Chromatic polynomials and bialgebras of graphs

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Abstract

The chromatic polynomial is characterized as the unique polynomial invariant of graphs, compatible with two interacting bialgebras structures: the first coproduct is given by partitions of vertices into two parts, the second one by a contraction-extraction process. This gives Hopf-algebraic proofs of Rota’s result on the signs of coefficients of chromatic polynomials and of Stanley’s interpretation of the values at negative integers of chromatic polynomials. We also consider chromatic symmetric functions and their noncommutative versions.

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Contents

1 Hopf algebraic structures on graphs 4
  1.1 The first coproduct ........................................ 4
  1.2 The second coproduct ..................................... 5
  1.3 Antipode ............................................... 7
  1.4 Cointeraction ........................................... 8

2 Chromatic polynomials 10
  2.1 Determination of \( \phi_0 \) ................................... 10
  2.2 Determination of \( \phi_1 \) ................................... 10
  2.3 Extraction and contraction of edges ....................... 12
  2.4 Lattices attached to graphs ................................. 14
  2.5 Applications ........................................... 17
  2.6 Values at negative integers ............................... 20

3 Chromatic symmetric functions 21
  3.1 Reminders on \( QSym \) .................................... 21
  3.2 Cointeraction and quasi-symmetric functions ............. 22
  3.3 Chromatic symmetric function ............................. 24
  3.4 Extension of \( \phi_0 \) ....................................... 26

4 Non-commutative versions 27
  4.1 Non-commutative Hopf algebra of graphs ................. 27
  4.2 Reminders on \( WQSym \) .................................. 29
  4.3 Non-commutative chromatic symmetric functions ........ 29
  4.4 Non-commutative version of \( F_0 \) ......................... 30
  4.5 From non-commutative to commutative .................... 31
Introduction

In graph theory, the chromatic polynomial, introduced by Birkhoff and Lewis [3] in order to treat the four color theorem, is a polynomial invariant attached to a graph; its values at $X = k$ give the number of valid colorings of the graph with $k$ colors, for all integer $k \geq 1$. Numerous results are known on this object, as for example the alternation signs of the coefficients, a result due to Rota [21], proved with the help of Möbius inversion in certain lattices.

Our aim here is to insert chromatic polynomial into the theory of combinatorial Hopf algebras, and to recover new proofs of these classical results. Our main tools, presented in the first section, will be a Hopf algebra $(\mathcal{H}_G, m, \Delta)$ and a bialgebra $(\mathcal{H}_G, m, \delta)$, both based on graphs. They share the same product, given by disjoint union; the first (cocommutative) coproduct, denoted by $\Delta$, is given by partitions of vertices into two parts; the second (not cocommutative) one, denoted by $\delta$, is given by a contraction-extraction process. For example:

$$\Delta(Y) = Y \otimes 1 + 1 \otimes Y + 3! \otimes \cdots + 3 \otimes 1,$$

$$\delta(Y) = \cdots \otimes Y + 3! \otimes 1 + Y \otimes \cdots .$$

$(\mathcal{H}_G, m, \Delta)$ is a Hopf algebra, graded by the cardinality of graphs, and connected, that is to say its connected component of degree 0 is reduced to the base field $\mathbb{Q}$: this is what is usually called a combinatorial Hopf algebra. On the other side, $(\mathcal{H}_G, m, \delta)$ is a bialgebra, graded by the degree defined by:

$$\text{deg}(G) = \# \{ \text{vertices of } G \} - \# \{ \text{connected components of } G \}.$$  

These two bialgebras are in cointeraction, a notion described in [7, 10, 18]: $(\mathcal{H}_G, m, \Delta)$ is a bialgebra-comodule over $(\mathcal{H}_G, m, \delta)$, see Theorem 7. Another example of interacting bialgebras is the pair $(\mathbb{Q}[X], m, \Delta)$ and $(\mathbb{Q}[X], m, \delta)$, where $m$ is the usual product of $\mathbb{Q}[X]$ and the two coproducts $\Delta$ and $\delta$ are defined by:

$$\Delta(X) = X \otimes 1 + 1 \otimes X,$$

$$\delta(X) = X \otimes X.$$  

This has interesting consequences, proved and used on quasi-posets in [10], listed here in Theorem 8. In particular:

1. We denote by $M_G$ the monoid of characters of $(\mathcal{H}_G, m, \delta)$. This monoid acts on the set $E_{\mathcal{H}_G \rightarrow \mathbb{Q}[X]}$ of Hopf algebra morphisms from $(\mathcal{H}_G, m, \Delta)$ to $(\mathbb{Q}[X], m, \Delta)$, via the map:

$$\leftarrow : \begin{cases} E_{\mathcal{H}_G \rightarrow \mathbb{Q}[X]} \times M_G & \rightarrow & E_{\mathcal{H}_G \rightarrow \mathbb{Q}[X]} \\ (\phi, \lambda) & \mapsto & \phi \leftarrow \lambda = (\phi \otimes \lambda) \circ \delta. \end{cases}$$

2. There exists a unique $\phi_0 \in E_{\mathcal{H}_G \rightarrow \mathbb{Q}[X]}$, homogeneous, such that $\phi_0(X) = X$. This morphism is attached to a character $\lambda_0 \in M_G$: for any graph $G$ with $n$ vertices:

$$\phi_0(G) = \lambda_0(G) X^n.$$  

3. There exists a unique $\phi_1 \in E_{\mathcal{H}_G \rightarrow \mathbb{Q}[X]}$, compatible with $m$, $\Delta$ and $\delta$. It is given by $\phi_1 = \phi_0 \leftarrow \lambda_0^{-1}$, where $\lambda_0^{-1}$ is the inverse of $\lambda_0$ for the convolution product of $M_G$.

The morphisms $\phi_0$ and $\phi_1$ and the attached characters are described in the second section. We first prove that, for any graph $G$, $\lambda_0(G) = 1$, and $\phi_1(G)$ is the chromatic polynomial $P_{\text{chr}}(G)$ (Proposition 9 and Theorem 11). This characterizes the chromatic polynomial as the unique polynomial invariant on graphs compatible with the product $m$ and both coproducts $\Delta$ and $\delta$. The character attached to the chromatic polynomial is consequently now called the chromatic character and denoted by $\lambda_{\text{chr}}$. The action of $M_G$ is used to prove that for any graph $G$:

$$P_{\text{chr}}(G) = \sum_{\sim} \lambda_{\text{chr}}(G|\sim) X^{d(\sim)},$$
where the sum is over a family of equivalences $\sim$ on the set of vertices of $G$, $\text{cl}(\sim)$ is the number of equivalence classes of $\sim$, and $G|\sim$ is a graph obtained by restricting $G$ to the classes of $\sim$ (Corollary 12). Therefore, the knowledge of the chromatic character implies the knowledge of the chromatic polynomial; we give a formula for computing this chromatic character on any graph with the notion (used in Quantum Field Theory) of forests, through the antipode of a quotient of $(H_G, m, \delta)$, see Proposition 13. We show how to compute the chromatic polynomial and the chromatic character of a graph by induction on the number of edges by an extraction-contraction of an edge in Proposition 15: we obtain an algebraic proof of this well-known result, which is classically obtained by a combinatorial study of colorings of $G$. As consequences, we obtain proofs of Rota’s result on the sign of the coefficients of a chromatic polynomial (Corollary 19) and of Stanley’s interpretation of values at negative integers of a chromatic polynomial in Corollary 24. The link with Rota’s proof is made via the lattice attached to a graph, defined in Proposition 16.

Using Aguiar, Bergeron and Sottile’s theory of combinatorial Hopf algebras [2], we prove that there exists a unique Hopf algebra morphism $F_{\text{chr}}$ from $\mathcal{H}_G$ to the Hopf algebra of quasisymmetric functions $\mathcal{QSym}$, compatible with the second coproduct $\delta$ of $\mathcal{QSym}$ (Theorems 27 and 28). This morphism sends any graph $G$ to its chromatic symmetric function, as defined by Stanley [24]. As a consequence, we obtain a diagram of Hopf algebra morphisms:

\[
\begin{array}{ccc}
\mathcal{H}_G & \xrightarrow{P_{\text{chr}}} & \mathbb{Q}[X] \\
\downarrow F_{\text{chr}} & & \downarrow \\
\mathcal{QSym} & & 
\end{array}
\]

where $H$ is given with the help of Hilbert polynomials (Proposition 25). This implies (Corollary 29) that the counit $\varepsilon'$ of $(H_G, m, \delta)$ can be seen as the exponential of an infinitesimal character of $H_G$, closely related to the character $\lambda_{\text{chr}}$ (namely, they coincide on connected graphs). We obtain also a lifting of $\phi_0$, giving a diagram of Hopf algebra morphisms:

\[
\begin{array}{ccc}
\mathcal{H}_G & \xrightarrow{\phi_0} & \mathbb{Q}[X] \\
\downarrow F_0 & & \downarrow \\
\mathcal{QSym} & & 
\end{array}
\]

For any graph $G$, via the action of $M_G$:

\[
F_{\text{chr}}(G) = \sum_{\sim} \lambda_{\text{chr}}(G|\sim)F_0(G/\sim).
\]

The last section deals with a non-commutative version of the chromatic symmetric function: the Hopf algebra of graphs is replaced by a non-commutative Hopf algebra of indexed graphs, and $\mathcal{QSym}$ is replaced by the Hopf algebra of packed words $\mathcal{WQSym}$. For any indexed graph $G$, its non-commutative chromatic symmetric function $F_{\text{chr}}(G)$ can also be seen as a symmetric formal series in non-commutative indeterminates (Theorem 35): we recover in this way the chromatic symmetric function introduced in [11] and related in [20] to MacMahon symmetric functions. We also obtain in this way a non-commutative version $F_0$ of $F_0$ (Proposition 36), also related to $F_{\text{chr}}$ by the action of the character $\lambda_{\text{chr}}$.

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Notations.

1. All the vector spaces in the text are taken over \( \mathbb{Q} \).
2. For any integer \( n \geq 0 \), we denote by \([n]\) the set \( \{1, \ldots, n\}\). In particular, \([0]\) = \emptyset.
3. The usual product of \( \mathbb{Q}[X] \) is denoted by \( m \). This algebra is given two bialgebra structures, defined by:

\[ \Delta(X) = X \otimes 1 + 1 \otimes X, \quad \delta(X) = X \otimes X. \]

Identifying \( \mathbb{Q}[X, Y] \) and \( \mathbb{Q}[X] \otimes \mathbb{Q}[Y] \), for any \( P \in \mathbb{Q}[X] \):

\[ \Delta(P)(X, Y) = P(X + Y), \quad \delta(P)(X, Y) = P(XY). \]

The counit of \( \Delta \) is given by:

\[ \forall P \in \mathbb{Q}[X], \quad \varepsilon(P) = P(0). \]

The counit of \( \delta \) is given by:

\[ \forall P \in \mathbb{Q}[X], \quad \varepsilon'(P) = P(1). \]

1 Hopf algebraic structures on graphs

We refer to [13] for classical results and vocabulary on graphs. Recall that a graph is a pair \( G = (V(G), E(G)) \), where \( V(G) \) is a finite set, and \( E(G) \) is a subset of the set of parts of \( V(G) \) of cardinality 2. In sections 1 and 2, we shall work with isoclasses of graphs, which we will simply call graphs. The set of graphs is denoted by \( G \). For example, here are graphs of cardinality \( \leq 4 \):

\[
\begin{align*}
1; \quad ; \quad 1, \ldots; \quad \forall, \forall, 1, \ldots; \quad \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \forall, \forall, 1, 1, 1, \ldots.
\end{align*}
\]

For any graph \( G \), we denote by \( |G| \) the cardinality of \( G \) and by \( cc(G) \) the number of its connected components. By convention, the empty graph \( 1 \) is considered as non connected.

A graph is totally disconnected if it has no edge.

We denote by \( \mathcal{H}_G \) the vector space generated by the set of graphs. The disjoint union of graphs gives it a commutative, associative product \( m \). As an algebra, \( \mathcal{H}_G \) is (isomorphic to) the free commutative algebra generated by connected graphs.

1.1 The first coproduct

Definition 1 Let \( G \) be a graph and \( I \subseteq V(G) \). The graph \( G|_I \) is defined by:

- \( V(G|_I) = I \).
- \( E(G|_I) = \{ \{x, y\} \in E(G) \mid x, y \in I \} \).

We refer to [1, 16, 26] for classical results and notations on bialgebras and Hopf algebras. The following Hopf algebra is introduced in [22]:

Proposition 2 We define a coproduct \( \Delta \) on \( \mathcal{H}_G \) by:

\[ \forall G \in \mathcal{G}, \quad \Delta(G) = \sum_{V(G)=I\sqcup J} G|_I \otimes G|_J. \]

Then \((\mathcal{H}_G, m, \Delta)\) is a graded, connected, cocommutative Hopf algebra. Its counit is given by:

\[ \forall G \in \mathcal{G}, \quad \varepsilon(G) = \delta_{G,1}. \]
Proof. If \( G, H \) are two graphs, then \( V(GH) = V(G) \sqcup V(H) \), so:

\[
\Delta(GH) = \sum_{V(G)=I \sqcup J, \ V(H)=K \sqcup L} GH|_{I \sqcup K} \otimes GH|_{J \sqcup L}
\]

\[
= \sum_{V(G)=I \sqcup J, \ V(H)=K \sqcup L} G|_I H|_K \otimes G|_J H|_L
\]

\[
= \Delta(G)\Delta(H).
\]

If \( G \) is a graph, and \( I \subseteq J \subseteq V(G) \), then \((G|_I)_J = G|_J \). Hence:

\[
(\Delta \otimes Id) \circ \Delta(G) = \sum_{V(G)=I \sqcup J, \ I = J \sqcup K} (G|_I)_J \otimes (G|_I)_K \otimes G|_L
\]

\[
= \sum_{V(G)=I \sqcup J, \ I = J \sqcup K} G|_I \otimes G|_K \otimes G|_L
\]

\[
= \sum_{V(G)=I \sqcup J, \ I = J \sqcup K} G|_I \otimes (G|_I)|_K \otimes (G|_I)|_L
\]

\[
= (Id \otimes \Delta) \circ \Delta(G).
\]

So \( \Delta \) is coassociative. It is obviously cocommutative. \( \square \)

Examples.

\[
\Delta(\cdot) = \cdot \otimes 1 + 1 \otimes \cdot,
\]

\[
\Delta(1) = 1 \otimes 1 + 1 \otimes 1 + 2 \otimes \cdot,
\]

\[
\Delta(\forall) = \forall \otimes 1 + 1 \otimes \forall + 3! \otimes \cdot + 3 \otimes 1,
\]

\[
\Delta(\forall \forall) = \forall \forall \otimes 1 + 1 \otimes \forall \forall + 2! \otimes \cdot + \ldots \otimes \cdot + 2 \otimes 1 + \ldots
\]

1.2 The second coproduct

Notations. Let \( V \) be a finite set \( \sim \) be an equivalence on \( V \).

- We denote by \( \pi_\sim : V \to V/\sim \) the canonical surjection.
- We denote by \( cl(\sim) \) the cardinality of \( V/\sim \).

Definition 3 Let \( G \) a graph, and \( \sim \) be an equivalence relation on \( V(G) \).

1. (Contraction). The graph \( V(G)/\sim \) is defined by:

\[
V(G/\sim) = V(G)/\sim,
E(G/\sim) = \{ \{\pi_\sim(x), \pi_\sim(y)\} \mid \{x, y\} \in E(G), \pi_\sim(x) \neq \pi_\sim(y)\}.
\]

2. (Extraction). The graph \( V(G)|\sim \) is defined by:

\[
V(G|\sim) = V(G),
E(G|\sim) = \{\{x, y\} \in E(G) \mid x \sim y\}.
\]

3. We shall write \( \sim \triangleleft G \) if, for any \( x \in V(G) \), \( G|_{\pi_\sim(x)} \) is connected.
Roughly speaking, $G/\sim$ is obtained by contracting each equivalence class of $\sim$ to a single vertex, and by deleting the loops and multiple edges created in the process; $G|\sim$ is obtained by deleting the edges which extremities are not equivalent, so is the product of the restrictions of $G$ to the equivalence classes of $\sim$.

We now define a coproduct on $\mathcal{H}_G$. This coproduct, which can also be found in [22], can also be deduced from a general operadic construction [27], see also [3]. A similar construction is defined on various families of oriented graphs in [18].

**Proposition 4** We define a coproduct $\delta$ on $\mathcal{H}_G$ by:

$$\forall G \in \mathcal{G}, \quad \delta(G) = \sum_{\sim \in \mathcal{G}} (G/\sim) \otimes (G|\sim).$$

Then $(\mathcal{H}_G, m, \delta)$ is a bialgebra. Its counit is given by:

$$\forall G \in \mathcal{G}, \quad \varepsilon(G) = \begin{cases} 1 & \text{if } G \text{ is totally disconnected,} \\ 0 & \text{otherwise.} \end{cases}$$

It is graded, putting:

$$\forall G \in \mathcal{G}, \quad \deg(G) = |G| - \text{cc}(G).$$

In particular, a basis of its homogeneous component of degree 0 is given by totally disconnected graphs, including 1.

**Proof.** Let $G, H$ be graphs and $\sim$ be an equivalence on $V(GH) = V(G) \cup V(H)$. We put $\sim' = \sim|_{V(G)}$ and $\sim'' = \sim|_{V(H)}$. The connected components of $GH$ are the ones of $G$ and $H$, so $\sim \sim GH$ if, and only if, the two following conditions are satisfied:

- $\sim' \sim G$ and $\sim'' \sim H$.
- If $x \sim y$, then $(x, y) \in V(G)^2 \cup V(H)^2$.

Note that the second point implies that $\sim$ is entirely determined by $\sim'$ and $\sim''$. Moreover, if this holds, $(GH)/\sim = (G/\sim')(H/\sim'')$ and $(GH)/\sim'' = (G|\sim')(H/\sim'')$, so:

$$\delta(GH) = \sum_{\sim' \sim G, \sim'' \sim H} (G/\sim')(H/\sim'') \otimes (G|\sim')(H/\sim'') = \delta(G)\delta(H).$$

Let $G$ be a graph. If $\sim \sim G$, the connected components of $G/\sim$ are the image by the canonical surjection of the connected components of $G$; the connected components of $G|\sim$ are the equivalence classes of $\sim$. If $\sim$ and $\sim'$ are two equivalences on $G$, we shall denote $\sim' \leq \sim$ if for all $x, y \in V(G)$, $x \sim' y$ implies $x \sim y$. Then:

$$(\delta \otimes Id) \circ \delta(G) = \sum_{\sim' \sim G, \sim\sim G/\sim} (G/\sim)/\sim' \otimes (G/\sim)|\sim' \otimes G|\sim$$

$$= \sum_{\sim' \sim G, \sim \leq \sim} (G/\sim)/\sim' \otimes (G/\sim)|\sim' \otimes G|\sim$$

$$= \sum_{\sim' \sim G, \sim \leq \sim} (G/\sim') \otimes (G|\sim')/\sim \otimes (G|\sim')|\sim$$

$$= \sum_{\sim' \sim G, \sim \leq \sim} (G/\sim') \otimes (G|\sim')/\sim \otimes (G|\sim')|\sim$$

$$= (Id \otimes \delta) \circ \delta(G).$$
So $\delta$ is coassociative.

We define two special equivalence relations $\sim_0$ and $\sim_1$ on $G$: for all $x, y \in V(G)$,

- $x \sim_0 y$ if, and only if, $x = y$.
- $x \sim_1 y$ if, and only if, $x$ and $y$ are in the same connected component of $G$.

Note that $\sim_0, \sim_1 \triangleleft G$. Moreover, if $\sim \triangleleft G$, $G/\sim$ is not totally disconnected, except if $\sim = \sim_1$; $G|\sim$ is not totally disconnected, except if $\sim = \sim_0$. Hence:

- If $G$ is totally disconnected, then $\delta(G) = G \otimes G$.
- Otherwise, denoting by $n$ the degree of $G$ and by $k$ its number of connected components:
  $$\delta(G) = n^k \otimes G + G \otimes n + \text{Ker}(\varepsilon') \otimes \text{Ker}(\varepsilon').$$

So $\varepsilon'$ is indeed the counit of $\delta$.

Let $G$ be a graph, with $n$ vertices and $k$ connected components (so of degree $n - k$). Let $\sim \triangleleft G$. Then:

1. $G/\sim$ has cardinality $cl(\sim)$ and $k$ connected components, so is of degree $cl(\sim) - k$.
2. $G|\sim$ has cardinality $n$ and $cl(\sim)$ connected components, so is of degree $n - cl(\sim)$.

Hence, $\deg(G/\sim) + \deg(G|\sim) = cl(\sim) - k + n - cl(\sim) = n - k = \deg(G)$: $\delta$ is homogeneous. □

Examples.

$$\delta(\cdot) = \cdot \otimes \cdot, \quad \delta(1) = \cdot \otimes 1 + 1 \otimes \cdot,$$

$$\delta(\nabla) = \cdot \otimes \nabla + 31 \otimes 1 + \nabla \otimes \cdot, \quad \delta(\nabla) = \cdot \otimes \nabla + 21 \otimes 1 + \nabla \otimes \cdot.$$

Remark. Let $G \in \mathcal{G}$. The following conditions are equivalent:

- $\varepsilon'(G) = 1$.
- $\varepsilon'(G) \neq 0$.
- $\deg(G) = 0$.
- $G$ is totally disconnected.

1.3 Antipode

$(\mathcal{H}_G, m, \delta)$ is not a Hopf algebra: the group-like element is has no inverse. However, the grading of $(\mathcal{H}_G, m, \delta)$ induced a graduation of $\mathcal{H}'_G = (\mathcal{H}_G, m, \delta)/\langle . - 1 \rangle$, which becomes a graded, connected bialgebra, hence a Hopf algebra; we denote its antipode by $S'$. Note that, as a commutative algebra, $\mathcal{H}'_G$ is freely generated by connected graphs different from $\cdot$.

The notations and ideas of the following definition and theorem come from Quantum Field Theory, where they are applied to Renormalization with the help of Hopf algebras of Feynman graphs; see for example [8, 9] for an introduction.

**Definition 5** Let $G$ be a connected graph, $G \neq \cdot$.

1. A forest of $G$ is a set $\mathcal{F}$ of subsets of $V(G)$, such that:
(a) \( V(G) \in \mathcal{F} \).
(b) If \( I, J \in \mathcal{F} \), then \( I \subseteq J \), or \( J \subseteq I \), or \( I \cap J = \emptyset \).
(c) For all \( I \in \mathcal{F} \), \( G|_I \) is connected.

The set of forests of \( G \) is denoted by \( \mathbb{F}(G) \).

2. Let \( \mathcal{F} \in \mathbb{F}(G) \); it is partially ordered by the inclusion. For any \( I \in \mathbb{F}(G) \), the relation \( \sim_I \) is the equivalence on \( I \) which classes are the maximal elements (for the inclusion) of \( \{ J \in \mathcal{F} \mid J \subseteq I \} \) (if this is non-empty), and singletons. We put:

\[
G_{\mathcal{F}} = \prod_{I \in \mathcal{F}} (G|_I)/\sim_I.
\]

**Examples.** The graph 1 has only one forest, \( \mathcal{F} = \{ 1 \} \); \( 1_{\mathcal{F}} = 1 \). The graph \( \nabla \) has four forests:

- \( \mathcal{F} = \{ \nabla \} \); in this case, \( \nabla_{\mathcal{F}} = \nabla \).
- Three forests \( \mathcal{F} = \{ \nabla, 1 \} \); for each of them, \( \nabla_{\mathcal{F}} = 11 \).

**Theorem 6** For any connected graph \( G \), \( G \neq \ast \), in \( \mathcal{H}_G' \):

\[
S(G) = \sum_{\mathcal{F} \in \mathbb{F}(G)} (-1)^{\sharp\mathcal{F}} G_{\mathcal{F}}.
\]

**Proof.** By induction on the number \( n \) of vertices of \( G \). If \( n = 2 \), then \( G = 1 \). As \( \delta'(1) = 1 \otimes 1 + 1 \otimes 1 \), \( S'(1) = -1 = -1_{\mathcal{F}} \), where \( \mathcal{F} = \{ 1 \} \) is the unique forest of 1. Let us assume the result at all ranks \( < n \). Then:

\[
S'(G) = -G - \sum_{\sim G} (G/\sim)S'(G|\sim)
\]

\[
= -G - \sum_{G/\sim = \{ I_1, \ldots, I_k \}} \sum_{\mathcal{F}_i \in \mathbb{F}(G|_{I_i})} (-1)^{\sharp\mathcal{F}_1 + \ldots + \sharp\mathcal{F}_k} (G|_{I_1})_{\mathcal{F}_1} \ldots (G|_{I_k})_{\mathcal{F}_k}
\]

\[
= -G - \sum_{\mathcal{F} \in \mathbb{F}(G), \mathcal{F} \neq \{ G \}} (-1)^{\sharp\mathcal{F}-1} G_{\mathcal{F}}
\]

\[
= \sum_{\mathcal{F} \in \mathbb{F}(G)} (-1)^{\sharp\mathcal{F}} G_{\mathcal{F}}.
\]

For the third equality, \( \mathcal{F} = \{ G \} \sqcup \mathcal{F}_1 \sqcup \ldots \sqcup \mathcal{F}_k \). \( \square \)

**1.4 Cointeraction**

**Theorem 7** With the coaction \( \delta \), \( (\mathcal{H}_G, m, \Delta) \) and \( (\mathcal{H}_G, m, \delta) \) are in cointeraction, that is to say that \( (\mathcal{H}_G, m, \Delta) \) is a \( (\mathcal{H}_G, m, \delta) \)-comodule bialgebra, or a Hopf algebra in the category of \( (\mathcal{H}_G, m, \delta) \)-comodules. In other words:

- \( \delta(1) = 1 \otimes 1 \).
- \( m^3_{2,4} \circ (\delta \otimes \delta) \circ \Delta = (\Delta \otimes \text{Id}) \circ \delta \), with:

\[
m^3_{2,4} : \{ \mathcal{H}_G \otimes \mathcal{H}_G \otimes \mathcal{H}_G \otimes \mathcal{H}_G \rightarrow \mathcal{H}_G \otimes \mathcal{H}_G \otimes \mathcal{H}_G \}
\]

\[
a_1 \otimes b_1 \otimes a_2 \otimes b_2 \rightarrow a_1 \otimes a_2 \otimes b_1 b_2.
\]

- For all \( a, b \in \mathcal{H}_G \), \( \delta(ab) = \delta(a)\delta(b) \).
• For all \( a \in \mathcal{H}_G \), \((\epsilon \otimes \text{Id}) \circ \delta(a) = \epsilon(a)1\).

**Proof.** The first and third points are already proved, and the fourth one is immediate for any \( a \in \mathcal{G} \). Let us prove the second point. For any graphs \( G, H \):

\[
(\Delta \otimes \text{Id}) \circ \delta(GH) = \sum_{\sim \in \mathcal{G}, V(G)/\sim = I \cup J} (G/\sim|_I \otimes G/\sim|_J) \sim = \sum_{V(G)=I \cup J', \sim' \in \mathcal{G}/_I \otimes \mathcal{G}/_J} (G|_{I'})/\sim' \otimes (G|_{J'})/\sim'' \otimes (G|_{J'})/\sim''
\]

\[
= m_{3,4} \circ (\delta \otimes \delta) \circ \Delta(G).
\]

For the second equality, \( I' = \pi^{-1}_1(I) \), \( I'' = \pi^{-1}_2(I) \), \( \sim' = \sim|_{I'} \) and \( \sim'' = \sim|_{I'} \). \( \square \)

We can apply the results of [10]:

**Theorem 8** We denote by \( M_G \) the monoid of characters of \( \mathcal{H}_G \).

1. Let \( \lambda \in M_G \). It is an invertible element if, and only if, \( \lambda(.) \neq 0 \).

2. Let \( B \) be a Hopf algebra, and \( E_{\mathcal{H}_G} \rightarrow B \) be the set of Hopf algebra morphisms from \( (\mathcal{H}_G, m, \Delta) \) to \( B \). Then \( M_G \) acts on \( E_{\mathcal{H}_G} \rightarrow B \) by:

\[
\begin{align*}
\left\{ E_{\mathcal{H}_G} \rightarrow B \times M_G \right\} & \rightarrow E_{\mathcal{H}_G} \rightarrow B \\
(\phi, \lambda) & \rightarrow \phi \leftarrow \lambda = (\phi \otimes \lambda) \circ \delta.
\end{align*}
\]

3. There exists a unique \( \phi_0 \in E_{\mathcal{H}_G} \rightarrow \mathbb{Q}[X] \), homogeneous, such that \( \phi_0(.) = X \); there exists a unique \( \lambda_0 \in M_G \) such that:

\[
\forall G \in \mathcal{G}, \phi_0(G) = \lambda_0(G)X^{|G|}.
\]

Moreover, the following map is a bijection:

\[
\left\{ \begin{array}{c}
M_G \\
\lambda
\end{array} \rightarrow E_{\mathcal{H}_G} \rightarrow \mathbb{Q}[X] \right\} \leftarrow \left\{ \begin{array}{c}
\phi_0
\end{array} \rightarrow \lambda
\end{array} \right\}.
\]

4. Let \( \lambda \in M_G \). There exists a unique element \( \phi \in E_{\mathcal{H}_G} \rightarrow \mathbb{Q}[X] \) such that:

\[
\forall x \in \mathcal{H}_G, \phi(x)(1) = \lambda(x).
\]

This morphism is \( \phi_0 \leftarrow (\lambda_0^{*} - 1) \ast \lambda \).

5. There exists a unique morphism \( \phi_1 : \mathcal{H}_G \rightarrow \mathbb{Q}[X] \), such that:

- \( \phi_1 \) is a Hopf algebra morphism from \( (\mathcal{H}_G, m, \Delta) \) to \( (\mathbb{Q}[X], m, \Delta) \).
- \( \phi_1 \) is a bialgebra morphism from \( (\mathcal{H}_G, m, \delta) \) to \( (\mathbb{Q}[X], m, \delta) \).

This morphism is the unique element of \( E_{\mathcal{H}_G} \rightarrow \mathbb{Q}[X] \) such that:

\[
\forall x \in \mathcal{H}_G, \phi_1(x)(1) = \epsilon'(x).
\]

Moreover, \( \phi_1 = \phi_0 \leftarrow \lambda^{*} - 1 \).

We shall determine \( \phi_0 \) and \( \phi_1 \) in the next section.
2 Chromatic polynomials

2.1 Determination of $\phi_0$

Proposition 9 For any graph $G$:

$$\phi_0(G) = X^{\lvert G \rvert}, \quad \lambda_0(G) = 1.$$ 

Proof. Let $\psi : \mathcal{H}_G \rightarrow \mathbb{Q}[X]$, sending any graph $G$ to $X^{\lvert G \rvert}$. It is a homogeneous algebra morphism. For any graph $G$, of degree $n$:

$$(\psi \otimes \psi) \circ \Delta(G) = \sum_{V(G) = I \sqcup J} X^{|I|} \otimes X^{|J|} = \sum_{i=0}^{n} \binom{n}{i} X^i \otimes X^{n-i} = \Delta(X^n) = \Delta \circ \psi(G).$$

So $\psi$ is a Hopf algebra morphism. As $\psi(\emptyset) = X$, $\psi = \phi_0$. \qed

2.2 Determination of $\phi_1$

Let us recall the definition of the chromatic polynomial, due to Birkhoff and Lewis [5]:

Definition 10 Let $G$ be a graph and $X$ a set.

1. A $X$-coloring of $G$ is a map $f : V(G) \rightarrow X$.

2. A $\mathbb{N}$-coloring of $G$ is packed if $f(V(G)) = [k]$, with $k \geq 0$. The set of packed colorings of $G$ is denoted by $\mathbb{P}C(G)$.

3. A valid $X$-coloring of $G$ by $X$ is a $X$-coloring $f$ such that if $\{i, j\} \in E(G)$, then $f(i) \neq f(j)$. The set of valid $X$-colorings of $G$ is denoted by $\mathbb{V}C(G, X)$; the set of packed valid colorings of $G$ is denoted by $\mathbb{P}V\mathbb{C}(G)$.

4. An independent subset of $G$ is a subset $I$ of $V(G)$ such that $G[I]$ is totally disconnected. We denote by $\mathbb{I}(G)$ the set of partitions $\{I_1, \ldots, I_k\}$ of $V(G)$ such that for all $p \in [k]$, $I_p$ is an independent subset of $G$.

5. For any $k \geq 1$, the number of valid $[k]$-colorings of $G$ is denoted by $P_{\text{chr}}(G)(k)$. This defines a unique polynomial $P_{\text{chr}}(G) \in \mathbb{Q}[X]$, called the chromatic polynomial of $G$.

Note that if $f$ is a $X$-coloring of a graph $G$, it is valid if, and only if, the partition of $V(G)$ $\{f^{-1}(x) \mid x \in f(V(G))\}$ belongs to $\mathbb{I}(G)$.

Theorem 11 The morphism $P_{\text{chr}} : \mathcal{H}_G \rightarrow \mathbb{Q}[X]$ is the morphism $\phi_1$ of Theorem 8.

Proof. It is immediate that, for any graphs $G$ and $H$, $P_{\text{chr}}(GH)(k) = P_{\text{chr}}(G)(k)P_{\text{chr}}(H)(k)$ for any $k$, so $P_{\text{chr}}(GH) = P_{\text{chr}}(G)P_{\text{chr}}(H)$: $P_{\text{chr}}$ is an algebra morphism. Let $G$ be a graph, and $k, l \geq 1$. We consider the two sets:

$$C = \mathbb{V}C(G, [k + l]),$$

$$D = \{(I, c', c'') \mid I \subseteq V(G), c' \in \mathbb{V}C(G[I],[k]), c'' \in \mathbb{V}C(G[V(G) \setminus I],[l])\}.$$ 

We define a map $\theta : C \rightarrow D$ by $\theta(c) = (I, c', c'')$, with:

- $I = \{x \in V(G) \mid c(x) \in [k]\}$.
- For all $x \in I$, $c'(x) = c(x)$.
- For all $x \notin I$, $c''(x) = c(x) - k$.
We define a map \( \theta' : D \rightarrow C \) by \( \theta(I, c', c'') = c \), with:

- For all \( x \in I \), \( c(x) = c'(x) \).
- For all \( x \notin I \), \( c(x) = c''(x) + k \).

Both \( \theta \) and \( \theta' \) are well-defined; moreover, \( \theta \circ \theta' = Id_D \) and \( \theta' \circ \theta = Id_C \), so \( \theta \) is a bijection. Via the identification of \( \mathbb{Q}[X] \otimes \mathbb{Q}[X] \) and \( \mathbb{Q}[X,Y] \):

\[
\Delta \circ P_{chr}(G)(k,l) = P_{chr}(G)(k + l)
= \sharp C
= \sharp D
= \sum_{I \subseteq V(G)} P_{chr}(G|_I)(k)P_{chr}(G|_{V(G) \setminus I})(l)
= (P_{chr} \otimes P_{chr})(\sum_{V(G) = I \cup J} G|_I \otimes G|_J)(k,l)
= (P_{chr} \otimes P_{chr}) \circ \Delta(G)(k,l).
\]

As this is true for all \( k,l \geq 1 \), \( \Delta \circ P_{chr}(G) = (P_{chr} \otimes P_{chr}) \circ \Delta(G) \). Moreover:

\[
\varepsilon(G) = \varepsilon \circ P_{chr}(G) = P_{chr}(G)(0) = \begin{cases} 1 & \text{if } G \text{ is empty,} \\ 0 & \text{otherwise.} \end{cases}
\]

So \( P_{chr} \in \mathcal{E}_{\mathcal{H}_G \rightarrow \mathbb{Q}[X]} \): For any graph \( G \):

\[
P_{chr}(G)(1) = \begin{cases} 1 & \text{if } G \text{ is totally disconnected,} \\ 0 & \text{otherwise;} \end{cases}
= \varepsilon'(G).
\]

So \( \phi_1 = P_{chr} \).

**Corollary 12** For any connected graph \( G \), we put:

\[
\lambda_{chr}(G) = \frac{dP_{chr}(G)}{dX}(0).
\]

We extend \( \lambda \) as an element of \( M_G \): for any graph \( G \), if \( G_1, \ldots, G_k \) are the connected components of \( G \),

\[
\lambda_{chr}(G) = \lambda_{chr}(G_1) \ldots \lambda_{chr}(G_k).
\]

Then \( \lambda_{chr} \) is an invertible element of \( M_G \), and \( \lambda_{chr}^{-1} = \lambda_0 \). Moreover:

\[
\forall G \in \mathcal{G}, \quad P_{chr}(G) = \sum_{\sim \in \mathcal{G}} \lambda_{chr}(G \sim)X^{cl(\sim)}.
\]

**Proof.** By Theorem 8, there exists a unique \( \lambda_{chr} \in M_G \), such that \( P_{chr} = \phi_0 \leftarrow \lambda_{chr} \). For any graph \( G \):

\[
P_{chr}(G) = (\phi_0 \otimes \lambda_{chr}) \circ \delta(G) = \sum_{\sim \in \mathcal{G}} \phi_0(G/\sim)\lambda_{chr}(G|_{\sim}) = \sum_{\sim \in \mathcal{G}} X^{cl(\sim)}\lambda_{chr}(G|_{\sim}).
\]

If \( G \) is connected, there exists a unique \( \sim \in \mathcal{G} \) such that \( cl(\sim) = 1 \): this is the equivalence relation such that for any \( x, y \in V(G) \), \( x \sim y \). Hence, the coefficient of \( X \) in \( P_{chr}(X) \) is \( \lambda(G|_{\sim}) = \lambda(G) \), so

\[
\lambda_{chr}(G) = \frac{dP_{chr}(G)}{dX}(0).
\]

By Theorem 8, \( \lambda_{chr} = \lambda_0^{-1} \).

The character \( \lambda_{chr} \) will be called the **chromatic character**.
Proposition 13 \( \lambda_{chr}(\cdot) = 1; \) if \( G \) is a connected graph, \( G \neq \cdot \), then:

\[
\lambda_{chr}(G) = \sum_{\mathcal{F} \in \mathcal{F}(G)} (-1)^{|\mathcal{F}|}.
\]

**Proof.** We have \( \lambda_{chr}(\cdot) = \lambda_{chr}^{* - 1}(\cdot) = 1 \), so both \( \lambda_{chr} \) and \( \lambda_{chr}^{* - 1} \) can be seen as characters on \( \mathcal{H}'_G \). Hence, for any connected graph \( G \), different from \( q \):

\[
\lambda_{chr}(G) = \lambda_{chr}^{* - 1} \circ S'(G) = \sum_{\mathcal{F} \in \mathcal{F}(G)} (-1)^{|\mathcal{F}|} \lambda_{chr}^{* - 1}(G_F) = \sum_{\mathcal{F} \in \mathcal{F}(G)} (-1)^{|\mathcal{F}|},
\]

as \( \lambda_{chr}^{* - 1}(H) = 1 \) for any graph \( H \).

**Examples.**

1. \[
\begin{array}{cccccccc}
G & \cdot & 1 & \checkmark & \checkmark & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\lambda_{chr}(G) & 1 & -1 & 2 & -6 & -4 & -2 & -3 & -1 & -1
\end{array}
\]

2. If \( G \) is a complete graph with \( n \) vertices, \( P_{chr}(G)(X) = X(X - 1) \ldots (X - n + 1) \), so \( \lambda_{chr}(G) = (-1)^{n-1}(n-1)! \).

2.3 Extraction and contraction of edges

**Definition 14** Let \( G \) be a graph and \( e \in E(G) \).

1. (Contraction of \( e \)). The graph \( G/e \) is \( G/\sim_e \), where \( \sim_e \) is the equivalence which classes are \( e \) and singletons.

2. (Subtraction of \( e \)). The graph \( G \setminus e \) is the graph \((V(G), E(G) \setminus \{e\})\).

3. We shall say that \( e \) is a bridge (or an isthmus) of \( G \) if \( cc(G \setminus e) > cc(G) \).

We now give an algebraic proof of the following well-known result:

**Proposition 15** For any graph \( G \), for any edge \( e \) of \( G \):

\[
P_{chr}(G) = P_{chr}(G \setminus e) - P_{chr}(G/e);
\]

\[
\lambda_{chr}(G) = \begin{cases} 
-\lambda_{chr}(G/e) & \text{if } e \text{ is a bridge,} \\
\lambda_{chr}(G \setminus e) - \lambda_{chr}(G/e) & \text{otherwise.} 
\end{cases}
\]

**Proof.** Let \( G \) be a graph, and \( e \in E(G) \). Let us prove that for all \( k \geq 1 \), \( P_{chr}(G)(k) = P_{chr}(G \setminus e)(k) - P_{chr}(G/e)(k) \). We proceed by induction on \( k \). If \( k = 1 \), \( P_{chr}(G)(1) = \varepsilon'(G) = 0 \). If \( G \) has only one edge, then \( G \setminus e \) and \( G/e \) are totally disconnected, and:

\[
P_{chr}(G \setminus e)(1) - P_{chr}(G/e)(1) = 1 - 1 = 0.
\]

Otherwise, \( G \setminus e \) and \( G/e \) have edges, and:

\[
P_{chr}(G \setminus e)(1) - P_{chr}(G/e)(1) = 0 - 0 = 0.
\]
Let us assume the result at rank $k$. Putting $e = \{x, y\}$:

\[
P_{\operatorname{chr}}(G \setminus e)(k + 1) - P_{\operatorname{chr}}(G/e)(k + 1)
= \sum_{V(G) = I \cup J} P_{\operatorname{chr}}((G \setminus e)_{I})(k)P_{\operatorname{chr}}((G \setminus e)_{J})(1)
- \sum_{V(G) = I \cup J, x \in I, y \in J} P_{\operatorname{chr}}((G/e)_{I})(k)P_{\operatorname{chr}}((G/e)_{J})(1)
+ \sum_{V(G) = I \cup J, (x, y) \in (I \times J) \cup (J \times I)} P_{\operatorname{chr}}((G/e)_{I})(k)P_{\operatorname{chr}}((G/e)_{J})(1)
\]

\[
= \sum_{V(G) = I \cup J, x \in I} P_{\operatorname{chr}}((G_{I}) \setminus e)(k)P_{\operatorname{chr}}(G_{J})(1) + \sum_{V(G) = I \cup J, x \in I, y \in J} P_{\operatorname{chr}}(G_{I})(k)P_{\operatorname{chr}}((G_{J}) \setminus e)(1)
- \sum_{V(G) = I \cup J, x \in I} P_{\operatorname{chr}}((G_{I}) \setminus e)(k)P_{\operatorname{chr}}(G_{J})(1) - \sum_{V(G) = I \cup J, x \in I, y \in J} P_{\operatorname{chr}}(G_{I})(k)P_{\operatorname{chr}}((G_{J}) \setminus e)(1)
+ \sum_{V(G) = I \cup J, (x, y) \in (I \times J) \cup (J \times I)} P_{\operatorname{chr}}(G_{I})(k)P_{\operatorname{chr}}(G_{J})(1)
\]

\[
= \sum_{V(G) = I \cup J, x \in I} P_{\operatorname{chr}}((G_{I}) \setminus e)(k)P_{\operatorname{chr}}(G_{J})(1) + \sum_{V(G) = I \cup J, x \in I, y \in J} P_{\operatorname{chr}}(G_{I})(k)P_{\operatorname{chr}}(G_{J})(1)
+ \sum_{V(G) = I \cup J, (x, y) \in (I \times J) \cup (J \times I)} P_{\operatorname{chr}}(G_{I})(k)P_{\operatorname{chr}}(G_{J})(1)
\]

\[
= P_{\operatorname{chr}}(G)(k + 1).
\]

So the result holds for all $k \geq 1$. Hence, $P_{\operatorname{chr}}(G) = P_{\operatorname{chr}}(G \setminus e) - P_{\operatorname{chr}}(G/e)$.

Let us assume that $G$ is connected. Note that $G/e$ is connected. If $e$ is a bridge, then $G \setminus e$ is not connected; each of its connected components belongs to the augmentation ideal of $H_{G}$, so their images belong to the augmentation ideal of $Q[X]$, that is to say $XQ[X]$; hence, $P_{\operatorname{chr}}(G \setminus e) \in X^{2}Q[X]$, so:

\[
\lambda_{\operatorname{chr}}(G) = \frac{dP_{\operatorname{chr}}(G)}{dX}(0) = \frac{dP_{\operatorname{chr}}(G \setminus e)}{dX}(0) - b\frac{dP_{\operatorname{chr}}(G/e)}{dX}(0) = 0 - \lambda_{\operatorname{chr}}(G/e).
\]

Otherwise, $G \setminus e$ is connected, and:

\[
\lambda_{\operatorname{chr}}(G) = \frac{dP_{\operatorname{chr}}(G)}{dX}(0) = \frac{dP_{\operatorname{chr}}(G \setminus e)}{dX}(0) - b\frac{dP_{\operatorname{chr}}(G/e)}{dX}(0) = \lambda_{\operatorname{chr}}(G \setminus e) - \lambda_{\operatorname{chr}}(G/e).
\]

If $G$ is not connected, we can write $G = G_{1}G_{2}$, where $G_{1}$ is connected and $e$ is an edge of
\[
\lambda_{\text{chr}}(G) = \lambda_{\text{chr}}(G_1)\lambda_{\text{chr}}(G_2)
\]
\[
= \begin{cases} 
-\lambda_{\text{chr}}(G_1/e)\lambda_{\text{chr}}(G_2) & \text{if } e \text{ is a bridge}, \\
\lambda_{\text{chr}}(G_1 \setminus e)\lambda_{\text{chr}}(G_2) - \lambda_{\text{chr}}(G_1/e)\lambda_{\text{chr}}(G_2) & \text{otherwise};
\end{cases}
\]
\[
= \begin{cases} 
-\lambda_{\text{chr}}((G_1/e)G_2) & \text{if } e \text{ is a bridge}, \\
\lambda_{\text{chr}}((G_1 \setminus e)G_2) - \lambda_{\text{chr}}((G_1/e)G_2) & \text{otherwise};
\end{cases}
\]
\[
= \begin{cases} 
-\lambda_{\text{chr}}(G/e) & \text{if } e \text{ is a bridge}, \\
\lambda_{\text{chr}}(G \setminus e) - \lambda_{\text{chr}}(G/e) & \text{otherwise}.
\end{cases}
\]

So the result holds for any graph \(G\). \(\square\)

2.4 Lattices attached to graphs

We here make the link with Rota’s methods for proving the alternation of signs in the coefficients of chromatic polynomials.

The following order is used to prove Proposition 4:

**Proposition 16** Let \(G\) be a graph. We denote by \(\mathcal{R}(G)\) the set of equivalences \(\sim\) on \(V(G)\), such that \(\sim \triangleleft G\). Then \(\mathcal{R}(G)\) is partially ordered by refinement:

\[
\forall \sim, \sim' \in \mathcal{R}(G), \quad \sim \leq \sim' \quad \text{if } (\forall x, y \in V(G), x \sim y \implies x \sim' y).
\]

In other words, \(\sim \leq \sim'\) if the equivalence classes of \(\sim'\) are disjoint unions of equivalence classes of \(\sim\). Then \((\mathcal{R}(G), \leq)\) is a bounded graded lattice. Its minimal element \(\sim_0\) is the equality; its maximal element \(\sim_1\) is the relation which equivalence classes are the connected components of \(\mathcal{R}(G)\).

**Proof.** Let \(\sim, \sim' \in \mathcal{R}(G)\). We define \(\sim \wedge \sim'\) as the equivalence which classes are the connected components of the subsets \(\text{Cl}(\sim) \cap \text{Cl}(\sim')\), \(x, y \in V(G)\). By its very definition, \(\sim \wedge \sim' \triangleleft G\), and \(\sim \wedge \sim' \leq \sim, \sim'\). If \(\sim'' \leq \sim, \sim' \leq \sim''\) in \(\mathcal{R}(G)\), then the equivalence classes of \(\sim\) and \(\sim'\) are disjoint union of equivalence classes of \(\sim''\), so their intersections also are; as the equivalence classes of \(\sim''\) are connected, the connected components of these intersections are also disjoint union of equivalence classes of \(\sim''\). This means that \(\sim'' \leq \sim \wedge \sim'\).

We define \(\sim \vee \sim'\) as the relation defined on \(V(G)\) in the following way: for all \(x, y \in V(G)\), \(x \sim \vee \sim' y\) if there exists \(x_1, x_1', \ldots, x_k, x_k' \in V(G)\) such that:

\[
x = x_1 \sim x_1' \sim x_2 \sim \ldots \sim x_k \sim x_k' = y.
\]

It is not difficult to prove that \(\sim \vee \sim'\) is an equivalence. Moreover, if \(x \sim y\), then \(x \sim \vee \sim' y\) \((x_1 = x, x_1' = y)\); if \(x \sim' y\), then \(x \sim \vee \sim' y\) \((x_1 = x_1' = x, x_2 = x_2' = y)\). Let \(C\) be an equivalence class of \(\sim \vee \sim'\), and let \(x, y \in C\). With the preceding notations, as the equivalence classes of \(\sim\) and \(\sim'\) are connected, for all \(p \in [k]\), there exists a path from \(x_p\) to \(x'_p\), formed of elements \(\sim\)-equivalent, hence \(\sim \vee \sim'\)-equivalent; for all \(p \in [k - 1]\), there exists a path from \(x'_p\) to \(x'_{p+1}\), formed of elements \(\sim'\)-equivalent, hence \(\sim \vee \sim'\)-equivalent. Concatenating these paths, we obtain a path from \(x\) to \(y\) in \(C\), which is connected. So \(\sim \sim' \in \mathcal{R}(G)\), and \(\sim \sim' \leq \sim \vee \sim'\). Moreover, if \(\sim, \sim' \leq \sim''\), then obviously \(\sim \vee \sim' \leq \sim''\). We proved that \(\mathcal{R}(G)\) is a lattice.

For any \(\sim \in \mathcal{R}(G)\), we put \(\text{deg}(G) = |G| - \text{cl}(\sim)\). Note that \(\text{deg}(\sim_0) = 0\). Let us assume that \(\sim\) is covered by \(\sim'\) in \(\mathcal{R}(G)\). We denote by \(C_1, \ldots, C_k\) the classes of \(\sim\). As \(\sim \leq \sim'\), the classes of \(\sim'\) are disjoint unions of \(C_p\), as \(\sim \neq \sim'\), one of them, denoted by \(C'\), contains at least two \(C_p\).
As $C'$ is connected, there is an edge in $C'$ connecting two different $C_p$; up to a reindexation, we assume that there exists an edge from $C_1$ to $C_2$ in $C'$. Then $C_1 \sqcup C_2$ is connected, and the equivalence $\sim''$ which classes are $C_1 \sqcup C_2, C_3, \ldots C_k$ satisfies $\sim \leq \sim'' \leq \sim'$. As $\sim'$ covers $\sim$, $\sim' = \sim''$, so $\deg(\sim') = |G| - k + 1 = \deg(\sim) + 1$.

**Remark.** This lattice is isomorphic to the one of [21]. The isomorphism between them sends a element $\sim \in \mathcal{R}(G)$ to the partition formed by its equivalence classes.

**Examples.** We represent $\sim \in \mathcal{R}(G)$ by $G \sim$. Here are examples of $\mathcal{R}(G)$, represented by their Hasse graphs:

\[
\begin{array}{c}
\vdots \\
| \\
\circ \\
\end{array}
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array}
\begin{array}{cccc}
\vdots \quad \vdots \quad \vdots \quad \vdots \\
\circ \quad \circ \quad \circ \quad \circ \\
\circ \quad \circ \quad \circ \quad \circ \\
\end{array}
\]

**Proposition 17** Let $G$ be a graph. We denote by $\mu_G$ the Möbius function of $\mathcal{R}(G)$.

1. If $\sim \leq \sim'$ in $\mathcal{R}(G)$, then the poset $[\sim, \sim']$ is isomorphic to $\mathcal{R}((G| \sim')/ \sim)$.

2. For any $\sim \leq \sim'$ in $\mathcal{R}(G)$, $\mu_G(\sim, \sim') = \lambda_{\text{chr}}((G| \sim')/ \sim)$. In particular:

\[
\mu_G(\sim_0, \sim_1) = \lambda_{\text{chr}}(G).
\]

**Proof.** Let $\sim \leq \sim' \in \mathcal{R}(G)$. If $\sim''$ is an equivalence on $V(G)$, then $\sim \leq \sim'' \leq \sim$ if, and only if, the following conditions are satisfied:

- $\sim''$ goes to the quotient $G/ \sim$, as an equivalence denoted by $\overline{\sim''}$.
- $\overline{\sim''} \in \mathcal{R}((G| \sim')/ \sim)$.

Hence, we obtain a map from $[\sim, \sim']$ to $\mathcal{R}((G| \sim')/ \sim)$, sending $\sim''$ to $\overline{\sim''}$. It is immediate that this is a lattice isomorphism.

Let $\sim \leq \sim' \in \mathcal{R}(G)$. As $[\sim, \sim']$ is isomorphic to the lattice $\mathcal{R}((G| \sim')/ \sim)$:

\[
\sum_{\sim \leq \sim'' \leq \sim'} \lambda_{\text{chr}}((G| \sim'')/ \sim) = \sum_{\overline{\sim''} \in \mathcal{R}((G| \sim')/ \sim)} \lambda_{\text{chr}}(((G| \sim')/ \sim)\overline{\sim''})
\]

\[
= P_{\text{chr}}((G| \sim')/ \sim)(1)
\]

\[
\begin{cases}
1 \text{ if } (G| \sim')/ \sim \text{ is totally disconnected,} \\
0 \text{ otherwise;}
\end{cases}
\]

\[
\begin{cases}
1 \text{ if } \sim = \sim', \\
0 \text{ otherwise.}
\end{cases}
\]

Hence, $\mu_G(\sim, \sim') = \lambda_{\text{chr}}((G| \sim')/ \sim)$. \qed

**Remark.** We now use the notion of incidence algebra of a family of posets exposed in [22]. We consider the family of posets:

\[
\{[\sim, \sim'] | G \in \mathcal{G}, \sim \leq \sim' \in \mathcal{R}(G)\}.
\]

It is obviously interval closed. We define an equivalence relation on this family as the one generated by $[\sim, \sim'] \equiv \mathcal{R}((G| \sim')/ \sim)$. The incidence bialgebra associated to this family is $(\mathcal{H}_G, m, \delta)$.
Proposition 18 Let $G$ be a graph.

1. Let $G_1, \ldots, G_k$ be the connected components of $G$. Then $\mathcal{R}(G) \approx \mathcal{R}(G_1) \times \ldots \times \mathcal{R}(G_k)$.

2. Let $e$ be a bridge of $G$. Then $\mathcal{R}(G) \approx \mathcal{R}(G/e) \times \mathcal{R}(1)$.

3. We consider the following map:

   \[ \zeta_G : \begin{cases} \mathcal{R}(G) & \to \mathcal{P}(E(G)) \setminus \{\emptyset\} \\ \sim & \to E(G \setminus \sim) \end{cases} \]

   This map is injective; for any $\sim, \sim' \in \mathcal{R}(G)$, $\sim \leq \sim'$ if, and only if, $\zeta_G(\sim) \subseteq \zeta_G(\sim')$. Moreover, $\zeta_G$ is bijective if, and only if, $G$ is a forest.

Proof. 1. If $G, H$ are graphs and $\sim$ is an equivalence on $V(GH)$, then $\sim \triangleleft GH$ if, and only if:

   - $\sim|_{V(G)} \triangleleft G$.
   - $\sim|_{V(H)} \triangleleft H$.
   - For any $x, y \in V(G) \cup V(H)$, $(x \sim y) \implies ((x, y) \in V(G)^2 \cup V(H)^2)$.

Hence, the map sending $\sim$ to $(\sim|_{V(G)}), \sim|_{V(H)})$ from $\mathcal{R}(GH)$ to $\mathcal{R}(G) \times \mathcal{R}(H)$ is an isomorphism; the first point follows.

2. Note that $\mathcal{R}(1) = \{\ldots, 1\}$, with $\ldots \leq 1$. By the first point, it is enough to prove it if $G$ is connected. Let us put $e = \{x', x''\}$, $G'$, respectively $G''$, the connected components of $G \setminus e$ containing $x'$, respectively $x''$. We define a map $\psi : \mathcal{R}(G/e) \times \mathcal{R}(1)$ to $\mathcal{R}(G)$ in the following way: if $\sim \triangleleft \mathcal{R}(G/e)$,

   - $\psi(\sim, 1) = \sim$, defined by $x \sim y$ if $\overline{x} \sim \overline{y}$. This is clearly an equivalence; moreover, $x' \sim x''$ if $x \sim y$, there exists a path from $\overline{x}$ to $\overline{y}$ in $G/e$, formed by vertices $\sim$-equivalent to $\overline{x}$ and $\overline{y}$. Adding edges $e$ if needed in this path, we obtain a path from $x$ to $y$ in $G$, formed by vertices $\sim$-equivalent to $x$ and $y$; hence, $\sim \triangleleft G$.

   - $\psi(\sim, \ldots) = \sim$, defined by $x \sim y$ if $\overline{x} \sim \overline{y}$ and $(x, y) \in (G')^2 \cup (G'')^2$. This is clearly an equivalence; moreover, we do not have $x' \sim x''$. If $x \sim y$, let us assume for example that both of them belong to $G'$. There is a path in $G \setminus e$ from $\overline{x}$ to $\overline{y}$, formed by vertices $\sim$-equivalent to $\overline{x}$ and $\overline{y}$. We choose such a path of minimal length. If this path contains vertices belonging to $G''$, as $e$ is a bridge of $G$, it has the form:

     \[ \overline{x} - \ldots - \overline{x'} - \ldots - \overline{y} - \ldots - \overline{y}. \]

     Hence, we can obtain a shorter path from $\overline{x}$ to $\overline{y}$; this is a contradiction. So all the vertices of this path belong to $G'$; hence, they are all $\sim$-equivalent. Finally, $\sim \triangleleft G$.

Let us assume that $\psi(\sim, 1) = \psi(\sim', 1) = \sim$. If $\overline{x} \sim \overline{y}$, then $x \sim y$, so $\overline{x} \sim \overline{y}$; by symmetry, $\overline{y} \sim \overline{y}$. Let us assume that $\psi(\sim, \ldots) = \psi(\sim', \ldots) = \sim$. If $\overline{x} \sim \overline{y}$:

   - If $x, y \in V(G')$ or $x, y \in V(G'')$, then $x \sim y$, so $\overline{x} \sim \overline{y}$.

   - If $(x, y) \in V(G') \times V(G'')$ or $(x, y) \in V(G'') \times V(G')$, up to a permutation we can assume that $x \in V(G')$ and $y \in V(G'')$. As $\sim \triangleleft G/e$, there exists a path from $\overline{x}$ to $\overline{y}$ formed by $\sim$-equivalent vertices. This path necessarily goes via $\overline{x'} = \overline{x''}$. Hence, $x \sim x'$ and $y \sim x''$, so $\overline{x} \sim \overline{x'}$ and $\overline{y} \sim \overline{x''}$, and finally $\overline{x} \sim \overline{y}$.
Let us assume the result at all ranks $< k$. We proceed by induction on the number of edges of $G$. If $k = 0$, there is nothing to prove. Let us assume the result at all ranks $< k$, with $k \geq 1$. Let $e$ be an edge of $G$. We shall apply the induction hypothesis to $G/e$ and $G \setminus e$. Note that $cc(G/e) = cc(G)$ and $|G/e| = |G| - 1$, so $deg(G/e) = deg(G) - 1$.

- If $e$ is a bridge, then:
  \[
  \lambda_{chr}(G) = -(-1)^{deg(G/e)}\tilde{\lambda}_{chr}(G/e) = (-1)^{deg(G)}\tilde{\lambda}_{chr}(G/e).
  \]
• If $e$ is not a bridge, then $cc(G \setminus e) = cc(G)$, and $|G \setminus e| = |G|$, so $\deg(G \setminus e) = \deg(G)$. Hence:

$$\lambda_{chr}(G/e) = (-1)^{\deg(G/e)}\tilde{\lambda}_{chr}(G \setminus e) - (-1)^{\deg(G/e)}\tilde{\lambda}_{chr}(G/e)$$

$$= (-1)^{\deg(G)}\tilde{\lambda}_{chr}(G \setminus e) + (-1)^{\deg(G)}\tilde{\lambda}_{chr}(G/e)$$

$$= (-1)^{\deg(G)}(\tilde{\lambda}_{chr}(G \setminus e) + \tilde{\lambda}_{chr}(G/e)).$$

So the result holds for all graph $G$.

If $G$ has no edge, then $\deg(G) = 0$ and $\lambda_{chr}(G) = \tilde{\lambda}_{chr}(G) = 1$. An easy induction on the number of edges proves that for any graph $G$, $\lambda_{chr}(G) \geq 1$.

2. By Corollary 12, for any $i$:

$$a_i = \sum_{\sim \in G, cl(\sim) = i} \lambda_{chr}(G/\sim) = \sum_{\sim \in G, cl(\sim) = i} (-1)^{|\sim| - i}\tilde{\lambda}_{chr}(G/\sim) = (-1)^{|\sim| - i}\sum_{\sim \in G, cl(\sim) = i} \tilde{\lambda}_{chr}(G/\sim).$$

As for any graph $H$, $\tilde{\lambda}_{chr}(H) > 0$, this is non-zero if, and only if, there exists a relation $\sim \in G$, such that $cl(\sim) = i$. If this holds, the sign of $a_i$ is $(-1)^{|\sim| - i}$. It remains to prove that there exists a relation $\sim \in G$, such that $cl(\sim) = i$ if, and only if, $cc(G) \leq i \leq |G|$.

$\implies$. If $\sim \in G$, with $cl(\sim) = i$, as the equivalence classes of $\sim$ are connected, each connected component of $G$ is a union of classes of $\sim$, so $i \geq cc(G)$. Obviously, $i \leq |G|$.

$\impliedby$. We proceed by decreasing induction on $i$. If $i = |G|$, then the equality of $V(G)$ answers the question. Let us assume that $cc(G) \leq i < |G|$ and that the result holds at rank $i + 1$. Let $\sim' \in G$, with $cl(\sim') = i + 1$. We denote by $I_1, \ldots, I_{i+1}$ the equivalence classes of $\sim'$. As $I_1, \ldots, I_{i+1}$ are connected, the connected components of $G$ are union of $I_k$; as $i + 1 > cc(G)$, one of the connected components of $G$, which we call $G'$, contains at least two equivalence classes of $\sim'$. As $G'$ is connected, there exists an edge in $G'$, relation two vertices into different equivalence classes of $\sim'$; up to a reindexation, we assume that they are $I_1$ and $I_2$. Hence, $I_1 \sqcup I_2$ is connected. We consider the relation $\sim$ which equivalence classes are $I_1 \sqcup I_2, I_3, \ldots, I_{i+1}$: then $\sim \in G$ and $cl(\sim) = i$.

3. For $i = |G| - 1$, we have to consider relations $\sim \in G$ such that $cl(\sim) = |G| - 1$. These equivalences are in bijection with edges, via the map $\zeta_G$ of Proposition 18. For such an equivalence, $G/\sim = 1 \cdot |G|^{-1}$, so $\lambda_{chr}(G/\sim) = -1$. Finally, $a_i = -|E(V)|$.

Remark. The result on the signs of the coefficients of $P_{chr}(G)$ is due to Rota [21], who proved it using the Möbius function of the poset of Proposition 18.

Corollary 20 Let $G$ be a graph; $|\lambda_{chr}(G)| = 1$ if, and only if, $G$ is a forest, that is to say that any edge of $G$ is a bridge.

Proof. $\impliedby$. We proceed by induction on the number of edges $k$ of $G$. If $k = 0$, $\lambda_{chr}(G) = 1$. If $k \geq 1$, let us choose an edge of $G$; it is a bridge and $G/e$ is also a forest, so $|\lambda_{chr}(G)| = |\lambda_{chr}(G/e)| = 1$.

$\implies$. If $G$ is not a forest, there exists an edge $e$ of $G$ which is not a bridge. Then:

$$|\lambda_{chr}(G)| = |\lambda_{chr}(G \setminus e)| + |\lambda_{chr}(G/e)| \geq 1 + 1 = 2.$$

So $|\lambda_{chr}(G)| \neq 1$. □

Lemma 21 If $G$ is a graph and $e$ is a bridge of $G$, then:

$$\lambda_{chr}(G) = -\lambda_{chr}(G \setminus e) = -\lambda_{chr}(G/e).$$
Let us assume the result at rank $k$, $k \geq 1$. Let $f$ be an edge of $G$ which is not a bridge of $G$. If $V(G)$ is connected, then $\lambda_{chr}(G) = 0$. Hence:

$$\lambda_{chr}(G) = -\lambda_{chr}(G \setminus e) = (-1)^{\text{deg}(G)}.$$

So the result holds for any bridge of any graph. \hfill \square

**Proposition 22**

1. Let $G$ and $H$ be two graphs, with $V(G) = V(H)$ and $E(G) \subseteq E(H)$. Then:

$$|\lambda_{chr}(G)| \leq |\lambda_{chr}(H)| + \text{cc}(G) - \text{cc}(H) - \sharp(E(H) - E(G)) \leq |\lambda_{chr}(H)|.$$

Moreover, if $\text{cc}(G) = \text{cc}(H)$, then $|\lambda_{chr}(G)| = |\lambda_{chr}(H)|$ if, and only if, $G = H$.

2. For any graph $G$, $|\lambda_{chr}(G)| \leq (|G| - 1)!$, with equality if, and only if, $G$ is complete.

**Proof.**

1. We put $k = \sharp(E(H) \subseteq E(G))$. There exists a sequence $e_1, \ldots, e_k$ of edges of $H$ such that:

$$G_0 = G, \quad G_k = H, \quad \forall i \in [k], G_{i-1} = G_i \setminus e_i.$$

For all $i$, $\text{cc}(G_i) = \text{cc}(G_{i-1}) + 1$ if $e_i$ is a bridge of $G_i$, and $\text{cc}(G_i) = \text{cc}(G_{i-1})$ otherwise. Hence, $\text{cc}(G) - \text{cc}(H) \leq k$. We denote by $I$ the set of indices $i$ such that $\text{cc}(G_i) = \text{cc}(G_{i-1})$; then $\sharp I = k - \text{cc}(G) + \text{cc}(H)$. Moreover:

$$|\lambda_{chr}(G_i)| = \begin{cases} |\lambda_{chr}(G_{i-1})| + |\lambda_{chr}((G_i)/e_i)| > |\lambda_{chr}(G_{i-1})| & \text{if } i \in I, \\
|\lambda_{chr}(G_{i-1})| & \text{if } i \notin I. \end{cases}$$

As a conclusion, $|\lambda_{chr}(G)| \leq |\lambda_{chr}(H)| - \sharp I = |\lambda_{chr}(H)| + \text{cc}(G) - \text{cc}(H) - k \leq |\lambda_{chr}(H)|$.

If $\text{cc}(G) = \text{cc}(H)$ and $|\lambda_{chr}(G)| = |\lambda_{chr}(H)|$, then $k = 0$, so $G = H$.

2. We put $n = |G|$. We apply the first point with $H$ the complete graph such that $V(H) = V(G)$. We already observed that $|\lambda_{chr}(H)| = (n-1)!$, so:

$$|\lambda_{chr}(G)| \leq (n-1)!.$$

If $G$ is not connected, there exist graphs $G_1$, $G_2$ such that $G = G_1G_2$, $n_1 = |G_1| < n$, $n_2 = |G_2| < n$. Hence:

$$|\lambda_{chr}(G)| = |\lambda_{chr}(G_1)||\lambda_{chr}(G_2) \leq (n_1-1)!(n_2-1)! \leq (n_1 + n_2 - 2) < (n-1)!.$$

If $G$ is connected, then $\text{cc}(G) = \text{cc}(H)$: if $|\lambda_{chr}(G)| = |\lambda_{chr}(H)|$, then $G = H$. \hfill \square

19
2.6 Values at negative integers

Theorem 23 Let $k \geq 1$ and $G$ a graph. Then $(-1)^{|G|} P_{\text{chr}}(G)(-k)$ is the number of families $((I_1, \ldots , I_k), O_1, \ldots , O_k)$ such that:

- $I_1 \cup \ldots \cup I_k = V(G)$ (note that one may have empty $I_p$’s).
- For all $1 \leq i \leq k$, $O_i$ is an acyclic orientation of $G|_{I_i}$.

In particular, $(-1)^{|P|} P_{\text{chr}}(G)(-1)$ is the number of acyclic orientations of $G$.

Proof. Note that for any graph $G$, $(-1)^{|G|} P_{\text{chr}}(G)(-X) = P_{-1}(G)(X)$. Moreover:

- If $G$ is totally disconnected, $P_{-1}(G)(1) = 1$.
- If $G$ has an edge $e$, $P_{-1}(G)(1) = P_{-1}(G \setminus e)(1) + P_{-1}(G/e)$.

An induction on the number of edges of $G$ proves that $P_{-1}(G)(1)$ is indeed the number of acyclic orientations of $G$. If $k \geq 2$:

$$P_{-1}(G)(k) = P_{-1}(G)(1 + \ldots + 1) = \Delta^{(k-1)} \circ P_{-1}(G)(1, \ldots , 1) = P_{-1}^\otimes \circ \Delta^{(k-1)}(G)(1, \ldots , 1) = \sum_{V(G) = I_1 \cup \ldots \cup I_k} P_{-1}(G|_{I_1})(1) \cdots P_{-1}(G|_{I_k})(1).$$

The case $k = 1$ implies the result. \qed

We recover the interpretation of Stanley [23]:

Corollary 24 Let $k \geq 1$ and $G$ a graph. Then $(-1)^{|G|} P_{\text{chr}}(G)(-k)$ is the number of pairs $(f, O)$ where

- $f$ is a map from $V(G)$ to $[k]$.
- $O$ is an acyclic orientation of $G$.
- If there is an oriented edge from $x$ to $y$ in $V(G)$ for the orientation $O$, then $f(x) \leq f(y)$.

Proof. Let $A$ be the set of families defined in Theorem 23, and $B$ be the set of pairs defined in Corollary 24. We define a bijection $\theta : A \longrightarrow B$ in the following way: if $((I_1, \ldots , I_k), O_1, \ldots , O_k) \in A$, we put $\theta((I_1, \ldots , I_k), O_1, \ldots , O_k) = (f, O)$, such that:

1. $f^{-1}(p) = I_p$ for any $p \in [k]$.
2. If $e = \{x, y\} \in E(G)$, we put $f(x) = i$ and $f(x) = j$. If $i = j$, then $e$ is oriented as in $O_i$. Otherwise, if $i < j$, $e$ is oriented from $i$ to $j$ if $i < j$ and from $j$ to $i$ if $i > j$.

Note that $O$ is indeed acyclic: if there is an oriented path from $x$ to $y$ in $G$ of length $\geq 1$, then $f$ increases along this path. If $f$ remains constant, as $O_{f(x)}$ is acyclic, $x \neq y$. Otherwise, $f(x) < f(y)$, so $x \neq y$. It is then not difficult to see that $\theta$ is bijective. \qed
3 Chromatic symmetric functions

3.1 Reminders on \texttt{QSym}

The Hopf algebra \texttt{QSym} [2, 12, 14, 17, 25] has a basis \((M_u)\) indexed by compositions, that is to say finite sequences of positive integers. Its product is given by quasi-shuffles. For example, if \(a, b, c, d > 0:\)

\[
M_a M_{bcd} = M_{abcd} + M_{bacd} + M_{bcda} + M_{a(b+c)d} + M_{b(a+c)d} + M_{ab(c+d)},
\]

\[
M_{ab} M_{cd} = M_{abcd} + M_{acbd} + M_{acdb} + M_{cadb} + M_{cadb} + M_{cdab}
\quad + M_{(a+c)bd} + M_{(a+c)db} + M_{a(b+c)d} + M_{ac(b+d)} + M_{ca(b+d)} + M_{(a+b)(c+d)}.
\]

Its coproduct is given by deconcatenation: for any composition \(w\),

\[
\Delta(M_w) = \sum_{w = u \circ v} M_u \otimes M_v.
\]

For any composition \(w\), we denote by \(|w|\) the sum of its letters; this induces a connected graduation of \texttt{QSym}. There exists a second coproduct \(\delta\), such that for any composition \(w\) of length \(n\):

\[
\delta(M_w) = \sum_{k=1}^{n} \sum_{w = w_1 \ldots w_k} M_{[w_1]| \ldots |w_k]} \otimes M_{w_1} \ldots M_{w_k}.
\]

The counit of this coproduct is denoted by \(\varepsilon'\); for any composition \(u\),

\[
\varepsilon'(M_u) = \begin{cases} 
1 & \text{if } u \text{ has only one letter,} \\
0 & \text{otherwise.}
\end{cases}
\]

With the coaction \(\delta\), \((\texttt{QSym}, m, \Delta)\) and \((\texttt{QSym}, m, \delta)\) are in cointeraction.

\texttt{QSym} admits a polynomial representation. Let \(X\) be a totally ordered alphabet; for any composition \(u = u_1 \ldots u_n\), we consider the element:

\[
\operatorname{rep}_X(M_u) = \sum_{x_1 < \ldots < x_n \text{ in } X} x_1^{u_1} \ldots x_n^{u_n} \in \mathbb{Q}[\mathbb{X}].
\]

We define in this way an algebra morphism \(\operatorname{rep}_X : \texttt{QSym} \rightarrow \mathbb{Q}[\mathbb{X}]\). Moreover, for any \(k \in \mathbb{N}\), the restriction of \(\operatorname{rep}_X\) to the \(k\)-th homogeneous component \(\texttt{QSym}_k\) of \texttt{QSym} is injective if, and only if, \(|X| \geq k\).

If \(X\) and \(Y\) are two totally ordered alphabets, \(X \sqcup Y\) is also totally ordered: for all \(x \in X, y \in Y, x \leq y\). We identify \(\mathbb{Q}[\mathbb{X} \sqcup \mathbb{Y}]\) with \(\mathbb{Q}[\mathbb{X}] \otimes \mathbb{Q}[\mathbb{Y}]\), via the continuous morphism sending \(x \in X\) to \(x \otimes 1\) and \(y \in Y\) to \(1 \otimes y\). For any \(a \in \texttt{QSym}\):

\[
\operatorname{rep}_{X \sqcup Y}(a) = (\operatorname{rep}_X \otimes \operatorname{rep}_Y) \circ \Delta(a).
\]

The cartesian product \(X \times Y\) is totally ordered by the lexicographic order: for any \(x, x' \in X, y, y' \in Y, xy \leq x'y'\) if, and only if, \((x < x')\) or \((x = x'\) and \(y \leq y')\). We identify \(\mathbb{Q}[\mathbb{X} \times \mathbb{Y}]\) with a subring of \(\mathbb{Q}[\mathbb{X}] \otimes \mathbb{Q}[\mathbb{Y}]\) through the continuous morphism sending \((x, y) \in X \times Y\) to \(x \otimes y\). For any \(a \in \texttt{QSym}\):

\[
\operatorname{rep}_{X \times Y}(a) = (\operatorname{rep}_X \otimes \operatorname{rep}_Y) \circ \delta(a).
\]

The Hopf algebra \texttt{QSym} contains the Hopf subalgebra \texttt{Sym} of symmetric functions; this subalgebra is generated by the elements:

\[
M_{\{u_1, \ldots, u_k\}} = \sum_{\sigma \in S_k} M_{a_{\sigma(1)} \ldots a_{\sigma(k)}}.
\]

Let us apply the results of [10] to \texttt{QSym}.
Proposition 25 For any \( k \geq 0 \), we denote by \( H_k \) the \( k \)-th Hilbert polynomial:

\[
H_k(X) = \frac{X(X-1)\ldots(X-k+1)}{k!}.
\]

Let us consider the map:

\[
H : \begin{cases} 
\text{QSym} & \rightarrow \mathbb{Q}[X] \\
M_{a_1\ldots a_k} & \rightarrow H_k.
\end{cases}
\]

Then \( H \) is the unique morphism from \( \text{QSym} \) to \( \mathbb{Q}[X] \) compatible with \( m, \Delta \) and \( \delta \).

**Proof.** By [10], such a morphism is unique. Let us prove that \( H \) is indeed compatible with \( m, \Delta \) and \( \delta \). For any finite totally ordered alphabet \( X \), of cardinality \( k \), for any \( a \in \text{QSym} \), by definition of the polynomial representation of \( \text{QSym} \):

\[
H(a)(k) = \text{rep}_X(a)_{\forall x \in X, x=1}.
\]

If \( a, b \in \text{QSym} \), for any \( k \geq 1 \), if \( X \) is a totally ordered alphabet of cardinality \( k \):

\[
H(ab)(k) = \text{rep}_X(ab)_{\forall x \in X, x=1} = \text{rep}_X(a)_{\forall x \in X, x=1}\text{rep}_X(b)_{\forall x \in X, x=1} = H(a)(k)H(b)(k).
\]

Hence, \( H(ab) = H(a)H(b) \). If \( a \in \text{QSym} \), for any \( k, l \geq 1 \), choosing totally ordered alphabets \( X \) and \( Y \) of respective cardinality \( k \) and \( l \):

\[
\Delta \circ H(a)(k,l) = H(a)(k+l)
\]

\[
= \text{rep}_X \vdash Y(a)_{\forall x \in X \vdash Y, x=1}
\]

\[
= (\text{rep}_X \otimes \text{rep}_Y) \circ \Delta(a)_{\forall x \in X \vdash Y, x=1}
\]

\[
= (H \otimes H) \circ \Delta(a)(k,l);
\]

\[
\delta \circ H(a)(k,l) = H(a)(kl)
\]

\[
= \text{rep}_X \times Y(a)_{\forall x \in X \times Y, x=1}
\]

\[
= (\text{rep}_X \otimes \text{rep}_Y) \circ \delta(a)_{\forall x \in X \times Y, x=1}
\]

\[
= (H \otimes H) \circ \delta(a)(k,l).
\]

Hence, \( \Delta \circ H = (H \otimes H) \circ \Delta \) and \( \delta \circ H = (H \otimes H) \circ \delta \). \( \Box \)

3.2 Cointeraction and quasi-symmetric functions

The following result is proved by Aguiar and Bergeron in [2]. It states that \( \text{QSym} \) is a terminal object in a suitable category of combinatorial Hopf algebras:

**Theorem 26** Let \((A, m, \Delta)\) be a graded, connected Hopf algebra, and \( \alpha \) be a character on \( A \). There exists a unique homogeneous Hopf algebra morphism \( \Phi_\alpha : (A, m, \Delta) \rightarrow (\text{QSym}, m, \Delta) \), such that \( \alpha \circ \Phi_\alpha = \varepsilon' \). For any \( a \in A \):

\[
\Phi_\alpha(a) = \varepsilon(a) + \sum_{k=1}^{\infty} \sum_{u_1, \ldots, u_k > 0} \alpha^{\otimes k} \circ (\pi_{u_1} \otimes \ldots \otimes \pi_{u_k}) \circ \Delta^{(k-1)}(a)M_{u_1, \ldots, u_k},
\]

where, for any \( j \geq 1 \), \( \pi_j \) is the canonical projection on the \( j \)-th homogeneous component \( A_j \) of \( A \).

**Theorem 27** Let \((A, m, \Delta)\) and \((A, m, \delta)\) be cointeracting bialgebras, such that \((A, m, \Delta)\) is a graded connected Hopf algebra.

1. There exists a unique morphism \( \Phi_1 : A \rightarrow \text{QSym} \) such that:
(a) $\Phi_1 : (A, m, \Delta) \rightarrow (\text{QSym}, m, \Delta)$ is a homogeneous morphism of Hopf algebras.

(b) $\Phi_1 : (A, m, \delta) \rightarrow (\text{QSym}, m, \delta)$ is a morphism of bialgebras.

Moreover, the morphism $\phi_1$ of Theorem 8 is $\phi_1 = \Phi_1 \circ H$.

2. Let us assume that for any $n \in \mathbb{N}$, $\delta(A_n) \subseteq A_n \otimes A_n$. Then for any character $\alpha$ on $A$, $\Phi_\alpha = \Phi_1 \leftarrow \alpha$.

**Proof.** 1. **Unicity.** Let us denote by $\varepsilon_\delta$ the counit of $(A, m, \delta)$. If $\Phi_1$ is such a morphism, then $\varepsilon' \circ \Phi_1 = \varepsilon_\delta$. By Theorem 26, $\Phi_1$ is unique.

**Existence.** Let $\Phi_1$ be the unique homogeneous Hopf algebra morphism from $(A, m, \Delta)$ to $(\text{QSym}, m, \Delta)$ such that $\varepsilon' \circ \Phi_1 = \varepsilon_\delta$. We shall use the polynomial representation of $\text{QSym}$. If $X, Y$ are totally ordered alphabets, for any $a \in A$, as $\Phi_1$ is compatible with $\Delta$:

$$rep_{X \times Y} \circ \Phi_1(a) = (rep_X \otimes rep_Y) \circ \Delta \circ \Phi_1(a) = (rep_X \otimes rep_Y) \circ (\Phi_1 \otimes \Phi_1) \circ \Delta(a).$$

Let us prove that for any finite totally ordered alphabet $X$, for any totally ordered alphabet $Y$, for any $a \in A$:

$$rep_{X \times Y} \circ \Phi_1(a) = (rep_X \otimes rep_Y) \circ (\Phi_1 \otimes \Phi_1) \circ \delta(a).$$

We proceed by induction on $n = |X|$. If $n = 1$, we identify $X \times Y$ and $Y$. Then:

$$(rep_X \otimes rep_Y) \circ (\Phi_1 \otimes \Phi_1) \circ \delta(a) = (\varepsilon' \otimes rep_Y) \circ (\Phi_1 \otimes \Phi_1) \circ \delta(a) = (\varepsilon_\delta \otimes rep_Y) \circ \Phi_1 \circ \delta(a) = rep_Y \circ \Phi_1(a).$$

Let us assume that the results holds for any totally ordered alphabet $X'$ such that $|X'| < |X|$, with $|X| \geq 2$. Let $x_n$ be the maximal element of $X$. We put $X' = X \setminus \{x_n\}$ and $X'' = \{x_n\}$, such that $X = X' \sqcup X''$. Then:

$$X \times Y = (X' \sqcup X'') \times Y = (X' \times Y) \sqcup (X'' \times Y),$$

so:

$$rep_{X \times Y} \circ \Phi_1(a) = rep_{(X' \times Y) \sqcup (X'' \times Y)} \circ \Phi_1(a) = (rep_X \otimes rep_{X'}) \circ (\Phi_1 \otimes \Phi_1) \circ \Delta(a).$$

Let $a \in A$. Let us choose a totally ordered alphabet $X$ of cardinality $n$ such that:

$$\delta(a) \in \bigoplus_{k,l \leq n} A_k \otimes A_l.$$

Then:

$$rep_X \circ \Phi_1(a) = (rep_X \otimes rep_X) \circ \delta \circ \Phi_1(a) = (rep_X \otimes rep_X) \circ (\Phi_1 \otimes \Phi_1) \circ \delta(a).$$
By injectivity of $\text{rep}_\mathbf{X}$ till degree $n$, as $|X| \geq n$, $\delta \circ \Phi_1(a) = (\Phi_1 \otimes \Phi_1) \circ \delta(a)$.

The morphism $\Phi_1 \circ H : A \rightarrow \text{QSym}$ is compatible with $m$, $\Delta$ and $\delta$ by composition. By unicity in Theorem 8, it is equal to $\phi_1$.

2. Let us consider $\Phi = \Phi_1 \leftarrow \alpha = (\Phi_1 \otimes \alpha) \circ \delta$; this is a Hopf algebra morphism. For any $n \in A_n$:

$\Phi(A_n) \subseteq (\Phi_1 \otimes \alpha) \circ \delta(A_n) \subseteq \Phi_1(A_n) \alpha(A_n) \subseteq A_n$,

so $\Phi_1$ is homogenous. Moreover:

$\epsilon' \circ \Phi = (\epsilon' \circ \Phi_1 \otimes \alpha) \circ \delta = (\epsilon_\delta \otimes \alpha) \circ \delta = \alpha$,

so $\Phi = \Phi_\alpha$. \hfill $\square$

3.3 Chromatic symmetric function

**Theorem 28** The following map is compatible with $m$, $\Delta$ and $\delta$:

$$F_{\text{chr}} : \begin{cases} \mathcal{H}_G & \rightarrow \text{QSym} \\ G \in \mathcal{G} & \rightarrow \sum_{f \in \text{PVC}(G)} M_{f^{-1}(1)[\ldots]f^{-1}(\text{max}(f))} \end{cases}$$

For any $G \in \mathcal{G}$, $F_{\text{chr}}(G)$ is the chromatic symmetric function of $G$ [24]. Moreover, the image of $F_{\text{chr}}$ is $\text{Sym}$, and $F_{\text{chr}} \circ H = P_{\text{chr}}$.

**Proof.** Let us prove that $F_{\text{chr}}$ is a Hopf algebra morphism. We shall use the polynomial representation of $\text{QSym}$. Choosing an infinite totally ordered alphabet $X$, for any indexed graph $G$:

$$\text{rep}_\mathbf{X} \circ F_{\text{chr}}(G) = \sum_{f \in \text{VC}(G,X)} \prod_{i \in V(G)} f(i).$$

If $G$ and $H$ are graphs, denoting that any valid coloring of $GH$ is obtained by the disjoint union a valid coloring of $G$ and a valid coloring of $H$:

$$\text{rep}_\mathbf{X} \circ F_{\text{chr}}(GH) = \sum_{f \in \text{VC}(G,X), g \in \text{VC}(H,X)} \prod_{i \in V(G)} f(i) \prod_{j \in V(H)} g(j)$$

$$= \text{rep}_\mathbf{X} \circ F_{\text{chr}}(G) \text{rep}_\mathbf{X} \circ F_{\text{chr}}(H)$$

$$= \text{rep}_\mathbf{X}(F_{\text{chr}}(G) F_{\text{chr}}(H)).$$

For any graph $G$:

$$(\text{rep}_\mathbf{X} \otimes \text{rep}_\mathbf{X}) \circ \Delta \circ F_{\text{chr}}(G) = \text{rep}_\mathbf{X} \Delta \mathbf{Y} \circ F_{\text{chr}}(G)$$

$$= \sum_{I \subseteq [n], f \in \text{VC}(G_i, X), g \in \text{VC}(G_{[n]\setminus I}, X)} \prod_{i \in I} f(i) \otimes \prod_{i \notin I} f(i)$$

$$= \sum_{I \subseteq [n]} \text{rep}_\mathbf{X} \circ F_{\text{chr}}(G|_I) \otimes \text{rep}_\mathbf{X} \circ F_{\text{chr}}(G|_{[n]\setminus I})$$

$$= (\text{rep}_\mathbf{X} \otimes \text{rep}_\mathbf{X}) \circ (F_{\text{chr}} \otimes F_{\text{chr}}) \circ \Delta(G).$$

So $F_{\text{chr}}$ is a Hopf algebra morphism from $(\mathcal{H}_G, m, \Delta)$ to $(\text{QSym}, m, \Delta)$, obviously homogeneous.
For any graph $G$:

$$
\varepsilon' \circ F_{\text{chr}}(G) = \begin{cases} 
1 & \text{if the constant coloring of } G \text{ by 1 is valid}, \\
0 & \text{otherwise}, 
\end{cases}
= \begin{cases} 
1 & \text{if } G \text{ is totally disconnected}, \\
0 & \text{otherwise}, 
\end{cases}
= \varepsilon'(G).
$$

By Theorem 27, $F_{\text{chr}}$ is the unique homogeneous morphism from $\mathcal{H}_G$ to $\textbf{QSym}$ compatible with $m$, $\Delta$ and $\delta$. Moreover, $F_{\text{chr}} \circ H = P_{\text{chr}}$.

As $\mathcal{H}_G$ is cocommutative, $F_{\text{chr}}(\mathcal{H}_G)$ is a cocommutative Hopf subalgebra of $\textbf{QSym}$, so is included in $\textbf{Sym}$, greatest cocommutative subalgebra of $\textbf{QSym}$. Let us prove the other inclusion.

Let $\pi_1, \ldots, \pi_k > 0$. For any $j$, let us fix a set $I_j$ of cardinality $I_j$. We define the graph of set of vertices $I = I_1 \sqcup \ldots \sqcup I_k$ by the following property: for all $x, y \in I$, there is an edge between $x$ and $y$ if, and only if, $x \in I_i$ and $y \in I_j$, with $i \neq j$. Note that the $I_j$'s are the maximal totally independent subsets of $G$; hence:

$$
F_{\text{chr}}(G) = M_{\{\pi_1, \ldots, \pi_k\}} + \text{a sum of terms } M_{\{\pi'_1, \ldots, \pi'_l\}} \text{ with } l > k.
$$

By a triangularity argument, we deduce that $\textbf{Sym} \subseteq F_{\text{chr}}(\mathcal{H}_G)$.

**Corollary 29** Let $\mu_{\text{chr}}$ be the infinitesimal character on $\mathcal{H}_G$ which coincides with $\lambda_{\text{chr}}$ on connected graphs: for any $G \in \mathcal{G}$,

$$
\mu_{\text{chr}}(G) = \begin{cases} 
\lambda_{\text{chr}}(G) & \text{if } G \text{ is connected}, \\
0 & \text{otherwise}. 
\end{cases}
$$

Denoting by $\exp_s$ the exponentiation map from infinitesimal characters to characters of $(\mathcal{H}_G, m, \Delta)$ and by $\ln_s$ its inverse:

$$
\varepsilon' = \exp_s(\mu_{\text{chr}}), \quad \mu_{\text{chr}} = \ln_s(\varepsilon').
$$

**Proof.** For any connected graph $G$:

$$
\mu_{\text{chr}}(G) = \lambda_{\text{chr}}(G) = \frac{\partial P_{\text{chr}}(G)}{\partial X}(0).
$$

If $G$ is not connected, then $P_{\text{chr}}(G) \in (1) \oplus \langle X^2 \rangle$, so:

$$
\frac{\partial P_{\text{chr}}(G)}{\partial X}(0) = 0 = \mu_{\text{chr}}(G).
$$

Consequently, if we put for any $x \in \mathcal{H}_G$:

$$
F_{\text{chr}}(x) = \sum_u \alpha_u(x) M_u,
$$
then, if \( x \) is homogeneous of degree \( n \), by Theorem 26:

\[
\mu_{\text{chr}}(x) = \sum_{|u| = n} \alpha_u(x) \frac{\partial H_{H_p(u)}}{\partial X}(0)
\]

\[
= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \sum_{u_1 + \ldots + u_k = n} \alpha_{u_1,\ldots,u_k}(x) (-1)^{k-1} \frac{\epsilon^{\otimes k} \circ (\pi_{u_1} \otimes \ldots \pi_{u_k}) \circ \Delta^{(k-1)}(x)}{k}
\]

\[
= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \epsilon^{\otimes k} \circ \left( \sum_{u_1 + \ldots + u_k = n} (\pi_{u_1} \otimes \ldots \pi_{u_k}) \right) \circ \Delta^{(k-1)}(x)
\]

\[
= \ln_{\#}(\epsilon + (\epsilon' - \epsilon))(x)
\]

So \( \mu_{\text{chr}} = \ln_{\#}(\epsilon') \), and \( \epsilon' = \exp_{\#}(\mu_{\text{chr}}) \). \( \square \)

### 3.4 Extension of \( \phi_0 \)

**Proposition 30** Let \( G \) be a graph and \( f \in \mathbb{PC}(G) \). We define the equivalence \( \sim_f \) in \( V(G) \) as the unique one which classes are the connected components of the subsets \( f^{-1}(x) \), \( x \in \{\max(f)\} \). Moreover, the coloring \( f \) induces a packed valid coloring \( \widetilde{f} \) of \( G/\sim_f \):

\[
\forall x \in V(G), \quad \widetilde{f}(x) = f(x).
\]

We put:

\[
M_f = M[\widetilde{f}]^{-1}(\ldots[\widetilde{f}]^{-1}(\max(\widetilde{f}))) \in \mathcal{QSym}.
\]

In other words, \( f^{-1}(i) \) is the number of connected components of the subgraph of \( G \) which vertices are the vertices of \( G \) colored by \( i \).

**Proof.** We have to prove that \( \widetilde{f} \) is a valid coloring of \( G/\sim_f \). Let \( \pi, \gamma \) be two vertices of \( G/\sim_f \), related by an edge (this implies that they are different); we assume that \( \widetilde{f}(\pi) = \widetilde{f}(\gamma) \). There exist \( x', y' \in V(G) \), such that \( x' \sim_f x \) and \( y' \sim_f y \), and \( x', y' \) are related by an edge in \( G \). By definition of \( \sim_f \), there exist vertices \( x' = x_1, \ldots, x_k = x \), \( y' = y_1, \ldots, y_k = y \) in \( G \) such that \( f(x_1) = \ldots = f(x_k) \), \( g(y_1) = \ldots = g(y_l) \), and for all \( p, q, x_p \) and \( x_{p+1} \), \( y_q \) and \( y_{q+1} \) are related by an edge in \( G \). Hence, there is a path in \( G \) from \( x \) to \( y \), such that for any vertex \( z \) on this path, \( f(z) = f(x) = f(y) \): this implies that \( x \sim_f y \), so \( \pi = \gamma \). This is a contradiction, so \( \widetilde{f} \) is valid. \( \square \)

**Proposition 31** Let us consider the following map:

\[
F_0 : \begin{cases} 
\mathcal{H}_G & \to & \mathcal{QSym} \\
G & \to & \sum_{f \in \mathbb{PC}(G)} M_f.
\end{cases}
\]

This is a Hopf algebra morphism, and \( F_0 \circ H = \phi_0 \). Moreover, in \( E_{\mathcal{H}_G} \to \mathcal{QSym} \):

\[
F_{\text{chr}} = F_0 \left( \lambda_{\text{chr}} \right).
\]
Proof. Let \( G \) be graph. By Proposition 30, we have a map:

\[
\theta : \left\{ \begin{array}{l}
\mathbb{P}C(G) \rightarrow \bigsqcup_{f \in \mathbb{P}C(G/ \sim)} \mathbb{P}C(G/ \sim) \\
f \mapsto f \in \mathbb{P}C(G/ \sim).
\end{array} \right.
\]

\( \theta \) is injective: if \( \theta(f) = \theta(g) \), then \( \sim_f = \sim_g \) and for any \( x \in V(G) \),

\[
f(x) = \overline{f}(x) = \overline{g}(x).
\]

Let us show that \( \theta \) is surjective. Let \( f \in \mathbb{P}C(G/ \sim) \), with \( \sim \circ G \). We define \( f \in \mathbb{P}C(G) \) by \( f(x) = \overline{f}(x) \) for any vertex \( x \). By definition of \( f \), the equivalence classes of \( \sim \) are included in sets \( f^{-1}(i) \), and are connected, as \( \sim \circ G \), so are included in equivalence classes of \( \sim_f \): if \( x \sim y \), then \( x \sim_f y \). Let us assume that \( x \sim_f y \). There exists a path \( x = x_1, \ldots, x_k = y \) in \( G \), such that \( f(x_1) = \ldots = f(x_k) \). So \( f(x_1) = \ldots = f(x_k) \). As \( f \) is a valid coloring of \( G/ \sim \), there is no edge between \( x_p \) and \( x_{p+1} \) in \( G/ \sim \) for any \( p \); this implies that \( x_p \sim x_{p+1} \) for any \( p \), so \( x = x_1 \sim x_k = y \). Finally, \( \sim = \sim_f \), so \( \theta(f) = \overline{f} \).

Using the bijection \( \theta \), we obtain:

\[
F_0(G) = \sum_{f \in \mathbb{P}C(G)} M_f
= \sum_{\sim \circ G} \sum_{f \in \mathbb{P}C(G/ \sim)} M_f
= \sum_{\sim \circ G} F_{\text{chr}}(G/ \sim)
= \sum_{\sim \circ G} F_{\text{chr}}(G/ \sim) \lambda_0(G/ \sim)
= (F_{\text{chr}} \leftarrow \lambda_0)(G).
\]

Therefore, \( F_0 = F_{\text{chr}} \leftarrow \lambda_0 \), so is a Hopf algebra morphism, taking its values in \( \text{Sym} \). Hence:

\[
H \circ F_0 = H \circ (F_{\text{chr}} \leftarrow \lambda_0) = (H \circ F_{\text{chr}}) \leftarrow \lambda_0 = P_{\text{chr}} \leftarrow \lambda_0 = \phi_0.
\]

Finally, \( F_{\text{chr}} = F_0 \leftarrow \lambda_0^{-1} = F_0 \leftarrow \lambda_{\text{chr}}. \)

Examples.

\[
F_0(\cdot) = M_1,
F_0(1) = 2M_{11} + M_1,
F_0(\vee) = 6M_{111} + 4M_{11} + M_{12} + M_{21} + M_1,
F_0(\wedge) = 6M_{111} + 6M_{11} + M_1.
\]

4 Non-commutative versions

4.1 Non-commutative Hopf algebra of graphs

Definition 32

1. An indexed graph is a graph \( G \) such that \( V(G) = [n] \), with \( n \geq 0 \). The set of indexed graphs is denoted by \( \mathcal{G} \).

2. Let \( G = ([n], E(G)) \) be an indexed graph and let \( I \subseteq [n] \). There exists a unique increasing bijection \( f : I \rightarrow [k] \), where \( k = \sharp I \). We denote by \( G_f \) the indexed graph defined by:

\[
G_f = ([k], \{\{f(x), f(y)\} \mid \{x, y\} \in E(G), x, y \in I\}).
\]

27
3. Let $G$ be an indexed graph and $\sim G$.
   (a) The graph $G|\sim$ is an indexed graph.
   (b) We order the elements of $V(G)/\sim$ by their minimal elements; using the unique increasing bijection from $V(G)/\sim$ to $[k]$, $G/\sim$ becomes an indexed graph.

4. Let $G = ([k], E(G))$ and $H = ([l], E(H))$ be indexed graphs. The indexed graph $GH$ is defined by:

$$V(GH) = [k + l],$$

$$E(GH) = E(G) \cup \{\{x + k, y + l\} \mid \{x, y\} \in E(H)\}.$$

The Hopf algebra $(\mathcal{H}_{G}, m, \Delta)$ is, as its commutative version, introduced in [22]:

**Theorem 33** 1. We denote by $\mathcal{H}_{G}$ the vector space generated by indexed graphs. We define a product $m$ and two coproducts $\Delta$ and $\delta$ on $\mathcal{H}_{G}$ in the following way:

$$\forall G, H \in \mathcal{G}, \ m(G \otimes H) = GH,$$

$$\forall G = ([n], E(G)) \in \mathcal{G}, \ \Delta(G) = \sum_{I \subseteq [n]} G_{[I]} \otimes G_{[n] \setminus I},$$

$$\forall G \in \mathcal{G}, \ \delta(G) = \sum_{\sim G} G/\sim \otimes G/\sim.$$

Then $(\mathcal{H}_{G}, m, \Delta)$ is a graded cocommutative Hopf algebra, and $(\mathcal{H}_{G}, m, \delta)$ is a bialgebra.

2. Let $\sim : \mathcal{H}_{G} \rightarrow \mathcal{H}_{G}$ be the surjection sending an indexed graph to its isoclass.
   (a) $\sim : (\mathcal{H}_{G}, m, \Delta) \rightarrow (\mathcal{H}_{G}, m, \Delta)$ is a surjective Hopf algebra morphism.
   (b) $\sim : (\mathcal{H}_{G}, m, \delta) \rightarrow (\mathcal{H}_{G}, m, \delta)$ is a surjective bialgebra morphism.
   (c) We put $\rho = (1d \circ \sim) \circ \delta : \mathcal{H}_{G} \rightarrow \mathcal{H}_{G} \otimes \mathcal{H}_{G}$. This defines a coaction of $(\mathcal{H}_{G}, m, \delta)$ on $\mathcal{H}_{G}$; moreover, $(\mathcal{H}_{G}, m, \Delta)$ is a Hopf algebra in the category of $(\mathcal{H}_{G}, m, \delta)$-comodules.

**Proof.** 1. Similar as the proof of Propositions 2 and 4.

2. Points (a) and (b) are immediate; point (c) is proved in the same way as Theorem 7. □

**Examples.**

$$\Delta(\cdot, 1) = \cdot \otimes 1 + 1 \otimes \cdot, \quad \Delta(1, \cdot) = 1 \otimes 1 + 1 \otimes 1 + \cdot \otimes \cdot,$$

$$\Delta(\mathcal{V}^{3}_{1}) = 2\mathcal{V}^{3}_{1} \otimes 1 + 1 \otimes \mathcal{V}^{3}_{1} + 3\cdot \otimes 1 + 31 \otimes \cdot,$$

$$\Delta(\mathcal{V}^{3}_{2}) = 2\mathcal{V}^{3}_{1} \otimes 1 + 1 \otimes \mathcal{V}^{3}_{1} + 21 \otimes \cdot + 2 \cdot \otimes 1 + \cdot \otimes \cdot \otimes \cdot;$$

$$\delta(\cdot, 1) = \cdot \otimes 1 + 1 \otimes \cdot,$$

$$\delta(1, \cdot) = 1 \otimes 1 + 1 \otimes 1 + \cdot \otimes \cdot,$$

$$\delta(\mathcal{V}^{3}_{1}) = \cdot \otimes \mathcal{V}^{3}_{1} + 1 \otimes (\cdot \otimes \cdot + 1 \otimes 1 + \cdot \otimes 1) + \mathcal{V}^{3}_{1} \otimes 1 \cdot 2 \cdot 3,$$

$$\delta(\mathcal{V}^{3}_{2}) = \cdot \otimes \mathcal{V}^{3}_{1} + 1 \otimes (\cdot \otimes \cdot + 1 \otimes 1 + \cdot \otimes 1) + \mathcal{V}^{3}_{1} \otimes 1 \cdot 2 \cdot 3.$$

**Remark.** $(\mathcal{H}_{G}, m, \Delta)$ is not a bialgebra in the category of $(\mathcal{H}_{G}, m, \delta)$-comodules, as shown in the following example:

$$(\Delta \otimes 1d) \circ \delta(\mathcal{V}^{3}_{1}) = \Delta(\cdot, 1) \otimes \mathcal{V}^{3}_{1} + \Delta(1, \cdot) \otimes (\cdot \otimes \cdot + 1 \otimes 1 + \cdot \otimes 1) + \Delta(\mathcal{V}^{3}_{1}) \otimes 1 \cdot 2 \cdot 3,$$

$$m_{2,4}^{3} \circ (\delta \otimes \delta) \circ \Delta(\mathcal{V}^{3}_{1}) = \Delta(\cdot, 1) \otimes \mathcal{V}^{3}_{1} + \Delta(1, \cdot) \otimes (\cdot \otimes \cdot + 1 \otimes 1 + \cdot \otimes 1) + \Delta(\mathcal{V}^{3}_{1}) \otimes 1 \cdot 2 \cdot 3.$$
4.2 Reminders on \textit{WQSym}

Let us recall the construction of \textit{WQSym} [19].

\textbf{Definition 34} \hspace{1em} 1. Let \( w \) be a word in \( \mathbb{N} \setminus \{0\} \). We shall say that \( w \) is packed if:
\[ \forall i, j \geq 0, \ (i \leq j \text{ and } j \text{ appears in } w) \implies (i \text{ appears in } w). \]

2. Let \( w = x_1 \ldots x_k \) a word in \( \mathbb{N} \). There exists a unique increasing bijection \( f \) from \( \{x_1, \ldots, x_k\} \) to \([l]\), with \( l \geq 0\); the packed word \( \text{Pack}(w) \) is \( f(x_1) \ldots f(x_k) \).

3. \( w = x_1 \ldots x_k \) a word in \( \mathbb{N} \setminus \{0\} \) and \( I \subseteq \mathbb{N} \setminus \{0\} \). The word \( w|_I \) is the word obtained by taking the letters of \( w \) which are in \( I \).

The Hopf algebra \textit{WQSym} has the set of packed words for basis. If \( w = w_1 \ldots w_k \) and \( w' = w'_1 \ldots w'_l \) are packed words, then:
\[ w \cdot w' = \sum_{w''=w''_1 \ldots w''_{k+l}, \text{Pack}(w''_1 \ldots w''_k)=w, \text{Pack}(w''_{k+1} \ldots w''_{k+l})=w'} w''. \]

For any packed word \( w \):
\[ \Delta(P_w) = \sum_{i=0}^{\max(w)} w|_I \otimes \text{Pack}(w|_{\max(w)}\setminus I). \]

This Hopf algebra admits a polynomial representation: we fix a finite totally ordered alphabet \( X \); the set of words in \( X \) is denoted by \( X^* \). For any packed word \( w \), we consider the element:
\[ \text{Rep}_{X}(P_w) = \sum_{w' \in X^*, \text{Pack}(w')=w} w' \in \mathbb{Q} \langle \langle X \rangle \rangle. \]

Then \( \text{Rep}_{X} \) is an algebra morphism from \textit{WQSym} to \( \mathbb{Q} \langle \langle X \rangle \rangle \). If \( X \) and \( Y \) are two totally ordered alphabets, we shall consider \( \mathbb{Q} \langle \langle X \rangle \rangle \otimes \mathbb{Q} \langle \langle Y \rangle \rangle \) as a quotient of \( \mathbb{Q} \langle \langle X \cup Y \rangle \rangle \). In this quotient, for any \( a \in \textit{WQSym} \):
\[ \text{Rep}_{X \cup Y}(a) = (\text{Rep}_{X} \otimes \text{Rep}_{Y}) \circ \Delta(a). \]

4.3 Non-commutative chromatic symmetric functions

\textbf{Notations.} A set partition is a partition of a set \([n]\), with \( n \geq 0 \). The set of set partitions is denoted by \( \mathcal{SP} \).

\textbf{Theorem 35} 1. For any packed word \( w \) of length \( n \) and of maximal \( k \), we denote by \( p(w) \) the set partition \( \{w^{-1}(1), \ldots, w^{-1}(k)\} \). For any set partition \( \varpi \in \mathcal{SP} \), we put:
\[ W_{\varpi} = \sum_{w \in PW, p(w)=\varpi} w. \]

These elements are a basis of a cocommutative Hopf subalgebra of \textit{WQSym}, denoted by \textit{WSym}.

2. The following map is a Hopf algebra morphism:
\[ F_{\text{chr}} : \begin{cases} \mathcal{H}_{\mathcal{G}} & \rightarrow \textit{WQSym} \\ G \in \mathcal{G} & \rightarrow \sum_{f \in \text{PVC}(G)} f(1) \ldots f(|G|). \end{cases} \]

The image of \( F_{\text{chr}} \) is \textit{WSym}. For any indexed graph \( G \):
\[ F_{\text{chr}}(G) = \sum_{\varpi \in \text{EP}(G)} W_{\varpi}. \]
Proof. 2. This is proved in the same way as Theorem 28, replacing the polynomial representation rep by Rep. The formula for $F_{\chr}(G)$ is immediate.

1. So $\WSym$ is a Hopf subalgebra of $\WQSym$, isomorphic to a quotient of $\mathcal{H}_g$, so is co-commutative.

Remark. The Hopf algebra $\WSym$, known as the Hopf algebra of word symmetric functions, is described and used in [4, 6, 15]. Here is a description of its product and coproduct, with immediate notations:

- For any set partitions $\varpi$, $\varpi'$ of respective degree $m$ and $n$:
  \[ W_{\varpi}W_{\varpi'} = \sum_{\varpi'' \in \mathcal{SP}, \deg(\varpi'') = k + l, \text{Pack}(\varpi''_{[k]}) = \varpi, \text{Pack}(\varpi''_{[k+l]}) = \varpi'} W_{\varpi''}. \]

- For any set partition $\varpi = \{P_1, \ldots, P_k\}$:
  \[ \Delta(P_w) = \sum_{I \subseteq [k]} W_{\text{Pack}(\{I_p | p \in I\})} \otimes W_{\text{Pack}(\{I_p | p \notin I\})}. \]

For example:

\[
\begin{align*}
W_{\{1,2\}}W_{\{1\}} &= W_{\{1,2,3\}} + W_{\{1,2,3\}}, \\
W_{\{1,2\}}W_{\{1\}} &= W_{\{1,2,3\}} + W_{\{1,2,3\}} + W_{\{1,3,2\}} + W_{\{1,2,3\}}, \\
\Delta(W_{\{1,3\}}W_{\{2\}}W_{\{4\}}) &= W_{\{1,3,2,4\}} \otimes 1 + W_{\{1,3,2\}} \otimes W_{\{1\}} + W_{\{1,2,3\}} \otimes W_{\{1\}} \\
&\quad + W_{\{1,2\}} \otimes W_{\{1,2\}} + W_{\{1,2\}} \otimes W_{\{1,2\}} + W_{\{1\}} \otimes W_{\{1,2,3\}} \\
&\quad + W_{\{1\}} \otimes W_{\{1,3,2\}} + 1 \otimes W_{\{1,3,2,4\}}.
\end{align*}
\]

4.4 Non-commutative version of $F_0$

We shall use the notations of Proposition 30. If $G$ be an indexed graph and $f \in \mathcal{PC}(G)$, then $G/\sim_f$ is an indexed graph; we denote its cardinality by $k$. We put:

\[ w_f = \bar{f}(1) \ldots \bar{f}(k). \]

Proposition 36 Let us consider the following map:

\[ F_0 : \begin{cases} 
\mathcal{H}_g & \to \WSym \\
G & \to \sum_{f \in \mathcal{PC}(G)} w_f.
\end{cases} \]

This is a Hopf algebra morphism. Moreover, in $\mathcal{E}_{\mathcal{H}_g} \to \WSym$:

\[ F_{\chr} = F_0 \leftarrow \lambda_{\chr}. \]

Proof. This is proved in the same way as Proposition 31. \qed

Examples.

\[
\begin{align*}
F_0(1) &= (1), \\
F_0(1^2) &= 2(11) + (1), \\
F_0(\bar{V}_3^1) &= 6(111) + 3(12) + 3(21) + (1), \\
F_0(\bar{V}_3^2) &= 6(111) + (122) + (211) + 2(12) + 2(21) + (1).
\end{align*}
\]
4.5 From non-commutative to commutative

As \( \mathbb{Q}[\langle X \rangle] \) is a quotient of \( \mathbb{Q}\langle\langle X \rangle\rangle \), this polynomial representations \( \text{Rep} \) of \( \text{WQSym} \) and \( \text{rep} \) of \( \text{QSym} \) induce a surjective Hopf algebra morphism:

\[
\pi : \begin{cases} 
\text{WQSym} & \longrightarrow \text{QSym} \\
W & \longrightarrow M_{[w^{-1}(1)], \ldots, [w^{-1}(\text{max}(w))]} 
\end{cases}
\]

Proposition 37 \( \pi \circ F_0 = F_0 \circ \varpi \) and \( \pi \circ F_{\text{chr}} = F_{\text{chr}} \circ \varpi \).

Proof. Immediate. \( \square \)

We obtain commutative diagrams of Hopf algebra morphisms:

\[
\begin{array}{ccc}
\text{WQSym} & \xrightarrow{\pi} & \text{QSym} \\
\downarrow F_{\text{chr}} & & \downarrow F_{\text{chr}} \\
\mathcal{H}_G & \xrightarrow{H} & \mathbb{Q}[X] \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
\text{WQSym} & \xrightarrow{\pi} & \text{QSym} \\
\downarrow \varphi_0 & & \downarrow \varphi_0 \\
\mathcal{H}_G & \xrightarrow{H} & \mathbb{Q}[X] \\
\end{array}
\]

References


32