From non-unitary wheeled PROPs to smooth amplitudes and generalised convolutions

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Abstract

We introduce the concept of TRAP (Traces and Permutations), which can roughly be viewed as a wheeled PROP (Products and Permutations) without unit. TRAPs are equipped with a horizontal concatenation and partial trace maps. Continuous morphisms on an infinite dimensional topological space and smooth kernels (respectively, smoothing operators) on a closed manifold form a TRAP but not a wheeled PROP. We build the free objects in the category of TRAPs as TRAPs of graphs and show that a TRAP can be completed to a unitary TRAP (or wheeled PROP). We further show that it can be equipped with a vertical concatenation, which on the TRAP of linear homomorphisms of a vector space, amounts to the usual composition. The vertical concatenation in the TRAP of smooth kernels gives rise to generalised convolutions. Graphs whose vertices are decorated by smooth kernels (respectively, smoothing operators) on a closed manifold form a TRAP. From their universal properties we build smooth amplitudes associated with the graph.

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Contents

Introduction

State of the art

PROPs (Products and Permutations)[1] provide an algebraic structure that allows to deal with operations with an arbitrary number of inputs and outputs. They generalise many other algebraic structures such as operads, which have one output and multiple inputs. PROPs appeared in [ML65] and later in the book [BV73] in the context of cartesian categories. Although operads stemmed from the study of iterated loop spaces in algebraic topology, see for example [May72], their origin can also be traced back to the earlier work [BV68][2].

An important asset of PROPs over operads is that they encompass algebraic structures such as bialgebras and Hopf algebras that lie outside the realm of operads or co-operads. This very fact is a motivation to consider PROPs in the context of renormalisation in quantum field theory [BS07]. We refer the reader to [Pir02] for the study of bialgebras in the PROPs framework and [Mar08, YJ15, ML65] for other classical examples of PROPs. In recent years, wheeled PROPs [Mer04, MMS09], which allow for loops, have played an important role in the context of deformation quantisation.

A central example of PROP is the PROP $\mathbf{Hom}_V$ of homomorphisms of a finite dimensional vector space V which we generalise to the PROP $\mathbf{Hom}_V^c$ of continuous homomorphisms of a nuclear Fréchet space V. Whereas the first is a wheeled PROP (Proposition 2.1.1), the latter is not unless the space V is finite dimensional (Theorem 2.2.8). It can nevertheless be interpreted

[1] The traditional notation is PROP, more recently prop.

[2] We thank B. Vallette for his enlightening comments on these historical aspects.

as a TRAP (Definition 1.1.1), which roughly speaking, amounts to a wheeled PROP without unit[3]. TRAPs introduced in this paper offer natural structures to host morphisms of infinite dimensional spaces (see Proposition 2.1.2) and are therefore expected to play a role in the context of renormalisation in quantum field theory.

Another class of important examples we consider are TRAPs of graphs (Proposition 3.2.1) of various types. In the context of deformation quantisation, the complex of oriented graphs whether directed or wheeled, plays an important role in the construction of a free PROP generated by a $\mathfrak{S} \times \mathfrak{S}^{op}$-module (see e.g [Mer04, Paragraph 2.1.3]). We will see that graphs play a similar role in the context of TRAPs.

Our first long term goal is to use the TRAP structure of graphs decorated by distribution (for example Green kernels) in order to build amplitudes as generalised convolutions (called P-amplitudes, see Definition 4.3.1) of kernels associated with the decorated graph. The expected singularities of the resulting amplitudes are immediate obstacles in defining such generalised convolutions. In this paper, we focus on the smooth setup, in which case the amplitudes are smooth.

Feynman rules and TRAPs

In space-time variables, a Feynman rule is expected to assign to a graph G with k incoming and l outgoing edges, an amplitude (it is actually a distribution) K_G in $k+l$ variables. Our second long term goal is to derive the existence and the properties of the map $G \mapsto K_G$ from a universal property of the PROP structure on graphs.

By means of blow-up methods, generalised convolutions of Green functions were built on a closed Riemannian manifold in [DZ21], with the goal of renormalising multiple loop amplitudes for Euclidean QFT on Riemannian manifolds. We hope to be able to simplify the intricate analytic aspects of the renormalisation procedure for multiple loop amplitudes, by adopting an algebraic point of view on amplitudes using TRAPs. There were earlier attempts to describe QFT theories in terms of PROPs (see for example [Ion07a, Ion07b]), yet to our knowledge, none with the focus we are putting on generalised convolutions to describe amplitudes.

We therefore expect the wheeled PROP of oriented graphs, briefly mentioned in [Ion07b], and more specifically TRAPs, their non-unitary counterparts which naturally arise in the infinite-dimensional set-up, to have concrete applications in the perturbative approach to quantum field theory. To our knowledge, this is yet an unexplored aspect of the theory. Filling in this gap is a long term goal we have in mind. A first step towards this goal is the study of the TRAP of smoothing symbols (Theorem 2.3.1), which like $\mathbf{Hom}_V^c$ is not a wheeled PROP due to the infinite dimensional spaces it involves.

TRAPs of graphs

TRAPs and unitary TRAPs entail two operations, the horizontal concatenation, and the partial trace map. The difference between TRAPs and unitary TRAPs is the existence of a unit for the trace in the latter. We define a TRAP structure on various families of graphs, which can be corolla ordered (Definition 3.1.1) or decorated (Proposition 3.2.1). The horizontal concatenation of this TRAP is the natural concatenation of graphs and the partial trace map consists in gluing together one of the inputs with one of the outputs, and therefore assigns to a graph G with k incoming and l outgoing edges a graph with $k-1$ incoming and $l-1$ outgoing edge. The set of corolla ordered graphs $\mathbf{CGr}^{\circlearrowleft}$ equipped with the partial trace map builds a unitary TRAP, and we prove that it is a free unitary TRAP (Theorem 3.3.1): this is the TRAP counterpart of a similar statement for free PROPs, described in terms of graphs without loops [Mar08, Proposition 57] and [Val03, Val04, YJ15]. More generally, the set of corolla ordered graphs $\mathbf{CGr}^{\circlearrowleft}(X)$ decorated

[3]To our knowledge, wheeled PROPs without units do not appear in the literature, which is why we allow ourselves to give them a shorter name.

by a set X on their vertices is the free unitary TRAP generated by X. These unitary TRAPs contain free nonunitary TRAPs, which are combinatorially described by particular graphs, which we call solar[4].

From TRAPs to unitary TRAPs

If V is a finite-dimensional vector space, then the PROP $\mathbf{Hom}_V$ of homomorphisms of V is a unitary TRAP, with the usual trace of endomorphisms. Its unit as a TRAP is the identity map of V. When V is not finite-dimensional, one cannot equip the whole PROP $\mathbf{Hom}_V$ with a structure of unitary TRAP. In this case, one has to restrict to smaller classes of homomorphisms, such as that of the homomorphisms of finite rank. This class no longer contains the identity, and we only obtain a TRAP and not a unitary TRAP (Proposition 2.1.2). To circumvent this difficulty, we construct for any TRAP P a unitary TRAP $\mathbf{uPGr}^{\circlearrowleft}(P)$ which contains P (Theorem 3.4.2). This object is characterised by a universal property (Proposition 3.5.1), which amounts to applying the left adjoint to the forgetful functor from the category of unitary TRAPs to the category of TRAPs. The existence of this functor comes from the inclusion of corolla ordered solar graphs, describing nonunitary free TRAPs, in the set of corolla ordered graphs, describing unitary free TRAPs. In particular, in $\mathbf{uPGr}^{\circlearrowleft}(P)$ an identity I is added, as well as its trace, symbolised by an abstract element $\mathcal{O}$, which is no longer an element of the base field $\mathbb{K}$.

The vertical concatenation

The vertical concatenation on wheeled PROPs previously considered in [YJ15, Definition 11.33] generalises the composition of morphisms. Indeed, for a finite-dimensional space V, the vertical concatenation of the TRAP $\mathbf{Hom}_V$ coincides with the usual composition of linear maps $f : V^{\otimes k} \longrightarrow V^{\otimes l}$ and the associativity of the vertical concatenation amounts to the Fubini property (Theorem 4.3.4, 2.).

When applied to a general unitary TRAP, this construction yields a functor from unitary TRAPs to PROPs (Proposition 4.1.2).

On graphs, the composition can roughly be described as follows. If G is a graph with k inputs and l outputs, and G' is a graph with l inputs and m outputs, $G' \circ G$ is obtained by gluing together the outgoing edges of G and the incoming edges of G' according to their indexation, giving a graph with k inputs and m outputs.

Extending this to the infinite dimensional setup requires the use of a completed tensor product $\widehat{\otimes}$ in order to have an isomorphism

$$\mathbf{Hom}^c_V(k,l) \cong \left(V'\right)^{\widehat{\otimes} k} \widehat{\otimes} V^{\widehat{\otimes} l},$$

where $\mathbf{Hom}^c_V(k,l)$ stands for the algebra of continuous morphisms from $V^{\widehat{\otimes} l}$ to $V^{\widehat{\otimes} k}$ (see Definition 2.2.7) and V' for the topological dual of a topological space V. This holds in the framework of Fréchet nuclear spaces which form a monoidal category under the completed tensor product (Lemma 2.2.4). On Fréchet nuclear spaces, the composition can indeed be described as a dual pairing, so it comes as no surprise that for a Fréchet nuclear space V, the vertical concatenation obtained from the nonunitary TRAP structure is the usual composition.

Generalised traces

A TRAP inherits a generalised trace defined on its elements with the same number of inputs and outputs. Roughly speaking, generalised traces are obtained by grafting the outputs to the inputs according to their indexation. These traces on TRAPs generalise the usual trace of morphisms, and they also enjoy a cyclicity property (Proposition 4.2.2).

[4] In [YJ15] such graphs are called ordinary.

When V is a space of smooth functions on a closed Riemannian manifold M, the associativity of the vertical concatenation amounts to the Fubini property (Theorem 4.3.4, 2.) and the generalised trace of a generalised kernel K with k inputs and k outputs is given by the integration of K along the small diagonal of M^k (Theorem 4.3.4, 3.).

Amplitude of a graph decorated by a TRAP

As mentioned above, our goal in the present paper is to provide an adequate algebraic and analytic framework in which we build generalised convolution functions associated with graphs decorated with smooth kernels. We show that these form a TRAP (Theorem 2.3.1), whose partial trace maps are given by a partial convolution.

When P is a TRAP, the universal property of the TRAP of corolla ordered graphs decorated by P gives rise to a canonical TRAP map, which associates to any such graph G an element of P which we call the P-amplitude associated with G (Definition 4.3.1). The P-amplitude commutes with both horizontal and vertical concatenation of P (Proposition 4.3.2). When applied to the TRAP of smooth generalised kernels, this construction generalises the usual convolution of kernels (Remark 4.3.3) and gives rise to smooth amplitudes (Theorem 4.3.4, 4.).

Unitary TRAPs and wheeled PROPs

A unitary TRAP is known in the literature under the name of wheeled PROP. In order to prove that the two notions coincide, we describe TRAPs and unitary TRAPs as algebras over a monad (see Definition 5.2.1) which generalises the notion of monoid to the frame of category theory. We state that unitary TRAPS are algebras over a monad $\Gamma^{\circlearrowleft}$ of graphs, described as an endofunctor of a category of modules over symmetric groups sending an object X to the free unitary TRAP of graphs $\mathbf{sol}\Gamma^{\circlearrowleft}(X)$ generated by X. When X is a unitary TRAP, $\Gamma^{\circlearrowleft}(X)$ inherits a contraction operation to X, which induces the monadic structure (Theorem 5.3.1). This monad $\Gamma^{\circlearrowleft}$ turns out to be the monad used to defined wheeled PROPs in the literature [YJ15, Corollary 11.35], thus relating our presentation of unitary TRAPs in terms of a family of sets with maps satisfying a set of axioms and the categorical presentation of wheeled PROPs in terms of algebras over a particular monad (Remark 5.3.1). A similar result holds for (nonunitary) TRAPs, replacing graphs by solar graphs introduced in Definition 3.1.1 (Theorem 5.3.1).

Openings

To sum up, by means of a TRAP structure, we were able to build generalised convolutions (respectively, traces) associated with graphs decorated with smooth kernels. As announced at the beginning of the introduction, we expect this algebraic approach to enable us to tackle non smooth kernels and thus to describe (non necessarily smooth) amplitudes as generalised convolutions of distribution kernels associated with graphs. At this stage these are open questions that we hope to address in future work.

There are other possible natural generalisations of the framework presented here, that are more algebraic in nature[5]. One could consider coloured TRAPs, whose input and output edges are coloured, and whose partial trace maps relate inputs and outputs of the same colour. Such structures are expected to play a role in QFTs with more than one type of particles (for example QED and QCD). Coloured TRAPs could also be relevent in the more geometric context of maps between different manifolds or in the context of modules over an algebra.

There are also potential generalisations of Theorem 2.3.1, which only requires that there be enough integral-like objects to define the partial trace maps, thereby hinting to the fact that more general spaces than the ones considered here should also carry TRAPs structures. Weakened versions of C^*-algebras such as inverse limits of C^*-algebras [Phi88] and locally multiplicative convex C^*-algebras [JJ06] would be worth investigating in that context.

[5]We thank Mark Johnson for pointing out the subsequent interesting questions to us.

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1 The category of TRAPs

1.1 Definition

Notation 1.1.1. For any $k \in \mathbb{N}_0$, we write $[k] := \{1, \cdots, k\}$. In particular, $[0] = \varnothing$. Let $\mathfrak{S}_k$ denote the symmetric group on k elements. An element $\sigma \in \mathfrak{S}_k$ sends $i \in [k]$ to $\sigma(i) \in [k]$.

Definition 1.1.1. *A **TRAP** is a family $P = (P(k,l))_{k,l\geqslant 0}$ of sets, equipped with the following structures:*

1. *P is a $\mathfrak{S} \times \mathfrak{S}^{op}$-module, that is to say, for any (k,l) in $\mathbb{N}_0^2$, $P(k,l)$ has a left $\mathfrak{S}_l-$ and a right $\mathfrak{S}_k^{op}$-action given by:*

$$\begin{cases} \mathfrak{S}_l \times P(k,l) & \longrightarrow & P(k,l) \\ (\sigma, p) & \longmapsto & \sigma \cdot p, \end{cases} \qquad \begin{cases} P(k,l) \times \mathfrak{S}_k & \longrightarrow & P(k,l) \\ (p, \tau) & \longmapsto & p \cdot \tau, \end{cases}$$

   such that for any (k,l) in $\mathbb{N}_0^2$, for any $(\sigma, \sigma', \tau, \tau') \in \mathfrak{S}_l^2 \times \mathfrak{S}_k^2$, for any $p \in P(k,l)$,

$$\mathrm{Id}_{[l]} \cdot p = p \cdot \mathrm{Id}_{[k]} = p,$$
$$\sigma \cdot (\sigma' \cdot p) = (\sigma\sigma') \cdot p, \qquad \sigma \cdot (p \cdot \tau) = (\sigma \cdot p) \cdot \tau, \qquad (p \cdot \tau) \cdot \tau' = p \cdot (\tau\tau').$$

2. *For any (k,l,k',l') in $\mathbb{N}_0^4$, there is a map*

$$* : \begin{cases} P(k,l) \times P(k',l') & \longrightarrow & P(k+k', l+l') \\ (p, p') & \longmapsto & p * p', \end{cases}$$

   *called the **horizontal concatenation**, such that:*

   (a) *(Associativity). For any (k,l,k',l',k'',l'') in $\mathbb{N}_0^6$, for any (p,p',p'') in $P(k,l)\times P(k',l')\times P(k'',l'')$,*

$$(p * p') * p'' = p * (p' * p'').$$

   (b) *(Unity). There exists $I_0 \in P(0,0)$ such that for any (k,l) in $\mathbb{N}_0^2$, for any p in $P(k,l)$,*

$$I_0 * p = p * I_0 = p.$$

   (c) *(Equivariance). For any (k,l,k',l') in $\mathbb{N}_0^4$, for any (p,p') in $P(k,l) \times P(k',l')$, for any $(\sigma, \tau, \sigma', \tau') \in \mathfrak{S}_l \times \mathfrak{S}_k \times \mathfrak{S}_{l'} \times \mathfrak{S}_{k'}$,*

$$(\sigma \cdot p \cdot \tau) * (\sigma' \cdot p' \cdot \tau') = (\sigma \otimes \sigma') \cdot (p * p') \cdot (\tau \otimes \tau'),$$

   where, for any $(\alpha, \beta) \in \mathfrak{S}_m \times \mathfrak{S}_n$, $\alpha \otimes \beta \in \mathfrak{S}_{m+n}$ is defined by

$$\alpha \otimes \beta(i) = \begin{cases} \alpha(i) \text{ if } i \leqslant m, \\ \beta(i-m) + m \text{ if } i > m. \end{cases}$$

(d) *(Commutativity). For any* (k,l,k',l') *in* $\mathbb{N}_0^4$*, for any* p *in* $P(k,l)$*,* p' *in* $P(k',l')$*,*

$$c_{l,l'} \cdot (p * p') = (p' * p) \cdot c_{k,k'},$$

where for any (m,n) *in* $\mathbb{N}_0^2$*,* $c_{m,n}$ *in* $\mathfrak{S}_{m+n}$ *is defined by:*

$$c_{m,n}(i) = \begin{cases} i+n \text{ if } i \leqslant m, \\ i-m \text{ if } i > m. \end{cases} \tag{1}$$

3. *For any* $k,l \geqslant 1$*, for any* $i \in [k]$*,* $j \in [l]$*, there is a map*

$$t_{i,j} : \begin{cases} P(k,l) & \longrightarrow & P(k-1,l-1) \\ p & \mapsto & t_{i,j}(p), \end{cases} \tag{2}$$

called the ***partial trace map****, such that:*

(a) *(Commutativity). For any* $k,l \geqslant 2$*, for any* $i \in [k]$*,* $j \in [l]$*,* $i' \in [k-1]$*,* $j' \in [l-1]$*,*

$$t_{i',j'} \circ t_{i,j} = \begin{cases} t_{i-1,j-1} \circ t_{i',j'} \text{ if } i' < i,\ j' < j, \\ t_{i,j-1} \circ t_{i'+1,j'} \text{ if } i' \geqslant i,\ j' < j, \\ t_{i-1,j} \circ t_{i',j'+1} \text{ if } i' < i,\ j' \geqslant j, \\ t_{i,j} \circ t_{i'+1,j'+1} \text{ if } i' \geqslant i,\ j' \geqslant j. \end{cases}$$

(b) *(Equivariance). For any* $k,l \geqslant 1$*, for any* $i \in [k]$*,* $j \in [l]$*,* $\sigma \in \mathfrak{S}_l$*,* $\tau \in \mathfrak{S}_k$*, for any* $p \in P(k,l)$*,*

$$t_{i,j}(\sigma \cdot p \cdot \tau) = l_j(\sigma) \cdot (t_{\tau(i),\sigma^{-1}(j)}(p)) \cdot r_i(\tau),$$

with the following notation: if $\alpha \in \mathfrak{S}_n$ *and* $q \in [n]$*, then* $(l_q(\alpha), r_q(\alpha)) \in \mathfrak{S}_{n-1}^2$ *are defined by*

$$l_q(\alpha)(s) = \begin{cases} \alpha(s) \text{ if } s < \alpha^{-1}(q) \text{ and } \alpha(s) < q, \\ \alpha(s)-1 \text{ if } s < \alpha^{-1}(q) \text{ and } \alpha(s) > q, \\ \alpha(s+1) \text{ if } s \geqslant \alpha^{-1}(q) \text{ and } \alpha(s+1) < q, \\ \alpha(s+1)-1 \text{ if } s \geqslant \alpha^{-1}(q) \text{ and } \alpha(s+1) > q, \end{cases}$$

$$r_q(\alpha)(s) = \begin{cases} \alpha(s) \text{ if } s < q \text{ and } \alpha(s) < \alpha(q), \\ \alpha(s)-1 \text{ if } s < q \text{ and } \alpha(s) > \alpha(q), \\ \alpha(s+1) \text{ if } s \geqslant q \text{ and } \alpha(s+1) < \alpha(q), \\ \alpha(s+1)-1 \text{ if } s \geqslant q \text{ and } \alpha(s+1) > \alpha(q). \end{cases}$$

In other words, if we represent α *by the word* $\alpha(1)\ldots\alpha(n)$*, then* $l_q(\alpha)$ *is represented by the word obtained by deleting the letter* q *and substracting* 1 *to the letters* $> q$*, whereas* $r_q(\alpha)$ *is represented by the word obtained by deleting the letter* $\alpha(q)$ *and substracting* 1 *to the letters* $> \alpha(q)$*. Note that* $r_q(\alpha) = l_{\alpha(q)}(\alpha)$*.*

(c) *(Compatibility with the horizontal concatenation). For any* $k,l,k',l' \geqslant 1$*, for any* $i \in [k+l]$*,* $j \in [k'+l']$*, for any* $p \in P(k,l)$*,* $p' \in P(k',l')$*:*

$$t_{i,j}(p * p') = \begin{cases} t_{i,j}(p) * p' \text{ if } i \leqslant k,\ j \leqslant l, \\ p * t_{i-k,j-l}(p') \text{ if } i > k,\ j > l. \end{cases}$$

The TRAP is ***unitary*** *if moreover there exists I in $P(1,1)$ such that for any $k,l \geqslant 1$, for any i in $[k+1]$, $j \in [l+1]$, for any p in $P(k,l)$:*

$$t_{1,j}(I * p) = (1, 2, \ldots, j-1) \cdot p \text{ if } j \geqslant 2,$$
$$t_{i,1}(I * p) = p \cdot (1, 2, \ldots, i-1)^{-1} \text{ if } i \geqslant 2,$$
$$t_{k+1,j}(p * I) = (j, j+1, \ldots, k)^{-1} \cdot p \text{ if } j \leqslant k,$$
$$t_{i,l+1}(p * I) = p \cdot (i, i+1, \ldots, l) \text{ if } i \leqslant l.$$

Remark 1.1.1. By commutativity of $*$, for any p in $P(0,0)$, for any (k,l) in $\mathbb{N}_0^2$, for any p' in $P(k,l)$:

$$p * p' = p' * p,$$

since $c_{0,k} = \mathrm{Id}_{[k]}$.

Remark 1.1.2. The abuse of notation $t_{i,j}$ is legitimate since a full notation such as $t_{i,j}^{k,l}$ is not necessary in practice. Indeed the indices k and l in $t_{i,j}(p)$ are entirely determined by the element p to which $t_{i,j}$ is applied.

More so, if P is unitary, $t_{i,j}(p)$ does not strongly depend on k and l determined by p: indeed, let $f : P(k,l) \longrightarrow P(k+1, l+1)$ be the map that sends p to $p * I$ then for $i \in [k]$ and $j \in [l]$, we have

$$t_{i,j} \circ f(p) = f \circ t_{i,j}(p),$$

which is the axiom 3.(c).

Remark 1.1.3. In Section 5 we will show that TRAPs can be described as algebras over a certain monad and we will use this to prove (Theorem 5.3.1) that the category of unitary TRAPs, defined below, is isomorphic to the category of wheeled PROPs introduced in [Mer06]. Our axiomatic approach is tailored to address analytic issues regarding products of singularities and their application to Feynman graphs in QFT.

We will use in Section 2 the axiomatic approach of Definition 1.1.1 to show that known analytic and geometric spaces carry TRAP structures. However, the categorical approach seems better suited for classification problems, for example regarding the solutions of the master equation in the BV formalism [MMS09, Mer10].

Definition 1.1.2. *We define a* **sub-TRAP** *of a TRAP $P = (P(k,l))_{k,l\geqslant 0}$ to be a $\mathfrak{S} \times \mathfrak{S}^{op}$-submodule $Q = (Q(k,l))_{k,l\geqslant 0}$ of P which contains the unit $I_0 \in P(0,0)$ and is stable under the partial trace map of P. If the TRAP P is unitary, then the sub-trap Q is unitary if it contains the unit $I \in P(1,1)$.*

Definition 1.1.3. *Let $P = (P(k,l))_{k,l\geqslant 0}$ and $Q = (Q(k,l))_{k,l\geqslant 0}$ be two TRAPs with partial trace maps $(t^P_{i,j})_{i,j\geqslant 0}$ and $(t^Q_{i,j})_{i,j\geqslant 0}$ respectively. A* ***morphism of TRAPs*** *is a family $\phi = (\phi(k,l))_{k,l\geqslant 0}$ of morphisms of $\mathfrak{S} \times \mathfrak{S}^{op}$-modules $\phi(k,l) : P(k,l) \longrightarrow Q(k,l)$ which are compatible with the horizontal concatenation, and the partial trace maps. More precisely, for any $(k,l,m,n) \in \mathbb{N}_0^4$:*

1. *For any (σ, p, τ) in $\mathfrak{S}_l \times P(k,l) \times \mathfrak{S}_k$, $\phi(k,l)(\sigma \cdot p \cdot \tau) = \sigma \cdot \phi(k,l)(p) \cdot \tau$.*
2. *$\phi(0,0)(I_0) = I_0$.*
3. *$\forall (p,q) \in P(k,l) \times P(n,m)$, $\phi(k+n, l+m)(p * q) = \phi(k,l)(p) * \phi(n,m)(q)$,*
4. *$\forall (p,i,j) \in P(k,l) \times [k] \times [l]$, $\phi(k-1, l-1) \circ t^P_{i,j}(p) = t^Q_{i,j} \circ \phi(k,l)(p)$.*

With a slight abuse of notation, we write $\phi(p)$ instead of $\phi(k,l)(p)$ for $p \in P(k,l)$. We denote by **TRAP** *the category of TRAPs and TRAPs morphisms.*

If P and Q are unitary TRAPs with units I_P and I_Q and $\phi : P \longrightarrow Q$ is a morphism of TRAPs, this morphism is **unitary** *if $\phi(1,1)(I_P) = I_Q$. We denote by* **uTRAP** *the subcategory of* **TRAP** *whose objects are unitary TRAPs and morphisms are unitary TRAP morphisms.*

In the following two Lemmas we identify conditions for a collection of $\mathfrak{S} \times \mathfrak{S}^{op}$-modules and linear maps between TRAPs to carry a TRAP structure and a be a TRAP morphism respectively.

Lemma 1.1.4. *Let $P = (P(k,l))_{k,l\geqslant 0}$ be a $\mathfrak{S} \times \mathfrak{S}^{op}$-module, equipped with a horizontal concatenation $*$ satisfying axioms 2. (a)-(d), and with maps $t_{i,j}$ satisfying axioms 3. (a)-(b).*

1. *Assuming that for any $k, l, k', l' \geqslant 1$, for any $p \in P(k,l)$, $p' \in P(k',l')$,*
$$t_{1,1}(p * p') = t_{1,1}(p) * p',$$
then Axiom 3.(c) is satisfied.

2. *Let $I \in P(1,1)$. We assume for any $k, l \geqslant 1$, for any $p \in P(k,l)$,*
$$t_{1,2}(I * p) = p.$$
Then I is a unit of P.

Proof. Let $p \in P(k,l)$ and $p' \in P(k',l')$. We take i in $[k+l]$, j in $[k'+l']$ and define the transpositions $\sigma = (1,j)$, $\tau = (1,i)$, with the convention $(1,1) = \mathrm{Id}$. We use the notation $\sigma_j := l_j(\sigma)$ and $\tau_i := r_i(\tau)$. Let us consider several cases.

- If $i \leqslant k$ and $j \leqslant l$, then:
$$\begin{aligned}
t_{i,j}(p * p') &= t_{i,j}(\sigma^2 \cdot (p * p') \cdot \tau^2) \\
&= \sigma_j \cdot t_{1,1}(\sigma \cdot (p * p') \cdot \tau) \cdot \tau_i \quad \text{by equivariance of } t_{i,j} \text{ (Axiom 3}b \\
&\qquad \text{of Definition 1.1.1)}, \\
&= \sigma_j \cdot (t_{1,1}((\sigma \cdot p \cdot \tau) * p')) \cdot \tau_i \quad \text{since } i \leqslant k,\ j \leqslant l \text{ and by Axiom 2}c \\
&\qquad \text{of Definition 1.1.1}, \\
&= \sigma_j \cdot (t_{1,1}(\sigma \cdot p \cdot \tau) * p') \cdot \tau_i \quad \text{by the assumption of the Lemma}, \\
&= (\sigma_j \cdot (t_{1,1}(\sigma \cdot p \cdot \tau) \cdot \tau_i)) * p' \quad \text{since } i \leqslant k,\ j \leqslant l \text{ and by Axiom 2}c \\
&\qquad \text{of Definition 1.1.1}, \\
&= t_{i,j}(p) * p' \quad \text{by equivariance of } t_{i,j}.
\end{aligned}$$

- If $i > k$ and $j > l$, using $c_{m,n}^{-1} = c_{n,m}$, and as before writing $(c_{l',l})_j := l_j(c_{l',l})$ and $(c_{k,k'})_i := r_i(c_{k,k'})$ we have
$$\begin{aligned}
t_{i,j}(p * p') &= t_{i,j}(c_{l',l} \cdot (p' * p) \cdot c_{k,k'}) \quad \text{by commutativity of } * \text{ (Axiom 2}d \\
&\qquad \text{of Definition 1.1.1)}, \\
&= (c_{l',l})_j \cdot t_{i-k,j-l}(p' * p) \cdot (c_{k,k'})_i \quad \text{by equivariance of } t_{i,j} \text{ (Axiom 3}b \\
&\qquad \text{of Definition 1.1.1)}, \\
&= c_{l'-1,l} \cdot (t_{i-k,j-l}(p') * p) \cdot c_{k,k'-1} \quad \text{By the first point}, \\
&= p * t_{i-k,j-l}(p') \quad \text{by commutativity of } *.
\end{aligned}$$
Thus Axiom 3.(c) is satisfied.

- Let us now take $j \geqslant 2$. In this case we have
$$\begin{aligned}
t_{1,j}(I * p) &= t_{1,j}((2,j)^2 \cdot (I * p)) \\
&= (2, \dots, j-1) \cdot t_{1,2}((2,j) \cdot (I * p)) \quad \text{by equivariance of } t_{1,j} \text{ (Axiom 3}b \\
&\qquad \text{of Definition 1.1.1)}, \\
&= (2, \dots, j-1) \cdot t_{1,2}((I * (1, j-1) \cdot p)) \quad \text{since } j \geqslant 2 \text{ and by Axiom 2}c \\
&\qquad \text{of Definition 1.1.1}, \\
&= (2, \dots, j-1) \cdot ((1, j-1) \cdot p) \quad \text{by the assumption of the Lemma}, \\
&= (2, \dots, j-1)(1, j-1) \cdot p \\
&= (1, \dots, j-1) \cdot p.
\end{aligned}$$

The other three relations are proved in the same way. Thus I is a unit of P. □

We can also simplify the axioms for morphisms of TRAPs.

Lemma 1.1.5. *Let $P=(P(k,l))_{k,l\geqslant 0}$ and $Q=(Q(k,l))_{k,l\geqslant 0}$ be two TRAPs and $\phi=(\phi(k,l))_{k,l\geqslant 0}$ be a family of set maps $\phi(k,l): P(k,l)\longrightarrow Q(k,l)$ satisfying Points 1-3 of Definition 1.1.3. Suppose further that for any $k,l\geqslant 1$, for any $p\in P(k,l)$*

$$t_{1,1}\circ\phi(p)=\phi\circ t_{1,1}(p).$$

Then ϕ is a map of TRAPs.

Proof. If i, j and p lies respectively in $[k]$, $[l]$, and $P(k,l)$, then

$$\begin{aligned}
\phi\circ t_{i,j}(p) &= \phi\circ t_{i,j}((1,j)^2\cdot p\cdot(1,i)^2)\\
&= \phi\left((1,j)\cdot t_{1,1}((1,j)\cdot p\cdot(1,i))\cdot(1,i)\right) \quad \text{by equivariance of } t_{i,j} \text{ (Axiom 3}b\\
&\qquad\qquad \text{of Definition 1.1.1),}\\
&= (1,j)\cdot\phi\circ t_{1,1}((1,j)\cdot p\cdot(1,i))\cdot(1,i) \quad \text{by Point 1 of Definition 1.1.3,}\\
&= (1,j)\cdot t_{1,1}\circ\phi((1,j)\cdot p\cdot(1,i))\cdot(1,i) \quad \text{by the assumption of the Lemma,}\\
&= t_{i,j}((1,j)\cdot\phi((1,j)\cdot p\cdot(1,i))\cdot(1,i)) \quad \text{by equivariance of } t_{i,j} \text{ (Axiom 3}b\\
&\qquad\qquad \text{of Definition 1.1.1),}\\
&= t_{i,j}\circ\phi(p) \quad \text{by Point 1 of Definition 1.1.3,,}
\end{aligned}$$

with the convention $(1,1)=\mathrm{Id}$. It follows that ϕ is a morphism of TRAPs. □

In particular, to show that a collection of linear maps between two TRAPs preserving the horizontal concatenation and the actions of the symmetry group is a morphism of TRAPs, it is enough to check the properties of Lemma 1.1.5.

1.2 Quotient of TRAPs

This paragraph prepares for the construction of an embedding of a TRAP P in a unitary TRAP.

Lemma 1.2.1. *Let Q be a TRAP and let $\sim$ be an equivalence relation on Q which is compatible with the TRAP-structure on Q in the following sense. For any two elements $x,x'\in Q$, such that $x\sim x'$:*

- *(Compatibility with the module-structure). For any $(\sigma,\tau)\in\mathfrak{S}_k\times\mathfrak{S}_l$, $\tau\cdot x\cdot\sigma\sim\tau\cdot x'\cdot\sigma$.*
- *(Compatibility with the horizontal concatenation). For any $y\in Q$, $x*y\sim x'*y$ and $y*x\sim y*x'$.*
- *(Compatibility with the partial trace maps). For any $(i,j)\in[k]\times[l]$, $t_{i,j}(x)\sim t_{i,j}(x')$.*

Then the quotient $Q/\sim$ is a TRAP with $[I_0]$ as unit for the concatenation product. If Q is unitary with unit $I\in Q(1,1)$ for the partial trace maps, then $Q/\sim$ is also unitary with $[I]$ as unit for the partial trace maps.

Proof. 1. By compatibility with the module structure, $Q/\sim$ is a $\mathfrak{S}\times\mathfrak{S}^{op}$-modules.

2. Using twice the compatibility with horizontal concatenation maps, we find that if $x\sim x'$ and $y\sim y'$, then $x*y\sim x'*y'$ by transitivity of $\sim$. Thus the horizontal concatenation $[x]*[y]:=[x*y]$ on $Q/\sim$ is well-defined. It fulfils properties 2.(a) to 2.(d) of Definition 1.1.1 by construction.

3. We defined the partial trace maps on the quotient to be $t_{i,j}([x]):=[t_{i,j}(x)]$. It is well defined by compatibility with the partial trace maps and has properties 3.(a) to 3.(c) of Definition 1.1.1 by construction.

Thus $Q/\sim$ is a TRAP. Finally, if Q is unitary with unit I, then $[I]$ endows the quotient $Q/\sim$ with a unit by construction and $Q/\sim$ is then a unitary TRAP. □

The following statement is a direct consequence.

Proposition 1.2.2. *Let P and Q be TRAPs and $\phi : Q \to P$ a TRAP-morphism. The relation*

$$x \sim x' \iff \phi(x) = \phi(x')$$

defines an equivalence relation compatible with the TRAP-structure on Q and $Q/\sim$ defines a TRAP.

Proof. Let x, x' in Q, such that $x \sim x'$.

- For any $(\sigma, \tau) \in \mathfrak{S}_k \times \mathfrak{S}_l$,

  $$\phi(\sigma \cdot x \cdot \tau) = \sigma \cdot \phi(x) \cdot \tau = \sigma \cdot \phi(x') \cdot \tau = \phi(\sigma \cdot x' \cdot \tau),$$

  where the first and last identities follow from the fact that ϕ is a morphism of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules. Thus $\sim$ is compatible with the module structure.

- For any y in Q we have

  $$\phi(x * y) = \phi(x) * \phi(y) = \phi(x') * \phi(y) = \phi(x' * y)$$

  where we have used the fact that ϕ is a morphism for the horizontal concatenation product (since ϕ is a morphism of TRAPs) and the fact that $\phi(x) = \phi(x')$. Thus $x * y \sim x' * y$. Similarly we show that $y * x \sim y * x'$ and $\sim$ is compatible with the horizontal concatenation.

- For any (i, j)
  sy in $[k] \times [l]$ we have

  $$\phi(t_{i,j}(x)) = t_{i,j}(\phi(x)) = t_{i,j}(\phi(x')) = \phi(t_{i,j}(x')),$$

  where the first and last identities follow from the fact that ϕ is a morphism of TRAPs.

  Thus $\sim$ is compatible with the partial trace maps. □

2 Fundamental examples

2.1 The Hom TRAP

Let us give a fundamental example of unitary TRAP:

Let V be a finite dimensional vector space and V^* its algebraic dual. We consider the family

$$\mathbf{Hom}_V = (\mathbf{Hom}_V(k,l))_{k,l\geqslant 0} := (\mathbf{Hom}(V^{\otimes k}, V^{\otimes l}))_{k,l\geqslant 0},$$

where for any $(k,l) \in \mathbb{N}_0^2$, $\mathbf{Hom}(V^{\otimes k}, V^{\otimes l})$ is the vector space of linear maps from $V^{\otimes k}$ to $V^{\otimes l}$. We shall identify $\mathbf{Hom}_V(k,l)$ and $V^{*\otimes k} \otimes V^{\otimes l}$ through the isomorphism

$$\theta_{k,l} : \left\{ \begin{array}{rcl} V^{*\otimes k} \otimes V^{\otimes l} & \longrightarrow & \mathbf{Hom}_V(k,l) \\ f_1 \dots f_k \otimes v_1 \dots v_l & \longmapsto & \left\{ \begin{array}{rcl} V^{\otimes k} & \longrightarrow & V^{\otimes l} \\ x_1 \dots x_k & \longmapsto & f_1(x_1) \dots f_k(x_k)\, v_1 \dots v_l, \end{array} \right. \end{array} \right.$$

where with some abuse of notation, we have set $f_1 \cdots f_k := f_1 \otimes \cdots \otimes f_k \in V^{*\otimes k}$ and $v_1 \cdots v_l := v_1 \otimes \cdots \otimes v_l \in V^{\otimes l}$. For any vector space W, the tensor power $W^{\otimes k}$ is a left $\mathfrak{S}_k$-module with the action defined by

$$\sigma \cdot w_1 \dots w_k = w_{\sigma^{-1}(1)} \dots w_{\sigma^{-1}(k)}.$$

Via the identification $\theta := (\theta_{k,l})_{k,l\geqslant 0}$, we can equip the family $\mathbf{Hom}_V = (\mathbf{Hom}(V^{\otimes k}, V^{\otimes l}))_{k,l\geqslant 0}$ with the structure of a $\mathfrak{S}_l \times \mathfrak{S}_k^{op}$-module by putting, for an $f \in \mathbf{Hom}(V^{\otimes k}, V^{\otimes l})$, for any $(\sigma, \tau) \in \mathfrak{S}_k \times \mathfrak{S}_l$:

$$\forall v_1 \dots v_k \in V^{\otimes l}, \qquad \tau \cdot f \cdot \sigma(v_1 \dots v_k) = \tau \cdot f(\sigma \cdot v_1 \dots v_k). \tag{3}$$

The horizontal concatenation is the usual tensor product of linear maps: if $f \in \mathbf{Hom}_V(k,l)$ and $g \in \mathbf{Hom}_V(k',l')$, then

$$f \otimes g : \left\{ \begin{array}{ccc} V^{\otimes(k+k')} & \longrightarrow & V^{\otimes(l+l')} \\ v_1 \dots v_{k+k'} & \longmapsto & f(v_1 \dots v_k) \otimes g(v_{k+1} \dots v_{k+k'}). \end{array} \right.$$

We define the following partial trace maps:

$$t_{i,j}(\theta_{k,l}(f_1 \dots f_k \otimes v_1 \dots v_l)) = f_i(v_j)\, \theta_{k-1,l-1}(f_1 \dots f_{i-1} f_{i+1} \dots f_k \otimes v_1 \dots v_{j-1} v_{j+1} \dots v_l). \tag{4}$$

Remark 2.1.1. Notice that for $p \in \mathbf{Hom}_V(1,1)$, $t_{1,1}(p)$ coincides with the usual trace of a linear map on V. Proposition 4.2.2 generalises the notion of trace to any element of $\mathbf{Hom}_V$ and to any TRAP.

Proposition 2.1.1. *For a finite dimensional $\mathbb{K}$-vector space V, the above construction equips $\mathbf{Hom}_V$ with a TRAP structure which is unitary, with unit given by the identity of V.*

Remark 2.1.2. We will see later in the paper (Theorem 5.3.1) that this implies that $\mathbf{Hom}_V$ is a wheeled PROP. Further details of $\mathbf{Hom}_V$ as a wheeled PROP can be found in [MMS09, Example 2.1.1]. In [DM19] (in particular Subsection 2.1, Section 6 and Section 7) the consequence of $\mathbf{Hom}_V$ carrying a wheeled PROP structure in the context of invariant theory are explored. $\mathbf{Hom}_V$ also appears for example in [KV21, Example 2.2] where questions of algebraic topology are studied.

Proof. Properties 2.(a)-(d) are trivially satisfied, with $I_0 = 1 \in \mathbb{K}$. Property 3.(a) is direct. Let us prove Property 3.(b).

$$\begin{aligned} t_{i,j}(\sigma \cdot \theta_{k,l}(f_1 \dots f_k \otimes v_1 \dots v_l) \cdot \tau) &= t_{i,j} \circ \theta_{k,l}(f_{\tau(1)} \dots f_{\tau(k)} \otimes v_{\sigma^{-1}(1)} \dots v_{\sigma^{-1}(l)}) \\ &= f_{\tau(i)}(v_{\sigma^{-1}(j)}) \theta_{k-1,l-1}(f_{\tau(1)} \dots f_{\tau(i-1)} f_{\tau(i+1)} \dots f_{\tau(k)} \\ &\otimes v_{\sigma^{-1}(1)} \dots v_{\sigma^{-1}(j-1)} v_{\sigma^{-1}(j+1)} \dots v_{\sigma^{-1}(l)}) \\ &= \sigma_j \cdot t_{\tau(i),\sigma^{-1}(j)} \theta_{k,l}(f_1 \dots f_k \otimes v_1 \dots v_l) \cdot \tau_i. \end{aligned}$$

Property 3.(c) is straightforward. Let us prove that $\mathbf{Hom}_V$ is unitary with the help of Lemma 1.1.4. Let us fix $(e_i)_{i\in I}$ a basis of V, then $(e_i^*)_{i\in I}$ is a basis of V^* and the identity map of V is

$$\mathrm{Id}_V = \theta_{1,1}\left(\sum_{i\in I} e_i \otimes e_i^*\right). \tag{5}$$

Then for any $p = \theta_{k,l}(f_1 \dots f_k \otimes v_1 \dots v_l) \in \mathbf{Hom}_V(k,l)$

$$\begin{aligned} t_{1,2}(\mathrm{Id}_V * \theta_{k,l}(f_1 \dots f_k \otimes v_1 \dots v_l)) &= \sum_{i\in I} t_{1,2} \circ \theta_{k+1,l+1}(e_i^* f_1 \dots f_k \otimes e_i v_1 \dots v_l) \\ &= \sum_{i\in I} \theta_{k,l}(f_1 \dots f_k \otimes e_i\, e_i^*(v_1) v_2 \dots v_l) \\ &= \theta_{k,l}(f_1 \dots f_k \otimes \mathrm{Id}_V(v_1) v_2 \dots v_l) \\ &= \theta_{k,l}(f_1 \dots f_k \otimes v_1 \dots v_l). \end{aligned}$$

So $\mathbf{Hom}_V$ is a unitary TRAP. □

When V is not finite-dimensional, θ is an injective, non surjective map. Its range is the subpsace $\mathbf{Hom}_V^{fr}$ of linear maps from $V^{\otimes k}$ to $V^{\otimes l}$ of finite rank. We can equip $\mathbf{Hom}_V^{fr}$ with a similar TRAP structure:

Proposition 2.1.2. *With the $\mathfrak{S} \times \mathfrak{S}^{op}$ action defined by (3), the usual tensor product of maps and the partial trace maps defined by (4), $\mathbf{Hom}_V^{fr}$ is a TRAP. It is unitary if, and only if, V is finite-dimensional.*

Proof. We skip the proof that $\mathbf{Hom}_V^{fr}$ can be equipped with a TRAP structure since it goes as for $\mathbf{Hom}_V$ when V is finite dimensional. Note that when V is finite-dimensional, then $\mathbf{Hom}_V^{fr} = \mathbf{Hom}_V$ is a unitary TRAP. We show the second part of the statement.

Let us assume that $\mathbf{Hom}_V^{fr}$ has a unit I. Then I has finite rank, let us fix a basis $(e_1, \dots, e_k)$ of $\mathrm{Im}(I)$. There exist $\lambda_1, \dots, \lambda_k \in V^*$ such that for any $v \in V$,

$$I(v) = \sum_{i=1}^{k} \lambda_i(v) e_i.$$

In other words,

$$I = \theta_{1,1}\left(\sum_{i=1}^{k} e_i \otimes \lambda_i\right).$$

Let $v \in V$, nonzero, and let $\lambda \in V^*$ such that $\lambda(v) = 1$. We consider $f = \theta_{1,1}(v \otimes \lambda)$. Then $f(v) = \lambda(v)v = v$. Moreover:

$$\begin{aligned} v = f(v) &= t_{1,2}(I * f)(v) \\ &= t_{1,2} \circ \theta_{2,2}\left(\sum_{i=1}^{k} e_i\, v \otimes \lambda_i \lambda\right)(v) \\ &= \theta_{1,1}\left(\sum_{i=1}^{k} \lambda_i(v)\, e_i \otimes \lambda\right)(v) \\ &= \sum_{i=1}^{k} \lambda(v)\lambda_i(v)\, e_i \\ &= \sum_{i=1}^{k} \lambda_i(v)\, e_i, \end{aligned}$$

so $v \in \mathrm{Vect}(e_1, \dots, e_k)$. Hence, $V \subseteq \mathrm{Im}(I)$, so V is finite-dimensional. □

We end this paragraph with an example of a TRAP similar to the TRAP $\mathbf{Hom}_V$ but of a more geometric nature.

Example 2.1.1 (The TRAP of tensors). Given a finite dimensional smooth manifold M and a point $x \in M$, we build the **Hom**-TRAP $\mathbf{Hom}_{T_xM}$ where T_xM is the tangent space to M at the point x. Given a pair $(p, q) \in \mathbb{N}_0^2$ we have, using the musical isomorphisms (see for example [Lee97, Chapter 3])

$$\mathbf{Hom}_{T_xM}(p, q) \cong (T_x^*M)^{\otimes p} \otimes T_xM^{\otimes q},$$

where we have set $V^{\otimes 0} = \mathbb{R}$. The partial trace maps are built by pairing cotangent and tangent vectors. We note that if M is equipped with a Riemannian metric, thanks to the musical isomorphisms $T_x^*M \ni \alpha \longmapsto \alpha^\sharp \in T_xM$ and $T_xM \ni v \longmapsto v^\flat \in T_x^*M$ between T_x^*M and T_xM, these dual pairings can be seen as contractions via the metric tensor.

This yields a smooth fibration $\mathbf{Hom}_{TM} := \{\mathbf{Hom}_{T_xM}, x \in M\}$ of TRAPs parametrised by M. For any $(p, q) \in \mathbb{Z}^2_{\geqslant 0}$, a smooth section of $\mathbf{Hom}_{TM}(p, q)$ defines a smooth (p, q) tensor on M.

2.2 The TRAP of continuous morphisms

We generalise the constructions of the previous paragraph, replacing the finite dimensional spaces $V^{\otimes k}$ in $\mathbf{Hom}_V$ by nuclear spaces. These nuclear spaces were defined in the seminal work [Gro54]. Most of the results stated here can be found in [Gro52, Gro54]. We also refer to the more recent presentation [Trè67].

We recall that:

- A topological vector space is **Fréchet** if it is Hausdorff, has its topology induced by a countable family of semi-norms and is complete with respect to this family of semi-norms.
- The topological dual E' of a locally convex topological vector space E can be endowed with various topologies, one of which is the **strong topology**, namely the topology of uniform convergence on the bounded domains of E. It is generated by the family of semi-norms of E' defined on any $f \in E'$ by $||f||_B := \sup_{x\in B} |f(x)|$ for any bounded set B of E. The topological dual E' endowed with this topology is called the **strong dual**.
- A topological vector space is called **reflexive** if $E'' = (E')' = E$, where E' is the topological dual of E endowed with the strong topology.

In the following E and F are two topological vector spaces and $\mathbf{Hom}^c(E,F)$ is the set of continuous linear maps from E to F.

Remark 2.2.1.
- When E and F are finite dimensional, we have $\mathbf{Hom}^c(E,F)$=$\mathbf{Hom}(E,F)$.
- As pointed out to us by Mark Johnson, a natural generalisation to consider in the context of Fréchet spaces are σ C^*-algebras, defined as inverse limits of C^*-algebras [Phi88], which however lie out of the scope of the present article.

In order to build the TRAP $\mathbf{Hom}^c_V$ for nuclear spaces, we need Grothendieck's completion of the tensor product, a notion we recall here in the set-up of locally convex topological $\mathbb{K}$-vector spaces.

Let E and F be two vector spaces. Recall that there exists a vector space $E\otimes F$, and a bilinear map $\phi : E \times F \longrightarrow E \otimes F$ such that for any vector space V and bilinear map $f : E \times F \longrightarrow V$, there is a unique linear map $\tilde{f} : E \otimes F \to V$ satisfying $f = \tilde{f} \circ \phi$. The space $E \otimes F$ is unique modulo isomorphism and is called the **tensor product** of E and F.

Given two topological vector spaces E and F, one can a priori equip $E\otimes F$ with several topologies, among which the **equicontinuous topology, or ϵ-topology** ([Trè67, Definition 43.1]) and the **projective topology**, or **π-topology** ([Trè67, Definition 43.2]), are of considerable importance. Their constructions are recalled in Appendix A. We denote by $E \otimes_\epsilon F$ (respectively, $E \otimes_\pi F$) the space $E \otimes F$ endowed with the ϵ-topology (respectively, the projective topology) and by $E\widehat{\otimes}_\epsilon F$ (respectively, $E\widehat{\otimes}_\epsilon F$) of $E \otimes_\epsilon F$ (respectively, $E \otimes_\epsilon F$) their completion with respect to the ϵ-topology (respectively, projective topology). These two spaces differ in general but coincide for nuclear spaces.

Definition 2.2.1. *[Gro54] A locally convex topological vector space E is **nuclear** if, and only if, for any locally convex topological vector space F,*

$$E\widehat{\otimes}_\epsilon F = E\widehat{\otimes}_\pi F =: E\widehat{\otimes} F$$

*holds, in which case $E\widehat{\otimes}F$ is called the **completed tensor product** of E and F.*

There are other equivalent definitions of nuclearity, see for example [GV64, HS08].

Remark 2.2.2. It was pointed out to us by Mark Johnson that such minimal and maximal tensor products, much used in the context of C^*-algebras, further extend to l.m.c. C^*-algebras, where l.m.c stands for locally multiplicative convex (see [JJ06] and references therein).

For Fréchet spaces, nuclearity is preserved under strong duality.

Proposition 2.2.2.
- *[Trè67, Proposition 50.6] A Fréchet space is nuclear if and only if its strong dual is nuclear.*
- *[Trè67, Proposition 36.5] A Fréchet nuclear space is reflexive.*

Many spaces relevant to renormalisation issues are Fréchet and nuclear. We list here some examples.

Example 2.2.1. Any finite dimensional vector space can be equipped with a norm and for any of these norms, they are trivially Banach, hence Fréchet and nuclear. If E and F are finite dimensional vector spaces we have $\mathbf{Hom}^c(E,F) = \mathbf{Hom}(E,F) \cong E^* \otimes F$, where $\mathbf{Hom}(E,F)$ stands for the space of F-valued linear maps on E and where the dual E^* is the **algebraic dual**.

Example 2.2.2. Let U be an open subset of $\mathbb{R}^n$. Take $E = C^\infty(U) =: \mathcal{E}(U)$. The topological dual is the space $E' = \mathcal{E}'(U)$ of distributions on U with compact support. Then E is Fréchet ([Trè67], pp. 86-89), and E' is nuclear ([Trè67], Corollary p. 530). By Proposition 2.2.2, E is also nuclear.

Remark 2.2.3. Note that the dual E' of a Fréchet space E is never a Fréchet space (for any of the natural topologies on E'), unless E is actually a Banach space (see for example [Köt69]). In particular, $\mathcal{E}'(U)$ is generally not Fréchet.

We now sum up various results of [Trè67] of importance for later purposes.

Proposition 2.2.3. *[Trè67, Equations (50.17)–(50.19)] Let E and F be two Fréchet spaces, with E nuclear. The following isomorphisms of topological vector spaces hold.*

$$E' \hat{\otimes} F \cong \mathbf{Hom}^c(E,F) \tag{6}$$
$$E \hat{\otimes} F \cong \mathbf{Hom}^c(E',F) \tag{7}$$
$$E' \hat{\otimes} F' \cong (E \hat{\otimes} F)' \cong \mathcal{B}^c(E \times F, \mathbb{K}). \tag{8}$$

with $\mathcal{B}^c(E \times F, \mathbb{K})$ the set of continuous bilinear maps $K : E \times F \longrightarrow \mathbb{K}$. Here the duals are endowed with the strong dual topology, $\mathbf{Hom}^c(E,F) \cong E' \otimes F$ with the topology of uniform convergence on the bounded subsets of E[6] and $\mathcal{B}^c(E \times F, \mathbb{K})$ with the topology of uniform convergence on products of bounded sets.

The stability of Fréchet nuclear spaces under completed tensor products follows from combining the definition of the completed tensor product with the fact that if E and F are two nuclear spaces then $E \hat{\otimes} F$ is a nuclear space ([Trè67, Equation (50.9)]). A stronger version of this Lemma is mentioned in [BB03, §6.f, page 182] which quotes [Gro54].

Lemma 2.2.4. *The completed tensor product $E \hat{\otimes} F$ of two Fréchet nuclear spaces is a Fréchet nuclear space.*

Proposition 2.2.5. *Let V be a Fréchet nuclear space. Then*

$$\left(V^{\hat{\otimes} k}\right)' \cong \left(V'\right)^{\hat{\otimes} k} \tag{9}$$

holds for any $k \geqslant 1$, where the duals are endowed with their strong topologies.

Proof. Let V be a Fréchet nuclear space. The case $k = 1$ is trivial. Then Equation (9) with $k = 2$ holds by Equation (8) with $E = F = V$. The cases $k \geqslant 2$ are proved by induction, using $E = V^{\hat{\otimes} k-1}$ and $F = V$. The induction holds by Lemma 2.2.4. □

[6] It is defined by a family of semi-norms p_{B,π_i} of the form $p_{B,\pi_i}(f) = \sup_{x \in B} \pi_i(f(x))$ when applied to some f in $\mathrm{Hom}^c(E,F)$, where B runs through the sets of all bounded subsets of E and π_i runs through a countable family of semi-norms which generate the topology of F. It gives back the strong topology on E' if F is the underlying field. It also carries other names such as "topology of bounded convergence" [Bou03, page III.14], [Sch71, p.81]

We denote by $\mathcal{D}'(M)$ the set of distributions on M and by $\mathcal{E}'(M)$, the set of distributions with compact support on a finite dimensional smooth manifold M, see for example [Hö89, Definition 6.3.3]. It is well-known (see for example [vdBC13, Exercise 2.3.2], [BDLGR18, p. 4]) that $\mathcal{E}(M)$ is a Fréchet nuclear space. It then follows from Proposition 2.2.2, that the space $\mathcal{E}'(M)$ is also nuclear.

Remark 2.2.4. (Compare with Remark 2.2.3). Note that the space $\mathcal{E}'(M)$ is *not* Fréchet since the dual of a Fréchet space F is Fréchet if and only if F is Banach (see for example [Köt69]) which is not the case of $\mathcal{E}(M)$.

One further useful result is:

Proposition 2.2.6. *Let M and N be two finite dimensional smooth manifolds. Then*

$$\mathbf{Hom}^c(\mathcal{E}'(M),\mathcal{E}(N))\cong\mathcal{E}(M)\,\widehat{\otimes}\,\mathcal{E}(N)\cong\mathcal{E}(M\times N)$$

holds.

Proof. The second isomorphism [Gro52, Chap. 5, p. 105] can be proved using a version of the Schwartz kernel theorem for smoothing operators [vdBC13, Theorem 2.4.5] by means of the identification $\mathbf{Hom}^c(\mathcal{E}'(M),\mathcal{E}(N))\cong\mathcal{E}(M\times N)$. The result then follows from (7) applied to $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ which are Fréchet nuclear spaces. □

Definition 2.2.7. *Let V be a Fréchet nuclear space. For any (k,l) in $\mathbb{N}_0^2$, we set*

$$\mathbf{Hom}_V^c(k,l)=\mathbf{Hom}^c(V^{\hat{\otimes}k},V^{\hat{\otimes}l})\cong(V')^{\hat{\otimes}k}\widehat{\otimes}V^{\hat{\otimes}l},$$

where, as before V' stands for the strong topological dual and the superscript "c" stands for continuous. Furthermore we set $\mathbf{Hom}_V^c:=(\mathbf{Hom}_V^c(k,l))_{k,l\geqslant 0}$.

For any $\sigma\in\mathfrak{S}_n$, let Θ_σ be the endomorphism of $V^{\otimes n}$ defined by

$$\Theta_\sigma(v_1\otimes\ldots\otimes v_n)=v_{\sigma^{-1}(1)}\otimes\ldots\otimes v_{\sigma^{-1}(n)}.$$

It extends to a continuous linear map $\overline{\Theta_\sigma}$ on the closure $V^{\hat{\otimes}n}$. For any $f\in\mathbf{Hom}_V^c(k,l)$, $\sigma\in\mathfrak{S}_l$, $\tau\in\mathfrak{S}_k$, we set:

$$\sigma\cdot f=\overline{\Theta_\sigma}\circ f,\qquad\qquad f\cdot\tau=f\circ\overline{\Theta_\tau}.$$

In the above definition, the superscript "c" stands for continuous. The family $\mathbf{Hom}_V^c$ carries a TRAP structure.

Theorem 2.2.8. *Let V be a Fréchet nuclear space. $\mathbf{Hom}_V^c$, with the action of $\mathfrak{S}\times\mathfrak{S}^{\mathrm{op}}$ described above, is a TRAP. Its horizontal concatenation is the usual topological tensor product of maps with $I_0:\mathbb{K}\longrightarrow\mathbb{K}$ given by the identity map of $\mathbb{K}$, its partial trace maps coincide with those of the TRAP $\mathbf{Hom}_V$ on elements of $(V')^{\hat{\otimes}k}\widehat{\otimes}V^{\hat{\otimes}l}$*

$$t_{i,j}(f_1\cdots f_k\otimes v_1\cdots v_l)=f_i(v_j)f_1\cdots f_{i-1}f_{i+1}\cdots f_k\otimes v_1\cdots v_{j-1}v_{j+1}\cdots v_l$$

with the same notations as in Subsection 2.1. It is unitary if and only if V is finite dimensional, in which case $I_1:V\longrightarrow V$ is the identity map of V.

Proof. The proof of the TRAP structure of $\mathbf{Hom}_V^c$ goes as in Proposition 2.1.1. The proof of the unital case is the same as the proof of Proposition 2.1.2. □

Example 2.2.3. For a finite dimensional vector space V, the TRAP $\mathbf{Hom}_V^c$ coincides with the TRAP $\mathbf{Hom}_V$.

Example 2.2.4. Let M be a smooth finite dimensional manifold. From Proposition 2.2.6 and Equation (9), it follows that the family $(\mathcal{K}_M(k,l))_{k,l\geqslant 0}$, with $\mathcal{K}_M(k,l)=(\mathcal{E}'(M))^{\hat{\otimes}k}\,\widehat{\otimes}\,\mathcal{E}(M)^{\hat{\otimes}l}$ defines a TRAP.

2.3 The TRAP $\mathcal{K}_M^\infty$ of smoothing pseudo-differential operators

We apply our results on TRAPs to tensor products of Fréchet spaces $\mathcal{E}(M)$ of smooth functions on a given smooth finite dimensional orientable closed manifold M and $\mu(z)$ a volume form on M. From now on, we work with vector space over $\mathbb{C}$. Recall from Proposition 2.2.6 that such spaces are stable under tensor products and morphisms in $\mathrm{Hom}^c(\mathcal{E}'(M), \mathcal{E}(N))$ are determined by smoothing kernels in $\mathcal{E}(M \times N)$.

We consider smooth kernels which stabilise $\mathcal{E}(M)$ and set, for $(k,l) \neq (0,0)$:

$$\mathcal{K}_M^\infty(k,l) := \mathcal{E}(M^k \times M^l) \cong \mathcal{E}(M)^{\hat{\otimes} k} \hat{\otimes} \mathcal{E}(M)^{\hat{\otimes} l}, \tag{10}$$

whose elements we refer to as smooth generalised kernels. We also set $\mathcal{K}_M^\infty(0,0) \cong \mathbb{C} \otimes \mathbb{C}$ and $\mathcal{K}_M^\infty := (\mathcal{K}_M^\infty(k,l))_{k,l \geqslant 0}$.

Theorem 2.3.1. *Let M be a smooth finite dimensional orientable closed manifold. The family of topological vector spaces $(\mathcal{K}_M^\infty(k,l))_{k,l\geqslant 0}$ can be equipped with the partial trace maps $t_{i,j} : \mathcal{K}_M^\infty(k,l) \longrightarrow \mathcal{K}_M^\infty(k-1,l-1)$ with $t_{i,j}(K_1 \otimes K_2)$ defined by*

$$\begin{aligned} t_{i,j}(K_1 \otimes K_2)(x_1, \cdots, x_{k-1}, y_1, \cdots, y_{l-1}) := \\ \int_M K_1(x_1, \cdots, x_{i-1}, z, x_i \cdots, x_{k-1})\, K_2(y_1, \cdots, y_{j-1}, z, y_j \cdots, y_k)\, d\mu(z), \end{aligned} \tag{11}$$

where $d\mu(z)$ is a volume form on M.

Together with the horizontal concatenation given by the tensor product of maps $(K_1 \otimes K_2) * (K_1' \otimes K_2') = K_a \otimes K_b$ with $K_a := K_1 \otimes K_1'$ and $K_b := K_2 \otimes K_2'$ this defines a TRAP, written $\mathcal{K}_M^\infty$, which we call the TRAP of **generalised smooth kernels** *on M.*

Remark 2.3.1. Note that the partial trace amounts to what one could call a partial convolution.

Proof. The unit $I_0 \in \mathcal{K}_M^\infty(0,0) \simeq \mathbb{C} \otimes \mathbb{C}$ of the horizontal concatenation $*$ is the constant map $f : \mathbb{C} \to \mathbb{C}$ defined by $f(x) = 1$. It is the unit of $*$ by bilinearity of the tensor product. The horizontal concatenation on the $\mathfrak{S} \times \mathfrak{S}^{op}$-module $(\mathcal{K}_M^\infty(k,l))_{k,l\geqslant 0}$ satisfies axioms 2. (a)-(d) of Definition 1.1.1 by the properties of the tensor product. We want to check that the maps $t_{i,j}$ are well-defined and satisfy axioms 3. (a)-(c).

The existence of the integral follows from the smoothness of K_1 and K_2 and the closedness of M. Therefore, by definition of $\mathcal{K}_M^\infty$, to show that $t_{i,j}(K_1 \otimes K_2) \in \mathcal{K}_M^\infty(k-1,l-1)$ and thus that $t_{i,j}$ is a partial trace, is enough to show that the function $t_{i,j}(K_1 \otimes K_2) : M^{k-1} \times M^{l-1} \longrightarrow \mathbb{C}$ is smooth. Since K_1 and K_2 are smooth, the map

$$(x_1, \cdots, x_{k-1}, y_1, \cdots, y_k) \longmapsto K_1(x_1, \cdots, x_{i-1}, z, x_i \cdots, x_{k-1})\, K_2(y_1, \cdots, y_{j-1}, z, y_j \cdots, y_k)$$

is infinitely differentiable for any $z \in M$. For $\vec{x} = (x_1, \cdots, x_k) \in M^k$ and $\vec{\alpha} = (\alpha_1, \cdots, \alpha_k) \in \mathbb{N}_0^k$ we use the short-hand notation

$$\partial_{\vec{x}}^{\vec{\alpha}} := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}.$$

Then, since M is compact, the partial derivatives

$$\partial_{\vec{x}}^{\vec{\alpha}} \partial_{\vec{y}}^{\vec{\beta}} K_1(x_1, \cdots, x_{i-1}, z, x_i \cdots, x_{k-1})\, K_2(y_1, \cdots, y_{j-1}, z, y_j \cdots, y_k)$$

are bounded uniformly in z and hence

$$\begin{aligned} &\int_M \partial_{\vec{x}}^{\vec{\alpha}} \partial_{\vec{y}}^{\vec{\beta}} K_1(x_1, \cdots, x_{i-1}, z, x_i \cdots, x_{k-1})\, K_2(y_1, \cdots, y_{j-1}, z, y_j \cdots, y_k)\, d\mu(z) \\ =&\partial_{\vec{x}}^{\vec{\alpha}} \partial_{\vec{y}}^{\vec{\beta}} \int_M K_1(x_1, \cdots, x_{i-1}, z, x_i \cdots, x_{k-1})\, K_2(y_1, \cdots, y_{j-1}, z, y_j \cdots, y_k)\, d\mu(z) \\ =&\partial_{\vec{x}}^{\vec{\alpha}} \partial_{\vec{y}}^{\vec{\beta}} t_{i,j}(K_1 \otimes K_2)(x_1, \cdots, x_{k-1}, y_1, \cdots, y_{l-1}). \end{aligned}$$

Therefore the map $t_{i,j}(K_1 \otimes K_2)(x_1, \cdots, x_{k-1}, y_1, \cdots, y_{l-1})$ is smooth.

Finally, to check Axiom 3.(c), by the first item of Lemma 1.1.4 it is enough to check the compatibility of the horizontal concatenation with the partial trace to show that $t_{1,1}(p * p') = t_{1,1}(p) * p'$ for any pair $(p, p') \in \mathcal{K}_M^\infty(k, l) \times \mathcal{K}_M^\infty(k', l')$ with $k, k', l, l' \geqslant 1$. Setting $p = K_1 \otimes K_2$ and $p' = K_1' \otimes K_2'$ we have, by definition of the partial trace maps and the horizontal concatenation

$$\begin{aligned}
&t_{1,1}(p * p')(x_1, \cdots, x_{k+k'-1}, y_1, \cdots, y_{l+l'-1}) \\
=&\int K_1(z, x_1, \cdots, x_{k-1}) K_1'(x_k, \cdots, x_{k+k'-1}) K_2(z, y_1, \cdots, y_{l-1}) K_2'(x_l, \cdots, x_{l+l'-1}) d\mu(z) \\
=&\left(\int K_1(z, x_1, \cdots, x_{k-1}) K_2(z, y_1, \cdots, y_{l-1}) d\mu(z) \right) K_1'(x_k, \cdots, x_{k+k'-1}) K_2'(x_l, \cdots, x_{l+l'-1}) \\
=&(t_{1,1}(p) * p')(x_1, \cdots, x_{k+k'-1}, y_1, \cdots, y_{l+l'-1}).
\end{aligned}$$

□

Remark 2.3.2. Notice that $\mathcal{K}_M^\infty$ is non unitary, since the map $f : M \times M \longrightarrow \mathbb{C}$ which could play the role of a vertical unity is a δ distribution supported on the diagonal. The simple examples considered here, namely $\mathcal{K}_M^\infty$ and the TRAP Hom_V^c speak for the fact that non-unitary TRAPs offer an appropriate framework to host infinite dimensional spaces. We expect non-unitary TRAPs to host more general distributions.

3 Free TRAPs

3.1 Various families of graphs

Here, we consider oriented multigraphs endowed with extra structures, in particular indexed input and output edges, loops, ordering and decorations. These extra structures make it difficult to implement the usual definition of multigraphs [Har69], where the edges form a multiset of pairs of vertices. In the literature on PROPs [Mar08, MMS09], graphs are defined as a set of half-edges (or flags), with an involution which tells us how to glue them together in order to obtain edges, and with a partition which defines the vertices. This definition does not take loops into account that is to say edges with no ends, yet loops enter our constructions in an essential way. Instead we borrow a definition from the theory of quivers [CB90, DW17], where two maps (called source and target, or alternatively tail and head) are given from the set of edges to the set of vertices. Our approach allows edges without source, or without target, or with neither source nor target. Among these are the inputs and outputs in the graph we consider, which we then index.

Definition 3.1.1. *A **graph** is a family $G = (V(G), E(G), I(G), O(G), IO(G), L(G), s, t, \alpha, \beta)$, where:*

1. *$V(G)$ (set of vertices), $E(G)$ (set of internal edges), $I(G)$ (set of input edges), $O(G)$ (set of output edges), $IO(G)$ (set of input-output edges) and $L(G)$ (set of loops, that is to say edges with no endpoints) are finite (possibly empty) sets.*

2. *$s : E(G) \sqcup O(G) \longrightarrow V(G)$ is a map (source map).*

3. *$t : E(G) \sqcup I(G) \longrightarrow V(G)$ is a map (target map).*

4. *$\alpha : I(G) \sqcup IO(G) \longrightarrow [i(G)]$ is a bijection, with $i(G) = |I(G)| + |IO(G)|$ (indexation of the input edges).*

5. *$\beta : O(G) \sqcup IO(G) \longrightarrow [o(G)]$ is a bijection, with $o(G) = |O(G)| + |IO(G)|$ (indexation of the output edges).*

*A **corolla ordered graph** is a graph G such that for any vertex v, the set of incoming edges $I(v)$ of v and the set of outgoing edges $O(v)$ of v are totally ordered and we shall denote both order relations by $\leqslant_v$.*

*A graph G is **solar** if $IO(G) = L(G) = \varnothing$.*

Remark 3.1.1. For a graph $G = (V(G), E(G), I(G), O(G), IO(G), L(G), s, t, \alpha, \beta)$, $O(G)$ and $I(G)$ will always respectively refer to the sets of outgoing and ingoing edges of G. On the other hand, for any $v \in V(G)$, $O(v)$ and $I(v)$ respectively refer to the set of outgoing and ingoing edges of the vertex v. In other words, for any $v \in V(G)$,

$$O(v) := \{e \in E(G) \sqcup O(G) | s(e) = v\}, \qquad I(v) := \{e \in E(G) \sqcup I(G) | t(e) = v.\}$$

We denote the cardinals of the sets $O(G)$, $I(G)$, $O(v)$ and $I(v)$ as $o(G)$, $i(G)$, $o(v)$ and $i(v)$ respectively.

For a solar graph (that is, such that $IO(G) = L(G) = \varnothing$), the terminology solar refers to its radiating aspect with rays around a central body. In [YJ15] such graphs are called ordinary.

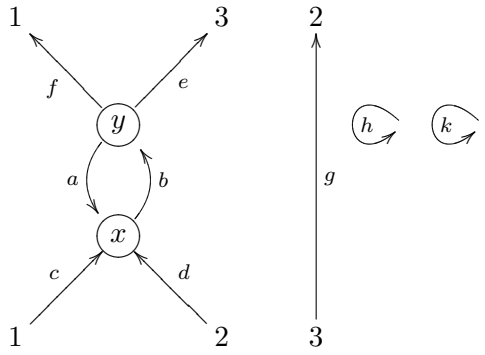
Example 3.1.1. Here is a graph G :

$$V(G) = \{x, y\}, \quad E(G) = \{a, b\}, \quad I(G) = \{c, d\}, \quad O(G) = \{e, f\}, \quad IO(G) = \{g\}, \quad L(G) = \{h, k\},$$

and:

$$s : \begin{cases} a \longmapsto y \\ b \longmapsto x \\ e \longmapsto y \\ f \longmapsto y, \end{cases} \quad t : \begin{cases} a \longmapsto x \\ b \longmapsto y \\ c \longmapsto x \\ d \longmapsto x, \end{cases} \quad \alpha : \begin{cases} c \longmapsto 1 \\ d \longmapsto 2 \\ g \longmapsto 3, \end{cases} \quad \beta : \begin{cases} e \longmapsto 3 \\ f \longmapsto 1 \\ g \longmapsto 2. \end{cases}$$

This is graphically represented as follows:

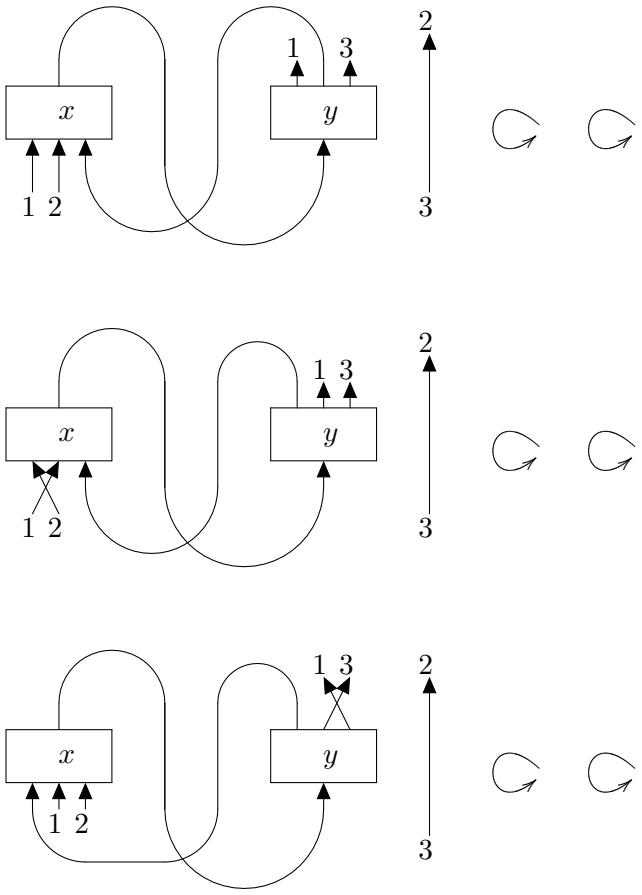


Note that this graph contains two loops, represented by  and  .

Graphically, if G is a corolla ordered graph, we shall represent the orders on the incoming and outgoing edges of a vertex by drawing box-shaped vertices, with the incoming and outgoing edges of any vertex ordered from left to right. For example, we distinguish the following two situations:

We note that the graph of Example 3.1.1 can be made corolla ordered in $3! \times 3! = 36$ ways, corresponding to the total orderings of the three incoming edges of x and of the three outgoing

edges of y. Here are three of them:



Definition 3.1.2. *Let G and G' be two graphs.*

1. *A **morphism** of graphs from G to G' is a family of maps $f = (f_V, f_E, f_I, f_O, f_{IO}, f_L)$ with:*

$$f_V : V(G) \longrightarrow V(G'), \qquad f_E : E(G) \longrightarrow E(G'), \qquad f_I : I(G) \longrightarrow I(G'),$$
$$f_O : O(G) \longrightarrow O(G'), \qquad f_{IO} : IO(G) \longrightarrow IO(G'), \qquad f_L : L(G) \longrightarrow L(G'),$$

such that:

$$s' \circ f_E = f_V \circ s_{|E(G)}, \qquad s' \circ f_O = f_V \circ s_{|O(G)},$$
$$t' \circ f_E = f_V \circ t_{|E(G)}, \qquad t' \circ f_I = f_V \circ t_{|I(G)},$$
$$\alpha' \circ f_I = \alpha_{|I(G)}, \qquad \alpha' \circ f_{IO} = \alpha_{|IO(G)},$$
$$\beta' \circ f_O = \beta_{|O(G)}, \qquad \beta' \circ f_{IO} = \beta_{|IO(G)}.$$

2. *An **isomorphism** of graphs from G to G' is a morphism of graphs $f = (f_V, f_E, f_I, f_O, f_{IO}, f_L)$ from G to G' such that all the structure maps are bijections.*

In other words, a morphism of graphs is an isomorphism if all the structure maps are bijection. Furthermore, an isomorphism of corolla ordered graphs is an isomorphism of graphs that preserves the orderings of ingoing and outgoing edges.

Definition 3.1.3. *Let G and G' be two corolla ordered graphs.*

1. *A **morphism** of corolla ordered graphs from G to G' is an **morphism** of graphs f from G to G' which preserves the order of incoming and outgoing edges that is, for any vertex of v:*

- *For any incoming edges e, e' of v, $e \leqslant_v e'$ in G if, and only if, $f(e) \leqslant_{f(v)} f(e')$ in G'.*
- *For any outgoing edges e, e' of v, $e \leqslant_v e'$ in G if, and only if, $f(e) \leqslant_{f(v)} f(e')$ in G'.*

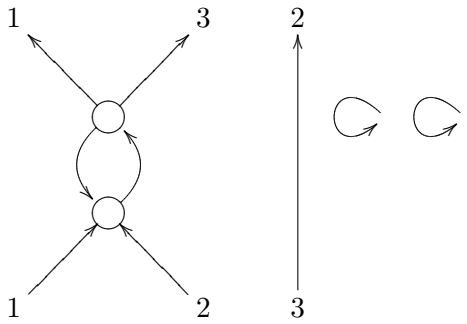
2. *An* **isomorphism** *of corolla ordered graphs from G to G' is a morphism of corolla ordered graphs from G to G' that is also an isomorphism of graphs from G to G'.*

3. *For any (k,l) in $\mathbb{N}_0^2$, we denote by $\mathbf{Gr}^{\circlearrowleft}(k,l)$ the set of the isoclasses of graphs G such that $i(G) = k$ and $o(G) = l$, that is $\mathbf{Gr}^{\circlearrowleft}(k,l)$ is the quotient space of graphs with k input edges and l output edges by the equivalence relation given by isomorphism. Similarly, we denote by $\mathbf{CGr}^{\circlearrowleft}(k,l)$ the set of isoclasses of corolla ordered graphs G such that $i(G) = k$ and $o(G) = l$.*

4. *The subset of $\mathbf{Gr}^{\circlearrowleft}(k,l)$ formed by isoclasses of solar graphs is denoted by $\mathbf{solGr}^{\circlearrowleft}(k,l)$ and the subset of $\mathbf{CGr}^{\circlearrowleft}(k,l)$ formed by isoclasses of solar corolla ordered graphs is denoted by $\mathbf{solCGr}^{\circlearrowleft}(k,l)$.*

In what follows, we shall write *graphs* for *isoclasses of graphs.*

Example 3.1.2. The isoclass of the graph of Example 3.1.1 is represented by:



We shall use later the two following special graphs:

Example 3.1.3.
1. We denote by $\mathcal{O}$ the graph with no vertex and with only one element in $L(G)$.
2. We denote by I the graph with no vertex and with only one element in $IO(I)$.

We will later define a monad structure on graphs and corolla ordered graphs (Proposition 5.2.4).

Throughout the paper, $X = (X(k,l))_{k,l\geqslant 0}$ is a family of sets.

Definition 3.1.4. *A graph decorated by $X = (X(k,l))_{k,l\geqslant 0}$ (or X-decorated graph, or simply decorated graph) is a couple (G, d_G) with G a graph as in Definition 3.1.1 and $d_G : V(G) \longrightarrow \bigsqcup_{k,l\geqslant 0} X(k,l)$ a map, such that for any vertex $v \in V(G)$, $d_G(v) \in X(i(v), o(v))$. We denote by $\mathbf{Gr}^{\circlearrowleft}(X)$ the set of graphs decorated by X. Similarly, we define X-decorated corolla ordered graphs which we denote by $\mathbf{CGr}^{\circlearrowleft}(X)$.*

We further write $\mathbf{Gr}^{\circlearrowleft}(X)(k,l)$ (respectively, $\mathbf{CGr}^{\circlearrowleft}(X)(k,l)$, $\mathbf{solGr}^{\circlearrowleft}(X)(k,l)$ and $\mathbf{solCGr}^{\circlearrowleft}(X)(k,l)$) the set of graphs (respectively, of corolla ordered graphs, solar graphs, solar corolla ordered graphs) decorated by X with k inputs (that is $|I(G)| = k$) and l ouputs (that is $|O(G)| = l$).

3.2 TRAPs of graphs

As before, $X = (X(k,l))_{k,l\geqslant 0}$ is a family of sets. We equip the set of graphs (possibly decorated by X) with a structure of TRAP. Let us first define an action of $\mathfrak{S} \times \mathfrak{S}^{op}$ on graphs. Let $G = (V(G), E(G), I(G), O(G), IO(G), L(G), s, t, \alpha, \beta) \in \mathbf{Gr}^{\circlearrowleft}(k,l)$, $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_l$. Then:

$$\tau \cdot G \cdot \sigma = (V(G), E(G), I(G), O(G), IO(G), L(G), s, t, \sigma^{-1} \circ \alpha, \tau \circ \beta).$$

If G is corolla ordered, then $\tau \cdot G \cdot \sigma$ is naturally corolla ordered; if G is X-decorated, then $\tau \cdot G \cdot \sigma$ is also X-decorated. Hence, this defines a structure of $\mathfrak{S} \times \mathfrak{S}^{op}$-module on $\mathbf{Gr}^{\circlearrowleft}$, $\mathbf{CGr}^{\circlearrowleft}$, $\mathbf{Gr}^{\circlearrowleft}(X)$ and $\mathbf{CGr}^{\circlearrowleft}(X)$ for any X.

We now define the **horizontal concatenation**. If G and G' are two graphs, we define a graph $G * G'$ in the following way:

$$V(G*G') = V(G) \sqcup V(G'), \quad E(G*G') = E(G) \sqcup E(G'), \quad L(G*G') = L(G) \sqcup L(G'),$$
$$I(G*G') = I(G) \sqcup I(G'), \quad O(G*G') = O(G) \sqcup O(G'), \quad IO(G*G') = IO(G) \sqcup IO(G').$$

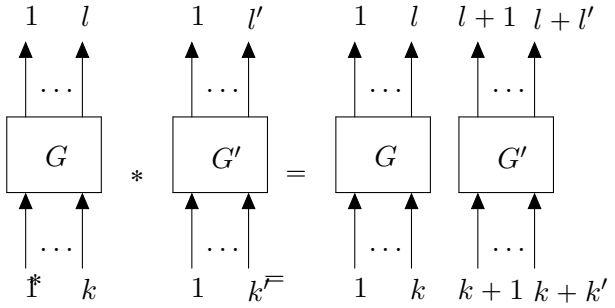
The source and target maps are given by:

$$s''_{|E(G)\sqcup O(G)} = s, \qquad s''_{|E(G')\sqcup O(G')} = s',$$
$$t''_{|E(G)\sqcup I(G)} = t, \qquad t''_{|E(G')\sqcup I(G')} = t'.$$

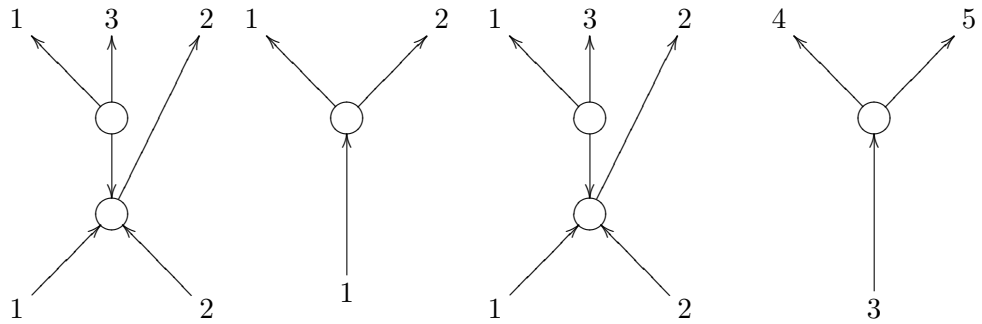
The indexations of the input and output edges are given by:

$$\alpha''_{|I(G)\sqcup IO(G)} = \alpha, \qquad \alpha''_{|I(G')\sqcup IO(G')} = i(G) + \alpha',$$
$$\beta''_{|O(G)\sqcup IO(G)} = \beta, \qquad \beta''_{|O(G')\sqcup IO(G')} = o(G) + \beta'$$

with an obvious abuse of notation in the definition of the second column. Notice that this product is not commutative in the usual sense for $G * G'$ and $G' * G$ might differ by the indexation of their input and output edges. However, it is commutative in the sense of Axiom 2.(d) of TRAPs. Roughly speaking, $G * G'$ is the disjoint union of G and G', the input and output edges of G' being indexed after the input and output edges of G.



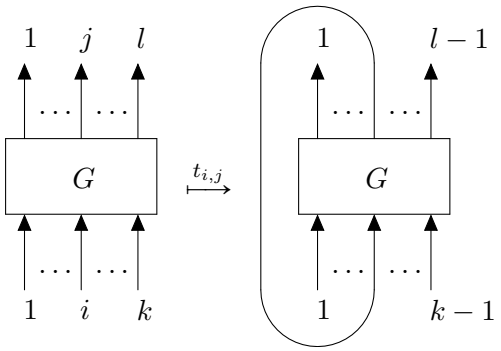
Example 3.2.1. Here is an example of horizontal concatenation :



Moreover, if G and H are corolla ordered graphs, then $G * H$ is naturally a corolla ordered graph. If G and H are X-decorated graphs, then $G * H$ is also naturally an X-decorated graph.

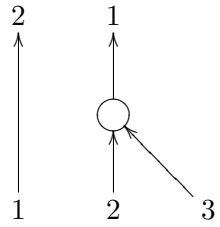
Let us finally define the partial trace maps. We only define the outline of their definition, and refer the reader to Appendix B for a rigorous definition. Let $G \in \mathbf{Gr}^{\circlearrowleft}(k,l)$, $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$. We set $e_i = \alpha_G^{-1}(i)$, $f_j = \beta_G^{-1}(j)$ and define $t_{i,j}(G)$ as the graph obtained by identifying the input of e_i with the output j of f_j. If $e_i \in I(G)$ and $f_j \in O(G)$, this creates an edge in $E(G)$. This case is illustrated in the figure below. Otherwise, we create an edge in $I(G)$, or $O(G)$ or $IO(G)$ or in $L(G)$. In all these cases, we then reindex in non-decreasing order, the inputs and the outputs of the obtained graph.

Graphically:

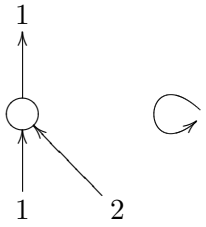


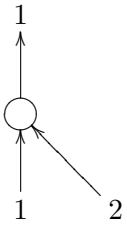
In particular, $t_{1,1}(I)$ is the graph $\mathcal{O}$ (see Example 3.1.3). As before, if G is corolla ordered, or if it is X-decorated, then $t_{i,j}(G)$ is corolla ordered, or X-decorated.

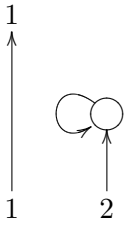
Example 3.2.2. Let G be the following graph:



Then:

$t_{1,2}(G) =$ 

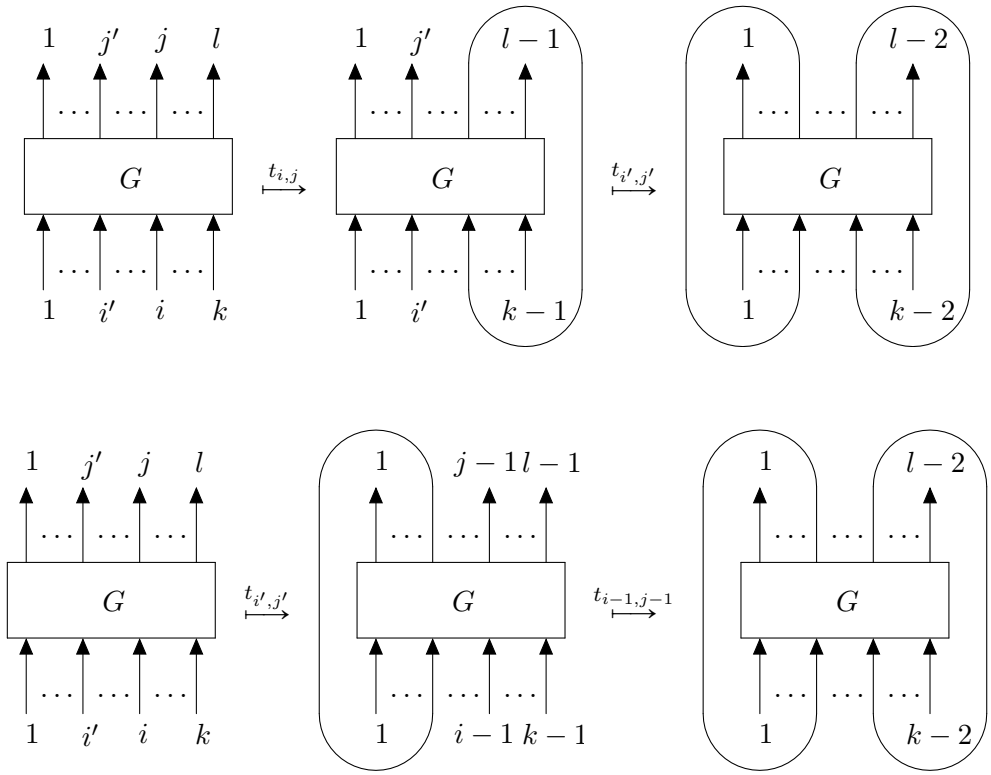
$t_{1,1}(G) = t_{2,2}(G) = t_{3,2}(G) =$ 

$t_{2,1}(G) = t_{3,1}(G) =$ 

Note that $t_{1,2}$ creates a loop when applied on G.

Proposition 3.2.1. *With this data,* $\mathbf{Gr}^{\circlearrowleft}$, $\mathbf{CGr}^{\circlearrowleft}$, $\mathbf{Gr}^{\circlearrowleft}(X)$ *and* $\mathbf{CGr}^{\circlearrowleft}(X)$ *are unitary TRAPs.*

Proof. We provide the proof for $\mathbf{CGr}^{\circlearrowleft}$. The proof is similar for the three other cases. Properties 1. and 2. follow directly from the symmetric group actions and the horizontal concatenation of graphs defined above. Let us give a graphical interpretation of the proof of Property 3.(a), when $i' < i$ and $j' < j$.



One can give similar graphical representations of the proofs for the remaining cases using the definitions given in Appendix B.

For Property 3.(b), let us consider a graph $p = G$. As the input edge indexed by i in $\sigma \cdot G \cdot \tau$ is the input edge of G indexed by $\tau(i)$ and the output edge indexed by j in $\sigma \cdot G \cdot \tau$ is the output edge of G indexed by $\sigma^{-1}(j)$, $G_1 = t_{i,j}(\sigma \cdot G \cdot \tau)$ is the graph obtained by gluing together the input indexed by $\tau(j)$ and the output indexed by $\sigma^{-1}(j)$, reindexing the input according to σ_i and the output edges by τ_j, so $G_1 = \sigma_i \cdot t_{\tau(i),\sigma^{-1}(j)}(G) \cdot \tau_j$.

Let us prove Property 3.(c). By Lemma 1.1.4, it is enough to prove it for $(p, p') = (G, G')$ a pair of graphs and $(i, j) = (1, 1)$. In this case, e_i and f_j are both edges of G, so $t_{1,1}(G * G') = t_{1,1}(G) * G'$.

The graph I is defined in Example 3.1.3. For any graph G with $|O(G)| \geqslant 1$,

$$t_{1,2}(I * G) = G.$$

By Lemma 1.1.4, I is a unit of $\mathbf{Gr}^{\circlearrowleft}$. □

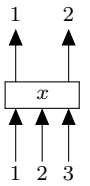
Corollary 3.2.2. $\mathbf{solGr}^{\circlearrowleft}$, $\mathbf{solCGr}^{\circlearrowleft}$, $\mathbf{solGr}^{\circlearrowleft}(X)$ *and* $\mathbf{solCGr}^{\circlearrowleft}(X)$ *are subTRAPs of* $\mathbf{Gr}^{\circlearrowleft}$, $\mathbf{CGr}^{\circlearrowleft}$, $\mathbf{Gr}^{\circlearrowleft}(X)$ *and* $\mathbf{CGr}^{\circlearrowleft}(X)$ *in the sense of Definition 1.1.2. They are non unitary.*

Proof. If G and H are solar, then $G * H$ is clearly also solar. If $G \in \mathbf{CGr}^{\circlearrowleft}(k, l)$ is solar, then for any $i \in [k]$ and $j \in [l]$, $t_{i,j}(G)$ is solar. Indeed, as $IO(G) = \varnothing$, $IO(t_{i,j}(G)) = \varnothing$; as $IO(G) = L(G) = \varnothing$, $L(t_{i,j}(G)) = \varnothing$. They are indeed non unitary, as the graph I is not solar, $IO(I)$ being nonempty. □

Remark 3.2.1. $\mathbf{Gr}^{\circlearrowleft}$, $\mathbf{CGr}^{\circlearrowleft}$, $\mathbf{Gr}^{\circlearrowleft}(X)$ and $\mathbf{CGr}^{\circlearrowleft}(X)$ admit other sub-TRAPs, for example with vertices with only a prescribed number of possible vertices. These sub-TRAPs might be of importance in the question of renormalisability of QFTs, but this question is far from the scope of this work and we therefore do not define rigorously these other objects.

3.3 Morphisms of TRAPs and free TRAPs

As before, $X = (X(k,l))_{k,l\geqslant 0}$ is a family of sets. It turns out that $\mathbf{solCGr}^{\circlearrowleft}(X)$ is the free TRAP generated by X. For any $x \in X(k,l)$, we identify x with the graph in $\mathbf{solCGr}^{\circlearrowleft}(k,l)(X)$ with one vertex decorated by x, k incoming edges, totally ordered according to their indices, and l outgoing edges, totally ordered according to their indices. For example, $x \in X(3,2)$ is identified with the corolla ordered graph



(12)

Theorem 3.3.1. *Let P be a TRAP and $\phi = (\phi(k,l))_{k,l\geqslant 0}$ be a map from X to P that is, for any $(k,l) \in \mathbb{N}_0^2$, $\phi(k,l) : X(k,l) \longrightarrow P(k,l)$ is a map. Then there exists a unique TRAP morphism $\Phi : \mathbf{solCGr}^{\circlearrowleft}(X) \longrightarrow P$, sending x to $\phi(x)$ for any $x \in X$. If moreover P is unitary, this morphism Φ uniquely extends as a unitary TRAP morphism from $\mathbf{CGr}^{\circlearrowleft}(X)$ to P.*

In other words, $\mathbf{solCGr}^{\circlearrowleft}(X)$ (respectively, $\mathbf{CGr}^{\circlearrowleft}(X)$) is the free TRAP (respectively, the free unitary TRAP, that is the free wheeled PROP) generated by X.

Remark 3.3.1. In practice we often have $P = X$ and $\phi = \mathrm{Id}$ which yields a map

$$\Phi : \mathbf{solCGr}^{\circlearrowleft}(X) \longrightarrow X \tag{13}$$

from decorated corolla ordered graphs to the space X of decorations.

Example 3.3.1. Here is a trivial yet enlightening example of how Φ acts on graphs: for $G = \mathcal{O}$, we have $G = t_{1,1}(I)$ and hence $\Phi(G) = t_{1,1}(I_P)$.

Proof. We provide here a sketch of the proof, and refer the reader to the appendix C for a full proof. Since $\mathbf{solCGr}^{\circlearrowleft}(X) \subseteq \mathbf{CGr}^{\circlearrowleft}(X)$, we take G in $\mathbf{CGr}^{\circlearrowleft}(k,l)(X)$ and treat simultaneously the case of solar graphs and the other. We define $\Phi(G)$ for any graph $G \in \mathbf{CGr}^{\circlearrowleft}(k,l)(X)$ by induction on the number N of internal edges of G.

If $N = 0$, then G can be written as

$$G = \mathcal{O}^{*p} * \sigma \cdot (I^{*q} * x_1 * \ldots * x_r) \cdot \tau,$$

where (p,q,r) lies in $\mathbb{N}_0^3$, (k_i,l_i) lies in $\mathbb{N}_0^2$ for any i, x_i in $X(k_i,l_i)$ and σ in $\mathfrak{S}_{q+k_1+\ldots+k_r}$, τ in $\mathfrak{S}_{q+l_1+\ldots+l_r}$. If G is solar, then $p = q = 0$ and this reduces to

$$G = \sigma \cdot (x_1 * \ldots * x_r) \cdot \tau.$$

We then set

$$\Phi(G) = \sigma \cdot (\phi(x_1) * \ldots * \phi(x_k)) \cdot \tau. \tag{14}$$

If G is not solar and if P is unitary, we denote by I_P the identity of P, and we put

$$\Phi(G) = t_{1,1}(I_P)^{*p} * \sigma \cdot (I_P^{*q} * \phi(x_1) * \ldots * \phi(x_r)) \cdot \tau.$$

We can prove that this does not depend on the choice of the decomposition of G, with the help of the TRAP axioms applied to P. Let us now assume that $\Phi(G')$ is defined for any graph with $N-1$ internal edges, for a given $N \geqslant 1$. Let G be a graph with N internal edges and let e be one of these edges. Let G_e be a graph obtained by cutting this edge in two, such that $G = t_{1,1}(G_e)$. We then set:

$$\Phi(G) = t_{1,1} \circ \Phi(G_e).$$

We can prove that this does not depend on the choice of e. It can then be shown that Φ defined as above is compatible with the partial trace maps. □

Since the ingoing and outgoing edges of each vertex of a corolla ordered graph are totally ordered, each corolla ordered graph $\mathbf{CGr}^{\circlearrowleft}$ is naturally acted upon by $\mathfrak{S} \times \mathfrak{S}^{op}$.

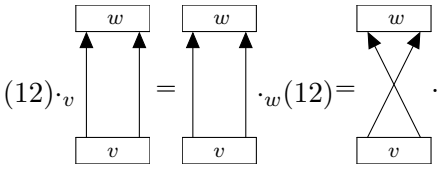
Definition 3.3.2. *For any corolla ordered graph $G \in \mathbf{CGr}^{\circlearrowleft}$ and any vertex $v \in V(G)$, there is a natural action of $\mathfrak{S}_{o(v)} \times \mathfrak{S}^{op}_{i(v)}$ induced by the action on the totally ordered edges in $O(v)$ and $I(v)$. The corolla ordered graph obtained from G by the action of (σ, τ) on the vertex v is denoted by*

$$\sigma \cdot_v G \cdot_v \tau.$$

A similar action can be built on a corolla ordered graph G decorated by a family of sets X:

$$\sigma \cdot_v (G, d_G) \cdot_v \tau := (\sigma \cdot_v G \cdot_v \tau, d_G).$$

Example 3.3.2.



In these pictures, the labelling of the edges outgoing (respectively, ingoing to) from the vertex v (respectively, w) are labelled from left to right.

Note that $\mathbf{Gr}^{\circlearrowleft}(X)$ is obtained from $\mathbf{CGr}^{\circlearrowleft}(X)$ by forgetting the total orders on the edges, which in fact is equivalent to the trivialisation of this action of symmetric groups on incoming and outgoing edges of any vertex. Hence:

Corollary 3.3.3. *Let P be a TRAP and $\phi = (\phi(k,l))_{k,l \geqslant 0}$ be a map from X to P. We assume that for any $x \in X(k,l)$, for any $(\sigma, \tau) \in \mathfrak{S}_k \otimes \mathfrak{S}_l$,*

$$\tau \cdot \phi(x) \cdot \sigma = \phi(x).$$

There exists a unique TRAP morphism $\Phi : \mathbf{solGr}^{\circlearrowleft}(X) \longrightarrow P$, sending x to $\phi(x)$ for any $x \in X$. If moreover P is unitary, this morphism Φ is uniquely extended as a unitary TRAP morphism from $\mathbf{Gr}^{\circlearrowleft}(X)$ to P.

We end this paragraph with the non corolla ordered counterpart of Remark 3.3.1:

Remark 3.3.2. In practice we often have $P = X$ and $\phi = \mathrm{Id}_P$ which yields a map

$$\Phi : \mathbf{solGr}^{\circlearrowleft}(X) \longrightarrow X \tag{15}$$

from decorated graphs to the space X of decorations.

3.4 Extending non-unitary TRAPs

In this section, we embed any TRAP P in a unitary TRAP denoted by $\mathbf{uPGr}^{\circlearrowleft}(P)$. We proceed in the following way:

- We start with the canonical TRAP morphism from the free TRAP $\mathbf{solCGr}^{\circlearrowleft}(P)$ generated by P to P.
- By Proposition 1.2.2, this defines an equivalence $\sim$ on $\mathbf{solCGr}^{\circlearrowleft}(P)$, compatible with the TRAP structure of $\mathbf{solCGr}^{\circlearrowleft}(P)$.
- We then extend this equivalence to $\mathbf{CGr}^{\circlearrowleft}(P)$, in such a way that it is compatible with the unitary TRAP structure of $\mathbf{CGr}^{\circlearrowleft}(P)$, as required in Lemma 1.2.1.

- Consequently, the quotient $\mathbf{CGr}^{\circlearrowleft}(P)/\sim$ is a unitary TRAP which contains P.

For this, we shall need the solar part of any corolla ordered graph G, which we now define:

Notation 3.4.1. Let $G \in \mathbf{CGr}^{\circlearrowleft}(P)(k,l)$. Then there exist a unique $(p,k',l') \in \mathbb{N}_0^3$, $k' \leqslant k$, $l' \leqslant l$, a unique solar graph $G' \in \mathbf{solCGr}^{\circlearrowleft}(P)(k',l')$, a unique pair of permutations $(\sigma,\tau) \in \mathfrak{S}_k \times \mathfrak{S}_l$ such that:

- $\sigma(1) < \ldots < \sigma(k')$ and $\sigma(k'+1) < \ldots < \sigma(k)$ (that is to say σ is a $(k', k-k')$-shuffle);
- $\tau(1) < \ldots < \tau(l')$ and $\tau(l'+1) < \ldots < \tau(l)$ (that is to say τ is a $(l', l-l')$-shuffle);
- $G = \mathcal{O}^p * \tau^{-1} \cdot (G' * I^{k-k'}) \cdot \sigma$.

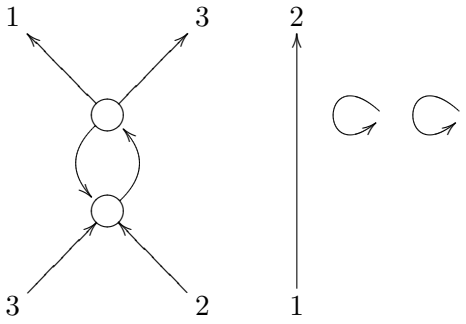
The graph G' is the **solar part** of G and is denoted by $\mathrm{sol}(G)$. We also set

$$\sigma := \mathrm{sh}_{in}(G), \qquad \tau := \mathrm{sh}_{out}(G), \qquad p := \mathrm{val}_{\mathcal{O}}(G).$$

Here sh stands for shuffle and val for valuation.

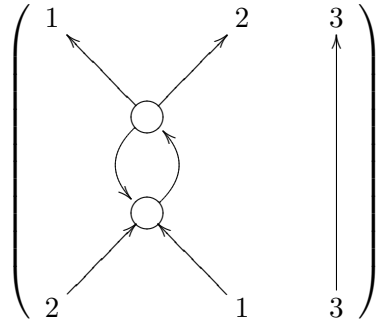
Remark 3.4.1. As the subsequent example will show, reindexing the ingoing and outgoing edges is useful to write the graph as a horizontal product of a solar graph, loops and I.

Example 3.4.1. Let G be the graph:



Then:

$$G = \mathcal{O}^2 * (132)^{-1} \cdot$$

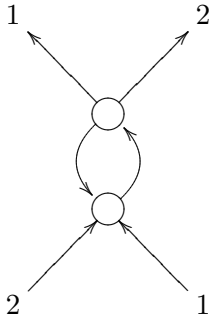


$$\cdot\,(231)$$

$$= \mathcal{O}^2 * (132)^{-1} \cdot *(G' * I) \cdot (231),$$

where G' is the following graph, which as a result is the solar part of G:

$G' =$



Moreover:

$$\mathrm{sh}_{in}(G) = (231), \qquad \mathrm{sh}_{out}(G) = (132), \qquad \mathrm{val}_{\mathcal{O}}(G) = 2.$$

Definition 3.4.1. *Let P be a TRAP and let us consider the unique TRAP morphism $\Phi : \mathbf{solCGr}^{\circlearrowleft}(P) \longrightarrow P$, extending the identity of P. We define a relation $\sim$ on $\mathbf{CGr}^{\circlearrowleft}(P)$ as follows: for $G, G' \in \mathbf{CGr}^{\circlearrowleft}(P)$,*

$$G \sim G' \Longleftrightarrow \begin{cases} \Phi(\mathrm{sol}(G)) = \Phi(\mathrm{sol}(G')), \\ \mathrm{sh}_{in}(G) = \mathrm{sh}_{in}(G'), \\ \mathrm{sh}_{out}(G) = \mathrm{sh}_{out}(G'), \\ \mathrm{val}_{\mathcal{O}}(G) = \mathrm{val}_{\mathcal{O}}(G'). \end{cases}$$

This is clearly an equivalence.

Roughly speaking, this equivalence identifies graphs with the same input-output edges and loops and which coincide after contraction of their components obtained from deleting input-output edges and loops.

Theorem 3.4.2. *Let P be a TRAP and let $\sim$ be the equivalence on $\mathbf{CGr}^{\circlearrowleft}(P)$ of Definition 3.4.1. The quotient $\mathbf{uPGr}^{\circlearrowleft}(P) := \mathbf{CGr}^{\circlearrowleft}(P)/\sim$ is a unitary TRAP, containing a sub-TRAP isomorphic to P through.*

Proof. • We first show the compatibility of the equivalence relation with the left and right actions of the symmetric group.

Let $G, G' \in \mathbf{CGr}^{\circlearrowleft}(P)$, such that $G \sim G'$. If $\sigma \in \mathfrak{S}_{k+l}$, there exists a unique triple $(\sigma_1, \sigma_2, \sigma') \in \mathfrak{S}_{k'} \times \mathfrak{S}_{k-k'} \times \mathfrak{S}_k$ such that:

- $\mathrm{sh}_{in}(G) \circ \sigma = \sigma' \circ (\sigma_1 \otimes \sigma_2)$.
- $\sigma'(1) < \ldots < \sigma'(k')$ and $\sigma'(k'+1) < \ldots < \sigma'(k)$.

Then $\mathrm{sol}(G \cdot \sigma) = \mathrm{sol}(G) \cdot \sigma_1$, $\mathrm{sh}_{in}(G \cdot \sigma) = \sigma'$ and $\mathrm{sh}_{out}(G \cdot \sigma) = sh_{out}(G)$. Obviously, $\mathrm{val}_{\mathcal{O}}(G \cdot \sigma) = \mathrm{val}_{\mathcal{O}}(G')$. A similar result holds for G'. We immediately obtain that:

$$\mathrm{sh}_{in}(G \cdot \sigma) = \mathrm{sh}_{in}(G' \cdot \sigma), \quad sh_o(G \cdot \sigma) = sh_o(G' \cdot \sigma), \quad \mathrm{val}_{\mathcal{O}}(G \cdot \sigma) = \mathrm{val}_{\mathcal{O}}(G' \cdot \sigma).$$

Recall that Φ is given in Definition 3.4.1. As it is a TRAP morphism:

$$\Phi(\mathrm{sol}(G \cdot \sigma)) = \Phi(\mathrm{sol}(G) \cdot \sigma_1) = \Phi(\mathrm{sol}(G)) \cdot \sigma_1 = \Phi(\mathrm{sol}(G')) \cdot \sigma_1 = \Phi(\mathrm{sol}(G' \cdot \sigma)),$$

So $G \cdot \sigma \sim G' \cdot \sigma$. Similarly, if $\tau \in \mathfrak{S}_l$, $\tau \cdot G \sim \tau \cdot G'$.

- Let us show the compatibility of the equivalence relation with the horizontal concatenation $*$ on the left and on the right.

  Let $H \in \mathbf{CGr}^{\circlearrowleft}(P)$. Then, by construction of the product $*$ (Paragraph 3.2):

$$\begin{aligned} \mathrm{sol}(G * H) &= \mathrm{sol}(G) * \mathrm{sol}(H), \\ \mathrm{sh}_{in}(G * H) &= \mathrm{sh}_{in}(G) * \mathrm{sh}_{in}(H), \\ \mathrm{sh}_{out}(G * H) &= \mathrm{sh}_{out}(G) * \mathrm{sh}_{out}(H), \\ \mathrm{val}_{\mathcal{O}}(G * H) &= \mathrm{val}_{\mathcal{O}}(G) + \mathrm{val}_{\mathcal{O}}(H). \end{aligned}$$

  A similar result holds for $G' * H$. As $G \sim G'$,

$$\mathrm{sh}_{in}(G * H) = \mathrm{sh}_{in}(G' * H), \quad sh_{out}(G * H) = sh_{out}(G' * H), \quad \mathrm{val}_{\mathcal{O}}(G * H) = \mathrm{val}_{\mathcal{O}}(G' * H).$$

Moreover, as Φ is a TRAP morphism:

$$\begin{aligned}\Phi(\mathrm{sol}(G * H)) &= \Phi(\mathrm{sol}(G) * \mathrm{sol}(H)) \\ &= \Phi(\mathrm{sol}(G)) * \Phi(\mathrm{sol}(H)) \\ &= \Phi(\mathrm{sol}(G')) * \Phi(\mathrm{sol}(H)) = \Phi(\mathrm{sol}(G' * H)).\end{aligned}$$

Hence, $G * H \sim G' * H$. Similarly, $H * G \sim H * G'$.

- We now check the compatibility of the equivalence relation with the partial trace maps.

  Let $i \in [k]$ and $j \in [l]$. We denote by e_i (respectively e'_i) the i-th input of G (respectively of G') and by f_j (respectively f'_j) the j-th output of G (respectively of G'). There are five possible cases:

  1. If $e_i = f_j \in IO(G)$, then $e'_i = f'_j \in IO(G')$. Moreover:

  $$\begin{aligned}\mathrm{sh}_{in}(t_{i,j}(G)) &= \mathrm{sh}_{in}(t_{i,j}(G')), \\ \mathrm{sh}_{out}(t_{i,j}(G)) &= \mathrm{sh}_{out}(t_{i,j}(G')), \\ \mathrm{val}_{\mathcal{O}}(t_{i,j}(G)) &= \mathrm{val}_{\mathcal{O}}(t_{i,j}(G')) = \mathrm{val}_{\mathcal{O}}(G) + 1, \\ \mathrm{sol}(t_{i,j}(G)) &= \mathrm{sol}(G), \\ \mathrm{sol}(t_{i,j}(G')) &= \mathrm{sol}(G').\end{aligned}$$

     As $G \sim G'$, $\Phi(\mathrm{sol}(G)) = \Phi(\mathrm{sol}(G'))$, so $t_{i,j}(G) \sim t_{i,j}(G')$.
  2. If $e_i, f_j \in IO(G)$, with $e_i \neq f_j$, then $e'_i, f'_j \in IO(G')$, with $e'_i \neq f'_j$. Moreover:

  $$\begin{aligned}\mathrm{sh}_{in}(t_{i,j}(G)) &= \mathrm{sh}_{in}(t_{i,j}(G')), \\ \mathrm{sh}_{out}(t_{i,j}(G)) &= \mathrm{sh}_{out}(t_{i,j}(G')), \\ \mathrm{val}_{\mathcal{O}}(t_{i,j}(G)) &= \mathrm{val}_{\mathcal{O}}(t_{i,j}(G')) = \mathrm{val}_{\mathcal{O}}(G), \\ \mathrm{sol}(t_{i,j}(G)) &= \mathrm{sol}(G), \\ \mathrm{sol}(t_{i,j}(G')) &= \mathrm{sol}(G').\end{aligned}$$

     So $t_{i,j}(G) \sim t_{i,j}(G')$.
  3. If $e_i \in IO(G)$ and $f_j \notin IO(G)$, then $e'_i \in IO(G')$ and $f_j \notin IO(G')$. Moreover:

  $$\begin{aligned}\mathrm{sh}_{in}(t_{i,j}(G)) &= \mathrm{sh}_{in}(t_{i,j}(G')), \\ \mathrm{sh}_{out}(t_{i,j}(G)) &= \mathrm{sh}_{out}(t_{i,j}(G')), \\ \mathrm{val}_{\mathcal{O}}(t_{i,j}(G)) &= \mathrm{val}_{\mathcal{O}}(t_{i,j}(G')) = \mathrm{val}_{\mathcal{O}}(G),\end{aligned}$$

     and there exists a permutation $\alpha \in \mathfrak{S}_{k'}$ such that

  $$\mathrm{sol}(t_{i,j}(G)) = \mathrm{sol}(G) \cdot \alpha, \qquad \mathrm{sol}(t_{i,j}(G')) = \mathrm{sol}(G') \cdot \alpha.$$

     As Φ is a TRAP morphism:

  $$\Phi(\mathrm{sol}(t_{i,j}(G))) = \Phi(\mathrm{sol}(G)) \cdot \alpha = \Phi(\mathrm{sol}(G')) \cdot \alpha = \Phi(\mathrm{sol}(G') \cdot \alpha) = \Phi(\mathrm{sol}(t_{i,j}(G'))),$$

     so $t_{i,j}(G) \sim t_{i,j}(G')$.
  4. The case where $e_i \notin IO(G)$ and $f_j \in IO(G)$ is treated similarly.
  5. If $e_i, f_j \notin IO(G)$, then $e_i, f_j \notin IO(G')$. Moreover:

  $$\begin{aligned}\mathrm{sh}_{in}(t_{i,j}(G)) &= \mathrm{sh}_{in}(t_{i,j}(G')), \\ \mathrm{sh}_{out}(t_{i,j}(G)) &= \mathrm{sh}_{out}(t_{i,j}(G')), \\ \mathrm{val}_{\mathcal{O}}(t_{i,j}(G)) &= \mathrm{val}_{\mathcal{O}}(t_{i,j}(G')) = \mathrm{val}_{\mathcal{O}}(G),\end{aligned}$$

and there exist $i' \in [k']$, $j' \in [l']$, such that

$$\mathrm{sol}(t_{i,j}(G)) = t_{i',j'}(\mathrm{sol}(G)), \qquad \mathrm{sol}(t_{i,j}(G')) = t_{i',j'}(\mathrm{sol}(G')).$$

As Φ is a TRAP morphism:

$$\begin{aligned}\Phi(\mathrm{sol}(t_{i,j}(G))) &= \Phi \circ t_{i',j'}(\mathrm{sol}(G)) \\ &= t_{i',j'} \circ \Phi(\mathrm{sol}(G)) \\ &= t_{i',j'} \circ \Phi(\mathrm{sol}(G')) = \Phi(\mathrm{sol}(t_{i,j}(G'))),\end{aligned}$$

so $t_{i,j}(G) \sim t_{i,j}(G')$.

By Lemma 1.2.1, $\mathbf{uPGr}^{\circlearrowleft}(P)$ is a unitary TRAP.

The canonical injection $\iota : P \longrightarrow \mathbf{CGr}^{\circlearrowleft}(P)$ induces a TRAP morphism $\iota' : P \longrightarrow \mathbf{uPGr}^{\circlearrowleft}(P)$, which we see as follow. If p, q lie in P, then in $\mathbf{CGr}^{\circlearrowleft}(P)$, $\iota(p) * \iota(q)$ and $\iota(p * q)$ are solar graphs, and:

$$\Phi(\iota(p) * \iota(q)) = \Phi \circ \iota(p) * \Phi \circ \iota(q) = p * q = \Phi \circ \iota(p * q),$$

so $\iota(p) * \iota(q) \sim \iota(p * q)$. Hence, $\iota'(p) * \iota'(q) = \iota'(p * q)$. If $p \in P(k, l)$, $i \in [k]$ and $j \in [l]$, then $t_{i,j} \circ \iota(p)$ and $\iota \circ t_{i,j}(p)$ are solar graphs in $\mathbf{CGr}^{\circlearrowleft}(P)$, and:

$$\Phi \circ t_{i,j} \circ \iota(p) = t_{i,j} \circ \Phi \circ \iota(p) = t_{i,j}(p) = \Phi \circ \iota \circ t_{i,j}(p),$$

so $t_{i,j} \circ \iota(p) \sim \iota \circ t_{i,j}(p)$, which implies that $t_{i,j} \circ \iota'(p) = \iota' \circ t_{i,j}(p)$: the map ι' is a TRAP morphism.

Let $p, q \in P$, such that $\iota'(p) = \iota'(q)$. Then $\iota(p) \sim \iota(q)$. As $\iota(p)$ and $\iota(q)$ are solar graphs,

$$p = \Phi \circ \iota(p) = \Phi \circ \iota(q) = q,$$

so ι' is injective. We have proved that the unitary TRAP $\mathbf{uPGr}^{\circlearrowleft}(P)$ contains a (non unitary) sub-TRAP isomorphic to P. □

We now identify the sub-TRAP $\iota'(P)$ of $\mathbf{uPGr}^{\circlearrowleft}(P)$ with P. Let us give a description of the $\mathfrak{S}_l \times \mathfrak{S}_k^{op}$-module $\mathbf{uPGr}^{\circlearrowleft}(P)(k, l)$. Its elements are obtained from the elements of $\mathbf{CGr}^{\circlearrowleft}(P)(k, l)$ by the contraction of their solar parts to an element of P. Moreover, the elements of $\mathbf{CGr}^{\circlearrowleft}(P)$ are obtained from their solar parts by adding copies of the unit I, corresponding to input-output edges, and copies of the trace $\mathcal{O}$ of the unit. Similarly, the elements of $\mathbf{uPGr}^{\circlearrowleft}(P)$ are obtained by adding copies of I and $\mathcal{O}$ to elements of P. The action of the symmetric groups on the copies of I and on P has to be taken in account: for any i, I^i generates a $\mathfrak{S}_i \times \mathfrak{S}_i^{op}$-module isomorphic to $\mathfrak{S}_i$, with its canonical $\mathfrak{S}_i \times \mathfrak{S}_i^{op}$-action. Therefore, we obtain that:

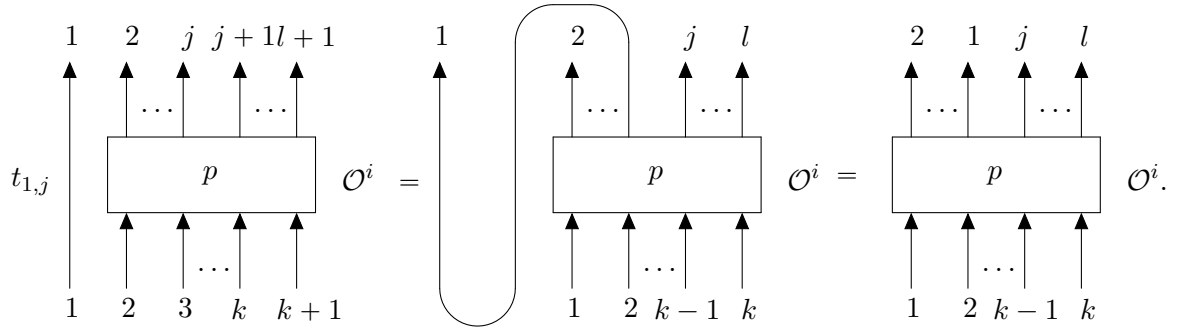
$$\mathbf{uPGr}^{\circlearrowleft}(P)(k, l) = \left(\bigsqcup_{i=0}^{\min(k,l)} \mathrm{ind}_{(\mathfrak{S}_i \otimes \mathfrak{S}_{l-i}) \times (\mathfrak{S}_i \otimes \mathfrak{S}_{k-i})^{op}}^{\mathfrak{S}_k \times \mathfrak{S}_l^{op}} \mathfrak{S}_i \times P(k - i, l - i) \right) \times \{\mathcal{O}^j, j \in \mathbb{N}_0\},$$

where ind is the induction of modules.

The partial trace maps can be computed with the help of the unitary TRAP axioms. For example, if $p \in P(k - 1, l - 1)$, if $j > 1$, then:

$$t_{1,j}(\mathrm{Id}_{[1]}, p, \mathcal{O}^i) = (\mathrm{Id}_{[0]}, (1 \dots j - 1) \cdot p, \mathcal{O}^i),$$

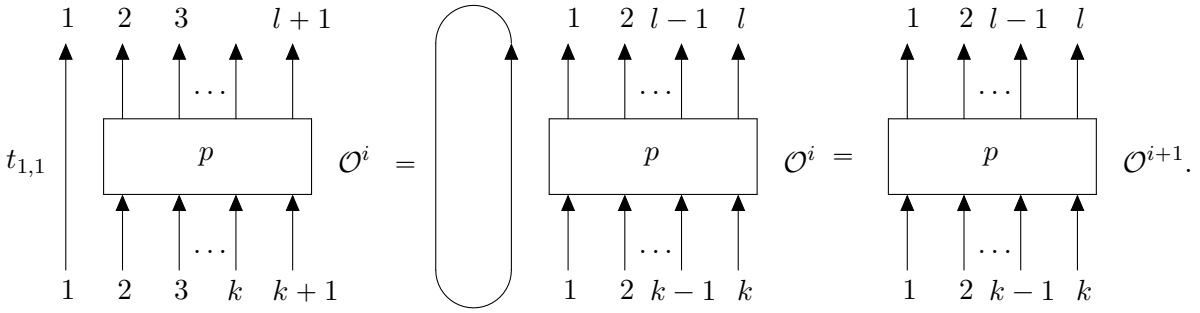
which is graphically represented by



and

$$t_{1,1}(\mathrm{Id}_{[1]}, p, \mathcal{O}^i) = (\mathrm{Id}_{[0]}, p, \mathcal{O}^{i+1}),$$

which is graphically represented by



3.5 A functor from TRAPs to unitary TRAPs

The unitary TRAP $\mathbf{uPGr}^{\circlearrowleft}(P)$ satisfies the following universal property:

Proposition 3.5.1. *Let P be a TRAP, Q a unitary TRAP and $\theta : P \longrightarrow Q$ be a TRAP morphism. There exists a unique unitary TRAP morphism $\Theta : \mathbf{uPGr}^{\circlearrowleft}(P) \longrightarrow Q$ extending θ.*

Proof. Uniqueness. Let θ be such a morphism. For any $G \in \mathbf{CGr}^{\circlearrowleft}(P)$, we denote by $[G]$ its class in $\mathbf{uPGr}^{\circlearrowleft}(P)$. Then:

$$G = \mathrm{sh}_{out}(G) \cdot (\mathrm{sol}(G) * I^{*k''}) \cdot \mathrm{sh}_{in}(G),$$

so

$$\Theta([G]) = \mathrm{sh}_{out}(G) \cdot (\theta(\mathrm{sol}(G)) * I_Q^{*k''}) \cdot \mathrm{sh}_{out}(G),$$

which entirely determines Θ.

Existence. Let $\overline{\Theta} : \mathbf{CGr}^{\circlearrowleft}(P) \longrightarrow Q$ be the unique unitary TRAP morphism such that $\Theta(p) = \theta(p)$ for any $p \in P$. Let $G, G' \in \mathbf{CGr}^{\circlearrowleft}(P)$, such that $G \sim G'$. Then:

$$G = \mathrm{sh}_{out}(G) \cdot (\mathrm{sol}(G) * I^{*p}) \cdot \mathrm{sh}_{in}(G),$$
$$G' = \mathrm{sh}_{out}(G) \cdot (\mathrm{sol}(G') * I^{*p}) \cdot \mathrm{sh}_{in}(G),$$

and $\Phi(\mathrm{sol}(G)) = \Phi(\mathrm{sol}(G'))$ in P, so:

$$\begin{aligned}
\overline{\Theta}(G) &= \mathrm{sh}_{out}(G) \cdot (\overline{\Theta}(\mathrm{sol}(G)) * I_Q^{*p}) \cdot \mathrm{sh}_{in}(G) \\
&= \mathrm{sh}_{out}(G) \cdot (\theta \circ \Phi(\mathrm{sol}(G)) * I_Q^{*p}) \cdot \mathrm{sh}_{in}(G) \\
&= \mathrm{sh}_{out}(G) \cdot (\theta \circ \Phi(\mathrm{sol}(G')) * I_Q^{*p}) \cdot \mathrm{sh}_{in}(G) \\
&= \mathrm{sh}_{out}(G) \cdot (\overline{\Theta}(\mathrm{sol}(G')) * I_Q^{*p}) \cdot \mathrm{sh}_{in}(G) \\
&= \overline{\Theta}(G').
\end{aligned}$$

Hence, $\overline{\Theta}$ induces a unitary TRAP morphism $\Theta : \mathbf{uPGr}^{\circlearrowleft}(P) \longrightarrow Q$, extending θ. □

In other words, $\mathbf{uPGr}^{\circlearrowleft}$ is a functor from the category of TRAPs to the category of unitary TRAPs, left adjoint of the forgetful functor from the category of unitary TRAPs to the category of TRAPs. This functor is the functor L of [YJ15, Theorem 12.1] (with the difference that in [YJ15], one works in the coloured setup). Notice that we have a more explicit and straightforward construction of this tensor than the one of [YJ15].

4 Compositions, generalised trace and convolution

4.1 Vertical concatenation in a TRAP

In the same way as wheeled PROPs are PROPs and are equipped with a second associative product [YJ15], TRAPs can be equipped with a natural operation, called the *vertical concatenation*. We start with the various TRAPs of graphs we introduced.

Let G and G' be two graphs such that $o(G) = i(G')$. We define a graph $G'' = G' \circ G$ in the following way:

$$\begin{aligned}
V(G'') &= V(G) \sqcup V(G'),\\
E(G'') &= E(G) \sqcup E(G') \sqcup \{(f,e) \in O(G) \times I(G') : \beta(f) = \alpha'(e)\},\\
I(G'') &= I(G) \sqcup \{(f,e) \in IO(G) \times I(G') : \beta(f) = \alpha'(e)\},\\
O(G'') &= O(G) \sqcup \{(f,e) \in O(G) \times IO(G') : \beta(f) = \alpha'(e)\},\\
IO(G'') &= \{(f,e) \in IO(G) \times IO(G') : \beta(f) = \alpha'(e)\},\\
L(G'') &= L(G) \sqcup L(G').
\end{aligned}$$

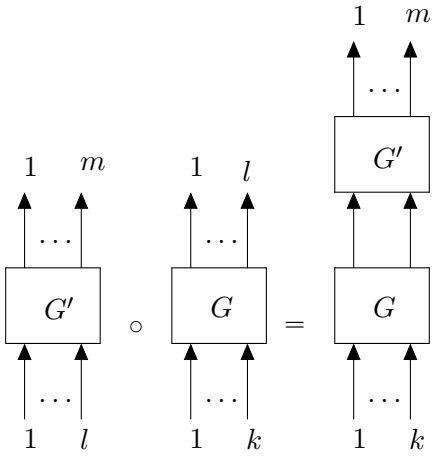
Its **source** and **target** maps are given by:

$$\begin{aligned}
s''_{|E(G)} &= s_{|E(G)}, & s''_{|E(G')} &= s'_{|E(G')}, & s''_{|O(G')} &= s'_{|O(G')}, & s''((f,e)) &= s(f),\\
t''_{|E(G)} &= t_{|E(G)}, & t''_{|E(G')} &= s'_{|E(G')}, & t''_{|I(G)} &= s_{|I(G)}, & s''((f,e)) &= t'(e).
\end{aligned}$$

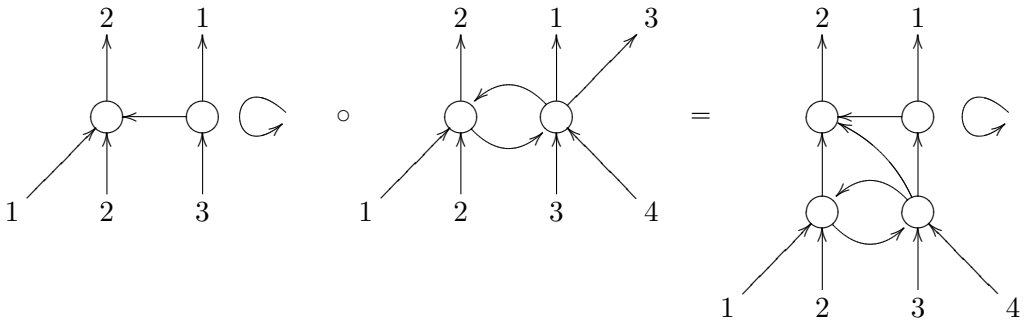
The indexations of its input and output edges are given by:

$$\begin{aligned}
\alpha''_{|I(G)} &= \alpha_{|I(G)}, & \alpha''((f,e)) &= \alpha(f),\\
\beta''_{|O(G')} &= \beta'_{|O(G')}, & \beta''((f,e)) &= \beta'(e).
\end{aligned}$$

Roughly speaking, $G' \circ G$ is obtained by gluing together the outgoing edges of G and the incoming edges of G' according to their indexation as depicted below.



Example 4.1.1. Here is an example of vertical concatenation :



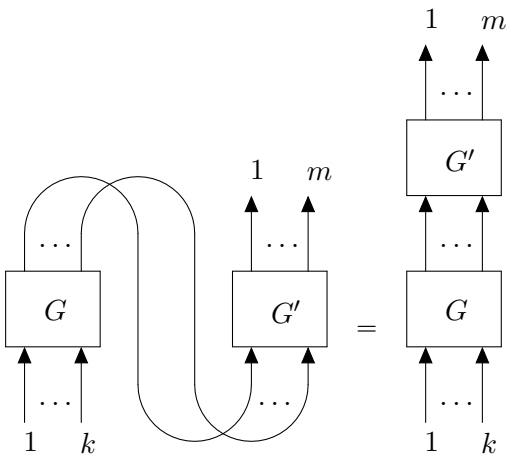
If G and G' are corolla ordered (respectively X-decorated) graphs, then $G \circ G'$ is naturally a corolla ordered (respectively X-decorated) graph. This operation $\circ$ is clearly associative. Moreover, denoting by I the identity graph, for any (k,l) in $\mathbb{N}^2$, for any graph G with k inputs and l outputs,

$$I^{*l} \circ G = G \circ I^{*k} = G.$$

The vertical concatenation can be described in terms of the horizontal concatenation and of the partial trace maps: If G is a graph with k inputs and l outputs, and G' a graph with l inputs and m outputs, then:

$$t_{k+1,1} \circ \ldots \circ t_{k+l-1,l-1} \circ t_{k+l,l}(G * G') = G \circ G',$$

or, graphically:



This construction can be generalised from TRAPs of graphs to an arbitrary TRAP:

Definition-Proposition 4.1.1. *Let P be a TRAP. We define a vertical concatenation*[7] $\circ$ *in the following way:*

$$\forall (k,l,m) \in \mathbb{N}_0^3,\ \forall p \in P(k,l),\ \forall q \in P(l,m), \quad q \circ p := t_{k+1,1} \circ \ldots \circ t_{k+l-1,l-1} \circ t_{k+l,l}(p * q).$$

This operation is associative: for any (k,l,m,n) in $\mathbb{N}_0^4$, for any (p,q,r) in $P(k,l) \times P(l,m) \times P(l,n)$,

$$r \circ (q \circ p) = (r \circ q) \circ p. \tag{16}$$

If the TRAP is unitary, then for any (k,l) in $\mathbb{N}_0^2$, for any p in $P(k,l)$, denoting by I_P the unit of P, then

$$I_P^{*l} \circ p = p \circ I_P^{*k} = p.$$

[7] When there is a risk of confusion, we will write $\circ_P$ for the vertical concatenation of a given TRAP P.

Proof. Recall that in Subsection 3.3 we identified any element p of the decorating set and the solar graph with one vertex decorated by p (see Equation (12)). Let $\alpha : \mathbf{solCGr}^{\circlearrowleft}(P) \longrightarrow P$ be the unique TRAP morphism such that $\alpha(p) = p$ for any $p \in P$ whose existence follows from Theorem 3.3.1 and more specifically from the case detailled in Remark 3.3.1. This is therefore a surjective TRAP morphism. As α respects the horizontal concatenation and the partial trace maps, for any graphs $G, G' \in \mathbf{solCGr}^{\circlearrowleft}(P)$ such that $G \circ G'$ is well-defined, $\alpha(G) \circ \alpha(G')$ is also well-defined and

$$\alpha(G) \circ \alpha(G') = \alpha(G \circ G').$$

Since the vertical concatenation is clearly associative in $\mathbf{solCGr}^{\circlearrowleft}(P)$, the vertical concatenation is associative in P. If P is unitary then again by Theorem 3.3.1, this morphism is extended as a unitary TRAP morphism from $\mathbf{CGr}^{\circlearrowleft}(P)$ to P, which we also denote by α. For any $p \in P(k,l)$, in $\mathbf{CGr}^{\circlearrowleft}(P)$:

$$I^{*l} \circ p = p \circ I^{*k} = p.$$

As $\alpha(I) = I_P$, in P:

$$\alpha(I^{*l} \circ p) = I_P^{*l} \circ p = p = \alpha(p \circ I^{*k}) = p \circ I_P^{*k}.$$ □

Remark 4.1.1. One could also define **partial vertical concatenations**, where only a subset of the outputs are glued to the inputs with the partial trace maps, in the spirit of [YJ15, Paragraph 3.3.3]. We do not pursue this course here since such partial vertical concatenations will play no role in the rest of the paper.

Example 4.1.2. Let V be a vector space and let $f = \theta(v_1 \dots v_l \otimes f_1 \dots f_k) \in \mathbf{Hom}_V^{fr}(k,l)$, $g = \theta(w_1 \dots w_m \otimes g_1 \dots g_l) \in \mathbf{Hom}_V^{fr}(l,m)$. Then, denoting by $\bullet$ the vertical concatenation of $\mathbf{Hom}_V^{fr}$:

$$\begin{aligned} g \bullet f &= g_1(v_1) \dots g_l(v_l)\theta(w_1 \dots w_m \otimes f_1 \dots f_k) \\ &= \theta(w_1 \dots w_m \otimes g_1 \dots g_l) \circ \theta(v_1 \dots v_l \otimes f_1 \dots f_k) \\ &= g \circ f. \end{aligned}$$

Hence, the vertical concatenation induced by the TRAP structure is the usual composition of linear maps. If V is not finite-dimensional, this composition does not have a unit, as Id_V is not of finite rank.

We end this subsection with a simple yet important proposition.

Proposition 4.1.2. *For any two TRAPs $P = (P(k,l))_{(k,l)\in\mathbb{N}_0^2}$ and Q, any TRAP morphism $\phi = (\phi(k,l))_{(k,l)\in\mathbb{N}_0^2} : P \longrightarrow Q$ is compatible with the vertical concatenations of P and Q.*

Proof. We need to show that for any TRAPs P and Q and any TRAP morphism $\phi : P \longrightarrow Q$ as in the statement of the proposition, for any (k,l,m) in $\mathbb{N}_0^3$, p_1 in $P(k,l)$ and p_2 in $P(l,m)$ we have

$$\phi(k,m)(p_2 \circ_P p_1) = \phi(k,l)(p_1) \circ_Q \phi(l,m)(p_2).$$

Using the definition of the vertical concatenation in the TRAP P and the third property of the Definition 1.1.3 of morphisms of TRAPs we have

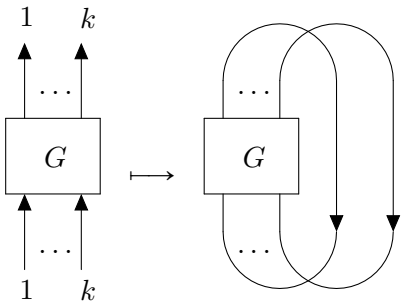
$$\phi(k,m)(p_2 \circ_P p_1) = t^Q_{k+1,1} \circ \dots \circ t^Q_{k+l,l}[\phi(k+l, m+l)(p_1 * p_2)]$$

with $t^Q_{i,j}$ the partial trace maps of the TRAP Q. Then using the second property of Definition 1.1.3 we obtain:

$$\phi(k,m)(p_2 \circ_P p_1) = t^Q_{k+1,1} \circ \dots \circ t^Q_{k+l,l}[\phi(k,l)(p_1) * \phi(l,m)(p_2)] = \phi(k,l)(p_1) \circ_Q \phi(l,m)(p_2).$$ □

4.2 The generalised trace on a TRAP

If G is a graph with the same number of inputs and outputs, we define its generalised trace by, roughly speaking, grafting any of its input to the output with the same index:



In particular, in $\mathbf{CGr}^{\circlearrowleft}(X)$, this construction applied to I gives $\mathcal{O}$. This construction preserves solar graphs, corolla ordered graphs and X-decorated graphs. Moreover, we can describe this construction in terms of the partial trace maps: if $G \in \mathbf{solCGr}^{\circlearrowleft}(X)(k,k)$, then its generalized trace is

$$t_{1,1} \circ \ldots \circ t_{k,k}(G) = t_{1,1} \circ \ldots \circ t_{1,1}(G).$$

These formulas have a meaning for any TRAP:

Definition 4.2.1. *Let P be a TRAP. For any p in $P(k,k)$, with k in $\mathbb{N}_0$, the generalised trace on P is defined as:*

$$\mathrm{Tr}_P(p) := t_{1,1} \circ \ldots \circ t_{k,k}(p) \in P(0,0).$$

In the case of the TRAPs $\mathbf{solCGr}^{\circlearrowleft}(X)$, we shall simply write Tr instead of $\mathrm{Tr}_{\mathbf{solCGr}^{\circlearrowleft}(X)}$.

Proposition 4.2.2. *Let P be a TRAP.*

1. *For any (k,l) in $\mathbb{N}_0^2$, for any (p,q) in $P(k,l) \times P(l,k)$,*

$$\mathrm{Tr}_P(p \circ q) = \mathrm{Tr}_P(q \circ p),$$

which justifies the terminology "trace".

2. *For any (k,l) in $\mathbb{N}_0^2$, for any (p,q) in $P(k,k) \times P(l,l)$,*

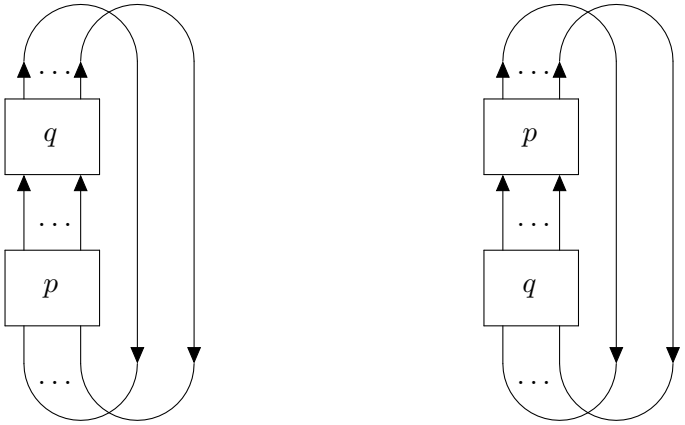
$$\mathrm{Tr}_P(p * q) = \mathrm{Tr}_P(p) * \mathrm{Tr}_P(q).$$

Proof. Let $\alpha : \mathbf{solCGr}^{\circlearrowleft}(P) \longrightarrow P$ be, as before in the proof of Definition-Proposition 4.1.1, the unique TRAP morphism which extends the identity map on P. Since α respects the partial trace maps, for any graph $G \in \mathbf{solCGr}^{\circlearrowleft}(P)(k,k)$,

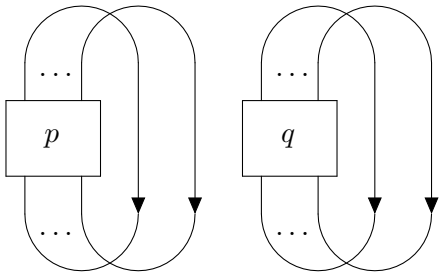
$$\alpha \circ \mathrm{Tr}(G) = \mathrm{Tr}_P \circ \alpha(G).$$

Let $p, q \in P(k,k)$. In $\mathbf{solGr}^{\circlearrowleft}(P)$, $\mathrm{Tr}(q \circ p)$ and $\mathrm{Tr}(p \circ q)$ are represented respectively by the

graphs



which are the same. Applying α, we obtain $\mathrm{Tr}_P(p \circ q) = \mathrm{Tr}_P(q \circ p)$. Moreover, the graph $\mathrm{Tr}(p * q)$ is represented by



which is also a graphical representation of $\mathrm{Tr}(p) * \mathrm{Tr}(q)$. Applying α, we obtain $\mathrm{Tr}_P(p * q) = \mathrm{Tr}_P(p) * \mathrm{Tr}_P(q)$. □

Example 4.2.1. Let V be a finite dimensional vector space and $f = \theta(v_1 \dots v_k \otimes f_1 \dots f_k) \in \mathbf{Hom}_V^{fr}(k,k)$. Identifying $\mathbf{Hom}_V(0,0)$ with $\mathbb{R}$, we obtain that

$$\mathrm{Tr}_{\mathbf{Hom}_V}(f) = f_1(v_1) \dots f_k(v_k),$$

which is the usual trace of linear endomorphisms of a finite-dimensional vector space. If V is not finite-dimensional, $\mathrm{Tr}_{\mathbf{Hom}_V^{fr}}$ is a direct generalisation of this trace for linear endomorphisms of finite rank.

4.3 Amplitudes and generalised convolutions

By Theorem 3.3.1 applied to $\phi = \mathrm{Id}_P$, we know that for any TRAP P there exists a canonical TRAP map $\Phi_P : \mathbf{solCGr}^{\circlearrowleft}(P) \longrightarrow P$ (see Remark 3.3.1).

Definition 4.3.1. *Let G be a graph decorated by a TRAP P. The* ***P-amplitude*** *associated to G is the image of G under Φ_P.*

When $P = \mathcal{K}_M^\infty$ is the TRAP of smooth generalised kernels over a smooth finite dimensional closed Riemannian manifold M of §2.3 (that is $P(k,l) := \mathcal{K}_M^\infty(k,l)$ with the r.h.s defined in (10)), we simply write Φ for Φ_P and call $\Phi(G)$ the ***smooth amplitudes*** *associated to $G \in \mathbf{solCGr}^{\circlearrowleft}(\mathcal{K}_M^\infty)$.*

Remark 4.3.1. The terminology P-amplitude is justified in so far as it associates to a graph an expression in P depending on the ingoing and outgoing edges of the graph in a similar way as an amplitude associated to a Feynman diagram depends on the external ingoing and outgoing momenta.

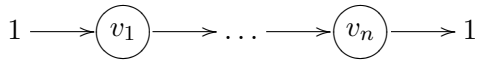
Remark 4.3.2. If we specialise to spaces $\mathcal{E}(M)^{\widehat{\odot}k} \hat{\otimes} \mathcal{E}(M)^{\widehat{\odot}l}$ which are symmetric in both sets of input and output variables, then ϕ can be extended to $\mathbf{solGr}^{\circlearrowleft}(X)$ (see Corollary 3.3.3).

The case of a path graph relates amplitudes to convolutions:

Remark 4.3.3. Let G be a path graph decorated by $X = (\mathcal{K}_X^\infty(k,l))_{k,l\geqslant 0}$, that is to say a graph such that $I(G) = O(G) = [1]$, $IO(G) = L(G) = \varnothing$, $V(G) = \{v_1, \cdots, v_n\}$, $E(G) = \{e_1, \cdots, e_{n-1}\}$ and the source and target maps defined by

$$\begin{aligned} & & s_G(1) = v_n, & & t_G(1) = v_1, \\ \forall i \in [n-1], & & s_G(e_i) = v_i, & & t_G(e_i) = v_{i+1}. \end{aligned}$$

Here is a graphical representation of this graph:



Let $P_i, i = 1, \cdots, n$ be smoothing pseudo-differential operator each of which is defined by the kernel K_i that decorates the vertex v_i. Then the generalised convolution associated to the graph G is the convolution $K_1 \star \cdots \star K_n$ of the kernels $K_1, \cdots, K_n$, which is the kernel of the smoothing pseudo-differential operator $P_1 \circ \cdots \circ P_n$. In this sense, P-amplitudes can be seen as a generalisation of the convolution of multiple smooth kernels.

Proposition 4.3.2. *For any TRAP P, the P-amplitude associated to a horizontal concatenation of graphs is the horizontal concatenation of their P-amplitudes: for any $G_1, G_2 \in \mathbf{solCGr}^{\circlearrowleft}(P)$,*

$$\Phi_P(G_1 * G_2) = \Phi_P(G_1) * \Phi_P(G_2),$$

and the same holds for the vertical concatenation: if $G_1 \circ G_2$ exists, then

$$\Phi_P(G_1 \circ G_2) = \Phi_P(G_1) \circ_P \Phi_P(G_2)$$

with $\circ_P$ the vertical concatenation of P.

Proof. This follows directly from the fact that Φ_P is a TRAP morphism and from Proposition 4.1.2. □

For any TRAP P, let $\iota_P : P \hookrightarrow \mathbf{solCGr}^{\circlearrowleft}(P)$ be the canonical embedding of P into the TRAP of P-decorated graphs that is, $\iota_P(p)$ is the solar graph with only one vertex decorated by p. We have the following simple yet useful lemma.

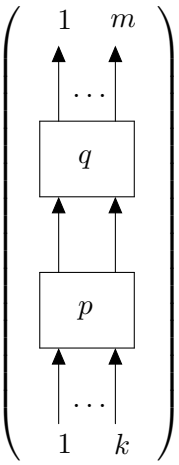
Lemma 4.3.3. *For any TRAP P the following diagram commutes:*

$$\begin{array}{ccc} P \times P & \hookrightarrow & \mathbf{solCGr}^{\circlearrowleft}(P) \times \mathbf{solCGr}^{\circlearrowleft}(P) \\ \downarrow{\scriptstyle \circ_P} & & \downarrow{\scriptstyle \circ} \\ P & \xleftarrow[\Phi_P]{} & \mathbf{solCGr}^{\circlearrowleft}(P) \end{array}$$

with $\circ_p$ the vertical concatenation of the TRAP P, the top arrow given by $\iota_P \times \iota_P$ and the obvious abuse of notation that vertical concatenations, if seen as maps, are not defined on the whole of their domains.

In words, the vertical concatenation of two elements p_1 and p_2 of P is the P-amplitude associated with the graph given by the vertical concatenation of two graphs with exactly one

vertex, each decorated by one p_i. Graphically, if $p \in P(k,l)$ and $q \in P(l,m)$:

Φ_P 

$= \Phi_P(p) \circ_P \Phi_P(q).$

Proof. Let P be a TRAP. Then for any p_1, p_2 in P such that $p_1 \circ_P p_2$ is well defined, $\iota_P(p_1) \circ \iota_P(p_2)$ is well-defined since ι_P respects the gradings and we have

$$\begin{aligned}\Phi_P(\iota_P(p_1) \circ \iota_P(p_2)) &= \Phi_P(\iota_P(p_1)) \circ_P \Phi(\iota_P(p_2)) \quad \text{by Proposition 4.3.2} \\ &= p_1 \circ_P p_2\end{aligned}$$

since for any TRAP P, $\Phi_P \circ \iota_P = \mathrm{Id}_P$ by definition of Φ_P (Equation (14) with $k = 1$ and $\phi = \mathrm{Id}_P$). □

Remark 4.3.4. Note that the vertical concatenation is *not* the same as the P-amplitude: the latter has a much larger domain.

Applying the above constructions to the TRAP of smooth kernels described in Theorem 2.3.1, whose partial traces (11) are given by integrations on the underlying manifold, easily yields the following statement. We use the notations of Paragraph 2.3: M is a smooth, finite dimensional orientable closed manifold and $\mu(z)$ is a volume form on M.

Theorem 4.3.4. *For the TRAP $(\mathcal{K}_M^\infty(k,l))_{k,l\geqslant 0}$, the following statements hold:*

1. *The vertical concatenation of two kernels corresponds to their **generalised convolution**:*

$$\begin{aligned}&\forall (k,l,m) \in \mathbb{N}_0^3,\ \forall K_1 \in \mathcal{K}_M^\infty(k,l),\ \forall K_2 \in \mathcal{K}_M^\infty(l,m), \forall (x_1,\cdots,x_k,z_1,\cdots,z_m) \in M^{k+m}, \\ &K_2 \circ K_1(x_1,\cdots,x_k,z_1,\cdots,z_m) \\ &= t_{k+1,1} \circ \ldots \circ t_{k+l-1,l-1} \circ t_{k+l,l}(K_1 \otimes K_2)(x_1,\cdots,x_k,z_1,\cdots,z_m) \\ &= \int_{M^l} K_1(x_1,\cdots,x_k,y_1,\cdots,y_l)\, K_2(y_1,\cdots,y_l,z_1,\cdots,z_m)\, d\mu(y_1)\cdots d\mu(y_l),\end{aligned}$$

   obtained by integrating along the diagonal $\Delta_M^l := \{(y_1,\cdots,y_l,y_1,\cdots,y_l), y_i \in M\} \subset M^{2l}$.

2. *The associativity property $K_3 \circ (K_2 \circ K_1) = (K_3 \circ K_2) \circ K_1$ (cfr. (16)) for any $K_3 \in \mathcal{K}_M^\infty(m,n)$, amounts to the Fubini property for the corresponding multiple integrals:*

$$\begin{aligned}&\int_{M^m} \left(\int_{M^l} K_1(\vec{x},\vec{y}_1) K_2(\vec{y}_1,\vec{y}_2) d\vec{\mu}(\vec{y}_1) \right) K_3(\vec{y}_2,\vec{z}) d\vec{\mu}(\vec{y}_2) \\ &= \int_{M^l} K_1(\vec{x},\vec{y}_1) \left(\int_{M^m} K_2(\vec{y}_1,\vec{y}_2) K_3(\vec{y}_2,\vec{z}) d\vec{\mu}(\vec{y}_2) \right) d\vec{\mu}(\vec{y}_1)\end{aligned} \tag{17}$$

   for any $\vec{x} \in M^k$ and $\vec{z} \in M^n$, where we use the short-hand notations $d\vec{\mu}(\vec{y}_i) := d\mu(y_1)\cdots d\mu(y_{l_i})$.

3. *The generalised trace of a generalised kernel* $K = K_1 \otimes K_2 \in \mathcal{K}_M^\infty(k,k)$ *is given by the integral along the small diagonal of* M^k*:*

$$\mathrm{Tr}_{K^\infty}(K) = \int_{M^k} K(x_1, \cdots, x_k, x_1, \cdots, x_k)\, d\mu(x_1) \cdots d\mu(x_k) \tag{18}$$

where we have set $K(\vec{x}, \vec{y}) := K_1(\vec{x})K_2(\vec{y})$ *and obeys the following cyclicity property:*

$$\mathrm{Tr}_{K^\infty}(\tilde{K} \circ K) = \mathrm{Tr}_{K^\infty}(K \circ \tilde{K})$$

for $K \in \mathcal{K}_M^\infty(k,l)$ *and* $\tilde{K} \in \mathcal{K}_M^\infty(l,k)$*.*

4. *The* $\mathcal{K}_M^\infty$*-amplitude is compatible with the horizontal and vertical concatenations in* $\mathcal{K}_M^\infty$*.*

Proof. We prove the assertions one by one.

1. The vertical concatenation $\circ$ of Definition-Proposition 4.1.1 applied to the TRAP $\mathcal{K}_M^\infty$ of smooth kernel of Theorem 2.3.1 gives the generalised convolution..
2. As proved in Definition-Proposition 4.1.1, the vertical concatenation $\circ$ of any TRAP is associative. Writing the explicit expression of each side of the equation $K_1 \circ (K_2 \circ K_3) = (K_1 \circ K_2) \circ K_3$ for the vertical concatenation of the TRAP $\mathcal{K}_M^\infty$ shows that the identity amounts to the Fubini property for multiple integrals as given by Equation (17).
3. By Equation (10), for any K in $\mathcal{K}_M^\infty(k,k)$, we can write $K = K_1 \otimes K_2$ with K_1 and K_2 in $\mathcal{E}^{\hat{\otimes} k}$. The generalised trace of Definition 4.2.1 for the TRAP $\mathcal{K}_M^\infty$ of smooth kernel of Theorem 2.3.1 combined with the partial traces of this TRAP given by Equation (11) yields Equation (18). The cyclicity property of Tr_{K^∞} follows from the cyclicity property of generalised traces (Proposition 4.2.2, item 1).
4. This follows from Proposition 4.3.2 applied to the generalised amplitude of Definition 4.3.1 for the TRAP $\mathcal{K}_M^\infty$ of smooth kernel discussed in Theorem 2.3.1.

□

5 Categorical interpretation

We describe TRAPs and unitary TRAPs as algebras over an endofunctor of the category of $\mathfrak{S} \times \mathfrak{S}^{op}-$modules, thus extending known results of [YJ15] on the categorial aspects of wheeled PROPs.

5.1 Two endofunctors in the category of $\mathfrak{S} \times \mathfrak{S}^{op}$modules

We consider graphs decorated by a $\mathfrak{S} \times \mathfrak{S}^{op}$-module $X = (X(k,l))_{k,l \geqslant 0}$ and use the action of the symmetric groups on the vertices of Definition 3.3.2 to define an endofunctor $\Gamma^{\circlearrowleft}$.

Definition 5.1.1. *We define a relation on* $\mathbf{CGr}^{\circlearrowleft}(X)(k,l)$ *by* $(G, d_G)\mathcal{R}_{k,l}(G', d_{G'})$ *for* (G, d_G) *and* $(G', d_{G'})$ *if there exists a vertex* v *of* G *and permutations* $\sigma \in \mathfrak{S}_{o(v)}$*,* $\tau \in \mathfrak{S}_{i(v)}$ *such that*

$$\sigma \cdot_v (G, d_G) \cdot_v \tau = (G, d_G^{\sigma,v,\tau})$$

with

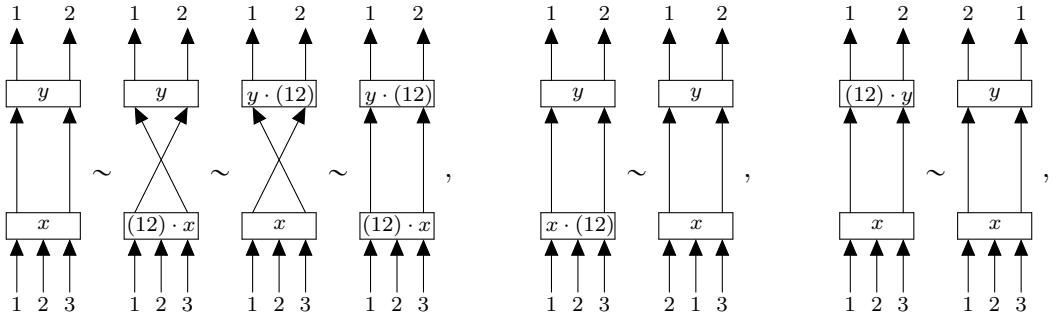
$$d_G^{\sigma,v,\tau}(v') = \begin{cases} d_G(v') & \text{for } v' \neq v \\ \sigma \cdot d_G(v) \cdot \tau & \text{otherwise.} \end{cases}$$

We denote by $\sim_{k,l}$ *the transitive closure of* $\mathcal{R}_{k,l}$ *which defines an equivalence. We further define*

$$\Gamma^{\circlearrowleft}(X)(k,l) := \frac{\mathbf{CGr}^{\circlearrowleft}(X)(k,l)}{\sim_{k,l}}, \qquad \mathbf{sol}\Gamma^{\circlearrowleft}(X)(k,l) := \frac{\mathbf{solCGr}^{\circlearrowleft}(X)(k,l)}{\sim_{k,l}}.$$

We further write $\Gamma^{\circlearrowleft}(X) := (\Gamma^{\circlearrowleft}(X)(k,l))_{k,l \geqslant 0}$ *and* $\mathbf{sol}\Gamma^{\circlearrowleft}(X) := (\mathbf{sol}\Gamma^{\circlearrowleft}(X)(k,l))_{k,l \geqslant 0}$.

Here is the type of relations we obtain graphically:



where $x \in X_{3,2}$ and $y \in X_{2,2}$.

It is easy to show that the family of equivalences $(\sim_{k,l})_{k,l\geqslant 0}$ is compatible with the action of $\mathfrak{S} \times \mathfrak{S}^{op}$, the partial trace maps and the horizontal concatenation in the sense of Lemma 1.2.1. The subsequent useful statement follows from Lemma 1.2.1.

Lemma 5.1.2. *Let X be a $\mathfrak{S} \times \mathfrak{S}^{op}$-module. $\Gamma^{\circlearrowleft}(X)$ is a unitary TRAP and $\mathbf{sol}\Gamma^{\circlearrowleft}(X)$ is a TRAP.*

Proof. By Corollary 3.2.2, $\mathbf{CGr}^{\circlearrowleft}(X)$ is a TRAP, and it is easy to see that the family $(\sim_{k,l})_{(k,l)\in\mathbb{N}_0^2}$ satisfies the conditions of Lemma 1.2.1. Hence, $\Gamma^{\circlearrowleft}(X)$ is a TRAP. The proof is similar for $\mathbf{sol}\Gamma^{\circlearrowleft}(X)$. □

Proposition 5.1.3. *Let X be $\mathfrak{S}\times\mathfrak{S}^{op}$ module, P be a TRAP and $\phi : X \longrightarrow P$ be a morphism of $\mathfrak{S}\times\mathfrak{S}^{op}$-modules. There exists a unique extension ϕ to a TRAP morphism $\Phi : \mathbf{sol}\Gamma^{\circlearrowleft}(X) \longrightarrow P$. If moreover P is unitary, this morphism extends to $\Gamma^{\circlearrowleft}(X)$.*

Proof. We know from Theorem 3.3.1 that $\mathbf{solCGr}^{\circlearrowleft}(X)$ is the free TRAP generated by the set X. Hence, ϕ is extended as a TRAP morphism $\overline{\Phi} : \mathbf{solCGr}^{\circlearrowleft}(X) \longrightarrow P$. Let $G, H \in \mathbf{solCGr}^{\circlearrowleft}(X)(k,l)$, with $(k,l) \in \mathbb{N}_0^2$. If $G\mathcal{R}_{k,l}H$, then, as ϕ is compatible with the actions of the symmetric groups, $\overline{\Phi}(G) = \overline{\Phi}(H)$. By transitive closure, if $G \sim_{k,l} H$, then $\overline{\Phi}(G) = \overline{\Phi}(H)$. Consequently, $\overline{\Phi}$ induces a TRAP morphism $\Phi : \mathbf{sol}\Gamma^{\circlearrowleft}(X) \longrightarrow P$, which extends ϕ. It is obviously unique, as X generates $\mathbf{sol}\Gamma^{\circlearrowleft}(X)$. The proof is similar for $\Gamma^{\circlearrowleft}(X)$. □

In other words, $\mathbf{sol}\Gamma^{\circlearrowleft}(X)$ is the free TRAP generated by the $\mathfrak{S} \times \mathfrak{S}^{op}$-module X and $\Gamma^{\circlearrowleft}(X)$ is the free unitary TRAP generated by the $\mathfrak{S} \times \mathfrak{S}^{op}$-module X.

Example 5.1.1. If X is a trivial $\mathfrak{S} \times \mathfrak{S}^{op}$-module, then $\Gamma^{\circlearrowleft}(X) = \mathbf{Gr}^{\circlearrowleft}(X)$ and $\mathbf{sol}\Gamma^{\circlearrowleft}(X) = \mathbf{solGr}^{\circlearrowleft}(X)$ as TRAPs. More generally, choosing for any graph G a corolla ordered graph $\overline{G}$ which underlying graph is G, we can prove that for any (k,l) in $\mathbb{N}_0^2$, the sets $\Gamma^{\circlearrowleft}(X)(k,l)$ and $\mathbf{Gr}^{\circlearrowleft}(X)(k,l)$ are in bijection, as well as $\mathbf{sol}\Gamma^{\circlearrowleft}(X)(k,l)$ and $\mathbf{solGr}^{\circlearrowleft}(X)(k,l)$ (but not in a canonical way), through the map sending the equivalence class of $\overline{G}$ to G.

The correspondence $P \to \Gamma^{\circlearrowleft}(P)$ defined above induces an endofunctor in the category $\mathbf{Mod}_{\mathfrak{S}}$ of $\mathfrak{S} \times \mathfrak{S}^{op}$-modules which we now introduce.

Definition 5.1.4. *Let $\mathbf{Mod}_{\mathfrak{S}}$ denote the category of $\mathfrak{S} \times \mathfrak{S}^{op}$-modules: its objects are families $X = (X(k,l))_{k,l\geqslant 0}$, such that for any (k,l) in $\mathbb{N}_0^2$, $X(k,l)$ is a $\mathfrak{S}_l \times \mathfrak{S}_k^{op}$-module; a morphism $\phi : X \longrightarrow Y$ is a family $(\phi(k,l))_{k,l\geqslant 0}$, where for any (k,l) in $\mathbb{N}_0^2$, $\phi(k,l) : X(k,l) \longrightarrow Y(k,l)$ is a morphism of $\mathfrak{S}_l \times \mathfrak{S}_k^{op}$-modules.*

By the functoriality of post-composition of morphisms of $\mathfrak{S} \times \mathfrak{S}^{op}$-modules, we obtain:

Definition-Proposition 5.1.5. *We define two endofunctors $\Gamma^{\circlearrowleft}$ and $\mathbf{sol}\Gamma^{\circlearrowleft}$ on the category $\mathbf{Mod}_{\mathfrak{S}}$ as follows. Both are defined on objects as in Definition 5.1.1. For a morphism $\phi : X \longrightarrow Y$ of $\mathfrak{S} \times \mathfrak{S}^{op}$-modules, the morphisms $\Gamma^{\circlearrowleft}(\phi) : \Gamma^{\circlearrowleft}(X) \longrightarrow \Gamma^{\circlearrowleft}(Y)$ and $\mathbf{sol}\Gamma^{\circlearrowleft}(\phi) : \mathbf{sol}\Gamma^{\circlearrowleft}(X) \longrightarrow \mathbf{sol}\Gamma^{\circlearrowleft}(Y)$ are defined by post-composing ϕ with the decoration map of Definition 3.1.4. That is, for $(G, d_G) \in \mathbf{CGr}^{\circlearrowleft}(X)$ and $(H, d_H) \in \mathbf{solCGr}^{\circlearrowleft}(X)$, we have:*

$$\Gamma^{\circlearrowleft}(\phi)(G, d_G) = (G, \phi \circ d_G), \qquad \mathbf{sol}\Gamma^{\circlearrowleft}(\phi)(H, d_H) = (H, \phi \circ d_H).$$

Proof. We need to prove that $\Gamma^{\circlearrowleft}$ and $\mathbf{sol}\Gamma^{\circlearrowleft}$ are indeed functors. Let X and Y be two $\mathfrak{S} \times \mathfrak{S}^{op}$-modules and $\varphi : X \longrightarrow Y$ a morphism of $\mathfrak{S}\times\mathfrak{S}^{op}$-modules and let $\mathbf{solCGr}^{\circlearrowleft}(\varphi) : \mathbf{solCGr}^{\circlearrowleft}(X) \longrightarrow \mathbf{solCGr}^{\circlearrowleft}(Y)$ be its pullback, defined by

$$\mathbf{solCGr}^{\circlearrowleft}(\varphi)(G, d_G) := (G, \varphi \circ d_G) \tag{19}$$

for any $G \in \mathbf{solCGr}^{\circlearrowleft}(X)$. It is easy to check that the induced morphism $\mathbf{sol}\Gamma^{\circlearrowleft}(\varphi) : \mathbf{sol}\Gamma^{\circlearrowleft}(P) \longrightarrow \mathbf{sol}\Gamma^{\circlearrowleft}(Q)$ is indeed a morphism of $\mathfrak{S}\times\mathfrak{S}^{op}$-modules, turning $\mathbf{sol}\Gamma^{\circlearrowleft}$ into an endofunctor of $\mathbf{Mod}_{\mathfrak{S}}$. The proof is similar for $\Gamma^{\circlearrowleft}$. □

5.2 Monad of graphs

We now endow the functor $\Gamma^{\circlearrowleft}$ with a monad structure.

We first recall basic definitions of category theory. In particular, for two functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$, a **natural transformation** $\eta : F \longrightarrow G$ between these two functors is given by maps $\eta_X : F(X) \longmapsto G(X)$ for each object X of $\mathcal{C}$ such that for any pair of objects $X, Y \in \mathrm{Obj}(\mathcal{C})$ and morphism $f : X \longmapsto Y \in \mathrm{Mor}(\mathcal{C})$ the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y). \end{array} \tag{20}$$

Let us now introduce the structure of monad, a terminology we borrow from [ML98]. A monad is the categorical equivalent of monoids.

Definition 5.2.1. *A **monad** (also called a triple) on a category $\mathcal{C}$ is given by an endofunctor $\Gamma \in \mathrm{End}(\mathcal{C})$ and two natural transformations $\mu : \Gamma \circ \Gamma \longrightarrow \Gamma$ and $\nu : \mathrm{Id}_{\mathcal{C}} \longrightarrow \Gamma$ which form an associative and unital monoid (Γ, μ, ν) in the unital monoid[8] $\mathrm{End}(\mathcal{C})$ of endofunctors of $\mathcal{C}$. This means that the multiplication $\mu : \Gamma \circ \Gamma \longrightarrow \Gamma$ and the unit morphism $\nu : \mathrm{Id}_{\mathcal{C}} \longrightarrow \Gamma$ should satisfy the axioms given by commutativity of the diagrams below for any object P of the category $\mathcal{C}$.*

$$\begin{array}{ccc} \Gamma\circ\Gamma\circ\Gamma(P) & \xrightarrow{\Gamma(\mu_P)} & \Gamma\circ\Gamma(P) \\ {\scriptstyle \mu_{\Gamma(P)}}\downarrow & & \downarrow{\scriptstyle \mu_P} \\ \Gamma\circ\Gamma(P) & \xrightarrow[\mu_P]{} & \Gamma(P) \end{array} \qquad \begin{array}{ccccc} \Gamma(P) & \xrightarrow{\Gamma(\nu_P)} & \Gamma\circ\Gamma(P) & \xleftarrow{\nu_{\Gamma(P)}} & \Gamma(P) \\ & {\scriptstyle \mathrm{Id}_{\mathcal{C}}}\searrow & {\scriptstyle \mu_P}\downarrow & \swarrow{\scriptstyle \mathrm{Id}_{\mathcal{C}}} & \\ & & \Gamma(P) & & \end{array} \tag{21}$$

We now recall the notion of Γ-algebra (see for example [MMS09, Definition 2.1.4]).

Definition 5.2.2. *Let $\mathcal{C}$ be a category. An algebra over a monad $\Gamma \in \mathrm{End}(\mathcal{C})$ or a **Γ-algebra** is an object P of $\mathcal{C}$ together with a structure morphism $\alpha : \Gamma(P) \to P$ such that the following*

[8] The terminology monoid is used in this definition with an obvious abuse of vocabulary since Γ and $\mathrm{End}(\mathcal{C})$ are not necessarily sets.

diagrams commute:

$$\begin{array}{ccc} \Gamma\circ\Gamma(P) & \xrightarrow{\Gamma(\alpha)} & \Gamma(P) \\ {\scriptstyle \mu_P}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \Gamma(P) & \xrightarrow[\alpha]{} & P \end{array} \qquad \begin{array}{ccc} P & \xrightarrow{\nu_P} & \Gamma(P) \\ {\scriptstyle \mathrm{Id}}\downarrow & \swarrow{\scriptstyle \alpha} & \\ P & & \end{array} \tag{22}$$

Let (P,α) and (Q,β) be two algebras over a fixed monad Γ. A morphism of Γ-algebras from P to Q is a morphism $\phi : P \longrightarrow Q$ in the category $\mathcal{C}$ such that the following diagram commutes:

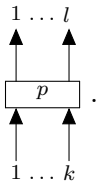
$$\begin{array}{ccc} \Gamma(P) & \xrightarrow{\alpha} & P \\ {\scriptstyle \Gamma(\phi)}\downarrow & & \downarrow{\scriptstyle \phi} \\ \Gamma(Q) & \xrightarrow[\beta]{} & Q \end{array} \tag{23}$$

We now define the natural transformations ν and μ in the case $\mathcal{C} = \mathbf{Mod}_{\mathfrak{S}}$ and $\Gamma = \Gamma^{\circlearrowleft}$. In this case, for any $\mathfrak{S}\times\mathfrak{S}^{op}$-module P, elements of $\Gamma^{\circlearrowleft}\circ\Gamma^{\circlearrowleft}(P)$ are graphs G whose vertices v are decorated by graphs G_v, consistently with the number of incoming and outgoing edges.

Definition 5.2.3. *1. For any $\mathfrak{S}\times\mathfrak{S}^{op}$-module P, let $\eta_P : P \longrightarrow \Gamma^{\circlearrowleft}(P)$ be the morphism of $\mathfrak{S}\times\mathfrak{S}^{op}$-modules which sends an element $p \in P(k,l)$ to the class of the graph $G(k,l)(p)$ with one vertex v decorated by p, k incoming edges indexed from left to right by $1,\dots,k$ and l outgoing edges indexed from left to right by $1,\dots,l$.*

2. For any $\mathfrak{S}\times\mathfrak{S}^{op}$-module P, let $\mu_P : \Gamma^{\circlearrowleft}\circ\Gamma^{\circlearrowleft}(P) \longrightarrow \Gamma^{\circlearrowleft}(P)$ be the morphism $\mathfrak{S}\times\mathfrak{S}^{op}$-modules which sends a graph $G \in \Gamma^{\circlearrowleft}\circ\Gamma^{\circlearrowleft}(P)$ to the graph $H \in \Gamma^{\circlearrowleft}(P)$ with $V(H) = \bigsqcup_{v\in V(G)} V(G_v)$ and whose edges are obtained by identifying, for any vertex v, the i-th incoming edges of v with the i-th incoming edge of G_v, and the j-th outgoing edge of v with the j-th outgoing edge of G_v.

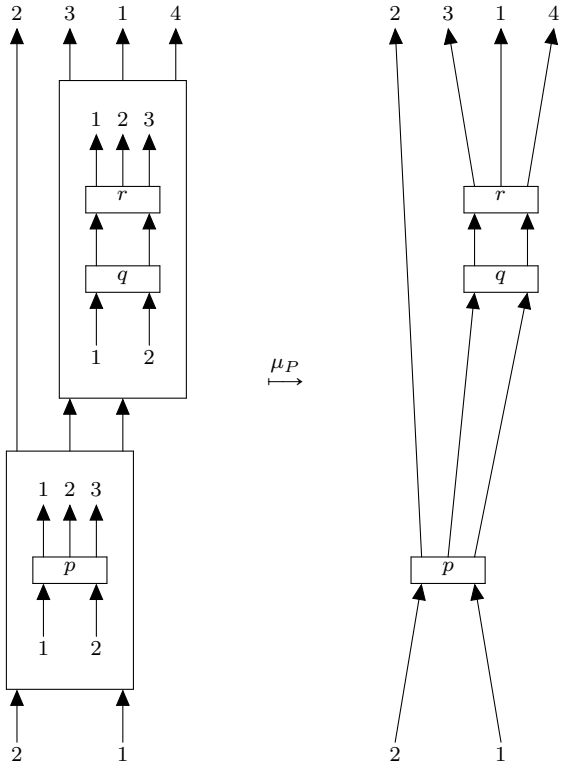
In simpler words, the map η_P sends an element $p \in P$ to the graph with one vertex, which is decorated by p and has the same numbers of input and output edges as p has inputs and outputs. In picture:

$\nu_P(p) =$ 

.

Furthermore, the map μ_P replaces vertices decorated by graphs (G, d_G) by (decorated) subgraphs. These subgraphs are exactly the graphs that were decorating the vertices of the original graph. To illustrate this graphically, we give an example in which μ_P sends the graph on the left

to the graph on the right:



where $p \in P(2,3)$, $q \in P(2,2)$ and $r \in P(2,3)$.

The families of morphisms η_P and μ_P define two natural transformations and we further obtain:

Proposition 5.2.4. *The triple $\Gamma^{\circlearrowleft} = (\Gamma^{\circlearrowleft}, \mu, \nu)$ is a monad in the category $\mathbf{Mod}_{\mathfrak{S}}$. Moreover, $\mathbf{sol}\Gamma^{\circlearrowleft} = (\mathbf{sol}\Gamma^{\circlearrowleft}, \mu_{|\mathbf{sol}\Gamma^{\circlearrowleft}}, \nu)$ is a sub-monad of $\Gamma^{\circlearrowleft}$.*

Proof. The associativity of μ is graphically immediate, as well as the fact that ν is a unit. The functor ν takes its values in $\mathbf{sol}\Gamma^{\circlearrowleft}$ and the composition of solar graphs is a solar graph, so $\mathbf{sol}\Gamma^{\circlearrowleft}$ is a submonad of $\Gamma^{\circlearrowleft}$. □

5.3 TRAPs versus wheeled PROPs

We can now state the main result of this section, which relates TRAPs and various known objects.

Theorem 5.3.1. *The categories of $\Gamma^{\circlearrowleft}$-algebras and of unitary TRAPs are isomorphic. Similarly, the categories of $\mathbf{sol}\Gamma^{\circlearrowleft}$-algebras and of TRAPs are isomorphic.*

Remark 5.3.1. Wheeled PROPs are defined (for example in [MMS09]) as $\Gamma^{\circlearrowleft}$-algebras. Thus Theorem 5.3.1 precisely says that wheeled PROPs and unitary TRAPs coincide, and that TRAPs can be viewed as non-unitary wheeled PROPs.

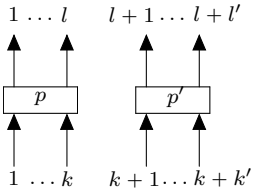
Proof. Let us start with the non-unitary case.

From Proposition 5.1.3, we know that $\mathbf{sol}\Gamma^{\circlearrowleft}(P)$ is the free TRAP generated by the $\mathfrak{S} \times \mathfrak{S}^{op}$-module P. If P is a TRAP, then the canonical TRAP morphism $\alpha_P : \mathbf{sol}\Gamma^{\circlearrowleft}(P) \longrightarrow P$ of Proposition 5.1.3 makes it a $\mathbf{sol}\Gamma^{\circlearrowleft}$-algebra. Furthermore, since $\mathbf{sol}\Gamma^{\circlearrowleft}$ is a functor by Definition-Proposition 5.1.5, for any TRAP morphism $\phi : P \longrightarrow Q$ we have the existence of $\mathbf{sol}\Gamma^{\circlearrowleft}(\phi) : \mathbf{sol}\Gamma^{\circlearrowleft}(P) \longrightarrow \mathbf{sol}\Gamma^{\circlearrowleft}(Q)$. Then by construction $\phi \circ \alpha_P = \alpha_Q \circ \mathbf{sol}\Gamma^{\circlearrowleft}(\phi)|_P$. By unicity of the

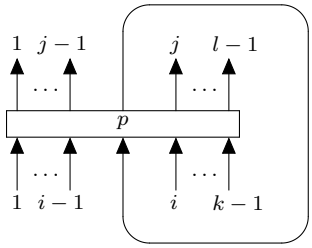
lift of $\phi \circ \alpha_P$ given by Theorem 3.3.1 we obtain $\phi \circ \alpha_P = \alpha_Q \circ \mathbf{sol}\Gamma^{\circlearrowleft}(\phi)|_P$, that is that Diagram (23) commutes. Thus we have defined a functor from the category of TRAPs to the category of $\mathbf{sol}\Gamma^{\circlearrowleft}$-algebras.

Conversely, if (P, α) is a $\mathbf{sol}\Gamma^{\circlearrowleft}$-algebra:

- For any $(p, p') \in P(k, l) \times P(k', l')$,we define $p * p'$ by applying α to the following graph:



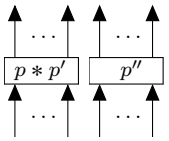
- For any $p \in P(k, l)$, for any $(i, j) \in [k] \times [l]$, we define $t_{i,j}(p)$ by the application of α to the following graph:



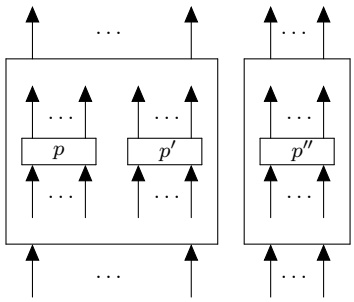
Let us prove some of the axioms of TRAPs for P. The others can be proved in the same way and are left to the reader.

1. holds by the functoriality of $\mathbf{sol}\Gamma^{\circlearrowleft}$.

2. (a): let $(p, p', p'') \in P(k, l) \times P(k', l') \times P(k'', l'')$. Then $(p * p') * p''$ is obtained by the application of α_P to the graph:



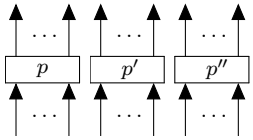
(For the sake of simplicity, we delete the indices of the input and output edges of this graph: they are always indexed from left to right). Hence, $(p * p') * p''$ is obtained by application of $\alpha \circ \Gamma^{\circlearrowleft}(\alpha)$ to the graph:



Note that for the second connected component of this graph, this comes from:

$$\alpha \circ \Gamma^{\circlearrowleft}(\alpha) \circ \Gamma^{\circlearrowleft}(\nu_P)(p'') = \alpha \circ \Gamma^{\circlearrowleft}(\alpha \circ \nu_P)(p'') = \alpha \circ \Gamma^{\circlearrowleft}(\mathrm{Id}_P)(p'') = \alpha(p'').$$

As $\alpha \circ \Gamma^{\circlearrowleft}(\alpha) = \alpha \circ \mu_P$, $(p * p') * p''$ is obtained by applying α to the graph:

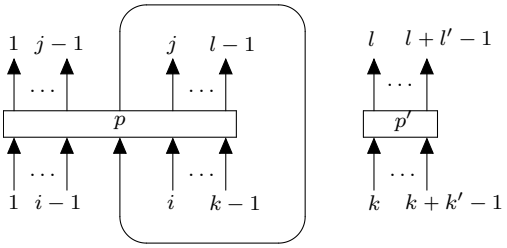


The same computation can be carried out for $p * (p' * p'')$, which gives the associativity of $*$.

2. (b): the unity I_0 of the concatenation product of graph is the empty graph, which is the image of the unity of P for the horizontal concatenation under α.

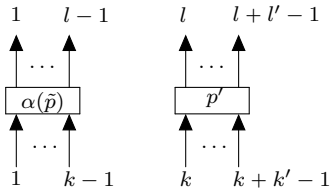
2. (c) holds trivially by definition of the horizontal concatenation product on P, the $\mathfrak{S} \times \mathfrak{S}^{op}$-module structure of P, and the fact that $\mathbf{solCGr}^{\circlearrowleft}(X)$ is a TRAP.

3. (c): for any $k, l, k', l' \geqslant 1$, for any $i \in [k]$, $j \in [l]$, for any $p \in P(k,l)$, $p' \in P(k',l')$, $t_{i,j}(p * p')$ is the image under α_P of the graph



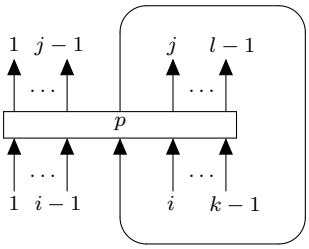
(24)

On the other hand, $t_{i,j}(p) * p'$ is the image under α_P of the graph



(25)

with $\tilde{p}$ the image under α_P of the graph



The images of the graphs (24) and (25) under α_P are identical by the commutativity of the first diagram of (22). The case $l + 1 \leqslant j \leqslant l + l'$ and $k + 1 \leqslant i \leqslant k + k'$ holds by the same argument.

Let us now focus on the unitary case.

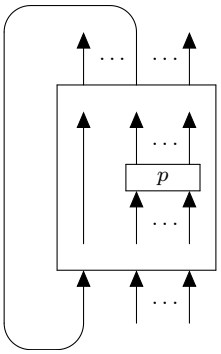
First if P is a unitary TRAP, then by Theorem 3.3.1, (P, μ_P) is a $\Gamma^{\circlearrowleft}$-algebra with exactly the same argument as in the non unitary case.

Conversely, let (P, α_P) be a $\Gamma^{\circlearrowleft}$-algebra. We then set

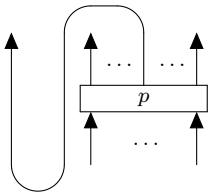
$$I := \alpha_P(I_1)$$

where I_1 is the graph with only one input-output edge.

Let $p \in P(k,l)$ and $2 \leqslant j \leqslant l+1$. Then $t_{1,j}(I * p)$ is obtained by applying $\alpha \circ \Gamma^{\circlearrowleft}(\alpha)$ to the graph:



where the curved edge relate the first edge at the bottom to the j-th edge on the top. As $\alpha \circ \Gamma^{\circlearrowleft}(\alpha) = \alpha \circ \mu_P$, $t_{1,j}(I * p)$ is obtained by application of α to the graph:



where the curved edge relate the first edge on the bottom to the j-th edge on the top (note that this edge is also the $(j-1)$-th outgoing the vertex decorated by p). As α is a $\mathfrak{S} \times \mathfrak{S}^{op}$ morphism, we obtain that this is $(1, \ldots, j-1) \cdot \alpha \circ \nu_P(p)$, that is to say $(1, \ldots, j-1) \cdot p$.

In this way, we define a functor from the category of $\mathbf{sol}\Gamma^{\circlearrowleft}$-algebras to the category of TRAPs. In the same way, we define a functor from the category of $\Gamma^{\circlearrowleft}$-algebras to the category of unitary TRAPs.

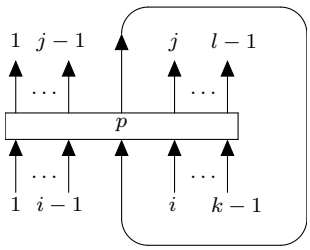
We obtain in this way two functors

$$\mathcal{F} : \mathbf{TRAP} \longrightarrow \mathbf{Alg}(\mathbf{sol}\Gamma^{\circlearrowleft}), \qquad \mathcal{G} : \mathbf{Alg}(\mathbf{sol}\Gamma^{\circlearrowleft}) \longrightarrow \mathbf{TRAP}.$$

Let P be a TRAP and P' the TRAP $\mathcal{G} \circ \mathcal{F}(P)$, with concatenation $*'$ and trace operators $t'_{i,j}$. We set $\mathcal{F}(P) := (P, \alpha_P)$: in other words, α_P is the TRAP morphism from $\mathbf{sol}\Gamma^{\circlearrowleft}(P)$ to P which is the identity on P. For any $p, q \in P$:

$$p *' q = \alpha_P(\nu_P(p) * \nu_P(q)) = p * q,$$

where in the middle term $*$ is the concatenation in the TRAP $\mathbf{sol}\Gamma^{\circlearrowleft}(P)$ and where we used that α_P is a TRAP morphism by Proposition 5.1.3. Therefore, $* = *'$. If $p \in P(k,l)$, $(i,j) \in [k] \times [l]$, then $t'_{i,j}$ is obtained by the application of α_P to the graph:



which is $t_{i,j}(\nu_P(p))$, where here $t_{i,j}$ is the trace operator of $\mathbf{sol}\Gamma^{\circlearrowleft}(P)$. As α_P is a TRAP morphism:

$$t'_{i,j}(p) = \alpha_P \circ t_{i,j} \circ \nu_P(p) = t_{i,j} \circ \alpha_P \circ \nu_P(p) = t_{i,j}(p),$$

so $P' = P$ and $\mathcal{G} \circ \mathcal{F}$ is the identity functor of **TRAP**.

Let now (P, α) be a $\mathbf{sol}\Gamma^{\circlearrowleft}$-algebra and (P', α') be the $\mathbf{sol}\Gamma^{\circlearrowleft}$-algebra $\mathcal{F} \circ \mathcal{G}(P)$. Both α and α' are TRAP morphisms from $\mathbf{sol}\Gamma^{\circlearrowleft}(P)$ to $\mathcal{G}(P)$; for any $p \in P$,

$$\alpha \circ \nu_P(p) = \alpha' \circ \nu_P(p) = p.$$

Since $\mathbf{sol}\Gamma^{\circlearrowleft}(P)$ viewed as a TRAP is generated by the elements $\nu_P(p)$, $\alpha = \alpha'$, it follows that $\mathcal{F} \circ \mathcal{G}$ is the identity functor of $\mathbf{Alg}(\mathbf{sol}\Gamma^{\circlearrowleft})$.

The proof is similar in the unitary case. □

Remark 5.3.1 gives a straightforward Corollary of Theorems 5.3.1 and 3.3.1, thus confirming previous statements.

Corollary 5.3.2. $\mathbf{CGr}^{\circlearrowleft}(X)$ *is the free wheeled PROP generated by* X.

Remark 5.3.2. The monad $\Gamma^{\circlearrowleft}$ contains an interesting sub-monad, formed by graphs without any oriented cycle (which includes also loops). This submonad is denoted by $\Gamma^{\uparrow}$. It is well-known that $\Gamma^{\uparrow}$ is the monad of PROPs [Mar08]. Hence, unitary TRAPs are PROPS. In particular, a unitary TRAP P inherits a vertical composition denoted by $\circ$, which is the one described in Definition-Proposition 4.1.1 in the more general frame of (non unitary) TRAPs.

A Appendix: topologies on tensor products

Tensor products ot topological spaces can be equipped with various topologies. A first possibility is the so-called **ϵ-topology**; [Trè67, Definition 43.1]. For two topological vector spaces E and F, one can show ([Trè67, Proposition 42.4]) the isomorphism of vector spaces $E \otimes F \cong \mathcal{B}^c(E'_\sigma \times F'_\sigma, \mathbb{K})$ where $\mathcal{B}^c(E'_\sigma \times F'_\sigma, \mathbb{K})$ denotes the space of continuous bilinear maps from $E'_\sigma \times F'_\sigma$ to $\mathbb{K}$ and E'_σ (respectively, F'_σ) the topological dual of E (respectively, F) for σ, the weak topology.

Recall that a bilinear map $f : E \times F \longrightarrow K$ is called separately continuous if, for any pair $(x, y) \in E \times F$, the maps $z \to f(x, z)$ and $z \to f(z, y)$ are continuous. We then clearly have that continuous bilinear maps build a linear subspace of the space $\mathcal{B}^{sc}(E \times F, \mathbb{K})$ of separately continuous bilinear maps.

The space $\mathcal{B}^{sc}(E \times F, \mathbb{K})$ can be equipped with the topology of uniform convergence on products of equicontinuous subsets of E'_σ with equicontinuous subsets of F'_σ. Recall that, for a topological space X and a topological vector space G, a set S of maps from X to G is said to be equicontinuous at $x_0 \in X$ if, for any $V \subseteq G$ neighbourhood of zero, there is some neighbourhood $V(x_0) \subseteq X$ of x_0, such that

$$\forall f \in S,\ x \in V(x_0) \Rightarrow f(x) - f(x_0) \in V.$$

In our case, G is $\mathbb{K}$ and X is E_σ (respectively, F_σ). This topology induces a topology on the subspace $\mathcal{B}^c(E'_\sigma \times F'_\sigma, \mathbb{K})$ and thus on $E \otimes F$. We denote by $E \otimes_\epsilon F$ the topological vector space obtained by endowing $E \otimes F$ with this topology.

There is another topology on $E \otimes F$ called the **projective topology**; [Trè67, Definition 43.2]. The projective topology is defined as the strongest locally convex topology on $E \otimes F$ such that the canonical map $\phi : E \times F \longrightarrow E \otimes F$ is continuous. We write $E \otimes_\pi F$ the topological vector space obtained by endowing $E \otimes F$ with this topology.

The neighbourhoods of zero of the projective topology can be simply described in terms of neighbourhoods of zero in E and V. A convex subset S of $E \otimes F$ containing zero is a neighbourhood of zero if it exist a neighbourhood U (respectively, V) of zero in E (respectively, F) such that $U \otimes V := \{u \otimes v | u \in U \wedge v \in V\} \subseteq S$.

Various topologies can be defined on the vector space $E \otimes F$ for E and F two topological vector spaces. However the projective topology and the ϵ-topology play an important special role since they allow to define nuclear spaces (see Definition 2.2.1).

B Appendix: definition of the partial trace maps on $\mathbf{Gr}^{\circlearrowleft}$

We give a rigorous definition of the partial trace maps on the space of graphs $\mathbf{Gr}^{\circlearrowleft}$, which were only loosely defined in the bulk of the article.

Let $G \in \mathbf{Gr}^{\circlearrowleft}(k,l)$ with $k,l \geqslant 1$, $i \in [k]$ and $j \in [l]$. We put $e_i = \alpha_G^{-1}(i)$ and $f_j = \beta_G^{-1}(j)$. We define the graph $G' = t_{i,j}(G)$ in the following way:

1. If $e_i \in I(G)$ and $f_j \in O(G)$, then:

$$
\begin{aligned}
V(G') &= V(G), & E(G') &= E(G) \sqcup \{(e_i, f_j)\},\\
I(G') &= I(G)\backslash\{e_i\}, & O(G') &= O(G)\backslash\{f_j\},\\
IO(G') &= IO(G), & L(G') &= L(G),\\
s_{G'}(e) &= \begin{cases} s_G(f_j) \text{ if } e = (e_i, f_j),\\ s_G(e) \text{ otherwise,}\end{cases} & t_{G'}(e) &= \begin{cases} t_G(e_i) \text{ if } e = (e_i, f_j),\\ t_G(e) \text{ otherwise,}\end{cases}\\
\alpha_{G'}(e) &= \begin{cases} \alpha_G(e) \text{ if } \alpha_G(e) < i,\\ \alpha_G(e) - 1 \text{ if } \alpha_G(e) \geqslant i,\end{cases} & \beta_{G'}(e) &= \begin{cases} \beta_G(e) \text{ if } \beta_G(e) < j,\\ \beta_G(e) - 1 \text{ if } \beta_G(e) \geqslant j.\end{cases}
\end{aligned}
$$

2. If $e_i \in IO(G)$ and $f_j \in O(G)$, then:

$$
\begin{aligned}
V(G') &= V(G), & E(G') &= E(G),\\
I(G') &= I(G), & O(G') &= O(G)\backslash\{f_j\} \sqcup \{(e_i, f_j,)\},\\
IO(G') &= IO(G)\backslash\{e_i\}, & L(G') &= L(G),\\
s_{G'}(e) &= \begin{cases} s_G(f_j) \text{ if } e = (e_i, f_j),\\ s_G(e) \text{ otherwise,}\end{cases} & t_{G'}(e) &= t_G(e),\\
\alpha_{G'}(e) &= \begin{cases} \alpha_G(e) \text{ if } \alpha_G(e) < i,\\ \alpha_G(e) - 1 \text{ if } \alpha_G(e) \geqslant i,\end{cases} & \beta_{G'}(e) &= \begin{cases} \beta_G(e_i) \text{ if } e = (e_i, f_j) \text{ and } \beta_G(e_i) < j,\\ \beta_G(e_i) - 1 \text{ if } e = (e_i, f_j) \text{ and } \beta_G(e_i) \geqslant j,\\ \beta_G(e) \text{ if } e \neq (e_i, f_j) \text{ and } \beta_G(e) < j,\\ \beta_G(e) - 1 \text{ if } e \neq (e_i, f_j) \text{ and } \beta_G(e) \geqslant j.\end{cases}
\end{aligned}
$$

3. If $e_i \in I(G)$ and $f_j \in IO(G)$, then:

$$
\begin{aligned}
V(G') &= V(G), & E(G') &= E(G),\\
I(G') &= I(G)\backslash\{e_i\} \sqcup \{(e_i, f_j)\}, & O(G') &= O(G),\\
IO(G') &= IO(G)\backslash\{f_j\}, & L(G') &= L(G),\\
s_{G'}(e) &= s_G(e), & t_{G'}(e) &= \begin{cases} t_G(e_i) \text{ if } e = (e_i, f_j),\\ t_G(e) \text{ otherwise,}\end{cases}\\
\alpha_{G'}(e) &= \begin{cases} \alpha_G(f_i) \text{ if } e = (e_i, f_j) \text{ and } \alpha_G(f_j) < i,\\ \alpha_G(f_i) - 1 \text{ if } e = (e_i, f_j) \text{ and } \alpha_G(f_j) \geqslant i,\\ \alpha_G(e) \text{ if } e \neq (e_i, f_j) \text{ and } \alpha_G(e) < i,\\ \alpha_G(e) - 1 \text{ if } e \neq (e_i, f_j) \text{ and } \alpha_G(e) \geqslant i,\end{cases} & \beta_{G'}(e) &= \begin{cases} \beta_G(e) \text{ if } \beta_G(e) < j,\\ \beta_G(e) - 1 \text{ if } \beta_G(e) \geqslant j.\end{cases}
\end{aligned}
$$

4. If $e_i \in IO(G)$, $f_j \in IO(G)$ and $e_i \neq f_j$, then:

$$\begin{aligned}
V(G') &= V(G), & E(G') &= E(G),\\
I(G') &= I(G), & O(G') &= O(G),\\
IO(G') &= \{(e_i, f_j)\} \sqcup IO(G)\backslash\{e_i, f_j\}, & L(G') &= L(G),\\
s_{G'}(e) &= s_G(e), & t_{G'}(e) &= t_G(e),
\end{aligned}$$

$$\alpha_{G'}(e) = \begin{cases} \alpha_G(f_i) \text{ if } e = (e_i, f_j) \text{ and } \alpha_G(f_j) < i,\\ \alpha_G(f_i) - 1 \text{ if } e = (e_i, f_j) \text{ and } \alpha_G(f_j) \geqslant i,\\ \alpha_G(e) \text{ if } e \neq (e_i, f_j) \text{ and } \alpha_G(e) < i,\\ \beta_G(e) - 1 \text{ if } e \neq (e_i, f_j) \text{ and } \alpha_G(e) \geqslant i, \end{cases}$$

$$\beta_{G'}(e) = \begin{cases} \beta_G(e_i) \text{ if } e = (e_i, f_j) \text{ and } \beta_G(e_i) < j,\\ \beta_G(e_i) - 1 \text{ if } e = (e_i, f_j) \text{ and } \beta_G(e_i) \geqslant j,\\ \beta_G(e) \text{ if } e \neq (e_i, f_j) \text{ and } \beta_G(e) < j,\\ \beta_G(e) - 1 \text{ if } e \neq (e_i, f_j) \text{ and } \beta_G(e) \geqslant j. \end{cases}$$

5. If $e_i \in IO(G)$, $f_j \in IO(G)$ and $e_i = f_j$, then:

$$\begin{aligned}
V(G') &= V(G), & E(G') &= E(G),\\
I(G') &= I(G), & O(G') &= O(G),\\
IO(G') &= IO(G)\backslash\{e_i, f_j\}, & L(G') &= L(G) \sqcup \{(e_i, f_j)\},\\
s_{G'}(e) &= s_G(e), & t_{G'}(e) &= t_G(e),
\end{aligned}$$

$$\alpha_{G'}(e) = \begin{cases} \alpha_G(e) \text{ if } \alpha_G(e) < i,\\ \alpha_G(e) - 1 \text{ if } \alpha_G(e) \geqslant i, \end{cases} \qquad \beta_{G'}(e) = \begin{cases} \beta_G(e) \text{ if } \beta_G(e) < j,\\ \beta_G(e) - 1 \text{ if } \beta_G(e) \geqslant j. \end{cases}$$

C Appendix: Freeness of $\mathbf{CGr}^{\circlearrowleft}(X)$

We now give a detailed proof of Theorem 3.3.1. We simultaneously prove the unitary and non-unitary cases. Let P be a TRAP or a unitary TRAP and let $\phi : X \longrightarrow P$ be a map.

Let us first prove the existence of Φ. We define $\Phi : \mathbf{CGr}^{\circlearrowleft}(X) \longrightarrow P$ by assigning to any graph $G \in \mathbf{CGr}^{\circlearrowleft}(X)(k,l)$, or $G \in \mathbf{solCGr}^{\circlearrowleft}(X)(k,l)$ for the non-unitary case, an element $\Phi(G) \in P(k,l)$. We proceed by induction on the number N of internal edges of G. If $N = 0$, then G can be written (non uniquely) as

$$G = \mathcal{O}^{*p} * \sigma \cdot (I^{*q} * G_1 * \ldots * G_r) \cdot \tau, \tag{26}$$

where p, q, r in $\mathbb{N}_0$ are unique, (k_i, l_i) in $\mathbb{N}_0^2$ for any i, unique up to a permutation, σ in $\mathfrak{S}_{q+k_1+\ldots+k_r}$, $\tau \in \mathfrak{S}_{q+l_1+\ldots+l_r}$ and $G_i \in X(k_i, l_i)$ for any i. Note that in the non-unitary case we necessarily have $p = q = 0$, and we have set in this case $H^{*0} = I_0 = \varnothing$, the empty graph, for any graph H. We then put:

$$\Phi(G) = t_{1,1}(I)^{*p} * \sigma \cdot (I^{*q} * \phi_{k_1,l_1}(G_1) * \ldots * \phi_{k_r,l_r}(G_r)) \cdot \tau,$$

where as before $x^{*0} = I_0$ (the unit for horizontal concatenation in the image P of Φ) for any $x \in P$; and I is now the unit of P in the unitary case.

Let us assume now that $\Phi(G')$ is defined for any graph with $N-1$ internal edges, for a given $N \geqslant 1$. Let G be a graph with N internal edges and let e be one of these edges. Let G_e be a graph obtained by cutting this edge in two:

- $V(G_e) = V(G)$.
- $E(G_e) = E(G)\backslash\{e\}$, $I(G_e) = I(G) \sqcup \{e\}$, $O(G_e) = O(G) \sqcup \{e\}$, $IO(G_e) = IO(G)$, $L(G_e) = L(G)$.
- $s_{G_e} = s_G$ and $t_{G_e} = t_G$.
- For any $e' \in I(G_e) \sqcup IO(G_e)$, for any $f' \in O(G_e) \sqcup IO(G_e)$:

$$\alpha_{G_e}(e') = \begin{cases} 1 \text{ if } e' = e, \\ \alpha_G(e') + 1 \text{ if } e' \neq e, \end{cases} \qquad \beta_{G_e}(f') = \begin{cases} 1 \text{ if } f' = e, \\ \beta_G(f') + 1 \text{ if } f' \neq e. \end{cases}$$

Notice that if $G \in \mathbf{solCGr}^{\circlearrowleft}(X)$ (that is $IO(G) = L(G) = \varnothing$) then G_e also lies in $\mathbf{solCGr}^{\circlearrowleft}(X)$. Then, as before, we can treat the unitary and non-unitary cases simultaneously. In both cases we have $G = t_{1,1}(G_e)$ and G_e has $N - 1$ internal edges. We then put:

$$\Phi(G) = t_{1,1} \circ \Phi(G_e). \tag{27}$$

Lemma C.1. *The map Φ is well-defined. Moreover, for any $\in \mathbf{solCGr}^{\circlearrowleft}(X)(k,l)$ or in $\mathbf{CGr}^{\circlearrowleft}(X)(k,l)$, with $(k,l) \in \mathbb{N}_0^2$, for any $\sigma \in \mathfrak{S}_l$, for any $\tau \in \mathfrak{S}_k$,*

$$\Phi(\sigma \cdot G \cdot \tau) = \sigma \cdot \Phi(G) \cdot \tau.$$

Proof. We proceed by induction on the number N of internal vertices. For $N = 0$, we have to show that $\Phi(G)$ does not depend on the choice of the decomposition (26) of G. Such a decomposition is determined modulo a permutation of the vertices and of the choice of σ and τ. Thus, we can go from one decomposition of G to any other one by means of a finite number of steps among the following two types:

1. We consider two decompositions of G of the form

$$G = \mathcal{O}^{*p} * \sigma \cdot (I^{*q} * G_1 * \ldots * G_i * G_{i+1} * \ldots * G_r) \cdot \tau,$$
$$G = \mathcal{O}^{*p} * \sigma' \cdot (I^{*q} * G_1 * \ldots * G_{i+1} * G_i * \ldots * G_r) \cdot \tau',$$

with

$$\sigma' = \sigma(\mathrm{Id}_{q+l_1+\ldots+l_{i-1}} \otimes c_{l_i,l_{i+1}} \otimes \mathrm{Id}_{l_{i+2}+\ldots+l_r}),$$
$$\tau' = (\mathrm{Id}_{q+k_1+\ldots+k_{i-1}} \otimes c_{k_{i+1},k_i} \otimes \mathrm{Id}_{k_{i+2}+\ldots+k_r})\tau.$$

Then, by commutativity of $*$:

$$\begin{aligned} &\sigma' \cdot (I^{*q} * \phi(G_1) * \ldots * \phi(G_r)) \cdot \tau' \\ &= \sigma \cdot (I^{*q} * \phi(G_1) * \ldots * c_{l_i,l_{i+1}} \cdot (\phi(G_{i+1}) * \phi(G_i)) \cdot c_{k_{i+1},k_i} * \ldots * \phi(G_r)) \cdot \tau \\ &= \sigma \cdot (I^{*q} * \phi(G_1) * \ldots * \phi(G_i) * \phi(G_{i+1}) * \ldots * \phi(G_r)) \cdot \tau. \end{aligned}$$

2. We consider two decompositions of G of the form

$$G = \mathcal{O}^{*p} * \sigma \cdot (I^{*q} * G_1 * \ldots * G_r) \cdot \tau,$$
$$G = \mathcal{O}^{*p} * \sigma' \cdot (I^{*q} * G_1 * \ldots * G_r) \cdot \tau',$$

with

$$\sigma' = \sigma(\sigma_0 \otimes \sigma_1 \otimes \ldots \otimes \sigma_r), \qquad \tau' = (\sigma_0^{-1} \otimes \tau_1 \otimes \ldots \otimes \tau_r)\tau,$$

with $\sigma_0 \in \mathfrak{S}_q$, $\sigma_i \in \mathfrak{S}_{k_i}$ and $\tau_i \in \mathfrak{S}_{l_i}$ if $i \geqslant 1$. Using the commutativity of $*$ and the invariance of the $x_{k,l}$, we find

$$\begin{aligned} &\sigma' \cdot (I^{*q} * \phi(G_1) * \ldots * \phi(G_r)) \cdot \tau' \\ &= \sigma \cdot (\sigma_0 \cdot I^{*q} \cdot \sigma_0^{-1} * \sigma_1 \cdot \phi(G_1) \cdot \tau_1 * \ldots * \sigma_r \cdot \phi(G_r) \cdot \tau_r) \cdot \tau \\ &= \sigma \cdot (I^{*q} * \phi(G_1) * \ldots * \phi(G_r)) \cdot \tau. \end{aligned}$$

Notice that setting $p = q = 0$ in these computations does not change the result.

Hence, $\Phi(G)$ is well-defined. Moreover, for any $\tau' \in \mathfrak{S}_k$, $\sigma' \in \mathfrak{S}_l$, a decomposition of G of the form

$$G = \mathcal{O}^{*p} * \sigma \cdot (I^{*q} * G_1 * \ldots * G_r) \cdot \tau,$$

give rise to a decomposition of $G' = \sigma' \cdot G \cdot \tau'$ given by

$$\mathcal{O}^{*p} * \sigma'\sigma \cdot (I^{*q} * G_1 * \ldots * G_r) \cdot \tau\tau',$$

and, by definition of $\Phi(G')$:

$$\begin{aligned}\Phi(G') &= t_{1,1}(I)^{*p} * \sigma'\sigma \cdot (I^{*q} * \phi(G_1) * \ldots * \phi(G_r)) \cdot \tau\tau' \\ &= \sigma' \cdot (t_{1,1}(I)^{*p} * \sigma \cdot (I^{*q} * \phi(G_1) * \ldots * \phi(G_r)) \cdot \tau) * \tau' \\ &= \sigma' \cdot \Phi(G) \cdot \tau'.\end{aligned}$$

Here again, the computations are valid in particular in the case $p = q = 0$, so for the non-unitary case.

Let us assume the result at rank $N - 1$ and let G be a graph with N internal edges. Let us prove that $\Phi(G)$ defined in (27) does not depend on the choice of e. If e' is another internal edge of G, then:

$$(G_e)_{e'} = (12) \cdot (G_{e'})_e \cdot (12),$$

which implies, by definition of $\Phi(G_e)$ and $\Phi(G_{e'})$:

$$\begin{aligned}t_{1,1} \circ \Phi(G_e) &= t_{1,1} \circ t_{1,1} \circ \Phi((G_e)_{e'}) \\ &= t_{1,1} \circ t_{1,1} \circ ((12) \cdot \Phi((G_{e'})_e) \cdot (12)) \\ &= t_{1,1} \circ t_{2,2} \circ \Phi((G_{e'})_e) \\ &= t_{1,1} \circ t_{1,1} \circ \Phi((G_{e'})_e) \\ &= t_{1,1} \circ \Phi(G_{e'}).\end{aligned}$$

So $\Phi(G)$ is well-defined. Let $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_l$. Then:

$$(\sigma \cdot G \cdot \tau)_e = ((1) \otimes \sigma) \cdot (G_e) \cdot ((1) \otimes \tau),$$

so:

$$\begin{aligned}\Phi(\sigma \cdot G \cdot \tau) &= t_{1,1} \circ \Phi((\sigma \cdot G \cdot \tau)_e) \\ &= t_{1,1}((1) \otimes \sigma) \cdot \Phi(G_e) \cdot ((1) \otimes \tau) \\ &= ((1) \otimes \sigma)_1 \cdot t_{1,1} \circ \Phi(G_e) \cdot ((1) \otimes \tau)_1 \\ &= \sigma \cdot \Phi(G) \cdot \tau,\end{aligned}$$

where, for $\sigma \in \mathfrak{S}_k$ we use σ_i for the permutation in $\mathfrak{S}_{k-1}$ defined by

$$\sigma_i(j) = \begin{cases} \sigma(j) & \text{if } j \leqslant i-1, \\ \sigma(j-1) & \text{if } j \geqslant i. \end{cases}$$

□

We have therefore defined a map $\Phi : \mathbf{CGr}^{\circlearrowleft}(X) \longrightarrow P$, or $\Phi : \mathbf{solCGr}^{\circlearrowleft}(X) \longrightarrow P$ in the non-unitary case, compatible with the action of the symmetric groups. It remains to prove that Φ is compatible with the horizontal concatenation $*$ and with the partial trace maps.

Lemma C.2. *For any graphs G, G',*

$$\Phi(G * G') = \Phi(G) * \Phi(G').$$

Proof. We proceed by induction on the number N of internal edges of $G * G'$. If $N = 0$, we put:

$$G = \mathcal{O}^{*p} * \sigma \cdot (I^{*q} * G_1 * \ldots * G_r) \cdot \tau,$$
$$G' = \mathcal{O}^{*p'} * \sigma' \cdots (I^{*q'} * G'_1 * \ldots * G'_{r'}) \cdot \tau'.$$

As before, if we are in the non-unitary case we set $p = q = p' = q' = 0$ and the whole discussion still holds. We obtain:

$$\begin{aligned} G * G' = \mathcal{O}^{*(p+p')} * (\sigma \otimes \sigma')(\mathrm{Id}_q \otimes c_{k_1+\ldots+k_r,q'} \otimes \mathrm{Id}_{k'_1+\ldots+k'_{r'}}) \\ \cdot (I^{q+q'} * G_1 * \ldots * G'_{r'}) \cdot (\mathrm{Id}_q \otimes c_{q',l_1+\ldots+l_r} \otimes \mathrm{Id}_{l'_1+\ldots+l'_{r'}})(\tau \otimes \tau'), \end{aligned}$$

which gives, by commutativity of $*$:

$$\begin{aligned} \Phi(G * G') &= t_{1,1}(I)^{*(p+p')} * (\sigma \otimes \sigma')(\mathrm{Id}_q \otimes c_{l_1+\ldots+l_r,q'} \otimes \mathrm{Id}_{l'_1+\ldots+l'_{r'}}) \\ &\quad \cdot (I^{q+q'} * \phi(G_1) * \ldots * \phi(G'_{r'})) \cdot (\mathrm{Id}_q \otimes c_{q',k_1+\ldots+k_r} \otimes \mathrm{Id}_{k'_1+\ldots+k'_{r'}})(\tau \otimes \tau') \\ &= t_{1,1}(I)^{*p} * \sigma \cdot (I^{*q} * \phi(G_1) * \ldots * \phi(G_r)) \cdot \tau \\ &\quad * t_{1,1}(I)^{*p'} * \sigma' \cdot (I^{*q'} * \phi(G'_1) * \ldots * \phi(G'_{r'})) \cdot \tau' \\ &= \Phi(G) * \Phi(G'). \end{aligned}$$

In the non-unitary case, the TRAP P has no unit P and one simply removes the terms with the identity I in the above computation and sets $p = q = p' = q = 0$. In this case, the result is the same as in the unitary case: $\Phi(G * G') = \Phi(G) * \Phi(G')$.

If $N \geqslant 1$, let us assume that the result holds at rank $N - 1$ and take an internal edge e of $G * G'$. If e is an internal edge of G, then $(G * G')_e = G_e * G'$, and:

$$\begin{aligned} \Phi(G * G') &= t_{1,1} \circ \Phi((G * G')_e) \\ &= t_{1,1} \circ \Phi(G_e * G') \\ &= t_{1,1}(\Phi(G_e) * \Phi(G')) \quad \text{by the induction hypothesis} \\ &= (t_{1,1} \circ \Phi(G_e)) * \Phi(G') \quad \text{by Axiom 3.(c) of Definition 1.1.1} \\ &= \Phi(G) * \Phi(G'). \end{aligned}$$

If e is an internal edge of G', we obtain similarly that $\Phi(G' * G) = \Phi(G') * \Phi(G)$. The result then follows from the commutativity of $*$ (Axiom 2.(d) of Definition 1.1.1). □

We still need to prove the compatibility of Φ with the partial trace maps.

Lemma C.3. *Let $G \in \mathbf{CGr}^{\circlearrowleft}(X)(k,l)$ or in $\mathbf{solCGr}^{\circlearrowleft}(k,l)$ with $(k,l) \in \mathbb{N}_0^2$, $i \in [k]$ and $j \in [l]$. Then:*

$$t_{i,j} \circ \Phi(G) = \Phi \circ t_{i,j}(G).$$

Proof. By Lemma 1.1.5, it is enough to prove that Φ is compatible with $t_{1,1}$. Let $G \in \mathbf{CGr}^{\circlearrowleft}(X)(k,l)$, or $G \in \mathbf{solCGr}^{\circlearrowleft}(X)(k,l)$ in the non-unitary case, $e_1 = \alpha^{-1}(1)$, $f_1 = \beta^{-1}(1)$. We set $G' = t_{1,1}(G)$ and $e = \{e_1, f_1\}$ to be the edge of G' created in the process. Notice that if $G \in \mathbf{solCGr}^{\circlearrowleft}(X)$ then $G' \in \mathbf{solCGr}^{\circlearrowleft}(X)$. There are five different cases (but only the first cases appear if $G \in \mathbf{solCGr}^{\circlearrowleft}(X)$):

1. If $e_1 \in I(G)$ and $f_1 \in O(G)$, then $e \in E(G')$ and $G'_e = G$. By construction of $\Phi(G')$:

$$\Phi \circ t_{1,1}(G) = \Phi(G') = t_{1,1} \circ \Phi(G'_e) = t_{1,1} \circ \Phi(G).$$

2. If $e_1 \in IO(G)$ and $f_1 \in O(G)$, let us put $j = \beta(e_1)$. Then there exists a graph H such that $(1,j) \cdot G = I * H$, hence

$$t_{1,1}(G) = t_{1,1}((1,j) \cdot (I * H)) = (1,\ldots,j) \cdot (t_{1,i}(I * H)) = (1,\ldots,j) \cdot H,$$

so:

$$\begin{aligned} t_{1,1} \circ \Phi(G) &= t_{1,1}((1,j) \cdot (I * \Phi(H)) \\ &= (1,j)(1,\dots,j-1) \cdot \Phi(H) \\ &= (1,\dots,j) \cdot \Phi(H) \\ &= \Phi((1,\dots,j) \cdot H) \\ &= \Phi \circ t_{1,1,}(G). \end{aligned}$$

3. If $e_1 \in I(G)$ and $f_1 \in IO(G)$ the computation is similar.

4. If $e_1, f_1 \in IO(G)$, with $e_1 \neq f_1$ the computation is similar.

5. If $e_1 = f_1$ in $IO(G)$, then $G = I * H$ for a certain graph G and $t_{1,1}(G) = \mathcal{O} * H$. Then:

$$\Phi \circ t_{1,1}(G) = \Phi(\mathcal{O}) * \Phi(H) = t_{1,1} \circ \Phi(I) * \Phi(H) = t_{1,1}(\Phi(I) * \Phi(H) = t_{1,1} \circ \Phi(G).$$

So Φ is compatible with the partial trace maps, both in the unitary and non-unitary cases. □

We have proved the existence of Φ. It remains to prove the unicity. In the non unitary case, any solar graph can be obtained from graphs with only one vertex, with the help of the horizontal concatenation and the partial trace maps, which allow to create the missing internal edges. Hence, $\mathbf{solCGr}^{\circlearrowleft}(X)$ is generated by X as a TRAP, which implies the unicity of Φ. In the unitary graph, any graph can be obtained from graphs with only one vertex and copies of the graph I, with the help of the horizontal concatenation and the partial trace maps, which allow to create the missing internal edges and the copies of $\mathcal{O}$ when applied to I. Hence, $\mathbf{CGr}^{\circlearrowleft}(X)$ is generated by X as a unitary TRAP, which implies the unicity of Φ.

References

[BB03] K. D. Bierstedt and J. Bonet, *Some aspects of the modern theory of Fréchet spaces*, RACSAM, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **97** (2003), no. 2, 159–188.

[BDLGR18] C. Brouder, V. Dang, C. Laurent-Gengoux, and K. Rejzner, *Properties of field functionals and characterization of local functionals*, J. Math. Phys. **59** (2018), no. 2, 023508, 47.

[Bou03] N. Bourbaki, *Elements of mathematics. Topological vector spaces. Chapters 1–5*, Berlin: Springer, 2003.

[BS07] C. Brouder and W. Schmitt, *Renormalization as a functor on bialgebras*, Journal of Pure and Applied Algebra **209** (2007), no. 2, 477–495.

[BV68] J. M. Boardman and R. M. Vogt, *Homotopy-everything H-spaces*, Bull. Amer. Math. Soc. **74** (1968), 1117–1122.

[BV73] ———, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, vol. 2003/30, Springer, 1973.

[CB90] W. Crawley-Boevey, *Lectures on representations of Quivers*, avalaible at `https://www.math.uni-bielefeld.de/~wcrawley/quivlecs.pdf`, 1990.

[DM19] H. Derksen and V. Makam, *Invariant theory and wheeled PROPs*, arXiv:1909.00443, 2019.

[DW17] H. Derksen and J. Weyman, *An introduction to quiver representations*, Graduate Studies in Mathematics, vol. 184, American Mathematical Society, Providence, RI, 2017.

[DZ21] V. Dang and B. Zhang, *Renormalization of Feynman amplitudes on manifolds by spectral zeta regularization and blow-ups*, J. Eur. Math. Soc. (JEMS) **23** (2021), no. 2, 503–556 (English).

[Gro52] A. Grothendieck, *Résumé des résultats essentiels dans la théorie des produits tensoriels topologiques et des espaces nucléaires*, Annales de l'Institut Fourier **4** (1952), 73–112 (fr). MR 15,879b

[Gro54] ———, *Produits tensoriels topologiques et espaces nucléaires*, Séminaire Bourbaki : années 1951/52 - 1952/53 - 1953/54, exposés 50-100, Séminaire Bourbaki, no. 2, Société mathématique de France, 1954, talk:69, pp. 193–200 (fr). MR 1609222

[GV64] I.M. Gelfand and N.Y. Vilenkin, *Generalized functions. vol. 4: Applications of harmonic analysis*, New York, 1964.

[Har69] F. Harary, *Graph theory*, Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London, 1969.

[HS08] T. Hida and S. Si, *Lectures on white noise functionals*, World Scientific, 2008.

[Hö89] L. Hörmander, *The analysis of linear partial differential operators i*, ISSN, Springer, 1989.

[Ion07a] L. Ionescu, *The Feynman Legacy*, Adv. Theor. Math. Phys. **1** (2007), 1–22.

[Ion07b] ———, *From operads and PROPs to Feynman processes*, JP J. Algebra Number Theory Appl. **7** (2007), no. 2, 261–283.

[JJ06] M. Joachim and M. Johnson, *Realizing Kasparov's KK-theory groups as the homotopy classes of maps of a Quillen model category*, Contemporary Math **399** (2006), no. 1, 163–197.

[Köt69] G. Köthe, *Topological vector spaces*, Grundlehren der mathematischen Wissenschaften, no. Bd. 2, Springer-Verlag, 1969.

[KV21] N. Kawazumi and C. Vespa, *On the wheeled prop of stable cohomology of aut(f_ n) with bivariant coefficients*, arXiv:2105.14497, 2021.

[Lee97] J. M. Lee, *Riemannian manifolds – an introduction to curvature*, Springer Graduate Texts in Mathematics, vol. 176, New York · Berlin · Heidelberg: Springer Verlag, 1997.

[Mar08] M. Markl, *Operads and PROPs*, Handbook of algebra. Vol. 5, Handb. Algebr., vol. 5, Elsevier/North-Holland, Amsterdam, 2008, pp. 87–140.

[May72] J. P. May, *The geometry of iterated loop spaces*, Springer Lecture Notes in Mathematics, vol. 271, Springer, 1972.

[Mer04] S. Merkulov, *PROP profile of deformation quantization and graph complexes with loops and wheels*, arXiv:0412257, 2004.

[Mer06] ———, *PROP profile of Poisson geometry*, Comm. Math. Phys. **262** (2006), no. 1, 117–135. MR 2200884

[Mer10] ———, *Wheeled props in algebra, geometry and quantization*, European Congress of Mathematics, Eur. Math. Soc., Zürich, 2010, pp. 83–114. MR 2648322

[ML65] S. Mac Lane, *Categorical algebra*, Bull. Amer. Math. Soc. **71** (1965), no. 1, 40–106.

[ML98] ———, *Categories for the working mathematician (second ed.)*, Graduate Texts in Mathematics., vol. 5, Springer, 1998.

[MMS09] M. Markl, S. Merkulov, and S. Shadrin, *Wheeled PROPs, graph complexes and the master equation*, J. Pure Appl. Algebra **213** (2009), no. 4, 496–535. MR 2483835

[Phi88] N. Ch. Phillips, *Inverse limits of C^*-algebras*, J. Operator Theory **19** (1988), no. 1, 159–195.

[Pir02] T. Pirashvili, *On the PROP corresponding to bialgebras*, Cah. Topol. Géom. Différ. Catég. **43** (2002), no. 3, 221–239.

[Sch71] H. Schaefer, *Topological Vector Spaces*, Graduate Texts in Mathematics, Springer-Verlag, 1971.

[Trè67] F. Trèves, *Topological vector spaces, distributions and kernels*, ISSN, Elsevier Science, 1967.

[Val03] B. Vallette, *Dualité de Koszul des PROPs*, Prépublication de l'Institut de Recherche Mathématique Avancée [Prepublication of the Institute of Advanced Mathematical Research], vol. 2003/30, Université Louis Pasteur, Département de Mathématique, Institut de Recherche Mathématique Avancée, Strasbourg, 2003, Dissertation, Université de Strasbourg I, Strasbourg, 2003.

[Val04] ———, *Koszul duality for PROPs*, Comptes Rendus Math. **338** (2004), no. 12, 909–914.

[vdBC13] E. van den Ban and M. Craivic, *Analysis on manifolds*, Lecture Notes, 2013.

[YJ15] D. Yau and M. W. Johnson, *A foundation for props, algebras and modules*, Mathematical Surveys and Monographs, vol. 203, American Mathematical Society, Providence, RI, 2015.