Free quadri-algebras and dual quadri-algebras

Loïc Foissy

Univ. Littoral Côte d'Opale, UR 2597 LMPA, Laboratoire de Mathématiques Pures et Appliquées
Joseph Liouville F-62100 Calais, France.
Email: foissy@univ-littoral.fr

Abstract

We study quadri-algebras and dual quadri-algebras. We describe the free quadri-algebra on one generator as a subobject of the Hopf algebra of permutations **FQSym**, proving a conjecture due to Aguiar and Loday, using that the operad of quadri-algebras can be obtained from the operad of dendriform algebras by both black and white Manin products. We also give a combinatorial description of free dual quadri-algebras. A notion of quadri-bialgebra is also introduced, with applications to the Hopf algebras **FQSym** and **WQSym**.

AMS classification. 16W10; 18D50; 16T05.

Keywords. Quadri-algebras; Koszul duality; Combinatorial Hopf algebras.

Contents

Introduction

An algebra with an associativity splitting is an algebra whose associative product $\star$ can be written as a sum of a certain number of (generally nonassociative) products, satisfying certain compatibilities. For example, dendriform algebras [7, 11] are equipped with two bilinear products $\prec$ and $\succ$, such that for all x, y, z:

$$\begin{aligned}(x \prec y) \prec z &= x \prec (y \prec z + y \succ z),\\ (x \succ y) \prec z &= x \succ (y \prec z),\\ (x \prec y + x \succ y) \succ z &= x \succ (y \succ z).\end{aligned}$$

Summing these axioms, we indeed obtain that $\star = \prec + \succ$ is associative. Another example is given by quadri-algebras, which are equipped with four products $\nwarrow$, $\swarrow$, $\searrow$ and $\nearrow$, in such a way that:

- $\leftarrow = \nwarrow + \swarrow$ and $\rightarrow = \searrow + \nearrow$ are dendriform products,
- $\uparrow = \nwarrow + \nearrow$ and $\downarrow = \swarrow + \searrow$ are dendriform products.

Shuffle algebras or the algebra of free quasi-symmetric functions **FQSym** are examples of quadri-algebras. No combinatorial description of the operad **Quad** of quadri-algebra is known, but a formula for its generating formal series is conjectured in [1] and proved in [19], as well as the koszulity of this operad, see also [14]. A description of **Quad** is given with the help of the black Manin product on nonsymmetric operads $\blacksquare$, namely **Quad** = **Dend** $\blacksquare$ **Dend**, where **Dend** is the nonsymmetric operad of dendriform algebras [1]. It is also suspected that the sub-quadri-algebra of **FQSym** generated by the permutation (12) is free. This conjecture is proved in [20]; we give here a different proof (Corollary 7). We use for this that **Quad** is also equal to **Dend** $\square$ **Dend**, where $\square$ is here the white Manin product (Corollary 5), and consequently can be seen as a suboperad of **Dend** $\otimes$ **Dend**: hence, free **Dend** $\otimes$ **Dend**-algebras contain free quadri-algebras, a result which is applied to **FQSym**. We also combinatorially describe the Koszul dual $\mathbf{Quad}^!$ of **Quad**, and prove its koszulity with the rewriting method of [2, 3, 10, 13].

The last section is devoted to a study of the compatibilities between the quadri-algebra structure of **FQSym** and its dual quadri-coalgebra structure: this leads to the notion of quadri-bialgebra (Definition 10). Another example of quadri-bialgebra is given by the Hopf algebra of packed words **WQSym**. It is observed that, unlike the case of dendriform bialgebras, there is no rigidity theorem for quadri-bialgebras; indeed:

- **FQSym** and **WQSym** are not free quadri-algebras, nor cofree quadri-coalgebras.
- **FQSym** and **WQSym** are not generated, as quadri-algebras, by their primitive elements, in the quadri-coalgebraic sense.

Acknowledgments. The research leading these results was partially supported by the French National Research Agency under the reference ANR-12-BS01-0017. I would like to thank Bruno Vallette for his precious comments, suggestions and help.

Notations 1.

1. We denote by K a commutative field. All the objects (vector spaces, algebras, coalgebras, operads...) of this text are taken over K.
2. For all $n \geqslant 1$, we denote by $[n]$ the set of integers $\{1, 2, \ldots, n\}$.

1 Reminders on quadri-algebras and operads

1.1 Definitions and examples of quadri-algebras

Definition 1. *1. A quadri-algebra is a family $(A, \nwarrow, \swarrow, \searrow, \nearrow)$, where A is a vector space and $\nwarrow$, $\swarrow$, $\searrow$, $\nearrow$ are products on A, such that for all $x, y, z \in A$:*

$$\begin{array}{lll} (x \nwarrow y) \nwarrow z = x \nwarrow (y \star z), & (x \nearrow y) \nwarrow z = x \nearrow (y \leftarrow z), & (x \uparrow y) \nearrow z = x \nearrow (y \rightarrow z), \\ (x \swarrow y) \nwarrow z = x \swarrow (y \uparrow z), & (x \searrow y) \nwarrow z = x \searrow (y \nwarrow z), & (x \downarrow y) \nearrow z = x \searrow (y \nearrow z), \\ (x \leftarrow y) \swarrow z = x \swarrow (y \downarrow z), & (x \rightarrow y) \swarrow z = x \searrow (y \swarrow z), & (x \star y) \searrow z = x \searrow (y \searrow z), \end{array}$$

where:

$$\leftarrow = \nwarrow + \swarrow, \qquad \rightarrow = \nearrow + \searrow, \qquad \uparrow = \nwarrow + \nearrow, \qquad \downarrow = \swarrow + \searrow,$$
$$\star = \nwarrow + \swarrow + \searrow + \nearrow = \leftarrow + \rightarrow = \uparrow + \downarrow.$$

These relations will be considered as the entries of a 3×3 matrix, and will be refered as relations $(1,1) \ldots (3,3)$.

[1] This product is denoted by $\square$ in [6, 12]. We shall not use this notation here, in order to avoid confusion between the two Manin products.

2. *A quadri-coalgebra is a family* $(C, \Delta_\nwarrow, \Delta_\swarrow, \Delta_\searrow, \Delta_\nearrow)$*, where* C *is a vector space and* $\Delta_\nwarrow$*,* $\Delta_\swarrow$*,* $\Delta_\searrow$*,* $\Delta_\nearrow$ *are coproducts on* C*, such that:*

$$\begin{array}{ll} (\Delta_\nwarrow \otimes Id) \circ \Delta_\nwarrow = (Id \otimes \Delta_*) \circ \Delta_\nwarrow, & (\Delta_\swarrow \otimes Id) \circ \Delta_\nwarrow = (Id \otimes \Delta_\uparrow) \circ \Delta_\swarrow, \\ (\Delta_\nearrow \otimes Id) \circ \Delta_\nwarrow = (Id \otimes \Delta_\leftarrow) \circ \Delta_\nearrow, & (\Delta_\searrow \otimes Id) \circ \Delta_\nwarrow = (Id \otimes \Delta_\nwarrow) \circ \Delta_\searrow, \\ (\Delta_\uparrow \otimes Id) \circ \Delta_\nearrow = (Id \otimes \Delta_\rightarrow) \circ \Delta_\nearrow; & (\Delta_\downarrow \otimes Id) \circ \Delta_\nearrow = (Id \otimes \Delta_\nearrow) \circ \Delta_\searrow; \end{array}$$

$$\begin{array}{c} (\Delta_\leftarrow \otimes Id) \circ \Delta_\swarrow = (Id \otimes \Delta_\downarrow) \circ \Delta_\swarrow, \\ (\Delta_\rightarrow \otimes Id) \circ \Delta_\swarrow = (Id \otimes \Delta_\swarrow) \circ \Delta_\searrow, \\ (\Delta_* \otimes Id) \circ \Delta_\searrow = (Id \otimes \Delta_\searrow) \circ \Delta_\searrow, \end{array}$$

with:

$$\Delta_\leftarrow = \Delta_\searrow + \Delta_\nearrow, \qquad \Delta_\rightarrow = \Delta_\nwarrow + \Delta_\swarrow, \qquad \Delta_\uparrow = \Delta_\nwarrow + \Delta_\nearrow, \qquad \Delta_\downarrow = \Delta_\swarrow + \Delta_\searrow,$$
$$\Delta_* = \Delta_\nwarrow + \Delta_\swarrow + \Delta_\searrow + \Delta_\nearrow.$$

Remark 1. 1. If A is a finite-dimensional quadri-algebra, then its dual A^* is a quadri-coalgebra, with $\Delta_\diamond = \diamond^*$ for all $\diamond \in \{\nwarrow, \swarrow, \searrow, \nearrow, \leftarrow, \rightarrow, \uparrow, \downarrow, \star\}$.

2. If C is a quadri-coalgebra (even not finite-dimensional), then C^* is a quadri-algebra, with $\diamond = \Delta_\diamond^*$ for all $\diamond \in \{\nwarrow, \swarrow, \searrow, \nearrow, \leftarrow, \rightarrow, \uparrow, \downarrow, \star\}$.

3. Let A be a quadri-algebra. Adding each row of the matrix of relations:

$$\begin{array}{c} (x \uparrow y) \uparrow z = x \uparrow (y \star z), \\ (x \downarrow y) \uparrow z = x \downarrow (y \uparrow z), \\ (x \star y) \downarrow z = x \downarrow (y \downarrow z). \end{array}$$

Hence, $(A, \uparrow, \downarrow)$ is a dendriform algebra. Adding each column of the matrix of relations:

$$(x \leftarrow y) \leftarrow z = x \leftarrow (y \star z), \quad (x \rightarrow y) \leftarrow z = x \rightarrow (y \leftarrow z), \quad (x \star y) \rightarrow z = x \rightarrow (y \rightarrow z).$$

Hence, $(A, \leftarrow, \rightarrow)$ is a dendriform algebra. The associative (non unitary) product associated to both these dendriform structures is $\star$.

4. Dually, if C is a quadri-coalgebra, $(C, \Delta_\uparrow, \Delta_\downarrow)$ and $(C, \Delta_\leftarrow, \Delta_\rightarrow)$ are dendriform coalgebras. The associated coassociative (non counitary) coproduct is Δ_*.

Example 1. 1. Let V be a vector space. As noticed in [1], the augmentation ideal of the tensor algebra $T(V)$ is given four products defined in the following way: for all $v_1, \ldots, v_k$, $v_{k+1}, \ldots, v_{k+l} \in V$, with $k, l \geqslant 1$,

$$\begin{array}{rcl} v_1 \ldots v_k \nwarrow v_{k+1} \ldots v_{k+l} &=& \displaystyle\sum_{\substack{\sigma \in Sh(k,l),\\ \sigma^{-1}(1)=1,\, \sigma^{-1}(k+l)=k}} v_{\sigma^{-1}(1)} \ldots v_{\sigma^{-1}(k+l)}, \\ v_1 \ldots v_k \swarrow v_{k+1} \ldots v_{k+l} &=& \displaystyle\sum_{\substack{\sigma \in Sh(k,l),\\ \sigma^{-1}(1)=k+1,\, \sigma^{-1}(k+l)=k}} v_{\sigma^{-1}(1)} \ldots v_{\sigma^{-1}(k+l)}, \\ v_1 \ldots v_k \searrow v_{k+1} \ldots v_{k+l} &=& \displaystyle\sum_{\substack{\sigma \in Sh(k,l),\\ \sigma^{-1}(1)=k+1,\, \sigma^{-1}(k+l)=k+l}} v_{\sigma^{-1}(1)} \ldots v_{\sigma^{-1}(k+l)}, \\ v_1 \ldots v_k \nearrow v_{k+1} \ldots v_{k+l} &=& \displaystyle\sum_{\substack{\sigma \in Sh(k,l),\\ \sigma^{-1}(1)=1,\, \sigma^{-1}(k+l)=k+l}} v_{\sigma^{-1}(1)} \ldots v_{\sigma^{-1}(k+l)}, \end{array}$$

where $Sh(k,l)$ is the set of (k,l)-shuffles, that is to say permutations $\sigma \in \mathfrak{S}_{k+l}$ such that $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(k+1) < \ldots < \sigma(k+l)$. The associated associative product is the usual shuffle product.

2. The augmentation ideal of the Hopf algebra **FQSym** of permutations introduced in [15] and studied in [5] is also a quadri-algebra, as mentioned in [1]. For all permutations $\alpha \in \mathfrak{S}_k$, $\beta \in \mathfrak{S}_l$, with $k, l \geqslant 1$:

$$\alpha \nwarrow \beta = \sum_{\substack{\sigma \in Sh(k,l),\\ \sigma^{-1}(1)=1,\, \sigma^{-1}(k+l)=k}} (\alpha \otimes \beta) \circ \sigma^{-1},$$
$$\alpha \swarrow \beta = \sum_{\substack{\sigma \in Sh(k,l),\\ \sigma^{-1}(1)=k+1,\, \sigma^{-1}(k+l)=k}} (\alpha \otimes \beta) \circ \sigma^{-1},$$
$$\alpha \searrow \beta = \sum_{\substack{\sigma \in Sh(k,l),\\ \sigma^{-1}(1)=k+1,\, \sigma^{-1}(k+l)=k+l}} (\alpha \otimes \beta) \circ \sigma^{-1},$$
$$\alpha \nearrow \beta = \sum_{\substack{\sigma \in Sh(k,l),\\ \sigma^{-1}(1)=1,\, \sigma^{-1}(k+l)=k+l}} (\alpha \otimes \beta) \circ \sigma^{-1}.$$

As **FQSym** is self-dual, its coproduct can also be split into four parts, making it a quadri-coalgebra. As the pairing on **FQSym** is defined by $\langle \sigma, \tau \rangle = \delta_{\sigma, \tau^{-1}}$ for any permutations σ, τ, we deduce that if $\sigma \in \mathfrak{S}_n$, $n \geqslant 1$, with the notations of [15]:

$$\Delta_{\nwarrow}(\sigma) = \sum_{\sigma^{-1}(1), \sigma^{-1}(n) \leqslant i < n} \mathrm{st}(\sigma(1)\ldots\sigma(i)) \otimes \mathrm{st}(\sigma(i+1)\ldots\sigma(n)),$$
$$\Delta_{\swarrow}(\sigma) = \sum_{\sigma^{-1}(n) \leqslant i < \sigma^{-1}(1)} \mathrm{st}(\sigma(1)\ldots\sigma(i)) \otimes \mathrm{st}(\sigma(i+1)\ldots\sigma(n)),$$
$$\Delta_{\searrow}(\sigma) = \sum_{1 \leqslant i < \sigma^{-1}(1), \sigma^{-1}(n)} \mathrm{st}(\sigma(1)\ldots\sigma(i)) \otimes \mathrm{st}(\sigma(i+1)\ldots\sigma(n)),$$
$$\Delta_{\nearrow}(\sigma) = \sum_{\sigma^{-1}(1) \leqslant i < \sigma^{-1}(n)} \mathrm{st}(\sigma(1)\ldots\sigma(i)) \otimes \mathrm{st}(\sigma(i+1)\ldots\sigma(n)).$$

The compatibilites between these products and coproducts will be studied in Proposition 11. For example:

$$\begin{array}{lll}
(12) \nwarrow (12) = (1342), & \Delta_{\nwarrow}((3412)) = (231) \otimes (1), & \Delta_{\nwarrow}((2143)) = (213) \otimes (1),\\
(12) \swarrow (12) = (3142) + (3412), & \Delta_{\swarrow}((3412)) = (12) \otimes (12), & \Delta_{\swarrow}((2143)) = 0,\\
(12) \searrow (12) = (3124), & \Delta_{\searrow}((3412)) = (1) \otimes (312), & \Delta_{\searrow}((2143)) = (1) \otimes (132),\\
(12) \nearrow (12) = (1234) + (1324), & \Delta_{\nearrow}((3412)) = 0, & \Delta_{\nearrow}((2143)) = (21) \otimes (21).
\end{array}$$

The dendriform algebra $(\mathbf{FQSym}, \leftarrow, \rightarrow)$ and the dendriform coalgebra $(\mathbf{FQSym}, \Delta_{\leftarrow}, \Delta_{\rightarrow})$ are decribed in [7, 8]; the dendriform algebra $(\mathbf{FQSym}, \uparrow, \downarrow)$ and the dendriform coalgebra $(\mathbf{FQSym}, \Delta_{\uparrow}, \Delta_{\downarrow})$ are decribed in [9]. Both dendriform algebras are free, and both dendriform coalgebras are cofree, by the dendriform rigidity theorem [7]. Note that **FQSym** is not free as a quadri-algebra, as $(1) \nwarrow (1) = 0$.

3. The dual of the Hopf algebra of totally assigned graphs [4] is a quadri-coalgebra.

1.2 Nonsymmetric operads

We refer to [13, 16, 19] for the usual definitions and properties of operads and nonsymmetric operads.

Notations 2. • Let V be a vector space. The free nonsymmetric operad generated in arity 2 by V is denoted by $\mathbf{F}(V)$. If we fix a basis $(v_i)_{i \in I}$ of V, then for all $n \geqslant 1$, a basis of $\mathbf{F}(V)_n$ is given by the set of planar binary trees with n leaves, whose $(n-1)$ internal vertices are decorated by elements of $\{v_i \mid i \in I\}$. The operadic composition is given by the grafting of trees on leaves. If V is finite-dimensional, then for all $n \geqslant 1$, $\mathbf{F}(V)_n$ is finite-dimensional, and:

$$\dim(\mathbf{F}(V)_n) = \frac{1}{n}\binom{2n-2}{n-1}\dim(V)^{n-1}.$$

- Let $\mathbf{P}$ be a nonsymmetric operad and V a vector space. A structure of $\mathbf{P}$-algebra on V is a family of maps:

$$\begin{cases} \mathbf{P}_n \otimes V^{\otimes n} & \longrightarrow & V \\ p \otimes v_1 \otimes \ldots \otimes v_n & \longrightarrow & p.(v_1, \ldots, v_n), \end{cases}$$

satisfying some compatibilities with the composition of $\mathbf{P}$.

- The free $\mathbf{P}$-algebra generated by the vector space V is, as a vector space:

$$F_{\mathbf{P}}(V) = \bigoplus_{n \geqslant 0} \mathbf{P}_n \otimes V^{\otimes n};$$

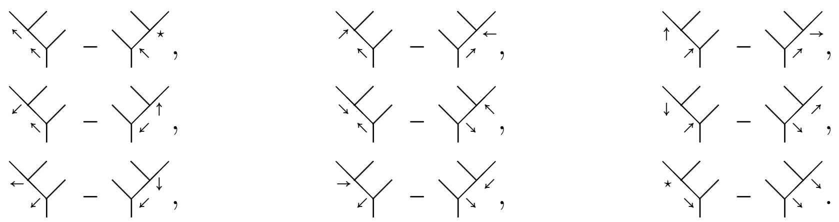
the action of $\mathbf{P}$ on $F_{\mathbf{P}}(V)$ is given by:

$$p.(p_1 \otimes w_1, \ldots, p_n \otimes w_n) = p \circ (p_1, \ldots, p_n) \otimes w_1 \otimes \ldots \otimes w_n.$$

- Let $\mathbf{P} = (\mathbf{P}_n)_{n \geqslant 1}$ be a nonsymmetric operad. It is quadratic if :
  - It is generated by $G_{\mathbf{P}} = \mathbf{P}_2$.
  - Let $\pi_{\mathbf{P}} : \mathbf{F}(G_{\mathbf{P}}) \longrightarrow \mathbf{P}$ be the canonical morphism from $\mathbf{F}(G_{\mathbf{P}})$ to $\mathbf{P}$; then its kernel is generated, as an operadic ideal, by $\mathrm{Ker}(\pi_{\mathbf{P}})_3 = \mathrm{Ker}(\pi_{\mathbf{P}}) \cap \mathbf{F}(G_{\mathbf{P}})_3$.

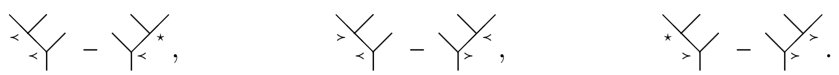
If $\mathbf{P}$ is binary and quadratic, we put $G_{\mathbf{P}} = \mathbf{P}_2$, and $R_{\mathbf{P}} = \mathrm{Ker}(\pi_{\mathbf{P}})_3$. By definition, these two spaces entirely determine $\mathbf{P}$, up to an isomorphism.

Example 2. 1. The nonsymmetric operad **Quad** of quadri-algebras is quadratic. It is generated by $G_{\mathbf{Quad}} = \mathrm{Vect}(\nwarrow, \swarrow, \searrow, \nearrow)$, and $R_{\mathbf{Quad}}$ is the linear span of the nine following elements:



As $\dim(F(G_{\mathbf{Quad}})_3) = 32$, $\dim(\mathbf{Quad}_3) = 32 - 9 = 23$.

2. The nonsymmetric operad **Dend** of dendriform algebras is quadratic. It is generated by $G_{\mathbf{Dend}} = \mathrm{Vect}(\prec, \succ)$, and $R_{\mathbf{Dend}}$ is the linear span of the three following elements:



The nonsymmetric-operad **Quad** of quadri-algebras, being quadratic, has a Koszul dual $\mathbf{Quad}^!$. The following formulas for the generating formal series of **Quad** and $\mathbf{Quad}^!$ has been conjectured in [1] and proved in [19], as well as the koszulity:

Proposition 2. *1. For all $n \geqslant 1$, dim$(\mathbf{Quad}_n) = \displaystyle\sum_{j=n}^{2n-1} \binom{3n}{n+1+j}\binom{j-1}{j-n}$. This is sequence A007297 in [18].*

2. For all $n \geqslant 1$, $\dim(\mathbf{Quad}^!_n) = n^2$.

3. The operad of quadri-algebras is Koszul.

2 The operad of quadri-algebras and its Koszul dual

2.1 Dual quadri-algebras

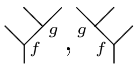
Algebras on $\mathbf{Quad}^!$ will be called dual quadri-algebras. This operad $\mathbf{Quad}^!$ is described in [19] in terms of the white Manin product. Let us give an explicit description.

Proposition 3. *A dual quadri-algebra is a family* $(A, \nwarrow, \swarrow, \searrow, \nearrow)$*, where* A *is a vector space and* $\nwarrow, \swarrow, \searrow, \nearrow : A \otimes A \longrightarrow A$*, such that for all* $x, y, z \in A$*:*

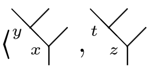
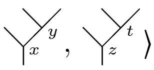
$$\begin{aligned}
&(x \nwarrow y) \nwarrow z = x \nwarrow (y \nwarrow z) = x \nwarrow (y \swarrow z) = x \nwarrow (y \searrow z) = x \nwarrow (y \nearrow z),\\
&(x \nearrow y) \nwarrow z = x \nearrow (y \nwarrow z) = x \nearrow (y \swarrow z),\\
&(x \nwarrow y) \nearrow z = (x \nearrow y) \nearrow z = x \nearrow (y \searrow z) = x \nearrow (y \nearrow z),\\
&(x \swarrow y) \nwarrow z = x \swarrow (y \nwarrow z) = x \swarrow (y \nearrow z),\\
&(x \searrow y) \nwarrow z = x \searrow (y \nwarrow z),\\
&(x \swarrow y) \nearrow z = (x \searrow y) \nearrow z = x \searrow (y \nearrow z),\\
&(x \nwarrow y) \swarrow z = (x \swarrow y) \swarrow z = x \swarrow (y \swarrow z) = x \swarrow (y \searrow y),\\
&(x \searrow y) \swarrow z = x(\nearrow y) \swarrow z = x \searrow (y \swarrow z),\\
&(x \nwarrow y) \searrow z = (x \swarrow y) \searrow z = (x \searrow y) \searrow z = (x \nearrow y) \searrow z = x \searrow (y \searrow z).
\end{aligned}$$

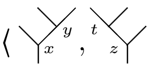
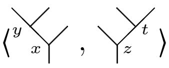
These groups of relations are denoted by $(1)^!, \ldots, (9)^!$*. Note that the four products* $\nwarrow, \swarrow, \searrow, \nearrow$ *are associative.*

Proof. We put $G = \mathrm{Vect}(\nwarrow, \swarrow, \searrow, \nearrow)$ and E the component of arity 3 of the free nonsymmetric operad generated by G, that is to say:

$E = \mathrm{Vect}\Big($  $\mid f, g \in \{\nwarrow, \swarrow, \searrow, \nearrow\}\Big).$

We give G a pairing, such that the four products form an orthonormal basis of G. This induces a pairing on E: for all $x, y, z, t \in G$,

$\langle$  $\rangle = \langle x, z\rangle\langle y, t\rangle,$ $\langle$  $\rangle = -\langle x, z\rangle\langle y, t\rangle,$

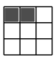
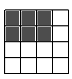
$\langle$  $\rangle = 0,$ $\langle$  $\rangle = 0.$

The quadratic nonsymmetric operad **Quad** is generated by $G = \mathrm{Vect}(\nwarrow, \swarrow, \searrow, \nearrow)$ and the subspace of relations R of E corresponding to the nine relations (1,1)...(3,3). The quadratic nonsymmetric operad $\mathbf{Quad}^!$ is generated by $G \approx G^*$ and the subspaces of relations $R^\perp$ of E. As $\dim(R) = 9$ and $\dim(E) = 32$, $\dim(R^\perp) = 23$. A direct verification shows that the 23 relations given in $(1)^!, \ldots, (9)^!$ are elements of $R^\perp$. As they are linearly independent, they form a basis of $R^\perp$. □

Notations 3. We consider:

$$\mathcal{R} = \bigsqcup_{n=1}^{\infty} [n]^2.$$

The element $(i, j) \in [n]^2 \subset \mathcal{R}$ will be denoted by $(i, j)_n$ in order to avoid the confusions. We graphically represent $(i, j)_n$ by putting in grey the boxes of coordinates (a, b), $1 \leqslant a \leqslant i$, $1 \leqslant b \leqslant j$, of a $n \times n$ array, the boxes $(1, 1)$, $(1, n)$, $(n, 1)$ and (n, n) being respectively up left, down left, up right and down right. For example:

$(2, 1)_3 =$ , $(1, 1)_2 =$ , $(3, 2)_4 =$ .

Proposition 4. *Let $A_{\mathcal{R}} = \mathrm{Vect}(\mathcal{R})$. We define four products $\nwarrow$, $\swarrow$, $\searrow$, $\nearrow$ on $A_{\mathcal{R}}$ by:*

$$\begin{aligned} (i,j)_p \nwarrow (k,l)_q &= (i,j)_{p+q}, & (i,j)_p \nearrow (k,l)_q &= (k+p,j)_{p+q},\\ (i,j)_p \swarrow (k,l)_q &= (i,p+l)_{p+q}, & (i,j)_p \searrow (k,l)_q &= (k+p,l+p)_{p+q}. \end{aligned}$$

Then $(A_{\mathcal{R}}, \nwarrow, \swarrow, \searrow, \nearrow)$ is a dual quadri-algebra. It is graded by putting the elements of $[n]^2 \in \mathcal{R}$ homogeneous of degree n, and the generating formal series of $A_{\mathcal{R}}$ is:

$$\sum_{n=1}^{\infty} n^2 X^n = \frac{X(1+X)}{(1-X)^3}.$$

Moreover, $A_{\mathcal{R}}$ is freely generated as a dual quadri-algebra by $(1,1)_1$.

Proof. Let us take $(i,j)_p$, $(k,l)_q$ and $(m,n)_r \in \mathcal{R}$. Then:

- Each computation in $(1)^!$ gives $(i,j)_{p+q+r}$.
- Each computation in $(2)^!$ gives $(p+k,j)_{p+q+r}$.
- Each computation in $(3)^!$ gives $(p+q+m,j)_{p+q+r}$.
- Each computation in $(4)^!$ gives $(i,p+l)_{p+q+r}$.
- Each computation in $(5)^!$ gives $(p+k,p+l)_{p+q+r}$.
- Each computation in $(6)^!$ gives $(p+q+m,p+l)_{p+q+r}$.
- Each computation in $(7)^!$ gives $(i,p+q+n)_{p+q+r}$.
- Each computation in $(8)^!$ gives $(p+k,p+q+n)_{p+q+r}$.
- Each computation in $(9)^!$ gives $(p+q+m,p+q+n)_{p+q+r}$.

So $A_{\mathcal{R}}$ is a dual quadri-algebra. We now prove that $A_{\mathcal{R}}$ is generated by $(1,1)_1$. Let B be the dual quadri-subalgebra of $A_{\mathcal{R}}$ generated by $(1,1)_1$, and let us prove that $(i,j)_n \in B$ by induction on n for all $(i,j)_n \in \mathcal{R}$. This is obvious in $n=1$, as then $(i,j)_n = (1,1)_1$. Let us assume the result at rank $n-1$, with $n>1$.

- If $i \geqslant 2$ and $j \leqslant n-1$, then $(1,1)_1 \nearrow (i-1,j)_{n-1} = (i,j)_n$. By the induction hypothesis, $(i-1,j)_{n-1} \in B$, so $(i,j)_n \in B$.
- If $i \leqslant n-1$ and $j \geqslant 2$, then $(1,1)_1 \swarrow (i,j-1)_{n-1} = (i,j)_n$. By the induction hypothesis, $(i,j-1)_{n-1} \in B$, so $(i,j)_n \in B$.
- Otherwise, ($i=1$ or $j=n$) and ($i=n$ or $j=1$), that is to say $(i,j)_n = (1,1)_n$ or $(i,j)_n = (n,n)_n$. We remark that $(1,1) \nwarrow (1,1)_{n-1} = (1,1)_n$ and $(1,1)_1 \searrow (n-1,n-1)_{n-1} = (n,n)_n$. By the induction hypothesis, $(1,1)_{n-1}$ and $(n-1,n-1)_n \in B$, so $(1,1)_n$ and $(n,n)_n \in B$.

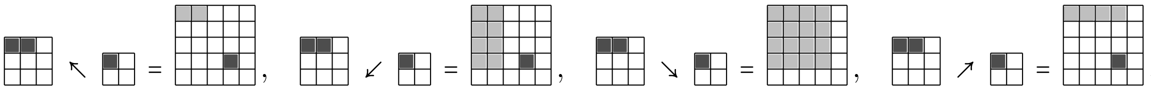
Finally, B contains $\mathcal{R}$, so $B = A_{\mathcal{R}}$.

Let C be the free $\mathbf{Quad}^!$-algebra generated by a single element x, homogeneous of degree 1. As a graded vector space:

$$C = \bigoplus_{n \geqslant 1} \mathbf{Quad}^!_n \otimes V^{\otimes n}.$$

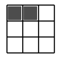
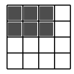
where $V = \mathrm{Vect}(x)$. So for all $n \geqslant 1$, by Proposition 2, $\dim(C_n) = n^2 = \dim(A_n)$. There exists a surjective morphism of $\mathbf{Quad}^!$-algebras θ from C to A, sending x to $(1,1)_1$. As x and $(1,1)_1$ are both homogeneous of degree 1, θ is homogeneous of degree 0. As A and C have the same generating formal series, θ is bijective, so A is isomorphic to C. □

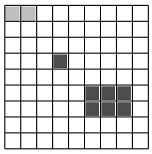
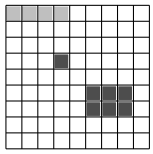
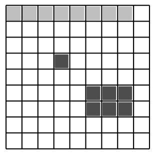
Example 3. Here are graphical examples of products. The result of the product is drawn in light gray:

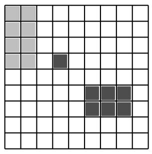
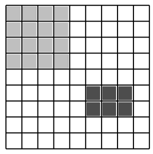
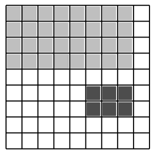


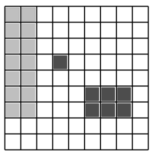
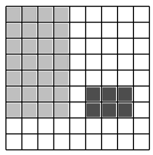
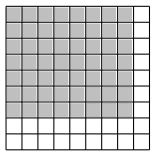
Roughly speaking, the products of $x \in [m]^2 \subset \mathcal{R}$ and $y \in [n]^2 \subset \mathcal{R}$ are obtained by putting x and y diagonally in a common array of size $(m+n) \times (m+n)$. This array is naturally decomposed in four parts denoted by nw, sw, se and ne according to their direction. Then:

1. $x \nwarrow y$ is given by the black boxes in the nw part.
2. $x \swarrow y$ is given by the boxes in the sw part which are simultaneously under a black box and to the left of a black box.
3. $x \searrow y$ is given by the black boxes in the se part.
4. $x \nearrow y$ is given by the boxes in the ne part which are simultaneously over a black box and to the right of a black box.

Here are the results of the nine relations applied to $x =$ , $y =$  and $z =$ :

$(1)^!$:  $(2)^!$:  $(3)^!$: 

$(4)^!$:  $(5)^!$:  $(6)^!$: 

$(7)^!$:  $(8)^!$:  $(9)^!$: 

Remark 2.
1. A description of the free $\mathbf{Quad}^!$-algebra generated by any set $\mathcal{D}$ is done similarly. We put:

$$\mathcal{R}(\mathcal{D}) = \bigsqcup_{n=1}^{\infty} [n]^2 \times \mathcal{D}^n.$$

The four products are defined by:

$$\begin{aligned}
((i,j)_p, d_1, \ldots, d_p) \nwarrow ((k,l)_q, e_1, \ldots, e_q) &= ((i,j)_{p+q}, d_1, \ldots, d_p, e_1, \ldots, e_q),\\
((i,j)_p, d_1, \ldots, d_p) \swarrow ((k,l)_q, e_1, \ldots, e_q) &= ((i,p+l)_{p+q} d_1, \ldots, d_p, e_1, \ldots, e_q),\\
((i,j)_p, d_1, \ldots, d_p) \searrow ((k,l)_q, e_1, \ldots, e_q) &= ((k+p,l+p)_{p+q} d_1, \ldots, d_p, e_1, \ldots, e_q)\\
((i,j)_p, d_1, \ldots, d_p) \nearrow ((k,l)_q, e_1, \ldots, e_q) &= ((k+p,j)_{p+q} d_1, \ldots, d_p, e_1, \ldots, e_q).
\end{aligned}$$

2. We can also deduce a combinatorial description of the nonsymmetric operad $\mathbf{Quad}^!$. As a vector space, $\mathbf{Quad}^!_n = \mathrm{Vect}([n]^2)$ for all $n \geqslant 1$. The composition is given by:

$$(i,j)_m \circ ((k_1,l_1)_{n_1}, \ldots, (k_n,l_n)_{n_m}) = (n_1 + \cdots + n_{i-1} + k_i, n_1 + \cdots + n_{j-1} + l_j)_{n_1+\cdots+n_m}.$$

In particular:

$$\nwarrow = (1,1)_2, \qquad \swarrow = (1,2)_2, \qquad \searrow = (2,2)_2, \qquad \nearrow = (2,1)_2.$$

Corollary 5. *We define a nonsymmetric operad* **Dias** *in the following way:*

- *For all* $n \geqslant 1$, $\mathbf{Dias}_n = \mathrm{Vect}([n])$. *The elements of* $[n] \subseteq \mathbf{Dias}_n$ *are denoted by* $(1)_n, \ldots, (n)_n$ *in order to avoid confusions.*
- *The composition is given by:*

$$(i)_m \circ ((j_1)_{n_1}, \ldots, (j_m)_{n_m}) = (n_1 + \cdots + n_{i-1} + j_i)_{n_1 + \cdots + n_m}.$$

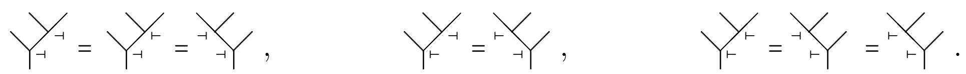
This is the nonsymmetric operad of associative dialgebras [11], that is to say algebras A with two products $\vdash$ and $\dashv$ such that for all $x, y, z \in A$:

$$\begin{aligned}
&x \dashv (y \dashv z) = x \dashv (y \vdash z) = (x \dashv y) \dashv z,\\
&(x \vdash y) \dashv z = x \vdash (y \dashv z),\\
&(x \dashv y) \vdash z = (x \vdash y) \vdash z = x \vdash (y \vdash z).
\end{aligned}$$

We denote by $\square$ and $\blacksquare$ the two Manin products on nonsymmetric-operads of [19]. Then:

$$\begin{aligned}
\mathbf{Quad}^! &= \mathbf{Dias} \otimes \mathbf{Dias} = \mathbf{Dias} \,\square\, \mathbf{Dias} = \mathbf{Dias} \blacksquare \mathbf{Dias},\\
\mathbf{Quad} &= \mathbf{Dend} \blacksquare \mathbf{Dend} = \mathbf{Dend} \,\square\, \mathbf{Dend}.
\end{aligned}$$

Proof. We denote by $\mathbf{Dias}'$ the nonsymmetric operad generated by $\dashv$ and $\vdash$ and the relations:



First, observe that:

$$\begin{aligned}
&(1)_2 \circ (I, (1)_2) = (1)_2 \circ (I, (2)_2) = (1)_2 \circ ((1)_2, I) = (1)_3,\\
&(1)_2 \circ ((2)_2, I) = (2)_2 \circ (I, (1)_2) = (2)_3,\\
&(2)_2 \circ (I, (2)_2) = (2)_2 \circ ((1)_2, I) = (2)_2 \circ ((2)_2, I) = (3)_3.
\end{aligned}$$

So there exists a morphism θ of nonsymmetric operad from $\mathbf{Dias}'$ to $\mathbf{Dias}$, sending $\dashv$ to $(1)_2$ and $\vdash$ to $(2)_2$. Note that $\theta(I) = (1)_1$.

Let us prove that θ is surjective. Let $n \geqslant 1$, $i \in [n]$, we show that $(i)_n \in Im(\theta)$ by induction on n. If $n \leqslant 2$, the result is obvious. Let us assume the result at rank $n-1$, $n \geqslant 3$. If $i = 1$, then:

$$(1)_2 \circ ((1)_1, (1)_{n-1}) = (1)_n.$$

By the induction hypothesis, $(1)_{n-1} \in Im(\theta)$, so $(1)_n \in Im(\theta)$. If $i \geqslant 2$, then:

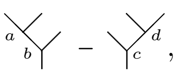
$$(2)_2 \circ ((1)_1, (i-1)_{n-1}) = (i)_n.$$

By the induction hypothesis, $(1)_{n-1} \in Im(\theta)$, so $(i)_n \in Im(\theta)$.

It is proved in [11] that $\dim(\mathbf{Dias}'_n) = \dim(\mathbf{Dias}_n) = n$ for all $n \geqslant 1$. As θ is surjective, it is an isomorphism. Moreover, let us consider the following map:

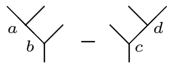
$$\begin{cases} \mathbf{Dias} \otimes \mathbf{Dias} & \longrightarrow & \mathbf{Quad}^! \\ (i)_n \otimes (j)_n & \longrightarrow & (i,j)_n. \end{cases}$$

It is clearly an isomorphism of nonsymmetric operads. It is proved in [19] that $\mathbf{Dias} \,\square\, \mathbf{Dias} = \mathbf{Quad}^!$. As $R_{\mathbf{Dias}}$ is the quadratic nonsymmetric algebra generated by $(1)_2$ and $(2)_2$ and the following relations:



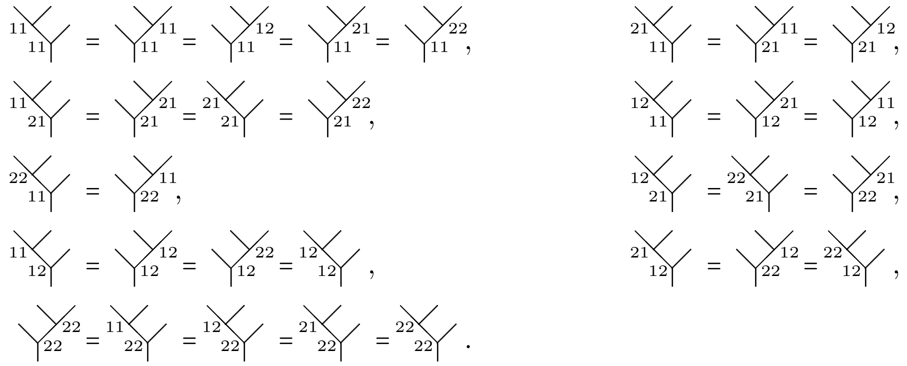
$$, (a,b,c,d) \in E = \left\{ \begin{array}{c} ((1)_2, (1)_2, (1)_2, (1)_2), ((1)_2, (1)_2, (1)_2, (2)_2), \\ ((2)_2, (1)_2, (2)_2, (1)_2), ((1)_2, (2)_2, (2)_2, (2)_2), \\ ((2)_2, (2)_2, (2)_2, (2)_2) \end{array} \right\}.$$

Dias ■ Dias is generated by $(1,1)_2$, $(1,2)_2$, $(2,1)_2$ and $(2,2)_2$ with the relations:



$$, (a,b,c,d) \in E',$$
$$E' = \{((a_1,a_2)_2,(b_1,b_2)_2,(c_1,c_2)_2,(d_1,d_2)_2) \mid (a_1,b_1,c_1,d_1),(a_2,b_2,c_2,d_2) \in E\}.$$

This gives 25 relations, which are not linearly independent, and can be regrouped in the following way:



where we denote ij instead of $(i,j)_2$. So **Dias■Dias** is isomorphic to $\mathbf{Quad}^!$ via the isomorphism given by:

$$\begin{cases} \mathbf{Quad}^! & \longrightarrow & \mathbf{Dias}\,\blacksquare\,\mathbf{Dias} \\ \nwarrow & \longrightarrow & (1,1)_2, \\ \swarrow & \longrightarrow & (1,2)_2, \\ \searrow & \longrightarrow & (2,2)_2, \\ \nearrow & \longrightarrow & (2,1)_2. \end{cases}$$

By Koszul duality, as $\mathbf{Dias}^! = \mathbf{Dend}$, we obtain the results for **Quad**. $\square$

2.2 Free quadri-algebra on one generator

As $\mathbf{Quad} = \mathbf{Dend}\square\mathbf{Dend}$, **Quad** is the suboperad of $\mathbf{Dend}\otimes\mathbf{Dend}$ generated by the component of arity 2. An explicit injection of **Quad** into $\mathbf{Dend}\otimes\mathbf{Dend}$ is given by:

Proposition 6. *The following defines a injective morphism of nonsymmetric operads:*

$$\Theta : \begin{cases} \mathbf{Quad} & \longrightarrow & \mathbf{Dend}\otimes\mathbf{Dend} \\ \nwarrow & \longrightarrow & \prec\otimes\prec \\ \swarrow & \longrightarrow & \prec\otimes\succ \\ \searrow & \longrightarrow & \succ\otimes\succ \\ \nearrow & \longrightarrow & \succ\otimes\prec. \end{cases}$$

Corollary 7. *The quadri-subalgebra of* $(\mathbf{FQSym},\nwarrow,\swarrow,\searrow,\nearrow)$ *generated by* (12) *is free.*

Proof. Both dendriform algebras $(\mathbf{FQSym},\downarrow,\uparrow)$ and $(\mathbf{FQSym},\leftarrow,\rightarrow)$ are free. So the $\mathbf{Dend}\otimes\mathbf{Dend}$-algebra $(\mathbf{FQSym}\otimes\mathbf{FQSym},\uparrow\otimes\leftarrow,\downarrow\otimes\leftarrow,\downarrow\otimes\rightarrow,\uparrow\otimes\rightarrow)$ is free. By restriction, the $\mathbf{Dend}\otimes\mathbf{Dend}$-subalgebra of $\mathbf{FQSym}\otimes\mathbf{FQSym}$ generated by $(1)\otimes(1)$ is free. By restriction, the quadri-subalgebra A of $\mathbf{FQSym}\otimes\mathbf{FQSym}$ generated by $(1)\otimes(1)$ is free.

Let B be the quadri-subalgebra of **FQSym** generated by (12) and let $\phi : A \longrightarrow B$ be the unique morphism sending $(1)\otimes(1)$ to (12). We denote by $\mathbf{FQSym}_{even}$ the subspace of **FQSym** formed by the homogeneous components of even degrees. It is clearly a quadri-subalgebra of **FQSym**. As $(12)\in\mathbf{FQSym}_{even}$, $A\subseteq\mathbf{FQSym}_{even}$. We consider the map:

$$\psi : \begin{cases} \mathbf{FQSym}_{even} & \longrightarrow & \mathbf{FQSym}\otimes\mathbf{FQSym} \\ \sigma\in\mathfrak{S}_{2n} & \longrightarrow & \begin{cases} \left(\frac{\sigma(1)-1}{2},\ldots,\frac{\sigma(n)-1}{2}\right)\otimes\left(\frac{\sigma(n+1)}{2},\ldots,\frac{\sigma(2n)}{2}\right) \\ \qquad \text{if } \sigma(1),\ldots,\sigma(n) \text{ are odd and } \sigma(n+1),\ldots,\sigma(2n) \text{ are even,} \\ 0 \text{ otherwise.} \end{cases} \end{cases}$$

Let $\sigma \in \mathfrak{S}_{2m}$, $\tau \in \mathfrak{S}_{2n}$. Let us prove that $\psi(\sigma \diamond \tau) = \psi(\sigma) \diamond \psi(\tau)$ for $\diamond \in \{\nwarrow, \swarrow, \searrow, \nearrow\}$.

First case. Let us assume that $\psi(\sigma) = 0$. There exists $1 \leqslant i \leqslant m$, such that $\sigma(i)$ is even, and an element $m+1 \leqslant j \leqslant m+n$, such that $\sigma(j)$ is odd. Let $\tau \in \mathfrak{S}_{2n}$. Let α be obtained by a shuffle of σ and $\tau[2n]$. If the letter $\sigma(i)$ appears in α in one of the position $1, \ldots, m+n$, then $\psi(\alpha) = 0$. Otherwise, the letter $\sigma(i)$ appears in one of the positions $m+n+1, \ldots, 2m+2n$, so $\sigma(j)$ also appears in one of these positions, as $i < j$, and $\psi(\alpha) = 0$. In both case, $\psi(\alpha) = 0$, and we deduce that $\psi(\sigma \diamond \tau) = 0 = \psi(\sigma) \diamond \psi(\tau)$.

Second case. Let us assume that $\psi(\tau) = 0$. By a similar argument, we show that $\psi(\sigma \diamond \tau) = 0 = \psi(\sigma) \diamond \psi(\tau)$.

Last case. Let us assume that $\psi(\sigma) \neq 0$ and $\psi(\tau) \neq 0$. We put $\sigma = (\sigma_1, \sigma_2)$ and $\tau = (\tau_1, \tau_2)$, where the letters of σ_1 and τ_1 are odd and the letters of σ_2 and τ_2 are even. Then $\psi(\sigma \nwarrow \tau)$ is obtained by shuffling σ and $\tau[2n]$, such that the first and last letters are letters of σ, and keeping only permutations such that the $(m+n)$ first letters are odd (and the $(m+n)$ last letters are even). These words are obtained by shuffling σ_1 and $\tau_1[2m]$ such that the first letter is a letter of σ_1, and by shuffling σ_2 and $\tau_2[2m]$, such that the last letter is a letter of σ_2. Hence:

$$\psi(\sigma \nwarrow \tau) = \psi(\sigma) \uparrow \otimes \leftarrow \psi(\tau) = \psi(\sigma) \nwarrow \psi(\tau).$$

The proof for the three other quadri-algebra products is similar.

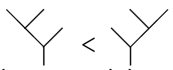
Consequently, ψ is a quadri-algebra morphism. Moreover, $\psi \circ \phi((1) \otimes (1)) = \psi(12) = (1) \otimes (1)$. As A is generated by $(1) \otimes (1)$, $\psi \circ \phi = Id_A$, so ϕ is injective, and A is isomorphic to B. $\square$

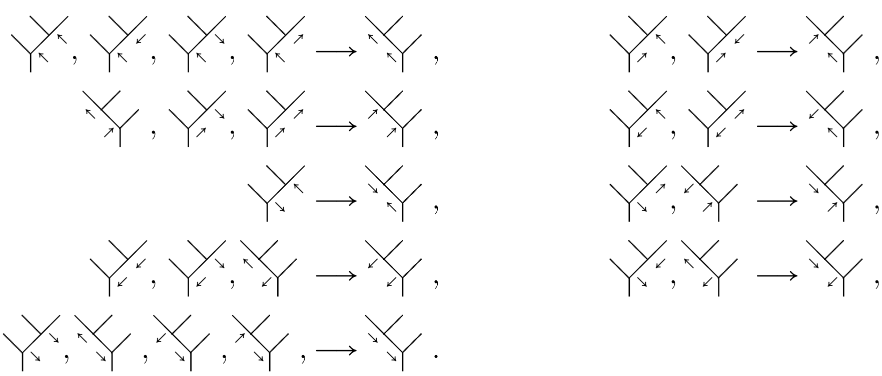
Remark 3. This result is also proved in [20], in a different way.

2.3 Koszulity of Quad

The koszulity of **Quad** is proved in [19] by the poset method. Let us give here a second proof, with the help of the rewriting method of [2, 3, 10, 13].

Theorem 8. *The operads* **Quad** *and* **Quad**$^!$ *are Koszul.*

Proof. By Koszul duality, it is enough to prove that **Quad**$^!$ is Koszul. We choose the order $\searrow < \nearrow < \swarrow < \nwarrow$ for the four operations, and the order  for the two planar binary trees of arity 3. Relations $(1)^!, \ldots, (9)^!$ give the following rewriting rule, defined by 23 elementary rewritings:



There are 156 critical monomials, and the 156 corresponding diagrams are confluent. Hence, **Quad**$^!$ is Koszul. We used a computer to find the critical monomials and to verify the confluence of the diagrams. $\square$

3 Quadri-bialgebras

3.1 Units and quadri-algebras

Let A, B be a vector spaces. We put $A\overline{\otimes}B = (K \otimes B) \oplus (A \otimes B) \oplus (A \otimes K)$. Clearly, if A, B, C are three vector spaces, $(A\overline{\otimes}B)\overline{\otimes}C = A\overline{\otimes}(B\overline{\otimes}C)$.

Proposition 9. *1. Let A be a quadri-algebra. We extend the four products on $A\overline{\otimes}A$ in the following way: if $a, b \in A$,*

$$\begin{array}{llll} a \nwarrow 1 = a, & a \nearrow 1 = 0, & 1 \nwarrow a = 0, & 1 \nearrow a = 0, \\ a \swarrow 1 = 0, & a \searrow 1 = 0, & 1 \swarrow a = 0, & 1 \searrow a = a. \end{array}$$

The nine relations defining quadri-algebras are true on $A\overline{\otimes}A\overline{\otimes}A$.

2. Let A, B be two quadri-algebras. Then $A\overline{\otimes}B$ is a quadri-algebra with the following products:

- *if $a, a' \in A \sqcup K$, $b, b' \in B \sqcup K$, with $(a, a') \notin K^2$ and $(b, b') \notin K^2$:*

$$\begin{array}{ll} (a \otimes b) \nwarrow (a' \otimes b') = (a \uparrow a') \otimes (b \leftarrow b'), & (a \otimes b) \nearrow (a' \otimes b') = (a \uparrow a') \otimes (b \rightarrow b'), \\ (a \otimes b) \swarrow (a' \otimes b') = (a \downarrow a') \otimes (b \leftarrow b'), & (a \otimes b) \searrow (a' \otimes b') = (a \downarrow a') \otimes (b \rightarrow b'). \end{array}$$

- *If $a, a' \in A$:*

$$\begin{array}{ll} (a \otimes 1) \nwarrow (a' \otimes 1) = (a \nwarrow a') \otimes 1, & (a \otimes 1) \nearrow (a' \otimes 1) = (a \nearrow a') \otimes 1, \\ (a \otimes 1) \swarrow (a' \otimes 1) = (a \swarrow a') \otimes 1, & (a \otimes 1) \searrow (a' \otimes 1) = (a \searrow a') \otimes 1. \end{array}$$

- *If $b, b' \in B$:*

$$\begin{array}{ll} (1 \otimes b) \nwarrow (1 \otimes b') = 1 \otimes (b \nwarrow b'), & (1 \otimes b) \nearrow (1 \otimes b') = 1 \otimes (b \nearrow b'), \\ (1 \otimes b) \swarrow (1 \otimes b') = 1 \otimes (b \swarrow b'), & (1 \otimes b) \searrow (1 \otimes b') = 1 \otimes (b \searrow b'). \end{array}$$

Proof. 1. It is shown by direct verifications.

2. As $(A, \uparrow, \downarrow)$ and $(B, \leftarrow, \rightarrow)$ are dendriform algebras, $A \otimes B$ is a **Dend** $\otimes$ **Dend**-algebra, so is a quadri-algebra by Proposition 6, with $\nwarrow = \uparrow \otimes \leftarrow$, $\swarrow = \downarrow \otimes \leftarrow$, $\searrow = \downarrow \otimes \rightarrow$ and $\nearrow = \uparrow \otimes \rightarrow$. The extension of the quadri-algebra axioms to $A\overline{\otimes}B$ is verified by direct computations. ☐

Remark 4. There is a second way to give $A\overline{\otimes}B$ a structure of quadri-algebra with the help of the associativity of $\star$:

$$\text{If } a \in A \text{ or } a' \in A,\ b, b' \in K \oplus B, \begin{cases} (a \otimes b) \nwarrow (a' \otimes b') & = (a \nwarrow a') \otimes (b \star b'), \\ (a \otimes b) \swarrow (a' \otimes b') & = (a \swarrow a') \otimes (b \star b'), \\ (a \otimes b) \searrow (a' \otimes b') & = (a \searrow a') \otimes (b \star b'), \\ (a \otimes b) \nearrow (a' \otimes b') & = (a \nearrow a') \otimes (b \star b'); \end{cases}$$

$$\text{if } b, b' \in K \oplus B, \begin{cases} (1 \otimes b) \nwarrow (1 \otimes b') & = 1 \otimes (b \nwarrow b'), \\ (1 \otimes b) \swarrow (1 \otimes b') & = 1 \otimes (b \swarrow b'), \\ (1 \otimes b) \searrow (1 \otimes b') & = 1 \otimes (b \searrow b'), \\ (1 \otimes b) \nearrow (1 \otimes b') & = 1 \otimes (b \nearrow b'). \end{cases}$$

$A \otimes K$ and $K \otimes B$ are quadri-subalgebras of $A\overline{\otimes}B$, respectively isomorphic to A and B.

3.2 Definitions and example of FQSym

Definition 10. *A quadri-bialgebra is a family* $(A, \nwarrow, \swarrow, \searrow, \nearrow, \tilde{\Delta}_{\nwarrow}, \tilde{\Delta}_{\swarrow}, \tilde{\Delta}_{\searrow}, \tilde{\Delta}_{\nearrow})$ *such that:*

- $(A \nwarrow, \swarrow, \searrow, \nearrow)$ *is a quadri-algebra.*

- $(A, \tilde{\Delta}_{\nwarrow}, \tilde{\Delta}_{\swarrow}, \tilde{\Delta}_{\searrow}, \tilde{\Delta}_{\nearrow})$ *is a quadri-coalgebra.*

- *We extend the four coproducts in the following way:*

$$\Delta_{\nwarrow} : \begin{cases} A & \longrightarrow A \otimes A \\ a & \longrightarrow \tilde{\Delta}_{\nwarrow}(a) + a \otimes 1, \end{cases} \qquad \Delta_{\nearrow} : \begin{cases} A & \longrightarrow A \otimes A \\ a & \longrightarrow \tilde{\Delta}_{\nearrow}(a), \end{cases}$$

$$\Delta_{\swarrow} : \begin{cases} A & \longrightarrow A \otimes A \\ a & \longrightarrow \tilde{\Delta}_{\swarrow}(a), \end{cases} \qquad \Delta_{\searrow} : \begin{cases} A & \longrightarrow A \otimes A \\ a & \longrightarrow \tilde{\Delta}_{\searrow}(a) + 1 \otimes a. \end{cases}$$

For all $a, b \in A$*:*

$$\begin{aligned}
\Delta_{\nwarrow}(a \nwarrow b) &= \Delta_{\uparrow}(a) \nwarrow \Delta_{\leftarrow}(b) & \Delta_{\nearrow}(a \nwarrow b) &= \Delta_{\uparrow}(a) \nwarrow \Delta_{\rightarrow}(b) \\
\Delta_{\nwarrow}(a \swarrow b) &= \Delta_{\uparrow}(a) \swarrow \Delta_{\leftarrow}(b) & \Delta_{\nearrow}(a \swarrow b) &= \Delta_{\uparrow}(a) \swarrow \Delta_{\rightarrow}(b) \\
\Delta_{\nwarrow}(a \searrow b) &= \Delta_{\uparrow}(a) \searrow \Delta_{\leftarrow}(b) & \Delta_{\nearrow}(a \searrow b) &= \Delta_{\uparrow}(a) \searrow \Delta_{\rightarrow}(b) \\
\Delta_{\nwarrow}(a \nearrow b) &= \Delta_{\uparrow}(a) \nearrow \Delta_{\leftarrow}(b) & \Delta_{\nearrow}(a \nearrow b) &= \Delta_{\uparrow}(a) \nearrow \Delta_{\rightarrow}(b)
\end{aligned}$$

$$\begin{aligned}
\Delta_{\swarrow}(a \nwarrow b) &= \Delta_{\downarrow}(a) \nwarrow \Delta_{\leftarrow}(b) & \Delta_{\searrow}(a \nwarrow b) &= \Delta_{\downarrow}(a) \nwarrow \Delta_{\rightarrow}(b) \\
\Delta_{\swarrow}(a \swarrow b) &= \Delta_{\downarrow}(a) \swarrow \Delta_{\leftarrow}(b) & \Delta_{\searrow}(a \swarrow b) &= \Delta_{\downarrow}(a) \swarrow \Delta_{\rightarrow}(b) \\
\Delta_{\swarrow}(a \searrow b) &= \Delta_{\downarrow}(a) \searrow \Delta_{\leftarrow}(b) & \Delta_{\searrow}(a \searrow b) &= \Delta_{\downarrow}(a) \searrow \Delta_{\rightarrow}(b) \\
\Delta_{\swarrow}(a \nearrow b) &= \Delta_{\downarrow}(a) \nearrow \Delta_{\leftarrow}(b) & \Delta_{\searrow}(a \nearrow b) &= \Delta_{\downarrow}(a) \nearrow \Delta_{\rightarrow}(b)
\end{aligned}$$

Remark 5. In other words, for all $a, b \in A$:

$$\begin{aligned}
\tilde{\Delta}_{\nwarrow}(a \nwarrow b) &= a'_{\uparrow} \uparrow b \otimes a''_{\uparrow} + a'_{\uparrow} \uparrow b'_{\leftarrow} \otimes a''_{\uparrow} \leftarrow b''_{\leftarrow},\\
\tilde{\Delta}_{\swarrow}(a \nwarrow b) &= a'_{\downarrow} \uparrow b \otimes a''_{\downarrow} + a'_{\downarrow} \uparrow b'_{\leftarrow} \otimes a''_{\downarrow} \leftarrow b''_{\leftarrow},\\
\tilde{\Delta}_{\searrow}(a \nwarrow b) &= a'_{\downarrow} \otimes a''_{\downarrow} \leftarrow b + a'_{\downarrow} \uparrow b'_{\rightarrow} \otimes a''_{\downarrow} \leftarrow b''_{\rightarrow},\\
\tilde{\Delta}_{\nearrow}(a \nwarrow b) &= a'_{\uparrow} \otimes a''_{\uparrow} \leftarrow b + a'_{\uparrow} \uparrow b'_{\rightarrow} \otimes a''_{\uparrow} \leftarrow b''_{\rightarrow},\\
\\
\tilde{\Delta}_{\nwarrow}(a \swarrow b) &= a'_{\uparrow} \downarrow b \otimes a''_{\uparrow} + a'_{\uparrow} \downarrow b'_{\leftarrow} \otimes a''_{\uparrow} \leftarrow b''_{\leftarrow},\\
\tilde{\Delta}_{\swarrow}(a \swarrow b) &= b \otimes a + b'_{\leftarrow} \otimes a \leftarrow b''_{\leftarrow} + a'_{\downarrow} \downarrow b \otimes a''_{\downarrow} + a'_{\downarrow} \downarrow b'_{\leftarrow} \otimes a''_{\downarrow} \leftarrow b''_{\leftarrow},\\
\tilde{\Delta}_{\searrow}(a \swarrow b) &= b'_{\rightarrow} \otimes a \leftarrow b''_{\rightarrow} + a'_{\downarrow} \downarrow b'_{\rightarrow} \otimes a''_{\downarrow} \leftarrow b''_{\rightarrow},\\
\tilde{\Delta}_{\nearrow}(a \swarrow b) &= a'_{\uparrow} \downarrow b'_{\rightarrow} \otimes a''_{\uparrow} \leftarrow b''_{\rightarrow},\\
\\
\tilde{\Delta}_{\nwarrow}(a \searrow b) &= a \downarrow b'_{\leftarrow} \otimes b''_{\leftarrow} + a'_{\uparrow} \downarrow b'_{\leftarrow} \otimes a''_{\uparrow} \rightarrow b''_{\leftarrow},\\
\tilde{\Delta}_{\swarrow}(a \searrow b) &= b'_{\leftarrow} \otimes a \rightarrow b''_{\leftarrow} + a'_{\downarrow} \downarrow b'_{\leftarrow} \otimes a''_{\downarrow} \rightarrow b''_{\leftarrow},\\
\tilde{\Delta}_{\searrow}(a \searrow b) &= b'_{\rightarrow} \otimes a \rightarrow b''_{\rightarrow} + a'_{\downarrow} \downarrow b'_{\rightarrow} \otimes a''_{\downarrow} \rightarrow b''_{\rightarrow},\\
\tilde{\Delta}_{\nearrow}(a \searrow b) &= a \downarrow b''_{\rightarrow} \otimes b''_{\rightarrow} + a'_{\uparrow} \downarrow b'_{\rightarrow} \otimes a''_{\uparrow} \rightarrow b''_{\rightarrow},\\
\\
\tilde{\Delta}_{\nwarrow}(a \nearrow b) &= a \uparrow b'_{\leftarrow} \otimes b''_{\leftarrow} + a'_{\uparrow} \uparrow b'_{\leftarrow} \otimes a''_{\uparrow} \rightarrow b''_{\leftarrow},\\
\tilde{\Delta}_{\swarrow}(a \nearrow b) &= a'_{\downarrow} \uparrow b'_{\leftarrow} \otimes a''_{\downarrow} \rightarrow b''_{\leftarrow},\\
\tilde{\Delta}_{\searrow}(a \nearrow b) &= a'_{\downarrow} \otimes a''_{\downarrow} \rightarrow b + a'_{\downarrow} \uparrow b'_{\rightarrow} \otimes a''_{\downarrow} \rightarrow b''_{\rightarrow},\\
\tilde{\Delta}_{\nearrow}(a \nearrow b) &= a \otimes b + a'_{\uparrow} \otimes a''_{\uparrow} \rightarrow b + a \uparrow b''_{\rightarrow} \otimes b''_{\rightarrow} + a'_{\uparrow} \uparrow b'_{\rightarrow} \otimes a''_{\uparrow} \rightarrow b''_{\rightarrow}.
\end{aligned}$$

Consequently, we obtain four dendriform bialgebras [7]:

$$(A, \leftarrow, \rightarrow, \Delta_{\leftarrow}, \Delta_{\rightarrow}), \quad (A, \downarrow^{op}, \uparrow^{op}, \Delta_{\downarrow}^{op}, \Delta_{\uparrow}^{op}), \quad (A, \rightarrow^{op}, \leftarrow^{op}, \Delta_{\uparrow}, \Delta_{\downarrow}), \quad (A, \uparrow, \downarrow, \Delta_{\rightarrow}^{op}, \Delta_{\leftarrow}^{op}).$$

Summing, we also obtain:

$$\begin{aligned}
\tilde{\Delta}(a \nwarrow b) &= a' \uparrow b \otimes a'' + a' \otimes a'' \leftarrow b + a' \uparrow b' \otimes a'' \leftarrow b'',\\
\tilde{\Delta}(a \swarrow b) &= b \otimes a + a' \downarrow b \otimes a'' + b' \otimes a \leftarrow b'' + a' \downarrow b' \otimes a'' \leftarrow b'',\\
\tilde{\Delta}(a \searrow b) &= a \downarrow b' \otimes b'' + b' \otimes a \rightarrow b'' + a' \downarrow b' \otimes a'' \rightarrow b'',\\
\tilde{\Delta}(a \nearrow b) &= a \otimes b + a \uparrow b' \otimes b'' + a' \otimes a'' \rightarrow b + a' \uparrow b' \otimes a'' \rightarrow b''.
\end{aligned}$$

Proposition 11. *The augmentation ideal of* **FQSym** *is a quadri-bialgebra.*

Proof. As an example, let us prove the last compatibility. Let σ, τ be two permutations, of respective length k and l. Then $\Delta_{\nearrow}(\sigma \nearrow \tau)$ is obtained by shuffling in all possible ways the words σ and the shifting $\tau[k]$ of τ, such that the first letter comes from σ and the last letter comes from $\tau[k]$, and then cutting the obtained words in such a way that 1 is in the left part and $k+l$ in the right part. Hence, the left part should contain letters coming from σ, including 1, and starts by the first letter of σ, and the right part should contain letters coming from $\tau[k]$, including $k+l$, and ends with the last letter of $\tau[k]$. there are four possibilities:

- The left part contains only letters from σ and the right part contains only letters form $\tau[k]$. This gives the term $\sigma \otimes \tau$.
- The left part contains only letters from σ, and the right part contains letters from σ and $\tau[k]$. This gives the term $\sigma'_{\uparrow} \otimes \sigma''_{\uparrow} \rightarrow \tau$.

- The left part contains letters from σ and $\tau[k]$, and the right part contains only letters form $\tau[k]$. This gives the term $\sigma \uparrow \tau'_{\rightarrow} \otimes \tau''_{\rightarrow}$.

- Both parts contains letters from σ and $\tau[k]$. This gives the term $\sigma'_{\uparrow} \uparrow \tau'_{\rightarrow} \otimes \sigma''_{\uparrow} \rightarrow \tau''_{\rightarrow}$.

So:

$$\Delta_{\nearrow}(\sigma \nearrow \tau) = \sigma \otimes \tau + \sigma'_{\uparrow} \otimes \sigma''_{\uparrow} \rightarrow \tau + \sigma \uparrow \tau'_{\rightarrow} \otimes \tau''_{\rightarrow} + \sigma'_{\uparrow} \uparrow \tau'_{\rightarrow} \otimes \sigma''_{\uparrow} \rightarrow \tau''_{\rightarrow}.$$

The other compatibilities are proved following the same lines. □

3.3 Other examples

Let $F_{\mathbf{Quad}}(V)$ be the free quadri-algebra generated by V. As it is free, it is possible to define four coproducts satisfying the quadri-bialgebra axioms in the following way: for all $v \in V$,

$$\tilde{\Delta}_{\nwarrow}(v) = \tilde{\Delta}_{\swarrow}(v) = \tilde{\Delta}_{\searrow}(v) = \tilde{\Delta}_{\nearrow}(v) = 0.$$

It is naturally graded by puting the elements of V homogeneous of degree 1.

Proposition 12. *For any vector space V, $F_{\mathbf{Quad}}(V)$ is a quadri-bialgebra.*

Proof. We only have to prove the nine compatibilities of quadri-coalgebras. We consider:

$$B_{(1,1)} = \{a \in F_{\mathbf{Quad}}(V) \mid (\Delta_{\nwarrow} \otimes Id) \circ \Delta_{\nwarrow}(a) = (Id \otimes \Delta) \circ \Delta_{\nwarrow}(a)\}.$$

First, for all $v \in V$:

$$(\Delta_{\nwarrow} \otimes Id) \circ \Delta_{\nwarrow}(v) = v \otimes 1 \otimes 1 = (Id \otimes \Delta) \circ \Delta_{\nwarrow}(v).$$

so $V \subseteq B_{(1,1)}$. If $a, b \in B_{(1,1)}$ and $\diamond \in \{\nwarrow, \swarrow, \searrow, \nearrow\}$:

$$\begin{aligned}
(\Delta_{\nwarrow} \otimes Id) \circ \Delta_{\nwarrow}(a \diamond b) &= ((\Delta_{\uparrow} \otimes Id) \circ \Delta_{\uparrow}(a)) \diamond ((\Delta_{\leftarrow} \otimes Id) \circ \Delta_{\leftarrow}(b)) \\
&= ((Id \otimes \Delta) \circ \Delta_{\uparrow}(a)) \diamond ((Id \otimes \Delta) \circ \Delta_{\leftarrow}(b)) \\
&= (Id \otimes \Delta)(\Delta_{\uparrow}(a) \diamond \Delta_{\leftarrow}(b)) \\
&= (Id \otimes \Delta) \circ \Delta_{\nwarrow}(a \diamond b).
\end{aligned}$$

So $a \diamond b \in B_{(1,1)}$, and $B_{(1,1)}$ is a quadri-subalgebra of $F_{\mathbf{Quad}}(V)$ containing V: $B_{(1,1)} = F_{\mathbf{Quad}}(V)$, and the quadri-coalgebra relation (1.1) is satisfied. The eight other relations can be proved in the same way. Hence, $F_{\mathbf{Quad}}(V)$ is a quadri-bialgebra. □

Remark 6.

1. We deduce that $(F_{\mathbf{Quad}}(V), \leftarrow, \rightarrow, \Delta_{\leftarrow}, \Delta_{\rightarrow})$ and $(F_{\mathbf{Quad}}(V), \uparrow, \downarrow, \Delta^{op}_{\rightarrow}, \Delta^{op}_{\leftarrow})$ are bidendriform bialgebras, in the sense of [7, 8]; consequently, $(F_{\mathbf{Quad}}(V), \leftarrow, \rightarrow)$ and $(F_{\mathbf{Quad}}(V), \uparrow, \downarrow)$ are free dendriform algebras.

2. When V is one-dimensional, here are the respective dimensions a_n, b_n and c_n of the homogeneous components, of the primitive elements, and of the dendriform primitive elements, of degree n, for these two dendriform bialgebras:

n	1	2	3	4	5	6	7	8	9	10
a_n	1	4	23	156	1 162	9 162	75 819	644 908	5 616 182	49 826 712
b_n	1	3	16	105	768	6 006	49 152	415 701	3 604 480	31 870 410
c_n	1	2	10	64	462	3 584	29 172	245 760	2 124 694	18 743 296

These are sequences A007297, A085614 and A078531 of [18].

3. Let V be finite-dimensional. The graded dual $F_{\mathbf{Quad}}(V)^*$ of $F_{\mathbf{Quad}}(V)$ is also a quadri-bialgebra. By the bidendriform rigidity theorem [7, 8], $(F_{\mathbf{Quad}}(V)^*, \leftarrow, \rightarrow)$ and $(F_{\mathbf{Quad}}(V)^*, \uparrow, \downarrow)$ are free dendriform algebras. Moreover, for any $x, y \in V$, nonzero, $x \nwarrow y$ and $x \searrow y$ are nonzero elements of $\mathrm{Prim}_{\mathbf{Quad}}(F_{\mathbf{Quad}}(V))$, which implies that $(F_{\mathbf{Quad}}(V)^*, \nwarrow, \swarrow, \searrow, \nearrow)$ is not generated in degree 1, so is not free as a quadri-algebra. Dually, the quadri-coalgebra $F_{\mathbf{Quad}}(V)$ is not cofree.

We now give a similar construction on the Hopf algebra of packed words **WQSym**, see [17] for more details on this combinatorial Hopf algebra.

Theorem 13. *For any nonempty packed word w of length n, we put:*

$$m(w) = \max\{i \in [n] \mid w(i) = 1\}, \qquad M(w) = \max\{i \in [n] \mid w(i) = \max(w)\}.$$

We define four products on the augmentation ideal of **WQSym** *in the following way: if u, v are packed words of respective lengths $k, l \geqslant 1$:*

$$u \nwarrow v = \sum_{\substack{\mathrm{pack}(w(1)\ldots w(k))=u,\\ \mathrm{pack}(w(k+1)\ldots w(k+l)=v,\\ m(w),M(w)\leqslant k}} w, \qquad u \nearrow v = \sum_{\substack{\mathrm{pack}(w(1)\ldots w(k))=u,\\ \mathrm{pack}(w(k+1)\ldots w(k+l)=v,\\ m(w)\leqslant k<M(w)}} w,$$

$$u \swarrow v = \sum_{\substack{\mathrm{pack}(w(1)\ldots w(k))=u,\\ \mathrm{pack}(w(k+1)\ldots w(k+l)=v,\\ M(w)\leqslant k<m(w)}} w, \qquad u \searrow v = \sum_{\substack{\mathrm{pack}(w(1)\ldots w(k))=u,\\ \mathrm{pack}(w(k+1)\ldots w(k+l)=v,\\ k<m(w),M(w)}} w.$$

Here, pack *denote the packing operation of words (see [17] for more details). We define four coproducts on the augmentation ideal of* **WQSym** *in the following way: if u is a packed word of length $n \geqslant 1$,*

$$\Delta_\nwarrow(u) = \sum_{u(1),u(n)\leqslant i<\max(u)} u_{|[i]} \otimes \mathrm{pack}(u_{|[\max(u)]\setminus[i]}),$$
$$\Delta_\swarrow(u) = \sum_{u(n)\leqslant i<u(1)} u_{|[i]} \otimes \mathrm{pack}(u_{|[\max(u)]\setminus[i]}),$$
$$\Delta_\searrow(u) = \sum_{1\leqslant i<u(1),u(n)} u_{|[i]} \otimes \mathrm{pack}(u_{|[\max(u)]\setminus[i]}),$$
$$\Delta_\nearrow(u) = \sum_{u(1)\leqslant i<u(n)} u_{|[i]} \otimes \mathrm{pack}(u_{|[\max(u)]\setminus[i]}).$$

We used the following notation: if u is a packed word and I is a set of integers, then $u_{|I}$ is the word (non necessarily packed) obtained by deleting of the letters of u which do not belong to I. These products and coproducts make **WQSym** *a quadri-bialgebra. The induced Hopf algebra structure is the usual one.*

Proof. For all packed words u, v of respective lengths $k, l \geqslant 1$:

$$u \star v = \sum_{\substack{\mathrm{pack}(w(1)\ldots w(k))=u,\\ \mathrm{pack}(w(k+1)\ldots w(k+l)=v}} w.$$

So $\star$ is the usual product of **WQSym**, and is associative. In particular, if u, v, w are packed words of respective lengths $k, l, n \geqslant 1$:

$$u \star (v \star w) = (u \star v) \star w = \sum_{\substack{\mathrm{pack}(x(1)\ldots x(k))=u,\\ \mathrm{pack}(x(k+1)\ldots x(k+l)=v,\\ \mathrm{pack}(x(k+l+1),\ldots,x(k+l+n))=w}} x.$$

Then each side of relations $(1,1)\dots(3,3)$ is the sum of the terms in this expression such that:

$$\begin{array}{lll} m(x), M(x) \leqslant k & m(x) \leqslant k < M(x) \leqslant k+l & m(x) \leqslant k < k+l < M(x) \\ M(x) \leqslant k < m(x) \leqslant k+l & k < m(x), M(x) \leqslant k+l & k < m(x) \leqslant k+l < M(x) \\ M(x) \leqslant k < k+l < m(x) & k < M(x) \leqslant k+l < m(x) & k+l < m(x), M(x) \end{array}$$

So $(\mathbf{WQSym}, \nwarrow, \swarrow, \searrow, \nearrow)$ is a quadri-algebra.

For all packed word u of length $n \geqslant 1$:

$$\tilde{\Delta}(u) = \sum_{1 \leqslant i < \max(u)} u_{|[i]} \otimes \mathrm{pack}(u_{|[\max(u)] \setminus [i]}).$$

So $\tilde{\Delta}$ is the usual coproduct of $\mathbf{WQSym}$ and is coassociative. Moreover:

$$(\tilde{\Delta} \otimes Id) \circ \tilde{\Delta}(u) = (Id \otimes \tilde{\Delta}) \circ \tilde{\Delta}(u) = \sum_{1 \leqslant i < j < \max(u)} u_{|[i]} \otimes \mathrm{pack}(u_{|[j] \setminus [i]}) \otimes \mathrm{pack}(u_{|[\max(u)] \setminus [j]}).$$

Then each side of relations $(1,1)\dots(3,3)$ is the sum of the terms in this expression such that:

$$\begin{array}{lll} u(1), u(n) \leqslant i & u(1) \leqslant i < u(n) \leqslant j & u(1) \leqslant i < j < u(n) \\ u(n) \leqslant i < u(1) \leqslant j & i < u(1), u(n) \leqslant j & i < u(1) \leqslant j < u(n) \\ u(n) \leqslant i < j < u(1) & i < u(n) \leqslant j < u(1) & j < u(1), u(n) \end{array}$$

So $(\mathbf{WQSym}, \Delta_\nwarrow, \Delta_\swarrow, \Delta_\searrow, \Delta_\nearrow)$ is a quadri-coalgebra.

Let us prove, as an example, one of the compatibilities between the products and the coproducts. If u, v are packed words of respective lengths $k, l \geqslant 1$, $\Delta_\nearrow(u \nearrow v)$ is obtained as follows:

- Consider all the packed words w such that $\mathrm{pack}(w(1)\dots w(k)) = u$, $\mathrm{pack}(w(k+1)\dots w(k+l)) = v$, such that $1 \notin \{w(k+1), \dots, w(k+l)\}$ and $\max(w) \in \{w(k+1), \dots, w(k+l)\}$.
- Cut all these words into two parts, by separating the letters into two parts according to their orders, such that the first letter of w in the left (smallest) part, and the last letter of w is in the right (greatest) part, and pack the two parts.

If $u' \otimes u''$ is obtained in this way, before packing, u' contains 1, so contains letters $w(i)$ with $i \leqslant k$, and u'' contains $\max(w)$, so contains letters $w(i)$, with $i > k$. Four cases are possible.

- u' contains only letters $w(i)$ with $i \leqslant k$, and u'' contains only letters $w(i)$ with $i > k$. Then $w = (u(1)\dots u(k)(v(1)+\max(u))\dots(v(l)+\max(u))$ and $u' \otimes u'' = u \otimes v$.
- u' contains only letters $w(i)$ with $i \leqslant k$, whereas u'' contains letters $w(i)$ with $i \leqslant k$ and letters $w(j)$ with $j > k$. Then u' is obtained from u by taking letters $< i$, with $i \geqslant u(1)$, and u'' is a term appearing in $\mathrm{pack}(u_{|[k] \setminus [i]}) \star v$, such that there exists $j > k - i$, with $u''(j) = \max(u'')$. Summing all the possibilities, we obtain $u'_\uparrow \otimes u''_\uparrow \rightarrow v$.
- u' contains letters $w(i)$ with $i \leqslant k$ and letters $w(j)$ with $j > k$, whereas u'' contains only letters $w(i)$ with $i > k$. With the same type of analysis, we obtain $u \uparrow v'_\rightarrow \otimes v''_\rightarrow$.
- Both u' and u'' contain letters $w(i)$ with $i \leqslant k$ and letters $w(j)$ with $j > k$. We obtain $u'_\uparrow \uparrow v'_\rightarrow \otimes u''_\uparrow \rightarrow v''_\rightarrow$.

Finally:

$$\Delta_\nearrow(u \nearrow v) = u \otimes v + u'_\uparrow \otimes u''_\uparrow \rightarrow v + u \uparrow v'_\rightarrow \otimes v''_\rightarrow + u'_\uparrow \uparrow v'_\rightarrow \otimes u''_\uparrow \rightarrow v''_\rightarrow.$$

The fifteen remaining compatibilites are proved following the same lines. $\square$

Example 4.

$$
\begin{aligned}
(12) \nwarrow (12) &= (1423),\\
(12) \swarrow (12) &= (1312) + (2312) + (2413) + (3412),\\
(12) \searrow (12) &= (1212) + (1213) + (2313) + (2314),\\
(12) \nearrow (12) &= (1223) + (1234) + (1323) + (1324).
\end{aligned}
$$

Corollary 14. *$(\mathbf{WQSym}, \rightarrow, \leftarrow)$ and $(\mathbf{WQSym}, \downarrow, \uparrow)$ are free dendriform algebras.*

Remark 7.

1. If A is a quadri-algebra, we put:

$$\mathrm{Prim}_{\mathbf{Quad}}(A) = \mathrm{Ker}(\tilde{\Delta}_{\nwarrow}) \cap \mathrm{Ker}(\tilde{\Delta}_{\swarrow}) \cap \mathrm{Ker}(\tilde{\Delta}_{\searrow}) \cap \mathrm{Ker}(\tilde{\Delta}_{\nearrow}).$$

   For any vector space V, $A = F_{\mathbf{Quad}}(V)$ is obviously generated by $\mathrm{Prim}_{\mathbf{Quad}}(A)$, as $V \subseteq \mathrm{Prim}_{\mathbf{Quad}}(A)$.

2. Let us consider the quadri-bialgebra **FQSym**. Direct computations show that:

$$
\begin{aligned}
\mathrm{Prim}_{\mathbf{Quad}}(\mathbf{FQSym})_1 &= \mathrm{Vect}(1),\\
\mathrm{Prim}_{\mathbf{Quad}}(\mathbf{FQSym})_2 &= (0),\\
\mathrm{Prim}_{\mathbf{Quad}}(\mathbf{FQSym})_3 &= (0),\\
\mathrm{Prim}_{\mathbf{Quad}}(\mathbf{FQSym})_4 &= \mathrm{Vect}((2413) - (2143), (2413) - (3412)).
\end{aligned}
$$

   Moreover, the homogeneous component of degree 4 of the quadri-subalgebra generated by $\mathrm{Prim}_{\mathbf{Quad}}(\mathbf{FQSym})$ has dimension 23, with basis:

$$
\begin{aligned}
&(1234), (1243), (1324), (1342), (1423), (1432), (2134), (2314), (2314), (2431),\\
&(3124), (3214), (3241), (3421), (4123), (4132), (4213), (4231), (4312), (4321),\\
&(2143) + (2413), (3142) + (3412), (2143) - (3142).
\end{aligned}
$$

   So **FQSym** is not generated by $\mathrm{Prim}_{\mathbf{Quad}}(\mathbf{FQSym})$, so is not isomorphic, as a quadri-bialgebra, to any $F_{\mathbf{Quad}}(V)$. A similar argument holds for **WQSym**.

References

[1] Marcelo Aguiar and Jean-Louis Loday, *Quadri-algebras*, J. Pure Appl. Algebra **191** (2004), no. 3, 205–221, arXiv:math/0309171.

[2] Vladimir Dotsenko and Anton Khoroshkin, *Gröbner bases for operads*, Duke Math. J. **153** (2010), no. 2, 363–396, arXiv:0812.4069.

[3] Vladimir Dotsenko and Bruno Vallette, *Higher Koszul duality for associative algebras*, Glasg. Math. J. **55** (2013), no. A, 55–74.

[4] G. H. E. Duchamp, L. Foissy, N. Hoang-Nghia, D. Manchon, and A. Tanasa, *A combinatorial non-commutative Hopf algebra of graphs*, Discrete Mathematics & Theoretical Computer Science **16** (2014), no. 1, 355–370, arXiv:1307.3928.

[5] Gérard Duchamp, Florent Hivert, and Jean-Yves Thibon, *Noncommutative symmetric functions. VI. Free quasi-symmetric functions and related algebras*, Internat. J. Algebra Comput. **12** (2002), no. 5, 671–717.

[6] Kurusch Ebrahimi-Fard and Li Guo, *On products and duality of binary quadratic regular operads*, J. Pure Appl. Algebra **200** (2005), no. 3, 293–317, arXiv:math/0407162.

[7] Loïc Foissy, *Bidendriform bialgebras, trees, and free quasi-symmetric functions*, J. Pure Appl. Algebra **209** (2007), no. 2, 439–459, arXiv:math/0505207.

[8] ______, *Primitive elements of the Hopf algebra of free quasi-symmetric functions*, Combinatorics and physics, Contemp. Math., vol. 539, Amer. Math. Soc., Providence, RI, 2011, pp. 79–88.

[9] Loïc Foissy and Frédéric Patras, *Natural endomorphisms of shuffle algebras*, Internat. J. Algebra Comput. **23** (2013), no. 4, 989–1009, arXiv:1311.1464.

[10] Eric Hoffbeck, *A Poincaré-Birkhoff-Witt criterion for Koszul operads*, Manuscripta Math. **131** (2010), no. 1-2, 87–110, arXiv:0709.2286.

[11] Jean-Louis Loday, *Dialgebras*, Dialgebras and related operads, Lecture Notes in Math., vol. 1763, Springer, Berlin, 2001, arXiv:math/0102053, pp. 7–66.

[12] Jean-Louis Loday, *Completing the operadic butterfly*, arXiv:math.RA/0409183, 2004.

[13] Jean-Louis Loday and Bruno Vallette, *Algebraic operads*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 346, Springer, Heidelberg, 2012.

[14] Sara Madariaga, *Gröbner-Shirshov bases for the non-symmetric operads of dendriform algebras and quadri-algebras*, J. Symbolic Comput. **60** (2014), 1–14.

[15] Clauda Malvenuto and Christophe Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra*, J. Algebra **177** (1995), no. 3, 967–982.

[16] Martin Markl, Steve Schnider, and Jim Stasheff, *Operads in Algebra, Topology and Physics*, American Mathematical Society, 2002.

[17] Jean-Christophe Novelli, Frédéric Patras, and Jean-Yves Thibon, *Natural endomorphisms of quasi-shuffle Hopf algebras*, Bull. Soc. Math. France **141** (2013), no. 1, 107–130, arXiv:1101.0725.

[18] N. J. A Sloane, *On-line encyclopedia of integer sequences*, http://oeis.org/.

[19] Bruno Vallette, *Manin products, Koszul duality, Loday algebras and Deligne conjecture*, Journal für die reine und angewandte Mathematik **620** (2008), 105–164, arXiv:math/0609002.

[20] Vincent Vong, *Combinatorial proofs of freeness of some $\mathcal{P}$-algebras*, Proceedings of FPSAC 2015, Discrete Math. Theor. Comput. Sci. Proc., Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2015, pp. 523–534. MR 3470891