Extension of the product of a post-Lie algebra and application to the SISO feedback transformation group

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Abstract

We describe the both post- and pre-Lie algebra $\mathfrak{g}_{SISO}$ associated to the affine SISO feedback transformation group. We show that it is a member of a family of post-Lie algebras associated to representations of a particular solvable Lie algebra. We first construct the extension of the magmatic product of a post-Lie algebra to its enveloping algebra, which allows to describe free post-Lie algebras and is widely used to obtain the enveloping of $\mathfrak{g}_{SISO}$ and its dual.

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Contents

Introduction

The affine SISO feedback transformation group G_{SISO} [4], which appears in Control Theory, can be seen as the character group of a Hopf algebra $\mathcal{H}_{SISO}$; let us start by a short presentation of this object (we slightly modify the notations of [4]).

1. First, let us recall some algebraic structures on noncommutative polynomials.
   (a) Let x_1, x_2 be two indeterminates. We consider the algebra of noncommutative polynomials $\mathbb{K}\langle x_1, x_2\rangle$. As a vector space, it is generated by words in letters x_1, x_2; its product is the concatenation of words; its unit, the empty word, is denoted by $\emptyset$.
   (b) $\mathbb{K}\langle x_1, x_2\rangle$ is a Hopf algebra with the concatenation product and the deshuffling coproduct $\Delta_{ш}$, defined by $\Delta_{ш}(x_i) = x_i \otimes \emptyset + \emptyset \otimes x_i$, for $i \in \{1, 2\}$.
   (c) $\mathbb{K}\langle x_1, x_2\rangle$ is also a commutative, associative algebra with the shuffle product ш: for example, if $i, j, k, l \in \{1, 2\}$,
$$\begin{aligned} x_i ш x_j &= x_ix_j + x_jx_i,\\ x_ix_j ш x_k &= x_ix_jx_k + x_ix_kx_j + x_kx_ix_j,\\ x_i ш x_jx_k &= x_ix_jx_k + x_jx_ix_k + x_jx_kx_i,\\ x_ix_j ш x_kx_l &= x_ix_jx_kx_l + x_ix_kx_jx_l + x_ix_kx_lx_j + x_kx_ix_jx_l + x_kx_ix_lx_j + x_kx_lx_ix_j. \end{aligned}$$
2. The vector space $\mathbb{K}\langle x_1, x_2\rangle^2$ is generated by words $x_{i_1} \ldots x_{i_k}\epsilon_j$, where $k \geq 0$, $i_1, \ldots, i_k, j \in \{1, 2\}$, and (ϵ_1, ϵ_2) denotes the canonical basis of $\mathbb{K}^2$.
3. As an algebra, $\mathcal{H}_{SISO}$ is equal to the symmetric algebra $S(\mathbb{K}\langle x_1, x_2\rangle^2)$; its product is denoted by μ and its unit by 1. Two coproducts Δ_* and $\Delta_\bullet$ are defined on $\mathcal{H}_{SISO}$. For all $h \in \mathcal{H}_{SISO}$, we put $\overline{\Delta}_*(h) = \Delta_*(h) - 1 \otimes h$ and $\overline{\Delta}_\bullet(h) = \Delta_\bullet(h) - 1 \otimes h$. Then:
   - For all $i \in \{1, 2\}$, $\overline{\Delta}_*(\emptyset\epsilon_i) = \emptyset\epsilon_i \otimes 1$.
   - For all $g \in \mathbb{K}\langle x_1, x_2\rangle$, for all $i \in \{1, 2\}$:
$$\begin{aligned} \overline{\Delta}_* \circ \theta_{x_1}(g\epsilon_i) &= (\theta_{x_1} \otimes Id) \circ \overline{\Delta}_*(g\epsilon_i) + (\theta_{x_2} \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{ш}(g)\epsilon_i \otimes \epsilon_2),\\ \overline{\Delta}_* \circ \theta_{x_2}(g\epsilon_i) &= (\theta_{x_2} \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{ш}(g)\epsilon_i \otimes \epsilon_1), \end{aligned}$$
     where $\theta_x(h\epsilon_i) = xh\epsilon_i$ for all $x \in \{x_1, x_2\}$, $h \in \mathbb{K}\langle x_1, x_2\rangle$, $i \in \{1, 2\}$. These are formulas of Lemma 4.1 of [4], with the notations $a_w = w\epsilon_2$, $b_w = w\epsilon_1$, $\theta_0 = \theta_{x_1}$, $\theta_1 = \theta_{x_2}$ and $\tilde{\Delta} = \overline{\Delta}_*$.
   - for all $g \in \mathbb{K}\langle x_1, x_2\rangle$:
$$\begin{aligned} \overline{\Delta}_\bullet(g\epsilon_1) &= (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{ш}(g)(\epsilon_1 \otimes \epsilon_1)),\\ \overline{\Delta}_\bullet(g\epsilon_2) &= \overline{\Delta}_*(g\epsilon_2) + (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{ш}(g)(\epsilon_2 \otimes \epsilon_1)). \end{aligned}$$

   This coproduct $\Delta_\bullet$ makes $\mathcal{H}_{SISO}$ a Hopf algebra, and Δ_* is a right coaction on this coproduct, that is to say:
$$(\Delta_\bullet \otimes Id) \circ \Delta_\bullet = (Id \otimes \Delta_\bullet) \circ \Delta_\bullet, \qquad (\Delta_* \otimes Id) \circ \Delta_* = (Id \otimes \Delta_\bullet) \circ \Delta_*.$$
4. After the identification of $\emptyset\epsilon_1$ with the unit of $\mathcal{H}_{SISO}$, we obtain a commutative, graded and connected Hopf algebra, in other words the dual of an enveloping algebra $\mathcal{U}(\mathfrak{g}_{SISO})$.

Our aim is to give a description of the underlying Lie algebra $\mathfrak{g}_{SISO}$. It turns out that it is both a pre-Lie algebra (or a Vinberg algebra [1], see [5] for a survey on these objects) and a post-Lie algebra [6, 10]: it has a Lie bracket ${}_a[-,-]$ and two nonassociative products $*$ and $\bullet$, such that for all $x, y, z \in \mathfrak{g}_{SISO}$:

$$x * {}_a[y,z] = (x*y)*z - x*(y*z) - (x*z)*y + x*(z*y),$$
$${}_a[x,y]*z = {}_a[x*z,y] + {}_a[x,y*z];$$

$$(x \bullet y) \bullet z - x \bullet (y \bullet z) = (x \bullet z) \bullet y - x \bullet (z \bullet y).$$

The Lie bracket on $\mathfrak{g}_{SISO}$ corresponding to G_{SISO} is ${}_a[-,-]_*$:

$$\forall x, y \in \mathfrak{g}_{SISO},\ {}_a[x,y]_* = {}_a[x,y] + x*y - y*x = x \bullet y - y \bullet x.$$

Let us be more precise on these structures. As a vector space, $\mathfrak{g}_{SISO} = \mathbb{K}\langle x_1, x_2\rangle^2$, and:

$$\forall f, g \in \mathbb{K}\langle x_1, x_2\rangle,\ \forall i, j \in \{1,2\}, \qquad {}_a[f\epsilon_i, g\epsilon_j] = \begin{cases} 0 \text{ if } i = j, \\ -f \sqcup\!\sqcup g\epsilon_2 \text{ if } i = 2 \text{ and } j = 1, \\ f \sqcup\!\sqcup g\epsilon_2 \text{ if } i = 1 \text{ and } j = 2. \end{cases}$$

The magmatic product $*$ is inductively defined. If $f, g \in \mathbb{K}\langle x_1, x_2\rangle$ and $i, j \in \{1,2\}$:

$$\emptyset\epsilon_i * g\epsilon_j = 0, \qquad x_2 f\epsilon_i * g\epsilon_1 = x_2(f\epsilon_i * g\epsilon_1) + x_2(f \sqcup\!\sqcup g)\epsilon_i,$$
$$x_1 f\epsilon_i * g\epsilon_j = x_1(f\epsilon_i * g\epsilon_j), \qquad x_2 f\epsilon_i * g\epsilon_2 = x_2(f\epsilon_i * g\epsilon_2) + x_1(f \sqcup\!\sqcup g)\epsilon_i.$$

The pre-Lie product $\bullet$, first determined in [4], is given by:

$$\forall f, g \in \mathbb{K}\langle x_1, x_2\rangle,\ \forall i, j \in \{1,2\},\ f\epsilon_i \bullet g\epsilon_j = (f \sqcup\!\sqcup g)\delta_{i,1}\epsilon_j + f\epsilon_i * g\epsilon_j.$$

We shall show here that this is a special case of a family of post-Lie algebras, associated to modules over certain solvable Lie algebras.

We start with general preliminary results on post-Lie algebras. We extend the now classical Oudom-Guin construction on prelie algebras [7, 8] to the post-Lie context in the first section: this is a result of [2] (Proposition 3.1), which we prove here in a different, less direct way; our proof allows also to obtain a description of free post-Lie algebras. Recall that if $(V, *)$ is a pre-Lie algebra, the pre-Lie product $*$ can be extended to $S(V)$ in such a way that the product defined by:

$$\forall f, g \in S(V),\ f \circledast g = \sum f * g^{(1)} g^{(2)}$$

is associative, and makes $S(V)$ a Hopf algebra, isomorphic to $\mathcal{U}(V)$. For any magmatic algebra $(V, *)$, we construct in a similar way an extension of $*$ to $T(V)$ in Proposition 1. We prove in Theorem 1 that the product $\circledast$ defined by:

$$\forall f, g \in T(V),\ f \circledast g = \sum f * g^{(1)} g^{(2)}$$

makes $T(V)$ a Hopf algebra. The Lie algebra of its primitive elements, which is the free Lie algebra $\mathcal{L}ie(V)$ generated by V, is stable under $*$ and turns out to be a post-Lie algebra (Proposition 2) satisfying a universal property (Theorem 2). In particular, if V is, as a magmatic algebra, freely generated by a subspace W, $\mathcal{L}ie(V)$ is the free post-Lie algebra generated by W (Corollary 1). Moreover, if $V = ([-,-], *)$ is a post-Lie algebra, this construction goes through the quotient defining $\mathcal{U}(V, [-,-])$, defining a new product $\circledast$ on it, making it isomorphic to the enveloping algebra of V with the Lie bracket defined by:

$$\forall x, y \in V,\ [x,y]_* = [x,y] + x*y - y*x.$$

For example, if $x_1, x_2, x_3 \in V$:

$$\begin{aligned} x_1 \circledast x_2 &= x_1x_2 + x_1 * x_2 \\ x_1 \circledast x_2x_3 &= x_1x_2x_3 + (x_1 * x_2)x_3 + (x_1 * x_3)x_2 + (x_1 * x_2) * x_3 - x_1 * (x_2 * x_3) \\ x_1x_2 \circledast x_3 &= x_1x_2x_3 + (x_1 * x_3)x_2 + x_1(x_2 * x_3). \end{aligned}$$

In the particular case where $[-,-] = 0$, we recover the Oudom-Guin construction.

The second section is devoted to the study of a particular solvable Lie algebra $\mathfrak{g}_a$ associated to an element $a \in \mathbb{K}^N$. As the Lie bracket of $\mathfrak{g}_a$ comes from an associative product, the construction of the first section holds, with many simplifications: we obtain an explicit description of $\mathcal{U}(\mathfrak{g}_a)$ with the help of a product $\blacktriangleleft$ on $S(\mathfrak{g}_a)$ (Proposition 6). A short study of $\mathfrak{g}_a$-modules when $a = (1, 0, \ldots, 0)$ (which is a generic case) is done in Proposition 8, considering $\mathfrak{g}_a$ as an associative algebra, and in Proposition 9, considering it as a Lie algebra. In particular, if $\mathbb{K}$ is algebraically closed, any $\mathfrak{g}_a$ modules inherits a natural decomposition in characteristic subspaces.

Our family of post-Lie algebras is introduced in the third section; it is reminescent of the construction of [3]. Let us fix a vector space V, $(a_1, \ldots, a_N) \in \mathbb{K}^N$ and a family $F_1, \ldots, F_N$ of endomorphisms of V. We define a product $*$ on $T(V)^N$, such that for all $f, g \in T(V)$, $x \in V$, $i, j \in \{1, \ldots, N\}$:

$$\begin{aligned} \emptyset\epsilon_i * g\epsilon_j &= 0, \\ xf\epsilon_i * g\epsilon_j &= x(f\epsilon_i * g\epsilon_j) + F_j(x)(f \sqcup\!\sqcup g)\epsilon_i, \end{aligned}$$

where $(\epsilon_1, \ldots, \epsilon_N)$ is the canonical basis of $\mathbb{K}^N$ and $\sqcup\!\sqcup$ is the shuffle product of $T(V)$. The Lie bracket of $T(V)^N$ that we shall use here is:

$$\forall f, g \in T(V),\ \forall i, j \in \{1, \ldots, N\},\ {}_a[f\epsilon_i, g\epsilon_j] = (f \sqcup\!\sqcup g)(a_i\epsilon_j - a_j\epsilon_i).$$

This Lie bracket comes from an associative product ${}_a\sqcup\!\sqcup$ defined by:

$$\forall f, g \in T(V),\ \forall i, j \in \{1, \ldots, N\}, f\epsilon_i\ {}_a\sqcup\!\sqcup\ g\epsilon_j = a_i(f \sqcup\!\sqcup g)\epsilon_j.$$

We put $\bullet = * + {}_a\sqcup\!\sqcup$. We prove in Theorem 3 the equivalence of the three following conditions:

- $(T(V)^N, \bullet)$ is a pre-Lie algebra.
- $(T(V)^N, {}_a[-,-], *)$ is a post-Lie algebra.
- $F_1, \ldots, F_N$ defines a structure of $\mathfrak{g}_a$-module on V.

If this holds, the construction of the first section allows to obtain two descriptions of the enveloping algebra of $\mathcal{U}(T(V)^N)$, respectively coming from the post-Lie product $*$ and from the pre-Lie product $\bullet$: the extensions of $*$ and of $\bullet$ are respectively described in Propositions 15 and 16. It is shown in Proposition 17 that the two associated descriptions of $\mathcal{U}(T(V)^N)$ are equal. For $\mathfrak{g}_{SISO}$, we take $a = (1, 0)$, $V = Vect(x_1, x_2)$ and:

$$F_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which indeed define a $\mathfrak{g}_{(1,0)}$-module. In order to relate this to the Hopf algebra $\mathcal{H}_{SISO}$ of [4], we need to consider the dual of the enveloping of $T(V)^N$. First, if $a = (1, 0, \ldots, 0)$, we observe that the decomposition of V as a $\mathfrak{g}_a$-module of the second section induces a graduation of the post-Lie algebra $T(V)^N$ (Proposition 18), unfortunately not connected: the component of degree 0 is 1-dimensional, generated by $\emptyset\epsilon_1$. Forgetting this element, that is, considering the augmentation ideal of the graded post-Lie algebra $T(V)^N$, we can dualize the product $\circledast$ of $S(T(V)^N)$ in order to obtain the coproduct of the dual Hopf algebra in an inductive way. For $\mathfrak{g}_{SISO}$, we indeed obtain the inductive formulas of $\mathcal{H}_{SISO}$, finally proving that the dual Lie algebra of this

Hopf algebra, which in some sense can be exponentiated to G_{SISO}, is indeed post-Lie and pre-Lie.

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Notations.

1. Let $\mathbb{K}$ be a commutative field. The canonical basis of $\mathbb{K}^n$ is denoted by $(\epsilon_1, \ldots, \epsilon_n)$.
2. For all $n \geq 1$, we denote by $[n]$ the set $\{1, \ldots, n\}$.
3. We shall use Sweeder's notations: if C is a coalgebra and $x \in C$,

$$\begin{aligned}
\Delta^{(1)}(x) = \Delta(x) &= \sum x^{(1)} \otimes x^{(2)}, \\
\Delta^{(2)}(x) = (\Delta \otimes Id) \circ \Delta(x) &= \sum x^{(1)} \otimes x^{(2)} \otimes x^{(3)}, \\
\Delta^{(3)}(x) = (\Delta \otimes Id \otimes Id) \circ (\Delta \otimes Id) \circ \Delta(x) &= \sum x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes x^{(4)}, \\
&\vdots
\end{aligned}$$

1 Extension of a post-Lie product

We first generalize the Oudom-Guin extension of a pre-Lie product in a post-Lie algebraic context, as done in [2]. Let us first recall what a post-Lie algebra is.

Definition 1 *1. A (right)* post-Lie algebra *is a family* $(\mathfrak{g}, \{-,-\}, *)$, *where* $\mathfrak{g}$ *is a vector space,* $\{-,-\}$ *and* $*$ *are bilinear products on* $\mathfrak{g}$ *such that:*

- $(\mathfrak{g}, \{-,-\})$ *is a Lie algebra.*
- *For all* $x, y, z \in \mathfrak{g}$*:*

$$x * \{y, z\} = (x * y) * z - x * (y * z) - (x * z) * y + x * (z * y), \tag{1}$$

$$\{x, y\} * z = \{x * z, y\} + \{x, y * z\}. \tag{2}$$

2. If $(\mathfrak{g}, \{-,-\}, *)$ *is post-Lie, we define a second Lie bracket on* $\mathfrak{g}$*:*

$$\forall x, y \in \mathfrak{g},\ \{x, y\}_* = \{x, y\} + x * y - y * x.$$

Note that if $\{-,-\}$ is 0, then $(\mathfrak{g}, *)$ is a (right) pre-Lie algebra, that is to say:

$$\forall x, y, z \in \mathfrak{g},\ (x * y) * z - x * (y * z) = (x * z) * y - x * (z * y). \tag{3}$$

1.1 Extension of a magmatic product

Let V be a vector space. We here use the tensor Hopf algebra $T(V)$. In this section, we shall denote the unit of $T(V)$ by 1. Its product is the concatenation of words, and its coproduct $\Delta_{\sqcup\!\sqcup}$ is the cocommutative deshuffling coproduct. For example, if $x_1, x_2, x_3 \in V$:

$$\begin{aligned}
\Delta_{\sqcup\!\sqcup}(x_1) &= x_1 \otimes 1 + 1 \otimes x_1, \\
\Delta_{\sqcup\!\sqcup}(x_1x_2) &= x_1x_2 \otimes 1 + x_1 \otimes x_2 + x_2 \otimes x_1 + 1 \otimes x_1x_2, \\
\Delta_{\sqcup\!\sqcup}(x_1x_2) &= x_1x_2x_3 \otimes 1 + x_1x_2 \otimes x_3 + x_1x_3 \otimes x_2 + x_2x_3 \otimes x_1 \\
&\quad + x_1 \otimes x_2x_3 + x_2 \otimes x_1x_3 + x_3 \otimes x_1x_2 + 1 \otimes x_1x_2x_3.
\end{aligned}$$

Its counit is denoted by ε: $\varepsilon(1) = 1$ and if $k \geq 1$ and $x_1, \ldots, x_k \in V$, $\varepsilon(x_1 \ldots x_k) = 0$.

Proposition 1 *Let V be a vector space and $* : V \otimes V \longrightarrow V$ be a magmatic product on V. Then $*$ can be uniquely extended as a map from $T(V) \otimes T(V)$ to $T(V)$ such that for all $f, g, h \in T(V)$, $x, y \in V$:*

- $f * 1 = f$.
- $1 * f = \varepsilon(f)1$.
- $x * (fy) = (x * f) * y - x * (f * y)$.
- $(fg) * h = \sum \left(f * h^{(1)}\right)\left(g * h^{(2)}\right)$.

Proof. *Existence.* We first inductively extend $*$ from $V \otimes T(V)$ to V. If $n \geq 0$, x, $y_1, \ldots, y_n \in V$, we put:

$$x * y_1 \ldots y_n = \begin{cases} x \text{ if } n = 0, \\ x * y_1 \text{ if } n = 1, \\ \underbrace{\underbrace{(x * (y_1 \ldots y_{n-1})}_{\in V} * \underbrace{y_n}_{\in V}}_{\in V} - \displaystyle\sum_{i=1}^{n-1} \underbrace{x * (\underbrace{y_1 \ldots (y_i * y_n) \ldots y_{n-1}}_{\in V^{\otimes(n-1)}})}_{\in V} \text{ if } n \geq 2. \end{cases}$$

This product is then extended from $T(V) \otimes T(V)$ to $T(V)$ in the following way:

- For all $f \in T(V)$, $1 * f = \varepsilon(f)1$.
- For all $n \geq 1$, for all $x_1, \ldots, x_n \in V$, $f \in T(V)$:

$$(x_1 \ldots x_n) * f = \sum \underbrace{(x_1 * f^{(1)})}_{\in V} \ldots \underbrace{(x_n * f^{(n)})}_{\in V} \in V^{\otimes n}.$$

Note that for all $n \geq 0$, $V^{\otimes n} * T(V) \subseteq V^{\otimes n}$, which induces the second point. Let us prove the first point with $f = x_1 \ldots x_n \in V^{\otimes n}$. If $n = 0$, $f * 1 = 1 * 1 = \varepsilon(1)1 = 1 = f$. If $n = 1$, $f \in V$, so $f * 1 = f$ by definition of the extension of $*$ on $V \otimes T(V)$. If $n \geq 2$:

$$f * 1 = (x_1 \ldots x_n) * 1 = (x_1 * 1) \ldots (x_n * 1) = x_1 \ldots x_n = f.$$

Let us prove the third point for $f = y_1 \ldots y_n$. Then:

$$x * (fy) = (x * f) * y - \sum x * (y_1 \ldots (y_i * y) \ldots y_n).$$

Moreover, as $\Delta_{\sqcup\!\sqcup}(y) = y \otimes 1 + 1 \otimes y$:

$$f * y = \sum_{i=1}^{n} (y_1 * 1) \ldots (y_i * y) \ldots (y_n * 1) = \sum_{i=1}^{n} y_1 \ldots (y_i * y) \ldots y_n.$$

So $x * (fy) = (x * f) * y - x * (f * y)$. Let us finally prove the last point for $f = x_1 \ldots x_k$ and $g = x_{k+1} \ldots x_{k+l}$. Then:

$$\begin{aligned} (fg) * h &= \sum \left(x_1 * h^{(1)}\right) \ldots \left(x_{k+l} * h^{(k+l)}\right) \\ &= \sum \left(x_1 * (h^{(1)})^{(1)}\right) \ldots \left(x_1 * (h^{(1)})^{(k)}\right) \left(x_{k+1} * (h^{(2)})^{(1)}\right) \ldots \left(x_{k+l} * (h^{(2)})^{(l)}\right) \\ &= \sum \left((x_1 \ldots x_k) * h^{(1)}\right) \left((x_{k+1} \ldots x_{k+l}) * h^{(2)}\right) \\ &= \sum \left(f * h^{(1)}\right) \left(g * h^{(2)}\right). \end{aligned}$$

We used the coassociativity of $\Delta_{\sqcup\!\sqcup}$ for the second equality.

Unicity. The first and third points uniquely determine $x*(x_1\ldots x_n)$ for $x,x_1,\ldots,x_n\in V$, by induction on n; the second and fourth points then uniquely determine $f*(x_1\ldots x_n)$ for all $f\in T(V)$ by induction on the length of f. $\Box$

Examples. If $x_1,x_2,x_3,x_4\in V$:

$$\begin{aligned}(x_1x_2)*x_3&=(x_1*x_3)x_2+x_1(x_2*x_3),\\ x_1*(x_2x_3)&=(x_1*x_2)*x_3-x_1*(x_2*x_3),\\ (x_1x_2x_3)*x_4&=(x_1*x_4)x_2x_3+x_1(x_2*x_4)x_3+x_1x_2(x_3*x_4),\\ (x_1x_2)*(x_3x_4)&=((x_1*x_3)*x_4)x_2-(x_1*(x_3*x_4))x_2+x_1((x_2*x_3)*x_4),\\ &\quad-x_1(x_2*(x_3*x_4))+(x_1*x_3)(x_2*x_4)+(x_1*x_4)(x_2*x_3),\\ x_1*(x_2x_3x_4)&=((x_1*x_2)*x_3)*x_4-(x_1*(x_2*x_3))*x_4-(x_1*(x_2*x_4))*x_3\\ &\quad+x_1*((x_2*x_4)*x_3)-(x_1*x_2)*(x_3*x_4)+x_1*(x_2*(x_3*x_4)).\end{aligned}$$

Lemma 1 *1. For all $k\in\mathbb{N}$, $V^{\otimes k}*T(V)\subseteq V^{\otimes k}$.*

2. For all $f,g\in T(V)$, $\varepsilon(f*g)=\varepsilon(f)\varepsilon(g)$.

3. For all $f,g\in T(V)$, $\Delta_{\sqcup\!\sqcup}(f*g)=\Delta_{\sqcup\!\sqcup}(f)*\Delta_{\sqcup\!\sqcup}(g)$.

4. For all $f,g\in T(V)$, $y\in V$, $f*(gy)=(f*g)*y-f*(g*y)$.

5. For all $f,g,h\in T(V)$, $(f*g)*h=\sum f*\left(\left(g*h^{(1)}\right)h^{(2)}\right)$.

Proof. 1. This was observed in the proof of Proposition 1.

2. From the first point, $Ker(\varepsilon)*T(V)+T(V)*Ker(\varepsilon)\subseteq Ker(\varepsilon)$, so if $\varepsilon(f)=0$ or $\varepsilon(g)=0$, then $\varepsilon(f*g)=0$. As $\varepsilon(1*1)=1$, the second point holds for all f,g.

3. We prove it for $f=x_1\ldots x_n$, by induction on n. If $n=0$, then $f=1$. Moreover, $\Delta_{\sqcup\!\sqcup}(1*g)=\varepsilon(g)\Delta_{\sqcup\!\sqcup}(1)=\varepsilon(g)1\otimes 1$, and:

$$\Delta_{\sqcup\!\sqcup}(f)*\Delta_{\sqcup\!\sqcup}(g)=\sum 1*g^{(1)}\otimes 1*g^{(2)}=\varepsilon\left(g^{(1)}\right)\varepsilon\left(g^{(2)}\right)1\otimes 1=\varepsilon(g)1\otimes 1.$$

If $n=1$, then $f\in V$. In this case, from the second point, $f*g\in V$, so $\Delta_{\sqcup\!\sqcup}(f*g)=f*g\otimes 1+1\otimes f*g$. Moreover:

$$\begin{aligned}\Delta_{\sqcup\!\sqcup}(f)*\Delta_{\sqcup\!\sqcup}(g)&=(f\otimes 1+1\otimes f)*\Delta_{\sqcup\!\sqcup}(g)\\ &=\sum f*g^{(1)}\otimes 1*g^{(2)}+\sum 1*g^{(1)}\otimes f*g^{(2)}\\ &=\sum f*g^{(1)}\otimes\varepsilon\left(g^{(2)}\right)1+\sum\varepsilon\left(g^{(1)}\right)1\otimes f*g^{(2)}\\ &=f*g\otimes 1+1\otimes f*g.\end{aligned}$$

If $n\geq 2$, we put $f_1=x_1\ldots x_{n-1}$ and $f_2=x_n$. By the induction hypothesis applied to f_1:

$$\begin{aligned}\Delta_{\sqcup\!\sqcup}(f*g)&=\sum\Delta_{\sqcup\!\sqcup}\left(\left(f_1*g^{(1)}\right)\left(f_2*g^{(2)}\right)\right)\\ &=\Delta_{\sqcup\!\sqcup}\left(f_1*g^{(1)}\right)\Delta_{\sqcup\!\sqcup}\left(f_2*g^{(2)}\right)\\ &=\sum\left(f_1^{(1)}*(g^{(1)})^{(1)}\right)\left(f_2^{(1)}*(g^{(2)})^{(1)}\right)\otimes\left(f_1^{(2)}*(g^{(1)})^{(2)}\right)\left(f_2^{(2)}*(g^{(2)})^{(2)}\right)\\ &=\sum(f_1f_2)^{(1)}*g^{(1)}\otimes(f_1f_2)^{(2)}*g^{(2)}\\ &=\Delta_{\sqcup\!\sqcup}(f)*\Delta_{\sqcup\!\sqcup}(g).\end{aligned}$$

We used the cocommutativity of $\Delta_{\sqcup\!\sqcup}$ for the fourth equality.

4. We prove it for $f = x_1 \dots x_n$, by induction on n. If $n = 0$, then $f = 1$ and:

$$1 * (gy) = (1 * g) * y - 1 * (g * y) = \varepsilon(g)\varepsilon(y) - \varepsilon(g * y) = 0.$$

For $n = 1$, this comes immediately from Proposition 1-3. If $n \geq 2$, we put $f_1 = x_1 \dots x_{n-1}$ and $f_2 = x_n$. The induction hypothesis holds for f_1. Moreover:

$$\begin{aligned}
f * (gy) &= \sum \left(f_1 * g^{(1)}\right)\left(f_2 * \left(g^{(2)} y\right)\right) + \sum \left(f_1 * \left(g^{(1)} y\right)\right)\left(f_2 * g^{(2)}\right)\\
&= \sum \left(f_1 * g^{(1)}\right)\left(\left(f_2 * g^{(2)}\right) * y\right) - \sum \left(f_1 * g^{(1)}\right)\left(f_2 * \left(g^{(2)} * y\right)\right)\\
&+ \sum \left(\left(f_1 * g^{(1)}\right) * y\right)\left(f_2 * g^{(2)}\right) - \sum \left(f_1 * \left(g^{(1)} * y\right)\right)\left(f_2 * g^{(2)}\right),\\
(f * g) * y &= \sum \left(\left(f_1 * g^{(1)}\right)\left(f_2 * g^{(2)}\right)\right) * y\\
&= \sum \left(\left(f_1 * g^{(1)}\right) * y\right)\left(f_2 * g^{(2)}\right) + \sum \left(f_1 * g^{(1)}\right)\left(\left(f_2 * g^{(2)}\right) * y\right),\\
f * (g * y) &= \sum \left(f_1 * (g * y)^{(1)}\right)\left(f_2 * (g * y)^{(2)}\right)\\
&= \sum \left(f_1 * \left(g^{(1)} * y\right)\right)\left(f_2 * g^{(2)}\right) + \sum \left(f_1 * g^{(1)}\right)\left(f_2 * \left(g^{(2)} * y\right)\right).
\end{aligned}$$

We use the third point for the third computation. So the result holds for all f.

5. We prove this for $h = z_1 \dots z_n$ and we proceed by induction on n. If $n = 0$, then $h = 1$ and $(f * g) * 1 = f * g$. Moreover, $\sum f * \left(\left(g * h^{(1)}\right) h^{(2)}\right) = f * ((g * 1)1) = (f * g)1 = f * g$. If $n = 1$, then $h \in V$, so $\Delta_{\sqcup\!\sqcup}(h) = h \otimes 1 + 1 \otimes h$. So:

$$\begin{aligned}
\sum f * \left(\left(g * h^{(1)}\right) h^{(2)}\right) &= f * ((g * h)1) + f * ((g * 1)h)\\
&= f * (g * h) + f * (gh)\\
&= f * (g * h) + (f * g) * h - f * (g * h)\\
&= (f * g) * h.
\end{aligned}$$

We use Proposition 1-3 for the third equality. If $n \geq 2$, we put $h_1 = z_1 \dots z_{n-1}$ and $h_2 = z_n$. From the fourth point:

$$\begin{aligned}
(f * g) * h &= ((f * g) * h_1) * h_2 - (f * g) * (h_1 * h_2)\\
&= \sum \left(f * \left(\left(g * h_1^{(1)}\right) h_1^{(2)}\right)\right) * h_2 - \sum f * \left(\left(g * (h_1 * h_2)^{(1)}\right)(h_1 * h_2)^{(2)}\right)\\
&= \sum f * \left(\left(\left(g * h_1^{(1)}\right) h_1^{(2)}\right) * h_2\right) + \sum f * \left(\left(g * h_1^{(1)}\right) h_1^{(2)} h_2\right)\\
&- \sum f * \left(\left(g * \left(h_1^{(1)} * h_2^{(1)}\right)\right)\left(h_1^{(2)} * h_2^{(2)}\right)\right)\\
&= \sum f * \left(\left(\left(g * h_1^{(1)}\right) * h_2\right) h_1^{(2)}\right) + \sum f * \left(\left(g * h_1^{(1)}\right)\left(h_1^{(2)} * h_2\right)\right)\\
&+ \sum f * \left(\left(g * h_1^{(1)}\right) h_1^{(2)} h_2\right) - \sum f * \left(\left(g * \left(h_1^{(1)} * h_2\right)\right) h_1^{(2)}\right)\\
&- \sum f * \left(\left(g * h_1^{(1)}\right)\left(h_1^{(2)} * h_2\right)\right)\\
&= \sum f * \left(\left(g * \left(h_1^{(1)} * h_2\right)\right) h_1^{(2)}\right) + \sum f * \left(\left(g * \left(h_1^{(1)} h_2\right)\right) h_1^{(2)}\right)\\
&+ \sum f * \left(\left(g * h_1^{(1)}\right)\left(h_1^{(2)} * h_2\right)\right) + \sum f * \left(\left(g * h_1^{(1)}\right) h_1^{(2)} h_2\right)\\
&- \sum f * \left(\left(g * \left(h_1^{(1)} * h_2\right)\right) h_1^{(2)}\right) - \sum f * \left(\left(g * h_1^{(1)}\right)\left(h_1^{(2)} * h_2\right)\right)\\
&= \sum f * \left(\left(g * \left(h_1^{(1)} h_2\right)\right) h_1^{(2)}\right) + \sum f * \left(\left(g * h_1^{(1)}\right) h_1^{(2)} h_2\right).
\end{aligned}$$

For the second equality, we used the induction hypothesis on h_1 and $h_1 * h_2 \in V^{\otimes(k-1)}$ by the first point; we used the third point for the third equality. As $\Delta_{\shuffle}(h_2) = h_2 \otimes 1 + 1 \otimes h_2$, $\Delta_{\shuffle}(h) = \sum h_1^{(1)} h_2 \otimes h_1^{(2)} + \sum h_1^{(1)} \otimes h_1^{(2)} h_2$, so the result holds for h. $\Box$

1.2 Associated Hopf algebra and post-Lie algebra

Theorem 1 *Let $*$ be a magmatic product on V. This product is extended to $T(V)$ by Proposition 1. We define a product $\circledast$ on $T(V)$ by:*

$$\forall f, g \in T(V),\ f \circledast g = \sum \left(f * g^{(1)}\right) g^{(2)}.$$

Then $(T(V), \circledast, \Delta_{\shuffle})$ is a Hopf algebra.

Proof. For all $f \in T(V)$:

$$1 \circledast f \sum \left(1 * f^{(1)}\right) f^{(2)} = \sum \varepsilon\left(f^{(1)}\right) f^{(2)} = f; \qquad \circledast 1 = (f * 1)1 = f.$$

For all $f, g, h \in T(V)$, by Lemma 1-5:

$$\begin{aligned}
(f \circledast g) \circledast h &= \sum \left(\left(f * g^{(1)}\right) g^{(2)}\right) \circledast h \\
&= \sum \left(\left(\left(f * g^{(1)}\right) g^{(2)}\right) * h^{(1)}\right) h^{(2)} \\
&= \sum \left(\left(f * g^{(1)}\right) * h^{(1)}\right) \left(g^{(2)} * h^{(2)}\right) h^{(3)} \\
&= \sum \left(f * \left(\left(g^{(1)} * h^{(1)}\right) h^{(2)}\right)\right) \left(g^{(2)} * h^{(3)}\right) h^{(4)}; \\
f \circledast (g \circledast h) &= \sum f \circledast \left(\left(g * h^{(1)}\right) h^{(2)}\right) \\
&= \sum \left(f * \left(\left(g^{(1)} * h^{(1)}\right) h^{(3)}\right)\right) \left(g^{(2)} * h^{(2)}\right) h^{(4)}.
\end{aligned}$$

As $\Delta_{\shuffle}$ is cocommutative, $(f \circledast g) \circledast h = f \circledast (g \circledast h)$, so $(T(V), \circledast)$ is a unitary, associative algebra.

For all $f, g \in T(V)$, by Lemma 1-3:

$$\begin{aligned}
\Delta_{\shuffle}(f \circledast g) &= \sum \Delta_{\shuffle}\left(\left(f * g^{(1)}\right) g^{(2)}\right) \\
&= \sum \left(f^{(1)} * \left(g^{(1)}\right)^{(1)}\right) \left(g^{(2)}\right)^{(1)} \otimes \left(f^{(2)} * \left(g^{(1)}\right)^{(2)}\right) \left(g^{(2)}\right)^{(2)} \\
&= \sum \left(f^{(1)} * \left(g^{(1)}\right)^{(1)}\right) \left(g^{(1)}\right)^{(2)} \otimes \left(f^{(2)} * \left(g^{(2)}\right)^{(1)}\right) \left(g^{(2)}\right)^{(2)} \\
&= \sum f^{(1)} \circledast g^{(1)} \otimes f^{(2)} \circledast g^{(2)}.
\end{aligned}$$

Note that we used the cocommutativity of $\Delta_{\shuffle}$ for the third equality. Hence, $(T(V), \circledast, \Delta_{\shuffle})$ is a Hopf algebra. $\Box$

Remark. By Lemma 1:

- For all $f, g, h \in T(V)$, $(f * g) * h = f * (g \circledast h)$: $(T(V), *)$ is a right $(T(V), \circledast)$-module.
- By restriction, for all $n \geq 0$, $(V^{\otimes n}, *)$ is a right $(T(V), \circledast)$-module. Moreover, for all $n \geq 0$, $(V^{\otimes n}, *) = (V, *)^{\otimes n}$ as a right module over the Hopf algebra $(T(V), \circledast, \Delta_{\shuffle})$.

Examples. Let $x_1, x_2, x_3 \in V$.

$$\begin{aligned} x_1 \circledast x_2 &= x_1x_2 + x_1 * x_2 \\ x_1 \circledast x_2x_3 &= x_1x_2x_3 + (x_1 * x_2)x_3 + (x_1 * x_3)x_2 + (x_1 * x_2) * x_3 - x_1 * (x_2 * x_3) \\ x_1x_2 \circledast x_3 &= x_1x_2x_3 + (x_1 * x_3)x_2 + x_1(x_2 * x_3). \end{aligned}$$

The vector space of primitive elements of $(T(V), \circledast, \Delta_{\sqcup\sqcup})$ is $\mathcal{L}ie(V)$. Let us now describe the Lie bracket induced on $\mathcal{L}ie(V)$ by $\circledast$.

Proposition 2 *1. Let $*$ be a magmatic product on V. The Hopf algebras $(T(V), \circledast, \Delta_{\sqcup\sqcup})$ and $(T(V), ., \Delta_{\sqcup\sqcup})$ are isomorphic, via the following algebra morphism:*

$$\phi_* : \left\{ \begin{array}{rcl} (T(V), ., \Delta_{\sqcup\sqcup}) & \longrightarrow & (T(V), \circledast, \Delta_{\sqcup\sqcup}) \\ x_1 \ldots x_k \in V^{\otimes k} & \longrightarrow & x_1 \circledast \ldots \circledast x_k. \end{array} \right.$$

2. $\mathcal{L}ie(V) * T(V) \subseteq \mathcal{L}ie(V)$. Moreover, $(\mathcal{L}ie(V), [-,-], *)$ is a post-Lie algebra. The induced Lie bracket on $\mathcal{L}ie(V)$ is denoted by $\{-,-\}_*$:

$$\forall f, g \in \mathcal{L}ie(V),\ \{f, g\}_* = [f, g] + f * g - g * f = fg - gf + f * g - g * f.$$

The Lie algebra $(\mathcal{L}ie(V), \{-,-\}_*)$ is isomorphic to $\mathcal{L}ie(V)$.

Proof. 1. There exists a unique algebra morphism $\phi_* : (T(V), .) \longrightarrow (T(V), \circledast)$, sending any $x \in V$ on itself. As the elements of V are primitive in both Hopf algebras, ϕ_* is a Hopf algebra morphism. As $V^{\otimes k} * T(V) \subseteq V^{\otimes k}$ for all $k \geq 0$, we deduce that for all $x_1, \ldots, x_{k+l} \in V$:

$$x_1 \ldots x_k \circledast x_{k+1} \ldots x_{k+l} = x_1 \ldots x_{k+l} + \text{a sum of words of length} < k + l.$$

Hence, if $x_1, \ldots, x_k \in V$:

$$\phi_*(x_1 \ldots x_k) = x_1 \circledast \ldots \circledast x_k = x_1 \ldots x_k + \text{a sum of words of length} < k.$$

Consequently:

- If $k \geq 0$ and $x_1, \ldots, x_k \in V$, an induction on k proves that $x_1 \ldots x_k \in \phi_*(T(V))$, so ϕ_* is surjective.
- If f is a nonzero element of $T(V)$, let us write $f = f_0 + \ldots + f_k$, with $f_i \in V^{\otimes i}$ for all i and $f_k \neq 0$. Then:
$$\phi_*(f) = f_k + \text{terms in } \mathbb{K} \oplus \ldots \oplus V^{\otimes(k-1)},$$
so $\phi_*(f) \neq 0$: ϕ_* is injective.

Hence, ϕ_* is an isomorphism.

2. We consider $A = \{f \in \mathcal{L}ie(V) \mid f * T(V) \subseteq \mathcal{L}ie(V)\}$. By Lemma 1-3, $V \subseteq A$. Let $f, g \in A$. For all $h \in T(V)$:

$$\begin{aligned} [f, g] * h &= (fg) * h - (gf) * h \\ &= \sum \left(f * h^{(1)}\right)\left(g * h^{(2)}\right) - \sum \left(g * h^{(1)}\right)\left(f * h^{(2)}\right) \\ &= \sum \left(f * h^{(1)}\right)\left(g * h^{(2)}\right) - \sum \left(g * h^{(2)})(f * h^{(1)}\right) \\ &= \sum \left[f * h^{(1)}, g * h^{(2)}\right]. \end{aligned}$$

We used the cocommutativity for the third equality. By hypothesis, $f*h^{(1)}, g*h^{(2)} \in \mathcal{L}ie(V)$, so $[f,g] \in A$. As A is a Lie subalgebra of $\mathcal{L}ie(V)$ containing V, it is equal to $\mathcal{L}ie(V)$.

Let $f,g,h \in \mathcal{L}ie(V)$. Then $g \circledast h = \sum \left(g*h^{(1)}\right)h^{(2)} = gh + g*h$. Similarly, $\sum \left(h*g^{(1)}\right)g^{(2)} = hg + h*g$, so, by Lemma 1-5:

$$\begin{aligned}
f*[g,h] &= f*(gh) - f*(hg)\\
&= \sum f*\left(\left(g*h^{(1)}\right)h^{(2)}\right) - f*(g*h) - \sum f*\left(\left(h*g^{(1)}\right)g^{(2)}\right) + f*(h*g)\\
&= (f*g)*h - f*(g*h) - (f*h)*g + f*(g*h).
\end{aligned}$$

Moreover:

$$\begin{aligned}
[f,g]*h &= (fg)*h - (gf)*h\\
&= (f*h)g + f(g*h) - (g*h)f - g(f*h)\\
&= [f*h,g] + [f,g*h].
\end{aligned}$$

So $\mathcal{L}ie(V)$ is a post-Lie algebra.

Consequently, $\{-,-\}_*$ is a second Lie bracket on $\mathcal{L}ie(V)$. In $(T(V),\circledast)$, if f and g are primitive:

$$f \circledast g - g \circledast f = fg + f*g - gf - g*f = \{f,g\}_*.$$

So, by the Cartier-Quillen-Milnor-Moore's theorem, $(T(V),\circledast,\Delta_{\sqcup\sqcup})$ is the enveloping algebra of $(\mathcal{L}ie(V),\{-,-\}_*)$. As it is isomorphic to the enveloping algebra of $\mathcal{L}ie(V)$, namely $(T(V),.,\Delta_{\sqcup\sqcup})$, these two Lie algebras are isomorphic. $\Box$

Let us give a combinatorial description of ϕ_*.

Proposition 3 *Let $(V,*)$ be a magmatic algebra, and $x_1,\ldots,x_k \in V$.*

- *Let $I = \{i_1,\ldots,i_p\} \subseteq [k]$, with $i_1 < \ldots < i_p$. We put:*
$$x_I^* = (\ldots((x_{i_1}*x_{i_2})*x_{i_3})*\ldots)*x_{i_p} \in V.$$
- *Let P be a partition of $[p]$. We denote it by $P = \{P_1,\ldots,P_p\}$, with the convention $\min(P_1) < \ldots < \min(P_p)$. We put:*
$$x_P^* = x_{P_1}^* \ldots x_{P_p}^* \in V^{\otimes p}.$$

Then:
$$\phi^*(x_1\ldots x_k) = \sum_{P \text{ partition of } [k]} x_P^*.$$

Proof. By induction on k. As $\phi_*(x) = x$ for all $x \in V$, it is obvious if $k=1$. Let us assume the result at rank k.

$$\begin{aligned}
\phi_*(x_1\ldots x_{k+1}) &= \phi_*(x_1\ldots x_k) \circledast x_{k+1}\\
&= \phi_*(x_1\ldots x_k)x_{k+1} + \phi_*(x_1\ldots x_k)*x_{k+1}\\
&= \sum_{P \text{ partition of } [k]} x_P^* x_{k+1} + \sum_{\substack{P=\{P_1,\ldots,P_p\}\\ \text{partition of } [k]}} \sum_{i=1}^{p} x_{P_1}^* \ldots (x_{P_i}^* * x_{k+1}) \ldots x_{P_p}^*\\
&= \sum_{\substack{P=\{P_1,\ldots,P_p\}\\ \text{partition of } [k]}} x^*_{\{P_1,\ldots,P_p,\{k+1\}\}} + \sum_{\substack{P=\{P_1,\ldots,P_p\}\\ \text{partition of } [k]}} \sum_{i=1}^{p} x^*_{\{P_1,\ldots,P_i\cup\{k+1\},\ldots,P_p\}}\\
&= \sum_{P \text{ partition of } [k+1]} x_P^*.
\end{aligned}$$

So the result holds for all k. $\square$

Examples. Let $x_1, x_2, x_3 \in V$.

$$\begin{aligned}
\phi_*(x_1) &= x_1,\\
\phi_*(x_1x_2) &= x_1x_2 + x_1 * x_2,\\
\phi_*(x_1x_2x_3) &= x_1x_2x_3 + (x_1 * x_2)x_3 + (x_1 * x_3)x_2 + x_1(x_2 * x_3) + (x_1 * x_2) * x_3.
\end{aligned}$$

Theorem 2 *Let $(V, *)$ be a magmatic algebra and let $(L, \{-,-\}, \star)$ be a post-Lie algebra. Let $\phi : (V, *) \longrightarrow (L, \star)$ be a morphism of magmatic algebras. There exists a unique morphism of post-Lie algebras $\overline{\phi} : \mathcal{L}ie(V) \longrightarrow L$ extending ϕ.*

Proof. Let $\psi : \mathcal{L}ie(V) \longrightarrow L$ be the unique Lie algebra morphism extending ϕ. Let us fix $h \in \mathcal{L}ie(V)$. We consider:

$$A_h = \{h \in \mathcal{L}ie(V) \mid \forall f \in \mathcal{L}ie(V),\ \psi(f * h) = \psi(f) \star \psi(h)\}.$$

If $f, g \in A_h$, then:

$$\begin{aligned}
\psi([f,g] * h) &= \psi([f * h, b] + [f, g * h])\\
&= \{\psi(f * h), \psi(g)\} + \{\psi(f), \psi(g * h)\}\\
&= \{\psi(f) \star \psi(h), \psi(g)\} + \{\psi(f), \psi(g) \star \psi(h)\}\\
&= \{\psi(f), \psi(g)\} \star \psi(h)\\
&= \psi([f,g]) \star \psi(h).
\end{aligned}$$

So $[f,g] \in A_h$: for all $h \in \mathcal{L}ie(V)$, A_h is a Lie subalgebra of $\mathcal{L}ie(V)$. Moreover, if $h \in V$, as $\psi_{|V} = \phi$ is a morphism of magmatic algebras, $V \subseteq A_h$; as a consequence, if $h \in V$, $A_h = \mathcal{L}ie(V)$.

Let $A = \{h \in \mathcal{L}ie(V) \mid A_h = \mathcal{L}ie(V)\}$. We put $\mathcal{L}ie(V)_n = \mathcal{L}ie(V) \cap V^{\otimes n}$; let us prove inductively that $\mathcal{L}ie(V)_n \subseteq A$ for all n. We already proved that $V \subseteq A$, so this is true for $n = 1$. Let us assume the result at all rank $k < n$. Let $h \in \mathcal{L}ie(V)_n$. We can assume that $h = [h_1, h_2]$, with $h_1 \in \mathcal{L}ie(V)_k$, $h_2 \in \mathcal{L}ie(V)_{n-k}$, $1 \leq k \leq n-1$. From Lemma 1 and Proposition 2, ${}_1f * h_2 \in \mathcal{L}ie(V)_k$ and $h_2 * h_1 \in \mathcal{L}ie(V)_{n-k}$, so the induction hypothesis holds for h_1, h_2, $h_1 * h_2$ and $h_2 * h_1$. Hence, for all $f \in T(V)$:

$$\begin{aligned}
\psi(f * h) &= \psi(f * [h_1, h_2])\\
&= \psi((f * h_1) * h_2 - f * (h_1 * h_2) - (f * h_2) * h_1 + f * (h_2 * h_1))\\
&= (\psi(f) \star \psi(h_1)) \star \psi(h_2) - \psi(f) \star (\psi(h_1) \star \psi(h_2))\\
&- (\psi(f) \star \psi(h_2)) \star \psi(h_1) + \psi(f) \star (\psi(h_2) \star \psi(h_1))\\
&= \psi(f) \star \{\psi(h_1), \psi(h_2)\}\\
&= \psi(f) \star \psi(h).
\end{aligned}$$

As a consequence, $\mathcal{L}ie(V)_n \subseteq A$. Finally, $A = \mathcal{L}ie(V)$, so for all $f, g \in \mathcal{L}ie(V)$, $\psi(f * g) = \psi(f) * \psi(g)$. $\square$

Corollary 1 *Let V be a vector space. The free magmatic algebra generated by V is denoted by $\mathcal{M}ag(V)$. Then $\mathcal{L}ie(\mathcal{M}ag(V))$ is the free post-Lie algebra generated by V.*

Proof. Let L be a post-Lie algebra and let ϕ be a linear map from V to L. From the universal property of $\mathcal{M}ag(V)$, there exists a unique morphism of magmatic algebras from $\mathcal{M}ag(V)$ to L extending ϕ; from the universal property of $\mathcal{L}ie(\mathcal{M}ag(V))$, this morphism can be uniquely extended as a morphism of post-Lie algebras from $\mathcal{L}ie(\mathcal{M}ag(V))$ to V. So $\mathcal{L}ie(\mathcal{M}ag(V))$ satisfies the required universal property to be a post-Lie algebra generated by V. $\square$

Remark. Describing the free magmatic algebra generated by V is terms of planar rooted trees with a grafting operation, we get back the construction of free post-Lie algebras of [6].

1.3 Enveloping algebra of a post-Lie algebra

Let $(V, \{-,-\}, *)$ be a post-Lie algebra. We extend $*$ onto $T(V)$ as previously in Proposition 1. The usual bracket of $\mathcal{L}ie(V) \subseteq T(V)$ is denoted by $[f,g] = fg - gf$, and should not be confused with the bracket $\{-,-\}$ of the post-Lie algebra V.

Lemma 2 *Let I be the two-sided ideal of $T(V)$ generated by the elements $xy - yx - \{x,y\}$, $x, y \in V$. Then $I * T(V) \subseteq I$ and $T(V) * I = (0)$.*

Proof. *First step.* Let us prove that for all $x, y \in V$, for all $h \in T(V)$:

$$\{x,y\} * h = \sum \left\{x * h^{(1)}, y * h^{(2)}\right\}.$$

Note that the second member of this formula makes sense, as $V * T(V) \subseteq V$ by Lemma 1.
We assume that $h = z_1 \dots z_n$ and we work by induction on n. If $n = 0$, then $h = 1$ and $\{x,y\} * 1 = \{x,y\} = \{x * 1, y * 1\}$. If $n = 1$, then $h \in V$, so $\Delta_{\sqcup\!\sqcup}(h) = h \otimes 1 + 1 \otimes h$.

$$\{x,y\} * h = \{x * h, y\} + \{x, y * h\} = \{x * h, y * 1\} + \{x * 1, y * h\} = \sum \{x * h^{(1)}, y * h^{(2)}\}.$$

If $n \geq 2$, we put $h_1 = z_1 \dots z_{n-1}$ and $h_2 = z_n$. The induction hypothesis holds for h_1, h_2 and $h_1 * h_2$:

$$\begin{aligned}
\{x,y\} * h &= (\{x,y\} * h_1) * h_2 - \{x,y\} * (h_1 * h_2)\\
&= \sum \left\{x * h_1^{(1)}, y * h_1^{(2)}\right\} * h_2 - \sum \left\{x * (h_1 * h_2)^{(1)}, y * (h_1 * h_2)^{(2)}\right\}\\
&= \sum \left\{\left(x * h_1^{(1)}\right) * h_2^{(1)}, \left(y * h_1^{(2)}\right) * h_2^{(2)}\right\} - \sum \left\{x * \left(h_1^{(1)} * h_2^{(1)}\right), y * \left(h_1^{(2)} * h_2^{(2)}\right)\right\}\\
&= \sum \left\{\left(x * h_1^{(1)}\right) * h_2, y * h_1^{(2)}\right\} + \sum \left\{x * h_1^{(1)}, \left(y * h_1^{(2)}\right) * h_2\right\}\\
&- \sum \left\{x * \left(h_1^{(1)} * h_2\right), y * h_1^{(2)}\right\} - \sum \left\{x * h_1^{(1)}, y * \left(h_1^{(2)} * h_2\right)\right\}\\
&= \sum \left\{\left(x * h_1^{(1)}\right) * h_2 - x * \left(h_1^{(1)} * h_2\right), y * h_1^{(2)}\right\}\\
&+ \sum \left\{x * h_1^{(1)}, \left(y * h_1^{(2)}\right) * h_2 - y * \left(h_1^{(2)} * h_2\right)\right\}\\
&= \sum \left\{x * \left(h_1^{(1)} h_2\right), y * h_1^{(2)}\right\} + \sum \left\{x * h_1^{(1)}, y * \left(h_1^{(2)} h_2\right)\right\}\\
&= \sum \left\{x * h^{(1)}, y * h^{(2)}\right\}.
\end{aligned}$$

Consequently, the result holds for all $h \in T(V)$.

Second step. Let $J = Vect(xy - yx - \{x,y\} \mid x, y \in V)$. For all $x, y \in V$, for all $h \in T(V)$, by the first step:

$$(xy - yx - \{x,y\}) * h = \sum \left(x * h^{(1)}\right)\left(y * h^{(2)}\right) - \left(y * h^{(1)}\right)\left(y * h^{(2)}\right) - \left\{x * h^{(1)}, y * h^{(2)}\right\} \in J.$$

So $J * T(V) \subseteq J$. If $g \in J$, $f_1, f_2, h \in T(V)$:

$$(f_1 g f_2) * h = \sum \left(f_1 * d^{(1)}\right) \underbrace{\left(g * h^{(2)}\right)}_{\in J} \left(f_2 * h^{(3)}\right) \in I.$$

So $I * T(V) \subseteq I$.

Let us prove that $T(V) * (T(V)JV^{\otimes n}) = (0)$ for all $n \geq 0$. We start with $n = 0$. First, $1 * (T(V)J) = \varepsilon(T(V)J) = (0)$. Let $x, y, z \in V$, $g \in T(V)$. Then:

$$\begin{aligned}
&x * (gyz - gzy - g\{y,z\}) \\
&= (x * (gy)) * z - x * ((gy) * z) - (x * (gz)) * y + x * ((gz) * y) \\
&- (x * g) * \{y,z\} + x * (g * \{y,z\}) \\
&= ((x * g) * y) * z - (x * (g * y) * z - x * ((g * z)y) \\
&- x * (g(y * z)) - ((x * g) * z) * y - (x * (g * z)) * y \\
&+ x * ((g * y)z) + x * (g(z * y)) - (x * g) * \{y,z\} + x * (g * \{y,z\}) \\
&= ((x * g) * y) * z - (x * (g * y)) * z - (x * (g * z)) * y + x * ((g * z) * y) \\
&- (x * g) * (y * z) + x * (g * (y * z)) - ((x * g) * z) * y + (x * (g * z)) * y \\
&(x * (g * y)) * z - x * ((g * y) * z) + (x * g) * (z * y) - x * (g * (z * y)) \\
&- (x * g) * \{y,z\} + x * (g * \{y,z\}) \\
&= x * ((g * z) * y) + x * (g * (y * z)) - x * ((g * y) * z) - x * (g * (z * y)) + x * (g * \{y,z\}) \\
&+ ((x * g) * y) * z - (x * g) * (y * z) - ((x * g) * z) * y + (x * g) * (z * y) - (x * g) * \{y,z\} \\
&= 0 + 0.
\end{aligned}$$

So $V * (T(V)J) = (0)$. As the elements of J are primitive, $T(V)J$ is a coideal. If $n \geq 1$, $x_1, \ldots, x_n \in V$ and $g \in T(V)J$, we put $\Delta_{\sqcup}^{(n-1)}(g) = \sum g^{(1)} \otimes \ldots \otimes g^{(n)}$, with at least one $g_i \in T(V)J$. Then $(x_1 \ldots x_n) * g = \sum (x_1 * g^{(1)}) \ldots (x_n * g^{(n)}) = 0$, so $T(V) * (T(V)J) = (0)$.

If $n \geq 1$, we take $f \in T(V)$, $g \in T(V)JV^{\otimes(n-1)}$ and $y \in V$. We put $g = g_1 g_2 g_3$, with $g_1 \in T(V)$, $g_2 \in J$, $g_3 \in V^{\otimes(n-1)}$. Then:

$$g * y = (g_1 * y) g_2 g_3 + g_1 \underbrace{(g_2 * y)}_{\in J * T(V) \subseteq J} g_3 + g_1 g_2 \underbrace{(g_3 * y)}_{\in V^{\otimes n}} \in T(V)JV^{\otimes n}.$$

So the induction hypothesis holds for g and for $g * y$. Then $f * (gy) = (f * g) * y - f * (g * y) = 0$. So $T(V) * I = (0)$. □

As a consequence, the quotient $T(V)/I$ inherits a magmatic product $*$. Moreover, I is a Hopf ideal, and this implies that it is also a two-sided ideal for $\circledast$. As $T(V)/I$ is the enveloping algebra $\mathcal{U}(V, \{-,-\})$, we obtain Proposition 3.1 of [2]:

Proposition 4 *Let $(\mathfrak{g}, \{-,-\}, *)$ be a post-Lie algebra. Its magmatic product can be uniquely extended to $\mathcal{U}(\mathfrak{g})$ such that for all $f, g, h \in \mathcal{U}(\mathfrak{g})$, $x, y \in \mathfrak{g}$:*

- $f * 1 = f$.
- $1 * f = \varepsilon(f)1$.
- $f * (gy) = (f * g) * y - f * (g * y)$.
- $(fg) * h = \sum \left(f * h^{(1)}\right)\left(g * h^{(2)}\right)$, *where* $\Delta(h) = \sum h^{(1)} \otimes h^{(2)}$ *is the usual coproduct of* $\mathcal{U}(\mathfrak{g})$.

We define a product $\circledast$ on $\mathcal{U}(\mathfrak{g})$ by $f * g = \sum \left(f * g^{(1)}\right) g^{(2)}$. Then $(\mathcal{U}(\mathfrak{g}), \circledast, \Delta)$ is a Hopf algebra, isomorphic to $\mathcal{U}(\mathfrak{g}, \{-,-\}_*)$.

Proof. By Cartier-Quillen-Milnor-Moore's theorem, $(\mathcal{U}(\mathfrak{g}), \circledast, \Delta)$ is an enveloping algebra; the underlying Lie algebra is $Prim(\mathcal{U}(\mathfrak{g})) = \mathfrak{g}$, with the Lie bracket defined by:

$$\{x, y\}_{\circledast} = x \circledast y - y \circledast x = xy + x * y - yx - y * x.$$

This is the bracket $\{-,-\}_*$. □

Remarks.

1. If $\mathfrak{g}$ is a post-Lie algebra with $\{-,-\}=0$, it is a pre-Lie algebra, and $\mathcal{U}(\mathfrak{g})=S(\mathfrak{g})$. We obtain again the Oudom-Guin construction [7, 8].

2. By Lemma 1, $(\mathcal{U}(\mathfrak{g}),*)$ is a right $(\mathcal{U}(\mathfrak{g}),\circledast)$-module. By restriction, $(\mathfrak{g},*)$ is also a right $(\mathcal{U}(\mathfrak{g}),\circledast)$-module.

1.4 The particular case of associative algebras

Let $(V,\triangleleft)$ be an associative algebra. The associated Lie bracket is denoted by $[-,-]_\triangleleft$. As $(V,0,\triangleleft)$ is post-Lie, the construction of the enveloping algebra of $(V,[-,-]_\triangleleft)$ can be done: we obtain a product $\triangleleft$ defined on $S(V)$ and an associative product $\blacktriangleleft$ making $(S(V),\blacktriangleleft,\Delta)$ a Hopf algebra, isomorphic to the enveloping algebra of $(V,[-,-]_\triangleleft)$.

Lemma 3 *If $x_1,\ldots,x_k,y_1,\ldots,y_l\in V$:*

$$x_1\ldots x_k\triangleleft y_1\ldots y_l=\sum_{\theta:[l]\hookrightarrow[k]}\left(\prod_{i\notin Im(\theta)}x_i\right)\left(\prod_{i=1}^k x_{\theta(i)}\triangleleft y_i\right),$$
$$x_1\ldots x_k\blacktriangleleft y_1\ldots y_l=\sum_{I\subseteq[l]}\sum_{\theta:I\hookrightarrow[k]}\left(\prod_{i\notin Im(\theta)}x_i\right)\left(\prod_{j\notin I}y_j\right)\left(\prod_{i\in I}x_{\theta(i)}\triangleleft y_i\right).$$

Proof. We first prove that for all $k\geq 2$, $x,y_1,\ldots,y_k\in V$, $x\triangleleft y_1\ldots y_k=0$. We proceed by induction on k. For $k=2$, $x\triangleleft y_1y_2=(x\triangleleft y_1)\triangleleft y_2-x\triangleleft(y_1\triangleleft y_2)=0$, as $\triangleleft$ is associative. Let us assume the result at rank k. Then:

$$x\triangleleft y_1\ldots y_{k+1}=(x\triangleleft y_1\ldots y_k)\triangleleft y_{k+1}-\sum_{i=1}^k x\triangleleft(y_1\ldots(y_i\triangleleft y_{k+1})\ldots y_k)=0.$$

Let us now prove the formula for $\triangleleft$.

$$x_1\ldots x_k\triangleleft y_1\ldots y_l=\sum_{[l]=I_1\sqcup\ldots\sqcup I_k}\left(x_1\triangleleft\prod_{i\in I_1}y_i\right)\ldots\left(x_k\triangleleft\prod_{i\in I_k}y_i\right).$$

Moreover, for all j:

$$x_j\triangleleft\prod_{i\in I_j}y_i=\begin{cases}x_j\text{ if }I_j=\emptyset,\\ x_j\triangleleft y_p\text{ if }I_j=\{p\},\\ 0\text{ otherwise.}\end{cases}$$

Hence:

$$\begin{aligned}x_1\ldots x_k\triangleleft y_1\ldots y_l&=\sum_{\substack{[l]=I_1\sqcup\ldots\sqcup I_k\\ \forall p,\ |I_p|\leq 1}}\left(x_1\triangleleft\prod_{i\in I_1}y_i\right)\ldots\left(x_k\triangleleft\prod_{i\in I_k}y_i\right)\\&=\sum_{\theta:[l]\hookrightarrow[k]}\left(\prod_{i\notin Im(\theta)}x_i\right)\left(\prod_{i=1}^k x_{\theta(i)}\triangleleft y_i\right).\end{aligned}$$

Finally:

$$x_1\ldots x_k\blacktriangleleft y_1\ldots y_l=\sum_{I\subseteq[l]}\left(\prod_{i\notin I}y_i\right)x_1\ldots x_k\triangleleft\left(\prod_{i\in I}y_i\right),$$

as announced. □

Examples. Let $x_1, x_2, y_2, y_2 \in V$.

$$\begin{aligned}
x_1 \blacktriangleleft y_1 &= x_1y_1 + x_1 \triangleleft y_1,\\
x_1x_2 \blacktriangleleft y_1 &= x_1x_2y_1 + (x_1 \triangleleft y_1)x_2 + x_1(x_2 \triangleleft y_1),\\
x_1 \blacktriangleleft y_1y_2 &= x_1y_1y_2 + (x_1 \triangleleft y_1)y_2 + (x_1 \triangleleft y_2)y_1,\\
x_1x_2 \blacktriangleleft y_1y_2 &= x_1x_2y_1y_2 + (x_1 \triangleleft y_1)x_2y_2 + (x_1 \triangleleft y_2)x_2y_1 + x_1(x_2 \triangleleft y_1)y_2\\
&\quad + x_1(x_2 \triangleleft y_2)y_1 + (x_1 \triangleleft y_1)(x_2 \triangleleft y_2) + (x_1 \triangleleft y_2)(x_2 \triangleleft y_1).
\end{aligned}$$

Remark. The number of terms in $x_1 \ldots x_k \triangleleft y_1 \ldots y_l$ is:

$$\sum_{i=0}^{\min(k,l)} \binom{l}{i}\binom{k}{i} i!,$$

see sequences A086885 and A176120 of [9].

2 A family of solvable Lie algebras

2.1 Definition

Definition 2 *Let us fix $a = (a_1, \ldots, a_N) \in \mathbb{K}^N$. We define an associative product $\triangleleft$ on $\mathbb{K}^N$:*

$$\forall i, j \in [N],\ \epsilon_i \triangleleft \epsilon_j = a_j \epsilon_i.$$

The associated Lie bracket is denoted by $[-,-]_a$:

$$\forall i, j \in [N],\ [\epsilon_i, \epsilon_j]_a = a_j\epsilon_i - a_i\epsilon_j.$$

This Lie algebra is denoted by $\mathfrak{g}_a$.

Remarks.

1. Let $A \in M_{N,M}(\mathbb{K})$, and $a \in \mathbb{K}^N$. The following map is a Lie algebra morphism:

$$\left\{ \begin{array}{rcl} \mathfrak{g}_{a.{}^tA} & \longrightarrow & \mathfrak{g}_a \\ x & \longrightarrow & Ax. \end{array} \right.$$

   Consequently, if $a \neq (0, \ldots, 0)$, g_a is isomorphic to $\mathfrak{g}_{(1,0,\ldots,0)}$.

2. These Lie algebras $\mathfrak{g}_a$ are characterized by the following property: if $\mathfrak{g}$ is a n-dimensional Lie algebra such that any 2-dimensional subspace of $\mathfrak{g}$ is a Lie subalgebra, there exists $a \in \mathbb{K}^n$ such that $\mathfrak{g}$ and $\mathfrak{g}_a$ are isomorphic.

Definition 3 *Let $A = T(V)^N$. The elements of A will be denoted by:*

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} = f_1\epsilon_1 + \ldots + f_N\epsilon_N.$$

For all $i, j \in [N]$, we define bilinear products ${}_i\sqcup\!\sqcup$ and $\sqcup\!\sqcup_j$:

$$\forall f, g \in T(V)^N, \qquad f \ {}_i\sqcup\!\sqcup\ g = \begin{pmatrix} f_i \sqcup\!\sqcup g_1 \\ \vdots \\ f_i \sqcup\!\sqcup g_N \end{pmatrix}, \qquad f \ \sqcup\!\sqcup_j\ g = \begin{pmatrix} f_1 \sqcup\!\sqcup g_j \\ \vdots \\ f_N \sqcup\!\sqcup g_j \end{pmatrix}.$$

In other words, if $f, g \in T(V)$, for all $k, l \in [N]$:

$$f\epsilon_k \ {}_i\sqcup\!\sqcup\ g\epsilon_l = \delta_{i,k}(f \sqcup\!\sqcup g)\epsilon_l, \qquad f\epsilon_k \ \sqcup\!\sqcup_j\ g\epsilon_l = \delta_{j,l}(f \sqcup\!\sqcup g)\epsilon_k.$$

If $a = (a_1, \ldots, a_N) \in \mathbb{K}^N$, we put ${}_a\sqcup\!\sqcup = a_1\, {}_1\sqcup\!\sqcup + \ldots + a_N\, {}_N\sqcup\!\sqcup$ and $\sqcup\!\sqcup_a = a_1 \sqcup\!\sqcup_1 + \ldots + a_N \sqcup\!\sqcup_N$.

Proposition 5 *Let $f, g \in \mathbb{K}^N$. For all $f, g, h \in A$:*

$$(f \sqcup_a g) \sqcup_b h = f \sqcup_a (g \sqcup_b h), \qquad (f \sqcup_a g)\, {}_b\sqcup\, h = f \sqcup_a (g\, {}_b\sqcup\, h),$$
$$(f\, {}_a\sqcup\, g) \sqcup_b h = f\, {}_a\sqcup\, (g \sqcup_b h), \qquad (f\, {}_a\sqcup\, g)\, {}_b\sqcup\, h = f\, {}_a\sqcup\, (g\, {}_b\sqcup\, h),$$
$$f \sqcup_a g = g\, {}_a\sqcup\, f.$$

Proof. Direct verifications, using the associativity and the commutativity of $\sqcup$. □

Definition 4 *Let $a \in \mathbb{K}^N$. We define a Lie bracket on A:*

$$\forall f, g \in A,\ {}_a[f, g] = f\, {}_a\sqcup\, g - g\, {}_a\sqcup\, f = g \sqcup_a f - f \sqcup_a g.$$

This Lie algebra is denoted by $\mathfrak{g}'_a$.

Remark. If A is an associative commutative algebra and $\mathfrak{g}$ is a Lie algebra, then $A \otimes \mathfrak{g}$ is a Lie algebra, with the following Lie bracket:

$$\forall f, g \in A,\ x, y \in \mathfrak{g},\ [f \otimes x, g \otimes y] = fg \otimes [x, y].$$

Then, as a Lie algebra, $\mathfrak{g}'_a$ is isomorphic to the tensor product of the associative commutative algebra $(T(V), \sqcup)$, and of the Lie algebra $\mathfrak{g}_{-a}$. Consequently, if $a \neq (0, \ldots, 0)$, $\mathfrak{g}'_a$ is isomorphic to $\mathfrak{g}'_{(1,0,\ldots,0)}$.

2.2 Enveloping algebra of $\mathfrak{g}_a$

Let us apply Lemma 3 to the Lie algebra $\mathfrak{g}_a$:

Proposition 6 *The symmetric algebra $S(\mathfrak{g}_a)$ is given an associative product $\blacktriangleleft$ such that for all $i_1, \ldots, i_k, j_1, \ldots, j_l \in [N]$:*

$$\epsilon_{i_1} \ldots \epsilon_{i_k} \blacktriangleleft \epsilon_{j_1} \ldots \epsilon_{j_l} = \sum_{I \subseteq [l]} k(k-1) \ldots (k - |I| + 1) \left(\prod_{q \in I} a_{j_q} \right) \left(\prod_{p \notin I} \epsilon_{j_p} \right) \epsilon_{i_1} \ldots \epsilon_{i_k}.$$

The Hopf algebra $(S(\mathfrak{g}_a), \blacktriangleleft, \Delta)$ is isomorphic to the enveloping algebra of $\mathfrak{g}_a$.

The enveloping algebra of $\mathfrak{g}_a$ has two distinguished bases, the Poincaré-Birkhoff-Witt basis and the monomial basis:

$$(\epsilon_{i_1} \blacktriangleleft \ldots \blacktriangleleft \epsilon_{i_k})_{k \geq 0,\, 1 \leq i_1 \leq \ldots \leq i_k \leq N}, \qquad (\epsilon_{i_1} \ldots \epsilon_{i_k})_{k \geq 0,\, 1 \leq i_1 \leq \ldots \leq i_k \leq N}.$$

Here is the passage between them.

Proposition 7 *Let us fix $n \geq 1$. For all $I = \{i_1 < \ldots < i_k\} \subseteq [n]$, we put:*

$$\lambda(I) = (i_1 - 1) \ldots (i_k - k), \qquad \mu(I) = (-1)^k (i_1 - 1) i_2 (i_3 + 1) \ldots (i_k + k - 2).$$

We use the following notation: if $[n] \setminus I = \{q_1 < \ldots < q_l\}$, $\overset{\blacktriangleleft}{\prod_{q \notin I}} \epsilon_{i_q} = \epsilon_{i_{q_1}} \blacktriangleleft \ldots \blacktriangleleft \epsilon_{i_{q_l}}$. Then:

$$\epsilon_{i_1} \blacktriangleleft \ldots \blacktriangleleft \epsilon_{i_n} = \sum_{I \subseteq [n]} \lambda(I) \left(\prod_{p \in I} a_{i_p} \right) \left(\prod_{q \notin I} \epsilon_{i_q} \right),$$
$$\epsilon_{i_1} \ldots \epsilon_{i_n} = \sum_{I \subseteq [n]} \mu(I) \left(\prod_{p \in I} a_{i_p} \right) \left(\overset{\blacktriangleleft}{\prod_{q \notin I}} \epsilon_{i_q} \right).$$

Proof. *First step.* Let us prove the first formula by induction on n. It is obvious if $n = 1$, as $\lambda(\emptyset) = 1$ and $\lambda(\{1\}) = 0$. Let us assume the result at rank n.

$$\begin{aligned}
\epsilon_{i_1} \blacktriangleleft \ldots \blacktriangleleft \epsilon_{i_{n+1}} &= \sum_{I \subseteq [n]} \lambda(I) \left(\prod_{p \in I} a_{i_p}\right) \left(\prod_{q \notin I} \epsilon_{i_q}\right) \blacktriangleleft \epsilon_{i_{n+1}} \\
&= \sum_{I \subseteq [n]} \lambda(I) \left(\prod_{p \in I} a_{i_p}\right) \left(\prod_{q \notin I} \epsilon_{i_q} \epsilon_{i_{n+1}} + (k - |I|) a_{i_{n+1}} \prod_{q \notin I} \epsilon_{i_q}\right) \\
&= \sum_{\substack{I \subseteq [n+1], \\ n+1 \notin I}} \lambda(I) \left(\prod_{p \in I} a_{i_p}\right) \left(\prod_{q \notin I} \epsilon_{i_p}\right) + \sum_{\substack{I \subseteq [n+1], \\ n+1 \in I}} \lambda(I) \left(\prod_{p \in I} a_{i_p}\right) \left(\prod_{q \notin I} \epsilon_{i_p}\right) \\
&= \sum_{I \subseteq [n+1]} \lambda(I) \left(\prod_{p \in I} a_{i_p}\right) \left(\prod_{q \notin I} \epsilon_{i_p}\right).
\end{aligned}$$

Second step. Let us prove that for all $I \subseteq [n]$, $\sum\limits_{J \subseteq I} \lambda(J)\mu(I \setminus J) = \delta_{I,\emptyset}$.

We put $I = \{i_1 < \ldots < i_k\}$ and we proceed by induction on k. As $\lambda(\emptyset) = \mu(\emptyset) = 1$, the result is obvious at rank $k = 0$ and $k = 1$. Let us assume the result at rank $k - 1$, with $k \geq 2$.

$$\begin{aligned}
\sum_{J \subseteq I} \lambda(J)\mu(I \setminus J) &= \sum_{\substack{J \subseteq I, \\ i_k \in J}} \lambda(J)\mu(I \setminus J) + \sum_{\substack{J \subseteq I, \\ i_k \notin J}} \lambda(J)\mu(I \setminus J) \\
&= \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J \cup \{i_k\})\mu(I \setminus \{i_k\} \setminus J) + \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J)\mu(I \setminus J) \\
&= \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J)(i_k - |J|)\mu(I \setminus \{i_k\} \setminus J) \\
&- \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J)\mu(I \setminus \{i_k\} \setminus J)(i_k + |I \setminus \{i_k\} \setminus J| + 1) \\
&= \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J)\mu(I \setminus \{i_k\} \setminus J)(i_k - |J| - i_k - |I| + 1 + |J| - 1) \\
&= -|I| \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J)\mu(I \setminus \{i_k\} \setminus J) \\
&= 0.
\end{aligned}$$

Therefore:

$$\begin{aligned}
\sum_{I \subseteq [n]} \mu(I) \left(\prod_{p \in I} a_{i_p}\right) \left(\prod_{q \notin I}^{\blacktriangleleft} \epsilon_{i_q}\right) &= \sum_{I \subseteq [n]} \sum_{J \subseteq [n] \setminus I} \mu(I)\lambda(J) \left(\prod_{p \in I} a_{i_p}\right) \left(\prod_{p \in J} a_{i_p}\right) \left(\prod_{q \in [n] \setminus I \setminus J} \epsilon_{i_q}\right) \\
&= \sum_{A \sqcup B \sqcup C = [n]} \mu(A)\lambda(B) \left(\prod_{p \in A \sqcup B} a_{i_p}\right) \left(\prod_{q \in C} \epsilon_{i_q}\right) \\
&= \sum_{I \sqcup J = [n]} \left(\underbrace{\sum_{I' \subseteq I} \lambda(I')\mu(I \setminus I')}_{=\delta_{I,\emptyset}}\right) \left(\prod_{p \in I} a_{i_p}\right) \left(\prod_{q \in J} \epsilon_{i_q}\right) \\
&= \epsilon_{i_1} \ldots \epsilon_{i_n},
\end{aligned}$$

which ends the proof. $\square$

2.3 Modules over $\mathfrak{g}_{(1,0,\ldots,0)}$

Proposition 8 *1. Let V be a module over the associative (non unitary) algebra $(\mathfrak{g}_{(1,0,\ldots,0)}, \lhd)$. Then $V = V^{(0)} \oplus V^{(1)}$, with:*

- *$\epsilon_1.v = v$ if $v \in V^{(1)}$ and $\epsilon_1.v = 0$ if $v \in V^{(0)}$.*
- *For all $i \geq 2$, $\epsilon_i.v \in V^{(0)}$ if $v \in V^{(1)}$ and $\epsilon_i.v = 0$ if $i \in V^{(0)}$.*

2. Conversely, let $V = V^{(1)} \oplus V^{(0)}$ be a vector space and let $f_i : V^{(1)} \longrightarrow V^{(0)}$ for all $2 \leq i \leq N$. One defines a structure of $(\mathfrak{g}_{(1,0,\ldots,0)}, \lhd)$-module over V:

$$\epsilon_1.v = \begin{cases} v \text{ if } v \in V^{(1)}, \\ 0 \text{ if } v \in V^{(0)}; \end{cases} \qquad \text{if } i \geq 2,\ \epsilon_i.v = \begin{cases} f_i(v) \text{ if } v \in V^{(1)}, \\ 0 \text{ if } v \in V^{(0)}. \end{cases}$$

Shortly:

$$\epsilon_1 : \begin{bmatrix} 0 & 0 \\ 0 & Id \end{bmatrix}, \qquad \forall i \geq 2,\ \epsilon_i : \begin{bmatrix} 0 & f_i \\ 0 & 0 \end{bmatrix}.$$

Proof. Note that in $\mathfrak{g}_{(1,0,\ldots,0)}$, $\epsilon_i \lhd \epsilon_j = \delta_{1,j}\epsilon_i$.

1. In particular, $\epsilon_1 \lhd \epsilon_1 = \epsilon_1$. If $F_1 : V \longrightarrow V$ is defined by $F_1(v) = \epsilon_1.v$, then:

$$F_1 \circ F_1(v) = \epsilon_1.(\epsilon_1.v) = (\epsilon_1 \lhd \epsilon_1).v = \epsilon.v = F_1(v),$$

so F_1 is a projection, which implies the decomposition of V as $V^{(0)} \oplus V^{(1)}$. Let $x \in V^{(1)}$ and $i \geq 2$. Then $F_1(\epsilon_i.v) = \epsilon_1.(\epsilon_i.v) = (\epsilon_1 \lhd \epsilon_i).v = 0$, so $\epsilon_i.v \in V^{(0)}$. Let $x \in V^{(0)}$. Then $\epsilon_i.v = (\epsilon_i \lhd \epsilon_1).v = \epsilon_i.F_1(v) = 0$, so $\epsilon_i.v = 0$.

2. Let $i \geq 2$ and $j \in [N]$. If $v \in V^{(1)}$:

$$\epsilon_1.(\epsilon_1.v) = v = \epsilon_1.v, \qquad \epsilon_i.(\epsilon_1.v) = f_i(v) = \epsilon_i.v, \qquad \epsilon_j.(\epsilon_i.v) = \epsilon_j.f_i(v) = 0.v.$$

If $v \in V^{(0)}$:

$$\epsilon_1.(\epsilon_1.v) = 0 = \epsilon_1.v, \qquad \epsilon_i.(\epsilon_1.v) = 0 = \epsilon_i.v, \qquad \epsilon_j.(\epsilon_i.v) = 0 = 0.v.$$

So V is indeed a $(\mathfrak{g}_{(1,0,\ldots,0)}, \lhd)$-module. □

Example. There are, up to an isomorphism, three indecomposable $(\mathfrak{g}_{(1,0)}, \lhd)$-modules:

ϵ_1	(0)	(1)	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
ϵ_2	(0)	(0)	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Proposition 9 *(We assume $\mathbb{K}$ algebraically closed). Let V be an indecomposable finite-dimensional module over the Lie algebra $\mathfrak{g}_{(1,0,\ldots,0)}$. There exists a scalar λ and a decomposition:*

$$V = V^{(0)} \oplus \ldots \oplus V^{(k)}$$

such that, for all $0 \leq p \leq k$:

- *$\epsilon_1 \left(V^{(p)}\right) \subseteq V^{(p)}$ and there exists $n \geq 1$ such that $(\epsilon_1 - (\lambda + p)Id)^n_{|V^{(p)}} = (0)$.*
- *If $i \geq 2$, $\epsilon_i \left(V^{(p)}\right) \subseteq V^{(p-1)}$, with the convention $V^{(-1)} = (0)$.*

Proof. First, observe that in the enveloping algebra of $\mathfrak{g}_{(1,0,\ldots,0)}$, if $i \geq 2$ and $\lambda \in \mathbb{K}$:

$$\epsilon_i \blacktriangleleft (\epsilon_1 - \lambda) = \epsilon_i \epsilon_1 + \epsilon_i - \lambda \epsilon_i = \epsilon_i \epsilon_1 + (1-\lambda)\epsilon_i = (\epsilon_1 - \lambda + 1) \blacktriangleleft \epsilon_i.$$

Therefore, for all $i \geq 2$, for all $n \in \mathbb{N}$, for all $\lambda \in \mathbb{K}$:

$$\epsilon_i \blacktriangleleft (\epsilon_1 - \lambda)^{\blacktriangleleft n} = (\epsilon_1 - \lambda + 1)^{\blacktriangleleft n} \blacktriangleleft \epsilon_i.$$

Let V be a finite-dimensional module over the Lie algebra $\mathfrak{g}_{(1,0,\ldots,0)}$. We denote by E_λ the characteristic subspace of eigenvalue λ for the action of ϵ_1. Let us prove that for all $\lambda \in \mathbb{K}$, if $i \geq 2$, $\epsilon_i(E_\lambda) \subseteq E_{\lambda-1}$. If $x \in E_\lambda$, there exists $n \geq 1$, such that $(\epsilon_1 - \lambda Id)^{\blacktriangleleft n}.v = 0$. Hence:

$$0 = \epsilon_i.((\epsilon_1 - \lambda Id)^n.v) = (\epsilon_1 - (\lambda - 1)Id)^n.(\epsilon_i.v),$$

so $\epsilon_i \in E_{\lambda-1}$.

Let us take now V an indecomposable module, and let Λ be the spectrum of the action of ϵ_1. The group $\mathbb{Z}$ acts on $\mathbb{K}$ by translation. We consider $\Lambda' = \Lambda + \mathbb{Z}$ and let Λ'' be a system of representants of the orbits of Λ'. Then:

$$V = \bigoplus_{\lambda \in \Lambda''} \underbrace{\left(\bigoplus_{n \in \mathbb{Z}} E_{\lambda+n} \right)}_{V_\lambda}.$$

By the preceding remarks, V_λ is a module. As V is indecomposable, Λ'' is reduced to a single element. As the spectrum of ϵ_1 is finite, it is included in a set of the form $\{\lambda, \lambda+1, \ldots, \lambda+k\}$. We then take $V^{(p)} = E_{\lambda+p}$ for all p. $\Box$

Example. Let us give the indecomposable modules of $\mathfrak{g}_{(1,0)}$ of dimension ≤ 3. For any $\lambda \in \mathbb{K}$:

ϵ_1	ϵ_2
(λ)	(0)
$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda+1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda+1 & 0 \\ 0 & 0 & \lambda+2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

ϵ_1	ϵ_2
$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda+1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda+1 & 1 \\ 0 & 0 & \lambda+1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Definition 5 *Let V be a module over the Lie algebra $\mathfrak{g}_a$. The associated algebra morphism is:*

$$\phi_V : \left\{ \begin{array}{rcl} \mathcal{U}(\mathfrak{g}_a) = (S(\mathfrak{g}_a), \blacktriangleleft) & \longrightarrow & End(V) \\ \epsilon_i & \longrightarrow & \left\{ \begin{array}{rcl} V & \longrightarrow & V \\ v & \longrightarrow & \epsilon_i.v. \end{array} \right. \end{array} \right.$$

For all $i_1, \ldots, i_k \in [N]$, we put $F_{i_1,\ldots,i_k} = \phi_V(\epsilon_{i_1} \ldots \epsilon_{i_k})$; this does not depend on the order on the indices i_p.

By Proposition 7:

Proposition 10 *For all $i_1,\ldots,i_n\in[N]$:*

$$F_{i_1}\circ\ldots\circ F_{i_n}=\sum_{\substack{I\subseteq[n],\\ I\setminus J=\{j_1<\ldots<j_l\}}}\lambda(I)\left(\prod_{p\in I}a_{i_p}\right)F_{i_{j_1},\ldots,i_{j_l}},$$

$$F_{i_1,\ldots,i_n}=\sum_{\substack{I\subseteq[n],\\ I\setminus J=\{j_1<\ldots<j_l\}}}\mu(I)\left(\prod_{p\in I}a_{i_p}\right)F_{i_{j_1}}\circ\ldots\circ F_{i_{j_l}}.$$

When V is a module over the associative algebra $(\mathfrak{g}_A,\triangleleft)$, these morphisms are easy to describe:

Proposition 11 *Let V be a module over the associative algebra $(\mathfrak{g}_a,\triangleleft)$; it is also a module over the Lie algebra $(\mathfrak{g}_a,[-,-]_a)$. For all $k\geq 2$, $i_1,\ldots,i_k\in[N]$, $F_{i_1,\ldots,i_k}=0$.*

Proof. As V is a module over the associative algebra $(\mathfrak{g}_a,\triangleleft)$, for any $i_1,i_2\in[N]$:

$$F_{i_1}\circ F_{i_2}=a_{i_2}F_{i_1}.$$

We proceed by induction on k. If $k=2$, $\epsilon_{i_1}\epsilon_{i_2}=\epsilon_{i_1}\blacktriangleleft\epsilon_{i_2}-a_{i_2}\epsilon_{i_1}$, so:

$$F_{i_1,i_2}=F_{i_1}\circ F_{i_2}-a_{i_2}F_{i_1}=a_{i_2}F_{i_1}-a_{i_2}F_{i_1}=0.$$

Let us assume the result at rank k. Then $\epsilon_1\ldots\epsilon_{i_{k+1}}=\epsilon_{i_1}\ldots\epsilon_{i_k}\blacktriangleleft\epsilon_{i_{k+1}}-ka_{i_{k+1}}\epsilon_{i_1}\ldots\epsilon_{i_k}$, and $F_{i_1,\ldots,i_{k+1}}=F_{i_1,\ldots,i_k}\circ F_{i_{k+1}}-ka_{i_{k+1}}F_{i_1,\ldots,i_k}=0$. $\square$

3 A family of post-Lie algebras

3.1 Reminders

We defined in [3] a family of pre-Lie algebras, associated to endomorphisms. Let us briefly recall this construction.

Proposition 12 *Let V be a vector space and $F:V\longrightarrow V$ be an endomorphism. We define a product $*$ on $T(V)$: for all $f,g\in T(V)$, for all $x\in V$,*

$$\emptyset * g=0,\qquad\qquad xf*g=x(f*g)+F(x)(f\sqcup g).$$

This product is pre-Lie. The pre-Lie algebra $(T(V),*)$ is denoted by $T(V,F)$. Moreover, for all $f,g,h\in T(V,F)$:

$$(f\sqcup g)*h=(f*h)\sqcup g+f\sqcup(g*h).$$

We also proved the following result:

Proposition 13 *Let $k,l\geq 0$.*

- *The set $Sh(k,l)$ of (k,l)-shuffles is the set of permutation $\sigma\in\mathfrak{S}_{k+l}$ such that $\sigma(1)<\ldots<\sigma(k)$ and $\sigma(k+1)<\ldots<\sigma(k+l)$.*
- *If $\sigma\in Sh(k,l)$, we put $m_k(\sigma)=\max\{i\in[k]\mid\sigma(1)=1,\ldots,\sigma(i)=i\}$. In particular, if $\sigma(1)\neq 1$, $m_k(\sigma)=0$.*

For all $x_1,\ldots,x_k$, $y_1,\ldots,y_l\in V$:

$$x_1\ldots x_k*y_1\ldots y_l=\sum_{\sigma\in Sh(k,l)}\sum_{p=1}^{m_k(\sigma)}\left(Id^{\otimes(p_1)}\otimes F\otimes Id^{\otimes(k+l-p)}\right)\sigma.(x_1\ldots x_ky_1\ldots y_l).$$

3.2 Construction

Let us fix a vector space V, a family of N endomorphisms $(F_1,\ldots,F_N)$ of V and $a=(a_1,\ldots,a_N)\in\mathbb{K}^N$. We define inductively a product $*$ on $T(V)^N$: for all $f,g\in T(V)^N$, $x\in V$, $i\in[N]$,

$$\emptyset\epsilon_i * g = 0, \qquad xf*g = x(f*g)+F_1(x)(f\sqcup_1 g)+\ldots+F_N(x)(f\sqcup_N g).$$

We define a second product $\bullet$ on $T(V)^N$:

$$\forall f,g\in T(V)^N,\ f\bullet g = f*g+f\ {}_a\sqcup\ g.$$

Examples. Let $x,y,z\in V$, $g\in T(V)$, $i,j\in[N]$. Then:

$$\begin{aligned}
x\epsilon_i * g\epsilon_j &= F_j(x)g\epsilon_j,\\
xy\epsilon_i * g\epsilon_j &= (xF_j(y)g+F_j(x)(y\sqcup g))\epsilon_i,\\
xyz\epsilon_i * g\epsilon_j &= (xyF_j(z)g+xF_j(y)(z\sqcup g)+F_j(x)(yz\sqcup g))\epsilon_i.
\end{aligned}$$

Proposition 14 *Let $x_1,\ldots,x_k,y_1,\ldots,y_l\in V$, $i,j\in[N]$.*

$$x_1\ldots x_k\epsilon_i * y_1\ldots y_l\epsilon_j = \sum_{\sigma\in Sh(k,l)}\sum_{p=1}^{m_k(\sigma)}\left(Id^{\otimes(p_1)}\otimes F_j\otimes Id^{\otimes(k+l-p)}\right)\sigma.(x_1\ldots x_ky_1\ldots y_l)\epsilon_i.$$

Proof. By induction on k. It is immediate if $k=0$, as both sides are equal to 0. Let us assume the result at rank $k-1$.

$$\begin{aligned}
x_1\ldots x_k\epsilon_i * y_1\ldots y_l\epsilon_j &= x_1(x_2\ldots x_k\epsilon_i * y_1\ldots y_l\epsilon_j)+F_j(x_1)(x_2\ldots x_k\sqcup y_1\ldots y_l)\epsilon_i\\
&= \sum_{\substack{\sigma\in Sh(k,l),\\ \sigma(1)=1}}\sum_{p=2}^{m_k(\sigma)}(Id^{\otimes(p_1)}\otimes F_j\otimes Id^{\otimes(k+l-p)})\sigma.(x_1\ldots x_ky_1\ldots y_l)\epsilon_i\\
&+\sum_{\substack{\sigma\in Sh(k,l),\\ \sigma(1)=1}}(F_j\otimes Id^{\otimes(k+l-1)})\sigma.(x_1\ldots x_ky_1\ldots y_l)\epsilon_i\\
&= \sum_{\substack{\sigma\in Sh(k,l),\\ \sigma(1)=1}}\sum_{p=1}^{m_k(\sigma)}(Id^{\otimes(p_1)}\otimes F_j\otimes Id^{\otimes(k+l-p)})\sigma.(x_1\ldots x_ky_1\ldots y_l)\epsilon_i\\
&= \sum_{\sigma\in Sh(k,l)}\sum_{p=1}^{m_k(\sigma)}(Id^{\otimes(p_1)}\otimes F_j\otimes Id^{\otimes(k+l-p)})\sigma.(x_1\ldots x_ky_1\ldots y_l)\epsilon_i,
\end{aligned}$$

so the result holds for all k. □

Remark. Let $*_j$ be the pre-Lie product of $T(V,F_j)$, described in [3]. For all $f,g\in T(V)$, for all $i,j\in[N]$:

$$f\epsilon_i * g\epsilon_j = (f *_j g)\epsilon_i.$$

Corollary 2 *For all $f,g,h\in T(V)^N$, for all $i\in[N]$:*

$$\begin{aligned}
(f\ {}_i\sqcup\ g)*h &= (f*h)\ {}_i\sqcup\ g+f\ {}_i\sqcup\ (g*h),\\
(f\sqcup_i g)*h &= (f*h)\sqcup_i g+f\sqcup_i(g*h),\\
(f\sqcup g)*h &= (f*h)\sqcup g+f\sqcup(g*h).
\end{aligned}$$

Proof. It is enough to prove these assertions for $f = f'\epsilon_k$, $g = g'\epsilon_l$ and $h = h'\epsilon_m$, with $f', g', h' \in T(V)$. For the first assertion:

$$\begin{aligned}
(f \;{}_i\!\sqcup\!\sqcup\; g) * h &= \delta_{i,k}(f' \sqcup\!\sqcup\, g'\epsilon_l) * h'\epsilon_m \\
&= \delta_{i,k}(f' \sqcup\!\sqcup\, g') *_m h'\epsilon_l \\
&= \delta_{i,k}((f' *_m h') \sqcup\!\sqcup\, g' + f' \sqcup\!\sqcup\, (g' *_m h'))\epsilon_l \\
&= (f * h) \;{}_i\!\sqcup\!\sqcup\; g + f \;{}_i\!\sqcup\!\sqcup\; (g * h).
\end{aligned}$$

The second point is deduced from the first one, as $\sqcup\!\sqcup_i = {}_i\!\sqcup\!\sqcup^{op}$. Finally:

$$\begin{aligned}
(f \;{}_i\!\sqcup\!\sqcup\; g) * h &= \delta_{k,l}(f' \sqcup\!\sqcup\, g'\epsilon_l) * h'\epsilon_m \\
&= \delta_{k,l}(f' \sqcup\!\sqcup\, g') *_m h'\epsilon_l \\
&= \delta_{k,l}((f' *_m h') \sqcup\!\sqcup\, g' + f' \sqcup\!\sqcup\, (g' *_m h'))\epsilon_l \\
&= (f * h) \sqcup\!\sqcup\, g + f \sqcup\!\sqcup\, (g * h).
\end{aligned}$$

So the last point holds. □

Theorem 3 *The following conditions are equivalent:*

1. $(T(V)^N, \bullet)$ *is a pre-Lie algebra.*
2. $\mathfrak{g}'_a = (T(V)^N, {}_a[-,-], *)$ *is a post-Lie algebra.*
3. V *is a module over the Lie algebra* $\mathfrak{g}_a$, *with the action given by* $\epsilon_i.v = F_i(v)$.

Proof. By Corollary 2, for all $f, g, h \in \mathfrak{g}'_a$, ${}_a[f,g] * h = {}_a[f * h, g] + {}_a[f, g * h]$.

1. $\Longleftrightarrow$ 2. Let $f, g, h \in \mathfrak{g}$.

$$\begin{aligned}
&(f \bullet g) \bullet h - f \bullet (g \bullet h) - (f \bullet h) \bullet g + f \bullet (h \bullet g) \\
&= (f * g) * h - f * (g * h) - (f * h) * g - f * (h * g) \\
&+ (f \;{}_a\!\sqcup\!\sqcup\; g) * h - f \;{}_a\!\sqcup\!\sqcup\; (g * h) - (f \;{}_a\!\sqcup\!\sqcup\; h) * g + f \;{}_a\!\sqcup\!\sqcup\; (h * g) \\
&+ (f * g) \;{}_a\!\sqcup\!\sqcup\; h - f * (g \;{}_a\!\sqcup\!\sqcup\; h) - (f * h) \;{}_a\!\sqcup\!\sqcup\; g + f * (h \;{}_a\!\sqcup\!\sqcup\; g) \\
&+ (f \;{}_a\!\sqcup\!\sqcup\; g) \;{}_a\!\sqcup\!\sqcup\; h - f \;{}_a\!\sqcup\!\sqcup\; (g \;{}_a\!\sqcup\!\sqcup\; h) - (f \;{}_a\!\sqcup\!\sqcup\; g) \;{}_a\!\sqcup\!\sqcup\; h + f \;{}_a\!\sqcup\!\sqcup\; (g \;{}_a\!\sqcup\!\sqcup\; h) \\
&= (f * g) * h - f * (g * h) - (f * h) * g - f * (h * g) \\
&+ f * (g \;{}_a\!\sqcup\!\sqcup\; h) - f * (h \;{}_a\!\sqcup\!\sqcup\; g) \\
&+ [(f \;{}_a\!\sqcup\!\sqcup\; g) * h - f \;{}_a\!\sqcup\!\sqcup\; (g * h) - (f * h) \;{}_a\!\sqcup\!\sqcup\; g] \\
&+ [(f \;{}_a\!\sqcup\!\sqcup\; h) * g - f \;{}_a\!\sqcup\!\sqcup\; (h * g) - (f * g) \;{}_a\!\sqcup\!\sqcup\; h] \\
&- [(f * h) \;{}_a\!\sqcup\!\sqcup\; g - f \;{}_a\!\sqcup\!\sqcup\; (h * g) - (f * g) \;{}_a\!\sqcup\!\sqcup\; h] \\
&+ [(f \;{}_a\!\sqcup\!\sqcup\; g) \;{}_a\!\sqcup\!\sqcup\; h - f \;{}_a\!\sqcup\!\sqcup\; (g \;{}_a\!\sqcup\!\sqcup\; h)] - [(f \;{}_a\!\sqcup\!\sqcup\; g) \;{}_a\!\sqcup\!\sqcup\; h - f \;{}_a\!\sqcup\!\sqcup\; (g \;{}_a\!\sqcup\!\sqcup\; h)] \\
&= (f * g) * h - f * (g * h) - (f * h) * g + f * (h * g) - f * {}_a[g, h].
\end{aligned}$$

So $(\mathfrak{g}'_a, \bullet)$ is pre-Lie if, and only if, $(\mathfrak{g}'_a, {}_a[-,-], *)$ is post-Lie.

2. $\Longrightarrow$ 3. Let $x, y, v \in V$ and $i, j, k \in [N]$. Then:

$$\begin{aligned}
x\epsilon_i * y\epsilon_j &= F_j(x)y\epsilon_i, & xy\epsilon_i * z\epsilon_k &= xF_k(y)z\epsilon_i + F_k(x)(y \sqcup\!\sqcup\, z)\epsilon_i. \\
x\epsilon_i * yz\epsilon_k &= F_k(x)yz\epsilon_i,
\end{aligned}$$

Hence:

$$\begin{aligned}
(x\epsilon_i * y\epsilon_j) * z\epsilon_k &= F_j(x) y\epsilon_i * z\epsilon_k \\
&= F_j(x)F_k(y)z\epsilon_i + F_k \circ F_j(x) y \sqcup z\epsilon_i, \\
x\epsilon_i * (y\epsilon_j * z\epsilon_k) &= x\epsilon_i * F_k(y)z\epsilon_j \\
&= F_j(x)F_k(y)z\epsilon_i, \\
x\epsilon_i \,{}_a[y\epsilon_j, z\epsilon_k] &= a_j x\epsilon_i * (y \sqcup z)\epsilon_k - a_k x\epsilon_i * (y \sqcup z)\epsilon_j \\
&= (a_j F_k(x)(y \sqcup z) - a_k F_j(x)(y \sqcup z))\epsilon_i.
\end{aligned}$$

The post-Lie relation (2) gives:

$$\begin{aligned}
&(a_j F_k(x) - a_k F_j(x))(y \sqcup z) \\
&= F_j(x)F_k(y)z + F_k \circ F_j(x)(y \sqcup z) - F_j(x)F_k(y)z - F_j \circ F_k(x)(y \sqcup z) \\
&= (F_j \circ F_k - F_k \circ F_j)(x)(y \sqcup z).
\end{aligned}$$

Let $y = z$ be a nonzero element of V. Then $y \sqcup z \neq 0$, and we obtain that for all $x \in V$, $a_j F_k(x) - a_k F_j(x) = (F_j \circ F_k - F_k \circ F_j)(x)$: V is a $\mathfrak{g}_a$-module.

3. $\Longrightarrow$ 2. Let us prove the post-Lie relation (2) for $f\epsilon_i$, g and h, with $f \in T(V)$, $i \in [N]$, $g, h \in \mathfrak{g}'_a$. We assume that f is a word and we proceed by induction on the length n of f. If $n = 0$, then $f = \emptyset$ and every term is 0 in the relation. Let us assume the result at rank $n-1$. We put $f = xf'$, with $x \in V$, and f' a word of length $n-1$.

$$\begin{aligned}
(f * g) * h &= x((f'\epsilon_i * g) * h) + \sum_{p=1}^{N} F_p(x)((f'\epsilon_i * g) \sqcup_p h) \\
&\quad + \sum_{p=1}^{N} F_p(x)((f'\epsilon_i \sqcup_p g) * h + \sum_{p,q=1}^{N} F_q \circ F_p(x)(f'\epsilon_i \sqcup_p g \sqcup_q h), \\
f * (g * h) &= x(f'\epsilon_i * (g * h)) + \sum_{p=1}^{N} F_p(x)(f'\epsilon_i \sqcup_p (g * h)), \\
\sum_{p=1}^{N} a_p f * (g \,{}_p\!\sqcup h) &= \sum_{p=1}^{N} a_p x(f'\epsilon_i * (g \,{}_p\!\sqcup h)) + \sum_{p,q=1}^{N} a_p F_q(x)(f'\epsilon_i \sqcup_q (g \,{}_p\!\sqcup h)).
\end{aligned}$$

We put:

$$P(f, g, h) = f * (g * h) - (f * g) * h + \sum_{p=1}^{N} a_p f * (g \,{}_p\!\sqcup h).$$

In order to prove the post-Lie relation (2), we have to prove that $P(f, g, h) = P(f, h, g)$. First:

$$\begin{aligned}
(f * g) * h &= x((f'\epsilon_i * g) * h) + \sum_{p=1}^{N} F_p(x)((f'\epsilon_i * g) \sqcup_p h) \\
&\quad + \sum_{p=1}^{N} F_p(x)((f'\epsilon_i \sqcup_p g) * h + \sum_{p,q=1}^{N} F_q \circ F_p(x)(f'\epsilon_i \sqcup_p g \sqcup_q h), \\
f * (g * h) &= x(f'\epsilon_i * (g * h)) + \sum_{p=1}^{N} F_p(x)(f'\epsilon_i \sqcup_p (g * h)), \\
\sum_{p=1}^{N} a_p f * (g \,{}_p\!\sqcup h) &= \sum_{p=1}^{N} a_p x(f'\epsilon_i * (g \,{}_p\!\sqcup h)) + \sum_{p,q=1}^{N} a_p F_q(x)(f'\epsilon_i \sqcup_q (g \,{}_p\!\sqcup h)).
\end{aligned}$$

Consequently:

$$\begin{aligned}
P(f,g,h) &= xP(f',g,h)\\
&\quad+\sum_{p=1}^{N} F_p(x)(-(f'\epsilon_i * g)\sqcup_p h - ((f'\epsilon_i \sqcup_p g) * h + f'\epsilon_i \sqcup_p (g*h))\\
&\quad+\sum_{p,q=1}^{N} a_p F_q(x)(f'\epsilon_i \sqcup_q g\, {}_p\sqcup h) - F_q\circ F_p(x)(f'\epsilon_i \sqcup_p g \sqcup_q h)\\
&= xP(f',g,h) - \sum_{p=1}^{N} F_p(x)((f'\epsilon_i * g)\sqcup_p h + (f'\epsilon_i * h)\sqcup_p g)\\
&\quad+\sum_{p,q=1}^{N} a_p F_q(x)(f'\epsilon_i \sqcup_q h \sqcup_p g) - F_q\circ F_p(x)(f'\epsilon_i \sqcup_p g \sqcup_q h)\\
&= xP(f',g,h) - \sum_{p=1}^{N} F_p(x)(f'\epsilon_i * g)\sqcup_p h + (f'\epsilon_i * h)\sqcup_p g)\\
&\quad+\sum_{p,q=1}^{N} (a_p F_q(x) - F_q\circ F_p(x))(f'\epsilon_i \sqcup_p g \sqcup_q h).
\end{aligned}$$

By the induction hypothesis, $P(f',g,h) = P(f',h,g)$, so the first row is symmetric in g,h. As V is a $\mathfrak{g}_a$-module, $a_pF_q - F_q\circ F_p = a_qF_p - F_p\circ F_q$, so the second row is symmetric in g,h, and finally $P(f,g,h) = P(f,h,g)$: $\mathfrak{g}'_a$ is a post-Lie algebra. □

Example. The post-Lie algebra $\mathfrak{g}_{SISO}$ is associated to $a=(1,0)$, $V = Vect(x_1,x_2)$ and:

$$F_1 = \begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}, \qquad F_2 = \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}.$$

As F_1 and F_2 define a module over the Lie algebra $\mathfrak{g}_{(1,0)}$, even in fact over the associative algebra $(\mathfrak{g}_{(1,0)},\triangleleft)$, we obtain indeed a post-Lie algebra. For all $f,g\in T(V)$, for all $i,j\in\{1,2\}$:

$$\begin{aligned}
\emptyset\epsilon_i * g\epsilon_j &= 0, & x_2 f\epsilon_i * g\epsilon_1 &= x_2(f\epsilon_i * g\epsilon_1) + x_2(f\sqcup g)\epsilon_i,\\
x_1 f\epsilon_i * g\epsilon_j &= x_1(f\epsilon_i * g\epsilon_j), & x_2 f\epsilon_i * g\epsilon_2 &= x_2(f\epsilon_i * g\epsilon_2) + x_1(f\sqcup g)\epsilon_i.
\end{aligned}$$

3.3 Extension of the post-Lie product

We now extend the post-Lie product of $\mathfrak{g}'_a$ to the enveloping algebra $\mathcal{U}(\mathfrak{g}'_a)$. As this Lie bracket is obtained from an associative product $\triangleleft = {}_a\sqcup$, we can see $\mathcal{U}(\mathfrak{g}'_a)$ as $(S(\mathfrak{g}'_a),\blacktriangleleft,\Delta)$. The post-Lie product $*$ is extended to $\mathcal{U}(\mathfrak{g}'_a)$, and we obtain a Hopf algebra $(\mathcal{U}(\mathfrak{g}),\circledast,\Delta)$, isomorphic to $\mathcal{U}(\mathfrak{g}'_a,\ {}_a[-,-]_*)$, with:

$$\forall f,g\in\mathfrak{g},\ {}_a[f,g]_* = {}_a[f,g] + f*g - g*f = f\ {}_a\sqcup\ g + f*g - g\ {}_a\sqcup\ f - g*f.$$

As $\bullet$ is a pre-Lie product, it can also be extended to $S(\mathfrak{g})$ and gives a product $\odot$, making $S(\mathfrak{g}'_a)$ a Hopf algebra isomorphic to $\mathcal{U}(\mathfrak{g}'_a,[-,-]_\bullet)$.

Remark. Let $f,g\in\mathfrak{g}'_a$.

$$\begin{aligned}
[f,g]_\bullet &= f\bullet g - g\bullet f\\
&= f\ {}_a\sqcup\ g + f*g - g\ {}_a\sqcup\ f - g*f\\
&= {}_a[f,g] + f*g - g*f\\
&= {}_a[f,g]_*.
\end{aligned}$$

So $[-,-]_\bullet = {}_a[-,-]_*$.

Lemma 4 *Let $f_1,\ldots,f_k,g\in\mathfrak{g}'_a$, $k\geq 1$.*

$$(f_1\blacktriangleleft\ldots\blacktriangleleft f_k)*g=\sum_{p=1}^k f_1\blacktriangleleft\ldots\blacktriangleleft(f_p*g)\blacktriangleleft\ldots\blacktriangleleft f_k,$$

$$(f_1\ldots f_k)*g=\sum_{p=1}^k f_1\ldots(f_p*g)\ldots f_k.$$

Proof. The first point comes by the very definition of $*$. For the second point, we proceed by induction on k. This is obvious if $k=1$. Let us assume the result at rank k, $k\geq 1$. Observe that:

$$f_1\ldots f_{k+1}=f_1\ldots f_k\blacktriangleleft f_{k+1}-\sum_{p=1}^k f_1\ldots(f_p\,{}_a\sqcup\!\sqcup\, f_{k+1})\ldots f_k,$$

so:

$$\begin{aligned}
f_1\ldots f_{k+1}*g&=(f_1\ldots f_k*g)\blacktriangleleft f_{k+1}+f_1\ldots f_k\blacktriangleleft(f_{k+1}*g)-\sum_{p=1}^k f_1\ldots(f_p\,{}_a\sqcup\!\sqcup\, f_{k+1})\ldots f_k*g\\
&=\sum_{p=1}^k f_1\ldots(f_p*g)\ldots f_k\blacktriangleleft f_{k+1}+f_1\ldots f_k\blacktriangleleft(f_{k+1}*g)\\
&-\sum_{p\neq q} f_1\ldots(f_p\,{}_a\sqcup\!\sqcup\, f_{k+1})\ldots(f_q*g)\ldots f_k-\sum_{p=1}^k f_1\ldots((f_p\,{}_a\sqcup\!\sqcup\, f_{k+1})*g)\ldots f_k\\
&=\sum_{p=1}^k f_1\ldots(f_p*g)\ldots f_kf_{k+1}+\sum_{p\neq q} f_1\ldots(f_p\,{}_a\sqcup\!\sqcup\, f_{k+1})\ldots(f_q*g)\ldots f_k\\
&+\sum_{p=1}^k f_1\ldots((f_p*g)\,{}_a\sqcup\!\sqcup\, f_{k+1})\ldots f_k-\sum_{p\neq q} f_1\ldots(f_p\,{}_a\sqcup\!\sqcup\, f_{k+1})\ldots(f_q*g)\ldots f_k\\
&-\sum_{p=1}^k f_1\ldots((f_p*g)\,{}_a\sqcup\!\sqcup\, f_{k+1})\ldots f_k-\sum_{p=1}^k f_1\ldots(f_p\,{}_a\sqcup\!\sqcup\,(f_{k+1}*g))\ldots f_k\\
&+f_1\ldots f_k(f_{k+1}*g)+\sum_{p=1}^k f_1\ldots(f_p\,{}_a\sqcup\!\sqcup\,(f_{k+1}*g))\ldots f_k\\
&=\sum_{p=1}^{k+1} f_1\ldots(f_p*g)\ldots f_{k+1}.
\end{aligned}$$

Finally, the result holds for all $k\geq 1$. $\Box$

The following result allows to compute $f*g_1\ldots g_k$ by induction on the length of f:

Proposition 15 *Let $x\in V$, $k\geq 1$, $f,g_1,\ldots,g_k\in T(V)^N$, $i\in[N]$.*

$$\emptyset\epsilon_i*(g_1\blacktriangleleft\ldots\blacktriangleleft g_k)=0,$$

$$xf*(g_1\blacktriangleleft\ldots\blacktriangleleft g_k)=\sum_{\substack{I=\{i_1<\ldots<i_l\}\subseteq[k],\\ j_1,\ldots,j_l\in[N]}} F_{j_l}\circ\ldots\circ F_{j_1}(x)\left(\left(f*\overset{\blacktriangleleft}{\prod_{i\notin I}}g_i\right)\sqcup\!\sqcup_{j_1}\,g_{i_1}\ldots\sqcup\!\sqcup_{j_l}\,g_{i_l}\right);$$

$$\emptyset\epsilon_i*(g_1\ldots g_k)=0,$$

$$xf*(g_1\ldots g_k)=\sum_{\substack{I=\{i_1<\ldots<i_l\}\subseteq[k],\\ j_1,\ldots,j_l\in[N]}} F_{j_1,\ldots,j_l}(x)\left(\left(f*\prod_{i\notin I}g_i\right)\sqcup\!\sqcup_{j_1}\,g_{i_1}\ldots\sqcup\!\sqcup_{j_l}\,g_{i_l}\right).$$

Proof. In order to ease the redaction, we put:

$$I_k = \{(I, j_1, \ldots, j_l) \mid I = \{i_1 < \ldots < i_l\} \subseteq [k],\ j_1, \ldots, j_l \in [N]\}.$$

We proceed by induction on k. It is immediate if $k = 1$. Let us assume the result at rank k, $k \geq 1$. Then:

$$\begin{aligned}
\emptyset\epsilon_i * (g_1 \blacktriangleleft \ldots \blacktriangleleft g_{k+1}) &= (\emptyset\epsilon_i * (g_1 \blacktriangleleft \ldots \blacktriangleleft g_k)) * g_{k+1} \\
&- \sum_{p=1}^{k} \emptyset\epsilon_i * (g_1 \blacktriangleleft \ldots \blacktriangleleft (g_p * g_{k+1}) \blacktriangleleft \ldots \blacktriangleleft g_k) \\
&= 0.
\end{aligned}$$

Moreover:

$$\begin{aligned}
&xf * (g_1 \blacktriangleleft \ldots \blacktriangleleft g_{k+1}) \\
&= (xf * (g_1 \blacktriangleleft \ldots \blacktriangleleft g_k)) * g_{k+1} - \sum_{p=1}^{k} xf * (g_1 \blacktriangleleft \ldots \blacktriangleleft (g_p * g_{k+1}) \blacktriangleleft \ldots \blacktriangleleft g_k \\
&= \sum_{I_k} F_{j_l} \circ \ldots \circ F_{j_1}(x) \left(\left(f * \overset{\blacktriangleleft}{\prod_{i \notin J \sqcup \{k+1\}}} g_i \right) ⧢_{j_1} g_{i_1} \ldots ⧢_{j_l} g_{i_l} \right) * g_{k+1} \\
&- \sum_{I_k} F_{j_l} \circ \ldots \circ F_{j_1}(x) \left(f * \left(\left(\overset{\blacktriangleleft}{\prod_{i \notin J \sqcup \{k+1\}}} g_i \right) * g_{k+1} \right) ⧢_{j_1} g_{i_1} \ldots ⧢_{j_l} g_{i_l} \right) \\
&- \sum_{p=1}^{k} \sum_{I_k} F_{j_l} \circ \ldots \circ F_{j_1}(x) \left(\left(f * \overset{\blacktriangleleft}{\prod_{i \notin J \sqcup \{k+1\}}} g_i \right) ⧢_{j_1} g_{i_1} \ldots ⧢_{j_p} (g_{i_p} * g_{k+1}) \ldots ⧢_{j_l} g_{i_l} \right) \\
&= \sum_{I_k} \sum_{j_{l+1} \in [N]} F_{j_l} \circ \ldots \circ F_{j_1}(x) \left(\left(f * \overset{\blacktriangleleft}{\prod_{i \notin J \sqcup \{k+1\}}} g_i \right) ⧢_{j_1} g_{i_1} \ldots ⧢_{j_l} g_{i_l} ⧢_{j_{l+1}} g_{k+1} \right) \\
&+ \sum_{I_k} F_{j_l} \circ \ldots \circ F_{j_1}(x) \\
&\qquad \left(\left(\left(f * \overset{\blacktriangleleft}{\prod_{i \notin J \sqcup \{k+1\}}} g_i \right) * g_{k+1} - f * \left(\left(\overset{\blacktriangleleft}{\prod_{i \notin J \sqcup \{k+1\}}} g_i \right) * g_{k+1} \right) \right) ⧢_{j_1} g_{i_1} \ldots ⧢_{j_l} g_{i_l} \right) \\
&= \sum_{I_k} \sum_{j_{l+1} \in [N]} F_{j_l} \circ \ldots \circ F_{j_1}(x) \left(\left(f * \overset{\blacktriangleleft}{\prod_{i \notin J \sqcup \{k+1\}}} g_i \right) ⧢_{j_1} g_{i_1} \ldots ⧢_{j_l} g_{i_l} ⧢_{j_{l+1}} g_{k+1} \right) \\
&+ \sum_{I_k} F_{j_l} \circ \ldots \circ F_{j_1}(x) \left(\left(f * \overset{\blacktriangleleft}{\prod_{i \notin J \sqcup \{k+1\}}} g_i \blacktriangleleft g_{k+1} \right) ⧢_{j_1} g_{i_1} \ldots ⧢_{j_l} g_{i_l} ⧢_{j_l} g_{k+1} \right) \\
&= \sum_{I_{k+1},\, k+1 \in I} F_{j_l} \circ \ldots \circ F_{j_1}(x) \left(\left(f * \overset{\blacktriangleleft}{\prod_{i \notin I}} g_i \right) ⧢_{j_1} g_{i_1} \ldots ⧢_{j_l} g_{i_l} \right) \\
&+ \sum_{I_{k+1},\, k+1 \notin I} F_{j_l} \circ \ldots \circ F_{j_1}(x) \left(\left(f * \overset{\blacktriangleleft}{\prod_{i \notin I}} g_i \right) ⧢_{j_1} g_{i_1} \ldots ⧢_{j_l} g_{i_l} \right) \\
&= \sum_{I_{k+1}} F_{j_l} \circ \ldots \circ F_{j_1}(x) \left(\left(f * \overset{\blacktriangleleft}{\prod_{i \notin I}} g_i \right) ⧢_{j_1} g_{i_1} \ldots ⧢_{j_l} g_{i_l} \right).
\end{aligned}$$

So, for all $F \in \mathcal{U}(\mathfrak{g})_+$, $\emptyset\epsilon_i * F = 0$. As $g_1 \dots g_k \in \mathcal{U}(\mathfrak{g})_+$, the first point holds. Let us prove the second point by induction on k. The result is immediate if $k = 1$. Let us assume the result at rank $k \geq 1$.

$$
\begin{aligned}
&xf * g_1 \dots g_{k+1} \\
&= xf * (g_1 \dots g_k * g_{k+1}) - \sum_{p=1}^{k} xf * (g_1 \dots (g_p \ {}_a\sqcup\!\sqcup\ g_{k+1}) \dots g_k) \\
&= (xf * g_1 \dots g_k) * g_{k+1} \sum_{p=1}^{k} xf * (g_1 \dots (g_p \ {}_a\sqcup\!\sqcup\ g_{k+1}) \dots g_k) - \sum_{p=1}^{k} xf * (g_1 \dots (g_p * g_{k+1}) \dots g_k) \\
&= \sum_{I_k} F_{j_1,\dots,j_l}(x) \left(\left(\left(f * \prod_{i \notin J \sqcup \{k+1\}} g_i \right) * g_{k+1} \right) \sqcup\!\sqcup_{j_1} g_{i_1} \dots \sqcup\!\sqcup_{j_l} g_{i_l} \right) \\
&+ \sum_{I_k} \sum_{p=1}^{k} F_{j_1,\dots,j_l}(x) \left(\left(f * \prod_{i \notin J \sqcup \{k+1\}} g_i \right) \sqcup\!\sqcup_{j_1} g_{i_1} \dots \sqcup\!\sqcup_{j_p} (g_{i_p} * g_{k+1}) \dots \sqcup\!\sqcup_{j_l} g_{i_l} \right) \\
&+ \sum_{I_k} \sum_{j_{l+1} \in [N]} F_{j_{l+1}} \circ F_{j_1,\dots,j_l}(x) \left(\left(f * \prod_{i \notin J \sqcup \{k+1\}} g_i \right) \sqcup\!\sqcup_{j_1} g_{i_1} \dots \sqcup\!\sqcup_{j_l} g_{i_l} \right) \\
&- \sum_{I_k} F_{j_1,\dots,j_l}(x) \left(f * \left(\left(\prod_{i \notin J \sqcup \{k+1\}} g_i \right) * g_{k+1} \right) \sqcup\!\sqcup_{j_1} g_{i_1} \dots \sqcup\!\sqcup_{j_l} g_{i_l} \right) \\
&- \sum_{I_k} \sum_{p=1}^{k} F_{j_1,\dots,j_l}(x) \left(\left(f * \prod_{i \notin J \sqcup \{k+1\}} g_i \right) \sqcup\!\sqcup_{j_1} g_{i_1} \dots \sqcup\!\sqcup_{j_p} (g_{i_p} * g_{k+1}) \dots \sqcup\!\sqcup_{j_l} g_{i_l} \right) \\
&- \sum_{I_k} \sum_{p=1}^{k} F_{j_1,\dots,j_l}(x) \left(\left(f * \prod_{i \notin J \sqcup \{k+1\}} g_i \right) \sqcup\!\sqcup_{j_1} g_{i_1} \dots \sqcup\!\sqcup_{j_p} (g_{i_p} \ {}_a\sqcup\!\sqcup\ g_{k+1}) \dots \sqcup\!\sqcup_{j_l} g_{i_l} \right) \\
&- \sum_{I_k} F_{j_1,\dots,j_l}(x) \left(f * \left(\left(\prod_{i \notin J \sqcup \{k+1\}} g_i \right) * g_{k+1} - \left(\prod_{i \notin J \sqcup \{k+1\}} g_i \right) g_{k+1} \right) \sqcup\!\sqcup_{j_1} g_{i_1} \dots \sqcup\!\sqcup_{j_l} g_{i_l} \right) \\
&= \sum_{I_k} F_{j_1,\dots,j_l}(x) \left(f * \left(\left(\prod_{i \notin J \sqcup \{k+1\}} g_i \right) g_{k+1} \right) \sqcup\!\sqcup_{j_1} g_{i_1} \dots \sqcup\!\sqcup_{j_l} g_{i_l} \right) \\
&+ \sum_{I_k} \sum_{j_{l+1} \in [N]} F_{j_{l+1}} \circ F_{j_1,\dots,j_l}(x) \left(\left(f * \left(\prod_{i \notin J \sqcup \{k+1\}} g_i \right) \right) \sqcup\!\sqcup_{j_1} g_{i_1} \dots \sqcup\!\sqcup_{j_l} g_{i_l} \sqcup\!\sqcup_{j_{l+1}} g_{i_{l+1}} \right) \\
&- \sum_{I_k} \sum_{p=1}^{k} \sum_{j \in [N]} a_j F_{j_1,\dots,j_l}(x) \left(\left(f * \left(\prod_{i \notin J \sqcup \{k+1\}} \right) \right) \right) \sqcup\!\sqcup_{j_1} g_{i_1} \dots \sqcup\!\sqcup_{j_p} (g_{k+1} \sqcup\!\sqcup_j g_p) \dots \sqcup\!\sqcup_{j_l} g_{i_l} \Bigg) \\
&= \sum_{I_k} F_{j_1,\dots,j_l}(x) \left(f * \left(\left(\prod_{i \notin J \sqcup \{k+1\}} g_i \right) g_{k+1} \right) \sqcup\!\sqcup_{j_1} g_{i_1} \dots \sqcup\!\sqcup_{j_l} g_{i_l} \right) \\
&+ \sum_{I_k} \sum_{j_{l+1} \in [N]} \left(F_{j_{l+1}} \circ F_{j_1,\dots,j_l} - \sum_{p=1}^{l} a_{j_p} F_{j_1,\dots,\widehat{j_p},\dots,j_{k+1}} \right)(x) \\
&\qquad \left(\left(f * \left(\prod_{i \notin J \sqcup \{k+1\}} g_i \right) \right) \sqcup\!\sqcup_{j_1} g_{i_1} \dots \sqcup\!\sqcup_{j_l} g_{i_l} \sqcup\!\sqcup_{j_{l+1}} g_{i_{l+1}} \right)
\end{aligned}
$$

$$= \sum_{I_{k+1},\, k+1\notin J} F_{j_1,\ldots,j_l}(x)\left(\left(f * \prod_{i\notin I} g_i\right) \sqcup_{j_1} g_{i_1} \ldots \sqcup_{j_l} g_{i_l}\right)$$
$$+ \sum_{I_{k+1},\, k+1\in J} F_{j_1,\ldots,j_l}(x)\left(\left(f * \prod_{i\notin I} g_i\right) \sqcup_{j_1} g_{i_1} \ldots \sqcup_{j_l} g_{i_l}\right)$$
$$= \sum_{I_{k+1}} F_{j_1,\ldots,j_l}(x)\left(\left(f * \prod_{i\notin I} g_i\right) \sqcup_{j_1} g_{i_1} \ldots \sqcup_{j_l} g_{i_l}\right).$$

Note that we used $F_{j_{l+1}} \circ F_{j_1,\ldots,j_l} = F_{j_1,\ldots,j_{l+1}} + \sum_{p=1}^{l} a_{j_p} F_{j_1,\ldots,\widehat{j_p},\ldots,j_{k+1}}$. □

Proposition 16 *Let $k \geq 1$, $f, g_1, \ldots, g_k \in T(V)^N$. Then:*

$$f \bullet g_1 \ldots g_k = f * g_1 \ldots g_k + \sum_{p=1}^{k} (f * g_1 \ldots g_{p-1} g_{p+1} \ldots g_k) \,{}_a\sqcup\, g_p.$$

Proof. We proceed by induction on k. This is obvious if $k = 1$. Let us assume the result at rank k, $k \geq 1$.

$$\begin{aligned}
&f \bullet g_1 \ldots g_{k+1}\\
&= (f \bullet g_1 \ldots g_k) \bullet g_{k+1} - \sum_{p=1}^{k} f \bullet (g_1 \ldots (g_p \bullet g_{k+1}) \ldots g_k)\\
&= (f * g_1 \ldots g_k) * g_{k+1} + (f * g_1 \ldots g_k) \,{}_a\sqcup\, g_{k+1}\\
&+ \sum_{p=1}^{k} ((f * g_1 \ldots g_{p-1} g_{p+1} \ldots g_k) \,{}_a\sqcup\, g_p) * g_{k+1} + \sum_{p=1}^{k} (f * g_1 \ldots g_{p-1} g_{p+1} \ldots g_k) \,{}_a\sqcup\, g_p \,{}_a\sqcup\, g_{k+1}\\
&- \sum_{p=1}^{k} f * (g_1 \ldots (g_p \bullet g_{k+1} \ldots g_k) - \sum_{p=1}^{k} (f * g_1 \ldots g_{p-1} g_{p+1} \ldots g_k) \sqcup (g_p \bullet g_{k+1})\\
&- \sum_{p\neq q} f * (g_1 \ldots (g_p \bullet g_{k+1}) \ldots \widehat{g_q} \ldots g_k) \,{}_a\sqcup\, g_q\\
&= (f * g_1 \ldots g_k) * g_{k+1} + (f * g_1 \ldots g_k) \,{}_a\sqcup\, g_{k+1} + \sum_{p=1}^{k} ((f * g_1 \ldots g_{p-1} g_{p+1} \ldots g_k) * g_{k+1}) \,{}_a\sqcup\, g_p\\
&+ \sum_{p=1}^{k} (f * g_1 \ldots g_{p-1} g_{p+1} \ldots g_k) \,{}_a\sqcup\, (g_p * g_{k+1} - g_p \bullet g_{k+1} + g_p \,{}_a\sqcup\, g_{k+1})\\
&- \sum_{p\neq q} f * (g_1 \ldots (g_p \bullet g_{k+1}) \ldots \widehat{g_q} \ldots g_k) \,{}_a\sqcup\, g_q - \sum_{p=1}^{k} f * (g_1 \ldots (g_p \bullet g_{k+1}) \ldots g_k)\\
&= (f * g_1 \ldots g_k) * g_{k+1} + (f * g_1 \ldots g_k) \,{}_a\sqcup\, g_{k+1}\\
&+ \sum_{p=1}^{k} \left((f * g_1 \ldots g_{p-1} g_{p+1} \ldots g_k) * g_{k+1} - \sum_{q\neq p} f * g_1 \ldots g_{p-1} g_{p+1} \ldots (g_k \bullet g_{k+1}) \ldots g_k \right) \,{}_a\sqcup\, g_p\\
&- \sum_{p=1}^{k} f * g_1 \ldots (g_p * g_{k+1}) \ldots g_k - \sum_{p=1}^{k} f * g_1 \ldots (g_p \,{}_a\sqcup\, g_{k+1}) \ldots g_k\\
&= f * \left(g_1 \ldots g_k \blacktriangleleft g_{k+1} - \sum_{p=1}^{k} g_1 \ldots (g_p \,{}_a\sqcup\, g_{k+1}) \ldots g_k \right) + \sum_{p=1}^{k+1} (f * g_1 \ldots g_{p-1} g_{p+1} \ldots g_{k+1}) \,{}_a\sqcup\, g_p
\end{aligned}$$

$$= f * g_1 \dots g_{k+1} + \sum_{p=1}^{k+1} (f * g_1 \dots g_{p-1} g_{p+1} \dots g_{k+1}) \,_a\sqcup\, g_p.$$

So the result holds for all $k \geq 1$. $\square$

Proposition 17 *On $S(\mathfrak{g}'_a)$, $\circledast = \odot$.*

Proof. Let $f, g \in S(\mathfrak{g}'_a)$; let us prove that $f \circledast g = f \odot g$. We assume that $f = f_1 \dots f_k$, $g = g_1, \dots, g_l$, with $f_1, \dots, f_k, g_1, \dots, g_l \in \mathfrak{g}'_a$, and we proceed by induction on k. If $k = 0$, then $f = 1$ and $f \circledast g = f \odot g = g$. Let us assume the result at all ranks $< k$. We proceed by induction on l. If $l = 0$, then $g = 1$ and $f \circledast g = f \odot g = f$. Let us assume the result at all ranks $< l$. We put:

$$\Delta(f) = f \otimes 1 + 1 \otimes f + f' \otimes f'', \qquad \Delta(g) = g \otimes 1 + 1 \otimes g + g' \otimes g''.$$

The induction hypothesis on k holds for f' and f'' and the induction hypothesis on l holds for g' and g''. From:

$$\Delta(f \circledast g - f \odot g) = f^{(1)} \circledast g^{(1)} \otimes f^{(2)} \circledast g^{(2)} - f^{(1)} \odot g^{(1)} \otimes f^{(2)} \odot g^{(2)},$$

these two induction hypotheses give:

$$\Delta(f \circledast g - f \odot g) = (f \circledast g - f \odot g) \otimes 1 + 1 \otimes (f \circledast g - f \odot g).$$

So $f \circledast g - f \odot g \in Prim(S(\mathfrak{g}'_a)) = \mathfrak{g}'_a$. Let π be the canonical projection on $\mathfrak{g}'_a$ in $S(\mathfrak{g}'_a)$. We obtain:

$$\begin{aligned}
\pi(f \circledast g) &= \pi\left(\sum_{I \subseteq [l]} \left(f * \prod_{i \in I} g_i\right) \blacktriangleleft \prod_{j \notin I} g_j\right) \\
&= \pi\left(\sum_{[l] = I_0 \sqcup \dots \sqcup I_k} \left(f_1 * \prod_{i \in I_1} g_i\right) \dots \left(f_k * \prod_{i \in I_k} g_i\right) \blacktriangleleft \prod_{i \in I_0} g_i\right) \\
&= \pi\left(\sum_{[l] = J_1 \sqcup \dots \sqcup J_k} \prod_{p=1}^{k} \left(f_p * \prod_{i \in J_k} g_i + \sum_{j_p \in J_p} \left(f_p * \prod_{i \in J_p \setminus \{j_p\}} g_i\right) \,_a\sqcup\, g_{j_p}\right)\right) \\
&= \pi\left(\sum_{[l] = J_1 \sqcup \dots \sqcup J_k} \left(\prod_{p=1}^{k} f_p \bullet \prod_{i \in J_p} g_i\right)\right) \\
&= \pi\left(\left(f_1 \bullet g^{(1)}\right) \dots \left(f_k \bullet g^{(k)}\right)\right) \\
&= \pi(f \bullet g) \\
&= \pi(f \odot g).
\end{aligned}$$

As $f \circledast g - f \odot g \in \mathfrak{g}'_a$, $f \circledast g = f \odot g$. $\square$

3.4 Graduation

We assume in this whole paragraph that $a = (1, 0, \dots, 0)$ and V is finite-dimensional. We decompose the $\mathfrak{g}_a$-module V as a direct sum of indecomposables. By Proposition 9, decomposing each indecomposables, we obtain a decomposition of V of the form:

$$V = V^{(0)} \oplus \dots \oplus V^{(k)},$$

with $F_1\left(V^{(p)}\right) \subseteq V^{(p)}$ and $F_i\left(V^{(p)}\right) \subseteq V^{(p-1)}$ for all $i \geq 2$, for all $p \in [k]$. We put $V_p = V^{(k+1-p)}$ for all $p \in [k+1]$. This defines a graduation of V, which induces a connected graduation of $T(V)$. For this graduation of V, F_1 is homogeneous of degree 0 and F_i is homogeneous of degree 1 for all $i \geq 2$. We define a graduation of $\mathfrak{g}'_a = T(V)^N$:

$$\forall n \geq 0,\ (\mathfrak{g}'_a)_n = T(V)_n\epsilon_1 \oplus \bigoplus_{i=2}^{N} T(V)_{n-1}\epsilon_i.$$

Let $v, w \in T(V)$, homogeneous of respective degree k and l. Let $i, j \geq 2$. Then:

- $v\epsilon_1$ is homogeneous of degree k.
- $v\epsilon_i$ is homogeneous of degree $k+1$.
- $w\epsilon_1$ is homogeneous of degree l.
- $w\epsilon_j$ is homogeneous of degree $l+1$.

As $v \sqcup\!\sqcup w$ is homogeneous of degree $k+l$:

- $v\epsilon_1 \ _{(1,0,\ldots,0)}\sqcup\!\sqcup\, w\epsilon_1 = v \sqcup\!\sqcup w\epsilon_1$ is homogeneous of degree $k+l$.
- $v\epsilon_1 \ _{(1,0,\ldots,0)}\sqcup\!\sqcup\, w\epsilon_j = v \sqcup\!\sqcup w\epsilon_j$ is homogeneous of degree $k+l+1$.
- $v\epsilon_i \ _{(1,0,\ldots,0)}\sqcup\!\sqcup\, w\epsilon_1 = 0$ is homogeneous of degree $k+l+1$.
- $v\epsilon_i \ _{(1,0,\ldots,0)}\sqcup\!\sqcup\, w\epsilon_j = 0$ is homogeneous of degree $k+l+2$.

Consequently, the product $_{(1,0,\ldots,0)}\sqcup\!\sqcup$ is homogeneous of degree 0. Proposition 14 implies that $*$ is homogeneous of degree 0; summing, $\bullet$ is also homogeneous of degree 0. Hence:

Proposition 18 *The decomposition of V in indecomposable $\mathfrak{g}_{(1,0,\ldots,0)}$-modules induces a graduation of the post-Lie algebra $\mathfrak{g}'_{(1,0,\ldots,0)}$.*

We put:

$$P(X) = \sum_{i=1}^{k+1} dim(V_p)X^p \in \mathbb{K}[X].$$

the formal series of $\mathfrak{g}'_{(1,0,\ldots,0)}$ is:

$$\begin{aligned} R(X) &= \sum_{p=1}^{\infty} dim((\mathfrak{g}'_{(1,0,\ldots,0)})_p)X^p \\ &= \frac{1}{1-P(X)} + (N-1)\frac{X}{1-P(X)} = \frac{1+(N-1)X}{1-P(X)}. \end{aligned}$$

Note that $R(0) = 1$: indeed, $(\mathfrak{g}'_{(1,0,\ldots,0)})_0 = Vect(\emptyset\epsilon_1)$. The augmentation ideal of $\mathfrak{g}'_{(1,0,\ldots,0)}$ is:

$$(\mathfrak{g}'_{(1,0,\ldots,0)})_+ = T(V)_+ \times T(V)^{N-1}.$$

This is a graded, connected post-Lie algebra.

Example. For the SISO case, $V_1 = Vect(x_2)$ and $V_2 = Vect(x_1)$. The formal series of $\mathfrak{g}_{SISO}$ is:

$$R_{SISO}(X) = \frac{1+X}{1-X-X^2} = 1 + 2X + 3X^2 + 5X^3 + 8X^4 + 13X^5 + \ldots$$

Hence, $(dim(\mathfrak{g}_{SISO})_n)_{n\geq 0}$ is the Fibonacci sequence A000045 [9]. For example:

$$\begin{aligned} (\mathfrak{g}_{SISO})_0 &= Vect(\emptyset\epsilon_1), \\ (\mathfrak{g}_{SISO})_1 &= Vect(x_2\epsilon_1, \emptyset\epsilon_2), \\ (\mathfrak{g}_{SISO})_2 &= Vect(x_1\epsilon_1, x_2x_2\epsilon_1, x_2\epsilon_2), \\ (\mathfrak{g}_{SISO})_3 &= Vect(x_1x_2\epsilon_1, x_2x_1\epsilon_1, x_2x_2x_2\epsilon_1, x_1\epsilon_2, x_2x_2\epsilon_2). \end{aligned}$$

4 Graded dual

We assume in this section that $a = (1, 0, \ldots, 0)$. The augmentation ideal of $\mathfrak{g}'_a$ is denoted by $(\mathfrak{g}'_a)_+$; recall that $(\mathfrak{g}'_a)_0 = Vect(\emptyset\epsilon_1)$.

- As $(\mathfrak{g}'_a)_+$ is a graded, connected Lie algebra, its enveloping algebra $\mathcal{U}((\mathfrak{g}'_a)_+)$ is a graded, connected Hopf algebra, and its graded dual also is. We denote it by $\mathcal{H}_V$.
- As an algebra, $\mathcal{H}_V$ is identified with $S((\mathfrak{g}'_a)^*)/\langle\emptyset\epsilon_1\rangle$. We identify $(\mathfrak{g}'_a)^*$ with $T(V^*)^N$ via the pairing:
$$\langle f_1 \ldots f_k\epsilon_i, x_1 \ldots x_l\epsilon_j\rangle = \delta_{i,j}\delta_{k,l} f_1(x_1) \ldots f_k(x_k).$$
- The coproduct dual of $\odot = \circledast$ is denoted by $\Delta_\bullet$.
- The dual of the product $\sqcup\!\sqcup_j$ defined on $\mathfrak{g}'_a$ is denoted by $\Delta_{\sqcup\!\sqcup_j}$, defined on $(\mathfrak{g}'_a)^* = T(V^*)^N$.
- We define a coproduct Δ_* on $S((\mathfrak{g}'_a)^*_+)$, dual of the right action $*$. Therefore, this is right coaction of $(\mathcal{H}_V, \Delta_\bullet)$ on itself:
$$(\Delta_* \otimes Id) \circ \Delta_* = (Id \otimes \Delta_\bullet) \circ \Delta_*.$$

Notations.

1. For all $y \in V^*$, we define $\theta_y : (\mathfrak{g}'_a)^* \longrightarrow (\mathfrak{g}'_a)^*$ by $\theta_y(f) = yf$.
2. For all $x \in (\mathcal{H}_V)_+$, we put $\overline{\Delta}_\bullet(x) = \Delta_\bullet(x) - 1 \otimes x$ and $\overline{\Delta}_*(x) = \Delta_*(x) - 1 \otimes x$. For all $g, f, f_1, \ldots, f_k \in (\mathfrak{g}'_a)^*_+$:
$$\langle\overline{\Delta}_*(g), f \otimes f_1 \ldots f_k\rangle = \langle g, f * f_1 \ldots f_k\rangle.$$

4.1 Deshuffling coproducts

Proposition 19 *For all $g \in T(V)$, for all $i \in [N]$, $\Delta_{\sqcup\!\sqcup_j}(g\epsilon_k) = \Delta_{\sqcup\!\sqcup}(g)(\epsilon_k \otimes \epsilon_j)$.*

Proof. Let $f_1, f_2 \in T(V)$, $i_1, i_2 \in [N]$.

$$\begin{aligned}
\langle\Delta_{\sqcup\!\sqcup_j}(g\epsilon_k), f_1\epsilon_{i_1} \otimes f_2\epsilon_{i_2}\rangle &= \langle g\epsilon_k, f_1\epsilon_{i_1} \sqcup\!\sqcup_j f_2\epsilon_{i_2}\rangle \\
&= \delta_{i_2,j}\langle g\epsilon_k, f_1 \sqcup\!\sqcup f_2\epsilon_{i_1}\rangle \\
&= \delta_{i_2,j}\delta_{i_1,k}\langle g, f_1 \sqcup\!\sqcup f_2\rangle \\
&= \delta_{i_2,j}\delta_{i_1,k}\langle\Delta_{\sqcup\!\sqcup}(g), f_1 \otimes f_2\rangle \\
&= \langle\Delta_{\sqcup\!\sqcup}(g)(\epsilon_k \otimes \epsilon_j), f_1\epsilon_{i_1} \otimes f_2\epsilon_{i_2}\rangle.
\end{aligned}$$

As the pairing is nondegenerate, we obtain the result. □

Notations. We define inductively, for $l \geq 0$, $j_1, \ldots, j_l \in [N]$:

$$\begin{cases} \Delta_{\sqcup\!\sqcup_\emptyset} = Id, \\ \Delta_{\sqcup\!\sqcup_{j_1,\ldots,j_l}} = \left(\Delta_{\sqcup\!\sqcup_{j_1}} \otimes Id^{\otimes(l-1)}\right) \circ \Delta_{\sqcup\!\sqcup_{j_2,\ldots,j_l}}. \end{cases}$$

For all $g \in T(V^*)$, for all $i \in [N]$:

$$\Delta_{\sqcup\!\sqcup_{j_1,\ldots,j_l}}(g\epsilon_k) = \Delta^{(l)}_{\sqcup\!\sqcup}(g)(\epsilon_k \otimes \epsilon_{j_1} \otimes \ldots \otimes \epsilon_{j_l});$$

for all $f_1, \ldots, f_l \in T(V)$:

$$\langle\Delta_{\sqcup\!\sqcup_{j_1,\ldots,j_l}}(g), f_1 \otimes \ldots \otimes f_{l+1}\rangle = \langle g, f_1 \sqcup\!\sqcup_{j_1} \ldots \sqcup\!\sqcup_{j_l} f_{l+1}\rangle.$$

4.2 Dual of the post-Lie product

Proposition 20 *In* $\mathcal{H}_V = S((\mathfrak{g}'_a)^*)/\langle \emptyset\epsilon_1\rangle$*:*

- *For all* $i \in [N]$, $\Delta_*(\emptyset\epsilon_i) = \emptyset\epsilon_i \otimes 1 + 1 \otimes \emptyset\epsilon_i$.
- *For all* $y \in V^*$, $g \in (\mathfrak{g}'_a)^*$*:*

$$\overline{\Delta}_* \circ \theta_y(g) = \sum_{l\geq 0} \sum_{j_1,\ldots,j_l \in [N]} (\theta_{F^*_{j_1,\ldots,j_l}(y)} \otimes \mu) \circ (\overline{\Delta}_* \otimes Id) \circ \Delta_{\sqcup\!\sqcup_{j_1,\ldots,j_l}}(g),$$

where we denote by μ *the sum of the iterated products of* $\mathcal{H}_V$*:*

$$\mu : \left\{ \begin{array}{rcl} T(\mathcal{H}_V) & \longrightarrow & \mathcal{H}_V \\ g_1 \otimes \ldots \otimes g_k & \longrightarrow & g_1 \ldots g_k. \end{array}\right.$$

Proof. The first point comes from $\emptyset\epsilon_i * \mathcal{U}(\mathfrak{g}'_a)_+ = (0)$.

In order to prove the formula, it is enough to prove that, for $f, f_1, \ldots, f_k \in \mathfrak{g}$:

$$\langle \overline{\Delta}_* \circ \theta_y(g), f \otimes f_1 \ldots f_k\rangle = \langle \sum_{l\geq 0} \sum_{j_1,\ldots,j_l \in [N]} (\theta_{F^*_{j_1,\ldots,j_l}(y)} \otimes \mu) \circ (\overline{\Delta}_* \otimes Id) \circ \Delta_{\sqcup\!\sqcup_{j_1,\ldots,j_l}}(g), f \otimes f_1 \ldots f_k\rangle,$$

or equivalently:

$$\langle \theta_y(g), f * f_1 \ldots f_k\rangle = \langle \sum_{l\geq 0} \sum_{j_1,\ldots,j_l \in [N]} (\theta_{F^*_{j_1,\ldots,j_l}(y)} \otimes \mu) \circ (\overline{\Delta}_* \otimes Id) \circ \Delta_{\sqcup\!\sqcup_{j_1,\ldots,j_l}}(g), f \otimes f_1 \ldots f_k\rangle,$$

If $f = \emptyset\epsilon_i$, both sides are equal to 0. Otherwise, we can assume that $f = xf'$, with $x \in V$ and $f' \in \mathfrak{g}$.

$$\begin{aligned}
&\langle \theta_y(g), f * f_1 \ldots f_k\rangle \\
&= \langle yg, \sum_{I=\{i_1<\ldots<i_l\}\subseteq[k]} \sum_{j_1,\ldots,j_l \in [N]} F_{j_1,\ldots,f_l}(x) \left(f' * \left(\prod_{i\notin I} f_i\right) \sqcup\!\sqcup_{j_1} f_{i_1} \ldots \sqcup\!\sqcup_{j_l} f_{i_l}\right)\rangle \\
&= \sum_{I=\{i_1<\ldots<i_l\}\subseteq[k]} \sum_{j_1,\ldots,j_l \in [N]} \langle y, F_{j_1,\ldots,j_l}(x)\rangle \langle \Delta_{\sqcup\!\sqcup_{j_1,\ldots,j_l}}(g), f' * \left(\prod_{i\notin I} f_i\right) \otimes f_{i_1} \ldots \otimes f_{i_l}\rangle \\
&= \sum_{j_1,\ldots,j_l \in [N]} \langle F^*_{j_1,\ldots,j_l}(y), x\rangle \langle (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id) \circ \Delta_{\sqcup\!\sqcup_{j_1,\ldots,j_l}}(g), f' \otimes f_1 \ldots f_k\rangle \\
&= \sum_{j_1,\ldots,j_l \in [N]} \langle (\theta_{F^*_{j_1,\ldots,j_l}(y)} \otimes Id) \circ (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id) \circ \Delta_{\sqcup\!\sqcup_{j_1,\ldots,j_l}}(g), xf' \otimes f_1 \ldots f_k\rangle,
\end{aligned}$$

which ends the proof. $\square$

In order to obtain a better description of the coproduct $\overline{\Delta}_*$, we are going to identify the following three objects:

$$\begin{array}{ccccc} & & S((\mathfrak{g}'_a)^*_+) & & \\ & \swarrow \sim & & \sim \searrow & \\ S((\mathfrak{g}'_a)^*)/\langle \emptyset\epsilon_1\rangle & & & & S((\mathfrak{g}'_a)^*)/\langle \emptyset\epsilon_1 - 1\rangle \end{array}$$

Both identification sends $x \in (\mathfrak{g}'_a)^*_+$ to its class. Let us reformulate Proposition 20 in the vector space $S((\mathfrak{g}'_a)^*)/\langle \emptyset\epsilon_1 - 1\rangle$:

$$\begin{aligned}
\overline{\Delta}_* \circ \theta_y(g\epsilon_k) &= \sum_{l\geq 0} \sum_{j_1,\ldots,j_l\in[N]} (\theta_{F^*_{j_1,\ldots,j_l}(y)} \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_\sqcup^{(l)}(g)\epsilon_k \otimes \epsilon_{j_1} \otimes \ldots \otimes \epsilon_{j_l}) \\
&- \sum_{l\geq 0} \sum_{j_1,\ldots,j_l\in[N]} (\theta_{F^*_{j_1,\ldots,j_l,1}(y)} \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_\sqcup^{(l+1)}(g)\epsilon_k \otimes \epsilon_{j_1} \otimes \ldots \otimes \epsilon_{j_l} \otimes \epsilon_1) \\
&= \sum_{l\geq 0} \sum_{j_1,\ldots,j_l\in[N]} (\theta_{F^*_{j_1,\ldots,j_l}(y)} \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_\sqcup^{(l)}(g)\epsilon_k \otimes \epsilon_{j_1} \otimes \ldots \otimes \epsilon_{j_l}) \\
&- \left(\sum_{l\geq 0} \sum_{j_1,\ldots,j_l\in[N]} (\theta_{F^*_{j_1,\ldots,j_l,1}(y)} \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_\sqcup^{(l)}(g)\epsilon_k \otimes \epsilon_{j_1} \otimes \ldots \otimes \epsilon_{j_l}) \right) (1 \otimes \emptyset\epsilon_1).
\end{aligned}$$

Finally, identifying in $S((\mathfrak{g}'_a)^*_+)$:

Proposition 21 *For all $j_1, \ldots, j_l \in [N]$, we put:*

$$G_{j_1,\ldots,j_l} = F_{j_1,\ldots,j_l} - F_{j_1,\ldots,j_l,1}.$$

In $S((\mathfrak{g}'_a)^*_+)/\langle \emptyset\epsilon_1 - 1\rangle$:

- *For all $i \in [N]$, $\overline{\Delta}_*(\emptyset\epsilon_i) = \emptyset\epsilon_i \otimes 1$.*
- *For all $y \in V^*$, for all $g \in (\mathfrak{g}'_a)^*_+$:*

$$\overline{\Delta}_* \circ \theta_y(g) = \sum_{l\geq 0} \sum_{j_1,\ldots,j_l\in[N]} (\theta_{G^*_{j_1,\ldots,j_l}(y)} \otimes \mu) \circ (\overline{\Delta}_* \otimes Id) \circ \Delta_{\sqcup j_1,\ldots,j_l}(g).$$

Example. For $\mathfrak{g}_{SISO}$, as V is a module over the associative algebra $(\mathfrak{g}_{(1,0)}, \triangleleft)$, if $l \geq 2$, $F_{j_1,\ldots,j_l} = 0$ by Proposition 11, so $G_{j_1,\ldots,j_l} = 0$. Moreover:

$$\begin{aligned}
F_\emptyset &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & F_1 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & F_2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \\
G_\emptyset = F_\emptyset - F_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & G_1 = F_1 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & G_2 = F_2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \\
G^*_\emptyset &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & G^*_1 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & G^*_2 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

The coproduct $\overline{\Delta}_*$ on $S((\mathfrak{g}_{SISO})^*_+)$ is given by:

- For all $i \in [2]$, $\overline{\Delta}_*(\emptyset\epsilon_i) = \emptyset\epsilon_i \otimes 1$.
- For all $g \in \mathbb{K}\langle x_1, x_2\rangle$, for all $i \in [2]$:

$$\begin{aligned}
\overline{\Delta}_* \circ \theta_{x_1}(g\epsilon_i) &= (\theta_{x_1} \otimes Id) \circ \overline{\Delta}_*(g\epsilon_i) + (\theta_{x_2} \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_\sqcup(g)\epsilon_i \otimes \epsilon_2), \\
\overline{\Delta}_* \circ \theta_{x_2}(g\epsilon_i) &= (\theta_{x_2} \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_\sqcup(g)\epsilon_i \otimes \epsilon_1).
\end{aligned}$$

These are formulas of Lemma 4.1 of [4], where $a_w = w\epsilon_2$, $b_w = w\epsilon_1$, $\theta_0 = \theta_{x_1}$, $\theta_1 = \theta_{x_2}$ and $\tilde{\Delta} = \overline{\Delta}_*$.

4.3 Dual of the pre-Lie product

Notations. We denote by $\Delta_{\ _1\sqcup\!\sqcup}$ the coproduct on $T_+(V^*)\otimes(V)^{N-1}$ dual to the product $\ _1\sqcup\!\sqcup$. As $\ _1\sqcup\!\sqcup\ =\ \sqcup\!\sqcup_1{}^{op}$, $\Delta_{\ _1\sqcup\!\sqcup} = \Delta^{cop}_{\sqcup\!\sqcup_1}$, and for all $g\in T(V)$, for all $i\in[N]$:

$$\Delta_{\ _1\sqcup\!\sqcup}(g\epsilon_i) = \Delta_{\sqcup\!\sqcup}(g)(\epsilon_1\otimes\epsilon_k).$$

Proposition 22 *In $S((\mathfrak{g}'_a)^*_+)/\langle\emptyset\epsilon_1\rangle$, for all $g\in(\mathfrak{g}'_a)^*_+$:*

$$\overline{\Delta}_\bullet(g) = \overline{\Delta}_*(g) + (Id\otimes\mu)\circ(\overline{\Delta}_*\otimes Id)\circ\Delta_{\ _1\sqcup\!\sqcup}(g).$$

Proof. Let $f, f_1,\ldots,f_k\in(\mathfrak{g}'_a)_+$.

$$\begin{aligned}
\langle\overline{\Delta}_\bullet(g), f\otimes f_1\ldots f_k\rangle &= \langle g, f\bullet f_1\ldots f_k\rangle\\
&= \langle g, f*f_1\ldots f_k + \sum_{p=1}^k (f*f_1\ldots\widehat{f_p}\ldots f_k)\ _1\sqcup\!\sqcup\ f_p\rangle\\
&= \langle\overline{\Delta}_*(g), f\otimes f_1\ldots f_k\rangle + \langle\Delta_{\ _1\sqcup\!\sqcup}(g), \sum_{p=1}^k f*f_1\ldots\widehat{f_p}\ldots f_k\otimes f_p\rangle\\
&= \langle\overline{\Delta}_*(g), f\otimes f_1\ldots f_k\rangle + \langle(\Delta_*\otimes Id)\circ\Delta_{\ _1\sqcup\!\sqcup}(g), \sum_{p=1}^k f\otimes f_1\ldots\widehat{f_p}\ldots f_k\otimes f_p\rangle\\
&= \langle\overline{\Delta}_*(g), f\otimes f_1\ldots f_k\rangle + \langle(Id\otimes\mu)\circ(\Delta_*\otimes Id)\circ\Delta_{\ _1\sqcup\!\sqcup}(g), f\otimes f_1\ldots f_k\rangle.
\end{aligned}$$

As $(\mathfrak{g}'_a, *)$ is pre-Lie, $\overline{\Delta}_\bullet(g)\in(\mathfrak{g}'_a)^*_+\otimes S((\mathfrak{g}'_a)^*_+)$ and the nondegeneracy of the pairing implies the formula. $\square$

Rewriting this formula in $S((\mathfrak{g}'_a)^*_+)/\langle\emptyset\epsilon_1 - 1\rangle$:

$$\begin{aligned}
\overline{\Delta}_\bullet(g\epsilon_1) &= \overline{\Delta}_*(g\epsilon_1) + (Id\otimes\mu)\circ(\overline{\Delta}_*\otimes Id)(\Delta_{\sqcup\!\sqcup}(g)(\epsilon_1\otimes\epsilon_1))\\
&= \overline{\Delta}_*(g\epsilon_1) + (Id\otimes\mu)\circ(\overline{\Delta}_*\otimes Id)((\Delta_{\sqcup\!\sqcup}(g) - g\otimes\emptyset)(\epsilon_1\otimes\epsilon_1))\\
&= \overline{\Delta}_*(g\epsilon_1)(1\otimes(1-\emptyset\epsilon_1)) + (Id\otimes\mu)\circ(\overline{\Delta}_*\otimes Id)(\Delta_{\sqcup\!\sqcup}(g)(\epsilon_1\otimes\epsilon_1))\\
&= (Id\otimes\mu)\circ(\overline{\Delta}_*\otimes Id)(\Delta_{\sqcup\!\sqcup}(g)(\epsilon_1\otimes\epsilon_1)).
\end{aligned}$$

Identifying in $S((\mathfrak{g}'_a)^*_+)$:

Proposition 23 *In $S((\mathfrak{g}'_a)^*_+)/\langle\emptyset\epsilon_1 - 1\rangle$, if $g\in T(V^*)$:*

$$\begin{aligned}
\overline{\Delta}_\bullet(g\epsilon_1) &= (Id\otimes\mu)\circ(\overline{\Delta}_*\otimes Id)(\Delta_{\sqcup\!\sqcup}(g)(\epsilon_1\otimes\epsilon_1)),\\
\textit{if } i\geq 2,\ \overline{\Delta}_\bullet(g\epsilon_i) &= \overline{\Delta}_*(g\epsilon_i) + (Id\otimes\mu)\circ(\overline{\Delta}_*\otimes Id)(\Delta_{\sqcup\!\sqcup}(g)(\epsilon_i\otimes\epsilon_1)),
\end{aligned}$$

with the convention $\emptyset\epsilon_1 = 1$. We put $\Delta_\bullet(g) = \overline{\Delta}_\bullet(g) + 1\otimes g$ for all $g\in(\mathfrak{g}'_a)^*_+$ and extend $\Delta_\bullet$ to $S((\mathfrak{g}'_a)^*_+)$ as an algebra morphism. This coproduct makes $S((\mathfrak{g}'_a)^*_+)$ a Hopf algebra, isomorphic to the graded dual of the enveloping algebra of $((\mathfrak{g}'_a)_+, [-,-]_*)$.

Remark. These are *mutatis mutandis* the formulas of Lemma 4.3 in [4].

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