sum_{\sim \lhd P} \sum_{w \in W^{str}_{P/\sim}} w = \sum_{\sim \lhd P} EHR^{str}(P/\sim) = EHR^{str}(\Theta(P)).$$

So $Ehr^{str} \circ \Theta = EHR$. $\square$

Examples. If $\{i, j\} = \{1, 2\}$ and $\{a, b, c\} = \{1, 2, 3\}$:

$$\begin{aligned}
\Theta(\bullet_1) &= \bullet_1, & \Theta^{-1}(\bullet_1) &= \bullet_1,\\
\Theta(\mathbf{I}^j_i) &= \mathbf{I}^j_i + \bullet_{i,j}, & \Theta^{-1}(\mathbf{I}^j_i) &= \mathbf{I}^j_i - \bullet_{i,j},\\
\Theta({}^b\mathbf{V}^c_a) &= {}^b\mathbf{V}^c_a + \mathbf{I}^c_{a,b} + \mathbf{I}^b_{a,c} + \bullet_{a,b,c}, & \Theta^{-1}({}^b\mathbf{V}^c_a) &= {}^b\mathbf{V}^c_a - \mathbf{I}^c_{a,b} - \mathbf{I}^b_{a,c} + \bullet_{a,b,c},\\
\Theta({}_b\mathbf{\Lambda}^a_c) &= {}_b\mathbf{\Lambda}^a_c + \mathbf{I}^{a,c}_b + \mathbf{I}^{a,b}_c + \bullet_{a,b,c}, & \Theta^{-1}({}_b\mathbf{\Lambda}^a_c) &= {}_b\mathbf{\Lambda}^a_c - \mathbf{I}^{a,c}_b - \mathbf{I}^{a,b}_c + \bullet_{a,b,c},\\
\Theta(\mathbf{I}^{c}_{b\,a}) &= \mathbf{I}^{c}_{b\,a} + \mathbf{I}^{b,c}_a + \mathbf{I}^c_{a,b} + \bullet_{a,b,c}, & \Theta^{-1}(\mathbf{I}^{c}_{b\,a}) &= \mathbf{I}^{c}_{b\,a} - \mathbf{I}^{b,c}_a - \mathbf{I}^c_{a,b} + \bullet_{a,b,c}.
\end{aligned}$$

Proposition 41 *Let us consider the following map:*

$$H : \left\{ \begin{array}{rcl} \mathbf{WQSym} & \longrightarrow & \mathbb{K}[X] \\ w \in \mathbf{PW} & \longrightarrow & H_{\max(w)}(X). \end{array} \right.$$

This is a surjective Hopf algebra morphism, making the following diagram commuting:

$$\begin{tikzcd}
 & \mathcal{H}_{\mathbf{QP}} \arrow[dl, "\lfloor\rfloor"'] \arrow[d, "\Theta"] \arrow[dr, "EHR"] & \\
\mathcal{H}_{\mathbf{qp}} \arrow[d, "\theta"'] \arrow[dr, "ehr"] & \mathcal{H}_{\mathbf{QP}} \arrow[r, two heads, "EHR^{str}"'] \arrow[dl, "\lfloor\rfloor"'] & \mathbf{WQSym} \arrow[dl, "H"] \\
\mathcal{H}_{\mathbf{qp}} \arrow[r, two heads, "ehr^{str}"'] & \mathbb{K}[X] &
\end{tikzcd}$$

Proof. Let $P \in \mathbf{QP}$. Then:

$$ehr(\lfloor P \rfloor) = \sharp W_P(k) H_k(X) = \sum_{w \in W_P} H_{\max(w)}(X) = \sum_{w \in W_P} H(w) = H \circ EHR(P).$$

So $ehr \circ \lfloor\rfloor = H \circ EHR$. Similarly, $ehr^{str} \circ \lfloor\rfloor = H \circ EHR^{str}$.

Let us prove that H is a Hopf algebra morphism. Let $w_1, w_2 \in \mathbf{WQSym}$. There exists $x_1, x_2 \in \mathcal{H}_{\mathbf{QP}}$, such that $w_1 = EHR(x_1)$ and $w_2 = EHR(x_2)$. Then:

$$\begin{aligned}
H(w_1 w_2) &= H(EHR(x_1)EHR(x_2))\\
&= H \circ EHR(x_1 x_2)\\
&= ehr(\lfloor x_1 x_2 \rfloor)\\
&= ehr(\lfloor x_1 \rfloor) ehr(\lfloor x_2 \rfloor)\\
&= H \circ EHR(x_1) H \circ EHR(x_2)\\
&= H(w_1) H(w_2).
\end{aligned}$$

Let $w \in \mathbf{WQSym}$. There exists $x \in \mathcal{H}_{\mathbf{QP}}$ such that $w = EHR(x)$.

$$\begin{aligned}\Delta \circ H(w) &= \Delta \circ H \circ EHR(x)\\ &= (H \otimes H) \circ (EHR \otimes EHR) \circ \Delta(x)\\ &= (H \otimes H) \circ \Delta \circ EHR(x)\\ &= (H \otimes H) \circ \Delta(w).\end{aligned}$$

So H is a Hopf algebra morphism. □

5.3 Compatibility with the other product and coproduct

Theorem 42 *We define a second coproduct δ on* $\mathbf{WQSym}$*:*

$$\forall w \in \mathbf{PW},\ \delta(w) = \sum_{(\sigma,\tau)\in A_w} (\sigma \circ w) \otimes (\tau \circ w),$$

where A_w is the set of pairs of packed words (σ, τ) of length $\max(w)$ such that:

- *σ is non-decreasing.*
- *If $1 \leq i < j \leq \max(w)$ and $\sigma(i) = \sigma(j)$, then $\tau(i) < \tau(j)$.*

Then $(\mathbf{WQSym}, m, \delta)$ is a bialgebra and EHR^{str} is a bialgebra morphism from $(\mathcal{H}_{\mathbf{QP}}, m, \delta)$ to $(\mathbf{WQSym}, m, \delta)$.

Proof. Let us prove that $\delta \circ EHR^{str} = (EHR^{str} \otimes EHR^{str}) \circ \delta$. Let $P \in \mathbf{QP}$. We consider the two following sets:

$$\begin{aligned}A &= \{(\sim, w_1, w_2) \mid \sim \lhd P, w_1 \in W^{str}_{P/\sim}, w_2 \in W^{str}_{P|\sim}\},\\ B &= \{(w, \sigma, \tau) \mid w \in W^{str}_P, (\sigma, \tau) \in A_w\}.\end{aligned}$$

Let $(\sim, w_1, w_2) \in A$. We put $I_p = w_1^{-1}(p)$ for all $1 \leq p \leq \max(w_1)$, and $w_2^{(p)}$ the standardization of the restriction of w_2 to I_p. We define w by:

$$w(i) = w_2^{(p)}(i) + \max w_1^{(2)} + \ldots + \max w_{p-1}^{(2)} \text{ if } i \in I_p.$$

Let us prove that $w \in W^{str}_P$. If $x \leq_P y$, then $x \leq_{P/\sim} y$, so $p = w_1(x) \leq w_2(y) = q$.

- If $p < q$, then $w(x) < w(y)$.
- If $p = q$, then $w_1(x) = w_2(y)$ and, as $x \leq_P y$, x and y are in the same connected component of $w^{-1}(p)$. So $x \sim_{w_1} y$, that is to say $x \sim y$ as $w_1 \in W^{str}_{P/\sim}$, and $x \leq_{P|\sim} y$, which implies that $w_2(x) \leq w_2(y)$ and finally $w(x) \leq w(y)$.

Let us assume that moreover $w(x) = w(y)$. Then $p = q$ and necessarily, $w_2(x) = w_2(y)$. As $w_2 \in W^{str}_{P|\sim}$, $x \sim_{P|\sim} y$, so $x \sim_P y$.

If $w(x) = w(y)$, then by definition of w, $w_1(x) = w_1(y)$. So there exists a unique $\sigma : [\max(w)] \longrightarrow [\max(w_1)]$, such that $w_1 = \sigma \circ w$. If $w(x) < w(y)$, then, by construction of w, $w_1(x) \leq w_1(y)$: σ is non-decreasing.

There exists a unique $\tau : [\max(w)] \longrightarrow [\max(w_2)]$, such that $w_2 = \tau \circ \sigma$. As $Pack(w_{|I_p}) = Pack((w_2)_{|I_p})$ for all p, τ is increasing on I_p.

To any $(\sim, w_1, w_2) \in A$, we associate $(w, \sigma, \tau) = F(\sim, w_1, w_2) \in B$, such that $w_1 = \sigma \circ \tau$, $w_2 = \tau \circ \sigma$, and $\sim = \sim_{\sigma\circ\tau}$. This defines a map $F : A \longrightarrow B$.

Let $(w,\sigma,\tau)\in B$. We put $G(w,\sigma,\tau)=(\sim,\sigma,\tau)=(\sim_{\sigma\circ w},\sigma\circ w,\tau\circ w)$. If $x\leq_P y$, then $w(x)\leq w(y)$, so $w_1(x)=\sigma\circ w(x)\leq \sigma\circ w(y)=w_1(y)$. If moreover $w_1(x)=w_1(y)$, then as $x\leq_P y$, x and y are in the same connected component of $w_1^{-1}(w_1(x))$, so $x\sim_{w_1} y$: $w_1\in W^{str}_{P/\sim}$.

If $x\leq_{P|\sim} y$, then $x\sim_{w_1} y$ and $x\leq_P y$, so $w_1(x)=w_1(y)$ and $w_(x)\leq w(y)$. By hypothesis on τ, $\tau\circ w(x)\leq \tau\circ w(y)$, so $w_2(x)\leq w_2(y)$. If moreover $w_2(x)=w_2(y)$, by hypothesis on τ, $w(x)=w(y)$. As $w\in W_P^{str}$, $x\sim_P y$, so $x\sim_{P|\sim} y$: $w_2\in W^{str}_{P|\sim}$.

We defined in this way a map $G:B\longrightarrow A$. If $(\sim,w_1,w_2)\in A$:

$$G\circ F(\sim,w_1,w_2)=G(w,\sigma,\tau)=(\sim_{\sigma\circ w},\sigma\circ w,\tau\circ w)=(\sim_{w_1},w_1,w_2)=(\sim,w_1,w_2).$$

So $G\circ F=Id_A$. If $(w,\sigma,\tau)\in B$:

$$F\circ G(w,\sigma,\tau)=F(\sim_{\sigma\circ w},\sigma\circ w,\tau\circ w)=(w,\sigma,\tau).$$

So $G\circ F=Id_B$: F and G are inverse bijections.

We obtain:

$$\begin{aligned}(EHR^{str}\otimes EHR^{str})\circ\delta(P)&=\sum_{(\sim,w_1,w_2)\in A} w_1\otimes w_2\\&=\sum_{(w,\sigma,\tau)\in B}\sigma\circ w\otimes\tau\circ w\\&=\sum_{w\in W_P^{str}}\delta(w)\\&=\delta\circ EHR^{str}(P).\end{aligned}$$

So EHR^{str} is compatible with δ.

As EHR^{str} is compatible with the product m and the coproduct δ, $Ker(EHR^{str})$ is a biideal of $(\mathcal{H}_{\mathbf{QP}},m,\delta)$, and $(\mathbf{WQSym},m,\delta)$ is identified with the quotient $\mathcal{H}_{\mathbf{QP}}/Ker(EHR^{str})$, so is a bialgebra. $\Box$

Examples.

$$\begin{aligned}&\delta(11)=(11)\otimes(11),\\&\delta(12)=(12)\otimes((11)+(12)+(21))+(11)\otimes(12),\\&\delta(21)=(21)\otimes((11)+(12)+(21))+(11)\otimes(21).\end{aligned}$$

This coproduct δ on **WQSym** is the internal coproduct of [15], dual to the product of the Solomon-Tits algebra.

Remarks.

1. The counit of $(\mathbf{WQSym},m,\delta)$ is given by:

$$\varepsilon_B(w)=\begin{cases}1 \text{ if } w=(1\ldots1),\\ 0 \text{ otherwise.}\end{cases}$$

2. There is no coproduct δ' on **WQSym** such that $(EHR\otimes EHR)\circ\delta=\delta'\circ EHR$. Indeed, if δ' is any coproduct on **WQSym**, for $x=\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}{}^{2}_{1}+\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}{}^{1}_{2}-\bullet_1\bullet_2-\bullet_{1,2}$:

$$\delta'\circ EHR(x)=\delta'(0)=0,$$

but:

$$\begin{aligned}
&(EHR\otimes EHR)\circ\delta(x)\\
&=(EHR\otimes EHR)((\mathbf{I}^{2}_{1}+\mathbf{I}^{1}_{2}-\bullet_1\bullet_2)\otimes\bullet_1\bullet_2+\bullet_{1,2}\otimes(\mathbf{I}^{2}_{1}+\mathbf{I}^{1}_{2}-\bullet_1\bullet_2-\bullet_{1,2}))\\
&=(11)\otimes(11).
\end{aligned}$$

Proposition 43 *$H:(\mathbf{WQSym},m,\delta)\longrightarrow(\mathbb{K}[X],m,\delta)$ is a bialgebra morphism.*

Proof. Let w be a packed word. We denote $k=\max(w)$. Let $a,b\in\mathbb{N}$.

$$\begin{aligned}
(H\otimes H)\circ\delta(w)(a,b)&=\sum_{(\sigma,\tau)\in A_w}H_{\max(\sigma\circ w)}(a)H_{\max(\tau\circ w)}(b)\\
&=\sum_{\sigma\,:\,[k]\twoheadrightarrow[l],\text{ non-decreasing}}\binom{a}{l}\binom{b}{|\sigma^{-1}(1)|}\cdots\binom{b}{|\sigma^{-1}(l)|}\\
&=\sum_{\substack{1\leq l\leq k,\\ i_1+\ldots+i_l=k}}\binom{a}{l}\binom{b}{i_1}\cdots\binom{b}{i_l}\\
&=\binom{ab}{k}\\
&=H_k(ab)\\
&=\delta(H(w))(a,b).
\end{aligned}$$

As this is true for any $a,b\in\mathbb{N}$, $(H\otimes H)\circ\delta(w)=\delta\circ H$. $\square$

Corollary 44 *ehr^{str} is the unique map from $\mathcal{H}_{\mathbf{qp}}$ to $\mathbb{K}[X]$ such that both $ehr^{str}:(\mathcal{H}_{\mathbf{qp}},m,\Delta)\longrightarrow(\mathbb{K}[X],m,\Delta)$ and $ehr^{str}:(\mathcal{H}_{\mathbf{qp}},m,\delta)\longrightarrow(\mathbb{K}[X],m,\delta)$ are bialgebra morphisms.*

Proof. We have a commutative diagram of surjective morphisms:

$$\begin{tikzcd}
\mathcal{H}_{\mathbf{QP}} \arrow[r, two heads, "EHR^{str}"] \arrow[d, "\Xi"'] & \mathbf{WQSym} \arrow[d, "H"'] \\
\mathcal{H}_{\mathbf{qp}} \arrow[r, two heads, "ehr^{str}"] & \mathbb{K}[X]
\end{tikzcd}$$

As the arrows EHR^{str}, Ξ and H are compatible with δ, necessarily the arrow ehr^{str} also is.

Let us consider an algebra morphism $\phi:\mathcal{H}_{\mathbf{qp}}\longrightarrow\mathbb{K}[X]$, compatible with both bialgebra structures. There exists $f\in M_{\mathbf{qp}}$, such that $\phi=\phi_0\leftarrow f$. Putting $g=\beta^{str}*f$, we obtain that $f=ehr^{str}\leftarrow g$. For any $x\in\mathcal{H}_{\mathbf{qp}}$, denoting by ε' the counit of $(\mathbb{K}[X],m,\delta)$ and using Sweedler's notation for δ:

$$\varepsilon_B(x)=\varepsilon'\circ\phi(x)=\varepsilon'(\phi(x^{(1)})g(x^{(2)}))=\varepsilon'\circ\phi(x^{(1)})g(x^{(2)})=\varepsilon_B(x^{(1)})g(x^{(2)})=g(x).$$

So $g=\varepsilon_B$, and $\phi=ehr^{str}\leftarrow\varepsilon_B=ehr^{str}$. $\square$

Definition 45 *Let $w=w_1\ldots w_k$ and $w'=w'_1\ldots w'_l$ be two packed words. We put:*

$$\begin{aligned}
w\downarrow w'&=w_1\ldots w_k(w'_1+\max(w))\ldots(w'_l+\max(w)),\\
w\circledast w'&=w_1\ldots w_k(w'_1+\max(w)-1)\ldots(w'_l+\max(w)-1),\\
w\lightning w'&=w\downarrow w'+w\circledast w'.
\end{aligned}$$

These three products are extended to $\mathbf{WQSym}$ by bilinearity.

Proposition 46 *For all $x, y \in \mathcal{H}_{\mathbf{QP}}$:*

$$EHR^{str}(x \downarrow y) = EHR^{str}(x) \downarrow EHR^{str}(y), \qquad EHR(x \downarrow y) = EHR(x) \not\downarrow EHR(y).$$

Proof. Let $P \in \mathbf{QP}(k)$ and $Q \in \mathbf{QP}(l)$. If $w = w_1 \ldots w_{k+l}$ is a packed word of length $k+l$:

$$\begin{aligned} w \in W^{str}_{P\downarrow Q} &\Longleftrightarrow w_1 \ldots w_k \in L^{str}_P, w_{k+1} \ldots w_{k+l} \in L^{str}_Q, w_1, \ldots, w_k < w_{k+1}, \ldots w_{k+l} \\ &\Longleftrightarrow w = w_P \downarrow w_Q, \text{ with } w_P \in W^{str}_P, w_Q \in W^{str}_P. \end{aligned}$$

So $W^{str}_{P\downarrow Q} = W^{str}_P \downarrow W^{str}_Q$, and:

$$EHR^{str}(P \downarrow Q) = \sum_{w_P \in W^{str}_P, w_Q \in W^{str}_Q} w_P \downarrow w_Q = EHR^{str}(P) \downarrow EHR^{str}(Q).$$

If $w = w_1 \ldots w_{k+l}$ is a packed word of length $k+l$:

$$\begin{aligned} w \in W_{P\downarrow Q} &\Longleftrightarrow w_1 \ldots w_k \in L_P, w_{k+1} \ldots w_{k+l} \in L_Q, w_1, \ldots, w_k \leq w_{k+1}, \ldots w_{k+l} \\ &\Longleftrightarrow w = (w_P \downarrow w_Q, \text{ with } w_P \in W_P, w_Q \in W_P) \\ &\quad \text{or } w = (w_P \circledast w_Q, \text{ with } w_P \in W_P, w_Q \in W_P). \end{aligned}$$

These two conditions are incompatible: in the first case,

$$\max(w_1 \ldots w_k) = \min(w_{k+1} \ldots w_{k+l}) - 1,$$

whereas in the second case,

$$\max(w_1 \ldots w_k) = \min(w_{k+1} \ldots w_{k+l}).$$

So $W_{P\downarrow Q} = (W_P \downarrow W_Q) \sqcup (W_P \circledast W_Q)$, and:

$$\begin{aligned} EHR(P \downarrow Q) &= \sum_{w_P \in W_P, w_Q \in W_Q} w_P \downarrow w_Q + w_P \circledast w_Q \\ &= EHR(P) \downarrow EHR(Q) + EHR(P) \circledast EHR(Q), \end{aligned}$$

so $EHR(P \downarrow Q) = EHR(P) \not\downarrow EHR(Q)$. $\Box$

Remark. As a consequence, $(\mathbf{WQSym}, \downarrow, \Delta)$ and $(\mathbf{WQSym}, \not\downarrow, \Delta)$ are infinitesimal bialgebras [11], As $(\mathcal{H}_{\mathbf{QP}}, \downarrow, \Delta)$ is [9, 10].

5.4 The non-commutative duality principle

Lemma 47 *The following map is an involution and a Hopf algebra automorphism:*

$$\Phi_{-1} : \left\{ \begin{array}{rcl} \mathbf{WQSym} & \longrightarrow & \mathbf{WQSym} \\ w & \longrightarrow & (-1)^{\max(w)} \displaystyle\sum_{\sigma : [\max(w)] \twoheadrightarrow [l], \text{ non-decreasing}} \sigma \circ w. \end{array} \right.$$

Proof. Using the surjective morphisms EHR^{str} and ehr^{str}, taking the quotients of the cointeracting bialgebras $(\mathcal{H}_{\mathbf{QP}}, m, \Delta)$ and $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$, we obtain that $(\mathbf{WQSym}, m, \Delta)$ and $(\mathbb{K}[X], m, \delta)$ are cointeracting bialgebras, with the coaction defined by:

$$\rho = (Id \otimes H) \circ \delta : \mathbf{WQSym} \longrightarrow \mathbf{WQSym} \otimes \mathbb{K}[X]$$

For any packed word w:

$$\rho(w) = \sum_{\sigma : [k] \twoheadrightarrow [l], \text{ non-decreasing}} \sigma \circ w \otimes H_{\max(Pack(w_{|(\sigma\circ w)^{-1}(1)}))}(X) \ldots H_{\max(Pack(w_{|(\sigma\circ w)^{-1}(l)}))}(X).$$

Using proposition 4, for any $\lambda \in \mathbb{K}$, considering the character:

$$ev_\lambda : \left\{ \begin{array}{rcl} \mathbb{K}[X] & \longrightarrow & \mathbb{K} \\ P & \longrightarrow & P(\lambda), \end{array} \right.$$

we obtain an endomorphism Φ_λ of $(\mathbf{WQSym}, m, \Delta)$ defined by $\Phi_\lambda = Id \leftarrow ev_\lambda$. if $\lambda \neq 0$, Φ_λ is invertible, of inverse $\Phi_{\lambda^{-1}}$. For any packed word w, denoting by k its maximum:

$$\Phi_\lambda(w) = \sum_{\sigma : [k] \twoheadrightarrow [l], \text{ non-decreasing}} H_{\max(Pack(w_{|(\sigma \circ w)^{-1}(1)}))}(\lambda) \ldots H_{\max(Pack(w_{|(\sigma \circ w)^{-1}(l)}))}(\lambda) \sigma \circ w.$$

In particular, for $\lambda = -1$, for any $p \in \mathbb{N}$:

$$H_p(-1) = \frac{(-1)(-2)\ldots(-k)}{k!} = (-1)^k.$$

Hence:

$$\begin{aligned} \Phi_{-1}(w) &= \sum_{\sigma : [k] \twoheadrightarrow [l], \text{ non-decreasing}} (-1)^{\max(Pack(w_{|(\sigma \circ w)^{-1}(1)}))+\ldots+\max(Pack(w_{|(\sigma \circ w)^{-1}(l)}))} \sigma \circ w \\ &= (-1)^k \sum_{\sigma : [k] \twoheadrightarrow [l], \text{ non-decreasing}} \sigma \circ w. \end{aligned}$$

Indeed, if $x \in (\sigma \circ w)^{-1}(p)$ and $y \in (\sigma \circ w)^{-1}(q)$, with $p < q$, then $\sigma \circ w(x) < \sigma \circ x(y)$; as σ is non-decreasing, $x < y$. So there exists $n_1 < n_2 < \ldots < n_l = k$ such that for all p, the values taken by w on $(\sigma \circ w)^{-1}(p)$ are $n_{p-1} + 1, \ldots, n_p$. Hence, the values taken by $Pack(w_{|(\sigma \circ w)^{-1}(p)})$ are $1, \ldots, n_p - n_{p-1}$, so:

$$\max(Pack(w_{|(\sigma \circ w)^{-1}(1)}))+\ldots+\max(Pack(w_{|(\sigma \circ w)^{-1}(l)})) = n_1+n_2-n_1+\ldots+n_l-n_{l-1} = n_l = k.$$

In particular, Φ_{-1} is an involution and a Hopf algebra automorphism of $(\mathbf{WQSym}, m, \Delta)$. $\square$

Theorem 48 *For any quasi-poset $P \in \mathbf{QP}$:*

$$EHR(P) = (-1)^{cl(P)} \Phi_{-1} \circ ERH^{str}(P), \qquad EHR^{str}(P) = (-1)^{cl(P)} \Phi_{-1} \circ ERH(P).$$

Proof. We shall use the following involution and Hopf algebra automorphism:

$$\Psi : \left\{ \begin{array}{rcl} \mathcal{H}_{\mathbf{QP}} & \longrightarrow & \mathcal{H}_{\mathbf{QP}} \\ p \in \mathbf{QP} & \longrightarrow & (-1)^{cl(P)} P. \end{array} \right.$$

Recall that the character ι of $\mathcal{H}_{\mathbf{QP}}$ sends any $P \in \mathbf{QP}$ to 1. By the duality principle:

$$\iota \circ \Psi(P) = (-1)^{cl(P)} = (-1)^{cl(P)} ehr(P)(1) = ehr^{str}(-1) = ev_{-1} \circ ehr^{str}(P).$$

So $\iota \circ \Psi = ev_{-1} \circ ehr^{str}$.

Let $P \in \mathbf{QP}$. Recalling that if $\sim \triangleleft P$, $cl(P|\sim) = cl(P)$:

$$\delta \circ \Psi(P) = (-1)^{cl(P)} \sum_{\sim \triangleleft P} P/\sim \otimes P|\sim = \sum_{\sim \triangleleft P} P/\sim \otimes (-1)^{cl(P|\sim)} P|\sim = (Id \otimes \Psi) \circ \delta(P).$$

So $\delta \circ \Psi = (Id \otimes \Psi) \circ \delta$. Hence, for any $x \in \mathcal{H}_{\mathbf{QP}}$:

$$\begin{aligned} EHR \circ \Psi(x) &= EHR^{str} \circ (Id \leftarrow \iota) \circ \Psi(x) \\ &= EHR^{str}(\Psi(x)_0) \iota \circ \Psi(x)_1 \\ &= EHR^{str}(x_0) \iota \circ \Psi(x_1) \\ &= EHR^{str}(x_0) ev_{-1} \circ ehr^{str}(x_1) \\ &= EHR^{str}(x^{(1)}) ev_{-1} \circ EHR^{str}(x^{(2)}) \\ &= EHR \leftarrow ev_{-1}(x) \\ &= (Id \leftarrow ev_{-1}) \circ EHR^{str}(x) \\ &= \Phi_{-1} \circ EHR^{str}(x), \end{aligned}$$

where we denote $\delta(x) = x^{(1)} \otimes x^{(2)}$ and $\rho(x) = x_0 \otimes x_1$. As Φ_{-1} and Ψ are involutions, $EHR^{str} \circ \Psi = \Phi_{-1} \circ EHR$. $\square$

In $E_{\mathbb{K}[X]\to\mathbb{K}[X]}$, putting $\phi_\lambda = Id \leftarrow ev_\lambda$, for any $P \in \mathbb{K}[X]$, $\phi_\lambda(P) = P(\lambda X)$. Moreover, as H is compatible with the coactions:

$$H \circ \Phi_\lambda = H \circ (Id \leftarrow ev_\lambda) = H \leftarrow ev_\lambda = (Id \leftarrow ev_\lambda) \circ H = \phi_\lambda \circ H,$$

so:

$$ehr \circ \Psi = H \circ EHR \circ \Psi = H \circ \Phi_{-1} \circ EHR^{str} = \phi_{-1} \circ H \circ EHR^{str} = \phi_{-1} \circ ehr^{str}.$$

In other words, for any $P \in \mathbf{QP}$, $(-1)^{cl(P)} ehr_P(X) = ehr_P^{str}(-X)$: we recover the duality principle.

We obtain the commutative diagram of Hopf algebra morphisms:

$$\begin{tikzcd}
\mathcal{H}_{\mathbf{QP}} \arrow[d, hook, "\Theta"'] \arrow[dr, "EHR"] \arrow[ddrr, bend left] & & & \\
\mathcal{H}_{\mathbf{QP}} \arrow[r, two heads, "EHR^{str}"] \arrow[d, hook, "\Psi"'] \arrow[ddrr, bend left] & \mathbf{WQSym} \arrow[d, hook, "\Phi_{-1}"'] \arrow[ddrr, "H"] \arrow[r, phantom, "\sqcup"] & {} & \\
\mathcal{H}_{\mathbf{QP}} \arrow[r, two heads, "EHR"] \arrow[ddrr, bend left] & \mathbf{WQSym} \arrow[ddrr, "H"] \arrow[r, phantom, "\sqcup"] & \mathcal{H}_{\mathbf{qp}} \arrow[d, hook, "\theta"'] \arrow[dr, "ehr"] & \\
& \sqcup & \mathcal{H}_{\mathbf{qp}} \arrow[r, two heads, "ehr^{str}"'] \arrow[d, hook, "\psi"'] & \mathbb{K}[X] \arrow[d, hook, "\phi_{-1}"'] \\
& & \mathcal{H}_{\mathbf{qp}} \arrow[r, two heads, "ehr"'] & \mathbb{K}[X]
\end{tikzcd}$$

Corollary 49 *For all $x, y \in \mathbf{WQSym}$:*

$$\Phi_{-1}(x \downarrow y) = \Phi_{-1}(x) \not\downarrow \Phi_{-1}(y) \qquad\qquad \Phi_{-1}(x \not\downarrow y) = \Phi_{-1}(x) \downarrow \Phi_{-1}(y).$$

Proof. If $P, Q \in \mathbf{QP}$, then $cl(P \downarrow Q) = cl(P) + cl(Q)$, so:

$$\Psi(P \downarrow Q) = (-1)^{cl(P)+cl(Q)} P \downarrow Q = \Psi(P) \downarrow \Psi(Q).$$

Let $x, y \in \mathbf{WQSym}$. There exist $x', y' \in \mathcal{H}_{\mathbf{QP}}$, such that $EHR^{str}(x') = x$ and $EHR^{str}(y') = y$. Hence, using the non-commutative duality principle:

$$\begin{aligned}
\Phi_{-1}(x \downarrow y) &= \Phi_{-1}(EHR^{str}(x') \downarrow EHR^{str}(y')) \\
&= \Phi_{-1} \circ EHR^{str}(x' \downarrow y') \\
&= \Phi_{-1} \circ EHR^{str} \circ \Psi(\Psi(x') \downarrow \Psi(y')) \\
&= EHR(\Psi(x') \downarrow \Psi(y')) \\
&= EHR \circ \Psi(x') \not\downarrow EHR \circ \Psi(y') \\
&= \Phi_{-1}(\Phi_{-1} \circ EHR \circ \Psi(x')) \not\downarrow \Phi_{-1}(\Phi_{-1} \circ EHR \circ \Psi(y')) \\
&= \Phi_{-1}(EHR^{str}(x')) \not\downarrow \Phi_{-1}(EHR^{str}(y')) \\
&= \Phi_{-1}(x) \not\downarrow \Phi_{-1}(y).
\end{aligned}$$

As Φ_{-1} is an involution, we obtain the second point. $\square$

5.5 Restriction to posets

In [9], the image of the restriction to $\mathcal{H}_{\mathbf{P}}$ of the map from $\mathcal{H}_{\mathbf{QP}}$ to **WQSym** defined by T-partitions is a Hopf subalgebra isomorphic to the Hopf algebra of permutations **FQSym** [12, 7]. This is not the case here:

Proposition 50 $EHR(\mathcal{H}_{\mathbf{P}}) = EHR^{str}(\mathcal{H}_{\mathbf{P}}) = \mathbf{WQSym}$.

Proof. Let w be a packed word of length n. We define a poset P on $[n]$ by:

$$\forall i, j \in [n],\ i \leq_P j \text{ if } (i = j) \text{ or} (w(i) < w(j)).$$

Note that if $i \leq_P j$, then $w(i) \leq w(j)$. If $i \leq_P j$ and $j \leq_P k$, then:

- if $i = j$ or $j = k$, obviously $i \leq_P k$.
- Otherwise, $w(i) < w(j)$ and $w(j) < w(k)$, so $w(i) < w(k)$ and $i \leq_P k$.

Let us assume that $i \leq_P j$ and $j \leq_P i$. Then $w(i) \leq w(j)$ and $w(j) \leq w(i)$, so $w(i) = w(j)$. As $i \leq_P j$, $i = j$. So P is indeed a poset.

Let w' be a packed word of length n. Let us prove that $w' \in W_P^{str}$ if, and only if, $w \leq w'$, where $\leq$ is the order on packed words of definition 23.

$\Longrightarrow$. Let us assume that $w' \in W_P^{str}$. If $w(i) < w(j)$, then $i \leq_P j$, so $w'(i) \leq w'(j)$. Moreover, if $w'(i) = w'(j)$, then $i \leq_P j$, so $i = j$ as P is a poset, and finally $w(i) = w(j)$: contradiction. So $w'(i) < w'(j)$, we shows that $w \leq w'$.

$\Longleftarrow$. Let us assume that $w' \leq w$. If $i \leq_P j$, then $i = j$ or $w(i) < w(j)$, so $w'(i) = w'(j)$ or $w'(i) < w'(j)$. If, moreover, $w'(i) = w'(j)$, then $i = j$; so $w' \in W_P^{str}$.

We obtain an element $P \in \mathcal{H}_{\mathbf{P}}$ such that:

$$EHR^{str}(P) = \sum_{w \leq w'} w'.$$

As this holds for any w, by a triangularity argument, $EHR^{str}(\mathcal{H}_{\mathbf{P}}) = \mathbf{WQSym}$. By the non-commutative duality principle:

$$EHR(\mathcal{H}_{\mathbf{P}}) = \Phi_{-1} \circ EHR^{str} \circ \Psi(\mathcal{H}_{\mathbf{P}}) = \Phi_{-1} \circ EHR^{str}(\mathcal{H}_{\mathbf{P}}) = \Phi_{-1}(\mathbf{WQSym}) = \mathbf{WQSym},$$

as Φ_{-1} is an automorphism of **WQSym**. $\square$

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