The operads of planar forests are Koszul

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ABSTRACT. We describe the Koszul dual of two quadratic operads on planar forests introduced to study the infinitesimal Hopf algebra of planar rooted trees and prove that these operads are Koszul.

KEYWORDS. Koszul quadratic operads, planar rooted trees.

AMS CLASSIFICATION. 05C05, 18D50.

Contents

Introduction

The Hopf algebra of planar rooted trees, described in [4, 6], is a non-commutative version of the Hopf algebra of rooted tree introduced in [1, 7, 8, 9] in the context of Quantum Field Theories and Renormalisation. An infinitesimal version of this object is introduced in [3], and is related to two operads on planar forests in [2]. These two operads, denoted by $\mathbb{P}_\searrow$ and $\mathbb{P}_\nearrow$, are presented in the following way:

1. $\mathbb{P}_\searrow$ is generated by m and $\searrow \in \mathbb{P}_\searrow(2)$, with relations:

$$\left\{\begin{array}{rcl} m\circ(\searrow,I) &=& \searrow\circ(I,m),\\ m\circ(m,I) &=& m\circ(I,m),\\ \searrow\circ(m,I) &=& \searrow\circ(I,\searrow). \end{array}\right.$$

2. $\mathbb{P}_{\nearrow}$ is generated by m and $\nearrow \in \mathbb{P}_{\nearrow}(2)$, with relations:

$$\left\{\begin{array}{rcl} m\circ(\nearrow,I) &=& \nearrow\circ(I,m),\\ m\circ(m,I) &=& m\circ(I,m),\\ \nearrow\circ(\nearrow,I) &=& \nearrow\circ(I,\nearrow). \end{array}\right.$$

The algebra of planar rooted trees is both the free $\mathbb{P}_{\nearrow}$- and $\mathbb{P}_{\searrow}$-algebra generated by $\centerdot$, with products $\nearrow$ and $\searrow$ given by certain graftings.

The operads $\mathbb{P}_{\nearrow}$ and $\mathbb{P}_{\searrow}$ are quadratic. Our aim in this note is to prove that they are both Koszul, in the sense of [5]. We describe their Koszul dual (it turns out that they are quotient of $\mathbb{P}_{\searrow}$ and $\mathbb{P}_{\nearrow}$) and the associated homology of $\mathbb{P}_{\nearrow}$- or $\mathbb{P}_{\searrow}$-algebras. We compute these homologies for free objects and prove that they are concentrated in degree 0. This proves that these operads are Koszul.

1 Operads of planar forests

1.1 Presentation

We work in this text with operads, whereas we worked in [2] with non-Σ-operads. In other terms, we replace the non-Σ-operads of [2] by their symmetrization [10].

Definition 1

1. $\mathbb{P}_{\searrow}$ is generated, as an operad, by m and $\searrow$, with the relations:

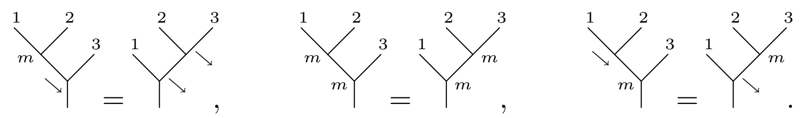
$$\left\{\begin{array}{rcl} \searrow\circ(m,I) &=& \searrow\circ(I,\searrow),\\ \searrow\circ(I,m) &=& m\circ(\searrow,I),\\ m\circ(m,I) &=& m\circ(I,m). \end{array}\right.$$

2. $\mathbb{P}_{\nearrow}$ is generated, as an operad, by m and $\nearrow$, with the relations:

$$\left\{\begin{array}{rcl} \nearrow\circ(\nearrow,I) &=& \nearrow\circ(I,\nearrow),\\ \nearrow\circ(I,m) &=& m\circ(\nearrow,I),\\ m\circ(m,I) &=& m\circ(I,m). \end{array}\right.$$

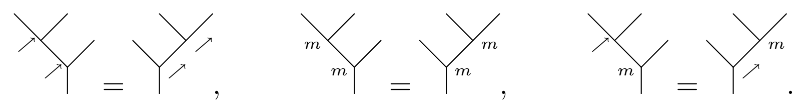
Remarks.

1. Graphically, the relations defining $\mathbb{P}_{\searrow}$ can be written in the following way:



2. We denote by $\tilde{\mathbb{P}}_{\searrow}$ the sub-non-Σ-operad of $\mathbb{P}_{\searrow}$ generated by m and $\searrow$. Then $\mathbb{P}_{\searrow}$ is the symmetrization of $\tilde{\mathbb{P}}_{\searrow}$.

3. Graphically, the relations of $\mathbb{P}^!_{\nearrow}$ can be written in the following way:



4. We denote by $\tilde{\mathbb{P}}_{\nearrow}$ the sub-non-Σ-operad of $\mathbb{P}_{\nearrow}$ generated by m and $\nearrow$. Then $\mathbb{P}_{\nearrow}$ is the symmetrization of $\tilde{\mathbb{P}}_{\nearrow}$.

Both of these non-Σ-operads admits a description in terms of planar forests [2]. In particular, the dimension of $\tilde{\mathbb{P}}_{\searrow}(n)$ and $\tilde{\mathbb{P}}_{\nearrow}(n)$ is given by the n-th Catalan number [11, 12]. Multiplying by a factorial, for all $n \geq 1$:

$$\dim \mathbb{P}_{\searrow}(n) = \dim \mathbb{P}_{\nearrow}(n) = \frac{(2n)!}{(n+1)!}.$$

In particular, $\dim \mathbb{P}_{\searrow}(2) = \dim \mathbb{P}_{\nearrow}(2) = 4$ and $\dim \mathbb{P}_{\searrow}(3) = \dim \mathbb{P}_{\nearrow}(3) = 30$.

1.2 Free algebras on these operads

We described in [2] the free $\mathbb{P}_{\searrow}$- and $\mathbb{P}_{\nearrow}$-algebras on one generators, using planar rooted trees. We here generalise (without proof) these results. Let $\mathcal{D}$ be any set. We denote by $\mathbf{T}^{\mathcal{D}}$ the set of planar trees decorated by $\mathcal{D}$ and by $\mathbf{F}^{\mathcal{D}}$ the set of non-empty planar forests decorated by $\mathcal{D}$.

1. The free $\mathbb{P}_{\searrow}$-algebra generated by $\mathcal{D}$ has the set $\mathbf{F}^{\mathcal{D}}$ as a basis. The product m is given by concatenation of forests. For all F, $G \in \mathbf{F}^{\mathcal{D}}$, the product $F \searrow G$ is obtained by grafting F on the root of G, on the left.

2. The free $\mathbb{P}_{\nearrow}$-algebra generated by $\mathcal{D}$ has the set $\mathbf{F}^{\mathcal{D}}$ as a basis. The product m is given by concatenation of forests. For all F, $G \in \mathbf{F}^{\mathcal{D}}$, the product $F \nearrow G$ is obtained by grafting F on the left leaf of G.

In both cases, we identified $d \in \mathcal{D}$ with $\bullet_d \in \mathbf{F}^{\mathcal{D}}$. Moreover, for all $F \in \mathbf{F}^{\mathcal{D}}$, $F \searrow \bullet_d = F \nearrow \bullet_d$ is the tree obtained by grafting the trees of F on a common root decorated by d: this tree will be denoted by $B_d(F)$.

2 The operad $\mathbb{P}_{\searrow}$ is Koszul

2.1 Koszul dual of $\mathbb{P}_{\searrow}$

(See [5, 10] for the notion of Koszul duality for quadratic operads). We denote by $\mathbb{P}^!_{\searrow}$ the Koszul dual of $\mathbb{P}_{\searrow}$.

Theorem 2 *The operad $\mathbb{P}^!_{\searrow}$ is generated by m and $\searrow \in \mathbb{P}^!_{\searrow}(2)$, with the relations:*

$$\left\{ \begin{array}{rcl} \searrow \circ(m, I) & = & \searrow \circ(I, \searrow), \\ m \circ (m, I) & = & m \circ (I, m), \\ m \circ (\searrow, I) & = & \searrow \circ(I, m), \\ \searrow \circ(\searrow, I) & = & 0, \\ m \circ (I, \searrow) & = & 0. \end{array} \right.$$

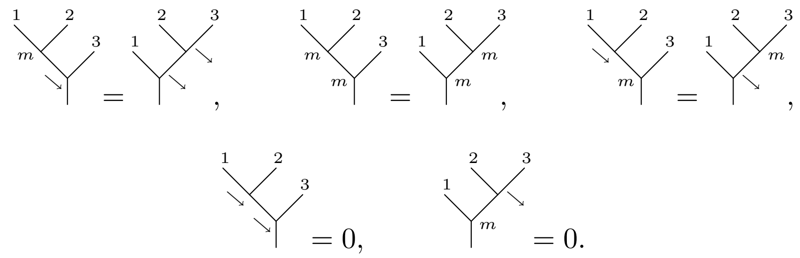
Proof. Let $\mathbb{P}(E)$ be the operad freely generated by the S_2-module freely generated by m and $\searrow$. Then $\mathbb{P}_{\searrow}$ can be written $\mathbb{P}_{\searrow} = \mathbb{P}(E)/(R)$, where R is a sub-S_3-module of $\mathbb{P}(E)(3)$. As $\dim(\mathbb{P}(E)) = 48$ and $\dim(\mathbb{P}_{\searrow}(3)) = 30$, $\dim(R) = 18$. So $\dim(R^{\perp}) = 48 - 18 = 30$. We then verify that the given relations for $\mathbb{P}^!_{\searrow}$ are indeed in $R^{\perp}$, that each of them generates a free S_3-module, which are in direct sum. So these relations generate entirely $\mathbb{P}(E)(3)$. $\square$

Remarks.

1. So $\mathbb{P}^!_{\searrow}$ is a quotient of $\mathbb{P}_{\searrow}$.

2. Moreover, $\mathbb{P}^!_{\searrow}$ is the symmetrisation of the non-Σ-operad $\tilde{\mathbb{P}}^!_{\searrow}$ generated by m and $\searrow$ and the relations:
$$\begin{cases} \searrow \circ(m,I) &=& \searrow \circ(I,\searrow),\\ m\circ(m,I) &=& m\circ(I,m),\\ m\circ(\searrow,I) &=& \searrow\circ(I,m),\\ \searrow\circ(\searrow,I) &=& 0,\\ m\circ(I,\searrow) &=& 0.\end{cases}$$
This is a general fact: the Koszul dual of the symmetrisation of a quadratic non-Σ operad is itself the symmetrisation of a certain quadratic non-Σ-operad.

3. Graphically, the relations defining $\mathbb{P}^!_{\searrow}$ can be written in the following way:



2.2 Free $\mathbb{P}^!_{\searrow}$-algebras

Let V be finite-dimensional vector space. We put:
$$\begin{cases} T_{\searrow}(V)(n) &=& \bigoplus_{k=1}^{n} V^{\otimes n} \text{ for all } n\geq 1,\\ T_{\searrow}(V) &=& \bigoplus_{n=1}^{\infty} T_{\searrow}(V)(n).\end{cases}$$
In order to distinguish the different copies of $V^{\otimes n}$, we put:
$$T(V)(n)=\bigoplus_{k=1}^{n}\underbrace{\left(A\otimes\ldots\otimes A\otimes\dot{A}\otimes A\otimes\ldots\otimes A\right)}_{\text{the } k\text{-th copy of } A \text{ is pointed.}}.$$
The elements of $A\otimes\ldots\otimes A\otimes\dot{A}\otimes A\otimes\ldots\otimes A$ will be denoted by $v_1\otimes\ldots\otimes v_{k-1}\otimes\dot{v}_k\otimes v_{k+1}\otimes\ldots\otimes v_n$. We define m and $\searrow$ over $T_{\searrow}(V)$ in the following way: for $v=v_1\otimes\ldots\otimes\dot{v}_k\otimes\ldots\otimes v_m$ and $w=w_1\otimes\ldots\otimes\dot{w}_l\otimes\ldots\otimes w_n$,
$$vw = \begin{cases} 0 \text{ if } l\neq 1,\\ v_1\otimes\ldots\otimes\dot{v}_k\otimes\ldots\otimes v_m\otimes w_1\otimes\ldots\otimes w_n \text{ if } l=1;\end{cases}$$
$$v\searrow w = \begin{cases} 0 \text{ if } k\neq 1,\\ v_1\otimes\ldots\otimes v_m\otimes w_1\otimes\ldots\otimes\dot{w}_l\otimes\ldots\otimes w_n \text{ if } k=1.\end{cases}$$

Lemma 3 *$T_{\searrow}(V)$ is a $\mathbb{P}^!_{\searrow}$-algebra generated by V.*

Proof. Let us first show that the relations of the $\mathbb{P}^!_{\searrow}$-algebras are satisfied. Let $u=u_1\otimes\ldots\otimes\dot{u}_j\otimes\ldots\otimes u_m$, $v=v_1\otimes\ldots\otimes\dot{v}_k\otimes\ldots\otimes v_n$ and $w=w_1\otimes\ldots\otimes\dot{w}_l\otimes\ldots\otimes w_p$.
$$\begin{array}{rcl}(uv)\searrow w &=& 0 \text{ if } j\neq 1 \text{ or } k\neq 1,\\ &=& u_1\otimes\ldots\otimes u_m\otimes v_1\otimes\ldots\otimes v_n\otimes w_1\otimes\ldots\otimes\dot{w}_l\otimes\ldots\otimes w_p \text{ if } j=k=1,\\ u\searrow(v\searrow w) &=& 0 \text{ if } j\neq 1 \text{ or } k\neq 1,\end{array}$$

$$
\begin{aligned}
&= u_1 \otimes \ldots \otimes u_m \otimes v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes \dot{w}_l \otimes \ldots \otimes w_p \text{ if } j=k=1,\\[1em]
(uv)w &= 0 \text{ if } k \neq 1 \text{ or } l \neq 1,\\
&= u_1 \otimes \ldots \otimes \dot{u}_j \ldots \otimes u_m \otimes v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_p \text{ if } k=l=1,\\
u(vw) &= 0 \text{ if } k \neq 1 \text{ or } l \neq 1,\\
&= u_1 \otimes \ldots \otimes \dot{u}_j \ldots \otimes u_m \otimes v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_p \text{ if } k=l=1,\\[1em]
(u \searrow v)w &= 0 \text{ if } j \neq 1 \text{ or } l \neq 1,\\
&= u_1 \otimes \ldots \otimes u_m \otimes v_1 \otimes \ldots \otimes \dot{v}_k \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_p \text{ if } j=l=1,\\
u \searrow (vw) &= 0 \text{ if } j \neq 1 \text{ or } l \neq 1,\\
&= u_1 \otimes \ldots \otimes u_m \otimes v_1 \otimes \ldots \otimes \dot{v}_k \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_p \text{ if } j=l=1,\\[1em]
(u \searrow v) \searrow w &= 0,\\[1em]
u(v \searrow w) &= 0.
\end{aligned}
$$

So $(T_\searrow(V), m, \searrow)$ is a $\mathbb{P}^!_\searrow$-algebra. Moreover, for all $v_1, \ldots, v_n \in V$:

$$v_1 \otimes \ldots \otimes \dot{v}_k \otimes \ldots \otimes v_n = (\dot{v}_1 \ldots \dot{v}_{k-1}) \searrow (\dot{v}_k \ldots \dot{v}_n).$$

Hence, $T_\searrow(V)$ is generated by V. □

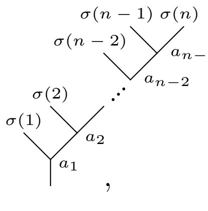
The $\mathbb{P}^!_\searrow$-algebra $T_\searrow(V)$ is also graded by putting V in degree 1. It is then a quotient of the free $\mathbb{P}^!_\searrow$-algebra generated by V, which is:

$$\bigoplus_{n=0}^{\infty} \tilde{\mathbb{P}}^!_\searrow(n) \otimes V^{\otimes n}.$$

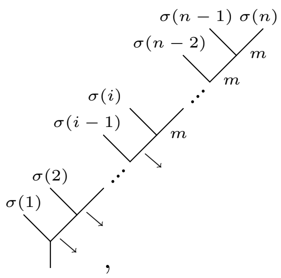
So, for all $n \in \mathbb{N}$, $\dim(\tilde{\mathbb{P}}^!_\searrow(n) \otimes V^{\otimes n}) \geq \dim(T_\searrow(V)(n))$, so $\dim(\tilde{\mathbb{P}}^!_\searrow(n)) \geq n$ and $\dim(\mathbb{P}^!_\searrow(n)) \geq nn!$.

Lemma 4 *For all $n \in \mathbb{N}$, $dim(\mathbb{P}^!_\searrow(n)) \leq nn!$.*

Proof. $\mathbb{P}^!_\searrow(n)$ is linearly generated by the binary trees with n indexed leaves, whose internal vertices are decorated by m and $\searrow$. By the four first relations of $\mathbb{P}^!_\searrow$, we obtain that $\mathbb{P}^!_\searrow(n)$ is generated by the trees of the following form:



with $\sigma \in S_n$, $a_1, \ldots a_{n-1} \in \{m, \searrow\}$. With the last relation, we deduce that $\mathbb{P}^!_\searrow(n)$ is generated by the trees of the following form:



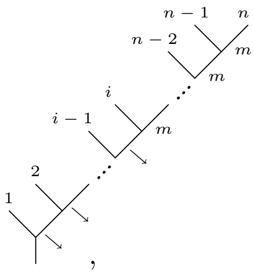
where $\sigma \in S_n$, $1 \leq i \leq n$. Hence, $dim(\mathbb{P}^!_{\searrow}(n)) \leq nn!$. □

As a consequence:

Theorem 5 *Let $n \geq 1$.*

1. $dim(\mathbb{P}^!_{\searrow}(n)) = nn!$.

2. $\mathbb{P}^!_{\searrow}(n)$ *is freely generated, as a S_n-module, by the following trees:*



,

where $1 \leq i \leq n$.

3. $T_{\searrow}(V)$ *is the free $\mathbb{P}^!_{\searrow}$-algebra generated by V.*

2.3 Homology of a $\mathbb{P}_{\searrow}$-algebra

Let us now describe the cofree $\mathbb{P}_{\searrow}$-algebra cogenerated by V. By duality, it is equal to $T_{\searrow}(V)$ as a vector space, with coproducts given in the following way: for $v = v_1 \otimes \ldots \otimes \dot{v}_k \otimes \ldots \otimes v_m$,

$$\begin{aligned} \Delta(v) &= \sum_{i=k}^{m-1} (v_1 \otimes \ldots \otimes \dot{v}_k \otimes \ldots \otimes v_i) \otimes (\dot{v}_{i+1} \otimes \ldots \otimes v_m), \\ \Delta_{\searrow}(v) &= \sum_{i=1}^{k-1} (\dot{v}_1 \otimes \ldots \otimes v_i) \otimes (v_{i+1} \otimes \ldots \otimes \dot{v}_k \otimes \ldots \otimes v_m). \end{aligned}$$

Let A be a $\mathbb{P}_{\searrow}$-algebra. The homology complex of A is given by the shifted cofree coalgebra $T_{\searrow}(V)[-1]$, with differential $d : T_{\searrow}(V)(n) \longrightarrow T_{\searrow}(V)(n-1)$, uniquely determined by the following conditions:

1. for all $a, b \in A$, $d(\dot{a} \otimes b) = ab$.

2. for all $a, b \in A$, $d(a \otimes \dot{b}) = a \searrow b$.

3. Let $\theta : T_{\searrow}(A) \longrightarrow T_{\searrow}(A)$ be the following application:

$$\theta : \left\{ \begin{array}{rcl} T_{\searrow}(A) & \longrightarrow & T_{\searrow}(A) \\ x & \longrightarrow & (-1)^{degree(x)} x \text{ for all homogeneous } x. \end{array} \right.$$

Then d is a θ-coderivation: for all $x \in T_{\searrow}(A)$,

$$\begin{aligned} \Delta(d(x)) &= (d \otimes Id + \theta \otimes Id) \circ \Delta(x), \\ \Delta_{\searrow}(d(x)) &= (d \otimes Id + \theta \otimes Id) \circ \Delta_{\searrow}(x). \end{aligned}$$

So, d is the application which sends the element $v_1 \otimes \ldots \otimes \dot{v_k} \otimes \ldots \otimes v_m$ to:

$$\begin{array}{l}\displaystyle\sum_{i=1}^{k-2}(-1)^{i-1} v_1 \otimes \ldots \otimes v_i v_{i+1} \otimes \ldots \otimes \dot{v_k} \otimes \ldots \otimes v_m \\ +(-1)^{k-2} v_1 \otimes \ldots \otimes \overbrace{v_{k-1} \searrow v_k}^{\cdot} \otimes \ldots \otimes v_m \\ +(-1)^{k-1} v_1 \otimes \ldots \otimes \overbrace{v_k v_{k+1}}^{\cdot} \otimes \ldots \otimes v_m \\ +\displaystyle\sum_{i=k+1}^{n-1}(-1)^{i-1} v_1 \otimes \ldots \otimes \dot{v_k} \otimes \ldots \otimes v_i v_{i+1} \otimes \ldots \otimes v_m.\end{array}$$

The homology of this complex will be denoted by $H_*^{\searrow}(A)$. More clearly, for all $n \in \mathbb{N}$:

$$H_n^{\searrow}(A) = \frac{Ker\left(d_{|T_{\searrow}(A)(n+1)}\right)}{Im\left(d_{|T_{\searrow}(A)(n+2)}\right)}.$$

Examples. Let $v_1, v_2, v_3 \in A$.

$$\left\{\begin{array}{rcl} d(v_1) & = & 0, \\ d(\dot{v_1} \otimes v_2) & = & v_1 v_2, \\ d(v_1 \otimes \dot{v_2}) & = & v_1 \searrow v_2, \\ d(\dot{v_1} \otimes v_2 \otimes v_3) & = & \overbrace{v_1 v_2}^{\cdot} \otimes v_3 - \dot{v_1} \otimes v_2 v_3, \\ d(v_1 \otimes \dot{v_2} \otimes v_3) & = & \overbrace{v_1 \searrow v_2}^{\cdot} \otimes v_3 - v_1 \otimes \overbrace{v_2 v_3}^{\cdot}, \\ d(v_1 \otimes v_2 \otimes \dot{v_3}) & = & v_1 v_2 \otimes \dot{v_3} - v_1 \otimes \overbrace{v_2 \searrow v_3}^{\cdot}. \end{array}\right.$$

So:

$$\left\{\begin{array}{rcl} d^2(\dot{v_1} \otimes v_2 \otimes v_3) & = & (v_1 v_2) v_3 - v_1 (v_2 v_3), \\ d^2(v_1 \otimes \dot{v_2} \otimes v_3) & = & (v_1 \searrow v_2) v_3 - v_1 \searrow (v_2 v_3), \\ d^2(v_1 \otimes v_2 \otimes \dot{v_3}) & = & (v_1 v_2) \searrow v_3 - v_1 \searrow (v_2 \searrow v_3). \end{array}\right.$$

So the nullity of d^2 on $T_{\searrow}(A)(3)$ is equivalent to the three relations defining $\mathbb{P}_{\searrow}$-algebras (this is a general fact [5]). In particular:

$$H_0^{\searrow}(A) = \frac{A}{A.A + A \searrow A}.$$

2.4 Homology of free $\mathbb{P}_{\searrow}$-algebras

The aim of this paragraph is to prove the following result:

Theorem 6 *let $N \geq 1$ and let A be the free $\mathbb{P}_{\searrow}$-algebra generated by D elements. Then $H_0^{\searrow}(A)$ is D-dimensional; if $n \geq 1$, $H_n^{\searrow}(A) = (0)$.*

Proof. *Preliminaries.* We put, for all $k, n \in \mathbb{N}^*$:

$$\left\{\begin{array}{rcl} C_n & = & T_{\searrow}(A)(n), \\ C_n^k & = & \underbrace{A \otimes \ldots \otimes \dot{A} \otimes \ldots \otimes A}_{A \text{ in position } k} \subseteq C_n \text{ if } k \leq n, \\ C_n^{\leq k} & = & \bigoplus_{i \leq k, n} C_n^i \subseteq C_n. \end{array}\right.$$

For all $k \in \mathbb{N}^*$, $C_*^{\leq k}$ is a subcomplex of C_*. In particular, $C_*^{\leq 1}$ is isomorphic to the complex defined by $C'_n = A^{\otimes n}$, with a differential defined by:

$$d' : \left\{ \begin{array}{rcl} A^{\otimes n} & \longrightarrow & A^{\otimes(n-1)} \\ v_1 \otimes \ldots \otimes v_n & \longrightarrow & \displaystyle\sum_{i=1}^{n-1} (-1)^{i-1} v_1 \otimes \ldots \otimes v_{i-1} \otimes v_i v_{i+1} \otimes v_{i+2} \otimes \ldots \otimes v_n. \end{array} \right.$$

The homology of C'_* is then the shifted Hochschild homology of A. As A is a free (non unitary) associative algebra, this homology is concentrated in degree 1. So, $Ker\left(d_{|C_n^{\leq 1}}\right) \subseteq Im(d)$ if $n \geq 2$.

First step. Let us fix $n \geq 2$ and let us show that $Ker\left(d_{|C_n^{\leq k}}\right) \subseteq Im(d)$ for all $1 \leq k \leq n-1$ by induction on k. For $k = 1$, this is already done. Let us assume that $2 \leq k < n$ and $Ker\left(d_{|C_n^{\leq k-1}}\right) \subseteq Im(d)$. Let $x = \displaystyle\sum_{i=1}^{k} x_i \in Ker\left(d_{|C_n^{\leq k}}\right)$, with $x_i \in C_n^i$. If $x_k = 0$, then $x \in Ker\left(d_{|C_n^{\leq k-1}}\right)$ by the induction hypothesis. Otherwise, we put:

$$x_k = \sum v_1 \otimes \ldots \otimes \dot{v_k} \otimes \ldots \otimes v_n.$$

We project $d(x)$ over C_{n-1}^k. we obtain:

$$\begin{array}{rcl} 0 & = & \displaystyle\sum_{i=1}^{k-1} \pi_k(d(x_i)) + \sum \sum_{i=1}^{k-2} (-1)^{i-1} \pi_k(v_1 \otimes \ldots \otimes v_{i-1} \otimes v_i v_{i+1} \otimes v_{i+2} \otimes \ldots \otimes \dot{v_k} \otimes \ldots \otimes v_n) \\ & & + \displaystyle\sum (-1)^{k-2} \pi_k(v_1 \otimes \ldots \otimes v_{k-2} \otimes \overbrace{v_{k-1} \searrow v_k}^{\cdot} \otimes v_{k+1} \otimes \ldots \otimes v_n) \\ & & + \displaystyle\sum (-1)^{k-1} \pi_k(v_1 \otimes \ldots \otimes v_{k-1} \otimes \overbrace{v_k v_{k+1}}^{\cdot} \otimes v_{k+2} \otimes \ldots \otimes v_n) \\ & & + \displaystyle\sum \sum_{i=k+1}^{n-1} (-1)^{i-1} \pi_k(v_1 \otimes \ldots \otimes \dot{v_k} \otimes \ldots \otimes v_{i-1} \otimes v_i v_{i+1} \otimes v_{i+2} \otimes \ldots \otimes v_n) \\ & = & 0 + 0 + 0 + \displaystyle\sum (-1)^{k-1} v_1 \otimes \ldots \otimes \otimes v_{k-1} \otimes \overbrace{v_k v_{k+1}}^{\cdot} \otimes v_{k+2} \otimes \ldots \otimes v_n \\ & & + \displaystyle\sum \sum_{i=k+1}^{n-1} (-1)^{i-1} v_1 \otimes \ldots \otimes \dot{v_k} \otimes \ldots \otimes v_{i-1} \otimes v_i v_{i+1} \otimes v_{i+2} \otimes \ldots \otimes v_n \\ & = & (-1)^{k-1} \displaystyle\sum v_1 \otimes \ldots \otimes v_{k-1} \otimes d'(v_k \otimes \ldots \otimes v_n). \end{array}$$

Hence, we can suppose that $d'(v_k \otimes \ldots \otimes v_n) = 0$. As $n - k + 1 \geq 2$ and the complex C'_* is exact in degree $n - k + 1 \geq 2$, there exists $\sum w_k \otimes \ldots \otimes w_{n+1} \in A^{\otimes(n-k+2)}$, such that:

$$d'\left(\sum w_k \otimes \ldots \otimes w_{n+1}\right) = v_k \otimes \ldots \otimes v_n.$$

We put $w = \sum v_1 \otimes \ldots \otimes v_{k-1} \otimes (\sum \dot{w_k} \otimes \ldots \otimes w_{n+1})$. Then $d(w) = x_k + C_n^{k-1}$, so $x - d(w) \in C_n^{k-1}$. As $Im(d) \subseteq Ker(d)$, $x - d(w) \in Ker\left(d_{|C_n^{\leq k-1}}\right) \subseteq Im(d)$ by the induction hypothesis. So, $x \in Im(d)$.

Second step. Let us show that $Ker\left(d_{|C_n^{\leq n}}\right) \subseteq Im(d)$ if $n \geq 3$. Let $x \in Ker\left(d_{|C_n^{\leq n}}\right)$. As before, we put $x = \displaystyle\sum_{i=1}^{n} x_i$, with $x_i \in C_n^i$ and:

$$x_n = \sum_i v_1^i \otimes \ldots \otimes \dot{v_k^i} \otimes \ldots \otimes v_n^i.$$

We can assume that the v_j^i's are homogeneous. Let us fix an integer N, greater than the degree of x_n and an integer M, smaller than $\max_i\{weight(v_n^i)\}$. Let us show by decreasing induction the following property: For all$x \in Ker\left(d_{|C_n^{\leq n}}\right)$ of weight $\leq N$ and such that $\max_i\{weight(v_n^i)\} \geq M$, then $x \in Im(d)$. If $M > N$, such an x is zero and the result is trivial. Let us assume the property at rank $M+1$ and let us prove it at rank M. Let A_M be the homogeneous component (for the weight of forests) of degree M of A. We project $d(x)$ over $A \otimes \ldots \otimes \dot{A}_M$. Then:

$$0 = \varpi_M(d(x)) = \sum_{i,\, weight(v_n^i)=M} d'(v_1^i \otimes \ldots \otimes v_{n-1}^i) \otimes \dot{v}_n^i.$$

Hence, we can suppose that, for all i such that $weight(v_n^i) = M$, $d'(v_1^i \otimes \ldots \otimes v_{n-1}^i) = 0$. As $n \geq 3$ and C'_* is exact at $n-1 \geq 2$, there exists $\sum_j w_1^{i,j} \otimes \ldots \otimes w_n^{i,j} \in A^{\otimes n}$ such that:

$$d'\left(\sum_j w_1^{i,j} \otimes \ldots \otimes w_n^{i,j}\right) = v_1^i \otimes \ldots \otimes v_{n-1}^i.$$

As d' is homogeneous for the weight, the weight of this element can be supposed smallest than the weight of $v_1^i \otimes \ldots \otimes v_{n-1}^i$. We then put $w = \sum_{i,\, poids(v_n^i)=M} \left(\sum_j w_1^{i,j} \otimes \ldots \otimes w_n^{i,j}\right) \otimes \dot{v}_n^i$. So $x - d(w)$ is in $x \in Ker\left(d_{|C_n^{\leq n}}\right)$, with weight $\leq N$, and satisfies the property on the v_n^i's for $M+1$. By induction hypothesis, $x - d(w) \in Im(d)$, so $x \in Im(d)$.

Hence, if $n \geq 3$, $Ker\left(d_{|C_n^{\leq n}}\right) \subseteq Im(d)$. As $C_n^{\leq n} = C_n$, for all $n \geq 3$, $d(C_{n+1}) \subseteq Ker\left(d_{|C_n}\right) \subseteq d(C_{n+1})$. Consequently, if $n \geq 2$, $H_n^{\searrow}(A) = (0)$.

Third step. We now compute $H_1^{\searrow}(A)$. We take an element $x \in C_2$ and show that it belongs to $Im(d)$. This x can be written under the form:

$$x = \sum_{F,G \in \mathbf{F}^{\mathcal{D}}-\{1\}} a_{F,G} F \otimes \dot{G} - \sum_{F,G \in \mathbf{F}^{\mathcal{D}}-\{1\}} b_{F,G} \dot{F} \otimes G.$$

So:

$$d(x) = \sum_{F,G \in \mathbf{F}^{\mathcal{D}}-\{1\}} a_{F,G} F \searrow G - \sum_{F,G \in \mathbf{F}-\{1\}} b_{F,G} FG.$$

Hence, the following assertions are equivalent:

1. $d(x) = 0$.
2. For all $H \in \mathbf{F}^{\mathcal{D}} - \{1\}$, $\sum_{F \searrow G = H} a_{F,G} = \sum_{FG=H} b_{F,G}$.

First case. For all $F, G \in \mathbf{F}^{\mathcal{D}} - \{1\}$, $a_{F,G} = 0$, that is to say $x \in \dot{A} \otimes A$. So $d(x) = d'(x')$. As C'_* is exact in degree 2, there exists $v_1 \otimes v_2 \otimes v_3 \in A^{\otimes 3}$ such that $d'(v_1 \otimes v_2 \otimes v_3) = \sum_{F,G} b_{F,G} F \otimes G$. Consequently, $d(\dot{v}_1 \otimes v_2 \otimes v_3) = \sum_{F,G} b_{F,G} \dot{F} \otimes G = x$.

Second case. $x = F_1 \otimes \dot{F}_2 - \dot{G}_1 \otimes G_2$, $F_1, F_2, G_1, G_2 \in \mathbf{F}^{\mathcal{D}}$, such that $F_1 \searrow F_2 = G_1 G_2 = H$. We put $H = t_1 \ldots t_n$ and $t_1 = B_d(s_1 \ldots s_m)$, $t_1, \ldots, t_n, s_1, \ldots, s_m \in \mathbf{T}^{\mathcal{D}}$. There exists

$i \in \{1, \ldots, n-1\}$ such that $G_1 = t_1 \ldots t_i$ and $G_2 = t_{i+1} \ldots t_n$; there exists $j \in \{1, \ldots, m-1\}$ such that $F_1 = s_1 \ldots s_j$ and $F_2 = B_d(s_{j+1} \ldots s_m) t_2 \ldots t_n$. Then:

$$\begin{array}{rl} & d(s_1 \ldots s_j \otimes \overbrace{B_d(s_{j+1} \ldots s_m) t_2 \ldots t_i}^{\cdot} \otimes t_{i+1} \ldots t_n) \\ = & \overbrace{(s_1 \ldots s_j) \searrow B_d(s_{j+1} \ldots s_m) t_2 \ldots t_i}^{\cdot} \otimes t_{i+1} \ldots t_n \\ & - s_1 \ldots s_j \otimes \overbrace{B_d(s_{j+1} \ldots s_m) t_2 \ldots t_i t_{i+1} \ldots t_n}^{\cdot} \\ = & \dot{G_1} \otimes G_2 - F_1 \otimes \dot{F_2}. \end{array}$$

So, $x \in Im(d)$.

Third case. We suppose now the following condition:

$$(a_{F,G} \neq 0) \Longrightarrow (G \notin \mathbf{T}^{\mathcal{D}}).$$

So, x can be written:

$$x = \sum_{F,G \in \mathbf{F}^{\mathcal{D}},\, t \in \mathbf{T}^{\mathcal{D}}} a_{F,tG} F \otimes \overbrace{tG}^{\cdot} - \sum_{F,G \in \mathbf{F}^{\mathcal{D}}} b_{F,G} \dot{F} \otimes G.$$

By the second case, $F \otimes \overbrace{tG}^{\cdot} - \overbrace{F \searrow t}^{\cdot} \otimes G \in Im(d) \subseteq Ker(d)$. So the following element belongs to $Ker(d)$:

$$\begin{array}{rl} & x - \displaystyle\sum_{F,G \in \mathbf{F}^{\mathcal{D}},\, t \in \mathbf{T}^{\mathcal{D}}} a_{F,tG} (F \otimes \overbrace{tG}^{\cdot} - \overbrace{F \searrow t}^{\cdot} \otimes G) \\ = & - \displaystyle\sum_{F,G \in \mathbf{F}^{\mathcal{D}}} b_{F,G} \dot{F} \otimes G + \sum_{F,G \in \mathbf{F}^{\mathcal{D}},\, t \in \mathbf{T}^{\mathcal{D}}} a_{F,tG} \overbrace{F \searrow t}^{\cdot} \otimes G. \end{array}$$

By the first case, this element belongs to $Im(d)$, so $x \in Im(d)$.

Fourth case. We suppose now the following condition:

$$(a_{F,G} \neq 0) \Longrightarrow (G \notin \mathbf{T}^{\mathcal{D}} \text{ ou } G = \bullet_d,\ d \in \mathcal{D}).$$

Let $H = B_d^+(t_1 \ldots t_n) \in \mathbf{T}^{\mathcal{D}}$, different from a single root. Then:

$$0 = \sum_{F \searrow G = H} a_{F,G} - \sum_{FG = H} b_{F,G} = \sum_{i=1}^{n} a_{t_1 \ldots t_i, B_d(t_{i+1} \ldots t_n)} - 0 = a_{t_1 \ldots t_n, \bullet_d} + 0 = a_{F, \bullet_d}.$$

Consequently, for all $F \in \mathbf{F}^{\mathcal{D}}$, $d \in \mathcal{D}$, $a_{F,\bullet_d} = 0$. By the third case, $x \in Im(d)$.

General case. The following element belongs to $Ker(d)$:

$$\begin{array}{rcl} x' & = & x + \displaystyle\sum_{F,G \in \mathbf{F}^{\mathcal{D}},\, d \in \mathcal{D}} a_{F,B_d(G)} d(F \otimes G \otimes \dot{\bullet}_d) \\ & = & x + \displaystyle\sum_{F,G \in \mathbf{F}^{\mathcal{D}},\, d \in \mathcal{D}} a_{F,B_d(G)} FG \otimes \dot{\bullet}_d - \sum_{F,G \in \mathbf{F}^{\mathcal{D}},\, d \in \mathcal{D}} a_{F,B_d(G)} F \otimes \overbrace{G \searrow \bullet_d}^{\cdot} \\ & = & x + \displaystyle\sum_{F,G \in \mathbf{F}^{\mathcal{D}},\, d \in \mathcal{D}} a_{F,B_d(G)} FG \otimes \dot{\bullet}_d - \sum_{F,G \in \mathbf{F}^{\mathcal{D}},\, d \in \mathcal{D}} a_{F,B_d(G)} F \otimes \dot{B_d(G)} \\ & = & \displaystyle\sum_{F \in \mathbf{F}^{\mathcal{D}},\, G \in \mathbf{F}^{\mathcal{D}} - \mathbf{T}^{\mathcal{D}}} a_{F,G} F \otimes \dot{G} + \sum_{F \in \mathbf{F}^{\mathcal{D}},\, d \in \mathcal{D}} a_{F,G} F \otimes \dot{\bullet}_d \\ & & - \displaystyle\sum_{F,G \in \mathbf{F}^{\mathcal{D}}} b_{F,G} \dot{F} \otimes G + \sum_{F,G \in \mathbf{F}^{\mathcal{D}},\, d \in \mathcal{D}} a_{F,B_d(G)} FG \otimes \dot{\bullet}_d. \end{array}$$

So x' satisfies the condition of the fourth case, so $x' \in Im(d)$. Hence, $x \in Im(d)$. This proves finally that $Ker(d_{|C_2}) = d(C_3)$, so $H_1^{\searrow}(A) = (0)$

It remains to compute $H_0^{\searrow}(A)$. This is equal to $A/(A.A + A \searrow A)$, so a basis of $H_0^{\searrow}(A)$ is given by the trees of weight 1, so $dim(H_0^{\searrow}(A)) = D$. □

As an immediate corollary:

Corollary 7 *The operad $\mathbb{P}_{\searrow}$ is Koszul.*

3 The operad $\mathbb{P}_{\nearrow}$ is Koszul

3.1 Koszul dual of $\mathbb{P}_{\nearrow}$

We denote by $\mathbb{P}^!_{\nearrow}$ the Koszul dual of $\mathbb{P}_{\nearrow}$.

Theorem 8 *The operad $\mathbb{P}^!_{\nearrow}$ is generated by m and $\nearrow \in \mathbb{P}^!_{\nearrow}(2)$, with the relations:*

$$\begin{cases} \nearrow \circ(\nearrow, I) & = & \nearrow \circ(I, \nearrow), \\ m \circ (m, I) & = & m \circ (I, m), \\ m \circ (\nearrow, I) & = & \nearrow \circ(I, m), \\ \nearrow \circ(m, I) & = & 0, \\ m \circ (I, \nearrow) & = & 0. \end{cases}$$

Proof. Similar as the proof of theorem 2. □

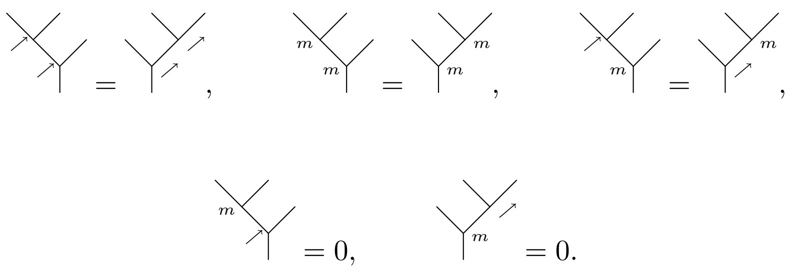
Remarks.

1. So $\mathbb{P}^!_{\nearrow}$ is a quotient of $\mathbb{P}_{\nearrow}$.

2. The operad $\mathbb{P}^!_{\nearrow}$ is the symmetrization of the non-Σ-operad $\tilde{\mathbb{P}}^!_{\nearrow}$, generated by m and $\nearrow$, with relations:

$$\begin{cases} \nearrow \circ(\nearrow, I) & = & \nearrow \circ(I, \nearrow), \\ m \circ (m, I) & = & m \circ (I, m), \\ m \circ (\nearrow, I) & = & \nearrow \circ(I, m), \\ \nearrow \circ(m, I) & = & 0, \\ m \circ (I, \nearrow) & = & 0. \end{cases}$$

3. Graphically, the relations of $\mathbb{P}^!_{\nearrow}$ can be written in the following way:



3.2 Free $\mathbb{P}^!_{\nearrow}$-algebras

Let V be finite-dimensional vector space. We put:

$$\begin{cases} T_{\nearrow}(V)(n) &= \displaystyle\bigoplus_{k=1}^{n} V^{\otimes n} \text{ for all } n \geq 1,\\ T_{\nearrow}(V) &= \displaystyle\bigoplus_{n=1}^{\infty} T_{\nearrow}(V)(n). \end{cases}$$

In order to distinguish the different copies of $V^{\otimes n}$, we put:

$$T(V)(n) = \bigoplus_{k=1}^{n} \left(\underbrace{A \oslash \ldots \oslash A \oslash A \otimes A \otimes \ldots \otimes A}_{(k-1) \text{ signs } \oslash} \right).$$

The elements of $A \oslash \ldots \oslash A \otimes \ldots \otimes A$ will be denoted by $v_1 \oslash \ldots \oslash v_k \otimes \ldots \otimes v_n$. We define m and $\nearrow$ over $T_{\nearrow}(V)$ in the following way: for $v = v_1 \oslash \ldots \oslash v_k \otimes \ldots \otimes v_m$ and $w = w_1 \oslash \ldots \oslash w_l \otimes \ldots \otimes w_n$,

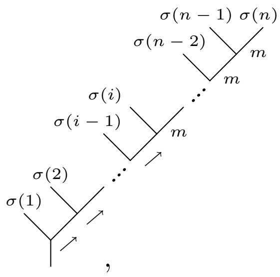
$$vw = \begin{cases} 0 \text{ if } l \neq 1,\\ v_1 \oslash \ldots \oslash v_k \otimes \ldots \otimes v_m \otimes w_1 \otimes \ldots \otimes w_n \text{ if } l = 1; \end{cases}$$

$$v \nearrow w = \begin{cases} 0 \text{ if } k \neq m-1,\\ v_1 \oslash \ldots \oslash v_m \oslash w_1 \oslash \ldots \oslash w_l \otimes \ldots \otimes w_n \text{ if } k = 1. \end{cases}$$

As for $\mathbb{P}_{\searrow}$, we can prove the following result:

Theorem 9 *Let $n \geq 1$.*

1. *$dim(\mathbb{P}^!_{\nearrow}(n)) = nn!$.*
2. *$\mathbb{P}^!_{\nearrow}(n)$ is freely generated, as a S_n-module, by the following trees:*



,

where $1 \leq i \leq n$.

3. *$T_{\nearrow}(V)$ is the free $\mathbb{P}^!_{\nearrow}$-algebra generated by V.*

3.3 Homology of a $\mathbb{P}_{\nearrow}$-algebra

Let us now describe the cofree $\mathbb{P}_{\nearrow}$-algebra cogenerated by V. By duality, it is equal to $T_{\nearrow}(V)$ as a vector space, with coproducts given in the following way: for $v = v_1 \oslash \ldots \oslash v_k \otimes \ldots \otimes v_m$,

$$\Delta(v) = \sum_{i=k}^{m-1} (v_1 \oslash \ldots \oslash v_k \otimes \ldots \otimes v_i) \otimes (v_{i+1} \otimes \ldots \otimes v_m),$$

$$\Delta_{\nearrow}(v) = \sum_{i=1}^{k-1} (v_1 \oslash \ldots \oslash v_i) \otimes (v_{i+1} \oslash \ldots \oslash v_k \otimes \ldots \otimes v_m).$$

Let A be a $\mathbb{P}_{\nearrow}$-algebra. The homology complex of A is given by the shifted cofree coalgebra $T_{\nearrow}(V)[-1]$, with differential $d : T_{\nearrow}(V)(n) \longrightarrow T_{\nearrow}(V)(n-1)$, uniquely determined by the following conditions:

1. for all $a, b \in A$, $d(a \otimes b) = ab$.
2. for all $a, b \in A$, $d(a \varnothing b) = a \nearrow b$.
3. Let $\theta : T_{\nearrow}(A) \longrightarrow T_{\nearrow}(A)$ be the following application:

$$\theta : \left\{ \begin{array}{rcl} T_{\nearrow}(A) & \longrightarrow & T_{\nearrow}(A) \\ x & \longrightarrow & (-1)^{degree(x)} x \text{ for all homogeneous } x. \end{array} \right.$$

Then d is a θ-coderivation: for all $x \in T_{\nearrow}(A)$,

$$\begin{array}{rcl} \Delta(d(x)) & = & (d \otimes Id + \theta \otimes Id) \circ \Delta(x), \\ \Delta_{\nearrow}(d(x)) & = & (d \otimes Id + \theta \otimes Id) \circ \Delta_{\nearrow}(x). \end{array}$$

So, d is the application which sends the element $v_1 \otimes \ldots \otimes \dot{v}_k \otimes \ldots \otimes v_m$ to:

$$\begin{array}{rl} & d(v_1 \varnothing \ldots \varnothing v_k \otimes \ldots \otimes v_n) \\ = & \displaystyle\sum_{i=1}^{k-1} (-1)^{i-1} v_1 \varnothing \ldots \varnothing v_{i-1} \varnothing v_i \nearrow v_{i+1} \varnothing v_{i+2} \varnothing \ldots \varnothing v_k \otimes \ldots \otimes v_n \\ & \displaystyle + \sum_{i=k}^{n-1} (-1)^{i-1} v_1 \varnothing \ldots \varnothing v_k \otimes \ldots \otimes v_{i-1} \otimes v_i v_{i+1} \otimes v_{i+2} \otimes \ldots \otimes v_n. \end{array}$$

This homology will be denoted by $H_*^{\nearrow}(A)$. More clearly, for all $n \in \mathbb{N}$:

$$H_n^{\nearrow}(A) = \frac{Ker\left(d_{|T_{\nearrow}(A)(n+1)}\right)}{Im\left(d_{|T_{\nearrow}(A)(n+2)}\right)}.$$

Examples. Let $v_1, v_2, v_3 \in A$.

$$\left\{ \begin{array}{rcl} d(v_1) & = & 0, \\ d(v_1 \otimes v_2) & = & v_1 v_2, \\ d(v_1 \varnothing v_2) & = & v_1 \nearrow v_2, \\ d(v_1 \otimes v_2 \otimes v_3) & = & v_1 v_2 \otimes v_3 - v_1 \otimes v_2 v_3, \\ d(v_1 \varnothing v_2 \otimes v_3) & = & v_1 \nearrow v_2 \otimes v_3 - v_1 \varnothing v_2 v_3, \\ d(v_1 \varnothing v_2 \varnothing v_3) & = & v_1 \nearrow v_2 \varnothing v_3 - v_1 \varnothing v_2 \nearrow v_3. \end{array} \right.$$

So:

$$\left\{ \begin{array}{rcl} d^2(v_1 \otimes v_2 \otimes v_3) & = & (v_1 v_2) v_3 - v_1 (v_2 v_3), \\ d^2(v_1 \varnothing v_2 \otimes v_3) & = & (v_1 \nearrow v_2) v_3 - v_1 \nearrow (v_2 v_3), \\ d^2(v_1 \varnothing v_2 \varnothing v_3) & = & (v_1 \nearrow v_2) \nearrow v_3 - v_1 \nearrow (v_2 \nearrow v_3). \end{array} \right.$$

So the nullity of d^2 on $T_{\nearrow}(A)(3)$ is equivalent to the three relations defining $\mathbb{P}_{\nearrow}$-algebras, as for $\mathbb{P}_{\searrow}$. In particular:

$$H_0^{\nearrow}(A) = \frac{A}{A.A + A \nearrow A}.$$

3.4 Homology of free $\mathbb{P}_{\nearrow}$-algebras

The aim of this paragraph is to prove the following result:

Theorem 10 *let $N \geq 1$ and let A be the free $\mathbb{P}_{\nearrow}$-algebra generated by D elements. Then $H_0^{\nearrow}(A)$ is D-dimensional; if $n \geq 1$, $H_n^{\nearrow}(A) = (0)$.*

Proof. *Preliminaries.* We put, for $k, n \in \mathbb{N}^*$:

$$\left\{\begin{array}{rcl} C'_n & = & T_{\nearrow}(A)(n), \\ C'^k_n & = & \underbrace{A \not\otimes \ldots \not\otimes A \otimes \ldots \otimes A}_{k-1 \text{ signs } \not\otimes} \subseteq C'_n \text{ if } k \leq n, \\ C'^{\leq k}_n & = & \bigoplus_{i \leq k,n} C'^i_n \subseteq C'_n. \end{array}\right.$$

For all $k \in \mathbb{N}^*$, $C'^{\leq k}_*$ is a subcomplex of C'_n. In particular, $C'^{\leq 1}_*$ is isomorphic to the complex defined by $C'_n = A^{\otimes n}$, with differential given by:

$$d' : \left\{\begin{array}{rcl} A^{\otimes n} & \longrightarrow & A^{\otimes (n-1)} \\ a_1 \otimes \ldots \otimes a_n & \longrightarrow & \displaystyle\sum_{i=1}^{n-1} (-1)^{i-1} a_1 \otimes \ldots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \ldots \otimes a_n. \end{array}\right.$$

Hence, the homology of C'_* is the (shifted) Hochschild homology of A. As A is a free (non unitary) associative algebra, this homology is concentrated in degree 1. So:

$$Ker\left(d_{|C'^{\leq 1}_n}\right) \subseteq Im(d) \text{ if } n \geq 2. \tag{1}$$

Moreover, C'_* admits a subcomplex defined by $C''_*(n) = A \not\otimes \ldots \not\otimes A$, with differential given by:

$$d : \left\{\begin{array}{rcl} C''_*(n) & \longrightarrow & C''_*(n-1) \\ v_1 \not\otimes \ldots \not\otimes v_n & \longrightarrow & \displaystyle\sum_{i=1}^{n-1} (-1)^{i-1} v_1 \not\otimes \ldots \not\otimes v_{i-1} \otimes v_i \nearrow v_{i+1} \not\otimes v_{i+2} \not\otimes \ldots \not\otimes v_n. \end{array}\right.$$

Hence, the homology of this subcomplex is the shifted Hochschild homology of the associative algebra $(A, \nearrow)$.

Lemma 11 *Every forest $F \in \mathbf{F}^{\mathcal{D}} - \{1\}$ can be uniquely written as $F_1 \nearrow \ldots \nearrow F_n$, where the F_i's are elements of $\mathbf{F}^{\mathcal{D}}$ of the form $F_i = \bullet_{d_i} G_i$.*

Proof. *Existence.* By induction on the weight of F. If $weight(F) = 1$, $F = \bullet_d$ and the result is obvious. If $weight(F) \geq 2$, we put $F = B_d^+(H_1)H_2$, with $weight(H_1) < weight(F)$. If $H_1 = 1$, the result is obvious. If $H_1 \neq 1$, we apply the induction hypothesis on H_1, so it can be written as $H_1 = F_1 \nearrow \ldots \nearrow F_n$, with $F_i = \bullet_{d_i} G_i$. We put $F_{n+1} = \bullet_d H_2$, so $F = F_1 \nearrow \ldots \nearrow F_{n+1}$.

Unicity. By induction on the weight of F. If $weight(F) = 1$, then $F = \bullet_d$ and this is obvious. If $weight(F) \geq 2$, we put $F = B_d(H_1)H_2$, with $weight(H_1) < weight(F)$. If $F = F_1 \nearrow \ldots \nearrow F_n$, then $F_n = \bullet_d H_2$ and $F_1 \nearrow \ldots \nearrow F_{n-1} = H_1$. Hence, F_n is unique. We conclude with the induction hypothesis. □

This lemma implies that $(A, \nearrow)$ is freely generated by forests of the form $\bullet_d G$. So:

$$Ker\left(d_{|C''_n}\right) \subseteq Im(d) \text{ if } n \geq 2. \tag{2}$$

First step. Let us fix $n \geq 2$. We show by induction on k the following property:

$$Ker\left(d_{|C'^{\leq k}_{n}}\right) \subseteq Im(d) \text{ for all } 1 \leq k \leq n-1.$$

For $k=1$, this is (1). Let us suppose $2 \leq k < n$ and $Ker\left(d_{|C'^{\leq k-1}_{n}}\right) \subseteq Im(d)$. Let $x = \sum\limits_{i=1}^{k} x_i \in Ker\left(d_{|C'^{\leq k}_{n}}\right)$, with $x_i \in C'^{i}_{n}$. If $x_k = 0$, then $x \in Ker\left(d_{|C'^{\leq k-1}_{n}}\right)$ and the induction hypothesis holds. We then suppose $x_k \neq 0$, and we put:

$$x_k = \sum v_1 \oslash \ldots \oslash v_k \otimes \ldots \otimes v_n.$$

Let us project $d(x)$ over C'^{k}_{n-1}. We get:

$$\begin{aligned}
&\sum_{i=1}^{k-1} \pi_k(d(x_i)) + \sum_{i=1}^{k-1}(-1)^{i-1} \sum \pi_k(v_1 \oslash \ldots \oslash v_i \nearrow v_{i+1} \oslash \ldots \oslash v_k \otimes \ldots \otimes v_n) \\
&+ \sum_{i=k}^{n-1}(-1)^{i-1} \sum \pi_k(v_1 \oslash \ldots \oslash v_k \otimes \ldots \otimes v_i v_{i+1} \otimes \ldots \otimes v_n) \\
= \;& 0 + 0 + (-1)^{k-1} \sum v_1 \oslash \ldots \oslash v_{k-1} \oslash d'(v_k \otimes \ldots \otimes v_n) \\
= \;& 0.
\end{aligned}$$

Hence, we can suppose $d'(v_k \otimes \ldots \otimes v_n) = 0$. As $n-k+1 \geq 2$, by (1), there exists an element $\sum w_k \otimes \ldots \otimes w_{n+1} \in A^{\otimes(n-k+2)}$, such that:

$$d'\left(\sum w_k \otimes \ldots \otimes w_{n+1}\right) = v_k \otimes \ldots \otimes v_n.$$

We put $w = \sum v_1 \oslash \ldots \oslash v_{k-1} \oslash (\sum w_k \otimes \ldots \otimes w_{n+1})$. Then, $d(w) = x_k + C'^{k-1}_{n}$, so $x - d(w) \in C'^{k-1}_{n}$. As $Im(d) \subseteq Ker(d)$, $x - d(w) \in Ker\left(d_{|C'^{\leq k-1}_{n}}\right) \subseteq Im(d)$ by the induction hypothesis. Hence, $x \in Im(d)$.

Second step. Let us show that, if $n \geq 3$, $Ker\left(d_{|C'^{\leq n}_{n}}\right) \subseteq Im(d)$. Take $x \in Ker\left(d_{|C^{\leq n}_{n}}\right)$, written as $x = \sum\limits_{i=1}^{n} x_i$, with $x_i \in C^{i}_{n}$ and $x_n = \sum\limits_{i} v_1^i \oslash \ldots \oslash v_n^i$. We can suppose the v_j^i's homogeneous. Let us fix an integer N, greater than the degree of x_n, and an integer M, smaller than $\min\limits_i\{weight(v_n^i)\}$. Let us show by a decreasing induction on M the following property: for all $x \in Ker\left(d_{|C'^{\leq n}_{n}}\right)$, of weight $\leq N$, and such that $\min\limits_i\{weight(v_n^i)\} \geq M$, then $x \in Im(d)$. If $M > N$, such an x is zero, and the result is obvious. Suppose the result at rank $M+1$ and let us show it at rank M. Let A_M be the homogeneous (for the weight) component of degree M of A and let us project $d(x)$ over $A \oslash \ldots \oslash A \oslash A_M$. We get:

$$0 = \varpi_M(d(x)) = \sum_{i,\, weight(v_n^i)=M} d(v_1^i \oslash \ldots \oslash v_{n-1}^i) \oslash v_n^i.$$

Hence, we can suppose that, for all i such that $weight(v_n^i) = M$, $d(v_1^i \oslash \ldots \oslash v_{n-1}^i) = 0$. As $n \geq 3$, by (2), there exists $\sum\limits_j w_1^{i,j} \oslash \ldots \oslash w_n^{i,j} \in C'_n$ such that:

$$d\left(\sum_j w_1^{i,j} \oslash \ldots \oslash w_n^{i,j}\right) = v_1^i \oslash \ldots \oslash v_{n-1}^i.$$

As d is homogeneous for the weight, we can suppose that the weight of this element is smaller than the weight of $v_1^i \otimes \ldots \otimes v_{n-1}^i$. We then put:

$$w = \sum_{i,\, weight(v_n^i)=M} \sum_j w_1^{i,j} \oslash \ldots \oslash w_n^{i,j} \oslash v_n^i.$$

So $x - d(w) \in Ker\left(d_{|C'^{\leq n}_n}\right)$, with a weight $\leq N$, and satisfies the property on the v_n^i's for $M+1$. By the induction hypothesis, $x - d(w) \in Im(d)$, so $x \in Im(d)$.

So, if $n \geq 2$, as $C'^{\leq n}_n = C'_n$, $H_n^{\nearrow}(A) = (0)$.

Third step. We now compute $H_1^{\nearrow}(A)$. We take an element $x \in C'_2$ and show that it belongs to $Im(d)$. This element can be written as:

$$x = \sum_{F,G \in \mathbf{F}^{\mathcal{D}} - \{1\}} a_{F,G} F \oslash G - \sum_{F,G \in \mathbf{F}^{\mathcal{D}} - \{1\}} b_{F,G} F \otimes G.$$

so:

$$d(x) = \sum_{F,G \in \mathbf{F}^{\mathcal{D}} - \{1\}} a_{F,G} F \nearrow G - \sum_{F,G \in \mathbf{F}^{\mathcal{D}} - \{1\}} b_{F,G} FG.$$

As a consequence, the following assertions are equivalent:

1. $d(x) = 0$.
2. for all $H \in \mathbf{F}^{\mathcal{D}} - \{1\}$, $\displaystyle\sum_{F \nearrow G = H} a_{F,G} = \sum_{FG=H} b_{F,G}$.

First case. For all $F, G \in \mathbf{F}^{\mathcal{D}} - \{1\}$, $a_{F,G} = 0$, that is to say $x \in A \otimes A$: then the result comes directly from (1).

Second case. $x = F_1 \oslash F_2 - G_1 \otimes G_2$, $F_1, F_2, G_1, G_2 \in \mathbf{F}^{\mathcal{D}}$, such that $F_1 \nearrow F_2 = G_1 G_2 = H$. We put $H = t_1 \ldots t_n$ et $t_1 = H_1 \nearrow \ldots \nearrow H_m$, $t_1, \ldots, t_n \in \mathbf{T}^{\mathcal{D}}$, the H_i's of the form $\bullet_{d_i} H'_i$ (lemma 11). Then there exists $i \in \{1, \ldots, n-1\}$, such that $G_1 = t_1 \ldots t_i$ and $G_2 = t_{i+1} \ldots t_n$; there exists $j \in \{1, \ldots, m-1\}$, such that $F_1 = H_1 \nearrow \ldots \nearrow H_j$ and $F_2 = (H_{j+1} \nearrow \ldots \nearrow H_m) t_2 \ldots t_n$. So:

$$\begin{aligned}
& d(H_1 \nearrow \ldots \nearrow H_j \oslash (H_{j+1} \nearrow \ldots \nearrow H_m) t_2 \ldots t_i \otimes t_{i+1} \ldots t_n) \\
= \; & (H_1 \nearrow \ldots \nearrow H_j) \nearrow (H_{j+1} \nearrow \ldots \nearrow H_m) t_2 \ldots t_i \otimes t_{i+1} \ldots t_n) \\
& - H_1 \nearrow \ldots \nearrow H_j \oslash (H_{j+1} \nearrow \ldots \nearrow H_m) t_2 \ldots t_i t_{i+1} \ldots t_n) \\
= \; & G_1 \otimes G_2 - F_1 \oslash F_2.
\end{aligned}$$

Hence, $x \in Im(d)$.

Third case. We suppose that the following condition holds:

$$(a_{F,G} \neq 0) \Longrightarrow (G \notin \mathbf{T}^{\mathcal{D}}).$$

So, x can be written as:

$$x = \sum_{F,G \in \mathbf{F}^{\mathcal{D}},\, t \in \mathbf{T}^{\mathcal{D}}} a_{F,tG} F \oslash tG - \sum_{F,G \in \mathbf{F}^{\mathcal{D}}} b_{F,G} F \otimes G.$$

By the second case, $F \varnothing tG - F \nearrow t \otimes G \in Im(d) \subseteq Ker(d)$. So, the following element belongs to $Ker(d)$:

$$
\begin{aligned}
& x - \sum_{F,G\in \mathbf{F}^{\mathcal{D}},\, t\in \mathbf{T}^{\mathcal{D}}} a_{F,tG}(F \varnothing tG - F \nearrow t \otimes G) \\
= & - \sum_{F,G\in \mathbf{F}^{\mathcal{D}}} b_{F,G} F\otimes G + \sum_{F,G\in \mathbf{F}^{\mathcal{D}},\, t\in \mathbf{T}^{\mathcal{D}}} a_{F,tG} F \nearrow t \otimes G.
\end{aligned}
$$

By the first case, this element belongs to $Im(d)$, so $x \in Im(d)$.

Fourth case. We suppose that the following condition holds:

$$(a_{F,G} \neq 0) \Longrightarrow (G \notin \mathbf{T}^{\mathcal{D}} \text{ or } G = \bullet_d,\ d \in \mathcal{D}).$$

Let $H \in \mathbf{F}^{\mathcal{D}} - \{1\}$. Let us write $B_d^+(H) = H_1 \nearrow \ldots \nearrow H_n$, $with H_i = \bullet_{d_i} H_i'$ for all i (lemma 11). As $B_d^+(H) \in \mathbf{T}^{\mathcal{D}}$, $H_n = \bullet_{d_n}$ and $H_1 \nearrow \ldots \nearrow H_{n-1} = H$. So:

$$
\begin{aligned}
0 &= \sum_{F\nearrow G = B_d(H)} a_{F,G} - \sum_{FG=B_d(H)} b_{F,G} \\
&= \sum_{i=1}^{n} a_{H_1\nearrow\ldots\nearrow H_i, H_{i+1}\nearrow\ldots\nearrow H_n} - 0 \\
&= a_{H_1\nearrow\ldots\nearrow H_{n-1},\bullet_d} + 0 \\
&= a_{H,\bullet_d}.
\end{aligned}
$$

(We used the condition on x for the third equality). So, for all $F \in \mathbf{F}^{\mathcal{D}}$, $d \in \mathcal{D}$, we obtain $a_{F,\bullet_d} = 0$. As a consequence, by the third case, $x \in Im(d)$.

General case. The following element belongs to $Ker(d)$:

$$
\begin{aligned}
x' &= x + \sum_{F,G\in \mathbf{F}^{\mathcal{D}},\, d\in\mathcal{D}} a_{F,B_d(G)} d(F \varnothing G \varnothing \bullet_d) \\
&= x + \sum_{F,G\in \mathbf{F}^{\mathcal{D}},\, d\in\mathcal{D}} a_{F,B_d(G)} F \nearrow G \varnothing \bullet_d - \sum_{F,G\in \mathbf{F}^{\mathcal{D}},\, d\in\mathcal{D}} a_{F,B_d(G)} F \varnothing G \nearrow \bullet_d \\
&= x + \sum_{F,G\in \mathbf{F}^{\mathcal{D}},\, d\in\mathcal{D}} a_{F,B_d(G)} F \nearrow G \varnothing \bullet_d - \sum_{F,G\in \mathbf{F}^{\mathcal{D}},\, d\in\mathcal{D}} a_{F,B_d(G)} F \varnothing B_d(G) \\
&= \sum_{F\in \mathbf{F}^{\mathcal{D}},\, G\in \mathbf{F}^{\mathcal{D}}-\mathbf{T}^{\mathcal{D}}} a_{F,G} F\varnothing G + \sum_{F\in \mathbf{F}^{\mathcal{D}},\, d\in\mathcal{D}} a_{F,G} F \varnothing \bullet_d \\
&\quad - \sum_{F,G\in \mathbf{F}^{\mathcal{D}}} b_{F,G} F \otimes G + \sum_{F,G\in \mathbf{F}^{\mathcal{D}},\, d\in\mathcal{D}} a_{F,B_d(G)} FG \varnothing \bullet_d.
\end{aligned}
$$

So, x' satisfies the condition of the fourth cas, so $x' \in Im(d)$. Hence, $x \in Im(d)$.

It remains to compute $H_0^{\nearrow}(A)$. This is equal to $A/(A.A + A \nearrow A)$, so a basis of $H_0^{\nearrow}(A)$ is given by the trees of weight 1, so $dim(H_0^{\nearrow}(A)) = D$. $\square$

As an immediate corollary:

Corollary 12 *The operad $\mathbb{P}_{\nearrow}$ is Koszul.*

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