# Realizations of Hopf algebras of graphs by alphabets

Loïc Foissy

*Fédération de Recherche Mathématique du Nord Pas de Calais FR 2956*
*Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville*
*Université du Littoral Côte d'Opale-Centre Universitaire de la Mi-Voix*
*50, rue Ferdinand Buisson, CS 80699, 62228 Calais Cedex, France*

*Email: foissy@univ-littoral.fr*

**Abstract**

We here give polynomial realizations of various Hopf algebras or bialgebras on Feynman graphs, graphs, posets or quasi-posets, that it to say injections of these objects into polynomial algebras generated by an alphabet. The alphabet here considered are totally quasi-ordered. The coproducts are given by doubling the alphabets; a second coproduct is defined by squaring the alphabets, and we obtain cointeracting bialgebras in the commutative case.

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## Contents

## Introduction

Some combinatorial Hopf algebras admit a polynomial realization, which gives an efficient way to prove the existence of the coproduct and more structures, see [8, 19, 15, 7, 17]. Let us explicit a well-known example. The algebra **QSym** of quasi-symmetric functions has a basis $(M_{a_1,\ldots,a_k})$ indexed by compositions, that is to say finite sequences of positive integers.

1. For any totally ordered alphabet $X$, let us consider the following elements of the ring of formal series $\mathbb{K}[[X]]$ generated by $X$:

$$M_{(a_1,\ldots,a_k)}(X) = \sum_{x_1<\ldots<x_k \text{ in } X} x_1^{a_1}\ldots x_n^{a_n}.$$

This defines a map from **QSym** to $\mathbb{K}[[X]]$, injective if, and only if, $X$ is infinite. This map is an algebra morphism. For example, if $a, b \geq 1$, for any totally ordered alphabet $X$:

$$\begin{aligned} M_{(a)}(X)M_{(b)}(X) &= \sum_{x,y\in X} x^a y^b \\ &= \sum_{x<y} x^a y^b + \sum_{y<x} x^a y^b + \sum_{x} x^{a+b} \\ &= M_{(a,b)}(X) + M_{(b,a)}(X) + M_{(a+b)}(X), \end{aligned}$$

and, in **QSym**:

$$M_{(a)}M_{(b)} = M_{(a,b)} + M_{(b,a)} + M_{(a+b)}.$$

2. If $X$ and $Y$ are totally ordered alphabets, then $X \sqcup Y$ is too, the elements of $X$ being smaller than the elements of $Y$. Identifying $\mathbb{K}[[X \sqcup Y]]$ with a subalgebra of $\mathbb{K}[[X]] \otimes \mathbb{K}[[Y]]$, we define a coproduct on **QSym** by:

$$\Delta(M_{(a_1,\ldots,a_k)})(X,Y) = M_{(a_1,\ldots,a_k)}(X \sqcup Y).$$

For example:

$$\begin{aligned} \Delta(M_{(a,b)})(X,Y) &= M_{(a,b)}(X \sqcup Y) \\ &= \sum_{x<y \text{ in} X} x^a y^b + \sum_{x<y \text{ in} Y} x^a y^b + \sum_{(x,y)\in X\times Y} x^a y^b \\ &= M_{(a,b)}(X) + M_{(a,b)}(Y) + M_{(a)}(X)M_{(b)}(Y), \end{aligned}$$

and, in **QSym**:

$$\Delta(M_{(a,b)}) = M_{(a,b)} \otimes 1 + 1 \otimes M_{(a,b)} + M_{(a)} \otimes M_{(b)}.$$

The coassociativity of $\Delta$ is easily obtained from the equality $(X \sqcup Y) \sqcup Z = X \sqcup (Y \sqcup Z)$.

3. If $X$ and $Y$ are totally ordered alphabets, then $XY = X \times Y$ is too, with the lexicographic order. We consider $\mathbb{K}[[XY]]$ as a subalgebra of $\mathbb{K}[[X]] \otimes \mathbb{K}[[Y]]$, identifying $(x, y)$ with $x \otimes y$. We can define a second coproduct on **QSym** by:

$$\delta(M_{(a_1,\ldots,a_k)})(X,Y) = M_{(a_1,\ldots,a_k)}(XY).$$

For example:

$$\begin{aligned} \delta(M_{(a,b)})(X,Y) &= M_{(a,b)}(XY) \\ &= \sum_{\substack{x<x' \text{ in} X,\\ y,y'\in Y}} x^a y^a x'^b y'^b + \sum_{x\in X,\, y<y' \text{ in} X,} x^a y^a x^b y'^b \\ &= M_{(a,b)}(X)(M_{(a,b)}(Y) + M_{(b,a)}(Y) + M_{(a+b)}(Y)) + M_{(a+b)}(X)M_{(a,b)}(Y), \end{aligned}$$

and, in **QSym**:

$$\delta(M_{(a,b)}) = M_{(a,b)} \otimes (M_{(a,b)} + M_{(b,a)} + M_{(a+b)}) + M_{(a+b)} \otimes M_{(a,b)}.$$

The coassociativity of $\delta$ is easily obtained from the equality $(XY)Z = X(YZ)$. Moreover:

$$(X \sqcup Y)Z = (XZ) \sqcup (YZ).$$

This implies that in **QSym**:

$$(\Delta \otimes Id) \circ \delta = m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta,$$

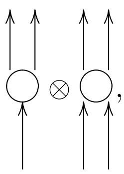
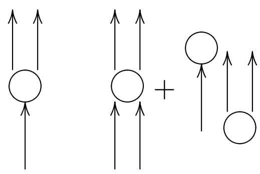
where $m_{1,3,24} : \mathbf{QSym}^{\otimes 4} \longrightarrow \mathbf{QSym}^{\otimes 3}$ sends $x \otimes y \otimes z \otimes t$ to $x \otimes z \otimes yt$. This means that the Hopf algebra $(\mathbf{QSym}, m, \Delta)$ is a Hopf algebra in the category of right comodules over the bialgebra $(\mathbf{QSym}, m, \delta)$, the coaction being $\delta$ itself: we call this a pair of bialgebras in cointeraction.

For other examples of such objects and applications, see [16, 18, 10, 9, 11].

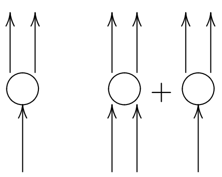
We here give other examples of cointeracting bialgebras coming from the manipulation of alphabets and polynomial realizations. We use here totally quasi-ordered alphabets, that is to say sets with a total transitive reflexive (but not necessarily antisymmetric) relation. The associated algebras $A_q(X)$ are slightly more complicated, see Definition 6. Their different sets of generators allows to polynomially realize Feynman graphs (one set for vertices, one set for internal edges, one set for incoming half-edges and a last one for outgoing half-edges); this gives a family of products $\cdot_q$ on the space $H_{\mathcal{FG}}$ generated by isoclasses of Feynman graphs, indexed by a scalar $q$ (Theorem 17). If $F$ and $G$ are two Feynman graphs, $F \cdot_q G$ is a sum of graphs obtained by gluing together vertices of $F$ and $G$; in particular, if $q = 0$, this is reduced to the disjoint union of $F$ and $G$. The trick of doubling the alphabet gives $\mathcal{H}_{\mathcal{FG}}$ a coproduct $\Delta$, given by ideals (Theorem 19), and the trick of squaring the alphabet gives it a second coproduct $\delta$; we obtain in this way a pair of cointeracting bialgebras (Corollary 20). For example, for the following graph:

$G =$  ,

we obtain:

$\Delta(G) = G \otimes 1 + 1 \otimes G +$  ,    $\delta(G) = G \otimes$  $\otimes G.$

The coproduct $\delta$ is similar to the Connes-Kreimer's one $\Delta_{CK}$ [2, 3, 4, 6, 5, 1, 16], but slightly different. For example:

$\Delta_{CK}(G) = G \otimes$  $\otimes G.$

We shall then consider several quotients of $A_q(X)$, leading to quotient bialgebras of $\mathcal{H}_{\mathcal{FG}}$. We obtain in this way a polynomial realization of a Hopf algebra of simple oriented graphs $\mathcal{H}_{\mathcal{SG}}$, and then a polynomial realization of the Hopf algebra on quasi-posets $\mathcal{H}_{\mathcal{QP}}$ of [12, 13, 14, 10]. Restricting to ordered alphabets, instead of quasi-ordered alphabets, we obtain quotients bialgebras, namely $H_{\mathcal{NCFG}}$ based on Feynman graphs with no cycle in Theorem 25 , $H_{\mathcal{NCSG}}$ on simple oriented graphs with no cycle in Theorem 33, and $H_{\mathcal{P}}$ on posets in Theorem 43, obtaining diagrams of Hopf algebras:

$$\begin{tikzcd}
(H_{\mathcal{FG}}, \cdot_q, \Delta) \arrow[r, two heads, "S"] \arrow[d, "T"'] & (H_{\mathcal{SG}}, \cdot_q, \Delta) \arrow[r, two heads, "P"] \arrow[d, "T"] & (H_{\mathcal{QP}}, \cdot_q, \Delta) \arrow[d, "T"] \\
(H_{\mathcal{NCFG}}, \cdot_q, \Delta) \arrow[r, two heads, "S"'] & (H_{\mathcal{NCSG}}, \cdot_q, \Delta) \arrow[r, two heads, "P"'] & (H_{\mathcal{P}}, \cdot_q, \Delta)
\end{tikzcd}$$

We also show that these Hopf algebra admit noncommutative versions, replacing the algebras $A_q(X)$ by a noncommutative analogue. The last paragraph is devoted to the description of the dual Hopf algebra of posets, using the notion of system of edge between two posets.

# 1 Operations on alphabets

All the proofs of this section are elementary and left to the reader.

## 1.1 Quasi-ordered alphabets

**Definition 1.** *A quasi-ordered alphabet is a pair $(X, \leq_X)$, where $X$ is an alphabet and $\leq_X$ is a total quasi-order on $X$, that is to say a relation on $X$ such that:*

$$\begin{array}{lll}
\forall i \in X, & i \leq_X i & \textit{(reflexivity)}, \\
\forall i, j, k \in X, & (i \leq_X j) \textit{ and } (j \leq_X k) \Longrightarrow i \leq_X k & \textit{(transitivity)}, \\
\forall i, j \in X, & i \leq_X j \textit{ or } j \leq_X i & \textit{(totality)}.
\end{array}$$

*If $\leq_X$ is an order, we shall say that $(X, \leq_X)$ is an ordered alphabet.*

*Notations* 1. Let $(X, \leq_X)$ be a quasi-ordered alphabet. We define an equivalence $\sim_X$ on X by:

$$\forall i, j \in X, \qquad i \sim_X j \text{ if } (i \leq_X j) \text{ and } (j \leq_X i).$$

For all $i, j \in X$, we shall denote $i <_X j$ if $i \leq_X j$ and not $j \leq_X i$.

## 1.2 Disjoint union

**Proposition 2.** *Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be two quasi-ordered alphabets. The set $X \sqcup Y$ is given a relation $\leq_{X \sqcup Y}$:*

$$\begin{array}{ll}
\forall i, j \in X \sqcup Y, & i \leq_{X \sqcup Y} j \textit{ if } (i, j \in X \textit{ and } i \leq_X j) \\
& \quad \textit{or } (i, j \in Y \textit{ and } i \leq_y j) \\
& \quad \textit{or } (i \in X,\ j \in Y).
\end{array}$$

*Then $(X \sqcup Y, \leq_{X \sqcup Y})$ is a quasi-ordered alphabet.*

*Remark* 1. If $X$ and $Y$ are ordered alphabets, then $X \sqcup Y$ is also ordered.

**Lemma 3.** *1. Let $X, Y$ be quasi-ordered alphabets.*

$$\forall i, j \in X \sqcup Y, \qquad i \sim_{X \sqcup Y} j \Longleftrightarrow (i, j \in X \textit{ and } i \sim_X j) \textit{ or } (i, j \in Y \textit{ and } i \sim_Y j).$$

*2. Let $X$, $Y$ and $Z$ be quasi-ordered alphabets. Then:*

$$(X \sqcup Y) \sqcup Z = X \sqcup (Y \sqcup Z).$$

### 1.3 Product

**Proposition 4.** *Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be two quasi-ordered alphabets. The set $XY = X \times Y$ is given a relation $\leq_{XY}$ in the following way:*

$$\forall i, i' \in X,\ j, j' \in Y, \qquad (i,j) \leq_{XY} (i',j') \text{ if } (i \sim_X i' \text{ and } j \leq_Y j') \text{ or } (i <_X i').$$

*Then $(XY, \leq_{XY})$ is a quasi-ordered alphabet, which we denote by $XY$.*

*Remark* 2. If $X$ and $Y$ are ordered alphabets, then $XY$ is also ordered, and $\leq_{XY}$ is the lexicographic order.

**Lemma 5.** *1. Let $X$ and $Y$ be quasi-ordered alphabets.*

$$\forall i, i' \in X,\ j, j' \in Y, \qquad (i,j) \sim_{XY} (i',j') \Longleftrightarrow (i \sim_X i') \text{ and } (j \sim_Y j').$$

*2. Let $X$, $Y$ and $Z$ be quasi-ordered alphabets. Then:*

$$(XY)Z = X(YZ), \qquad (X \sqcup Y)Z = (XZ) \sqcup (YZ), \qquad X(Y \sqcup Z) = (XY) \sqcup (XZ).$$

## 2 Algebras attached to alphabets

### 2.1 Definition

**Definition 6.** *Let $X$ be a quasi-ordered alphabet and let $q \in \mathbb{K}$. We put:*

$$A_q(X) = \frac{\mathbb{K}[x_i, i \in X]\,[[x_{i,j}, i,j \in X, i \leq_X j]]\,[[x_{-\infty,j}, j \in X]]\,[[x_{i,+\infty}, i \in X]]}{\langle x_i^2 = q x_i, i \in X\rangle},$$
$$\mathbf{A}_q(X) = \frac{\mathbb{K}\langle x_i, i \in X\rangle\,[[x_{i,j}, i,j \in X, i \leq_X j]]\,[[x_{-\infty,j}, j \in X]]\,[[x_{i,+\infty}, i \in X]]}{\langle x_i P x_i = q x_i P, i \in X,\ P \in \mathbf{A}_q(X)\rangle}.$$

*Both of them are given their usual topology of rings of formal series.*

Elements of $A_q(X)$ are formal infinite spans of monomials

$$M = \prod_{i \in X} x_i^{\epsilon_i} \prod_{i \leq_X j} x_{i,j}^{\alpha_{i,j}} \prod_{j \in X} x_{-\infty,j}^{\beta_j} \prod_{i \in X} x_{i,+\infty}^{\gamma_i},$$

where $\epsilon_i \in \{0,1\}$, $\alpha_{i,j}, \beta_j, \gamma_i \in \mathbb{N}$, with only a finite number of them non-zero. Elements of $\mathbf{A}_q(X)$ are formal infinite spans of monomials

$$M = x_{i_1} \ldots x_{i_k} \prod_{i \leq_X j} x_{i,j}^{\alpha_{i,j}} \prod_{j \in X} x_{-\infty,j}^{\beta_j} \prod_{i \in X} x_{i,+\infty}^{\gamma_i},$$

where $i_1, \ldots, i_k$ are elements of $X$, all distinct, $\alpha_{i,j}, \beta_j, \gamma_i \in \mathbb{N}$, with only a finite number of them non-zero.

### 2.2 Doubling the alphabets

**Proposition 7.** *Let $X, Y$ be two quasi-ordered alphabets. We define a continuous algebra morphism $\Delta_{X,Y}$ from $A_q(X \sqcup Y)$ to $A_q(X) \otimes A_q(Y)$ or from $\mathbf{A}_q(X \sqcup Y)$ to $\mathbf{A}_q(X) \otimes \mathbf{A}_q(Y)$ by:*

$$\Delta_{X,Y}(x_i) = \begin{cases} x_i \otimes 1 \text{ if } i \in X, \\ 1 \otimes x_i \text{ if } i \in Y; \end{cases} \qquad \Delta_{X,Y}(x_{i,j}) = \begin{cases} x_{i,j} \otimes 1 \text{ if } i,j \in X, \\ 1 \otimes x_{i,j} \text{ if } i,j \in Y, \\ x_{i,\infty} \otimes x_{-\infty,j} \text{ if } i \in X,\ j \in Y; \end{cases} \tag{1}$$

$$\Delta_{X,Y}(x_{i,\infty}) = \begin{cases} x_{i,\infty} \otimes 1 \text{ if } i \in X, \\ 1 \otimes x_{i,\infty} \text{ if } i \in Y; \end{cases} \qquad \Delta_{X,Y}(x_{-\infty,j}) = \begin{cases} x_{-\infty,j} \otimes 1 \text{ if } j \in X, \\ 1 \otimes x_{-\infty,j} \text{ if } j \in Y. \end{cases}$$

*Proof.* In the commutative case, we have to check that for any $i \in X \sqcup Y$, $\Delta_{X,Y}(x_i^2 - qx_i) = 0$. Indeed:

$$\Delta_{X,Y}(x_i^2 - qx_i) = \begin{cases} (x_i^2 - qx_i) \otimes 1 = 0 \text{ if } i \in X, \\ 1 \otimes (x_i^2 - qx_i) = 0 \text{ if } i \in Y. \end{cases}$$

So $\Delta_{X,Y}$ is well-defined. The proof in the noncommutative case is similar. $\square$

**Proposition 8.** *Let $X$, $Y$ and $Z$ be quasi-ordered alphabets. Then:*

$$(\Delta_{X,Y} \otimes Id) \circ \Delta_{X \sqcup Y,Z} = (Id \otimes \Delta_{Y,Z}) \circ \Delta_{X,Y \sqcup Z},$$

*seen as morphisms from $A_q(X \sqcup Y \sqcup Z)$ to $A_q(X) \otimes A_q(Y) \otimes A_q(Z)$, or from $\mathbf{A}_q(X \sqcup Y \sqcup Z)$ to $\mathbf{A}_q(X) \otimes \mathbf{A}_q(Y) \otimes \mathbf{A}_q(Z)$.*

*Proof.* It is enough to apply these two algebra morphisms on generators. We find:

$$(\Delta_{X,Y} \otimes Id) \circ \Delta_{X \sqcup Y,Z}(x_i) = (Id \otimes \Delta_{Y,Z}) \circ \Delta_{X,Y \sqcup Z}(x_i) = \begin{cases} x_i \otimes 1 \otimes 1 \text{ if } i \in X, \\ 1 \otimes x_i \otimes 1 \text{ if } i \in Y, \\ 1 \otimes 1 \otimes x_i \text{ if } i \in Z. \end{cases}$$

When applied to $x_{i,j}$, we find for both of them:

| $i \setminus j$ | $\in X$ | $\in Y$ | $\in Z$ | $\infty$ |
|---|---|---|---|---|
| $-\infty$ | $x_{-\infty,j} \otimes 1 \otimes 1$ | $1 \otimes x_{-\infty,j} \otimes 1$ | $1 \otimes 1 \otimes x_{-\infty,j}$ | $\times$ |
| $\in X$ | $x_{i,j} \otimes 1 \otimes 1$ | $x_{i,\infty} \otimes x_{-\infty,j} \otimes 1$ | $x_{i,\infty} \otimes 1 \otimes x_{-\infty,j}$ | $x_{i,\infty} \otimes 1 \otimes 1$ |
| $\in Y$ | $\times$ | $1 \otimes x_{i,j} \otimes 1$ | $1 \otimes x_{i,\infty} \otimes x_{-\infty,j}$ | $1 \otimes x_{i,\infty} \otimes 1$ |
| $\in Z$ | $\times$ | $\times$ | $1 \otimes 1 \otimes x_{i,j}$ | $1 \otimes 1 \otimes x_{i,\infty}$ |

So these morphisms are equal. $\square$

## 2.3 Squaring the alphabets

**Proposition 9.** *Let $X, Y$ be two nonempty quasi-ordered alphabets. There exists a unique continuous algebra morphism $\delta_{X,Y}$ from $A_q(XY)$ to $A_{q_1}(X) \otimes A_{q_2}(Y)$, or from $\mathbf{A}_q(XY)$ to $\mathbf{A}_{q_1}(X) \otimes \mathbf{A}_{q_2}(Y)$ such that:*

$$\begin{aligned} &\delta_{X,Y}(x_{(i,i')}) = x_i \otimes x_{i'}, && \delta_{X,Y}(x_{-\infty,(j,j')}) = x_{-\infty,j} \otimes x_{-\infty,j'}, \\ &\delta_{X,Y}(x_{(i,i'),\infty}) = x_{i,\infty} \otimes x_{i',\infty}, && \delta_{X,Y}(x_{(i,i'),(j,j')}) = \begin{cases} x_{i,j} \otimes x_{i',\infty} x_{-\infty,j'} \text{ if } i <_X j, \\ 1 \otimes x_{i',j'} \text{ if } i \sim_X j \text{ and } i' \leq_Y j'. \end{cases} \end{aligned} \tag{2}$$

*if, and only if, $q = q_1 q_2$.*

*Proof.* In the commutative case, we have to check that for any $(i,i') \in XY$, $\delta_{X,Y}(x_{(i,i')}^2 - qx_{(i,i')}) = 0$. We compute:

$$\begin{aligned} \delta_{X,Y}(x_{(i,i')}^2 - qx_{(i,i')}) &= (x_i \otimes x_{i'})^2 - qx_i \otimes x_{i'} \\ &= x_i^2 \otimes x_{i'}^2 - qx_i \otimes x_{i'} \\ &= q_1 q_2 x_i \otimes x_{i'} - qx_i \otimes x_{i'} \\ &= (q_1 q_2 - q) x_i \otimes x_{i'}. \end{aligned}$$

So this holds if, and only if, $q_1 q_2 = q$. The noncommutative case is proved in the same way. $\square$

*Remark* 3. In particular, if $q = q_1 = q_2$, $\delta_{X,Y}$ exists if, and only if, $q = 1$ or $q = 0$.

**Lemma 10.** *Let $q_1, q_2, q_3 \in \mathbb{K}$ and let $X$, $Y$ and $Z$ be quasi-ordered alphabets. The following diagrams commute:*

$$\begin{tikzcd}
A_{q_1q_2q_3}(XYZ) \arrow[r, "\delta_{X,YZ}"] \arrow[d, "\delta_{XY,Z}"'] & A_{q_1q_2}(XY) \otimes A_{q_3}(Z) \arrow[d, "\delta_{X,Y}\otimes Id"] \\
A_{q_1}(X) \otimes A_{q_2q_3}(YZ) \arrow[r, "Id\otimes\delta_{Y,Z}"'] & A_{q_1}(X) \otimes A_{q_2}(Y) \otimes A_{q_3}(Z)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{A}_{q_1q_2q_3}(XYZ) \arrow[r, "\delta_{X,YZ}"] \arrow[d, "\delta_{XY,Z}"'] & \mathbf{A}_{q_1q_2}(XY) \otimes \mathbf{A}_{q_3}(Z) \arrow[d, "\delta_{X,Y}\otimes Id"] \\
\mathbf{A}_{q_1}(X) \otimes \mathbf{A}_{q_2q_3}(YZ) \arrow[r, "Id\otimes\delta_{Y,Z}"'] & \mathbf{A}_{q_1}(X) \otimes \mathbf{A}_{q_2}(Y) \otimes \mathbf{A}_{q_3}(Z)
\end{tikzcd}$$

*Proof.* As these two maps are algebra morphisms, it is enough to prove that they coincide on generators of $A_{q_1q_2q_3}(XYZ)$ or $\mathbf{A}_{q_1q_2q_3}(XYZ)$. Let $i, i' \in X$, $j, j' \in Y$, $k, k' \in Z$.

- Both send $x_{(i,j,k)}$ to $x_i \otimes x_j \otimes x_k$.
- Both send $x_{(i,j,k),\infty}$ to $x_{i,\infty} \otimes x_{j,\infty} \otimes x_{k,\infty}$.
- Both send $x_{-\infty,(i',j',k')}$ to $x_{-\infty,i'} \otimes x_{-\infty,j'} \otimes x_{-\infty,k'}$.
- Both send $x_{(i,j,k),(i',j'k')}$ to $\begin{cases} x_{i,i'} \otimes x_{j,\infty}x_{-\infty,j'} \otimes x_{k,\infty}x_{-\infty,k'} \text{ if } i <_X i', \\ 1 \otimes x_{j,j'} \otimes x_{k,\infty}x_{-\infty,k'} \text{ if } i \sim_X i' \text{ and } j <_Y j', \\ 1 \otimes 1 \otimes x_{k,k'} \text{ if } i \sim_X i' \text{ and } j \sim_Y j'. \end{cases}$

So $(\delta_{X,Y} \otimes Id) \circ \delta_{XY,Z} = (Id \otimes \delta_{Y,Z}) \circ \delta_{X,YZ}$. $\square$

**Lemma 11.** *Let $q_1, q_2 \in \mathbb{K}$ and let $X$, $Y$, $Z$ be quasi-ordered alphabets. The following diagram commutes:*

$$\begin{tikzcd}
A_{q_1q_2}((X \sqcup Y)Z) = A_{q_1q_2}(XZ \sqcup YZ) \arrow[r, "\Delta_{XZ,YZ}"] \arrow[dd, "\delta_{X\sqcup Y,Z}"'] & A_{q_1q_2}(XZ) \otimes A_{q_1q_2}(YZ) \arrow[d, "\delta_{XZ}\otimes\delta_{YZ}"] \\
 & A_{q_1}(X) \otimes A_{q_2}(Z) \otimes A_{q_1}(Y) \otimes A_{q_2}(Z) \arrow[d, "m_{1,3,24}"] \\
A_{q_1}(X \sqcup Y) \otimes A_{q_2}(Z) \arrow[r, "\Delta_{X,Y}\otimes Id"'] & A_{q_1}(X) \otimes A_{q_1}(Y) \otimes A_{q_2}(Z)
\end{tikzcd}$$

*where:*

$$m_{1,3,24} : \left\{ \begin{array}{rcl} A_{q_1}(X) \otimes A_{q_2}(Z) \otimes A_{q_1}(Y) \otimes A_{q_2}(Z) & \longrightarrow & A_{q_1}(X) \otimes A_{q_1}(Y) \otimes A_{q_2}(Z) \\ x \otimes z_1 \otimes y \otimes z_2 & \longrightarrow & x \otimes y \otimes z_1z_2. \end{array} \right.$$

*Proof. First step.* Let us prove that $m_{1,3,24}$ is an algebra morphism. Let $X = x \otimes z_1 \otimes y \otimes z_2$ and $X' = x' \otimes z_1' \otimes y' \otimes z_2'$ in $A_{q_1}(X) \otimes A_{q_2}(Z) \otimes A_{q_1}(Y) \otimes A_{q_2}(Z)$.

$$\begin{aligned} m_{1,3,24}(XX') &= m_{1,3,24}(xx' \otimes z_1z_1' \otimes yy' \otimes z_2z_2') \\ &= xx' \otimes yy' \otimes z_1z_1'z_2z_2'; \\ m_{1,3,24}(X)m_{1,3,24}(X') &= (x \otimes y \otimes z_1z_2)(x' \otimes y' \otimes z_1'z_2') \\ &= xx' \otimes yy' \otimes z_1z_2z_1'z_2'. \end{aligned}$$

As $A_{q_1q_2}(Z)$ is commutative, $m_{1,3,24}(XX') = m_{1,3,24}(X)m_{1,3,24}(X')$.

*Second step.* By composition, both $(\delta_{X,Y} \otimes Id) \circ \delta_{X\sqcup Y,Z}$ and $(Id \otimes \delta_{Y,Z}) \circ \delta_{X,Y\sqcup Z}$ are algebra morphisms: it is enough to prove that they coincide on the generators of $A_{q_1q_2}((X \sqcup Y)Z)$ or $\mathbf{A}_{q_1q_2}((X \sqcup Y)Z)$. Let $i, i' \in X$, $j, j' \in Y$ and $k, k' \in Z$.

- Both send $x_{(i,k)}$ to $x_i \otimes 1 \otimes x_k$ and $x_{j,k}$ to $1 \otimes x_j \otimes x_k$.
- Both send $x_{(i,k),\infty}$ to $x_{i,\infty} \otimes 1 \otimes x_{k,\infty}$ and $x_{(j,k),\infty}$ to $1 \otimes x_{j,\infty} \otimes x_{k,\infty}$.
- Both send $x_{-\infty,(i',k')}$ to $x_{-\infty,i'} \otimes 1 \otimes x_{-\infty,k'}$ and $x_{-\infty,(j',k')}$ to $1 \otimes x_{-\infty,j'} \otimes x_{-\infty,k'}$.
- Both send $x_{(i,k),(i',k')}$ to $\begin{cases} x_{i,i'} \otimes 1 \otimes x_{k,\infty}x_{-\infty,k'} \text{ if } i <_X i', \\ 1 \otimes 1 \otimes x_{k,k'} \text{ if } i \sim_X i'. \end{cases}$
- Both send $x_{(j,k),(j',k')}$ to $\begin{cases} 1 \otimes x_{j,j'} \otimes x_{k,\infty}x_{-\infty,k'} \text{ if } j <_Y j', \\ 1 \otimes 1 \otimes x_{k,k'} \text{ if } j \sim_Y j'. \end{cases}$
- Both send $x_{(i,k),(j',k')}$ to $x_{i,\infty} \otimes x_{-\infty,j} \otimes x_{k,\infty}x_{-\infty,k'}$.

Therefore, they are equal. □

*Remark* 4. This does not work for morphisms

$$\mathbf{A}_{q_1q_2}((X \sqcup Y)Z) = \mathbf{A}_{q_1q_2}((XZ) \sqcup (YZ)) \longrightarrow \mathbf{A}_{q_1}(X) \otimes \mathbf{A}_{q_1}(Y) \otimes \mathbf{A}_{q_2}(Z).$$

But, if we put $\rho_{X,Y} = (Id \otimes p_Y) \circ \delta_{X,Y} : \mathbf{A}_{q_1q_2}(X \sqcup Y) \longrightarrow \mathbf{A}_{q_1}(X) \otimes A_{q_2}(Y)$, where $p_Y$ is the canonical surjection from $\mathbf{A}_{q_2}(Y)$ to $A_{q_2}(Y)$, then

$$m_{1,3,24} \circ (\rho_{X,Z} \otimes \rho_{Y,Z}) \circ \Delta_{XZ,YZ} = (\Delta_{X,Y} \otimes Id) \circ \rho_{X\sqcup Y,Z},$$

seen as morphisms

$$\mathbf{A}_{q_1q_2}((X \sqcup Y)Z) = \mathbf{A}_{q_1q_2}((XZ) \sqcup (YZ)) \longrightarrow \mathbf{A}_{q_1}(X) \otimes \mathbf{A}_{q_1}(Y) \otimes A_{q_2}(Z).$$

The proof is identical to the one of Lemma 11.

# 3 Feynman graphs

## 3.1 Definition

**Definition 12.** *A Feynman graph $G$ is given by:*

- *A non-empty, finite set $HE(G)$ of half-edges, with a map $type_G : HE(G) \longrightarrow \{out, in\}$.*
- *A non-empty, finite set $V(G)$ of vertices.*
- *An incidence map for half-edges, that is to say an involution $i_G : HE(G) \longrightarrow HE(G)$.*
- *A source map for half-edges, that is to say a map $s_G : HE(G) \longrightarrow V(G)$.*

*The incidence rule must be respected:*

$$\forall e \in HE(G), \qquad (i_G(e) \neq e) \Longrightarrow type_G(e) \neq type_G \circ i_G(e).$$

*The set of external half-edges of $G$ is:*

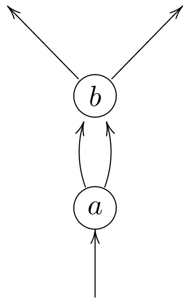
$$Ext(G) = \{e \mid e \in HE, i_G(e) = e\}.$$

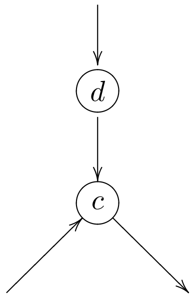
*The set of Feynman graphs is denoted by $\mathcal{FG}$.*

We shall use graphical representations of Feynman graphs: vertices are represented by $\bigcirc$, half-edges of type *out* are represented by $\bigcirc\!\!\longrightarrow$ , half-edges of type *in* by $\longrightarrow\!\!\bigcirc$; the incidence map glues two such half-edges to obtain an oriented internal edge of the Feynman graph $\bigcirc\!\!\longrightarrow\!\!\bigcirc$. In the sequel, we shall write *Feynman graph* instead of *isoclass of Feynman graphs.* We shall also consider *ordered Feynman graphs*, that is to say Feynman graphs such that the set of vertices is given a total order.

*Remark* 5. We restraint ourselves to Feynman graphs with a unique type of edges. It is possible to do the same for several type of edges, adding generators to the algebras associated to totally quasi-ordered alphabets.

*Example* 1. Here are examples of Feynman graphs:

$G =$ 


$H =$ 


**Definition 13.** *Let $G_1, \ldots, G_k$ be Feynman graphs and let $\sigma : V(G_1) \sqcup \ldots \sqcup V(G_k) \twoheadrightarrow C$ be a surjective map. We define a Feynman graph $G = \sigma(G_1, \ldots, G_k)$ in the following way:*

- $V(\sigma(G_1, \ldots, G_k)) = C$.
- *The set of half-edges of $\sigma(G_1, \ldots, G_k)$ is:*

$$HE(G) = \bigsqcup_{i=1}^{k} HE(G_i) \setminus \{e \in HE(G_i) \mid i_{G_i}(e) \neq e,\ \sigma \circ s_{G_i}(e) = \sigma \circ s_{G_i} \circ i_{G_i}(e)\}.$$

- *If $e \in HE(G_i) \cap HE(\sigma(G_1, \ldots, G_k))$, then:*

$$type_G(e) = type_{G_i}(e), \qquad i_G(e) = i_{G_i}(e), \qquad s_G(e) = \sigma \circ s_{G_i}(e).$$

Roughly speaking, $G \sqcup_\sigma H$ is obtained by identifying the vertices of the disjoint union of Feynman graphs $G_1 \ldots G_k$ with the same image by $\sigma$, and deleting the loops created in this process. In particular, if $\sigma = Id_{V(G_1) \sqcup \ldots \sqcup V(G_k)}$, or more generally if $\sigma$ is bijective, then $\sigma(G_1, \ldots, G_k)$ is (isomorphic to) the disjoint union $G_1 \ldots G_k$.

*Remark* 6. If $G_1, \ldots, G_k$ are ordered Feynman graphs, then $G_1 \ldots G_k$ is also ordered: if $x, y \in V(G_1) \sqcup \ldots \sqcup V(G_k)$,

$$x \leq y \text{ if } (x \in V(G_i),\ y \in V(G_j),\ i < j) \text{ or } (x, y \in V(G_i), x \leq y).$$

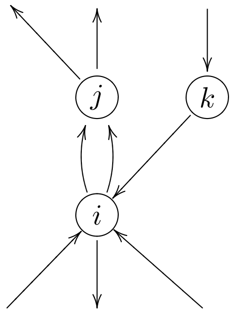
We deduce a total order on $\sigma(G_1, \ldots, G_k)$ in this way: for any $x, y \in C$,

$$x \leq y \text{ if } \min(\sigma^{-1}(x)) \leq \min(\sigma^{-1}(y)).$$

*Example* 2. If $G$ and $H$ are the Feynman graphs of Example 1 and:

$$\sigma : \begin{cases} a & \mapsto & i \\ b & \mapsto & j \\ c & \mapsto & i \\ d & \mapsto & k \end{cases}$$

then:

$F \sqcup_\sigma G =$ 

We shall use the following particular case:

**Definition 14.** *Let $G$ and $H$ be Feynman graphs, $A \subseteq V(G)$ and $\sigma : A \longrightarrow V(H)$ an injection. We define an equivalence $\sim$ on $V(G) \sqcup V(H)$ by:*

$$\forall a, b \in V(G) \sqcup V(H),\ a \sim_\sigma b \text{ if } (a = b) \text{ or } (a \in A \text{ and } b = \sigma(a)) \text{ or } (b \in A \text{ and } a = \sigma(b)).$$

*Let $\pi$ be the canonical surjection from $V(G) \sqcup V(H)$ to $(V(G) \sqcup V(H))/\sim$. The Feynman graph $\pi(G, H)$ is denoted by $G \sqcup_\sigma H$.*

In particular, if $A = \emptyset$, $G \sqcup_\sigma H$ is the disjoint union $GH$.

### 3.2 Monomials and Feynman graphs

**Definition 15.** *1. Let $M$ be a monomial of $A_q(X)$:*

$$M = \prod_{i \in X} x_i^{\epsilon_i} \prod_{i \leq_X j} x_{i,j}^{\alpha_{i,j}} \prod_{j \in X} x_{-\infty,j}^{\beta_j} \prod_{i \in X} x_{i,+\infty}^{\gamma_i}.$$

*(a) We shall say that $M$ is* admissible *if:*

$$\begin{array}{ll} \forall i, j \in X, & (\alpha_{i,j} \geq 1) \Longrightarrow (\epsilon_i = \epsilon_j = 1), \\ \forall i \in X, & (\alpha_{i,\infty} \geq 1) \Longrightarrow (\epsilon_i = 1), \\ \forall j \in X, & (\alpha_{-\infty,j} \geq 1) \Longrightarrow (\epsilon_j = 1). \end{array}$$

*(b) If $M$ is admissible, we attach to $M$ a Feynman graph $fg(M)$, defined in this way:*

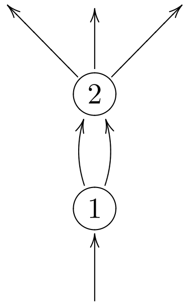
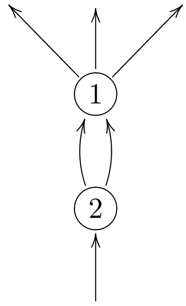
- *The set of vertices of $fg(M)$ is the set of elements $i \in X$, such that $\epsilon_i = 1$.*
- *The number of internal edges $i \longrightarrow j$ in $fg(M)$ is $\alpha_{i,j}$.*
- *The number of external edges $\longrightarrow j$ in $fg(M)$ is $\beta_j$.*
- *The number of external edges $i \longrightarrow$ in $fg(M)$ is $\gamma_i$.*

*2. Let $M$ be a monomial of $\mathbf{A}_q(X)$:*

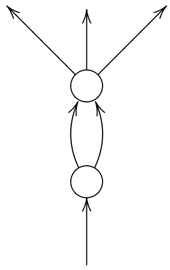
$$M = x_{i_1} \ldots x_{i_k} \prod_{i \leq_X j} x_{i,j}^{\alpha_{i,j}} \prod_{j \in X} x_{-\infty,j}^{\beta_j} \prod_{i \in X} x_{i,+\infty}^{\gamma_i}.$$

*We shall say that $M$ is admissible if its image $\overline{M}$ in $A_q(X)$ (which is a monomial) is admissible. The Feynman graph $fg(\overline{M})$ attached to the image of $M$ in the quotient $A_q(X)$ of $\mathbf{A}_q(M)$ is ordered: the set of its vertices is $\{i_1, \ldots, i_k\}$, totally ordered by $i_1 < \ldots < i_k$. This ordered Feynman graph is denoted by $fg(M)$.*

*Example* 3. Let $a, b \in X$, $a \leq_X b$, $a \neq b$. In $\mathbf{A}_q(X)$:

$$fg(x_a x_b x_{-\infty,a} x_{a,b}^2 x_{b,\infty}^3) =$$  , $$fg(x_b x_a x_{-\infty,a} x_{a,b}^2 x_{b,\infty}^3) =$$  .

In $A_q(X)$:

$$fg(x_a x_b x_{-\infty,a} x_{a,b}^2 x_{b,\infty}^3) =$$  .

**Lemma 16.** *Let $M$ and $N$ be two admissible monomials in $A_q(X)$ or in $\mathbf{A}_q(X)$. We put $A = V(fg(M)) \cap V(fg(N))$ and $\sigma : A \longrightarrow V(fg(N))$ be the canonical injection. There exists an admissible monomial $P$, such that in $A_q(X)$:*

$$MN = q^{|A|} P.$$

*Moreover:*

$$fg(P) = \pi_\sigma(fg(M), fg(N)).$$

*Proof.* We work in $A_q(X)$; the proof for $\mathbf{A}_q(X)$ is similar. We put:

$$M = \prod_{i \in B} x_i \prod_{i \leq_X j} x_{i,j}^{\alpha_{i,j}} \prod_{j \in X} x_{-\infty,j}^{\beta_j} \prod_{i \in X} x_{i,+\infty}^{\gamma_i},$$
$$M = \prod_{i \in C} x_i \prod_{i \leq_X j} x_{i,j}^{\alpha'_{i,j}} \prod_{j \in X} x_{-\infty,j}^{\beta'_j} \prod_{i \in X} x_{i,+\infty}^{\gamma'_i},$$

then, as $x_i^2 = q x_i$ in $A_q(X)$:

$$P = \prod_{i \in B \cup C} x_i^{\epsilon_i} \prod_{i \leq_X j} x_{i,j}^{\alpha_{i,j} + \alpha'_{i,j}} \prod_{j \in X} x_{-\infty,j}^{\beta_j + \beta'_j} \prod_{i \in X} x_{i,+\infty}^{\gamma_i + \gamma'_i},$$

and $MN = q^{|A|} P$. $\square$

**Theorem 17.** *1. Let $G$ be a Feynman graph and let $X$ a quasi-ordered alphabet. We put:*

$$M_G(X) = \sum_{\substack{M \text{ admissible monomial}, \\ fg(M) \approx G}} M \in A_q(X),$$
$$H_{\mathcal{FG}}(X) = Vect(M_G(X) \mid G \text{ Feynman graph}).$$

*Then $H_{\mathcal{FG}}(X)$ is a subalgebra of $A_q(X)$. For any Feynman graphs $G$ and $H$:*

$$M_G(X) M_H(X) = \sum_{\sigma : V(G) \supseteq A \hookrightarrow V(H)} q^{|A|} M_{G \sqcup_\sigma H}(X).$$

*Moreover, there exists a quasi-ordered alphabet $X$, such that the elements $M_G(X)$ are linearly independent in $A_q(X)$.*

2. (a) *Let $G$ be an ordered Feynman graph and let $X$ a quasi-ordered alphabet. We put:*

$$\mathbf{M}_G(X) = \sum_{\substack{M \text{ admissible monomial,} \\ fg(M) \approx G}} M \in \mathbf{A}_q(X),$$
$$\mathbf{H}_{\mathcal{FG}}(X) = Vect(\mathbf{M}_G(X) \mid G \text{ ordered Feynman graph}).$$

*Then $\mathbf{H}_{\mathcal{FG}}(X)$ is a subalgebra of $\mathbf{A}_q(X)$. For any ordered Feynman graphs $G$ and $H$:*

$$\mathbf{M}_G(X)\mathbf{M}_H(X) = \sum_{\sigma: V(G) \supseteq A \hookrightarrow V(H)} q^{|A|} \mathbf{M}_{G \sqcup_\sigma H}(X).$$

*There exists a quasi-ordered alphabet $X$, such that the elements $\mathbf{M}_G(X)$ are linearly independent in $\mathbf{A}_q(X)$.*

*Proof.* 1. By Lemma 16, $M_G(X)M_H(X)$ is a linear sums of terms $q^{|A|}P$, with $fg(P) = \pi_\sigma(F, G)$. It remains to show that all such terms are obtained, which is immediate. So $H_{\mathcal{FG}}(X)$ is a subalgebra of $A_q(X)$.

Let $X$ be an infinite set. We give it a relation $\leq_X$ by:

$$\forall x, y \in X, \qquad\qquad x \leq_X y,$$

making it a quasi-ordered alphabet. Let $G$ be a Feynman graph. For any $i, j \in V(G)$, we define:

- $\alpha_{i,j}$ is the number of internal edges $i \longrightarrow j$ in $G$.
- $\beta_j$ is the number of external edges $\longrightarrow j$ in $G$.
- $\gamma_i$ is the number of external edges $i \longrightarrow$ in $G$.

Let $\tau : V(G) \longrightarrow X$ be an injection: this exists, as $V(G)$ is finite and $X$ is infinite. We consider:

$$M = \prod_{i \in V(G)} x_{\tau(i)} \prod_{i,j \in V(G)} x_{\tau(i),\tau(j)}^{\alpha_{i,j}} \prod_{j \in V(G)} x_{-\infty,\tau(j)}^{\beta_j} \prod_{i \in V(G)} x_{\tau(i),\infty}^{\gamma_i}.$$

Obviously, $M$ is admissible and $fg(M) \approx G$, so $M_G(X)$ is non-zero. As the elements of $M_G(X)$ are all non-zero and their support are disjoint, they are linearly independent.

2. Similar proof. $\square$

*Remark* 7. In particular, if $q = 0$:

$$M_G(X)M_H(X) = M_{GH}(X), \qquad\qquad \mathbf{M}_G(X)\mathbf{M}_H(X) = \mathbf{M}_{G.H}(X).$$

## 3.3 The first coproduct

**Definition 18.** *Let $G$ be a Feynman graph, and let $A \subseteq V(G)$.*

1. *We define a Feynman graph $G_{|A}$ by the following:*
   - *The set of vertices of $G_{|A}$ is $A$.*
   - *The set of half-edges of $G_{|A}$ is the set of half-edges $e \in HE(G)$ such that $s_G(e) \in A$.*

- *For all half-edge $e$ of $G_{|A}$:*

$$\begin{aligned}
type_{G_{|A}}(e) &= type_G(e),\\
s_{G_{|A}}(e) &= s_G(e),\\
i_{G_{|A}}(e) &= \begin{cases} i_G(e) \text{ if } s_G(i_G(e)) \in A,\\ e \text{ otherwise.}\end{cases}
\end{aligned}$$

2. *We shall say that $A$ is an* ideal *of $G$ if for all $i, j \in V(G)$, if $i \in A$ and if there is an edge from $i$ to $j$ in $G$, then $j \in A$. The set of ideals of $G$ is denoted by $I(G)$.*

Note that if $G$ is an ordered Feynman graph, then for any $A$, $G_{|A}$ is also ordered.

**Theorem 19.** *Let $G$ be a Feynman graph. For any quasi-ordered alphabets $X$ and $Y$:*

$$\begin{aligned}
\Delta_{X,Y}(M_G(X \sqcup Y)) &= \sum_{A \in I(G)} M_{G_{|V(G)\setminus A}}(X) \otimes M_{G_{|A}}(Y),\\
\Delta_{X,Y}(\mathbf{M}_G(X \sqcup Y)) &= \sum_{A \in I(G)} \mathbf{M}_{G_{|V(G)\setminus A}}(X) \otimes \mathbf{M}_{G_{|A}}(Y).
\end{aligned}$$

*Proof.* We consider the two following sets:

- $\mathcal{A}$ is the set of triples $(M, M_1, M_2)$, where $M$, $M_1$ and $M_2$ are monomials of respectively $A_q(X \sqcup Y)$, $A_q(X)$ and $A_q(Y)$, such that $\Delta_{X,Y}(M) = M_1 \otimes M_2$ and $fg(M) = G$.
- $\mathcal{B}$ is the set of triples $(A, M_1, M_2)$, where $A$ is an ideal of $G$, $M_1$ is an admissible monomial of $A_q(X)$ such that $fg(M_1) = G_{|V(G)\setminus A}$, $M_2$ is an admissible monomial of $A_q(Y)$ such that $fg(M_2) = G_{|A}$.

Let $(M, M_1, M_2) \in \mathcal{A}$. Let $A$ be the set of vertices of $fg(M)$ belonging to $Y$. By Definition of $\Delta_{X,Y}$, as $\Delta_{X,Y}(M) = M_1 \otimes M_2$, $fg(M_1) = G_{|V(G)\setminus A}$ and $fg(M_2) = G_{|A}$. Moreover, if $i \in A$ and if there is an edge from $i$ to $j$ in $G$, this means that $x_{i,j}$ appears in $M$, so $i \leq_{X \sqcup Y} j$. As $i \in Y$, $j \in Y$, so $j \in A$: $A \in I(G)$. We define in this way a map:

$$\begin{cases} \mathcal{A} & \longrightarrow & \mathcal{B}\\ (M, M_1, M_2) & \longrightarrow & (Vert(fg(M)) \cap Y, M_1, M_2)\end{cases}$$

It is not difficult to prove that it is a bijection. Hence:

$$\begin{aligned}
\Delta_{X,Y}(M_G(X \sqcup Y)) &= \sum_{(M,M_1,M_2)\in\mathcal{A}} M_1 \otimes M_2\\
&= \sum_{(A,M_1,M_2)\in\mathcal{B}} M_1 \otimes M_2\\
&= \sum_{A \in I(G)} M_{G_{|V(G)\setminus A}}(X) \otimes M_{G_{|A}}(Y).
\end{aligned}$$

The proof in $\mathbf{A}_q(X \sqcup Y)$ is similar. □

**Corollary 20.** 1. *Let $H_{\mathcal{FG}}$ be the vector space generated by the set of Feynman graphs and let $q \in \mathbb{K}$. It is given a Hopf algebra structure: for any Feynman graphs $G$ and $H$,*

$$G._q H = \sum_{\sigma: V(G) \supseteq A \hookrightarrow V(H)} q^{|A|} G \sqcup_\sigma H \tag{3}$$

$$\Delta(G) = \sum_{A \in I(G)} G_{|V(G)\setminus A} \otimes G_{|A}. \tag{4}$$

2. *Let $\mathbf{H}_{\mathcal{FG}}$ be the vector space generated by the set of ordered Feynman graphs and let $q \in \mathbb{K}$. It is given a Hopf algebra structure by (3)-(4).*

*Proof.* 1. For any quasi-ordered alphabet $X$, there is linear map:

$$\Theta_X : \left\{ \begin{array}{rcl} H_{\mathcal{FG}} & \longrightarrow & H_{\mathcal{FG}}(X) \\ G & \longrightarrow & M_G(X). \end{array} \right.$$

By Theorems 17 and 19, for any quasi-ordered alphabets $X$ and $Y$, for any $a, b \in H_{\mathcal{FG}}$:

$$\begin{aligned} \Theta_X(a)\Theta_X(b) &= \Theta_X(ab), \\ (\Theta_X \otimes \Theta_Y) \circ \Delta(a) &= \Delta_{X,Y} \circ \Theta_{X \sqcup Y}(a). \end{aligned}$$

Choosing quasi-ordered alphabets $X$, $Y$ and $Z$, such that $\Theta_X$, $\Theta_Y$ and $\Theta_Z$ (Proposition 17), we obtain that $H_{\mathcal{FG}}$ is an algebra. For all $a, b \in H_{\mathcal{FG}}$:

$$\begin{aligned} (\Theta_X \otimes \Theta_Y) \circ \Delta(a._q b) &= \Delta_{X,Y} \circ \Theta_{X \sqcup Y}(ab) \\ &= \Delta_{X,Y}(\Theta_{X \sqcup Y}(a)\Theta_{X \sqcup Y}(b)) \\ &= \Delta_{X,Y} \circ \Theta_{X \sqcup Y}(a)\Delta_{X,Y} \circ \Theta_{X \sqcup Y}(b) \\ &= (\Theta_X \otimes \Theta_Y) \circ \Delta(a)(\Theta_X \otimes \Theta_Y) \circ \Delta(b) \\ &= (\Theta_X \otimes \Theta_Y)(\Delta(a)\Delta(b)). \end{aligned}$$

As $\Theta_X$ and $\Theta_Y$ are injective, $\Theta_X \otimes \Theta_Y$ is too, so $\Delta(ab) = \Delta(a)\Delta(b)$. Moreover:

$$\begin{aligned} (\Theta_X \otimes \Theta_Y \otimes \Theta_Z) \circ (\Delta \otimes Id) \circ \Delta &= (\Delta_{X,Y} \otimes Id) \circ \Delta_{X \sqcup Y, Z} \circ \Theta_{X \sqcup Y \sqcup Z} \\ &= (Id \otimes \Delta_{Y,Z}) \circ \Delta_{X, Y \sqcup Z} \circ \Theta_{X \sqcup Y \sqcup Z} \\ &= (\Theta_X \otimes \Theta_Y \otimes \Theta_Z) \circ (Id \otimes \Delta) \circ \Delta. \end{aligned}$$

By the injectivity of $\Theta_X \otimes \Theta_Y \otimes \Theta_Z$, $\Delta$ is coassociative, so $H_{\mathcal{FG}}$ is a bialgebra.

Observe that $(H_{\mathcal{FG}}, m, \Delta)$ is filtered by the cardinality of the set of vertices of Feynman graphs, even graded if $q = 0$; as its components of degree 0 is reduced to $\mathbb{K}$, it is connected. Hence, it is a Hopf algebra.

2. Similar proof. □

### 3.4 The second coproduct

**Definition 21.** *Let $G$ be a Feynman graph, and let $\sim$ be an equivalence on $V(G)$.*

1. *We shall say that $\sim$ is $G$-compatible if the following assertions hold:*
   - *If $A \subseteq V(G)$ is an equivalence class of $\sim$, then $G_{|A}$ is a connected Feynman graph.*
   - *If there is a path $x_1 \to x_2 \to \ldots \to x_k$ in $G$, with $x_1 \sim x_k$, then $x_1 \sim x_2 \sim \ldots \sim x_k$.*

   *The set of $G$-compatible equivalences will be denoted by $CE(G)$.*

2. *If $A_1, \ldots, A_k$ are the equivalence classes of $\sim$, we denote by $G_{|\sim}$ the disjoint union of the Feynman graphs $G_{|A_i}$, $1 \leq i \leq k$.*

3. *We denote by $G/\sim$ the following Feynman graph:*
   - *The set of vertices of $G/\sim$ is $V(G)$.*
   - *The half-edges of $G/\sim$ are the half-edges of $G$ such that $i(e) = e$ or not $s(e) \sim s(i(e))$.*

- *For any half-edge $e$ of $G/\sim$:*

$$\begin{aligned} type_{G/\sim}(e) &= type_G(e),\\ s_{G/\sim}(e) &= s_G(e),\\ i_{G/\sim}(e) &= \begin{cases} i_G(e) \text{ if not } s(e)\sim s(i(e)),\\ 0 \text{ otherwise.}\end{cases}\end{aligned}$$

Roughly speaking, $G_{|\sim}$ is obtained by the deletion in $G$ of all the internal edges whose two extremities are not $\sim$-equivalent, whereas $G/\sim$ is obtained the deletion in $G$ of all the internal edges whose two extremities are $\sim$-equivalent.

*Remark* 8. 1. If $G$ is ordered, then, as $V(G_{|\sim}) = V(G/\sim) = V(G)$, the Feynman graphs $G_{|\sim}$ and $G/\sim$ are also ordered.

2. The Feynman graph $G/\sim$ is not the usual contraction of $G$ according to a subgraph used by Connes and Kreimer to define a coproduct on Feynman graphs [2, 3, 4, 6, 5, 1, 16], as here all the vertices are conserved.

*Notations* 2. Let $q\in\mathbb{K}$. For any Feynman graph $G$, if $G_1,\ldots,G_k$ are its connected components, we put:

$$\psi_q(G) = G_1\cdot_q\ldots\cdot_q G_k\in A_q(G).$$

In particular, $\psi_0(G)=G$.

**Proposition 22.** *Let $X$, $Y$ be two quasi-ordered alphabets.*

1. *We assume that $q=q_1q_2$. We consider $\delta_{X,Y}: A_q(XY)\longrightarrow A_{q_1}(X)\otimes A_{q_2}(Y)$. For any Feynman graph $G$:*

$$\delta_{X,Y}(M_G(XY)) = \sum_{\sim\in CE(G)} M_{\psi_{q_1}(G/\sim)}(X)\otimes M_{\psi_{q_2}(G_{|\sim})}(Y).$$

2. *We consider $\delta_{X,Y}:\mathbf{A}_0(XY)\longrightarrow\mathbf{A}_0(X)\otimes\mathbf{A}_0(Y)$. For any ordered Feynman graph $G$, in $\mathbf{A}_0(X)$:*

$$\delta_{X,Y}(\mathbf{M}_G(XY)) = \sum_{\sim\in CE(G)}\mathbf{M}_{G/\sim}(X)\otimes\mathbf{M}_{G_{|\sim}}(Y).$$

*Proof.* 1. Let $M$ be a monomial of $A_q(XY)$, such that $fg(M)=G$. The set of vertices of $fg(M)$ is denoted by $V(G)=\{(i_1,j_1),\ldots,(i_k,j_k)\}$. We define an equivalence on $V(G)$ by:

$$(i_p,j_p)\sim(i_q,j_q) \text{ if } i_p\sim_X i_q.$$

Let us assume that there is a path $(i_{p_1},j_{p_1})\to\ldots\to(i_{p_k},j_{p_k})$ in $G$, with $i_{p_1}\sim_X j_{p_1}$. Then, as $G=fg(M)$, in $XY$:

$$(i_{p_1},j_{p_1})\leq_{XY}\ldots\leq_{XY}(i_{p_k},j_{p_k}),$$

so, in $X$:

$$i_{p_1}\leq_X\ldots\leq_X i_{p_k}.$$

As $i_{p_1}\sim_X i_{p_k}$, we obtain that $i_{p_1}\sim_X\ldots\sim_X i_{p_k}$, and finally:

$$(i_{p_1},j_{p_1})\sim\ldots\sim(i_{p_k},j_{p_k}).$$

Moreover, $\delta_{X,Y}(M)=M_1\otimes M_2$, with, by Definition of $\delta_{X,Y}$, $fg(M_1)=G/\sim$ and $fg(M_2)=G_{|\sim}$. Let $\sim'$ be the equivalence which equivalent classes are the connected components of $G_{|\sim}$. Then $\sim'\in CE(G)$ and:

$$G_{|\sim}=G_{|\sim'}, \qquad G/\sim\, = G/\sim'. \tag{5}$$

Moreover, $\sim'$ is the unique element of $CE(G)$ such that (5) holds. Finally, we proved that $\delta_{X,Y}$ is a sum of terms $M_1 \otimes M_2$, such that there exists $\sim' \in CE(G)$, such that $fg(M_1)$ is a monomial of $\psi_{q_1}(G/\sim')$ and $fg(M_2)$ is a monomial of $\psi_{q_2}(G_{|\sim'})$. It is not difficult to see that all these terms are obtained.

2. Similar proof. $\square$

**Corollary 23.** *1. We define two coproducts on $H_{\mathcal{FG}}$ for any Feynman graph $G$ by:*

$$\delta(G) = \sum_{\sim \in CE(G)} (G/\sim) \otimes G_{|\sim}, \tag{6}$$

$$\delta_1(G) = \sum_{\sim \in CE(G)} \psi_1(G/\sim) \otimes \psi_1(G_{|\sim}).$$

*Then $(H_{\mathcal{FG}}, ._0, \delta)$ and $(H_{\mathcal{FG}}, ._1, \delta_1)$ are bialgebras. For any $q_1, q_2 \in \mathbb{K}$, we define a coaction by:*

$$\rho_{q_1,q_2}(G) = \sum_{\sim \in CE(G)} \psi_{q_1}(G/\sim) \otimes \psi_{q_2}(G_{|\sim}).$$

*If $(q_1, q_2) = (q, 1)$ with $q \in \mathbb{K}$, or $(0,0)$, then $(H_{\mathcal{FG}}, ._{q_1}, \Delta)$ is a bialgebra in the category of right $(H_{\mathcal{FG}}, ._{q_2}, \delta_{q_2})$-comodules by the coaction $\rho_{q_1,q_2}$, that is to say:*

- *For all $x, y \in H_{\mathcal{FG}}$, $\rho_{q_1,q_2}(x ._{q_1} y) = \rho_{q_1,q_2}(x) ._{q_1,q_2} \rho_{q_1,q_2}(y)$, where:*

$$._{q_1,q_2} : \left\{ \begin{array}{rcl} H_{\mathcal{FG}}^{\otimes 4} & \longrightarrow & H_{\mathcal{FG}}^{\otimes 2} \\ x \otimes y \otimes z \otimes t & \longrightarrow & x ._{q_1} z \otimes y ._{q_2} z. \end{array} \right.$$

- $\rho_{q_1,q_2}(1) = 1 \otimes 1$.
- $(\Delta \otimes Id) \circ \rho_{q_1,q_2} = m_{1,3,24} \circ (\rho_{q_1,q_2} \otimes \rho_{q_1,q_2}) \circ \Delta$*, where:*

$$m_{1,3,24} : \left\{ \begin{array}{rcl} H_{\mathcal{FG}}^{\otimes 4} & \longrightarrow & H_{\mathcal{FG}}^{\otimes 3} \\ x \otimes y \otimes z \otimes t & \longrightarrow & x \otimes z \otimes y ._{q_2} t. \end{array} \right.$$

- *For all $x \in H_{\mathcal{FG}}$, $(\varepsilon_\Delta \otimes Id) \circ \rho_{q_1,q_2}(x) = \varepsilon_\Delta(x) 1$.*

*Note that $\rho_{0,0} = \delta$.*

*2. Similarly, (6) defines a coproduct $\delta$ on $\mathbf{A}_q(0)$, making it a bialgebra on the category of right $(H_{\mathcal{FG}}, ._0, \delta)$-comodules.*
*makes of $(H_{\mathcal{FG}}, ._0, \delta)$-comodules,*

*Proof.* We take $(q_1, q_2) = (q, 1)$ or $(0,0)$. Hence, $q_1 q_2 = q_1$. For any quasi-ordered alphabets $X$, $Y$, $(\Theta_X \otimes \Theta_Y) \circ \delta = \delta_{X,Y} \circ \Theta_{XY}$. The proof that $(H_{\mathcal{FG}}, ._0, \delta)$, $(\mathbf{H}_{\mathcal{FG}}, ._0, \delta)$, $(H_{\mathcal{FG}}, ._1, \delta_1)$ and $(\mathbf{H}_{\mathcal{FG}}, ._1, \delta_1)$ are bialgebras is similar to the proof of corollary 20. The only non-trivial remaining assertion to prove is point 3. Let us take $X, Y, Z$ alphabets such that $\Theta_X$, $\Theta_Y$ and $\Theta_Z$ are injective.

$$\begin{aligned} (\Theta_X \otimes \Theta_Y \otimes \Theta_Z) \circ (\Delta \otimes Id) \circ \delta &= (\Delta_{X,Y} \otimes Id) \circ \delta_{X \sqcup Y, Z} \circ \Theta_{XYZ} \\ &= m_{1,3,24} \circ (\delta_{X,Z} \otimes \delta_{Y,Z}) \circ \Delta_{XZ,YZ} \circ \Theta_{XYZ} \\ &= (\Theta_X \otimes \Theta_Y \otimes \Theta_Z) \circ m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta. \end{aligned}$$

We conclude by the injectivity of $\Theta_X \otimes \Theta_Y \otimes \Theta_Z$. $\square$

### 3.5 Restriction to ordered alphabets

Let $G$ be a Feynman graph. A *cycle* in $G$ is a sequence of vertices $(i_1, \ldots, i_k)$, with $k \geq 2$, all distinct, such that there exist internal edges $e_1, \ldots, e_k$, with:

$$i_1 \xrightarrow{e_1} i_2 \xrightarrow{e_2} \ldots \xrightarrow{e_{k-1}} i_k \xrightarrow{e_k} i_1$$

**Lemma 24.** *Let $G$ be a Feynman graph. The following conditions are equivalent:*

1. *For any ordered alphabet, $M_G(X) = 0$.*
2. *$G$ has a cycle.*

*Moreover, there exists an ordered alphabet $X$, such that for any Feynman graph with no cycle, $M_G(X) \neq 0$.*

*Proof.* 2. $\Longrightarrow$ 1. Let us assume that $G$ has a cycle $(i_1, \ldots, i_k)$, with $M_G(X) \neq 0$. There exists $j_1, \ldots, k_k \in X$, all distinct, such that $x_{j_1,j_2} \ldots x_{j_{k-1},j_k} x_{j_k,j_1}$ appears in $M_G(X)$, so $j_1 \leq_X \ldots \leq_X j_{k-1} \leq_X j_k \leq_X j_1$. As $\leq_X$ is an order, $j_1 = \ldots = j_k$, which is a contradiction. So $M_G(X) = 0$.

1. $\Longrightarrow$ 2. Let us prove that there exists a total order $\leq$ on $V(G)$, such that if there is an edge from $i$ to $j$ in $G$, then $i \leq j$. We proceed by induction on $|V(G)|$. If $|V(G)| = 1$, this is obvious. If $|V(G)| \geq 2$, as $G$ has no cycle, it has a source $v_1$, that is to say a vertex with no internal incoming edge. The Feynman graph $G'$ obtained from $G$ by deleting $v_0$ and all the attached half-edges has no cycle, so, by the induction hypothesis, the set of its vertices inherits a total order $v_2 \leq \ldots \leq v_k$, compatible with the internal edges of $G'$. We give $V(G)$ a total order by $v_1 \leq v_2 \leq \ldots \leq v_k$; as $v_1$ is a source, this order is compatible with the internal edges of $G$.

We take $X = \mathbb{N}$, with its usual order. There exists a monomial of $A_q(X)$ of the form:

$$M = x_1 \ldots x_k \prod_{1 \leq i \leq j \leq n} x_{i,j}^{\alpha_{i,j}} \prod_{1 \leq i \leq n} x_{i,\infty}^{\beta_i} \prod_{1 \leq j \leq n} x_{-\infty,j}^{\gamma_j},$$

such that $fg(M) = G$: $\alpha_{i,j}$ is the number of internal edges between between $v_i$ and $v_j$, $\beta_i$ is the number of incoming external edges in $v_i$, and $\gamma_j$ is the number of external half-edges outgoing from $v_j$, if $v_1 \leq \ldots \leq v_k$ is the previously defined order on the set of vertices of $G$. So $M_G(X) \neq 0$. $\Box$

Observe that if $G$ has no cycle and $\sim \in CE(G)$, then both $G_{|\sim}$ and $G/\sim$ has no cycle. Hence, considering only ordered alphabets, we obtain a Hopf algebra and a bialgebra of Feynman graphs with no cycle:

**Theorem 25.** *Let $H_{\mathcal{NCFG}}$ be the vector space generated by the set of Feynman graphs with no cycle and let $q \in \mathbb{K}$.*

1. *It is given a Hopf algebra structure: for any Feynman graphs $G$ and $H$ with no cycle,*

$$G._q H = \sum_{\substack{\sigma: V(G) \supseteq A \hookrightarrow V(H), \\ G \sqcup_\sigma H \text{ with no cycle}}} q^{|A|} G \sqcup_\sigma H, \tag{7}$$

$$\Delta(G) = \sum_{A \in I(G)} G_{|V(G) \setminus A} \otimes G_{|A}. \tag{8}$$

2. *We define a second coproduct by:*

$$\delta(G) = \sum_{\sim \in CE(G)} (G/\sim) \otimes (G_{|\sim}). \tag{9}$$

*Then $(H_{\mathcal{NCFG}}, ._0, \delta)$ is a bialgebra. Moreover, $(H_{\mathcal{NCFG}}, ._0, \Delta)$ is a bialgebra in the category of $(H_{\mathcal{NCFG}}, ._0, \delta)$-comodules.*

3. *Let us consider the map:*

$$T : \left\{ \begin{array}{rcl} H_{\mathcal{FG}} & \longrightarrow & H_{\mathcal{NCFG}} \\ G & \longrightarrow & \begin{cases} G \text{ if } G \text{ has no cycle,} \\ 0 \text{ otherwise.} \end{cases} \end{array} \right.$$

*It is a Hopf algebra morphism from* $(H_{\mathcal{FG}}, ._q, \Delta)$ *to* $(H_{\mathcal{NCFG}}, ._q, \Delta)$ *and from* $(H_{\mathcal{FG}}, ._0, \delta)$ *to* $(H_{\mathcal{NCFG}}, ._0, \delta)$.

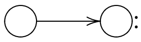
Here is its non-commutative version:

**Theorem 26.** *Let* $\mathbf{H}_{\mathcal{NCFG}}$ *be the vector space generated by the set of ordered Feynman graphs with no cycle and let* $q \in \mathbb{K}$.

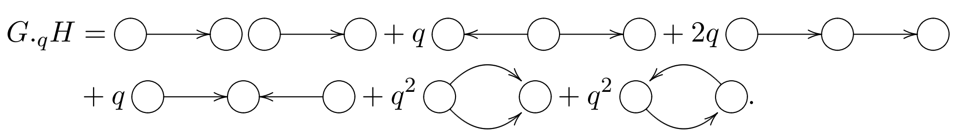
1. *It is given a Hopf algebra structure by (7)-(8).*
2. *We define a second coproduct by (9). Then* $(\mathbf{H}_{\mathcal{NCFG}}, ._0, \delta)$ *is a bialgebra. Moreover,* $(\mathbf{H}_{\mathcal{NCFG}}, ._0, \Delta)$ *is a bialgebra in the category of* $(H_{\mathcal{NCFG}}, ._0, \delta)$*-comodules.*
3. *Let us consider the map:*

$$T : \left\{ \begin{array}{rcl} \mathbf{H}_{\mathcal{FG}} & \longrightarrow & \mathbf{H}_{\mathcal{NCFG}} \\ G & \longrightarrow & \begin{cases} G \text{ if } G \text{ has no cycle,} \\ 0 \text{ otherwise.} \end{cases} \end{array} \right.$$

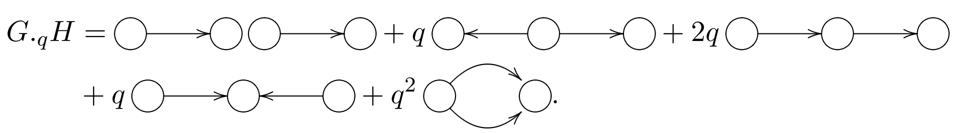
*It is a Hopf algebra morphism from* $(\mathbf{H}_{\mathcal{FG}}, ._q, \Delta)$ *to* $(\mathbf{H}_{\mathcal{NCFG}}, ._q, \Delta)$ *and from* $(\mathbf{H}_{\mathcal{FG}}, ._0, \delta)$ *to* $(\mathbf{H}_{\mathcal{NCFG}}, ._0, \delta)$.

*Remark* 9. There is a canonical injection $\iota : H_{\mathcal{NCFG}} \longrightarrow H_{\mathcal{FG}}$. This injection $\iota$ is compatible with $._q$ if, and only if, $q = 0$: indeed, as $._0$ is the disjoint union, it is compatible with $\iota$. If $q \neq 0$, taking $G = H =$ :

- In $H_{\mathcal{FG}}$:



- In $H_{\mathcal{NCFG}}$:



However, $\iota$ is compatible with $\Delta$ and $\delta$.

# 4 Quotients of Feynman graphs

## 4.1 Simple oriented graphs

Let $X$ be a quasi-ordered alphabet. We consider the following quotients of $A_q(X)$ and $\mathbf{A}_q(X)$:

$$A'_q(X) = \frac{A_q(X)}{\langle x_{i,\infty} - 1, x_{-\infty,j} - 1, x_{i,j}^2 - x_{i,j}, x_{i,i} - 1, i, j \in X \rangle},$$
$$\mathbf{A}'_q(X) = \frac{\mathbf{A}_q(X)}{\langle x_{i,\infty} - 1, x_{-\infty,j} - 1, x_{i,j}^2 - x_{i,j}, x_{i,i} - 1, i, j \in X \rangle}.$$

Elements of $A'_q[X]$ are formal spans of monomials

$$M = \prod_{i \in X} x_i^{\epsilon_i} \prod_{i \leq_X j} x_{i,j}^{\alpha_{i,j}},$$

where the $\epsilon_i, \alpha_{i,j} \in \{0,1\}$, with only a finite number of them non-zero. Elements of $\mathbf{A}'_q[X]$ are formal spans of monomials

$$M = x_{i_1} \ldots x_{i_k} \prod_{i \leq_X j} x_{i,j}^{\alpha_{i,j}},$$

where $i_1, \ldots, i_k$ are elements of $X$, all distinct, $\alpha_{i,j} \in \{0,1\}$, with only a finite number of them non-zero. The canonical surjections from $A_q(X)$ to $A'_q(X)$ or form $\mathbf{A}_q(X)$ to $\mathbf{A}'_q(X)$ are both denoted by $\varpi'_X$.

**Proposition 27.** *Let $X, Y$ be two quasi-ordered alphabets, and $q, q_1, q_2 \in \mathbb{K}$, such that $q = q_1 q_2$.*

1. *There exist unique algebra morphisms $\Delta_{X,Y} : A'_q(X \sqcup Y) \longrightarrow A'_q(X) \otimes A'_q(Y)$ and $\delta_{X,Y} : A'_q(XY) \longrightarrow A'_{q_1}(X) \otimes A'_{q_2}(Y)$, such that the following diagrams commute:*

$$\begin{tikzcd}
A_q(X \sqcup Y) \arrow[r, "\Delta_{X,Y}"] \arrow[d, "\varpi'_{X \sqcup Y}"'] & A_q(X) \otimes A_q(Y) \arrow[d, "\varpi'_X \otimes \varpi'_Y"] \\
A'_q(X \sqcup Y) \arrow[r, "\Delta_{X,Y}"'] & A'_q(X) \otimes A'_q(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
A_q(XY) \arrow[r, "\delta_{X,Y}"] \arrow[d, "\varpi'_{XY}"'] & A_{q_1}(X) \otimes A_{q_2}(Y) \arrow[d, "\varpi'_X \otimes \varpi'_Y"] \\
A'_q(XY) \arrow[r, "\delta_{X,Y}"'] & A'_{q_1}(X) \otimes A'_{q_2}(Y)
\end{tikzcd}$$

*They are given in the following way: if $i, i' \in X$, $j, j' \in Y$,*

$$\begin{aligned}
\Delta_{X,Y}(x_i) &= x_i \otimes 1, & \Delta_{X,Y}(x_{i,i'}) &= x_{i,i'} \otimes 1, \\
\Delta_{X,Y}(x_j) &= 1 \otimes x_j, & \Delta_{X,Y}(x_{j,j'}) &= 1 \otimes x_{j,j'}, \\
& & \Delta_{X,Y}(x_{i,j}) &= 1 \otimes 1,
\end{aligned}$$

$$\delta_{X,Y}(x_{(i,j)}) = x_i \otimes x_j, \qquad \delta_{X,Y}(x_{(i,j),(i',j')}) = \begin{cases} x_{i,i'} \otimes 1 \text{ if } i <_X i', \\ 1 \otimes x_{j,j'} \text{ if } i \sim_X i'. \end{cases}$$

2. *The same assertions hold if one replaces $A_q$ and $A'_q$ by $\mathbf{A}_q$ and $\mathbf{A}'_q$ everywhere.*

*Proof.* Immediate verifications. □

**Definition 28.** *Let $G$ be a Feynman graph. We denote by $S(G)$ the simple oriented graph obtained by the following procedure:*

1. *Delete all the external edges of $G$.*
2. *Delete the loops, that is to say internal edges with two identical extremities.*
3. *If $G$ has several edges from $i$ to $j$, where $i, j \in V(G)$, keep only one edge from $i$ to $j$ in $S(G)$.*

**Lemma 29.** *Let $G$ and $H$ be two Feynman graphs. The following conditions are equivalent:*

1. *$\varpi'_X(M_G(X)) = \varpi'_X(M_H(X))$ for any quasi-ordered alphabet $X$.*
2. *$S(G) = S(H)$.*

*Proof.* 2. $\Longrightarrow$ 1. Let $X$ be a quasi-ordered alphabet. As $S(G)=S(H)$, we can assume that $V(G)=V(H)$. We denote by $E$ the set of pairs $(i,j)$ such that there is an edge from $i$ to $j$ in $G$ and in $H$, with $i\neq j$. There exist non-zero scalars $\alpha_{i,j}$, $\alpha'_{i,j}$, for $(i,j)\in E$, and scalars $\alpha_{i,}$, $\alpha'_i$, $\beta_i$, $\beta'_i$, $\gamma_j$, $\gamma'_j$, and a set of injections $\Lambda$ from $V(G)$ to $X$ such that:

$$\begin{aligned}
M_G(X)&=\sum_{\sigma\in\Lambda}\prod_{i\in V(G)}x_{\sigma(i)}\prod_{(i,j)\in E}x_{\sigma(i),\sigma(j)}^{\alpha_{i,j}}\prod_{i\in V(G)}x_{\sigma(i),\sigma(i)}^{\alpha_i}\prod_{i\in V(G)}x_{\sigma(i),\infty}^{\beta_i}\prod_{i\in V(G)}x_{-\infty,\sigma(j)}^{\gamma_j},\\
M_H(X)&=\sum_{\sigma\in\Lambda}\prod_{i\in V(G)}x_{\sigma(i)}\prod_{(i,j)\in E}x_{\sigma(i),\sigma(j)}^{\alpha'_{i,j}}\prod_{i\in V(G)}x_{\sigma(i),\sigma(i)}^{\alpha'_i}\prod_{i\in V(G)}x_{\sigma(i),\infty}^{\beta'_i}\prod_{i\in V(G)}x_{-\infty,\sigma(j)}^{\gamma'_j}.
\end{aligned}$$

Their image under $\varpi'_X$ are both equal to:

$$\sum_{\sigma\in\Lambda}\prod_{i\in V(G)}x_{\sigma(i)}\prod_{(i,j)\in E}x_{\sigma(i),\sigma(j)}.$$

1. $\Longrightarrow$ 2. Let us choose an alphabet $X$, such that $M_G(X)$ and $M_H(X)$ are non-zero. Let $M$ be a monomial of $M_G(X)$:

$$M=\prod_{i\in X}x_i^{\epsilon_i}\prod_{i\leq_X j}x_{i,j}^{\alpha_{i,j}}\prod_{j\in X}x_{-\infty,j}^{\beta_j}\prod_{i\in X}x_{i,+\infty}^{\gamma_i}.$$

Then:

$$\varpi'_X(M)=\prod_{i\in X}x_i^{\epsilon_i}\prod_{i\leq_X j,\, i\neq j}x_{i,j}^{\tilde{\alpha}_{i,j}},$$

where $\tilde{\alpha}'_{i,j}=1$ if $\alpha_{i,j}\neq 0$ and 0 otherwise. This monomial appears in $M_H(X)$, so there is a monomial $M'$ in $M_H(X)$, of the form:

$$M'=\prod_{i\in X}x_i^{\epsilon_i}\prod_{i\leq_X j}x_{i,j}^{\alpha'_{i,j}}\prod_{j\in X}x_{-\infty,j}^{\beta'_j}\prod_{i\in X}x_{i,+\infty}^{\gamma'_i},$$

with, if $i\neq j$, $\alpha'_{i,j}\neq 0$ if, and only if, $\tilde{\alpha}_{i,j}=1$ if, and only if, $\alpha_{i,j}\neq 0$. This implies that $S(G)=S(H)$. $\square$

For any simple graph $G$, we denote $M_G(X)=\varpi'_X(M_H(X))$ and $\mathbf{M}_G(X)=\varpi'_X(\mathbf{M}_H(X))$, where $H$ is any Feynman graph such that $S(H)=G$ (for example, $H=G$). These elements, if they are all non-zero, form a basis of a subalgebra of $A'_q(X)$ or $\mathbf{A}'_q(X)$. We obtain a quotient of $H_{\mathcal{FG}}$ and $\mathbf{H}_{\mathcal{FG}}$ based on simple graphs. To sum up:

**Theorem 30.** *Let $H_{\mathcal{SG}}$ be the vector space generated by the set of simple graphs and let $q\in\mathbb{K}$.*

1. *It is given a Hopf algebra structure: for any simple graphs $G$ and $H$,*

$$G._qH=\sum_{\sigma:V(G)\supseteq A\hookrightarrow V(H)}q^{|A|}S(G\sqcup_\sigma H),\tag{10}$$

$$\Delta(G)=\sum_{A\in I(G)}S(G_{|V(G)\setminus A})\otimes S(G_{|A}).\tag{11}$$

2. *We define a second coproduct by:*

$$\delta(G)=\sum_{\sim\in CE(G)}S(G/\sim)\otimes S(G_{|\sim}).\tag{12}$$

*Then $(H_{\mathcal{SG}},._0,\delta)$ is a bialgebra. Moreover, $(H_{\mathcal{SG}},._0,\Delta)$ is a bialgebra in the category of $(H_{\mathcal{SG}},._0,\delta)$-comodules.*

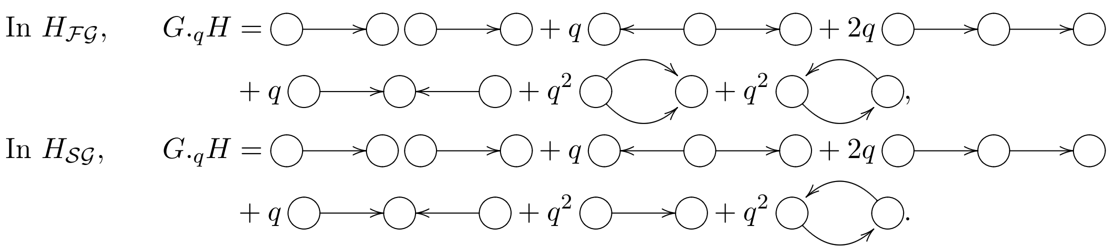
3. *Let us consider the map:*
$$S:\left\{\begin{array}{rcl} H_{\mathcal{FG}} & \longrightarrow & H_{\mathcal{SG}}\\ G & \longrightarrow & S(G).\end{array}\right.$$
*It is a Hopf algebra morphism from $(H_{\mathcal{FG}},\cdot_q,\Delta)$ to $(H_{\mathcal{SG}},\cdot_q,\Delta)$ and from $(H_{\mathcal{FG}},\cdot_0,\delta)$ to $(H_{\mathcal{SG}},\cdot_0,\delta)$.*

It has a non-commutative version:

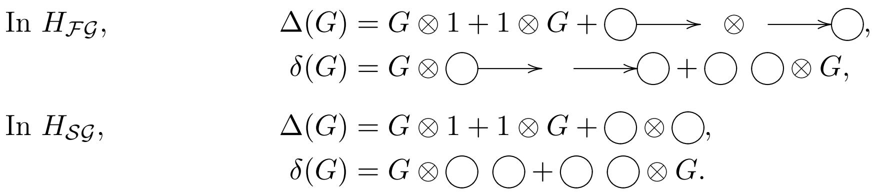
**Theorem 31.** *Let $\mathbf{H}_{\mathcal{SG}}$ be the vector space generated by the set of ordered simple graphs and let $q\in\mathbb{K}$.*

1. *It is given a Hopf algebra structure by (10)-(11).*
2. *We define a second coproduct by (12). Then $(\mathbf{H}_{\mathcal{SG}},\cdot_0,\delta)$ is a bialgebra. Moreover, $(\mathbf{H}_{\mathcal{SG}},\cdot_0,\Delta)$ is a bialgebra in the category of $(H_{\mathcal{SG}},\cdot_0,\delta)$-comodules.*
3. *Let us consider the map:*
$$S:\left\{\begin{array}{rcl} \mathbf{H}_{\mathcal{FG}} & \longrightarrow & \mathbf{H}_{\mathcal{SG}}\\ G & \longrightarrow & S(G).\end{array}\right.$$
*It is a Hopf algebra morphism from $(\mathbf{H}_{\mathcal{FG}},\cdot_q,\Delta)$ to $(\mathbf{H}_{\mathcal{SG}},\cdot_q,\Delta)$ and from $(\mathbf{H}_{\mathcal{FG}},\cdot_0,\delta)$ to $(\mathbf{H}_{\mathcal{SG}},\cdot_0,\delta)$.*

*Remark* 10. As simple graphs are Feynman graphs, there is a canonical injection $\kappa: H_{\mathcal{SG}}\longrightarrow\mathcal{FG}$. It is compatible with $\cdot_q$ if, and only if, $q=0$. Indeed, the disjoint union product $\cdot_0$ is compatible with $\kappa$. If $q\neq 0$, taking $G=H=\bigcirc\!\!\longrightarrow\!\!\bigcirc$:




Hence, $\kappa$ is not compatible with $\cdot_q$. Moreover, $\kappa$ is not compatible with $\Delta$ and $\delta$. For example, if $G=\bigcirc\!\!\longrightarrow\!\!\bigcirc$:




## 4.2 Simple graphs with no cycle

**Lemma 32.** *Let $G$ be a simple graph. The following conditions are equivalent:*

1. *For any ordered alphabet, $M_G(X)=0$.*
2. *$G$ has a cycle.*

*Moreover, there exists an ordered alphabet $X$, such that for any simple graph with no cycle, $M_G(X)\neq 0$.*

*Proof.* Similar to the proof of Lemma 24. $\square$

Consequently, we obtain Hopf algebras structures on simple graphs with no cycle:

**Theorem 33.** *Let $H_{\mathcal{NCSG}}$ be the vector space generated by the set of simple graphs with no cycle and let $q \in \mathbb{K}$. It is given a Hopf algebra structure: for any simple graphs with no cycle $G$ and $H$,*

$$G \cdot_q H = \sum_{\substack{\sigma : V(G) \supseteq A \hookrightarrow V(H), \\ G \sqcup_\sigma H \text{ with no cycle}}} q^{|A|} S(G \sqcup_\sigma H), \tag{13}$$

$$\Delta(G) = \sum_{A \in I(G)} S(G_{|V(G)\setminus A}) \otimes S(G_{|A}). \tag{14}$$

*Let us consider the map:*

$$T : \begin{cases} H_{\mathcal{SG}} & \longrightarrow & H_{\mathcal{NCSG}} \\ G & \longrightarrow & \begin{cases} G \text{ if } G \text{ has no cycle,} \\ 0 \text{ otherwise.} \end{cases} \end{cases}$$

*It is a Hopf algebra morphism from $(H_{\mathcal{SG}}, \cdot_q, \Delta)$ to $(H_{\mathcal{NCSG}}, \cdot_q, \Delta)$.*

Here is its non-commutative version:

**Theorem 34.** *Let $\mathbf{H}_{\mathcal{NCSG}}$ be the vector space generated by the set of ordered simple graphs with no cycle and let $q \in \mathbb{K}$. It is given a Hopf algebra structure by (13)-(14). Let us consider the map:*

$$T : \begin{cases} \mathbf{H}_{\mathcal{FG}} & \longrightarrow & \mathbf{H}_{\mathcal{NCFG}} \\ G & \longrightarrow & \begin{cases} G \text{ if } G \text{ has no cycle,} \\ 0 \text{ otherwise.} \end{cases} \end{cases}$$

*It is a Hopf algebra morphism from $(\mathbf{H}_{\mathcal{SG}}, \cdot_q, \Delta)$ to $(\mathbf{H}_{\mathcal{NCSG}}, \cdot_q, \Delta)$ and from $(\mathbf{H}_{\mathcal{SG}}, \cdot_0, \delta)$ to $(\mathbf{H}_{\mathcal{NCSG}}, \cdot_0, \delta)$.*

### 4.3 quasiposets

Let $X$ be a quasi-ordered alphabet. We consider the following quotients of $A'_q(X)$ and $\mathbf{A}'_q(X)$:

$$A''_q(X) = \frac{A'_q(X)}{\langle x_{i,j} x_{j,k}(x_{i,k} - 1), i \leq_X j \leq_X k \rangle},$$
$$\mathbf{A}''_q(X) = \frac{\mathbf{A}'_q(X)}{\langle x_{i,j} x_{j,k}(x_{i,k} - 1), i \leq_X j \leq_X k \rangle}.$$

The canonical surjection from $A'_q(X)$ to $A''_q(X)$ and from $\mathbf{A}'_q(X)$ to $\mathbf{A}''_q(X)$ are both denoted by $\varpi''_X$.

**Definition 35.** *Let $G$ be a simple oriented graph, which set of vertices is denoted by $V(G)$. We define a quasi-order on $V(G)$, by:*

$$\forall i, j \in V(G),\ i \leq_G j \text{ if there exists a path from } i \text{ to } j \text{ in } G.$$

Note that any quasi-poset, that is to say any pair $P = (V(P), \leq_P)$, where $V(P)$ is a finite set and $\leq_P$ is a quasi-order on $V(P)$, can be obtained in this way: consider the arrow diagram $G$ of $P$, which is a simple graph such that $\leq_G = \leq_P$.

**Lemma 36.** *Let $X$ be a quasi-ordered alphabet. For any simple graphs $G$, $H$, the following conditions are equivalent:*

1. *$\varpi''_X(M_G(X)) = \varpi''_X(M_H(X))$ for any quasi-ordered alphabet $X$.*

2. *The quasi-posets $(V(G),\leq_G)$ and $(V(H),\leq_H)$ are isomorphic.*

*Proof.* We define a congruence on the set of monomials of $A'_q(X)$ by $x_{i,j}x_{j,k}x_{i,k} \equiv x_{i,j}x_{j,k}$. Then two monomials of $A'_q(X)$ have the same image under $\varpi''_X$ if, and only if, they are congruent. At the level of graphs, this gives that for any quasi-ordered alphabet $X$, $\varpi''_X(M_G(X)) = \varpi''_X(M_H(X))$ if, and only if, one can go from $G$ to $H$ by a sequence of transformations:

$$\left(\begin{tikzcd} i \arrow[r] \arrow[rr, bend left] & j \arrow[r] & k \end{tikzcd}\right) \longleftrightarrow \left(\begin{tikzcd} i \arrow[r] & j \arrow[r] & k \end{tikzcd}\right)$$

that is to say if, and only if, $\leq_G$ and $\leq_H$ are isomorphic. $\square$

**Proposition 37.** *1. Let $X,Y$ be two quasi-ordered alphabets. There exists a unique algebra morphism $\Delta_{X,Y} : A''_q(X\sqcup Y) \longrightarrow A''_q(X)\otimes A''_q(Y)$ such that the following diagram commutes:*

$$\begin{tikzcd} A'_q(X\sqcup Y) \arrow[r, "\Delta_{X,Y}"] \arrow[d, "\varpi'_{X\sqcup Y}"'] & A'_q(X)\otimes A'_q(Y) \arrow[d, "\varpi'_X\otimes\varpi'_Y"] \\ A''_q(X\sqcup Y) \arrow[r, "\Delta_{X,Y}"'] & A''_q(X)\otimes A''_q(Y) \end{tikzcd}$$

2. *The same assertion hold, after replacing $A_q$ and $A'_q$ by $\mathbf{A}_q$ and $\mathbf{A}'_q$ everywhere.*

*Proof.* 1. If $i \leq_{X\sqcup Y} j \leq_{X\sqcup Y} k$, in $A'_q(X\sqcup Y)$:

$$\Delta_{X,Y}(x_{i,j}x_{j,k}(x_{j,k}-1)) = \begin{cases} x_{i,j}x_{j,k}(x_{j,k}-1)\otimes 1 \text{ if } i,k\in X,\\ 1\otimes x_{i,j}x_{j,k}(x_{j,k}-1) \text{ if } i,k\in Y,\\ 0 \text{ if } i\in i\in X, k\in Y. \end{cases}$$

So $\Delta_{X,Y}$ is defined from $A''_q(X\sqcup Y)$ to $A''_q(X)\otimes A''_q(Y)$.

2. Similar proof. $\square$

*Remark* 11. Unfortunately, this does not work for $\delta_{X,Y}$, except if $X$ is a totally ordered alphabet.

For any quasi-poset $P$, we denote $M_P(X) = \varpi''_X(M_G(X))$ and $\mathbf{M}_G(X) = \varpi''_X(\mathbf{M}_G(X))$, where $G$ is any simple graph such that $\leq_G = \leq_P$, for example the arrow graph or the Hasse graph of $\leq_P$. These elements, if all non-zero, are a basis of a subalgebra of $A''_q(X)$ or $\mathbf{A}''_q(X)$. We obtain a quotient of $H_{\mathcal{SG}}$ and $\mathbf{H}_{\mathcal{SG}}$ based on quasi-posets. We shall need the following definitions to describe it:

**Definition 38.** *Let $P,Q$ be two quasi-posets, $A\subseteq V(P)$ and $\sigma : A \hookrightarrow V(Q)$ be an injective map. We consider the quotient, already used in Definition 14:*

$$V(P)\sqcup_\sigma V(Q) = (V(P)\sqcup V(Q))/(a=\sigma(a), a\in A).$$

*We define a relation $\mathcal{R}$ on $V(P)\sqcup V(Q)$ by:*

$$\forall i,j\in V(P)\sqcup_\sigma V(Q),\ i\mathcal{R}j \text{ if } (i,j\in V(P), i\leq_P j) \text{ or } (i,j\in V(Q), i\leq_Q j).$$

*The transitive closure of $\mathcal{R}$ is denoted by $\leq_{P\sqcup_\sigma Q}$, and $P\sqcup_\sigma Q = (V(P)\sqcup_\sigma V(Q), \leq_{P\sqcup_\sigma Q})$ is a quasi-poset.*

Note that if $P$ and $Q$ are ordered quasi-posets, then $P\sqcup_\sigma Q$ is also an ordered quasi-poset.

**Definition 39.** *Let $P$ be a quasi-poset and let $A\subseteq V(P)$.*

1. *We denote by $P_{|A}$ the quasi-poset $(A,(\leq_P)_{|A})$. Note that if $P$ is a poset, $P_{|A}$ is too.*

2. *We shall say that $A$ is an ideal (or an open set) of $P$ if:*

$$\forall i, j \in V(P), \qquad (i \in A \text{ and } i \leq_P j) \Longrightarrow j \in A.$$

*The set of ideals of $P$ is denoted by $I(P)$.*

**Theorem 40.** *Let $H_{\mathcal{QP}}$ be the vector space generated by the set of quasi-posets and let $q \in \mathbb{K}$.*

1. *It is given a Hopf algebra structure: for any quasi-posets $P, Q$,*

$$P._q Q = \sum_{\sigma: V(P) \supseteq A \hookrightarrow V(Q)} q^{|A|} P \sqcup_\sigma Q, \tag{15}$$

$$\Delta(G) = \sum_{A \in I(P)} P_{|V(P) \setminus A} \otimes P_{|A}. \tag{16}$$

2. *Let us consider the map:*

$$P : \left\{ \begin{array}{rcl} H_{\mathcal{SG}} & \longrightarrow & H_{\mathcal{QP}} \\ G & \longrightarrow & p(G) = (V(G), \leq_G) \end{array} \right.$$

*It is a Hopf algebra morphism from $(H_{\mathcal{SG}}, ._q, \Delta)$ to $(H_{\mathcal{QP}}, ._q, \Delta)$.*

Here is its non-commutative version:

**Theorem 41.** *Let $\mathbf{H}_{\mathcal{QP}}$ be the vector space generated by the set of ordered quasi-posets and let $q \in \mathbb{K}$.*

1. *It is given a Hopf algebra structure by (15)-(16).*

2. *Let us consider the map:*

$$P : \left\{ \begin{array}{rcl} \mathbf{H}_{\mathcal{SG}} & \longrightarrow & \mathbf{H}_{\mathcal{QP}} \\ G & \longrightarrow & p(G) = (V(G), \leq_G) \end{array} \right.$$

*It is a Hopf algebra morphism from $(\mathbf{H}_{\mathcal{SG}}, ._q, \Delta)$ to $(\mathbf{H}_{\mathcal{QP}}, ._q, \Delta)$.*

The Hopf algebras $(H_{\mathcal{QP}}, ._0, \Delta)$ and $\mathbf{H}_{\mathcal{QP}}, ._0, \Delta)$ are introduced and studied in [12, 13, 14, 10].

## 4.4 Posets

**Lemma 42.** *Let $P$ be a quasi-poset. The following conditions are equivalent:*

1. *For any ordered alphabet $X$, $M_P(X) = 0$.*

2. *$P$ is a poset.*

*Moreover, there exists an ordered alphabet $X$, such that for any poset $P$, $M_P(X) \neq 0$.*

*Proof.* Similar to the proof of Lemma 24. □

Consequently, we obtain a Hopf algebra structure on posets.

**Theorem 43.** *Let $H_{\mathcal{P}}$ be the vector space generated by the set of posets and let $q \in \mathbb{K}$.*

1. *It is given a Hopf algebra structure: for any posets $P$ and $Q$,*

$$P._q Q = \sum_{\substack{\sigma: V(P) \supseteq A \hookrightarrow V(Q), \\ P \sqcup_\sigma Q \text{ poset}}} q^{|A|} S(P \sqcup_\sigma Q), \tag{17}$$

$$\Delta(P) = \sum_{A \in I(P)} P_{|V(P) \setminus A} \otimes P_{|A}. \tag{18}$$

2. *Let us consider the map:*

$$P:\left\{\begin{array}{rcl} H_{\mathcal{NCSG}} & \longrightarrow & H_{\mathcal{P}} \\ G & \longrightarrow & P(G)=(V(G),\leq_G). \end{array}\right.$$

*It is a Hopf algebra morphism from* $(H_{\mathcal{NCSG}},\cdot_q,\Delta)$ *to* $(H_{\mathcal{P}},\cdot_q,\Delta)$.

Here is its non-commutative version:

**Theorem 44.** *Let* $\mathbf{H}_{\mathcal{P}}$ *be the vector space generated by the set of ordered posets and let* $q\in\mathbb{K}$.

1. *It is given a Hopf algebra structure by (17)-(18).*
2. *Let us consider the map:*

$$P:\left\{\begin{array}{rcl} \mathbf{H}_{\mathcal{NCSG}} & \longrightarrow & \mathbf{H}_{\mathcal{P}} \\ G & \longrightarrow & P(G)=(V(G),\leq_G). \end{array}\right.$$

*It is a Hopf algebra morphism from* $(\mathbf{H}_{\mathcal{NCSG}},\cdot_q,\Delta)$ *to* $(\mathbf{H}_{\mathcal{P}},\cdot_q,\Delta)$.

*Remark* 12. Using Remark 11, we could use totally ordered alphabets to define a second coproduct on $H_{\mathcal{P}}$. We would obtain the coproduct given for any poset $P$ of order $n$ by:

$$\delta(P)=P\otimes\bullet^n,$$

which is coassocative, with a right counit but no left counit.

### 4.5 Dual product

We now describe the dual product of $\Delta$ on $H_{\mathcal{P}}$. We identify $H_{\mathcal{P}}$ and its graded dual through the symmetric pairing defined for any pair $(P,Q)$ of posets by:

$$\langle P,Q\rangle=s_P\delta_{P,Q},$$

where $s_P$ is the number of automorphisms of $P$.

**Proposition 45.** *Let us define a coproduct* $\blacktriangle$ *on* $\mathcal{H}_{\mathcal{P}}$ *in the following way: If* $P$ *is a poset, denoting by* $P_1,\ldots,P_k$ *its connected components,*

$$\blacktriangle(P)=\sum_{I\subseteq[k]}\prod_{i\in I}P_i\otimes\prod_{i\notin I}P_i.$$

*Then* $\blacktriangle$ *is coassociative and counitary, and for any* $x,y,z\in H_{\mathcal{P}}$*:*

$$\langle x,y\cdot_0 z\rangle=\langle\blacktriangle(x),y\otimes z\rangle.$$

*Proof.* The coassociativity of $\blacktriangle$ is immediate. Its counit is the map defined by $\varepsilon(P)=\delta_{P,1}$ for any poset $P$. Let $P$, $Q$, $R$ be three posets. Let us denote by $P_1,\ldots,P_k$ the different isoclasses of connected components of $P$, $Q$ and $R$. There exist $\alpha=(\alpha_i)_{i\in[k]}$, $\beta=(\beta_i)_{i\in[k]}$, $\gamma=(\gamma_i)_{i\in[k]}$, such that:

$$P=\prod_{i\in[k]}P_i^{\alpha_i},\qquad Q=\prod_{i\in[k]}P_i^{\beta_i},\qquad R=\prod_{i\in[k]}P_i^{\gamma_i}.$$

Moreover:

$$s_P=\prod_{i\in[k]}s_{P_i}^{\alpha_i}\alpha_i!.$$

Hence:

$$\langle P, QR\rangle = \delta_{\alpha,\beta+\gamma} \prod_{i\in[k]} s_{P_i}^{\alpha_i}\alpha_i!.$$

Moreover:

$$\begin{aligned}
\blacktriangle(P) &= \sum_{\alpha'+\alpha''=\alpha} \prod_{i\in[k]} \frac{\alpha_i!}{\alpha_i'!\alpha_i''!} \prod_{i\in[k]} P_i^{\alpha_i'} \otimes \prod_{i\in[k]} P_i^{\alpha_i''},\\
\langle \blacktriangle(P), Q\otimes R\rangle &= \sum_{\alpha'+\alpha''=\alpha} \prod_{i\in[k]} \frac{\alpha_i!}{\alpha_i'!\alpha_i''!} \alpha_i'!\alpha_i''! s_{P_i}^{\alpha_i'+\alpha_i''} \delta_{\alpha',\beta}\delta_{\alpha'',\gamma}\\
&= \prod_{i\in[k]} \alpha_i! s_{P_i}^{\alpha_i} \delta_{\alpha,\beta+\gamma}.
\end{aligned}$$

Finally, $\langle \blacktriangle(P), Q\otimes R\rangle = \langle P, QR\rangle$. $\square$

**Definition 46.** *Let $P = (V(P), \leq_P)$ and $Q = (V(Q), \leq_Q)$. We denote by $\mathcal{P}(V(Q))$ the set of subsets of $V(Q)$. A* system of edges *from $P$ to $Q$ is a map $\Theta : V(P) \longrightarrow \mathcal{P}(V(Q))$ such that:*

1. *For any $x, y \in V(P)$, such that $x <_P y$, then for any $x' \in \Theta(x)$, $y' \in \Theta(y)$, we do not have $y' \leq_Q x'$.*
2. *For any $x \in V(P)$, for any $x', x'' \in \Theta(x)$, then $x' \leq_Q x''$ if, and only if, $x' = x''$.*

**Proposition 47.** *Let $P$, $Q$ be two posets and $\Theta$ be a system of edges from $P$ to $Q$. We define an order on $V(P) \sqcup V(Q)$ in the following way:*

- *For any $x, y \in V(P)$, $x \leq y$ if $x \leq_P y$.*
- *For any $x, y \in V(Q)$, $x \leq y$ if $x \leq_Q y$.*
- *For any $x \in V(P)$, $y \in V(Q)$, $x \leq y$ if there exists $x' \in V(P)$, $y' \in \Theta(x')$, such that $x \leq_P x'$ and $y' \leq_Q y$.*

*This poset is denoted by $P \sqcup_\Theta Q$. Moreover, $V(Q)$ is an ideal of $P \sqcup_\Theta Q$ and the edges of the Hasse graph of $P \sqcup_\Theta Q$ are:*

- *The edges of the Hasse graph of $P$,*
- *The edges of the Hasse graph of $Q$,*
- *The edges $(x, x')$, with $x \in V(P)$ and $x' \in \Theta(x)$.*

*Proof.* Note that if $x \leq y$, we cannot have $x \in V(Q)$ and $y \in V(P)$.

$\leq$ is obviously reflexive. If $x \leq y$ and $y \leq x$, then both $x$ and $y$ belong to $V(P)$, or both belong to $V(Q)$. Hence, $x \leq_P y$ and $y \leq_P x$, or $x \leq_Q y$ and $y \leq_Q x$. In both cases, $x = y$, so $\leq$ is antisymmetric. Let us assume that $x \leq y$ and $y \leq z$. Four cases hold.

1. $(x, y, z) \in V(P)^3$. Then $x \leq_P y \leq_P z$, so $x \leq z$.
2. $(x, y, z) \in V(P)^2 \times V(Q)$. There exist $y' \in V(P)$, $z' \in \Theta(x')$, such that $x \leq_P y \leq_P y'$ and $z' \leq_Q z$. So $x \leq z$.
3. $(x, y, z) \in V(P) \times V(Q)^2$. There exist $x' \in V(P)$, $y' \in \Theta(x')$, such that $x \leq_P x'$ and $y' \leq_Q y \leq_Q z$. So $x \leq z$.
4. $(x, y, z) \in V(Q)^3$. Then $x \leq_Q y \leq_Q z$, so $x \leq z$.

Hence, $\leq$ is an order.

Let $(x, y)$ be an edge of the Hasse graph of $P \sqcup_\Theta Q$, that is to say:

- $x < y$.
- If $x \leq z \leq y$, then $z \in \{x, y\}$.

Three cases are possible.

1. $(x, y) \in V(P)^2$. As $(P \sqcup Q)_{|V(P)} = P$, $(x, y)$ is an edge of the Hasse graph of $P$.
2. $(x, y) \in V(Q)^2$. As $(P \sqcup Q)_{|V(Q)} = Q$, $(x, y)$ is an edge of the Hasse graph of $Q$.
3. $(x, y) \in V(P) \times V(Q)$. Then there exists $x' \in V(P)$, $y' \in \Theta(x')$, such that $x \leq_P x'$ and $y' \leq_Q y$. By definition, $x' < y'$, so $x \leq x' < y' \leq y$. As $(x, y)$ is an edge, $x = x'$ and $y = y'$, so $y \in \Theta(x)$.

Conversely:

- Let $(x, y)$ be an edge of the Hasse graph of $P$. If $x \leq z \leq y$, necessarily $z \in V(P)$ as $y \in V(P)$, so $x \leq_P z \leq_P y$, which implies $z \in \{x, y\}$.
- Similarly, if $(x, y)$ is an edge of the Hasse graph of $Q$, then it is an edge of the Hasse graph of $P \sqcup_\Theta Q$.
- If $x \in V(P)$ and $y \in \Theta(x)$, then $x < y$. If $x \leq z \leq y$, two cases are possible.
  1. If $z \in V(P)$, there exists $z' \in V(P)$, $y' \in \Theta(z')$, such that $x \leq_P z \leq_P z'$, $y' \leq_Q y$. So $x \leq z \leq z' < y' \leq y$. As $(x, y)$ is an edge, $x = z = z'$ and $y' = y$.
  2. Similarly, if $z \in V(Q)$, $z = y$.

Obviously, $V(Q)$ is an ideal of $P \sqcup_\Theta Q$. $\square$

**Lemma 48.** *Let $P_1, P_2, P$ be three posets. We consider the two following sets:*

- *$C(P_1, P_2, P)$ is the set of triple $(I, \phi_1, \phi_2)$, where $I$ is an ideal of $P$, $\phi_1$ is an isomorphism from $P_1$ to $P_{|V(P) \setminus I}$ and $\phi_2$ is an isomorphism from $P_2$ to $P_{|I}$.*
- *$D(P_1, P_2, P)$ is the set of pairs $(\Theta, \phi)$, where $\Theta$ is a system of edges from $P_1$ to $P_2$ and $\phi$ is an isomorphism from $P_1 \sqcup_\Theta P_2$ to $P$.*

*Then $C(P_1, P_2, P)$ and $D(P_1, P_2, P)$ are in bijection.*

*Proof.* We shortly denote $C = C(P_1, P_2, P)$ and $D = D(P_1, P_2, P)$.

*First step.* Let $F : C \longrightarrow D$ defined by $F(I, \phi_1, \phi_2) = (\Theta, \phi)$, with:

- $\phi$ is defined by $\phi_{|V(P_1)} = \phi_1$ and $\phi_{|V(P_2)} = \phi_2$.
- For any $x \in V(P_1)$, $\Theta(x)$ is the set of $y \in V(P_2)$ such that $(\phi_1(x), \phi_2(y))$ is an edge of the Hasse graph of $P$.

Let us prove that $F$ is well-defined.

Let $x, y \in V(P_1)$, $x' \in \Theta(x)$, $y' \in \Theta(y)$, such that $x <_{P_1} y$ and $y' \leq_{P_2} x'$. Then $\phi_1(x) <_P \phi_1(y)$, $\phi_2(y') \leq_P \phi_2(x')$, $(\phi_1(x), \phi_2(x'))$ and $(\phi_1(y), \phi_1(y'))$ are edges of the Hasse graph of $P$. We obtain:

$$\phi_1(x) <_P \phi_1(y) <_P \phi_2(y') \leq_P \phi_2(x').$$

This contradicts that $(\phi_1(x), \phi_2(x'))$ is an edge of the Hasse graph of $P$.

Let $x \in V(P_1)$, $x', x'' \in \Theta(x)$, with $x' \leq_{P_2} x''$. Then $\phi_1(x) <_P \phi_2(x') \leq \phi_2(x'')$. As $(\phi_1(x), \phi_2(x''))$ is an edge, $\phi_2(x') = \phi_2(x'')$, so $x' = x''$. We proved that $\Theta$ is a system of edges from $P_1$ to $P_2$.

By the preceding lemma, the image by $\phi$ of the edges of $P_1 \sqcup_\Theta P_2$ are the edges of $P$, so $\phi$ is an isomorphism. We proved that $F$ is well-defined.

*Second step.* Let $G : D \longrightarrow C$ defined by $G(\Theta, \phi) = (I, \phi_1, \phi_2)$, where:

- $I = \phi(V(P_2))$.
- $\phi_1 = \phi_{|V(P_1)}$ and $\phi_2 = \phi_{|V(P_2)}$.

By the preceding lemma, $G$ is well-defined.

*Last step.* Let $(I, \phi_1, \phi_2) \in C$. We put $F(I, \phi_1, \phi_2) = (\Theta, \phi)$ and $G(\Theta, \phi) = (I', \phi'_1, \phi'_2)$. Then $I' = \phi(V(P_2)) = I$, $\phi'_1 = \phi_{|V(P_1)} = \phi_1$ and similarly, $\phi'_2 = \phi_2$. So $G \circ F = Id_C$.

Let $(\Theta, \phi) \in D$. We put $G(\Theta, \phi) = (I, \phi_1, \phi_2)$ and $F(I, \phi_1, \phi_2) = (\Theta', \phi')$. For any $x \in V(P_1)$, $y \in V(P_2)$:

$$\begin{aligned} y \in \Theta'(x) &\Longleftrightarrow (\phi_1(x), \phi_2(y)) \text{ edge of the Hasse graph of } P\\ &\Longleftrightarrow (\phi(x), \phi(y)) \text{ edge of the Hasse graph of } P\\ &\Longleftarrow (x, y) \text{ edge of the Hasse graph of } P \sqcup_\Theta Q\\ &\Longleftrightarrow y \in \Theta(x). \end{aligned}$$

Therefore, $\Theta' = \Theta$. Moreover, for $i \in \{1, 2\}$, $\phi'_{|V(P_i)} = \phi_i = \phi_{|V(P_i)}$, so $\phi' = \phi$. Hence, $F \circ G = Id_D$. □

**Theorem 49.** *We define a product $\star$ on $H_\mathcal{P}$ in the following way: for any posets $P, Q$,*

$$P \star Q = \sum_{\substack{\Theta \text{ system of edges}\\ \text{from } P \text{ to } Q}} P \sqcup_\Theta Q.$$

*Then $(\mathcal{H}_\mathcal{P}, \star, \blacktriangle)$ is a Hopf algebra. Moreover, for any $x, y, z \in H_\mathcal{P}$:*

$$\langle x \star y, z \rangle = \langle x \otimes y, \Delta(z) \rangle.$$

*Proof.* Let $P_1, P_2, P$ be posets. By the preceding lemma:

$$\begin{aligned} \langle P_1 \otimes P_2, \Delta(P) \rangle &= \sum_{I \text{ ideal of } P} \langle P_1, P_{|V(P) \setminus I} \rangle \langle P_2, P_{|I} \rangle\\ &= \sharp C(P_1, P_2, P)\\ &= \sharp D(P_1, P_2, P)\\ &= \sum_{\substack{\Theta \text{ system of edges}\\ \text{from } P_1 \text{ to } P_2}} \langle P_1 \sqcup_\Theta P_2, P \rangle\\ &= \langle P_1 \star P_2, P \rangle. \end{aligned}$$

As $\langle -, - \rangle$ is non degenerate and $(H_\mathcal{P}, m, \Delta)$ is a Hopf algebra, dually $(\mathcal{H}_\mathcal{P}, \star, \blacktriangle)$ is a Hopf algebra.

Let $x, y, z \in H_\mathcal{P}$. For any $t \in H_\mathcal{P}$:

$$\begin{aligned} \langle (x \star y) * z, t \rangle &= \langle x \otimes y \otimes z, (\Delta \otimes Id) \circ \delta(t) \rangle\\ &= \langle x \otimes y \otimes z, m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta(t) \rangle\\ &= \langle x \otimes z' \otimes y \otimes z'', (\delta \otimes \delta) \circ \Delta(t) \rangle\\ &= \langle (x * z') \star (x * z''), t \rangle. \end{aligned}$$

We conclude by the non degeneracy of $\langle -, - \rangle$. □

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