Examples of Com-PreLie Hopf algebras

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ABSTRACT. We gives examples of Com-PreLie bialgebras, that is to say bialgebras with a preLie product satisfying certain compatibilities. Three families are defined on shuffle algebras: one associated to linear endomorphisms, one associated to linear form, one associated to preLie algebras. We also give all graded preLie product on $\mathbb{K}[X]$, making this bialgebra a Com-PreLie bialgebra, and classify all connected cocommutative Com-PreLie bialgebras.

KEYWORDS. Com-PreLie bialgebras; PreLie algebras; connected cocommutative bialgebras.

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Contents

Introduction

The composition of Fliess operators [6] gives a group structure on set of noncommutative formal series $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle$ in two variables x_0 and x_1. For example, let us consider the following formal series:

$$A = a_\emptyset + a_0x_0 + a_1x_1 + a_{00}x_0^2 + a_{01}x_0x_1 + a_{10}x_1x_0 + a_{11}x_1^2 + \dots,$$
$$B = b_\emptyset + b_0x_0 + b_1x_1 + b_{00}x_0^2 + b_{01}x_0x_1 + b_{10}x_1x_0 + b_{11}x_1^2 + \dots,$$
$$B = c_\emptyset + c_0x_0 + c_1x_1 + c_{00}x_0^2 + c_{01}x_0x_1 + c_{10}x_1x_0 + c_{11}x_1^2 + \dots;$$

if $C = A.B$, then:

$$\begin{aligned}
c_\emptyset &= a_\emptyset + b_\emptyset,\\
c_0 &= a_0 + b_0 + a_1b_\emptyset,\\
c_{00} &= a_{00} + b_{00} + a_{01}b_\emptyset + a_{10}b_\emptyset + a_{11}b_\emptyset^2 + a_1b_0,\\
c_{01} &= a_{01} + b_{01} + a_{11}b_\emptyset + a_1b_1,\\
c_{10} &= a_{10} + b_{10} + a_{11}b_\emptyset,\\
c_{11} &= a_{11} + b_{11}.
\end{aligned}$$

This quite complicated structure can be more easily described with the help of the Hopf algebra of coordinates of this group; this leads to a Lie algebra structure on the algebra $\mathbb{K}\langle x_0, x_1\rangle$ of noncommutative polynomials in two variables, which is in a certain sense the infinitesimal structure associated to the group of Fliess operators. As explained in [3], this Lie bracket comes from a nonassociative, preLie product $\bullet$. For example:

$$\begin{aligned}
x_0x_0 \bullet x_0 &= 0, & x_0x_0 \bullet x_1 &= 0,\\
x_0x_1 \bullet x_0 &= x_0x_0x_0, & x_0x_1 \bullet x_1 &= x_0x_0x_1,\\
x_1x_0 \bullet x_0 &= 2x_0x_0x_0, & x_1x_0 \bullet x_1 &= x_0x_0x_1 + x_0x_1x_0,\\
x_1x_1 \bullet x_0 &= x_1x_0x_0 + x_0x_1x_0 + x_0x_0x_1, & x_1x_1 \bullet x_1 &= x_1x_0x_1 + 2x_0x_1x_1.
\end{aligned}$$

Moreover, $\mathbb{K}\langle x_0, x_1\rangle$ is naturally a Hopf algebra with the shuffle product $\sqcup\!\sqcup$ and the deconcatenation coproduct Δ, and it turns out that there exists compatibilities between this Hopf-algebraic structure and the preLie product $\bullet$:

- For all $a, b, c \in A$, $(a \sqcup\!\sqcup b) \bullet c = (a \bullet c) \sqcup\!\sqcup b + a \sqcup\!\sqcup (b \bullet c)$.
- For all $a, b \in A$, $\Delta(a \bullet b) = a^{(1)} \otimes a^{(2)} \bullet b + a^{(1)} \bullet b^{(1)} \otimes a^{(2)} \sqcup\!\sqcup b^{(2)}$, with Sweedler's notation.

this is a Com-PreLie bialgebra (definition 1). Moreover, the shuffle bracket can be induced by the half-shuffle product $\prec$, and there is also a compatibility between $\prec$ and $\bullet$:

- For all $a, b, c \in A$, $(a \prec b) \bullet c = (a \bullet c) \prec b + a \prec (b \bullet c)$.

we obtain a Zinbiel-PreLie bialgebra.

Our aim in the present text is to give examples of other Com-PreLie algebras or bialgebras. We first introduce three families, all based on the shuffle Hopf algebra $T(V)$ associated to a vector space V.

1. The first family $T(V, f)$, introduced in [4], is parametrized by linear endomorphism of V. For example, if $x_1, x_2, x_3 \in V$, $w \in T(V)$:

$$\begin{aligned}
x_1 \bullet w &= f(x_1)w,\\
x_1x_2 \bullet w &= x_1f(x_2)w + f(x_1)(x_2 \sqcup\!\sqcup w),\\
x_1x_2x_3 \bullet w &= x_1x_2f(x_3)w + x_1f(x_2)(x_3 \sqcup\!\sqcup w) + f(x_1)(x_2x_3 \sqcup\!\sqcup w).
\end{aligned}$$

   In particular, if $V = Vect(x_0, x_1)$, $f(x_0) = 0$ and $f(x_1) = x_0$, we recover in this way the Com-PreLie bialgebra of Fliess operators.

2. The second family $T(V,f,\lambda)$ is indexed by pairs (f,λ), where f is a linear form on V and λ is a scalar. For example, if $x,y_1,y_2,y_3 \in V$ and $w \in T(V)$:

$$\begin{aligned} xw \bullet y_1 &= f(x) w ⧢ y_1, \\ xw \bullet y_1 y_2 &= f(x)(w ⧢ y_1 y_2 + \lambda f(y_1) w ⧢ y_2), \\ xw \bullet y_1 y_2 y_3 &= f(x)(w ⧢ y_1 y_2 y_3 + \lambda f(y_1) w ⧢ y_2 y_3 + \lambda^2 f(y_1) f(y_2) w ⧢ y_3). \end{aligned}$$

We obtain a Com-PreLie algebra, but generally not a Com-PreLie bialgebra. Nevertheless, the subalgebra $coS(V)$ generated by V is a Com-PreLie bialgebra. Up to an isomorphism, the symmetric algebra becomes a Com-PreLie bialgebra, denoted by $S(V,f,\lambda)$.

3. If $\star$ is a preLie product on V, then it can be extended in a product on $T(V)$, making it a Com-PreLie bialgebra denoted by $T(V,\star)$. For example, if $x_1,x_2,x_3,y \in V$, $w \in T(V)$.

$$\begin{aligned} x_1 \bullet yw &= (x_1 \star y) w, \\ x_1 x_2 \bullet yw &= (x_1 \star y)(x_2 ⧢ w) + x_1 (x_2 \star y) w, \\ x_1 x_2 x_3 \bullet yw &= (x_1 \star y)(x_2 x_3 ⧢ w) + x_1 (x_2 \star y)(x_3 ⧢ w) + x_1 x_2 (x_3 \star y) w. \end{aligned}$$

These examples answer some questions on Com-PreLie bialgebras. According to proposition 4, if A is a Com-PreLie bialgebra, the map f_A defined by $f_A(x) = x \bullet 1_A$ is an endomorphism of $Prim(A)$; if $f_A = 0$, then $Prim(A)$ is a PreLie subalgebra of A. Then:

- If $A = T(V,f)$, then $f_A = f$, which proves that any linear endomorphim can be obtained in this way.
- If $A = T(V,\star)$, then $f_A = 0$ and the preLie product on $Prim(A)$ is $\star$, which proves that any preLie product can be obtained in this way.

The next section is devoted to the algebra $\mathbb{K}[X]$. We first classify preLie products making it a graded Com-PreLie algebra: this gives four families of Com-PreLie algebras described in theorem 18, including certain cases of $T(V,f)$. Only a few of them are compatible with the coproduct of $\mathbb{K}[X]$ (proposition 23). The last paragraph gives a classification of all connected, cocommutative Com-PreLie bialgebras (theorem 24): up to an isomorphism these are the $S(V,f,\lambda)$ and examples on $\mathbb{K}[X]$.

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Notations.

1. $\mathbb{K}$ is a commutative field of characteristic zero. All the objects (vector spaces, algebras, coalgebras, preLie algebras...) in this text will be taken over $\mathbb{K}$.
2. Let A be a bialgebra.
   (a) We shall use Swwedler's notation $\Delta(a) = a^{(1)} \otimes a^{(2)}$ for all $a \in A$.
   (b) We denote by A_+ the augmentation ideal of A, and by $\tilde{\Delta}$ the coassociative coproduct defined by:
   $$\tilde{\Delta} : \left\{ \begin{array}{rcl} A_+ & \longrightarrow & A_+ \otimes A_+ \\ a & \longrightarrow & \Delta(a) - a \otimes 1_A - 1_A \otimes a. \end{array} \right.$$
   We shall use Sweedler's notation $\tilde{\Delta}(a) = a' \otimes a''$ for all $a \in A_+$.

1 Com-PreLie and Zinbiel-PreLie algebras

1.1 Definitions

Definition 1 *1. A* Com-PreLie algebra *[8] is a family $A = (A, ⧢, \bullet)$, where A is a vector space and $⧢$ and $\bullet$ are bilinear products on A, such that:*

(a) $(A, ⧢)$ is an associative, commutative algebra.

(b) $(A, \bullet)$ is a (right) preLie algebra, that is to say, for all $a, b, c \in A$:

$$(a \bullet b) \bullet c - a \bullet (b \bullet c) = (a \bullet c) \bullet b - a \bullet (c \bullet b).$$

(c) For all $a, b, c \in A$, $(a ⧢ b) \bullet c = (a \bullet c) ⧢ b + a ⧢ (b \bullet c)$.

2. A Com-PreLie bialgebra *is a family $(A, ⧢, \bullet, \Delta)$, such that:*

(a) $(A, ⧢, \bullet)$ is a unitary Com-PreLie algebra.

(b) $(A, ⧢, \Delta)$ is a bialgebra.

(c) For all $a, b \in A$, $\Delta(a \bullet b) = a^{(1)} \otimes a^{(2)} \bullet b + a^{(1)} \bullet b^{(1)} \otimes a^{(2)} ⧢ b^{(2)}$.

We shall say that A is unitary *if the associative algebra $(A, ⧢)$ has a unit.*

3. A Zinbiel-PreLie algebra *is a family $A = (A, \prec, \bullet)$, where A is a vector space and $\prec$ and $\bullet$ are bilinear products on A, such that:*

(a) $(A, \prec)$ is a Zinbiel algebra (or shuffle algebra, [9, 7, 5]) that is to say, for all $a, b, c \in A$:

$$(a \prec b) \prec c = a \prec (b \prec c + c \prec b).$$

(b) $(A, \bullet)$ is a preLie algebra.

(c) For all $a, b, c \in A$, $(a \prec b) \bullet c = (a \bullet c) \prec b + a \prec (b \bullet c)$.

4. A Zinbiel-PreLie bialgebra *is a family $(A, ⧢, \prec, \bullet, \Delta)$ such that:*

(a) $(A, ⧢, \bullet, \Delta)$ is a Com-PreLie bialgebra.

(b) $(A_+, \prec, \bullet)$ is a Zinbiel-PreLie algebra, and for all $x, y \in A_+$, $x \prec y + y \prec x = x ⧢ y$.

(c) For all $a, b \in A_+$:

$$\tilde{\Delta}(a \prec b) = a' \prec b' \otimes a'' ⧢ b'' + a' \prec b \otimes a'' + a' \otimes a'' ⧢ b + a \prec b' \otimes b'' + a \otimes b.$$

Remarks.

1. If $(A, ⧢, \bullet, \Delta)$ is a Com-PreLie bialgebra, then for any $\lambda \in \mathbb{K}$, $(A, ⧢, \lambda\bullet, \Delta)$ also is.

2. If A is a Zinbiel-preLie algebra, then the product $⧢$ defined by $a ⧢ b = a \prec b + b \prec a$ is associative and commutative, and $(A, ⧢, \bullet)$ is a Com-PreLie algebra. Moreover, if A is a Zinbiel-PreLie bialgebra, it is also a Com-PreLie bialgebra.

3. If A is a Zinbiel-PreLie bialgebra, the product $⧢$ is entirely determined by $\prec$: we can omit $⧢$ in the description of a Zinbiel-PreLie bialgebra.

4. If A is a Zinbiel-PreLie bialgebra, we extend $\prec$ by $a \prec 1_A = a$ and $1_A \prec a = 0$ for all $a \in A_+$. Note that $1_A \prec 1_A$ is not defined.

5. If A is a Com-Prelie bialgebra, if $a, b \in A_+$:

$$\begin{aligned}
\tilde{\Delta}(a \bullet 1_A) &= a' \otimes a'' \bullet 1_A + a' \bullet 1_A \otimes a'', \\
\tilde{\Delta}(a \bullet b) &= a' \otimes a'' \bullet b + a \bullet 1_A \otimes b + a \bullet b' \otimes b'' \\
&\quad + a' \bullet 1_A \otimes a'' \sqcup\!\sqcup b + a' \bullet b \otimes a'' + a' \bullet b' \otimes a'' \sqcup\!\sqcup b'',
\end{aligned}$$

as we shall prove later (lemma 3) that $1_A \bullet c = 0$ for all $c \in A$.

Associative algebras are preLie. However, Com-PreLie algebras are rarely associative:

Proposition 2 *Let $A = (A, ш, \bullet)$ be a Com-PreLie algebra, such that for all $x \in A$, $x \sqcup\!\sqcup x = 0$ if, and only if, $x = 0$. If $\bullet$ is associative, then it is zero.*

Proof. Let $x, y \in A$.

$$\begin{aligned}
((x \sqcup\!\sqcup x) \bullet y) \bullet y &= 2((x \bullet y) \sqcup\!\sqcup x) \bullet y \\
&= 2((x \bullet y) \bullet y) \sqcup\!\sqcup x + 2(x \bullet y) \sqcup\!\sqcup (x \bullet y) \\
&= 2(x \bullet (y \bullet y)) \sqcup\!\sqcup x + 2(x \bullet y) \sqcup\!\sqcup (x \bullet y) \\
&= (x \sqcup\!\sqcup x) \bullet (y \bullet y) + 2(x \bullet y) \sqcup\!\sqcup (x \bullet y).
\end{aligned}$$

Hence, $(x \bullet y) ш (x \bullet y) = 0$. As A is a domain, $x \bullet y = 0$. □

Hence, in our examples below, which are integral domains (shuffle algebras or symmetric algebras), the preLie product is associative if, and only if, it is zero. Here is another example, where $\bullet$ is associative. We take $A = Vect(1, x)$, with the products defined by:

ш	1	x
1	1	x
x	x	0

$\bullet$	1	x
1	0	0
x	0	x

If the characteristic of the base field $\mathbb{K}$ is 2, this is a Com-PreLie bialgebra, with the coproduct defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$.

1.2 Linear endomorphism on primitive elements

Lemma 3 *1. Let A be a Com-PreLie algebra. For all $a \in A$, $1_A \bullet a = 0$.*

2. Let A be a Com-PreLie bialgebra, with counit ε. For all $a, b \in A$, $\varepsilon(a \bullet b) = 0$.

Proof. 1. Indeed, $1_A \bullet a = (1_A.1_A) \bullet a = (1_A \bullet a).1_A + 1_A.(1_A \bullet a) = 2(1_A \bullet a)$, so $1_A \bullet a = 0$.

2. For all $a, b \in A$:

$$\begin{aligned}
\varepsilon(a \bullet b) &= (\varepsilon \otimes \varepsilon) \circ \Delta(a \bullet b) \\
&= \varepsilon(a^{(1)})\varepsilon(a^{(2)} \bullet b) + \varepsilon(a^{(1)} \bullet b^{(1)})\varepsilon(a^{(2)} \sqcup\!\sqcup b^{(2)}) \\
&= \varepsilon(a^{(1)})\varepsilon(a^{(2)} \bullet b) + \varepsilon(a^{(1)} \bullet b^{(1)})\varepsilon(a^{(2)})\varepsilon(b^{(2)}) \\
&= \varepsilon(a \bullet b) + \varepsilon(a \bullet b),
\end{aligned}$$

so $\varepsilon(a \bullet b) = 0$. □

Remark. Consequently, if a is primitive:

$$\Delta(a \bullet b) = 1_A \otimes a \bullet b + a \bullet b^{(1)} \otimes b^{(2)}.$$

So the map $b \longrightarrow a \bullet b$ is a 1-cocycle for the Cartier-Quillen cohomology [1].

If A is a Com-PreLie bialgebra, we denote by $Prim(A)$ the space of its primitive elements:

$$Prim(A) = \{a \in A \mid \Delta(a) = a \otimes 1 + 1 \otimes a\}.$$

We define an endomorphism of $Prim(A)$ in the following way:

Proposition 4 *Let A be a Com-PreLie bialgebra.*

1. *If $x \in Prim(A)$, then $x \bullet 1_A \in Prim(A)$. We denote by f_A the map:*

$$f_A : \left\{ \begin{array}{rcl} Prim(A) & \longrightarrow & Prim(A) \\ a & \longrightarrow & a \bullet 1_A. \end{array} \right.$$

2. *If $f_A = 0$, then $Prim(A)$ is a preLie subalgebra of A.*

Proof. 1. Indeed, if a is primitive:

$$\begin{aligned} \Delta(a \bullet 1_A) &= a \otimes 1_A \bullet 1_A + 1_A \otimes a \bullet 1_A + a \bullet 1_A \otimes 1_A ш 1_A + 1_A \bullet 1_A \otimes a ш 1_A \\ &= 0 + 1_A \otimes 1_A \bullet a + a \bullet 1_A \otimes 1_A + 0, \end{aligned}$$

so $a \bullet 1_A$ is primitive.

2. Let $a, b \in Prim(A)$.

$$\begin{aligned} \Delta(a \bullet b) &= a \otimes 1_A \bullet b + 1_A \otimes a \bullet b + 1_A \bullet 1_A \otimes a ш b + a \bullet 1_A \otimes b + 1_A \bullet b \otimes a + a \bullet b \otimes 1_A \\ &= 1_A \otimes a \bullet b + a \bullet b \otimes 1_A. \end{aligned}$$

So $a \bullet b \in Prim(A)$. □

2 Examples on shuffle algebras

Let V be a vector space and let $f : V \longrightarrow V$ be any linear map. The tensor algebra $T(V)$ is given the shuffle product ш, the half-shuffle $\prec$ and the deconcatenation coproduct Δ, making it a bialgebra. Recall that these products can be inductively defined in the following way: if $x, y \in V$, $u, v \in T(V)$:

$$\left\{ \begin{array}{rcl} 1 \prec yv &=& 0, \\ xu \prec v &=& x(u \prec v + v \prec u), \end{array} \right. \qquad \left\{ \begin{array}{rcl} 1 ш v &=& 0, \\ xu ш yv &=& x(u ш yv) + y(xu ш v). \end{array} \right.$$

For any $x_1, \ldots, x_n \in V$:

$$\Delta(x_1 \ldots x_n) = \sum_{i=0}^{n} x_1 \ldots x_i \otimes x_{i+1} \ldots x_n.$$

For all linear map $F : V \longrightarrow W$, we define the map:

$$T(F) : \left\{ \begin{array}{rcl} T(V) & \longrightarrow & T(W) \\ x_1 \ldots x_n & \longrightarrow & F(x_1) \ldots F(x_n). \end{array} \right.$$

This a Hopf algebra morphism from $T(V)$ to $T(W)$.

The subalgebra of $(T(V), ш)$ generated by V is denoted by $coS(V)$. It is the largest cocommutative Hopf subalgebra of $(T(V), ш, \Delta)$; it is generated by the symmetric tensors of elements of V.

2.1 Com-PreLie algebra attached to a linear endomorphism

We described in [4] a first family of Zinbiel-PreLie bialgebras; coming from a problem of composition of Fliess operators in Control Theory. Let f be an endomorphism of a vector space V. We define a bilinear product $\bullet$ on $T(V)$ inductively on the length of words in the following way: if $x \in V$, $v, w \in T(V)$,

$$1 \bullet w = 0, \qquad xv \bullet w = x(v \bullet w) + f(x)(v \sqcup\!\sqcup w).$$

Then $(T(V), \prec, \bullet, \Delta)$ is a Zinbiel-PreLie bialgebra, denoted by $T(V, f)$. Moreover, $f_{T(V,f)} = f$.

Examples. If $x_1, x_2, x_3 \in V$, $w \in T(V)$:

$$\begin{aligned}
x_1 \bullet w &= f(x_1)w,\\
x_1x_2 \bullet w &= x_1 f(x_2)w + f(x_1)(x_2 \sqcup\!\sqcup w),\\
x_1x_2x_3 \bullet w &= x_1x_2 f(x_3)w + x_1 f(x_2)(x_3 \sqcup\!\sqcup w) + f(x_1)(x_2x_3 \sqcup\!\sqcup w).
\end{aligned}$$

More generally, if $x_1, \ldots, x_n \in V$ and $w \in T(V)$:

$$x_1 \ldots x_n \bullet w = \sum_{i=1}^{n} x_1 \ldots x_{i-1} f(x_i)(x_{i+1} \ldots x_n \sqcup\!\sqcup w).$$

This construction is functorial: let V and W be two vector spaces, f an endomorphism of V and g an endomorphism of W; let $F : V \longrightarrow W$, such that $g \circ F = F \circ f$. Then $T(F)$ is a morphism of Zinbiel-PreLie bialgebras from $T(V, f)$ to $T(W, g)$.

Proposition 5 *Let $\blacklozenge$ be a preLie product on $(T(V), \sqcup\!\sqcup, \Delta)$, making it a Com-PreLie bialgebra, such that for all $k, l \in \mathbb{N}$, $V^{\otimes k}$
blacklozenge$V^{\otimes l} \subseteq V^{\otimes(k+l)}$. There exists a $f \in End(V)$, such that $(T(V), \sqcup\!\sqcup, \blacklozenge, \Delta) = T(V, f)$.*

Proof. Let $f = f_{T(V)}$. We denote by $\bullet$ the preLie product of $T(V, f)$. Let us prove that for any $x = x_1 \ldots x_k, y = y_1 \ldots y_l \in T(V)$, $x \bullet y = x \blacklozenge y$. If $k = 0$, we obtain $1 \bullet y = 1 \blacklozenge y = 0$. We now treat the case $l = 0$. We proceed by induction on k. It is already done for $k = 0$. If $k = 1$, then $x \in V$ and $x \bullet 1 = f(x) = x \blacklozenge 1$. Let us assume the result at all ranks $< k$, with $k \geq 2$. Then, as the length of x' andx'' is $< k$:

$$\begin{aligned}
\Delta(x \bullet 1) &= x^{(1)} \otimes x^{(2)} \bullet 1 + x^{(1)} \bullet 1 \otimes x^{(2)}\\
&= 1 \otimes x \bullet 1 + x' \bullet 1 \otimes 1 + x' \otimes x'' \bullet 1 + x \otimes 1 \otimes 1\\
&= 1 \otimes x \bullet 1 + x' \blacklozenge 1 \otimes 1 + x' \otimes x'' \blacklozenge 1 + x \otimes 1 \otimes 1\\
&= \Delta(x \blacklozenge 1) + (x \bullet y - x \blacklozenge y) \otimes 1 + 1 \otimes (x \bullet y - x \blacklozenge y).
\end{aligned}$$

We deduce that $x \bullet 1 - x \blacklozenge 1$ is primitive, so belongs to V. As it is homogeneous of length $k \geq 2$, it is zero, and $x \bullet 1 = x \blacklozenge 1$.

We can now assume that $k, l \geq 1$. We proceed by induction on $k + l$. There is nothing left to do for $k + l = 0$ or 1. Let us assume that the result is true at all rank $< k + l$, with $k + l \geq 2$. Then, using the induction hypothesis, as x' and x'' have lengths $< k$ and y' has a length $< l$:

$$\begin{aligned}
\Delta(x \bullet y) &= 1 \otimes x \bullet y + x' \otimes x'' \bullet y + x \otimes 1 \bullet y + x \bullet 1 \otimes y + x' \bullet 1 \otimes x'' \sqcup\!\sqcup y + 1 \bullet 1 \otimes x \sqcup\!\sqcup y\\
&\quad + x \bullet y \otimes 1 + x' \bullet y \otimes x'' + 1 \bullet y \otimes x + x \bullet y' \otimes y'' + x' \bullet y' \otimes x'' \sqcup\!\sqcup y'' + 1 \bullet y' \otimes x \sqcup\!\sqcup y''\\
&= 1 \otimes x \bullet y + x' \otimes x'' \blacklozenge y + x \otimes 1 \blacklozenge y + x \blacklozenge 1 \otimes y + x' \blacklozenge 1 \otimes x'' \sqcup\!\sqcup y + 1 \blacklozenge 1 \otimes x \sqcup\!\sqcup y\\
&\quad + x \bullet y \otimes 1 + x' \blacklozenge y \otimes x'' + 1 \blacklozenge y \otimes x + x \blacklozenge y' \otimes y'' + x' \blacklozenge y' \otimes x'' \sqcup\!\sqcup y'' + 1 \blacklozenge y' \otimes x \sqcup\!\sqcup y''\\
&= \Delta(x \blacklozenge y) + (x \bullet y - x \blacklozenge y) \otimes +1 \otimes (x \bullet y - x \blacklozenge y).
\end{aligned}$$

We deduce that $x \bullet y - x \blacklozenge y$ is primitive, hence belongs to V. As it belongs to $V^{\otimes(k+l)}$ and $k + l \geq 2$, it is zero. Finally, $x \bullet y = x \blacklozenge y$. $\Box$

Proposition 6 *The Com-PreLie bialgebras $T(V,f)$ and $T(W,g)$ are isomorphic if, and only if, there exists a linear isomorphism $F : V \longrightarrow W$, such that $g \circ F = F \circ f$.*

Proof. If such an F exists, by functoriality $T(F)$ is an isomorphism from $T(V,f)$ to $T(W,g)$. Let us assume that $\phi : T(V,f) \longrightarrow T(V,g)$ is an isomorphism of Com-PreLie bialgebras. Then $\phi(1) = 1$, and ϕ induces an isomorphism from $V = Prim(T(V))$ to $W = Prim(T(W))$, denoted by F. For all $x \in V$:

$$\phi(x \bullet 1) = \phi(f(x)) = F \circ f(x) = F(x) \bullet 1 = g \circ F(x).$$

So such an F exists. $\Box$

2.2 Com-PreLie algebra attached to a linear form

Let V be a a vector space, $f : V \longrightarrow \mathbb{K}$ be a linear form, and $\lambda \in \mathbb{K}$.

Theorem 7 *Let $\bullet$ be the product on $T(V)$ such that for all $x_1, \ldots, x_m, y_1, \ldots, y_n \in V$:*

$$x_1 \ldots x_m \bullet y_1 \ldots y_n = \sum_{i=0}^{n-1} \lambda^i f(x_1) f(y_1) \ldots f(y_i) x_2 \ldots x_m ⧢ y_{i+1} \ldots y_n.$$

Then $(T(V), ⧢, \bullet)$ is a Com-PreLie algebra. It is denoted by $T(V,f,\lambda)$.

Examples. If $x_1, x_2, x_3 \in V$, $w \in T(V)$:

$$\begin{aligned} x_1 \bullet w &= f(x_1) w, \\ x_1 x_2 \bullet w &= x_1 f(x_2) w + f(x_1)(x_2 ⧢ w), \\ x_1 x_2 x_3 \bullet w &= x_1 x_2 f(x_3) w + x_1 f(x_2)(x_3 ⧢ w) + f(x_1)(x_2 x_3 ⧢ w). \end{aligned}$$

In particular if $x_1 = \ldots = x_n = y_1 = \ldots = y_n = x$:

Lemma 8 *Let $x \in V$. We put $f(x) = \nu$ and $\mu = \lambda f(x)$. Then, for all $m, n \geq 0$, in $T(V,f,\lambda)$:*

$$x^m \bullet x^n = \nu \sum_{j=m}^{m+n-1} \mu^{m+n-j-1} \binom{j}{m-1} x^j.$$

The proof of theorem 7 will use definition 9 and lemma 10:

Definition 9 *Let ∂ and ϕ be the linear maps defined by:*

$$\partial : \left\{ \begin{array}{rcl} T(V) & \longrightarrow & T(V) \\ 1 & \longrightarrow & 0, \\ x_1 \ldots x_n & \longrightarrow & f(x_1) x_2 \ldots x_n, \end{array} \right. \qquad \phi : \left\{ \begin{array}{rcl} T(V) & \longrightarrow & T(V) \\ 1 & \longrightarrow & 0, \\ x_1 \ldots x_n & \longrightarrow & \displaystyle\sum_{i=0}^{n-1} \lambda^i f(x_1) \ldots f(x_i) x_{i+1} \ldots x_n. \end{array} \right.$$

Lemma 10 *1. For all $u, v \in T(V)$:*

(a) $\partial(u ⧢ v) = \partial(u) ⧢ v + u ⧢ \partial(v)$.

(b) $\partial \circ \phi(u) ⧢ \phi(v) - \phi(\partial(u) ⧢ \phi(v)) = \partial \circ \phi(v) ⧢ \phi(u) - \phi(\partial(v) ⧢ \phi(u))$.

2. For all $u \in T(V,f,\lambda)$:

$$\Delta \circ \partial(u) = (\partial \otimes Id) \circ \Delta(u), \qquad \Delta \circ \phi(u) = (\phi \otimes Id) \circ \Delta(u) + 1 \otimes \phi(u).$$

Proof. 1. (a) This is obvious if $u=1$ or $v=1$, as $\partial(1)=0$. Let us assume that u,v are nonempty words. We put $v=xu'$,$v=yv'$, with $x,y\in V$. Then:

$$\begin{aligned}\partial(u⧢v)&=\partial(x(u'⧢v)+y(u⧢v'))\\&=f(x)u'⧢v+f(y)u⧢v'\\&=(f(x)u')⧢v+u⧢(f(y)v')\\&=\partial(u)⧢v+u⧢\partial(v).\end{aligned}$$

1. (b) Let us take $u=x_1\ldots x_m$ and $y=y_1\ldots y_n$ be two words of $T(V)$ of respective lengths m and n. First, observe that $\phi(\partial u⧢\phi(v))$ is a linear span of terms:

$$\lambda^{i+j-1}f(x_1)\ldots f(x_i)f(y_1)\ldots f(y_j)x_{i+1}\ldots x_m⧢y_{j+1}\ldots y_m,$$

with $1\leq i\leq m$, $0\leq j\leq n$, $(i,j)\neq(0,0)$. Let us compute the coefficient of such a term:

- If $j<n$, it is $\displaystyle\sum_{p=0}^{j}\binom{i-1+j-p}{i-1}=\sum_{p=i-1}^{i+j-1}\binom{p}{i-1}=\binom{i+j}{i}$.
- If $j=n$, its is $\displaystyle\sum_{p=0}^{n-1}\binom{i-1+j-p}{i-1}=\sum_{p=i}^{i+j-1}\binom{p}{i-1}=\sum_{p=i-1}^{i+j-1}\binom{p}{i-1}-1=\binom{i+j}{i}-1.$

We obtain:

$$\begin{aligned}\phi(\partial u⧢\phi(v))=&\sum_{i=1}^{m}\sum_{j=0}^{n}\lambda^{i+j-1}\binom{i+j}{i}f(x_1)\ldots f(x_i)f(y_1)\ldots f(y_j)x_{i+1}\ldots x_m⧢y_{j+1}\ldots y_n\\&-\sum_{i=1}^{m-1}\lambda^{i+n-1}f(x_1)\ldots f(x_i)f(y_1)\ldots f(y_n)x_{i+1}\ldots x_m\\&-\lambda^{m+n-1}\binom{m+n}{m}f(x_1)\ldots f(x_m)f(y_1)\ldots f(y_n)\\=&\sum_{i=1}^{m}\sum_{j=1}^{n}\lambda^{i+j-1}\binom{i+j}{i}f(x_1)\ldots f(x_i)f(y_1)\ldots f(y_j)x_{i+1}\ldots x_m⧢y_{j+1}\ldots y_n\\&+\sum_{i=1}^{m}\lambda^{i-1}f(x_1)\ldots f(x_i)x_{i+1}\ldots x_m⧢y_1\ldots y_n\\&-\sum_{i=1}^{m-1}\lambda^{i+n-1}f(x_1)\ldots f(x_i)f(y_1)\ldots f(y_n)x_{i+1}\ldots x_m\\&-\lambda^{m+n-1}\binom{m+n}{m}f(x_1)\ldots f(x_m)f(y_1)\ldots f(y_n).\end{aligned}$$

Moreover:

$$\begin{aligned}\partial\circ\phi(u)⧢\phi(v)=&\sum_{i=1}^{m}\sum_{j=0}^{n-1}\lambda^{i+j-1}\binom{i+j}{i}f(x_1)\ldots f(x_i)f(y_1)\ldots f(y_j)x_{i+1}\ldots x_m⧢y_{j+1}\ldots y_n\\=&\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\lambda^{i+j-1}\binom{i+j}{i}f(x_1)\ldots f(x_i)f(y_1)\ldots f(y_j)x_{i+1}\ldots x_m⧢y_{j+1}\ldots y_n\\&+\sum_{j=1}^{n-1}\lambda^{j+m-1}f(x_1)\ldots f(x_m)f(y_1)\ldots f(y_j)y_{j+1}\ldots y_n\\&+\sum_{i=1}^{m}\lambda^{i-1}f(x_1)\ldots f(x_i)x_{i+1}\ldots x_m⧢y_1\ldots y_n.\end{aligned}$$

Hence:

$$\begin{aligned}
&\partial \circ \phi(u) ⧢ \phi(v) - \phi(\partial u ⧢ \phi(v)) \\
&= \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \lambda^{i+j-1} \binom{i+j}{i} f(x_1) \ldots f(x_i) f(y_1) \ldots f(y_j) x_{i+1} \ldots x_m ⧢ y_{j+1} \ldots y_n \\
&- \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda^{i+j-1} \binom{i+j}{i} f(x_1) \ldots f(x_i) f(y_1) \ldots f(y_j) x_{i+1} \ldots x_m ⧢ y_{j+1} \ldots y_n \\
&+ \lambda^{m+n-1} \binom{m+n}{m} f(x_1) \ldots f(x_m) f(y_1) \ldots f(y_n) \\
&+ \sum_{j=1}^{n-1} \lambda^{j+m-1} f(x_1) \ldots f(x_m) f(y_1) \ldots f(y_j) y_{j+1} \ldots y_n \\
&+ \sum_{i=1}^{m-1} \lambda^{i+n-1} f(x_1) \ldots f(x_i) f(y_1) \ldots f(y_n) x_{i+1} \ldots x_m.
\end{aligned}$$

The three first rows are symmetric in u and v, whereas the sum of the fourth and fifth rows is symmetric in u and v. So $\partial \circ \phi(u) ⧢ \phi(v) - \phi(\partial u ⧢ \phi(v))$ is symmetric in u and v.

2. Let us take $u = x_1 \ldots x_n$, with $x_1, \ldots, x_n \in V$. Then:

$$\begin{aligned}
\Delta \circ \partial(u) &= f(x_1) \sum_{i=1}^{n} x_2 \ldots x_i \otimes x_{i+1} \ldots x_n \\
&= \sum_{i=1}^{n} \partial(x_1 \ldots x_i) \otimes x_{i+1} \ldots x_n + \partial(1) \otimes x_1 \ldots x_n \\
&= \sum_{i=0}^{n} \partial(x_1 \ldots x_i) \otimes x_{i+1} \ldots x_n \\
&= (\partial \otimes Id) \circ \Delta(u).
\end{aligned}$$

Moreover:

$$\begin{aligned}
\Delta \circ \phi(u) &= \sum_{i=0}^{n-1} \lambda^i f(x_1) \ldots f(x_i) \Delta(x_{i+1} \ldots x_n) \\
&= \sum_{i=0}^{n-1} \sum_{j=i}^{n} \lambda^i f(x_1) \ldots f(x_i) x_{i+1} \ldots x_j \otimes x_{j+1} \ldots x_n \\
&= \sum_{j=0}^{n} \sum_{i=0}^{j} \lambda^i f(x_1) \ldots f(x_i) x_{i+1} \ldots x_j \otimes x_{j+1} \ldots x_n - \lambda^n f(x_1) \ldots f(x_n) \otimes 1 \\
&= \sum_{j=0}^{n} \phi(x_1 \ldots x_j) \otimes x_{j+1} \ldots x_n + \sum_{j=0}^{n-1} \lambda^j f(x_1) \ldots f(x_j) \otimes x_{j+1} \ldots x_n \\
&= \sum_{j=0}^{n} \phi(x_1 \ldots x_j) \otimes x_{j+1} \ldots x_n + 1 \otimes \left(\sum_{j=0}^{n-1} \lambda^j f(x_1) \ldots f(x_j) x_{j+1} \ldots x_n \right) \\
&= (\phi \otimes Id) \circ \Delta(u) + 1 \otimes \phi(u).
\end{aligned}$$

□

Proof. (Theorem 7). By definition, for all $u, v \in T(V)$:

$$u \bullet v = \partial(u) ⧢ \phi(v).$$

Let $u,v,w\in T(V)$. By lemma 10-1:

$$\begin{aligned}(u⧢v)\bullet w&=\partial(u⧢v)⧢\phi(w)\\&=\partial(u)⧢v⧢\phi(w)+u⧢\partial(v)⧢\phi(w)\\&=\partial(u)⧢\phi(w)⧢v+u⧢\partial(v)⧢\phi(w)\\&=(u\bullet w)⧢v+u⧢(v\bullet w).\end{aligned}$$

Moreover:

$$\begin{aligned}(u\bullet v)\bullet w-u\bullet(v\bullet w)&=(\partial(u)⧢\phi(v))\bullet w-u\bullet(\partial(v)⧢\phi(w))\\&=\partial(\partial(u)⧢\phi(v))⧢\phi(w)-\partial(u)⧢\phi(\partial(v)⧢\phi(w))\\&=\partial^2(u)⧢\phi(v)⧢\phi(w)+\partial(u)⧢\left(\partial\circ\phi(v)⧢\phi(w)-\phi(\partial(v)⧢\phi(w))\right).\end{aligned}$$

By lemma 10-2, this is symmetric in v and w. Consequently, $T(V,f,\lambda)$ is Com-PreLie. □

This construction is functorial. Let (V,f) and (W,g) be two spaces equipped with a linear form and let $F:V\longrightarrow W$ be a map such that $g\circ F=f$. Then $T(F)$ is a Com-PreLie algebra morphism from $T(V,f,\lambda)$ to $T(W,g,\lambda)$.

Proposition 11 *$(coS(V),⧢,\bullet,\Delta)$ is a Com-PreLie bialgebra, denoted by $coS(V,f,\lambda)$.*

Proof. Let us first prove that $coS(V)$ is stable under $\bullet$. It is enough to prove that it is stable under ∂ and ϕ. Let us first consider ∂. As it is a derivation for $⧢$, it is enough to prove that $\partial(V)\subseteq coS(V)$, which is obvious as $\partial(V)\subset\mathbb{K}$. Let us now consider ϕ. Let $x_1,\ldots,x_k\in V$.

$$\begin{aligned}\phi(x_1⧢\ldots⧢x_k)&=\sum_{\sigma\in\mathfrak{S}_k}\phi(x_{\sigma(1)}\ldots x_{\sigma(k)})\\&=\sum_{i=0}^{k-1}\sum_{\sigma\in\mathfrak{S}_k}\mu^i f(x_{\sigma(1)})\ldots f(x_{\sigma(i)})x_{\sigma(i+1)}\ldots x_{\sigma(k)}\\&=\sum_{i=0}^{k-1}\sum_{1\leq k_1<\ldots<k_i\leq k}i!\mu^i\prod_{j=1}^{i}f(x_{k_i})x_1⧢\widehat{x_{k_1}}⧢\ldots⧢\widehat{x_{k_i}}⧢\ldots⧢x_k.\end{aligned}$$

This is an element of $coS(V)$, so $coS(V)$ is stable under $\bullet$.

Let us prove now the compatibility between $\bullet$ and the coproduct of $coS(V)$. As $coS(V)$ is cocommutative, lemma 10 implies that for all $u\in coS(V)$:

$$\Delta\circ\partial(u)=\partial(u^{(1)})\otimes u^{(2)}=\partial(u^{(2)})\otimes u^{(1)}=u^{(1)}\otimes\partial(u^{(2)}).$$

Let us consider $u,v\in coS(V)$. Then, by lemma 10:

$$\begin{aligned}\Delta(u\bullet v)&=\Delta(\partial(u)⧢\phi(v))\\&=(\Delta\circ\partial(u))⧢\Delta\circ\phi(v)\\&=(\Delta\circ\partial u)⧢(\Phi(v^{(1)})\otimes v^{(2)}+1\otimes\phi(v))\\&=\partial(u^{(1)})⧢\Phi(v^{(1)})\otimes u^{(2)}⧢v^{(2)}+u^{(1)}⧢1\otimes\partial(u^{(2)})⧢\phi(v)\\&=u^{(1)}\bullet v^{(1)}\otimes u^{(2)}⧢v^{(2)}+u^{(1)}\otimes u^{(2)}\bullet v.\end{aligned}$$

So $coS(V)$ is a Com-PreLie bialgebra. □

Note that $f_{coS(V,f,\lambda)}=0$. The preLie product induced on $Prim(coS(V))=V$ is given by $x\star y=f(x)y$.

Corollary 12 *Let V be a vector space, $f \in V^*$, $\lambda \in \mathbb{K}$. We give $S(V)$ its usual product m and coproduct $\mathbf{\Delta}$, defined by $\mathbf{\Delta}(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$, and the product $\bullet$ defined by:*

1. $1 \bullet x = 0$ *for any* $x \in S(V)$.
2. $x \bullet x_1 \dots x_k = \displaystyle\sum_{I \subsetneq \{1,\dots,k\}} |I|!\lambda^{|I|} f(x) \prod_{i \in I} f(x_i) \prod_{i \notin I} x_i$, *for all* $x, x_1, \dots, x_k \in V$.
3. $x_1 \dots x_k \bullet x = \displaystyle\sum_{i=1}^{k} x_1 \dots (x_i \bullet x) \dots x_k$ *for any* $x_1, \dots, x_k \in V$, $x \in S(V)$.

Then $(S(V), m, \bullet, \mathbf{\Delta})$ is a Com-PreLie bialgebra, denoted by $S(V, f, \lambda)$.

Proof. There is a Hopf algebra isomorphism:

$$\theta : \left\{ \begin{array}{rcl} (S(V), m, \mathbf{\Delta}) & \longrightarrow & (coS(V), ш, \Delta) \\ v \in V & \longrightarrow & v. \end{array} \right.$$

Let $v, x_1, \dots, x_k \in V$.

$$\begin{aligned} \theta(v) \bullet \theta(x_1 \dots x_k) &= v \bullet x_1 ш \dots ш x_k \\ &= f(v) ш \phi(x_1 ш \dots ш x_k) \\ &= f(v) \sum_{i=0}^{k-1} \sum_{1 \leq k_1 < \dots < k_i \leq k} i! \mu^i \prod_{j=1}^{i} f(x_{k_i}) x_1 ш \widehat{x_{k_1}} ш \dots ш \widehat{x_{k_i}} ш \dots ш x_k \\ &= \theta \left(\sum_{I \subsetneq \{1,\dots,k\}} |I|! \mu^i \prod_{i \in I} f(x_i) \prod_{i \notin I} x_i \right). \end{aligned}$$

Therefore, as $coS(V)$ is a Com-PreLie algebra, $S(V)$ is also a Com-PreLie bialgebra. □

Proposition 13 *Let us assume that $f \neq 0$. Then:*

1. *$(T(V), \prec, \bullet)$ is a Zinbiel-PreLie algebra if, and only if, $dim(V) = 1$.*
2. *$(T(V), ш, \bullet, \Delta)$ is a Com-PreLie bialgebra if, and only if, $dim(V) = 1$.*

Proof. 1. $\Longrightarrow$. Let $y \in V$, such that $f(y) = 1$. Note that $y \neq 0$. Let $x \in V$, such that $f(x) = 0$. Then:

$$\begin{aligned} (x \prec y) \bullet y = xy \bullet y = f(x) y ш y = 0, \\ (x \bullet y) \prec y + x \prec (y \bullet y) = f(x) y \prec y + x \prec f(y) y = 0 + f(y) x \prec y = xy. \end{aligned}$$

As $T(V, f, \lambda)$ is Zinbiel-PreLie, $xy = 0$. As $y \neq 0$, $x = 0$; we obtain that f is injective, so $dim(V) = 1$.

1. $\Longrightarrow$. We use the notations of lemma 8. It is enough to prove that for all $k, l, m \geq 0$, $(x^k \prec x^l) \bullet x^m = (x^k \bullet x^m) \prec x^l + x^k \prec (x^l \bullet x^m)$.

$$(x^k \prec x^l) \bullet x^m = \lambda \sum_{j=k+l}^{k+l+m-1} \mu^{k+l+m-j-1} \binom{j}{k+l-1} \binom{k+l-1}{k-1} x^j,$$

and:

$$\begin{aligned}
&(x^k \bullet x^m) \prec x^l + x^k \prec (x^l \bullet x^m)\\
&= \lambda \sum_{j=k}^{k+m-1} \mu^{k+m-j-1} \binom{j}{k-1} x^j \prec x^l + \lambda \sum_{j=l}^{l+m-1} \mu^{l+m-1-j} \binom{j}{k-1} x^k \prec x^j\\
&= \lambda \sum_{j=k}^{k+m-1} \mu^{k+m-j-1} \binom{j}{k-1}\binom{j+l-1}{j-1} x^{l+j} + \lambda \sum_{j=l}^{l+m-1} \mu^{l+m-1-j} \binom{j}{k-1}\binom{k+j-1}{k-1} x^{k+j}\\
&= \lambda \sum_{j=k+l}^{k+l+m-1} \mu^{k+l+m-j-1} \binom{j-l}{k-1}\binom{j-1}{j-l-1} x^j + \lambda \sum_{j=k+l}^{k+l+m-1} \mu^{k+l+m-j-1} \binom{j-k}{l-1}\binom{j-1}{k-1} x^j.
\end{aligned}$$

Moreover, a simple computation proves that:

$$\binom{j-l}{k-1}\binom{j-1}{j-l-1} + \binom{j-k}{l-1}\binom{j-1}{k-1} = \binom{j}{k+l-1}\binom{k+l-1}{k-1}.$$

So $T(V, f, \lambda)$ is Zinbiel-PreLie.

2. $\Longrightarrow$. Let us choose $z \in V$, nonzero, and $x \in V$ such that $f(x) = 1$. Then:

$$\Delta(xy \bullet z) = \Delta(f(x)y \sqcup\!\sqcup z) = xy \bullet z \otimes 1 + 1 \otimes xy \bullet z + y \otimes z + z \otimes y,$$

whereas:

$$\begin{aligned}
&(xy)^{(1)} \otimes (xy)^{(2)} \bullet z + (xy)^{(1)} \bullet z^{(1)} \otimes (xy)^{(2)} \sqcup\!\sqcup z^{(2)}\\
&= xy \otimes 1 \bullet z + x \otimes y \bullet z + 1 \otimes xy \bullet z\\
&+ xy \bullet z \otimes 1 + xy \bullet 1 \otimes z + x \bullet z \otimes y + x \bullet 1 \otimes y \sqcup\!\sqcup z + 1 \bullet z \otimes xy + 1 \bullet 1 \otimes xy \sqcup\!\sqcup z\\
&= xy \bullet z \otimes 1 + 1 \otimes xy \bullet z + f(y)x \otimes z + z \otimes y.
\end{aligned}$$

So, for all $y \in V$, $f(y)x \otimes z = y \otimes z$. As $z \neq 0$, $f(y)x = y$: $V = Vect(x)$ is one-dimensional.

$\Longleftarrow$. In this case, $T(V) = coS(V)$, so is a Com-PreLie bialgebra. □

Proposition 14 *The Com-PreLie bialgebras $coS(V, f, \lambda)$ and $coS(W, g, \mu)$ are isomorphic if, and only if, one of the following assertion holds:*

1. *$dim(V) = dim(W)$, and f and g are both zero.*
2. *$dim(V) = dim(W)$, $\lambda = \mu$ and f and g are both nonzero.*

Proof. If $dim(V) = dim(W)$, and f and g are both zero, then $\bullet = 0$ in both these Com-preLie bialgebras. Take any linear isomorphism F from V to W, then the restriction of $T(F)$ as an algebra morphism from $coS(V)$ to $coS(W)$ is an isomorphism of Com-PreLie bialgebras.

If $dim(V) = dim(W)$, $\lambda = \mu$ and f and g are both nonzero, there exists an isomorphism $F : V \longrightarrow W$ such that $g \circ F = f$. By functoriality, $T(V, f, \lambda)$ and $T(W, g, \lambda)$ are isomorphic via $T(F)$. The restriction of $T(F)$ induces an isomorphism from $coS(V, f, \lambda)$ to $coS(W, g, \lambda)$.

Let us assume that $\phi : coS(V, f, \lambda) \longrightarrow coS(W, g, \mu)$ is an isomorphism of Com-PreLie bialgebras. It induces an isomorphism from $Prim(coS(V)) = V$ to $Prim(coS(W)) = W$, denoted by F: consequently, $dim(V) = dim(W)$. Let us choose $y \in V$, nonzero. For all $x \in V$:

$$\phi(x \bullet y) = \phi(f(x)y) = f(x)F(y) = \phi(x) \bullet \phi(y) = F(x) \bullet F(y) = g \circ F(x)F(y).$$

As F is an isomorphism, for all $x \in V$, $f(x) = g \circ F(x)$. So f and g are both zero or are both nonzero. Let us assume that they are nonzero. We choose $x \in V$, such that $f(x) = 1$. Then:

$$\phi(x^2) = \phi\left(\frac{x \sqcup\!\sqcup x}{2}\right) = \frac{\phi(x) \sqcup\!\sqcup \phi(x)}{2} = F(x)^2.$$

Hence:

$$\begin{aligned} \phi(x) \bullet \phi(x^2) &= F(x) \bullet F(x)^2 \\ &= g \circ F(x)F(x)^2 + \mu g \circ F(x)^2 F(x) \\ &= F(x)^2 + \mu F(x). \end{aligned} \qquad \begin{aligned} \phi(x \bullet x^2) &= \phi(f(x)x^2 + \lambda f(x)^2 x) \\ &= F(x)^2 + \lambda F(x). \end{aligned}$$

As $x \neq 0$, $F(x) \neq 0$, so $\lambda = \mu$. □

2.3 Com-PreLie algebra associated to a preLie algebra

Theorem 15 *Let $(V, \star)$ be a preLie algebra. We define a product on $T(V)$ by:*

$$x_1 \ldots x_k \bullet y_1 \ldots y_l = \sum_{i=1}^{k} x_1 \ldots x_{i-1}(x_i \star y_1)(x_{i+1} \ldots x_l \sqcup\!\sqcup y_2 \ldots y_l),$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_l \in V$; by convention, this is equal to 0 if $k = 0$ or $l = 0$. Then $(T(V), \prec, \bullet, \Delta)$ is a Zinbiel-PreLie bialgebra, denoted by $T(V, \star)$.

Examples. Let $x_1, x_2, x_3, y \in V$, $w \in T(V)$.

$$\begin{aligned} x_1 \bullet yw &= (x_1 \star y)w, \\ x_1x_2 \bullet yw &= (x_1 \star y)(x_2 \sqcup\!\sqcup w) + x_1(x_2 \star y)w, \\ x_1x_2x_3 \bullet yw &= (x_1 \star y)(x_2x_3 \sqcup\!\sqcup w) + x_1(x_2 \star y)(x_3 \sqcup\!\sqcup w) + x_1x_2(x_3 \star y)w. \end{aligned}$$

Proof. First, remark that for all $x, y \in V$, for all $u, v \in T(V)$:

$$xu \bullet yv = (x \star y)u \sqcup\!\sqcup v + x(u \bullet yv).$$

Let us prove that for all $a, b, c \in T(V)$, $(a \prec b) \bullet c = (a \bullet c) \prec b + a \prec (b \bullet c)$. This is obvious if one of a, b, c is equal to 1, as $1 \bullet d = d \bullet 1 = 0$ for all d. We now assume that a, b, c are nonempty words of respective lengths k, l and m, and we proceed by induction on $k + l + m$. There is nothing to do if $k + l + m \leq 2$. Let us assume the result at rank $k + l + m - 1$. We put $a = xu$, $b = v$, $c = zw$, avec $x, z \in V$.

$$\begin{aligned} (xu \prec v) \bullet zw &= (x(u \sqcup\!\sqcup v)) \bullet zw \\ &= x \star z(u \sqcup\!\sqcup v \sqcup\!\sqcup w) + x((u \sqcup\!\sqcup v) \bullet zw) \\ &= x \star z(u \sqcup\!\sqcup v \sqcup\!\sqcup w) + x((u \bullet zw) \sqcup\!\sqcup v + u \sqcup\!\sqcup (v \bullet zw)) \\ &= (x \star z(u \sqcup\!\sqcup w)) \prec v + x(u \bullet zw) \prec v + xu \prec (v \bullet zw) \\ &= (xu \bullet v) \prec zw + xu \prec (v \bullet zw). \end{aligned}$$

Let us now prove that for all $a, b, c \in T(V)$, $a \bullet (b \bullet c) - (a \bullet b) \bullet c = a \bullet (c \bullet b) - (a \bullet c) \bullet b$. If one of a, b, c is equal to 1, this is obvious. We now assume that a, b, c are nonempty words of respective lengths k, l and m, and we proceed by induction on $k + l + m$. There is nothing to do if $k + l + m \leq 2$. Let us assume the result at rank $k + l + m - 1$. We put $a = xu$, $b = yv$, $c = zw$, avec $x, y, z \in V$.

$$\begin{aligned} (xu \bullet yv) \bullet zw &= (x \star y(u \sqcup\!\sqcup v)) \bullet zw + (x(u \bullet yv)) \bullet zw \\ &= (x \star y) \star z(u \sqcup\!\sqcup v \sqcup\!\sqcup w) + x \star y((u \sqcup\!\sqcup v) \bullet zw) \\ &\quad + x \star z((u \bullet yv) \sqcup\!\sqcup w) + x((u \bullet yv) \bullet zw); \\ xu \bullet (yv \bullet zw) &= xu \bullet (y \star z(v \sqcup\!\sqcup w) + y(v \bullet zw)) \\ &= x \star (y \star z)u \sqcup\!\sqcup v \sqcup\!\sqcup w + xu \bullet (y \star z(v \sqcup\!\sqcup w)) \\ &\quad + x \star y(u \sqcup\!\sqcup (v \bullet zw)) + x(u \bullet y(v \bullet zw)). \end{aligned}$$

Hence:

$$\begin{aligned}(xu\bullet yv)\bullet zw - xu\bullet(yv\bullet zw) &= ((x\star y)\star z - x\star(y\star z))(u\sqcup\!\sqcup v\sqcup\!\sqcup w)\\ &\quad + x\star y((u\sqcup\!\sqcup v)\bullet zw) + x\star z((u\bullet yv)\sqcup\!\sqcup w)\\ &\quad + x((u\bullet yv)\bullet zw - u\bullet(yv\bullet zw)).\end{aligned}$$

As $\star$ is preLie and $\sqcup\!\sqcup$ is commutative, the first row is symmetric in yv and zw. The second row is obviously symmetric in yv and zw,and by the induction hypothesis, the last row also is. So the preLie relation is satisfied for xu, yv and zw.

Let us prove the compatibility with the coproduct. Let $a, b \in T(V)$. Let us prove that:

$$\Delta(a\bullet b) = a^{(1)}\otimes a^{(2)}\bullet b + a^{(1)}\bullet b^{(1)}\otimes a^{(2)}\sqcup\!\sqcup b^{(2)}.$$

This is immediate if a or b is equal to 1. We now assume that a and b are nonempty words of respective lengths k and l, and we proceed by induction on $k+l$. There is nothing to do if $k+l\leq 1$. Let us assume the result at rank $k+l-1$. We put $a = xu$ and $b = yv$, $x, y \in V$.

$$\begin{aligned}\Delta(xu\bullet yv) &= \Delta(x\star y(u\sqcup\!\sqcup v) + x(u\bullet yv))\\ &= (x\star y)u^{(1)}\sqcup\!\sqcup v^{(1)}\otimes u^{(2)}\sqcup\!\sqcup v^{(2)} + 1\otimes x\star y(u\sqcup\!\sqcup v)\\ &+ xu^{(1)}\otimes u^{(2)}\bullet yv + x(u^{(1)}\bullet yv^{(1)})\otimes u^{(2)}\sqcup\!\sqcup v^{(2)}\\ &+ xu^{(1)}\bullet 1\otimes u^{(2)}\sqcup\!\sqcup yv + 1\otimes x(u\bullet yv)\\ &= xu^{(1)}\otimes u^{(2)}\bullet yv + 1\otimes xu\bullet yv\\ &+ (x\star y)u^{(1)}\sqcup\!\sqcup v^{(1)}\otimes u^{(2)}\sqcup\!\sqcup v^{(2)}\\ &= (xu)^{(1)}\otimes(xu)^{(2)}\bullet yv\\ &+ (xu)^{(1)}\bullet(yv)^{(1)}\otimes(xu)^{(2)}\sqcup\!\sqcup(yv)^{(2)}.\end{aligned}$$

So $T(V,\star)$ is indeed a Zinbiel-PreLie bialgebra. $\Box$

This is also a functorial construction. If $F : (V,\star)\longrightarrow(W,\star)$ is a preLie algebra morphism, then $T(F)$ is a Zinbiel-PreLie algebra morphism.

Note that $f_{T(V,\star)} = 0$. The preLie product induced on $Prim(T(V)) = V$ is given by $\star$.

Proposition 16 *Let $\blacklozenge$ be a product on $T(V)$, such that $(T(V),\sqcup\!\sqcup,\bullet,\Delta)$ is a Zinbiel-PreLie bialgebra, with $V^{\otimes k}\blacklozenge V^{\otimes l}\subseteq V^{\otimes(k+l-1)}$ for all $k, l\in\mathbb{N}$. There exists a preLie product $\star$ on V, such that $(T(V),\prec,$*
$blacklozenge,\Delta) = T(V,\star)$.

Proof. By hypothesis, $V\blacklozenge V\subseteq V$: V is a preLie subalgebra of $T(V)$. We denote its preLie product by $\star$, and by $\bullet$ the preLie product of $T(V,\star)$. Let us prove that for any $x = x_1\dots x_k, y = y_1\dots y_l\in T(V)$, $x\bullet y = x\blacklozenge yl$. If $k=0$, we obtain $1\bullet y = 1'\bullet y = 0$. We now treat the case $l=0$: let us prove that $x\blacklozenge 1 = 0$ by induction on k. It is already done for $k=0$. If $k=1$, then $x\in V$, so $x\blacklozenge 1\in\mathbb{K}$ by homogeneity. Moreover, $\varepsilon(x\blacklozenge 1) = 0$, so $x\blacklozenge 1 = 0$. Let us assume the result at rank $k-1$, with $k\geq 2$. We put $u = x_2\dots x_k$. Then:

$$x\blacklozenge 1 = (x_1\prec u)\blacklozenge 1 = (x_1\blacklozenge 1)\prec u + x_1\prec(u\blacklozenge 1) = 0+0 = 0.$$

We can now assume that $k, l\geq 1$. We proceed by induction on $k+l$. There is nothing to do for $k+l = 0$ or 1. If $k+l = 2$, then $k = l = 1$, and $x\bullet y = x\star y = x\blacklozenge y$. Let us assume that the result is true at all rank $< k+l$, with $k+l \geq 3$. Then, using the induction hypothesis, as x' and x'' have lengths $< k$ and y' has a length $< l$:

$$\begin{aligned}\Delta(x\bullet y) &= 1\otimes x\bullet y + x'\otimes x''\bullet y + x\otimes 1\bullet y + x\bullet 1\otimes y + x'\bullet 1\otimes x''\sqcup\!\sqcup y + 1\bullet 1\otimes x\sqcup\!\sqcup y\\ &\quad + x\bullet y\otimes 1 + x'\bullet y\otimes x'' + 1\bullet y\otimes x + x\bullet y'\otimes y'' + x'\bullet y'\otimes x''\sqcup\!\sqcup y'' + 1\bullet y'\otimes x\sqcup\!\sqcup y''\\ &= 1\otimes x\bullet y + x'\otimes x''\blacklozenge y + x\otimes 1\blacklozenge y + x\blacklozenge 1\otimes y + x'\blacklozenge 1\otimes x''\sqcup\!\sqcup y + 1\blacklozenge 1\otimes x\sqcup\!\sqcup y\\ &\quad + x\bullet y\otimes 1 + x'\blacklozenge y\otimes x'' + 1\blacklozenge y\otimes x + x\blacklozenge y'\otimes y'' + x'\blacklozenge y'\otimes x''\sqcup\!\sqcup y'' + 1\blacklozenge y'\otimes x\sqcup\!\sqcup y''\\ &= \Delta(x\blacklozenge y) + (x\bullet y - x\blacklozenge y)\otimes +1\otimes (x\bullet y - x\blacklozenge y).\end{aligned}$$

We deduce that $x\bullet y - x\blacklozenge y$ is primitive, so belongs to V. As it belongs to $V^{\otimes(k+l-1)}$ and $k+l-1\geq 2$, it is zero. So $x\bullet y = x\blacklozenge y$. □

Proposition 17 *1. Let $(V,\star)$ and $(V',\star')$ be two preLie algebras. The Com-PreLie bialgebras $T(V,\star)$ and $T(V',\star')$ are isomorphic if, and only if, the preLie algebras $(V,\star)$ and $(V',\star')$ are isomorphic.*

2. Let $(V,\star)$ be a preLie algebra and $g: W \longrightarrow W$ be an endomorphism. The Com-PreLie bialgebras $T(V,\star)$ and $T(W,g)$ are isomorphic if, and only if, $dim(V) = dim(W)$, $\star = 0$ and $f = 0$.

Proof. 1. If $F: V \longrightarrow V'$ is a preLie algebra isomorphism, by functoriality, $T(F)$ is an isomorphism from $T(V,\star)$ to $T(V',\star')$. Let us assume that $\phi: T(V,\star) \longrightarrow T(V',\star')$ is an isomorphism. It induces by restriction an isomorphism F from $Prim(T(V)) = V$ to $Prim(T(V')) = V'$. Moreover, for all $x,y\in V$:

$$\phi(x\bullet y) = \phi(x\star y) = F(x\star y) = \phi(x)\bullet\phi(y) = F(x)\bullet F(y) = F(x)\star' F(y).$$

So $(V,\star)$ and $(V',\star')$ are isomorphic.

2. If $dim(V) = dim(W)$, $\star = 0$ and $f = 0$, then both preLie product of $T(V,\star)$ and $T(W,g)$ are zero. Let $F: V \longrightarrow W$ be an isomorphism. Then $T(F)$ is an isomorphism from $T(V,\star)$ to $T(W,g)$. Conversely, if $\phi: T(V,\star)\to T(W,g)$ is an isomorphism, it induces an isomorphism F from $Prim(T(V)) = V$ to $Prim(T(W)) = W$. As $\phi(1) = 1$, for all $x\in V$:

$$\phi(x\bullet 1) = 0 = \phi(x)\bullet\phi(1) = F(x)\bullet 1 = g\circ F(x).$$

As F is an isomorphism, $g = 0$, so the preLie product of $T(W,g)$ is zero. By isomorphism, the preLie product $\star$ of $T(V,\star)$ is zero. □

3 Examples on $\mathbb{K}[X]$

Our aim in this section is to give all preLie products on $\mathbb{K}[X]$, making it a graded Com-PreLie algebra. We shall prove the following result:

Theorem 18 *1. The following objects are Zinbiel-PreLie algebras:*

(a) Let $N\geq 1$, $\lambda, a, b\in\mathbb{K}$, $a\neq 0$, $b\notin\mathbb{Z}_-$. We put $\mathfrak{g}^{(1)}(N,\lambda,a,b) = (\mathbb{K}[X], m, \bullet)$, with:

$$X^i\bullet X^j = \begin{cases} i\lambda X^i \text{ if } j=0,\\ a\frac{i}{\frac{j}{N}+b}X^{i+j} \text{ if } j\neq 0 \text{ and } N\mid j,\\ 0 \text{ otherwise.}\end{cases}$$

(b) *Let $N \geq 1$, $\lambda, \mu \in \mathbb{K}$, $\mu \neq 0$. We put $\mathfrak{g}^{(2)}(N,\lambda,\mu) = (\mathbb{K}[X], m, \bullet)$, with:*

$$X^i \bullet X^j = \begin{cases} i\lambda X^i \text{ if } j = 0, \\ i\mu X^{i+N} \text{ if } j = N, \\ 0 \text{ otherwise.} \end{cases}$$

(c) *Let $N \geq 1$, $\lambda, \mu \in \mathbb{K}$, $\mu \neq 0$. We put $\mathfrak{g}^{(3)}(N,\lambda,\mu) = (\mathbb{K}[X], m, \bullet)$, with:*

$$X^i \bullet X^j = \begin{cases} i\lambda X^i \text{ if } j = 0, \\ i\mu X^{i+j} \text{ if } j \neq 0 \text{ and } N \mid j, \\ 0 \text{ otherwise.} \end{cases}$$

(d) *Let $\lambda \in \mathbb{K}$. We put $\mathfrak{g}^{(4)}(\lambda) = (\mathbb{K}[X], m, \bullet)$, with:*

$$X^i \bullet X^j = \begin{cases} i\lambda X^i \text{ if } j = 0, \\ 0 \text{ otherwise.} \end{cases}$$

In particular, the preLie product of $\mathfrak{g}^{(4)}(0)$ is zero.

2. *Moreover, if $\bullet$ is a product on $\mathbb{K}[X]$, such that $\mathfrak{g} = (\mathbb{K}[X], m, \bullet)$ is a graded Com-PreLie algebra, Then $\mathfrak{g}$ is one of the preceding examples.*

Remark. If $\lambda = \frac{a}{b}$, in $\mathfrak{g}^{(1)}(N,\lambda,a,b)$, for all $i, j \in \mathbb{N}$:

$$X^i \bullet X^j = \begin{cases} \frac{ai}{\frac{j}{N}+b} \text{ if } N \mid j, \\ 0 \text{ otherwise.} \end{cases}$$

We put $\mathfrak{g}^{(1)}(N,a,b) = \mathfrak{g}^{(1)}(N, \frac{a}{b}, a, b)$.

It is possible to prove that all these Com-PreLie algebras are not isomorphic. However, they can be isomorphic as Lie algebras. Let us first recall some notations on the Faà di Bruno Hopf algebra [2]:

- $\mathfrak{g}_{FdB}$ has a basis $(e_i)_{i\geq 1}$, and for all $i, j \geq 1$, $[e_i, e_j] = (i-j)e_{i+j}$.
- Let $\alpha \in \mathbb{K}$. The right $\mathfrak{g}_{FdB}$-module has a basis $V_\alpha = Vect(f_i)_{i\geq 1}$, and the right action of $\mathfrak{g}_{FdB}$ is defined by $f_i.e_j = (i+\alpha)e_{i+j}$.

Proposition 19 *Let $N \geq 1$, $\lambda, \lambda', \mu, a, b \in \mathbb{K}$, $\mu, a \neq 0$, $b \notin \mathbb{Z}_-$. Then, as Lie algebras:*

$$\mathfrak{g}^{(1)}(N,\lambda,a,b)_+ \approx \mathfrak{g}^{(3)}(N,\lambda,\mu)_+ \approx \left(V_{-\frac{1}{N}} \oplus \ldots \oplus V_{-\frac{N-1}{N}}\right) \rtimes \mathfrak{g}_{FdB}.$$

Proof. We first work in $\mathfrak{g}^{(1)}(N,\lambda,a,b)$. For all $i \geq 1$, for all $1 \leq r \leq N-1$, we put $E_i = \frac{i+b}{Na} X^{Ni}$ and $F_i^{(r)} = X^{N(i-1)+r}$. Then $(E_i)_{i\geq 1} \cup \bigcup_{r=1}^{N-1} \left(F_i^{(r)}\right)_{i\geq 1}$ is a basis of $\mathfrak{g}^{(1)}(N,\lambda,a,b)_+$, and, for all $i, j \geq 1$, for all $1 \leq r, s \leq N-1$:

$$[E_i, E_j] = (i-j)E_{i+j}, \qquad [F_i^{(r)}, F_j^{(s)}] = 0, \qquad [F_i^{(r)}, E_j] = \left(i + \frac{r-N}{N}\right) F_{i+j}^{(r)}.$$

Hence, this Lie algebra is isomorphic to $\left(V_{-\frac{N-1}{N}} \oplus \ldots \oplus V_{-\frac{1}{N}}\right) \rtimes \mathfrak{g}_{FdB}$. The proof is similar for $\mathfrak{g}^{(3)}(N,\lambda',\mu)$, with $E_i = \frac{1}{N\mu} X^{Ni}$ and $F_i^{(r)} = X^{N(i-1)+r}$. □

Consequently, we can describe the group corresponding to these Lie algebras.

1. G_{FdB} is the group of formal diffeomorphisms of $\mathbb{K}$ tangent to the identity:

$$G_{FdB} = (\{X + a_1 X^2 + a_2 X^3 + \ldots \mid a_2, a_2, \ldots \in \mathbb{K}\}, \circ).$$

2. For all $\alpha \in \mathbb{K}$, we define a right G_{FdB}-module $\mathbb{V}_\alpha$: as a vector space, this is $\mathbb{K}[[X]]_+$. The action is given by $P.Q = \left(\frac{Q(X)}{X}\right)^\alpha P \circ Q(X)$ for all $P \in \mathbb{V}_\alpha$ and $Q \in G_{FdB}$.

Then the group corresponding to our Lie algebras $\mathfrak{g}^{(1)}(N, \lambda, a, b)_+$ and $\mathfrak{g}^{(3)}(N, \lambda, \mu)_+$ is:

$$\left(\mathbb{V}_{-\frac{1}{N}} \oplus \ldots \oplus \mathbb{V}_{-\frac{N-1}{N}}\right) \rtimes G_{FdB}.$$

Let us conclude this paragraph with the description of the Lie algebra associated to $\mathfrak{g}^{(2)}(N, \lambda, \mu)$.

Proposition 20 *The Lie algebra $\mathfrak{g}^{(2)}(N, \lambda, \mu)_+$ admits a decomposition $\mathfrak{g}^{(2)}(N, \lambda, \mu)_+ \approx V^{\oplus N} \rtimes \mathfrak{g}_0$, where:*

- *$\mathfrak{g}_0$ is an abelian, one-dimensional, Lie algebra, generated by an element z.*
- *V is a right $\mathfrak{g}_0$-module, with a basis $(f_i)_{i \geq 0}$, and the right action defined by $f_i.z = f_{i+1}$.*

Proof. The Lie bracket of $\mathfrak{g}^{(2)}(N, \lambda, \mu)_+$ is given by:

$$[X^i, X^j] = \begin{cases} 0 \text{ if } i, j \neq N, \mu i X^{i+N} \text{ if } i \neq N, j = N. \end{cases}$$

We put $\mathfrak{g}_0 = Vect(X^N)$. The N-copies of V are given by:

- For $1 \leq r < N$, $V^{(r)} = Vect\left(\mu^i \prod_{j=1}^{i-1}(r + jN)X^{r+iN} \mid i \geq 0\right)$.
- $V^{(N)} = Vect\left(\mu^i N^i (i+1)! X^{(i+2)N} \mid i \geq 0\right)$.

□

3.1 Graded preLie products on $\mathbb{K}[X]$

We now look for all preLie products on $\mathbb{K}[X]$, making it a graded Com-PreLie algebra. Let $\bullet$ be a such a product. By homogeneity, for all $i, j \geq 0$, there exists a scalar $\lambda_{i,j}$ such that:

$$X^i \bullet X^j = \lambda_{i,j} X^{i+j}.$$

Moreover, for all $i, j, k \geq 0$:

$$\begin{aligned} X^{i+j} \bullet X^k &= \lambda_{i+j,k} X^{i+j+k} \\ &= (X^i X^j) \bullet X^k \\ &= (X^i \bullet X^k) X^j + X^i (X^j \bullet X^k) \\ &= (\lambda_{i,k} + \lambda_{j,k}) X^{i+j+k}. \end{aligned}$$

Hence, $\lambda_{i+j,k} = \lambda_{i,k} + \lambda_{j,k}$. Putting $\lambda_k = \lambda_{1,k}$ for all $k \geq 0$, we obtain:

$$X^i \bullet X^j = i \lambda_j X^{i+j}.$$

Lemma 21 *For all $k \geq 0$, let $\lambda_k \in \mathbb{K}$. We define a product $\bullet$ on $\mathbb{K}[X]$ by:*

$$X^i \bullet X^j = i \lambda_j X^{i+j}.$$

Then $(\mathbb{K}[X], m, \bullet)$ is Com-PreLie if, and only if, for all $j, k \geq 1$:

$$(j\lambda_k - k\lambda_j)\lambda_{j+k} = (j - k)\lambda_j \lambda_k.$$

Proof. Let $i, j, k \geq 0$. Then:

$$X^i \bullet (X^j \bullet X^k) - (X^i \bullet X^j) \bullet X^k = (ij\lambda_k\lambda_{j+k} - i(i+j)\lambda_j\lambda_k)X^{i+j+k}.$$

Hence:

$$\begin{aligned}
\bullet \text{ is preLie} &\Longleftrightarrow \forall i,j,k \geq 0, ij\lambda_k\lambda_{j+k} - i(i+j)\lambda_j\lambda_k = ik\lambda_j\lambda_{j+k} - i(i+k)\lambda_j\lambda_k \\
&\Longleftrightarrow \forall j,k \geq 0, (j\lambda_k - k\lambda_j)\lambda_{j+k} = (j-k)\lambda_j\lambda_k \\
&\Longleftrightarrow \forall j,k \geq 1, (j\lambda_k - k\lambda_j)\lambda_{j+k} = (j-k)\lambda_j\lambda_k,
\end{aligned}$$

as this relation is trivially satisfied if $j = 0$ or $k = 0$. □

Lemma 22 *Let $\bullet$ be a product on $\mathbb{K}[X]$, making it a graded Com-PreLie algebra. Then $(\mathbb{K}[X], \prec, \bullet)$ is a Zinbiel-PreLie algebra.*

Proof. Let us take $i, k, k \geq 0$, $(i, j) \neq (0, 0)$. Then:

$$\begin{aligned}
(X^i \bullet X^k) \prec X^j + X^i \prec (X^j \bullet X^k) &= \lambda_k(iX^{i+k} \prec X^j + jX^i \prec X^{j+k}) \\
&= \lambda_k \left(\frac{i(i+k)}{i+j+k} + \frac{ij}{i+j+k}\right) X^{i+j+k} \\
&= i\lambda_k X^{i+j+k} \\
&= (i+j)\lambda_k \frac{i}{i+j} X^{i+j+k}, \\
(X^i \prec X^j) \bullet X^k &= \frac{i}{i+j} X^{i+j} \bullet X^k \\
&= \frac{i}{i+j}(i+j)\lambda_k X^{i+j+k} \\
&= i\lambda_k X^{i+j+k}.
\end{aligned}$$

So $\mathbb{K}[X]$ is Zinbiel-PreLie. □

Proof. (Theorem 18, first part). Let us first prove that the objects defined in theorem 18 are indeed Zinbiel-PreLie algebras. By lemma 22, it is enough to prove that they are Com-PreLie algebras. We shall use lemma 21 in all cases.

1. For all $j \geq 1$, $\lambda_j = a\frac{1}{\frac{j}{N}+b}$ if $N \mid j$ and 0 otherwise. If j or k is not a multiple of N, then:

$$(j\lambda_k - k\lambda_j)\lambda_{j+k} = (j-k)\lambda_j\lambda_k = 0.$$

If $j = Nj'$ and $k = Nk'$, with j', k' integers, then:

$$\begin{aligned}
(j\lambda_k - k\lambda_j)\lambda_{j+k} &= Na^2 \left(\frac{j'}{k'+b} - \frac{k'}{j'+b}\right) \frac{1}{j'+k'+b} \\
&= Na^2 \frac{j'^2 - k'^2 + b(j'-k')}{(j'+b)(k'+b)(j'+k'+b)} \\
&= Na^2(j'-k') \frac{j'+k'+b}{(j'+b)(k'+b)(j'+k'+b)} \\
a^2(j-k)&\frac{1}{(j'+b)(k'+b)} \\
&= (j-k)\lambda_j\lambda_k.
\end{aligned}$$

2. In this case, $\lambda_j = \mu$ if $j = N$ and 0 otherwise. Hence, for all $j, k \geq 1$:

$$(j\lambda_k - k\lambda_j)\lambda_{j+k} = \mu^2(j\delta_{k,N} - k\delta_{j,N})\delta_{j+k,N} = 0,$$
$$(j-k)\lambda_j\lambda_k = \mu^2(j-k)\delta_{j,N}\delta_{k,N} = 0.$$

3. Here, for all $j \geq 1$, $\lambda_j = \mu$ if $N \mid j$ and 0 otherwise. Then:

$$(j\lambda_k - k\lambda_j)\lambda_{j+k} = \begin{cases} \mu^2(j-k) \text{ if } N \mid j, k, \\ 0 \text{ otherwise;} \end{cases} \qquad (j-k)\lambda_j\lambda_k = \begin{cases} \mu^2(j-k) \text{ if } N \mid j, k, \\ 0 \text{ otherwise.} \end{cases}$$

4. In this case, for all $j \geq 1$, $\lambda_j = 0$ and the result is trivial. □

3.2 Classification of graded preLie products on $\mathbb{K}[X]$

We now prove that the preceding examples cover all the possible cases.

Proof. (Theorem 18, second part). We put $X^i \bullet X^j = i\lambda_j X^{i+j}$ for all $i, j \geq 0$, and we put $\lambda = \lambda_0$. If for all $j \geq 1$, $\lambda_j = 0$, then $\mathfrak{g} = \mathfrak{g}^{(4)}(\lambda)$. If this is not the case, we put:

$$N = \min\{j \geq 1 \mid \lambda_j \neq 0\}.$$

First step. Let us prove that if i is not a multiple of N, then $\lambda_i = 0$.We put $i = qN + r$, with $0 < r < N$, and we proceed by induction on q. By definition of N, $\lambda_1 = \ldots = \lambda_{N-1} = 0$, which is the result for $q = 0$. Let us assume the result at rank $q - 1$, with $q > 0$. We put $j = i - N$ and $k = N$. By the induction hypothesis, $\lambda_j = 0$. Then, by lemma 21:

$$(i - N)\lambda_N\lambda_i = 0.$$

As $i \neq N$ and $\lambda_N \neq 0$, $\lambda_i = 0$. It is now enough to determine λ_{iN} for all $i \geq 1$.

Second step. Let us assume that $\lambda_{2N} = 0$. Let us prove that $\lambda_{iN} = 0$ for all $i \geq 2$, by induction on i. This is obvious if $i = 2$. Let us assume the result at rank $i - 1$, with $i \geq 3$, and let us prove it at rank i. We put $j = (i - 1)N$ and $k = N$. By the induction hypothesis, $\lambda_j = 0$. Then, by lemma 21:

$$(i - 2)N\lambda_N\lambda_i N = 0.$$

As $i \geq 3$ and $\lambda_N \neq 0$, $\lambda_{iN} = 0$. As a conclusion, if $\lambda_{2N} = 0$, putting $\mu = \lambda_N$, $\mathfrak{g} = \mathfrak{g}^{(2)}(N, \lambda, \mu)$.

Third step. We now assume that $\lambda_{2N} \neq 0$. We first prove that $\lambda_{iN} \neq 0$ for all $i \geq 1$. This is obvious if $i = 1, 2$. he result at rank $i - 1$, with $i \geq 3$, and let us prove it at rank i. We put $j = (i - 1)N$ and $k = N$. Then, by lemma 21:

$$(j\lambda_N - N\lambda_j)\lambda_{iN} = (i - 2)N\lambda_j\lambda_N.$$

By the induction hypothesis, $\lambda_j \neq 0$. Moreover, $i > 2$ and $\lambda_N \neq 0$, so $\lambda_{iN} \neq 0$.

For all $j \geq 1$, we put $\mu_j = \dfrac{\lambda_{kN}}{\lambda_N}$: this is a nonzero scalar, and $\mu_1 = 1$. Let us prove inductively that:

$$\mu_k = \frac{\mu_2}{(k-1) - (k-2)\mu_2}, \qquad \mu_2 \neq \frac{k-1}{k-2} \text{ if } k \neq 2.$$

If $k = 1$, $\mu_1 = 1 = \frac{\mu_2}{0-(-1)\mu_2}$, and $\mu_2 \neq 0$ as $\lambda_{2N} \neq 0$; if $k = 2$, $\mu_2 = \frac{\mu_2}{1-0\mu_2}$. Let us assume the result at rank $k - 1$, with $k \geq 3$. By lemma 21, with $j = (k - 1)N$ and $k = N$:

$$((k-1)N\lambda_N - \lambda_N\mu_{k-1})\lambda_N\mu_k = (k-2)N\mu_{k-1}\mu_1\lambda_N^2,$$
$$\mu_k(k - 1 - \mu_{k-1}) = (k-2)\mu_{k-1}.$$

As $\mu_{k-1} \neq 0$ and $k > 2$, $k-1-\mu_{k-1} \neq 0$. Moreover, by the induction hypothesis:

$$\begin{aligned} k-1-\mu_{k-1} &= k-1-\frac{\mu_2}{(k-2)-(k-3)\mu_2} \\ &= \frac{(k-1)(k-2)-((k-1)(k-3)+1)\mu_2}{(k-2)-(k-3)\mu_2} \\ &= (k-2)\frac{(k-1)-(k-2)\mu_2}{(k-2)-(k-3)\mu_2}. \end{aligned}$$

As this is nonzero, $\mu_2 \neq \frac{k-1}{k-2}$. We finally obtain:

$$\mu_k = (k-2)\mu_{k-1}\frac{1}{k-2}\frac{(k-2)-(k-3)\mu_2}{(k-1)-(k-2)\mu_2} = \frac{\mu_2}{(k-1)-(k-2)\mu_2}.$$

Finally, for all $k \geq 1$:

$$\lambda_{kN} = \frac{\lambda_N \mu_2}{(k-1)-(k-2)\mu_2} = \frac{\lambda_N \mu_2}{(1-\mu_2)k + 2\mu_2 - 1}.$$

Last step. If $\mu_2 = 1$, then for all $k \geq 1$, $\lambda_{kN} = \lambda_N$: this is $\mathfrak{g}^{(3)}(N, \lambda, \lambda_N)$. If $\mu_2 \neq 1$, we put $b = \frac{2\mu_2 - 1}{1-\mu_2}$.

- As $\mu_2 \neq 0$, $b \neq -1$;
- $b \neq -2$;
- for all $k \geq 3$, $\mu_2 \neq \frac{k-1}{k-2}$, so $b \neq -k$.

This gives that $b \notin \mathbb{Z}_-$. Moreover, for all $k \geq 1$:

$$\lambda_{kN} = \frac{\frac{\lambda_N \mu_2}{1-\mu_2}}{k+b}.$$

We take $a = \frac{\lambda_N \mu_2}{1-\mu_2}$, and we obtain $\mathfrak{g}^{(1)}(N, \lambda, a, b)$. □

Proposition 23 *Among the examples of theorem 18, the Com-PreLie bialgebras (or equivalently the Zinbiel-PreLie bialgebras) are* $\mathfrak{g}^{(1)}(1, a, 1)$ *for all* $a \neq 0$ *and* $g^{(4)}(0)$.

Proof. Note that $\mathfrak{g}^{(1)}(1,0,1) = \mathfrak{g}^{(4)}(0)$. Let us first prove that $\mathfrak{g}(1,a,1)$ is a Zinbiel-PreLie bialgebra for all $a \in \mathbb{K}$. Let us take V one-dimensional, generated by x, with $f = aId$. We work in $T(V,f)$. Let us prove that $x^k \bullet x^l = a\binom{k+l}{k-1}x^{k+l}$ by induction on k. It is obvious if $k = 0$, as $\binom{k+l}{-1} = 0$. Let us assume the result at rank $k-1$.

$$\begin{aligned} x^k \bullet x^l &= x(x^{k-1} \sqcup\!\sqcup x^l) + f(x)x^{k-1} \sqcup\!\sqcup x^l \\ &= a\left(\binom{k+l-1}{k-1} + \binom{k+l-1}{k-1}\right)x^{k+l} \\ &= a\binom{k+l}{k-1}x^{k+l}. \end{aligned}$$

The Zinbiel product of $T(V)$ is given by:

$$x^k \prec x^l = \binom{k+l-1}{k-1}x^{k+l},$$

for all $k, l \geq 1$. There is an isomorphism of Hopf algebras:

$$\Theta : \left\{ \begin{array}{rcl} \mathbb{K}[X] & \longrightarrow & T(V) \\ X & \longrightarrow & x. \end{array} \right.$$

For all $n \geq 0$, $\Theta(X^n) = x^{ш n} = n!x^n$. For all $k, l \geq 0$:

$$\Theta(X^k) \bullet \Theta(X^l) = a\binom{k+l}{k-1} k!l!x^{k+l} \qquad \Theta(X^k) \prec \Theta(X^l) = \binom{k+l-1}{k-1} k!l!x^{k+l}$$
$$= \frac{ak}{l+1}\Theta(X^{k+l}); \qquad = \frac{k}{k+l}\Theta(X^{k+l}).$$

Consequently, $\mathfrak{g}^{(1)}(1, a, 1)$ is isomorphic, as a Zinbiel-PreLie bialgebra to $T(V, f)$ (so is indeed a Zinbiel-PreLie bialgebra).

Let $\mathfrak{g}$ be one of the examples of theorem 18. First:

$$\begin{aligned}\Delta(X \bullet X) &= X \otimes 1 \bullet X + 1 \otimes X \bullet X \\ &\quad + X \bullet X \otimes 1 + X \bullet 1 \otimes X + 1 \bullet X \otimes X + 1 \bullet 1 \otimes X^2 \\ \lambda_1(1 \otimes X^2 + 2X \otimes X + X^2 \otimes 1) &= \lambda_1 1 \otimes X^2 + \lambda X \otimes X + \lambda_1 X^2 \otimes 1.\end{aligned}$$

This gives $\lambda = 2\lambda_1$. In particular, if $\mathfrak{g} = \mathfrak{g}^{(4)}(\lambda)$, then $\lambda = 2\lambda_1 = 0$: this is $\mathfrak{g}^{(4)}(0)$. In the other cases, N exists. By definition of N, $X \bullet X^k = 0$ if $1 \leq k \leq N-1$. We obtain:

$$\begin{aligned}\Delta(X \bullet X^N) &= 1 \otimes X \bullet X^N + X \otimes 1 \bullet X^N + \sum_{k=0}^{N} \binom{N}{k} (X \bullet X^k \otimes X^{N-k} + 1 \bullet X^k \otimes X^{n-k+1}) \\ \lambda_N \Delta(X^{N+1}) &= 1 \otimes X \bullet X^N + \lambda X \otimes X^N + 1 \otimes X \bullet X^N.\end{aligned}$$

If $\lambda = 0$, we obtain that X^{N+1} is primitive, so $N+1 = 1$: absurd, $N \geq 1$. So $\lambda \neq 0$. The cocommutativity of Δ implies that $N = 1$.

$$\begin{aligned}\Delta(X \bullet X^2) &= \lambda_2(X^3 \otimes 1 + 3X^2 \otimes X + 3X \otimes X^2 + 1 \otimes X^3) \\ &= 1 \otimes X \bullet X^2 + 2\lambda_1 X^2 \otimes X + \lambda_0 X \otimes X^2 + 1 \otimes X \bullet X^2\end{aligned}$$

Hence, $3\lambda_2 = 2\lambda_1$.

- If $\mathfrak{g} = \mathfrak{g}^{(3)}(1, \lambda, \mu)$, we obtain $3\mu = 2\mu$, so $\mu = 0$: this is a contradiction.
- If $\mathfrak{g} = \mathfrak{g}^{(2)}(1, \lambda, \mu)$, we obtain $0 = 2\mu$, so $\mu = 0$: this is a contradiction.

So $\mathfrak{g} = \mathfrak{g}^{(1)}(1, \lambda, a, b)$. We obtain:

$$3\frac{a}{2+b} = 2\frac{a}{1+b},$$

so $b = 1$. Then $\lambda_0 = 2\lambda_1 = \frac{2a}{2} = a = \frac{a}{b}$, so $\mathfrak{g} = \mathfrak{g}^{(1)}(1, a, 1)$. □

4 Cocommutative Com-PreLie bialgebras

We shall prove the following theorem:

Theorem 24 *Let A be a connected, cocommutative Com-PreLie bialgebra. Then one of the following assertions holds:*

1. *There exists a linear form $f : V \longrightarrow \mathbb{K}$ and $\lambda \in \mathbb{K}$, such that A is isomorphic to $S(V, f, \lambda)$.*
2. *There exists $\lambda \in \mathbb{K}$ such that A is isomorphic to $\mathfrak{g}^{(1)}(1, \lambda, 1)$.*

First, observe that if A is a cocommutative, commutative, connected Hopf algebra: by the Cartier-Quillen-Milnor-Moore theorem, it is isomorphic to the enveloping Hopf algebra of an abelian Lie algebra, so is isomorphic to $S(V)$ as a Hopf algebra, where $V = Prim(A)$. If $V = (0)$, the first point holds trivially.

4.1 First case

We assume in this paragraph that V is at least 2-dimensional.

Lemma 25 *Let A be a connected, cocommutative Com-PreLie algebra, such that the dimension of $Prim(A)$ is at least 2. Then $f_A = 0$, and there exists a map $F : A \otimes A \longrightarrow A$, such that:*

1. *For all $x, y \in A_+$, $x \bullet y = F(x \otimes y')y'' + F(x \otimes 1)y$.*
2. *For all $x_1, x_2 \in A$, $F(x_1x_2 \otimes y) = F(x_1 \otimes y)x_2 + x_1F(x_2 \otimes y)$.*
3. *$F(Prim(A) \otimes A) \subseteq \mathbb{K}$.*

Proof. We assume that $A = S(V)$ as a bialgebra, with its usual product and coproduct $\boldsymbol{\Delta}$, and that $dim(V) \geq 2$. Let $x, y \in V$. Then:

$$\boldsymbol{\Delta}(x \bullet y) = x \bullet y \otimes 1 + 1 \otimes x \bullet y + f_A(x) \otimes y.$$

By cocommutativity, for all $x, y \in V$, $f_A(x)$ and y are colinear. Let us choose y_1 and $y_2 \in V$, non colinear. Then $f_A(x)$ is colinear to y_1 and y_2, so belongs to $Vect(y_1) \cap Vect(y_2) = (0)$. Finally, $f_A = 0$.

We now construct linear maps $F_i : V \otimes S^i(V) \longrightarrow \mathbb{K}$, such that for all $k \geq 0$, putting:

$$F^{(k)} = \bigoplus_{i=0}^{k} F_i : \bigoplus_{i=0}^{k} V \otimes S^i(V) \longrightarrow \mathbb{K},$$

for all $x, y_1, \ldots, y_{k+1} \in V$:

$$x \bullet y_1 \ldots y_{k+1} = F^{(k)}(x \otimes (y_1 \ldots y_{k+1})') \otimes (y_1 \ldots y_{k+1})'' + F^{(k)}(x \otimes 1)y_1 \ldots y_{k+1}.$$

We proceed by induction on k. Let us first construct $F^{(0)}$. Let $x, y \in V$.

$$\boldsymbol{\Delta}(x \bullet y^2) = 1 \otimes x \bullet y^2 + x \bullet y^2 \otimes 1 + 2x \bullet y \otimes y.$$

By cocommutativity, $x \bullet y$ and y are colinear, so there exists a linear map $g : V \longrightarrow \mathbb{K}$ such that $x \bullet y = g(x)y$. We the take $F^{(0)}(x \otimes 1) = g(x)$. For all $x, y \in V$, $x \bullet y = F(x \otimes 1)y$, so the result holds for $k = 0$.

Let us assume that $F^{(0)}, \ldots, F^{(k-2)}$ are constructed for $k \geq 2$. Let $x, y_1, \ldots, y_k \in V$. For all $I \subseteq [k] = \{1, \ldots, k\}$, we put $y_I = \prod_{i \in I} y_i$. Then:

$$\tilde{\boldsymbol{\Delta}}(y_1 \ldots y_k) = \sum_{I \sqcup J = [k], I, J \neq 1} y_I \otimes y_J,$$

and:

$$\begin{aligned}\boldsymbol{\Delta}(x \bullet y_1 \ldots y_k) &= 1 \otimes x \bullet y_1 \ldots y_k + x \bullet y_1 \ldots y_k \otimes 1 + \sum_{[k] = I \sqcup J, J \neq 1} x \bullet y_I \otimes y_J\\ &= 1 \otimes x \bullet y_1 \ldots y_k + x \bullet y_1 \ldots y_k \otimes 1 + \sum_{I \sqcup J \sqcup K = [k], J, K \neq 1} F^{(k-2)}(x \otimes y_I) \otimes y_J \otimes y_K.\end{aligned}$$

We put:

$$P(x, y_1 \ldots y_k) = x \bullet y_1 \ldots y_k - \sum_{I \sqcup J = [k], |J| \geq 2} F^{(k-2)}(x \otimes y_I)y_J.$$

The preceding computation shows that $P(x, y_1 \ldots, y_k)$ is primitive, so belongs to V. Let $y_{k+1} \in V$.

$$\begin{aligned}\tilde{\mathbf{\Delta}}(x \bullet y_1 \ldots y_{k+1}) &= \sum_{I \sqcup J \sqcup K=[k+1], K \neq 1, |J| \geq 2} \underbrace{F^{(k-2)}(x \otimes y_I) y_J}_{\in S_{\geq 2}(V)} \otimes y_K \\ &+ P(x, y_1 \ldots y_k) \otimes y_{k+1} + \sum_{i=1}^{k} P(x, y_1 \ldots y_{i-1} y_{i+1} \ldots y_{k+1}) \otimes y_i.\end{aligned}$$

By cocommutativity, considering the projection on $V \otimes V$, we deduce that $P(x, y_1 \ldots y_k) \in Vect(y_1, \ldots, y_k, y_{k+1})$ for all nonzero $y_{k+1} \in V$. In particular, for $y_1 = y_{k+1}$, $P(x \otimes y_1 \ldots y_k) \in Vect(y_1, \ldots, y_k)$. By multilinearity, there exists $F'_1, \ldots, F'_k \in (V \otimes S_{k-1}(V))^*$, such that for all $x, y_1, \ldots, y_k \in V$:

$$P(x, y_1 \ldots y_k) = F'_1(x \otimes y_2 \ldots y_k) y_1 + \ldots + F'_k(x \otimes y_1 \ldots y_{k-1}) y_k.$$

By symmetry in $y_1, \ldots, y_k$, $F'_1 = \ldots = F'_k = F_{k-1}$. Then:

$$\begin{aligned}x \bullet y_1 \ldots y_k &= \sum_{I \sqcup J=[k], |J| \geq 2} F^{(k-2)}(x \otimes y_I) y_J + \sum_{I \sqcup J=[k], |J|=1} F_{k-1}(x \otimes y_I) y_J \\ &= \sum_{I \sqcup J=[k], |J| \geq 1} F^{(k-1)}(x \otimes y_I) y_J \\ &= F^{(k-1)}(x \otimes (y_1 \ldots y_k)')(y_1 \ldots y_k)'' + F(x \otimes 1) y_1 \ldots y_k.\end{aligned}$$

We defined a map $F : V \otimes S(V) \longrightarrow K$, such that for all $x \in V$, $b \in S_+(V)$,

$$x \bullet b = F(x \otimes b')b'' + F(x \otimes 1)b.$$

We extend F in a map from $S(V) \otimes S(V)$ to $S(V)$ by:

- $F(1 \otimes b) = 0$.
- For all $x_1, \ldots, x_k \in V$, $F(x_1 \ldots x_k \otimes b) = \sum_{i=1}^{k} x_1 \ldots x_{i-1} F(x_1 \otimes b) x_{i+1} \ldots x_k$.

This map F satisfies points 2 and 3. Let us consider:

$$B = \{a \in A \mid \forall b \in S_+(V), a \bullet b = F(a \otimes b')b'' + F(a \otimes 1)b\}.$$

As $1 \bullet b = 0$ for all $b \in S(V)$, $1 \in B$. By construction of F, $V \subseteq B$. Let $a_1, a_2 \in B$. For any $b \in S_+(V)$:

$$\begin{aligned}a_1 a_2 \bullet b &= (a_1 \bullet b) a_2 + a_1 (a_2 \bullet b) \\ &= F(a_1 \otimes b') a_2 b'' + a_1 F(a_2 \bullet b') b'' + F(a_1 \otimes 1) a_2 b + a_1 F(a_2 \otimes 1) b \\ &= F(a_1 a_2 \otimes b') b'' + F(a_1 a_2 \otimes 1) b.\end{aligned}$$

So $a_1 a_2 \in B$. Hence, B is a subalgebra of $S(V)$ containing V, so is equal to $S(V)$: F satisfies the first point. □

Remarks.

1. In this case, for all primitive element v, the 1-cocycle of the bialgebra A defined by $L(x) = a \bullet x$ is the coboundary associated to the linear form defined by $f(x) = -F(a \otimes x)$

2. In particular, the preLie product of two elements x, y of $Prim(A)$ si given by:

$$x \bullet y = F(x \otimes 1)y.$$

Lemma 26 *With the preceding hypothesis, let us assume that $F(x \otimes 1) = 0$ for all $x \in Prim(A)$. Then $\bullet = 0$.*

Proof. We assume that $A = S(V)$ as a bialgebra. By hypothesis, for all $a \in A$, $F(a \otimes 1) = 0$, so $a \bullet 1 = 0$. This implies that for all $a, b \in S_+(V)$:

$$\tilde{\mathbf{\Delta}}(a \bullet b) = a \bullet b' \otimes b'' + a' \bullet b' \otimes a''b'' + a' \bullet b \otimes a'' + a' \otimes a'' \bullet b.$$

Let us prove the following assertion by induction on N: for all $k < N$, for all $x, y_1, \ldots, y_k \in V$, $x \bullet y_1 \ldots y_k = 0$. By hypothesis, this is true for $N = 1$. Let us assume the result at a certain rank $N \geq 2$. Let us choose $x, y_1, \ldots, y_N \in V$. Then, by the induction hypothesis:

$$\tilde{\mathbf{\Delta}}(x \bullet y_1 \ldots y_N) = 0 + 0 + 0 + 0 = 0.$$

So $x \bullet y_1 \ldots y_N$ is primitive.

Up to a factorization, we can write any $x \bullet y_1 \ldots y_N$ as a linear span of terms of the form $z_1 \bullet z_1^{\beta_1} \ldots z_n^{\beta_n}$, with $z_1, \ldots, z_n$ linearly independent, $\beta_1, \ldots, \beta_n \in \mathbb{N}$, with $\beta_1 + \ldots + \beta_n = N$. If $n = 1$, as $dim(V) \geq 2$ we can choose any z_2 linearly independent with z_1 and take $\beta_2 = 0$. It is now enough to consider $z_1 \bullet z_1^{\beta_1} \ldots z_n^{\beta_n}$, with $n \geq 2$, $z_1, \ldots, z_n$ linearly independent, $\beta_1, \ldots, \beta_n \in \mathbb{N}$, $\beta_1 + \ldots + \beta_n = N$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$, such that $\alpha_1 + \ldots + \alpha_n = N + 1$.

$$\tilde{\mathbf{\Delta}}(z_1 \bullet z_1^{\alpha_1} \ldots z_n^{\alpha_n}) = \sum_{i=1}^{n} \alpha_i z_1 \bullet z_1^{\alpha_1} \ldots z_i^{\alpha_i - 1} \ldots z_n^{\alpha_n} \otimes z_i,$$

$$\begin{aligned}\tilde{\mathbf{\Delta}}\left(\frac{z_1^2}{2} \bullet z_1^{\alpha_1} \ldots z_n^{\alpha_n}\right) &= \sum_{i=1}^{n} \alpha_i (z_1 \bullet z_1^{\alpha_1} \ldots z_i^{\alpha_i - 1} \ldots z_n^{\alpha_n}) z_1 \otimes z_i \\ &+ \sum_{i=1}^{n} \alpha_i z_1 \bullet z_1^{\alpha_1} \ldots z_i^{\alpha_i - 1} \ldots z_n^{\alpha_n} \otimes z_1 z_i \\ &+ z_1 \bullet z_1^{\alpha_1} \ldots z_n^{\alpha_n} \otimes z_1 + z_1 \otimes z_1 \bullet z_1^{\alpha_1} \ldots z_n^{\alpha_n},\end{aligned}$$

$$\begin{aligned}(\tilde{\mathbf{\Delta}} \otimes Id) \circ \tilde{\mathbf{\Delta}}\left(\frac{z_1^2}{2} \bullet z_1^{\alpha_1} \ldots z_n^{\alpha_n}\right) &= \sum_{i=1}^{n} \alpha_i z_1 \bullet z_1^{\alpha_1} \ldots z_i^{\alpha_i - 1} \ldots z_n^{\alpha_n} \otimes z_1 \otimes z_i \\ &+ \sum_{i=1}^{n} \alpha_i z_1 \otimes z_1 \bullet z_1^{\alpha_1} \ldots z_i^{\alpha_i - 1} \ldots z_n^{\alpha_n} \otimes z_i \\ &+ \sum_{i} \alpha_i z_1 \bullet z_1^{\alpha_1} \ldots z_i^{\alpha_i - 1} \ldots z_n^{\alpha_n} \otimes z_i \otimes z_1.\end{aligned}$$

The cocommutativity implies that for all $1 \leq i \leq n$, $\alpha_i z_1 \bullet z_1^{\alpha_1} \ldots z_i^{\alpha_i - 1} \ldots z_n^{\alpha_n}$ and z_i are colinear. We first choose $\alpha_1 = \beta_1 + 1$, $\alpha_i = \beta_i$ for all $i \geq 2$, and we obtain for $i = 1$ that $z_1 \bullet z_1^{\beta_1} \ldots z_n^{\beta_n} \in Vect(z_1)$. We then choose $\alpha_n = \beta_n + 1$ and $\alpha_i = \beta_i$ for all $i \leq n - 1$, and we obtain for $i = n$ that $z_1 \bullet z_1^{\beta_1} \ldots z_n^{\beta_n} \in Vect(z_n)$. Finally, as $n \geq 2$, $z_1 \bullet z_1^{\beta_1} \ldots z_n^{\beta_n} \in Vect(z_1) \cap Vec(z_2) = (0)$; the hypothesis is true at trank N.

We proved that for all $x \in V$, for all $b \in S(V)$, $x \bullet b = 0$. By the derivation property of $\bullet$, as V generates $S(V)$, for all $a, b \in S(V)$, $a \bullet b = 0$. $\square$

Lemma 27 *Under the preceding hypothesis, Let us assume that* $F(Prim(A) \otimes \mathbb{K}) \neq (0)$. *Then* A *is isomorphic to a certain* $S(V, f, \lambda)$*, with* $V = Prim(A)$ *and* $f(x) = F(x \otimes 1)$ *for all* $x \in V$.

Proof. We assume that $A = S(V)$ as a bialgebra. Let $a, b, c \in S_+(V)$. Then:

$$\begin{aligned}\tilde{\mathbf{\Delta}}([a,b]) = \; & a' \otimes a'' \bullet b + a \bullet b' \otimes b'' + a' \bullet b \otimes a'' \\ & - b' \otimes b'' \bullet a - b \bullet a' \otimes a'' - b' \otimes a \otimes b'' + [a', b'] \otimes a''b'',\end{aligned}$$

where $[-,-]$ is the Lie bracket associated to $\bullet$. Hence:

$$\begin{aligned}(a \bullet b) \bullet c &= F(a \otimes 1) b \bullet c + F(a \otimes b') b'' \bullet c \\ &= F(a \otimes 1) F(b \otimes 1) c + F(a \otimes 1) F(b \otimes c') c'' \\ &+ F(a \otimes b') F(b'' \otimes 1) c + F(a \otimes b') F(b'' \otimes c') c'',\end{aligned}$$

$$\begin{aligned}(a \bullet c) \bullet b &= F(a \otimes 1) F(c \otimes 1) b + F(a \otimes 1) F(c \otimes b') b'' \\ &+ F(a \otimes c') F(c'' \otimes 1) b + F(a \otimes c') F(c'' \otimes b') b'',\end{aligned}$$

$$\begin{aligned}a \bullet [b, c] &= F(a \otimes 1) F(b \otimes 1) c + F(a \otimes 1) F(b \otimes c') c'' - F(a \otimes 1) F(c \otimes 1) b \\ &- F(a \otimes 1) F(c \otimes b') b'' + F(a \otimes b') F(b'' \otimes 1) c + F(a \otimes b') F(b'' \otimes c') c'' \\ &- F(a \otimes c') F(c'' \otimes 1) b - F(a \otimes c') F(c'' \otimes b') b'' + F(a \otimes F(b \otimes 1) c') c'' \\ &+ F(a \otimes F(b \otimes c') c'') c''' - F(a \otimes F(c \otimes 1) b') b'' - F(a \otimes F(c \otimes b') b'') b''' \\ &+ F(a \otimes F(b' \otimes 1) c) b'' + F(a \otimes F(b' \otimes c') c'') b'' - F(a \otimes F(c' \otimes 1) b) c'' \\ &+ F(a \otimes F(c' \otimes b') b'') c'' + F(a \otimes F(b' \otimes 1) c') b'' c'' + F(a \otimes F(b' \otimes c') c'') b'' c''' \\ &- F(a \otimes F(c' \otimes 1) b') b'' c''' - F(a \otimes F(c' \otimes b') b'') b''' c''.\end{aligned}$$

The preLie relation implies that:

$$\begin{aligned}0 = \; & F(a \otimes F(b \otimes 1) c') c'' + F(a \otimes F(b \otimes c') c'') c''' - F(a \otimes F(c \otimes 1) b') b'' \\ & - F(a \otimes F(c \otimes b') b'') b''' + F(a \otimes F(b' \otimes 1) c) b'' + F(a \otimes F(b' \otimes c') c'') b'' \\ & - F(a \otimes F(c' \otimes 1) b) c'' + F(a \otimes F(c' \otimes b') b'') c'' + F(a \otimes F(b' \otimes 1) c') b'' c'' \\ & + F(a \otimes F(b' \otimes c') c'') b'' c''' - F(a \otimes F(c' \otimes 1) b') b'' c''' - F(a \otimes F(c' \otimes b') b'') b''' c''.\end{aligned}$$

For $a = x \in V$, $b = y \in V$, as $F(V \otimes S(V)) \subset \mathbb{K}$, this simplifies to:

$$F(x \otimes c') F(y \otimes 1) c'' + F(y \otimes c') F(x \otimes c'') c''' = F(x \otimes F(c' \otimes 1) y) c''. \tag{1}$$

Let $x_1, \ldots, x_k \in V$, linearly independent, $\alpha_1, \ldots, \alpha_k \in \mathbb{N}$, with $\alpha_1 + \ldots + \alpha_N \geq 1$. We take $c = x_1^{\alpha_1+1} \ldots x_k^{\alpha_k}$ and $d = x_1^{\alpha_1} \ldots x_k^{\alpha_k}$. The coefficient of x_1 in (1), seen as an equality between two polynomials in $x_1, \ldots, x_k$, gives:

$$(\alpha_1 + 1)(F(x \otimes d) F(y \otimes 1) + F(y \otimes d') F(x \otimes d'')) = (\alpha_1 + 1) F(x \otimes F(d \otimes 1) y).$$

Hence, for all $x, y \in V$, for all $c \in S_+(V)$:

$$F(x \otimes c) F(y \otimes 1) + F(y \otimes c') F(x \otimes c'') = F(x \otimes F(c \otimes 1) y). \tag{2}$$

We put $f(x) = F(x \otimes 1)$ for all $x \in V$. If $f = 0$, by lemma 26, $\bullet = 0$, so A is isomorphic to $S(V, 0, 0)$. Let us assume that $f \neq 0$ and let us choose $y \in V$, such that $f(y) = 1$. If $z_1, \ldots, z_k \in Ker(f)$, then:

$$F(z_1 \ldots z_k \otimes 1) = \sum_{i=1}^{k} z_1 \ldots g(z_i) \ldots z_k = 0.$$

Consequenlty, if $c \in S_+(Ker(f)) \subseteq S_+(V)$, (2) gives:

$$F(x \otimes c') + F(y \otimes c')F(x \otimes c'') = 0.$$

An easy induction on the length of c proved that for all $c \in S_+(Ker(g))$, $F(x \otimes c) = 0$ for all $x \in V$. So there exists linear forms $g_k \in V^*$, such that for all $x, y_1, \ldots, y_k \in V$:

$$F(x \otimes y_1 \ldots y_k) = g_k(x) f(y_1) \ldots f(y_k).$$

In particular, $g_0 = f$. The preLie product is then given by:

$$x \bullet y_1 \ldots y_k = \sum_{i=1}^{k-1} g_i(x) \sum_{1 \leq j_1 < \ldots < j_i \leq k} y_1 \ldots f(y_{j_1}) \ldots f(y_{j_i}) \ldots y_k.$$

Let $x, y, z_1, \ldots, z_k \in V$.

$$\begin{aligned}
x \bullet (y \bullet z_1 \ldots z_k) &= x \bullet \sum_{i=0}^{k-1} g_i(y) \sum_{j_1, \ldots, j_i} z_i \ldots f(z_{j-1}) \ldots f(z_{j_i}) \ldots z_l \\
&= \sum_{i=0}^{k-1} g_{l-i-1}(x) g_i(x) \binom{l-1}{i} \sum_{j=1}^{k} f(z_1) \ldots f(z_{j-1}) z_j f(z_{j+1}) \ldots f(z_k) + S_{\geq 2}(V), \\
(x \bullet y) \bullet z_1 \ldots z_k &= f(x) y \bullet z_1 \ldots z_k \\
&= f(x) g_{k-1}(y) \sum_{j=1}^{k} f(z_1) \ldots f(z_{j-1}) z_j f(z_{j+1}) \ldots f(z_k) + S_{\geq 2}(V), \\
x \bullet (z_1 \ldots z_k \bullet y) &= \sum_{i=1}^{k} f(y_i) x \bullet z_1 \ldots z_{i-1} z_{i+1} \ldots z_k y \\
&= k g_{k-1}(x) f(z_1) \ldots f(z_k) y \\
&+ (k-1) f(x) g_{k-1}(y) \sum_{j=1}^{k} f(z_1) \ldots f(z_{j-1}) z_j f(z_{j+1}) \ldots f(z_k) + S_{\geq 2}(V), \\
(x \bullet z_1 \ldots z_k) \bullet y &= \sum_{i=0}^{k-1} g_i(x) \sum_{j_1, \ldots, j_i} z_1 \ldots f(z_{j_1}) \ldots f(z_{j_i}) \ldots z_k \bullet y \\
&= k f(x) g_{k-1}(y) \sum_{j=1}^{k} f(z_1) \ldots f(z_{j-1}) z_j f(z_{j+1}) \ldots f(z_k) + S_{\geq 2}(V).
\end{aligned}$$

Let us choose $z_1 = \ldots = z_k = z$, such that $f(z) = 1$. Then:

$$\sum_{j=1}^{k} f(z_1) \ldots f(z_{j-1}) z_j f(z_{j+1}) \ldots f(z_k) = kz \neq 0.$$

The preLie relation implies:

$$f(x) g_{k-1}(y) + (k-1) g_{k-1}(x) f(y) - \sum_{i=0}^{k-1} g_i(y) g_{k-i-1}(x) \binom{k-1}{i} = 0,$$

so, for all $l \geq 1$:

$$l g_l(x) f(y) = \sum_{i=1}^{l} g_i(y) g_{l_i}(x) \binom{l}{i}. \tag{3}$$

Let us choose x such that $f(x) = 1$. Let us consider $y \in Ker(f)$, and let us prove that $g_i(y) = 0$ for all $i \geq 0$. As $g_0 = f$, this is obvious for $i = 0$. Let us assume the result at all rank $< l$, with $l \geq 1$. Then (3) gives:

$$0 = \sum_{i=1}^{l-1} g_i(y) g_{l_i}(x) \binom{l}{i} + g_l(y) f(x) = g_l(y).$$

Consequently, for all $l \geq 1$, there exists a scalar λ_l such that $g_l = \lambda_l f$. If $f(x) = f(y) = 1$, equation (3) gives, for all $l \geq 1$:

$$l\lambda_l = \sum_{i=1}^{l} \lambda_i \lambda_{l-i} \binom{l}{i} = \sum_{i=1}^{l-1} \lambda_i \lambda_{l-i} \binom{l}{i} + \lambda_l,$$

so, for all $l \geq 2$:

$$\lambda_l = \frac{1}{l-1} \sum_{i=1}^{l-1} \lambda_i \lambda_{l-i} \binom{l}{i}.$$

An induction proves that $\lambda_l = l! \lambda_1^l$ for all $l \geq 1$. Putting $\lambda_1 = \lambda$, for all $x, x_1, \ldots, x_n \in V$:

$$x \bullet x_1 \ldots x_k = \sum_{I \subsetneq \{1,\ldots,k\}} |I|! \lambda^{|I|} f(x) \prod_{i \in I} f(x_i) \prod_{i \notin I} x_i.$$

This is the preLie product of $S(V, f, \lambda)$. □

4.2 Second case

We now assume that V is one-dimensional. So $S(V)$ and $\mathbb{K}[X]$ are isomorphic as bialgebras. Let us describe all the preLie products on $\mathbb{K}[X]$ making it a Com-PreLie bialgebra.

Proposition 28 *Let $\lambda, \mu \in \mathbb{K}$. We define:*

$$X^k \bullet X^l = \lambda k l! \sum_{i=k}^{k+l-1} \frac{\mu^{k+l-i-1}}{(i-k+1)!} X^i.$$

Then $(\mathbb{K}[X], m, \prec, \Delta)$ is a Zinbiel-PreLie algebra denoted by $\mathfrak{g}'(\lambda, \mu)$.

Proof. If $\lambda = 0$, $\bullet = 0$ and the result is obvious. Let us assume that $\lambda \neq 0$. Let V one-dimensional, $x \in V$, nonzero, and let $f \in V^*$ defined by $f(x) = \frac{\mu}{\lambda}$. In $T(V, f, \lambda)$, by lemma 8, for all $k, l \geq 0$:

$$x^k \bullet x^l = \lambda \sum_{i=k}^{k+l-1} \mu^{k+l-i-1} \binom{i}{k-1} x^j.$$

Let us consider the Hopf algebra isomorphism:

$$\Theta : \begin{cases} \mathbb{K}[X] & \longrightarrow & T(V) \\ X & \longrightarrow & x. \end{cases}$$

For all $k, l \geq 0$:

$$\begin{aligned} \Theta(X^k) \bullet \Theta(X^l) &= \lambda \sum_{i=k}^{k+l-1} \mu^{k+l-i-1} \frac{i!k!l!}{(k-1)!(i-k+1)!} x^i \\ &= \lambda k l! \sum_{i=k}^{k+l-1} \frac{\mu^{k+l-1-i}}{(i-k+1)!} \Theta(X^i). \end{aligned}$$

By proposition 13, $T(V, f, \lambda)$ is a Zinbiel-PreLie bialgebra, so is $\mathfrak{g}'(\lambda, \mu)$. □

Proposition 29 *Let $\bullet$ a preLie product on $\mathbb{K}[X]$ such that $(\mathbb{K}[X],m,\bullet,\Delta)$ is a Com-PreLie bialgebra. Then $(\mathbb{K}[X],m,\bullet,\Delta)=\mathfrak{g}^{(1)}(1,\lambda,1)$ for a certain $\lambda\in\mathbb{K}$, or $\mathfrak{g}'(\lambda,\mu)$ for a certain $(\lambda,\mu)\in\mathbb{K}^2$.*

Proof. Let $\pi:\mathbb{K}[X]\longrightarrow\mathbb{K}[X]$ be the canonical projection on $Vect(X)$:

$$\pi:\left\{\begin{array}{rl}\mathbb{K}[X] & \longrightarrow \mathbb{K}[X]\\ X^k & \longrightarrow \delta_{k,1}X.\end{array}\right.$$

For all $k\geq 0$, we put $\pi(X\bullet X^k)=\lambda_k X$.

We shall use the map $\varpi=m\circ(\pi\otimes Id)\circ\Delta$. For all $k\geq 0$:

$$\varpi(X^k)=m\circ(\pi\otimes Id)\left(\sum_{i=0}^{k}\binom{k}{i}X^i\otimes X^{k-i}\right)=m(kX\otimes X^{k-1})=kX^k.$$

First step. We fix $l\geq 0$. For all $P,Q\in\mathbb{K}[X]$, $\varepsilon(P\bullet Q)=0$; hence, we can write:

$$X\bullet X^l=\sum_{i=1}^{\infty}a_iX^i.$$

Then:

$$\begin{aligned}\varpi(X\bullet X^l)&=\sum_{i=1}^{\infty}ia_iX^i\\ &=m\circ(\pi\otimes Id)\circ\Delta(X\bullet X^l)\\ &=m\circ(\pi\otimes Id)\left(1\otimes X\bullet X^l+\sum_{i=0}^{l}\binom{l}{i}X\bullet X^i\otimes X^{l-i}\right)\\ &=m\left(\sum_{i=0}^{l}\binom{l}{i}\lambda_iX\otimes X^{l-i}\right)\\ &=\sum_{i=0}^{l}\binom{l}{i}\lambda_iX^{l-i+1}\\ &=\sum_{j=1}^{l+1}\binom{l}{l-j+1}\lambda_{l-j+1}X^j.\end{aligned}$$

Hence:

$$X\bullet X^l=\sum_{j=1}^{l+1}\binom{l}{l-j+1}\frac{\lambda_{l-j+1}}{j}X^j.$$

By derivation, for all $k\geq 0$, $X^k\bullet X^l=kX^{k-1}(X\bullet X^l)$, so for all $k,l\geq 0$:

$$X^k\bullet X^l=\sum_{j=1}^{l+1}k\binom{l}{l-j+1}\frac{\lambda_{l-j+1}}{j}X^{j+k-1}.$$

Second step. In particular, for all $k \geq 0$, $X^k \bullet 1 = k\lambda_0 X^k$, and $X \bullet X = \frac{\lambda_0}{2}X^2 + \lambda_1 X$. Hence:

$$\begin{aligned} X \bullet (X \bullet 1) - (X \bullet X) \bullet 1 &= \frac{\lambda_0^2}{2}X^2 + \lambda_0\lambda_1 X - \frac{\lambda_0}{2}X^2 \bullet 1 - \lambda_1 X \bullet 1 \\ &= \frac{\lambda_0^2}{2}X^2 + \lambda_0\lambda_1 X - \lambda_0^2 X^2 - \lambda_0\lambda_1 X \\ &= -\frac{\lambda_0^2}{2}X^2; \end{aligned}$$

$$\begin{aligned} X \bullet (1 \bullet X) - (X \bullet 1) \bullet X &= 0 - \lambda_0 X \bullet X \\ &= -\frac{\lambda_0^2}{2}X^2 - \lambda_0\lambda_1 X. \end{aligned}$$

By the preLie relation, $\lambda_0\lambda_1 = 0$. We shall now study three cases:

$$\textbf{1.}\ \begin{cases} \lambda_0 & \neq 0, \\ \lambda_1 & = 0; \end{cases} \qquad \textbf{2.}\ \begin{cases} \lambda_0 & = 0, \\ \lambda_1 & = 0; \end{cases} \qquad \textbf{3.}\ \begin{cases} \lambda_0 & = 0, \\ \lambda_1 & \neq 0; \end{cases}$$

Third step. First case: $\lambda_0 \neq 0$, $\lambda_1 = 0$. Let us prove that $\lambda_k = 0$ for all $k \geq 1$ by induction on k. It is obvious if $k = 1$. Let us assume that $\lambda_1 = \ldots = \lambda_{k-1} = 0$. Then $X \bullet X^k = \frac{\lambda_0}{k+1}X^{k+1} + \lambda_k X$, and:

$$\begin{aligned} X \bullet (X^k \bullet 1) - (X \bullet X^k) \bullet 1 &= k\lambda_0 \left(\frac{\lambda_0}{k+1}X^{k+1} + \lambda_k X\right) - \left(\frac{\lambda_0}{k+1}X^{k+1} + \lambda_k X\right) \bullet 1 \\ &= \frac{k}{k+1}\lambda_0^2 X^{k+1} + \lambda_0\lambda_k X - \lambda_0^2 X^{k+1} - \lambda_0\lambda_k X \\ &= \frac{-1}{k+1}\lambda_0^2 X^{k+1}; \end{aligned}$$

$$\begin{aligned} X \bullet (1 \bullet X^k) - (X \bullet 1) \bullet X^k &= 0 - \lambda_0 \left(\frac{\lambda_0}{k+1}X^{k+1} + \lambda_k X\right) \\ &= \frac{-1}{k+1}\lambda_0^2 X^{k+1} - \lambda_0\lambda_k X. \end{aligned}$$

By the preLie relation, $\lambda_0\lambda_k = 0$. As $\lambda_0 \neq 0$, $\lambda_k = 0$.

Finally, $X^k \bullet X^l = \lambda_0 \frac{k}{l+1} X^{k+l}$ for all $k, l \geq 0$: this is the preLie product of $\mathfrak{g}^{(1)}(1, \lambda_0, 1)$.

Fourth step. Second case: $\lambda_0 = \lambda_1 = 0$. Let us prove that $\lambda_k = 0$ for all $k \geq 0$. It is obvious if $k = 0, 1$. Let us assume that $\lambda_0 = \ldots = \lambda_{k-1} = 0$, with $k \geq 2$. Then $X^i \bullet X^j = 0$ for all $j < k$, $i \geq 0$. Hence:

$$X \bullet (X^{k+1} \bullet X^{k-1}) = (X \bullet X^{k+1}) \bullet X^{k-1} = (X \bullet X^{k-1}) \bullet X^{k+1} = 0.$$

By the preLie relation, $X \bullet (X^{k-1} \bullet X^{k+1}) = 0$. Moreover:

$$\begin{aligned}
X \bullet (X^{k-1} \bullet X^{k+1}) &= X \bullet \left(\sum_{j=1}^{k-2} \binom{k+1}{k+2-j} (k-1) \frac{\lambda_{k+2-j}}{j} X^{k+2-j} \right) \\
&= X \bullet \left((k-1)\lambda_{k+1} X^{k-1} + (k+1)(k-1) \frac{\lambda_k}{2} X^k \right) \\
&= 0 + \frac{(k-1)(k+1)}{2} \lambda_k X \bullet X^k \\
&= \frac{(k-1)(k+1)}{2} \lambda_k \left(\sum_{j=1}^{k-1} \binom{k}{k+1-j} \frac{\lambda_{k+1-j}}{j} X^j \right) \\
&= \frac{(k-1)(k+1)}{2} \lambda_k^2 X + 0.
\end{aligned}$$

Hence, $\lambda_k = 0$.

We finally obtain by the first step $X^k \bullet X^l = 0$ for all $k, l \geq 0$: this is the trivial preLie product of $\mathfrak{g}^{(4)}(0)$.

Fifth step. Last case: $\lambda_0 = 0$, $\lambda_1 \neq 0$. Let us prove that $\lambda_k = \dfrac{k!}{2^{k-1}} \dfrac{\lambda_2^{k-1}}{\lambda_1^{k-2}}$ for all $k \geq 1$. It is obvious if $k = 1$ or $k = 2$. Let us assume the result at all rank $< k$, with $k \geq 2$.

$$\begin{aligned}
\pi((X \bullet X) \bullet X^k) &= \pi(\lambda_1 X \bullet X^k) \\
&= \lambda_1 \lambda_k X;
\end{aligned}$$

$$\begin{aligned}
\pi(X \bullet (X \bullet X^k)) &= \pi \left(\sum_{j=1}^{k} \binom{k}{k+1-j} \frac{\lambda_{k+1-j}}{j} X \bullet X^j \right) \\
&= \sum_{j=1}^{k} \binom{k}{k+1-j} \frac{\lambda_{k+1-j}\lambda_j}{j} X \\
&= \left(\lambda_k \lambda_1 + \sum_{j=2}^{k-1} \frac{1}{j} \binom{k}{k+1-j} \frac{(k+1-j)!j!}{2^{k-j+j-1}} \frac{\lambda_2^{k-j+j-1}}{\lambda_1^{k-j-1+j-2}} + \frac{k}{k} \lambda_1 \lambda_k \right) X \\
&= \left(2\lambda_1 \lambda_k + \sum_{j=2}^{k-1} \frac{k!}{2^{k-1}} \frac{\lambda_2^{k-1}}{\lambda_1^{k-3}} \right) X \\
&= \left(2\lambda_1 \lambda_k + (k-2) \frac{k!}{2^{k-1}} \frac{\lambda_2^{k-1}}{\lambda_1^{k-3}} \right) X;
\end{aligned}$$

$$\begin{aligned}
\pi((X \bullet X^k) \bullet X) &= \sum_{j=1}^{k} \binom{k}{k+1-j} \frac{\lambda_{k+1-j}}{j} \pi(X^j \bullet X) \\
&= \sum_{j=1}^{k} \binom{k}{k+1-j} \frac{\lambda_{k+1-j}}{j} \pi(j\lambda_1 X^j) \\
&= \lambda_1 \lambda_k X + 0;
\end{aligned}$$

$$\begin{aligned}
\pi(X \bullet (X^k \bullet X)) &= k\lambda_1 \pi(X \bullet X^k) \\
&= k\lambda_1 \lambda_k X.
\end{aligned}$$

By the preLie relation:

$$\lambda_1\lambda_k - 2\lambda_1\lambda_k - (k-2)\frac{k!}{2^{k-1}}\frac{\lambda_2^{k-1}}{\lambda_1^{k-3}} = \lambda_1\lambda_k - k\lambda_1\lambda_k,$$

which gives, as $\lambda_1 \neq 0$ and $k \geq 3$, $\lambda_k = \frac{k!}{2^{k-1}}\frac{\lambda_2^{k-1}}{\lambda_1^{k-2}}$. Finally, the first step gives, for all $k, l \geq 0$, with $\lambda = \lambda_1$ and $\mu = \frac{\lambda_2}{2\lambda_1}$:

$$\begin{aligned}
X^k \bullet X^l &= \sum_{j=1}^{k+1} k\binom{l}{l+1-j}\frac{\lambda_{l+1-j}}{j}X^{j+k-1}\\
&= \sum_{j=1}^{k} k\frac{l!(l+1-j)!}{(l+1-j)!(j-1)!j2^{l-j}}\frac{\lambda_2^{l-j}}{\lambda_1^{l-1-j}}X^{j+k-1}\\
&= \lambda k l! \sum_{j=1}^{k}\frac{\mu^{l-j}}{j!}X^{j+k-1}\\
&= \lambda k l! \sum_{i=k}^{k+l-1}\frac{\mu^{k+l-i-1}}{(i-k+1)!}X^i.
\end{aligned}$$

This is the preLie product of $\mathfrak{g}'(\lambda, \mu)$. □

As $\mathfrak{g}'(\lambda, \mu)$ is a special case of $S(V, f, \lambda)$, this ends the proof of theorem 24.

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