Cocommutative Com-PreLie bialgebras

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Abstract

A Com-PreLie bialgebra is a commutative bialgebra with an extra preLie product satisfying some compatibilities with the product and coproduct. We here give a classification of connected, cocommutative Com-PreLie bialgebras over a field of characteristic zero: we obtain a main family of symmetric algebras on a space V of any dimension, and another family available only if V is one-dimensional.

We also explore the case of Com-PreLie bialgebras over a group algebra and over a tensor product of a group algebra and of a symmetric algebra.

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Contents

Introduction

Com-PreLie bialgebras, introduced in [1, 2], are commutative bialgebras with an extra preLie product, compatible with the product and coproduct: see Definition 1 below. They appeared in Control Theory: the Lie algebra of the group of Fliess operators [4] naturally owns a Com-PreLie bialgebra structure, and its underlying bialgebra is a shuffle Hopf algebra. Free (non unitary) Com-PreLie bialgebras were also described, in terms of partionned rooted trees.

We here give examples of cocommutative Com-PreLie bialgebras, and in particular, we classify all connected cocommutative (as a coalgebra) Com-PreLie bialgebras. We first introduce in Theorem 2 a family $S(V, f, \lambda)$ of cocommutative and connected Com-PreLie bialgebras, where V is a vector space, f a linear form on V and λ a scalar; these objects are classified up to isomorphism in Proposition 4. As a bialgebra, $S(V, f, \lambda)$ is the usual symmetric algebra on V and, for any $x, x_1, \ldots, x_k \in V$:

$$x \bullet x_1 \ldots x_k = \sum_{I \subsetneq [k]} |I|! \lambda^{|I|} f(x) \prod_{i \in I} f(x_i) \prod_{i \notin I} x_i.$$

Secondly, we give in Theorem 5 all homogeneous prelie products on the polynomial algebra $\mathbb{K}[X]$, making it a Com-PreLie algebra: we obtain four families. Among them, only a few satisfies the compatibility with the coproduct: we only obtain a one-parameter family $\mathfrak{g}^{(1)}(1, a, 1)$, where a is a scalar, see Proposition 8. For any $k, l \in \mathbb{N}$, in $\mathfrak{g}^{(1)}(1, a, 1)$:

$$X^k \bullet X^l = \frac{k}{l+1} X^{k+l}.$$

The underlying Lie algebras of these preLie algebras are described in Proposition 9 as semi-direct products of abelian or Faà di Bruno Lie algebras. We prove in Theorem 10 that these examples cover all the connected cocommutative cases. Namely, if A is a cocommutative Com-PreLie bialgebra, connected as a coalgebra, then it is isomorphic to $S(V, f, \lambda)$ or to $\mathfrak{g}^{(1)}(1, a, 1)$ (we should precise here that we work on a field of characteristic zero).

We then turn to the non connected case and start with preLie products on group algebras. We prove that if G is an abelian group, then any preLie product $\bullet$ on $\mathbb{K}G$ making it a Com-PreLie bialgebra is given, for any $g, h \in G$, by:

$$g \bullet h = \lambda(g, h)(g - gh),$$

where $(\lambda(g, h))_{g,h \in G}$ is a family of scalars satisfying certain conditions exposed in Theorem 19. These conditions imply that if G is a finite group, then $\bullet = 0$. If $G = \mathbb{Z}$, we prove in Theorem 21 that there exist two families of preLie products on the Laurent polynomial algebra $\mathbb{K}[X, X^{-1}]$ making it a Com-PreLie bialgebra.

We end by several results on the Hopf algebra $\mathbb{K}|G] \otimes S(V)$, where G is an abelian group and V a vector space. In particular, we give in Theorem 26 all possible preLie products making it a Com-PreLie bialgebra, with the extra conditions that $S(V)$ is a non trivial PreLie subalgebra, isomorphic to $S(V, f, \lambda)$.

This text is organized in six sections. The first one gives reminders and definitions on Com-PreLie bialgebras and Zinbiel-PreLie bialgebras. The second one is devoted to the existence of Com-PreLie bialgebras $S(V, f, \lambda)$, and the third one to the classification of homogeneous preLie products on $\mathbb{K}[X]$. The theorem of classification of connected cocommutative Com-PreLie bialgebras is proved in the fourth section. The study of preLie products on a group algebra is done in the fifth section and the last one deals with the general case $\mathbb{K}[G] \otimes S(V)$.

Notations 1.   1. We denote by $\mathbb{K}$ a commutative field of characteristic zero. All the objects (vector spaces, algebras, coalgebras, prelie algebras...) in this text will be taken over $\mathbb{K}$.

2. For all $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, \ldots, n\}$.

3. Let V be a vector space. We denote by $S(V)$ the symmetric algebra of V. It is a Hopf algebra, with the coproduct defined by:

$$\forall v \in V, \qquad\qquad \Delta(v) = v \otimes 1 + 1 \otimes v.$$

1 Com-PreLie and Zinbiel-PreLie algebras

Notations 2.

Definition 1. *1. A Com-PreLie algebra [6] is a family $A = (A, \cdot, \bullet)$, where A is a vector space and $\cdot$, $\bullet$ are bilinear products on A, such that:*

$$\forall a, b \in A, \qquad\qquad a \cdot b = b \cdot a,$$
$$\forall a, b, c \in A, \qquad\qquad (a \cdot b) \cdot c = a \cdot (b \cdot c),$$
$$\forall a, b, c \in A, \quad (a \bullet b) \bullet c - a \bullet (b \bullet c) = (a \bullet c) \bullet b - a \bullet (c \bullet b) \quad \textit{(preLie identity)},$$
$$\forall a, b, c \in A, \qquad\qquad (a \cdot b) \bullet c = (a \bullet c) \cdot b + a \cdot (b \bullet c) \quad \textit{(Leibniz identity)}.$$

In particular, $(A, \cdot)$ is an associative, commutative algebra and $(A, \bullet)$ is a right preLie algebra. We shall say that a Com-Prelie algebra is unitary if the associative algebra $(A, \cdot)$ has a unit, which will be denoted by 1.

2. A Com-PreLie bialgebra is a family $(A, \cdot, \bullet, \Delta)$, such that:

(a) $(A, \cdot, \bullet)$ is a unitary Com-PreLie algebra.

(b) $(A, \cdot, \Delta)$ is a bialgebra.

(c) For all $a, b \in A$:

$$\Delta(a \bullet b) = a^{(1)} \otimes a^{(2)} \bullet b + a^{(1)} \bullet b^{(1)} \otimes a^{(2)} \cdot b^{(2)},$$

with Sweedler's notation $\Delta(x) = x^{(1)} \otimes x^{(2)}$.

3. A Zinbiel-PreLie algebra is a family $A = (A, \prec, \bullet)$, where A is a vector space and $\prec$, $\bullet$ are bilinear products on A, such that:

$$\forall a, b, c \in A, \qquad\qquad (a \prec b) \prec c = a \prec (b \prec c + c \prec b) \quad \textit{(Zinbiel identity)},$$
$$\forall a, b, c \in A, \quad (a \bullet b) \bullet c - a \bullet (b \bullet c) = (a \bullet c) \bullet b - a \bullet (c \bullet b) \quad \textit{(preLie identity)},$$
$$\forall a, b, c \in A, \qquad\qquad (a \prec b) \bullet c = (a \bullet c) \prec b + a \cdot (b \prec c) \quad \textit{(Leibniz identity)}.$$

In particular, $(A, \prec)$ is a Zinbiel algebra (or half-shuffle algebra) [3, 5, 7]. The product $\cdot$ defined on A by $a \cdot b = a \prec b + b \prec a$ is associative and commutative, and $(A, \cdot, \prec)$ is a Com-PreLie algebra.

4. A Zinbiel-PreLie bialgebra is a family $(A, \cdot, \prec, \bullet, \Delta)$ such that:

(a) $(A, \cdot, \bullet, \Delta)$ is a Com-PreLie bialgebra. We denote by A_+ the augmentation ideal of A, and by $\tilde{\Delta}$ the coassociative coproduct defined by:

$$\tilde{\Delta} : \begin{cases} A_+ & \longrightarrow & A_+ \otimes A_+ \\ a & \longrightarrow & \Delta(a) - a \otimes 1 - 1 \otimes a. \end{cases}$$

(b) $(A_+, \prec, \bullet)$ is a Zinbiel-PreLie algebra, and the restriction of $\cdot$ on A_+ is the commutative product induced by $\prec$: for all $x, y \in A_+$, $x \prec y + y \prec x = x \cdot y$.

(c) For all $a, b \in A_+$, with Sweedler's notation $\tilde{\Delta}(x) = x' \otimes x''$, for all $a, b \in A_+$:

$$\tilde{\Delta}(a \prec b) = a' \prec b' \otimes a'' \cdot b'' + a' \prec b \otimes a'' + a' \otimes a'' \cdot b + a \prec b' \otimes b'' + a \otimes b.$$

Remark 1.   1. If $(A, \cdot, \bullet, \Delta)$ is a Com-PreLie bialgebra, then for any $\lambda \in \mathbb{K}$, $(A, \cdot, \lambda\bullet, \Delta)$ also is.

2. If $(A, \prec, \bullet, \Delta)$ is Zinbiel-PreLie bialgebra, denoting $\cdot$ the product induced by $\prec$, $(A, \cdot, \bullet, \Delta)$ is a Com-PreLie bialgebra.

3. If A is a Zinbiel-PreLie bialgebra, we extend $\prec$ to $A_+ \otimes A + A \otimes A_+$ by $a \prec 1 = a$ and $1 \prec a = 0$ for all $a \in A_+$. Note that $1 \prec 1$ is not defined.

4. If $(A, \cdot, \bullet)$ is a unitary Com-PreLie algebra, for any $x \in A$:

$$1 \bullet x = (1 \cdot 1) \bullet x = (1 \bullet x) \cdot 1 + 1 \cdot (1 \bullet x) = 2(1 \bullet x).$$

Hence, for any $x \in A$, $1 \bullet x = 0$.

5. If $(A, \cdot, \bullet, \Delta)$ is a Com-PreLie bialgebra, we denote by $Prim(A)$ the subspace of primitive elements of A. For any $x \in Prim(A)$:

$$\Delta(x \bullet 1) = x \otimes 1 \bullet 1 + 1 \otimes x \bullet 1 + 1 \otimes x \bullet 1 + x \otimes 1 \bullet 1$$
$$= 1 \otimes x \bullet 1 + 1 \otimes x \bullet 1.$$

So $x \bullet 1 \in Prim(A)$. We shall consider the map:

$$f_A : \begin{cases} Prim(A) & \longrightarrow & Prim(A) \\ x & \longrightarrow & x \bullet 1. \end{cases}$$

2   Examples on symmetric algebras

Our goal in this section is to prove the following theorem:

Theorem 2. *Let V be a vector space, $f \in V^*$, $\lambda \in \mathbb{K}$. We give $S(V)$ the product $\bullet$ defined by:*

$$\forall x \in S(V), \qquad\qquad\qquad 1 \bullet x = 0,$$
$$\forall x, x_1, \ldots, x_k \in V, \qquad x \bullet x_1 \ldots x_k = \sum_{I \subsetneq [k]} |I|! \lambda^{|I|} f(x) \prod_{i \in I} f(x_i) \prod_{i \notin I} x_i,$$
$$\forall x_1, \ldots, x_k \in V, \forall x \in S(V), \qquad x_1 \ldots x_k \bullet x = \sum_{i=1}^{k} x_1 \ldots (x_i \bullet x) \ldots x_k.$$

Then $(S(V), m, \bullet, \Delta)$ is a Com-PreLie bialgebra, denoted by $S(V, f, \lambda)$.

2.1   Two operators

We shall consider the two following operators:

$$\partial : \begin{cases} S(V) & \longrightarrow & S(V) \\ x_1 \ldots x_k & \longrightarrow & \sum_{i=1}^{k} x_1 \ldots x_{i-1} f(x_i) x_{i+1} \ldots x_k, \end{cases}$$

$$\phi : \begin{cases} S(V) & \longrightarrow & S(V) \\ x_1 \ldots x_k & \longrightarrow & \sum_{I \subsetneq [k]} \lambda^{|I|} |I|! \prod_{i \in I} f(x_i) \prod_{i \notin I} x_i, \end{cases}$$

where $x_1, \ldots, x_k$ are elements of V.

Lemma 3. *1. For any $u, v, w \in S(V)$:*

$$\partial(uv) = \partial(u)v + u\partial(v),$$
$$\partial \circ \phi(v)\phi(w) - \phi(\partial(v)\phi(w)) = \partial \circ \phi(w)\phi(v) - \phi(\partial(w)\phi(v)).$$

2. For any $u \in S(V)$:

$$\Delta \circ \partial(u) = (\partial \otimes Id) \circ \Delta(u) = (Id \otimes \partial) \circ \Delta(u),$$
$$\Delta \circ \phi(u) = (\phi \otimes Id) \circ \Delta(u) + 1 \otimes \phi(u).$$

Proof. 1. The fact that δ is a derivation is immediate. Let us prove the second assertion. We consider $v = x_1 \ldots x_k$ and $w = y_1 \ldots y_l$, with $x_1, \ldots, x_k$ and $y_1, \ldots, y_l \in V$. Then:

$$\partial \circ \phi(v)\phi(w) = \sum_{\substack{I \subsetneq [k],\, i_0 \notin I, \\ J \subsetneq [l]}} \lambda^{|I|+|J|}|I|!|J|! \prod_{i \in I \sqcup \{i_0\}} f(x_i) \prod_{j \in J} f(y_j) \prod_{i \notin I \sqcup \{i_0\}} x_i \prod_{j \notin J} y_j$$

$$= \sum_{\substack{\emptyset \subsetneq I \subseteq [k], \\ J \subsetneq [l]}} \lambda^{|I|+|J|-1}|I|!|J|! \prod_{i \in I} f(x_i) \prod_{j \in J} f(y_j) \prod_{i \notin I} x_i \prod_{j \notin J} y_j$$

$$= \underbrace{\sum_{\substack{\emptyset \subsetneq I \subsetneq [k], \\ \emptyset \subsetneq J \subsetneq [l]}} \lambda^{|I|+|J|-1}|I|!|J|! \prod_{i \in I} f(x_i) \prod_{j \in J} f(y_j) \prod_{i \notin I} x_i \prod_{j \notin J} y_j}_{=\varphi_1(v,w)}$$

$$+ \sum_{\emptyset \subsetneq I \subsetneq [k]} \lambda^{|I|-1}|I|! \prod_{i \in I} f(x_i) \prod_{i \notin I} x_i \prod_{j \in [l]} y_j$$

$$+ \sum_{\emptyset \subsetneq J \subsetneq [l]} \lambda^{k+|J|-1}k!|J|! \prod_{i \in [k]} f(x_i) \prod_{j \in J} f(y_j) \prod_{j \notin J} y_j$$

$$+ \lambda^{k-1}k! \prod_{i \in [k]} f(x_i) \prod_{j \in [l]} y_j.$$

Observe that $\varphi_1(v, w) = \varphi_1(w, v)$.

$$\phi(\partial(v)\phi(w))$$
$$= \sum_{\substack{i_0 \in [k],\, I \subseteq [k] \setminus \{i_0\} \\ J' \subseteq [l],\, J'' \subsetneq [l] \setminus J, \\ I \neq [k] \setminus \{i_0\} \text{ or } J'' \neq [l] \setminus J}} \lambda^{|I|+|J'|+|J''|}|J'|!(|I|+|J''|)! \prod_{i \in I \sqcup \{i_0\}} f(x_i) \prod_{j \in J' \sqcup J''} f(y_j) \prod_{i \notin I \sqcup \{i_0\}} x_i \prod_{j \notin J' \sqcup J''} y_j$$

$$= \sum_{\substack{\emptyset \subsetneq I \subseteq [k], \\ J \subseteq [l]}} \lambda^{|I|+|J|-1}|I| \left(\sum_{\substack{J = J' \sqcup J'', \\ J' \neq [l]}} |J'|!(|J''|+|I|-1)! \right) \prod_{i \in I} f(x_i) \prod_{j \in J} f(y_j) \prod_{i \notin I} x_i \prod_{j \notin J} y_j.$$

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For any $I \subseteq [k]$, $J \subseteq [l]$:

$$|I|\left(\sum_{J=J' \sqcup J''} |J'|!(|J''| + |I| - 1)!\right) = |I| \sum_{k=0}^{|J|} \binom{|J|}{k} k!(|I| + |J| - 1)!$$

$$= |I|!|J|! \sum_{k=0}^{|J|} \binom{k}{|I| - 1}$$

$$= |I|!|J|! \sum_{k=|I|-1}^{|I|+|J|-1} \binom{k}{|I| - 1}$$

$$= |I|!|J|! \binom{|I| + |J|}{|I|}$$

$$= (|I| + |J|)!;$$

$$|I|\left(\sum_{\substack{J=J' \sqcup J'', \\ J' \neq [l]}} |J'|!(|J''| + |I| - 1)!\right) = \begin{cases} (|I| + |J|)! \text{ if } J \neq [l], \\ (|I| + |J|)! - |I|!|J|! \text{ if } J = [l]. \end{cases}$$

This gives:

$$\phi(\partial(v)\phi(w)) = \underbrace{\sum_{\substack{\emptyset \subsetneq I \subset [k], \\ \emptyset \subsetneq J \subset [l], \\ I \neq [k] \text{ or } J \neq [l]}} \lambda^{|I|+|J|-1}(|I| + |J|)! \prod_{i \in I} f(x_i) \prod_{j \in J} f(y_j) \prod_{i \notin I} x_i \prod_{j \notin J} y_j}_{= \varphi_2(v,w)}$$

$$+ \sum_{\emptyset \subsetneq I \subsetneq [k]} \lambda^{|I|-1}|I|! \prod_{i \in I} f(x_i) \prod_{i \notin I} x_i \prod_{j \in [l]} y_j$$

$$+ \lambda^{k-1}k! \prod_{i \in [k]} f(x_i) \prod_{j \in [l]} y_j$$

$$- \sum_{\emptyset \subsetneq I \subsetneq [k]} \lambda^{l+|I|-1}l!|I|! \prod_{i \in I} f(x_i) \prod_{j \in [l]} f(y_j) \prod_{i \notin I} y_j.$$

Note that $\varphi_2(v, w) = \varphi_2(w, v)$. Finally:

$$\partial \circ \phi(v)\phi(w) - \phi(\partial(v)\phi(w)) = \varphi_1(v, w) - \varphi_2(v, w)$$

$$+ \sum_{\emptyset \subsetneq J \subsetneq [l]} \lambda^{k+|J|-1}k!|J|! \prod_{i \in [k]} f(x_i) \prod_{j \in J} f(y_j) \prod_{j \notin J} y_j$$

$$+ \sum_{\emptyset \subsetneq I \subsetneq [k]} \lambda^{l+|I|-1}l!|I|! \prod_{i \in I} f(x_i) \prod_{j \in [l]} f(y_j) \prod_{i \notin I} y_j.$$

This is symmetric in v, w.

2. Let us consider $A = \{u \in S(V) \mid \Delta \circ \partial u = (\partial \otimes Id) \circ \Delta(u)\}$. As $\partial(1) = 0$, $1 \in A$. If $x \in V$:

$$\Delta \circ \partial(x) = f(x)1 \otimes 1 = \partial(x) \otimes 1 + \partial(1) \otimes x = (\partial \otimes Id) \circ \Delta(x),$$

so $V \subseteq A$. Let $u, v \in A$.

$$\Delta \circ \partial(uv) = \Delta(\partial(u)v + u\partial(v))$$

$$= \partial(u^{(1)})v^{(1)} \otimes u^{(2)}v^{(2)} + u^{(1)}\partial(v^{(1)}) \otimes u^{(2)}v^{(2)}$$

$$= \partial(u^{(1)}v^{(1)}) \otimes u^{(2)}v^{(2)}$$

$$= (\partial \otimes Id) \circ \Delta(uv).$$

We proved that A is a subalgebra of $S(V)$ containing V, so $A = S(V)$.

Let us denote by $\tau : S(V) \otimes S(V)$ by $\tau(a \otimes b) = b \otimes a$. As Δ is cocommutative:

$$\begin{aligned}
\Delta \circ \partial &= \tau \circ \Delta \circ \partial \\
&= \tau \circ (\partial \otimes Id) \circ \Delta \\
&= (Id \otimes \partial) \circ \tau \circ \Delta \\
&= (Id \otimes \partial) \circ \Delta.
\end{aligned}$$

Let $u = x_1 \ldots x_k \in S(V)$.

$$\begin{aligned}
\Delta \circ \phi(u) &= \sum_{\substack{[k]=I \sqcup J \sqcup K, \\ J \sqcup K \neq \emptyset}} \lambda^{|I|} |I|! \prod_{i \in I} f(x_i) \prod_{j \in J} x_j \otimes \prod_{k \in K} x_k \\
&= \sum_{\substack{[k]=I \sqcup J \sqcup K, \\ J \neq \emptyset}} \lambda^{|I|} |I|! \prod_{i \in I} f(x_i) \prod_{j \in J} x_j \otimes \prod_{k \in K} x_k + \sum_{\substack{[k]=I \sqcup K, \\ K \neq \emptyset}} \lambda^{|I|} |I|! \prod_{i \in I} f(x_i) 1 \otimes \prod_{k \in K} x_k \\
&= \sum_{[k]=I \sqcup K} \phi\left(\prod_{i \in I} x_i \right) \otimes \prod_{k \in K} x_k + 1 \otimes \phi(u) \\
&= (\phi \otimes Id) \circ \Delta(u) + 1 \otimes \phi(u),
\end{aligned}$$

which ends this proof. $\qquad\square$

2.2 Proof of Theorem 2

To start with, observe that for any $u, v \in S(V)$:

$$u \bullet v = \partial(u)\phi(v).$$

1. We first prove the Leibniz identity. Let us take $u, v, w \in S(V)$. As ∂ is a derivation:

$$(uv) \bullet w = \partial(uv)\phi(w) = \partial(u)v\phi(w) + u\partial(v)\phi(w) = (u \bullet w)v + u(v \bullet w).$$

2. Let us now prove the preLie identity. If $u, v, w \in S(V)$:

$$\begin{aligned}
(u \bullet v) \bullet w - u \bullet (v \bullet w) &= \partial(\partial(u)\phi(v))\phi(w) - \partial(u)\phi(\partial(v)\phi(w)) \\
&= \partial^2(u)\phi(v)\phi(w) + \partial(u)(\partial \circ \phi(v)\phi(w) - \phi(\partial(v)\phi(w))).
\end{aligned}$$

By Lemma 3, this is symmetric in v, w.

3. Let us finish by the compatibility with the coproduct. For any $u, v \in S(V)$, by Lemma 3:

$$\begin{aligned}
\Delta(u \bullet v) &= \Delta(\partial(u)\phi(v)) \\
&= \Delta \circ \partial(u)(\phi(v^{(1)}) \otimes v^{(2)} + 1 \otimes \phi(v)) \\
&= \partial(u^{(1)})\phi(v^{(1)}) \otimes u^{(2)}v^{(2)} + u^{(1)} \otimes \partial(u^{(2)})\phi(v) \\
&= u^{(1)} \bullet v^{(1)} \otimes u^{(2)}v^{(2)} + u^{(1)} \otimes u^{(2)} \bullet v.
\end{aligned}$$

Hence, $S(V, f, \lambda)$ is indeed a Com-PreLie bialgebra.

2.3 Isomorphisms

Proposition 4. *Let V, W be two vector spaces, f and g be linear forms of respectively V and W, and $\lambda, \mu \in \mathbb{K}$. The Com-PreLie bialgebras $S(V, f, \lambda)$ and $S(W, g, \mu)$ are isomorphic if, and only if, one of the two following assertions holds:*

1. $f = g = 0$ and $\dim(V) = \dim(W)$.

2. $\lambda = \mu$ and there exists a linear bijection $\psi : V \longrightarrow W$ such that $g \circ \psi = f$.

Proof. If the first assertion holds, then both preLie products on $S(V, f, \lambda)$ and $S(W, g, \mu)$ are zero. Any linear isomorphism between V and W, extended as an algebra isomorphism, is a Com-PreLie bialgebra isomorphism.

If the second assertion holds, the extension of ψ as an algebra isomorphism is a Com-PreLie bialgebra isomorphism.

Let $\Psi : S(V, f, \lambda) \longrightarrow S(W, g, \mu)$ be an isomorphism. It is a coalgebra isomorphism, so the restriction ψ of Ψ to $Prim(S(V)) = V$ is a bijection to $Prim(S(W)) = W$. As Ψ is an algebra morphism, it is the extension of ψ as an algebra morphism from $S(V)$ to $S(W)$.

Let $x, y \in V$. Then:

$$\begin{aligned}
\Psi(x \bullet y) = \Psi(f(x)y) &= f(x)\psi(y) \\
&= \Psi(x) \bullet \Psi(y) \\
&= g \circ \psi(x)\psi(y).
\end{aligned}$$

Choosing a nonzero y, this proves that $f = g \circ \psi$. As a consequence, $f = 0$ if, and only if, $g = 0$.

Let $x, y, z \in V$. Then:

$$\begin{aligned}
\Psi(x \bullet yz) &= f(x)\psi(y)\psi(z) + \lambda f(x)f(y)\psi(z) + \lambda f(x)f(z)\psi(y) \\
&= \psi(x) \bullet \psi(y)\psi(z) \\
&= f(x)\psi(y)\psi(z) + \mu f(x)f(y)\psi(z) + \mu f(x)f(z)\psi(y)
\end{aligned}$$

If $f \neq 0$, let us choose $x = y = z$ such that $f(x) = 1$. Then $2\lambda\psi(x) = 2\mu\psi(x)$, so $\lambda = \mu$. $\qquad \square$

Remark 2. If $\bullet$ is the product of $S(V, f, \lambda)$ and $\mu \neq 0$, the Com-PreLie biagebra $(S(V), m, \mu\bullet, \Delta)$ is $S(V, \mu f, \lambda/\mu)$.

3 Examples on $\mathbb{K}[X]$

Our aim in this section is to give all preLie products on $\mathbb{K}[X]$, making it a graded Com-PreLie algebra.

Recall that $\mathbb{K}[X]$ is given a Zinbiel product $\prec$, defined by:

$$\forall i, j \geq 1, \qquad\qquad X^i \prec X^j = \frac{i}{i+j} X^{i+j}.$$

The associated product is the usual product of $\mathbb{K}[X]$.

We shall prove the following result:

Theorem 5. *The following objects are Zinbiel-PreLie algebras:*

1. *Let $N \geq 1$, $\lambda, a, b \in \mathbb{K}$, $a \neq 0$, $b \notin \mathbb{Z}_-$. We put $\mathfrak{g}^{(1)}(N, \lambda, a, b) = (\mathbb{K}[X], m, \bullet)$, with:*

$$X^i \bullet X^j = \begin{cases} i\lambda X^i & \text{if } j = 0, \\ a\dfrac{i}{\frac{j}{N}+b}X^{i+j} & \text{if } j \neq 0 \text{ and } N \mid j, \\ 0 & \text{otherwise.} \end{cases}$$

2. *Let $N \geq 1$, $\lambda, \mu \in \mathbb{K}$, $\mu \neq 0$. We put $\mathfrak{g}^{(2)}(N, \lambda, \mu) = (\mathbb{K}[X], m, \bullet)$, with:*

$$X^i \bullet X^j = \begin{cases} i\lambda X^i & \text{if } j = 0, \\ i\mu X^{i+N} & \text{if } j = N, \\ 0 & \text{otherwise.} \end{cases}$$

3. *Let $N \geq 1$, $\lambda, \mu \in \mathbb{K}$, $\mu \neq 0$. We put $\mathfrak{g}^{(3)}(N, \lambda, \mu) = (\mathbb{K}[X], m, \bullet)$, with:*

$$X^i \bullet X^j = \begin{cases} i\lambda X^i & \text{if } j = 0, \\ i\mu X^{i+j} & \text{if } j \neq 0 \text{ and } N \mid j, \\ 0 & \text{otherwise.} \end{cases}$$

4. *Let $\lambda \in \mathbb{K}$. We put $\mathfrak{g}^{(4)}(\lambda) = (\mathbb{K}[X], m, \bullet)$, with:*

$$X^i \bullet X^j = \begin{cases} i\lambda X^i & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the preLie product of $\mathfrak{g}^{(4)}(0)$ is zero.

Moreover, if $\bullet$ is a product on $\mathbb{K}[X]$, such that $\mathfrak{g} = (\mathbb{K}[X], m, \bullet)$ is a graded Com-PreLie algebra, Then $\mathfrak{g}$ is one of the preceding examples.

Remark 3. If $\lambda = \frac{a}{b}$, in $\mathfrak{g}^{(1)}(N, \lambda, a, b)$:

$$X^i \bullet X^j = \begin{cases} \dfrac{ai}{\frac{j}{N}+b}X^{i+j} & \text{if } N \mid j, \\ 0 & \text{otherwise.} \end{cases}$$

We denote $\mathfrak{g}^{(1)}(N, a, b) = \mathfrak{g}^{(1)}(N, \frac{a}{b}, a, b)$.

3.1 Graded preLie products on $\mathbb{K}[X]$

In this paragraph, we look for all graded preLie products on $\mathbb{K}[X]$, making it a Com-PreLie algebra. Let $\bullet$ be a homogeneous product on $\mathbb{K}[X]$, making it a graded Com-PreLie algebra. For all $i, j \geq 0$, there exists a scalar $\lambda_{i,j}$ such that:

$$X^i \bullet X^j = \lambda_{i,j}X^{i+j}.$$

Moreover, for all $i, j, k \geq 0$:

$$\begin{aligned} X^{i+j} \bullet X^k &= \lambda_{i+j,k}X^{i+j+k} \\ &= (X^i X^j) \bullet X^k \\ &= (X^i \bullet X^k)X^j + X^i(X^j \bullet X^k) \\ &= (\lambda_{i,k} + \lambda_{j,k})X^{i+j+k}. \end{aligned}$$

Hence, $\lambda_{i+j,k} = \lambda_{i,k} + \lambda_{j,k}$. Putting $\lambda_k = \lambda_{1,k}$ for all $k \geq 0$, we obtain:

$$X^i \bullet X^j = i\lambda_j X^{i+j}.$$

Lemma 6. *For all $k \geq 0$, let $\lambda_k \in \mathbb{K}$. We define a product $\bullet$ on $\mathbb{K}[X]$ by:*

$$X^i \bullet X^j = i\lambda_j X^{i+j}.$$

Then $(\mathbb{K}[X], m, \bullet)$ is Com-PreLie if, and only if, for all $j, k \geq 1$:

$$(j\lambda_k - k\lambda_j)\lambda_{j+k} = (j-k)\lambda_j\lambda_k.$$

Proof. Let $i, j, k \geq 0$. Then:

$$X^i \bullet (X^j \bullet X^k) - (X^i \bullet X^j) \bullet X^k = (ij\lambda_k\lambda_{j+k} - i(i+j)\lambda_j\lambda_k)X^{i+j+k}.$$

So $\bullet$ is preLie if, and only if:

$$\begin{aligned}
&\forall i, j, k \geq 0, ij\lambda_k\lambda_{j+k} - i(i+j)\lambda_j\lambda_k = ik\lambda_j\lambda_{j+k} - i(i+k)\lambda_j\lambda_k \\
\Longleftrightarrow &\forall j, k \geq 0, (j\lambda_k - k\lambda_j)\lambda_{j+k} = (j-k)\lambda_j\lambda_k \\
\Longleftrightarrow &\forall j, k \geq 1, (j\lambda_k - k\lambda_j)\lambda_{j+k} = (j-k)\lambda_j\lambda_k,
\end{aligned}$$

as the identity is trivially satisfied if $j = 0$ or $k = 0$. $\qquad\square$

Lemma 7. *Let $\bullet$ be a product on $\mathbb{K}[X]$, making it a graded Com-PreLie algebra. Then $(\mathbb{K}[X], \prec, \bullet)$ is a Zinbiel-PreLie algebra.*

Proof. Let us take $i, k, k \geq 0$, $(i, j) \neq (0, 0)$. Then:

$$\begin{aligned}
(X^i \bullet X^k) \prec X^j + X^i \prec (X^j \bullet X^k) &= \lambda_k(iX^{i+k} \prec X^j + jX^i \prec X^{j+k}) \\
&= \lambda_k\left(\frac{i(i+k)}{i+j+k} + \frac{ij}{i+j+k}\right)X^{i+j+k} \\
&= i\lambda_k X^{i+j+k} \\
&= (i+j)\lambda_k \frac{i}{i+j}X^{i+j+k}, \\
(X^i \prec X^j) \bullet X^k &= \frac{i}{i+j}X^{i+j} \bullet X^k \\
&= \frac{i}{i+j}(i+j)\lambda_k X^{i+j+k} \\
&= i\lambda_k X^{i+j+k}.
\end{aligned}$$

So $\mathbb{K}[X]$ is Zinbiel-PreLie. $\qquad\square$

Proof. (Theorem 5-1). Let us first prove that the objects defined in Theorem 5 are indeed Zinbiel-PreLie algebras. By Lemma 7, it is enough to prove that they are Com-PreLie algebras. We shall use Lemma 6 in all cases.

1. For all $j \geq 1$, $\lambda_j = a\frac{1}{\frac{j}{N}+b}$ if $N \mid j$ and 0 otherwise. If j or k is not a multiple of N, then:

$$(j\lambda_k - k\lambda_j)\lambda_{j+k} = (j-k)\lambda_j\lambda_k = 0.$$

If $j = Nj'$ and $k = Nk'$, with $j', k' \in \mathbb{N}$, then:

$$\begin{aligned}
(j\lambda_k - k\lambda_j)\lambda_{j+k} &= Na^2\left(\frac{j'}{k'+b} - \frac{k'}{j'+b}\right)\frac{1}{j'+k'+b} \\
&= Na^2\frac{j'^2 - k'^2 + b(j'-k')}{(j'+b)(k'+b)(j'+k'+b)} \\
&= Na^2(j'-k')\frac{j'+k'+b}{(j'+b)(k'+b)(j'+k'+b)} \\
& a^2(j-k)\frac{1}{(j'+b)(k'+b)} \\
&= (j-k)\lambda_j\lambda_k.
\end{aligned}$$

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2. In this case, $\lambda_j = \mu$ if $j = N$ and 0 otherwise. Hence, for all $j, k \geq 1$:

$$(j\lambda_k - k\lambda_j)\lambda_{j+k} = \mu^2(j\delta_{k,N} - k\delta_{j,N})\delta_{j+k,N} = 0,$$
$$(j - k)\lambda_j\lambda_k = \mu^2(j - k)\delta_{j,N}\delta_{k,N} = 0.$$

3. Here, for all $j \geq 1$, $\lambda_j = \mu$ if $N \mid j$ and 0 otherwise. Then:

$$(j\lambda_k - k\lambda_j)\lambda_{j+k} = \begin{cases} \mu^2(j - k) & \text{if } N \mid j, k, \\ 0 & \text{otherwise;} \end{cases} \qquad (j - k)\lambda_j\lambda_k = \begin{cases} \mu^2(j - k) & \text{if } N \mid j, k, \\ 0 & \text{otherwise.} \end{cases}$$

4. In this case, for all $j \geq 1$, $\lambda_j = 0$ and the result is trivial. $\qquad\square$

3.2 Classification of graded preLie products on $\mathbb{K}[X]$

We now prove that the preceding examples cover all the possible cases.

Proof. (Theorem 5-2). We put $X^i \bullet X^j = i\lambda_j X^{i+j}$ for all $i, j \geq 0$ and we put $\lambda = \lambda_0$. If for all $j \geq 1$, $\lambda_j = 0$, then $\mathfrak{g} = \mathfrak{g}^{(4)}(\lambda)$. If this is not the case, we put:

$$N = \min\{j \geq 1 \mid \lambda_j \neq 0\}.$$

First step. Let us prove that if i is not a multiple of N, then $\lambda_i = 0$. If i is not a multiplie of N, we put $i = qN + r$, with $0 < r < N$, and we proceed by induction on q. If $q = 0$, by definition of N, $\lambda_1 = \ldots = \lambda_{N-1} = 0$. Let us assume the result at rank $q - 1$, with $q > 0$. We put $j = i - N$ and $k = N$. By the induction hypothesis, $\lambda_j = 0$. Then, by Lemma 6:

$$(i - N)\lambda_N\lambda_i = 0.$$

As $i \neq N$ and $\lambda_N \neq 0$, $\lambda_i = 0$. It is now enough to determine λ_{iN} for all $i \geq 1$.

Second step. Let us assume that $\lambda_{2N} = 0$. Let us prove that $\lambda_{iN} = 0$ for all $i \geq 2$, by induction on i. This is obvious if $i = 2$. Let us assume the result at rank $i - 1$, with $i \geq 3$, and let us prove it at rank i. We put $j = (i-1)N$ and $k = N$. By the induction hypothesis, $\lambda_j = 0$. Then, by Lemma 6:

$$(i - 2)N\lambda_N\lambda_iN = 0.$$

As $i \geq 3$ and $\lambda_N \neq 0$, $\lambda_{iN} = 0$. As a conclusion, if $\lambda_{2N} = 0$, putting $\mu = \lambda_N$, $\mathfrak{g} = \mathfrak{g}^{(2)}(N, \lambda, \mu)$.

Third step. We now assume that $\lambda_{2N} \neq 0$. We first prove that $\lambda_{iN} \neq 0$ for all $i \geq 1$. This is obvious if $i = 1, 2$. Let us assume the result at rank $i - 1$, with $i \geq 3$, and let us prove it at rank i. We put $j = (i - 1)N$ and $k = N$. Then, by Lemma 6:

$$(j\lambda_N - N\lambda_j)\lambda_{iN} = (i - 2)N\lambda_j\lambda_N.$$

By the induction hypothesis, $\lambda_j \neq 0$. Moreover, $i > 2$ and $\lambda_N \neq 0$, so $\lambda_{iN} \neq 0$.

For all $j \geq 1$, we put $\mu_j = \frac{\lambda_{kN}}{\lambda_N}$: this is a nonzero scalar, and $\mu_1 = 1$. Let us prove inductively that:

$$\mu_k = \frac{\mu_2}{(k - 1) - (k - 2)\mu_2}, \qquad \mu_2 \neq \frac{k - 1}{k - 2} \text{ if } k \neq 2.$$

If $k = 1$, $\mu_1 = 1 = \frac{\mu_2}{0 - (-1)\mu_2}$, and $\mu_2 \neq 0$ as $\lambda_{2N} \neq 0$; if $k = 2$, $\mu_2 = \frac{\mu_2}{1 - 0\mu_2}$. Let us assume the result at rank $k - 1$, with $k \geq 3$. By Lemma 6, with $j = (k - 1)N$ and $k = N$:

$$((k - 1)N\lambda_N - \lambda_N\mu_{k-1})\lambda_N\mu_k = (k - 2)N\mu_{k-1}\mu_1\lambda_N^2,$$
$$\mu_k((k - 1) - \mu_{k-1}) = (k - 2)\mu_{k-1}.$$

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Moreover, by the induction hypothesis:

$$(k-1) - \mu_{k-1} = k - 1 - \frac{\mu_2}{(k-2) - (k-3)\mu_2}$$
$$= \frac{(k-1)(k-2) - ((k-1)(k-3) + 1)\mu_2}{(k-2) - (k-3)\mu_2}$$
$$= (k-2)\frac{(k-1) - (k-2)\mu_2}{(k-2) - (k-3)\mu_2}.$$

As $\mu_{k-1} \neq 0$ and $k > 2$, this is nonzero, so $\mu_2 \neq \frac{k-1}{k-2}$. We finally obtain:

$$\mu_k = (k-2)\mu_{k-1}\frac{1}{k-2}\frac{(k-2) - (k-3)\mu_2}{(k-1) - (k-2)\mu_2} = \frac{\mu_2}{(k-1) - (k-2)\mu_2}.$$

Finally, for all $k \geq 1$:

$$\lambda_{kN} = \frac{a\mu_2}{(k-1) - (k-2)\mu_2} = \frac{\lambda_N\mu_2}{(1 - \mu_2)k + 2\mu_2 - 1}.$$

Last step. If $\mu_2 = 1$, then for all $k \geq 1$, $\lambda_{kN} = \lambda_N$: this is $\mathfrak{g}^{(3)}(N, \lambda, \lambda_N)$. If $\mu_2 \neq 1$, we put $b = \frac{2\mu_2 - 1}{1 - \mu_2}$. As $\mu_2 \neq 0$, $b \neq -1$. As for all $k \geq 3$, $\mu_2 \neq \frac{k-1}{k-2}$, $b \neq -k$; and $b \neq -2$, so $b \notin \mathbb{Z}_-$. Moreover, for all $k \geq 1$:

$$\lambda_{kN} = \frac{\frac{\lambda_N\mu_2}{1 - \mu_2}}{k + b}.$$

We take $a = \frac{\lambda_N\mu_2}{1 - \mu_2}$, and we obtain $\mathfrak{g}^{(1)}(N, \lambda, a, b)$. $\qquad\square$

Proposition 8. *Among the examples of Theorem 5, the Com-PreLie bialgebras (or equivalently the Zinbiel-PreLie bialgebras) are $g^{(4)}(0)$ and $\mathfrak{g}^{(1)}(1, a, 1)$, with $a \neq 0$.*

Proof. Note that $\mathfrak{g}^{(1)}(1, 0, 1) = \mathfrak{g}^{(4)}(0)$. Let us first prove that $\mathfrak{g}(1, a, 1)$ is a Zinbiel-PreLie bialgebra for all $a \in \mathbb{K}$. By the first remark following Definition 1, it is enough to consider $\mathfrak{g}(1, 1, 1)$. We consider:

$$A = \{x \in \mathfrak{g}(1, 1, 1) \mid \forall y \in \mathfrak{g}(1, 1, 1), \ \Delta(x \bullet y) = x^{(1)} \otimes x^{(2)} \bullet y + x^{(1)} \bullet y^{(1)} \otimes x^{(2)}y^{(2)}\}.$$

Firstly, $1 \in A$: for any $y \in \mathfrak{g}(1, 1, 1)$,

$$\Delta(1 \bullet y) = 0 = 1 \otimes 1 \bullet y + 1 \bullet y^{(1)} \otimes 1y^{(2)}$$

Let $x_1, x_2 \in A$. For any $y \in \mathfrak{g}(1, 1, 1)$, by the Leibniz identity:

$$\Delta((x_1x_2) \bullet y) = \Delta(x_1 \bullet y)\Delta(x_2) + \Delta(x_1)\Delta(x_2 \bullet y)$$
$$= x_1^{(1)}x_2^{(1)} \otimes (x_1^{(2)} \bullet y)x_2^{(2)} + (x_1^{(1)} \bullet y^{(1)})x_2^{(1)} \otimes x_1^{(2)}x_2^{(2)}y^{(2)}$$
$$+ x_1^{(1)}x_2^{(1)} \otimes x_1^{(2)}(x_2^{(2)} \bullet y) + x_1^{(1)}(x_2^{(1)} \bullet y^{(1)}) \otimes x_1^{(2)}x_2^{(2)}y^{(2)}$$
$$= x_1^{(1)}x_2^{(1)} \otimes (x_1^{(2)}x_2^{(2)}) \bullet y + (x_1^{(1)}x_2^{(1)}) \bullet y^{(1)} \otimes x_1^{(2)}x_2^{(2)}y^{(2)}$$
$$= (x_1x_2)^{(1)} \otimes (x_1x_2)^{(2)} \bullet y + (x_1x_2)^{(1)}bullety^{(1)} \otimes (x_1x_2)^{(2)}y^{(2)}.$$

So $x_1x_2 \in A$: A is a subalgebra of $\mathbb{K}[X]$. Hence, it is enough to prove that $X \in A$. Let $n \geq 0$,

let us consider $y = X^n$.

$$\Delta(X \bullet y) = \frac{1}{1+n}\Delta(X^{n+1})$$

$$= \sum_{k=0}^{n+1} \frac{n!}{k!(n+1-k)!} X^k \otimes X^{n+1-k};$$

$$X^{(1)} \bullet y^{(1)} \otimes X^{(2)}y^{(2)} = X \bullet y^{(1)} \otimes y^{(2)} + 0$$

$$= \sum_{k=0}^{n} \frac{n!}{(k+1)!(n-k)} X^{k+1} \otimes X^{n-k}$$

$$= \sum_{k=1}^{n+1} \frac{n!}{k!(n+1-k)!} X^k \otimes X^{n+1-k};$$

$$X^{(1)} \otimes X^{(2)} \bullet y = 1 \otimes X \bullet y + 0$$

$$= \frac{n!}{0!(n+1-0)!} X^0 \otimes X^{n+1-0}.$$

This proves that $X \in A$, so $\mathfrak{g}(1,1,1)$ is a Zinbiel-PreLie bialgebra.

Let $\mathfrak{g}$ be one of the examples of Theorem 5. Firstly:

$$\Delta(X \bullet X) = X \otimes 1 \bullet X + 1 \otimes X \bullet X$$
$$+ X \bullet X \otimes 1 + X \bullet 1 \otimes X + 1 \bullet X \otimes X + 1 \bullet 1 \otimes X^2$$
$$\lambda_1(1 \otimes X^2 + 2X \otimes X + X^2 \otimes 1) = \lambda_1 1 \otimes X^2 + \lambda X \otimes X + \lambda_1 X^2 \otimes 1.$$

This gives $\lambda_0 = 2\lambda_1$. In particular, if $\mathfrak{g} = \mathfrak{g}^{(4)}(\lambda)$, then $\lambda = 2\lambda_1 = 0$: this is $\mathfrak{g}^{(4)}(0)$. In the other cases, N exists. By definition of N, $X \bullet X^k = 0$ if $1 \leq k \leq N-1$. We obtain:

$$\Delta(X \bullet X^N) = 1 \otimes X \bullet X^N + X \otimes 1 \bullet X^N + \sum_{k=0}^{N} \binom{N}{k}(X \bullet X^k \otimes X^{N-k} + 1 \bullet X^k \otimes X^{n-k+1})$$

$$\lambda_N \Delta(X^{N+1}) = 1 \otimes X \bullet X^N + \lambda X \otimes X^N + 1 \otimes X \bullet X^N.$$

If $\lambda = 0$, we obtain that X^{N+1} is primitive, as $\lambda_N = 0$, so $N+1 = 1$: absurd, $N \geq 1$. So $\lambda \neq 0$,. The cocommutativity of Δ implies that $N = 1$.

$$\Delta(X \bullet X^2) = \lambda_2(X^3 \otimes 1 + 3X^2 \otimes X + 3X \otimes X^2 + 1 \otimes X^3)$$
$$= 1 \otimes X \bullet X^2 + 2\lambda_1 X^2 \otimes X + \lambda_0 X \otimes X^2 + 1 \otimes X \bullet X^2$$

Hence, $3\lambda_2 = 2\lambda_1$.

- If $\mathfrak{g} = \mathfrak{g}^{(3)}(1, \lambda, \mu)$, we obtain $3\mu = 2\mu$, so $\mu = 0$: contradiction.

- If $\mathfrak{g} = \mathfrak{g}^{(2)}(1, \lambda, \mu)$, we obtain $0 = 2\mu$, so $\mu = 0$: contradiction.

So $\mathfrak{g} = \mathfrak{g}^{(1)}(1, \lambda, a, b)$. We obtain:

$$3\frac{a}{2+b} = 2\frac{a}{1+b},$$

so $b = 1$. Then $\lambda_0 = 2\lambda_1 = \frac{2a}{2} = a = \frac{a}{b}$, so $\mathfrak{g} = \mathfrak{g}^{(1)}(1, a, 1)$. $\qquad\square$

3.3 Underlying Lie algebras

We aim in this paragraph to describe the underlying Lie algebras of the preLie algebras of Theorem 5. Let us first recall the construction of of the semi-direct sum of two Lie algebras. Let $\mathfrak{g}, \mathfrak{h}$ be two Lie algebras and let $\tau : \mathfrak{h} \longrightarrow Der(\mathfrak{g})^{op}$ be a Lie algebra morphism, where $Der(\mathfrak{h})^{op}$ is the opposite of the Lie algebra of derivations of the Lie algebra $\mathfrak{h}$. Then $\mathfrak{g} \oplus \mathcal{H}$ is given a Lie bracket in the following way: if $x, x' \in \mathfrak{g}$, $y, y' \in \mathfrak{h}$,

$$[x + y, x' + y'] = [x, x']_{\mathfrak{g}} - \tau(y).x' + \tau(y').x + [y, y']_{\mathfrak{h}}.$$

This Lie algebra is denoted by $\mathfrak{g} \oplus_{\tau} h$. Here are the examples we shall use in the sequel:

1. Let $\mathfrak{g}$ be a graded preLie algebra. Then the abelian Lie algebra $\mathbb{K}$ acts on $\mathfrak{g}$ by derivation: if $x \in \mathfrak{g}$ is homogeneous of degree n, then $\tau(1)(x) = nx$. The associated semi-direct sum is denoted by $\mathfrak{g} \oplus_{deg} \mathbb{K}$.

2. Let $\mathfrak{g}$ be a Lie algebra and let m be a right $\mathfrak{g}$-module; the action of $\mathfrak{g}$ over m is denoted by m. Considering m as an abelian Lie algebra, we obtain a semi-direct product $m \oplus_{\tau} g$. For any $x, x' \in m$, $y, y' \in \mathfrak{g}$:

$$[x + y, x' + y'] = x.y' - x'.y + [y, y'].$$

We shall use the Faà di Bruno Lie algebra $\mathfrak{g}_{FdB}$: as a vector space it has a basis $(e_i)_{i \geq 1}$, and its Lie bracket is given by:

$$\forall k, l \geq 1, \qquad\qquad [e_k, e_l] = (k - l)e_{k+l}.$$

This is the Lie algebra of the group of formal diffeomorphisms $\{x + a_1 x^2 + \ldots\} \subseteq \mathbb{K}[[x]]$, with the composition of formal series. For any $\lambda \in \mathbb{K}$, the right $\mathfrak{g}_{FdB}$-module has a basis $(f_k)_{k \geq 0}$ and:

$$\forall k, l \geq 1, \qquad\qquad f_k.e_l = (k + \lambda)f_{k+l}.$$

Any $\mathfrak{g}$ described in Theorem 5 can be decomposed into a semi-direct sum $\mathfrak{g}_+ \oplus \mathfrak{g}_0$, where $\mathfrak{g}_0 = Vect(1)$ and $\mathfrak{g}_+ = Vect(X^k, k \geq 1)$. The action of $\mathfrak{g}_0$ over $\mathfrak{g}_+$ is given by the product $\bullet$. As a consequence, if $\lambda = 0$, this is a trival action and $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_+ \oplus \mathbb{K}$; otherwise, $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_+ \oplus_{deg} \mathbb{K}$. Let us now describe $\mathfrak{g}_+$.

Proposition 9. *Let $\mathfrak{g}$ be one of the Com-PreLie algebras of Theorem 5 and let $\mathfrak{g}_+$ its augmentation ideal.*

1. *If $\mathfrak{g} = \mathfrak{g}^{(1)}(N, \lambda, a, b)$ or $\mathfrak{g}^{(3)}(N, \lambda, \mu)$ then, as a Lie algebra:*

$$\mathfrak{g} \approx \left(V_{\frac{1}{N}} \oplus \ldots \oplus V_{\frac{N-1}{N}} \right) \oplus_{\tau} \mathfrak{g}_{FdB}.$$

2. *If $\mathfrak{g} = \mathfrak{g}^{(2)}(N, \lambda, \mu)$, let us put $\mathfrak{g}_1 = Vect(f_k, 1 \leq k \neq N)$ be an abelian Lie algebra and let τ be the action of $\mathbb{K}$ on $\mathfrak{g}_1$ given by $k.1 = f_{k+N}$ for all k. Then:*

$$\mathfrak{g}_+ \approx \mathfrak{g}_1 \oplus_{\tau} \mathfrak{g}_2.$$

3. *If $\mathfrak{g} = \mathfrak{g}^{(4)}(\lambda)$, then $\mathfrak{g}_+$ is abelian.*

Proof. The cases 2 and 3 are immediate. Let us consider the case $\mathfrak{g} = \mathfrak{g}^{(1)}(N, \lambda, a, b)$. We put $\mathfrak{g}_0 = Vect(X^{Nk}, k \geq 1)$ and for all $i \in [N-1]$, $\mathfrak{g}_i = Vect(X^{Nk+i}, k \geq 1)$. For all $k \geq 1$, we put

$e_k = \frac{k+b}{Na} X^k$; then $(e_k)_{k \geq 1}$ is a basis of $\mathfrak{g}_0$ and, for any $k, l \geq 1$:

$$\begin{aligned}
[e_k, e_l] &= \frac{(k+b)(l+b)}{N^2 a^2} (X^{kN} \bullet X^{lN} - X^{lN} \bullet X^{kN}) \\
&= \frac{(k+b)(l+b)}{N^2 a^2} a \left(\frac{kN}{l+b} - \frac{lN}{k+b} \right) X^{(k+l)N} \\
&= (k-l) \frac{k+l+b}{Na} X^{(k+l)N} \\
&= (k-l) e_{k+l}.
\end{aligned}$$

So $\mathfrak{g}_0$ is isomorphic to $\mathfrak{g}_{FdB}$. By definition of the preLie product, $\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_{N-1}$ is an abelian Lie algebra. Moreover, if $i \in [N-1]$, $k \geq 0$, $l \geq 1$:

$$\begin{aligned}
[X^{kN+i}, e_l] &= X^{kN+i} \bullet e_l + 0 \\
&= \frac{kN+i}{N} X^{(k+l)N+i} \\
&= \left(k + \frac{i}{N} \right) X^{(k+l)N+i}.
\end{aligned}$$

So $\mathfrak{g}_i$ is a right $\mathfrak{g}_0$-module, isomorphic to $V_{\frac{i}{N}}$. The result follows. The proof for $\mathfrak{g}^{(3)}(N, \lambda, \mu)$ is similar. $\qquad\square$

4 Cocommutative Com-PreLie bialgebras

We now prove the following theorem:

Theorem 10. *Let A be a connected, cocommutative Com-PreLie bialgebra. Then one of the following assertions holds:*

1. *There exists a linear form $f : Prim(A) \longrightarrow \mathbb{K}$ and $\lambda \in \mathbb{K}$, such that A is isomorphic to $S(V, f, \lambda)$.*

2. *There exists $a \in \mathbb{K}$ such that A is isomorphic to $\mathfrak{g}^{(1)}(1, a, 1)$.*

First, observe that A is a cocommutative, commutative, connected Hopf algebra: by the Cartier-Quillen-Milnor-Moore theorem, it is isomorphic to the enveloping Hopf algebra of an abelian Lie algebra, so is isomorphic to $S(V)$ as a Hopf algebra, where $V = Prim(A)$. If $V = (0)$, the first point holds trivially.

4.1 First case

We assume in this paragraph that V is at least 2-dimensional.

Lemma 11. *Let A be a connected, cocommutative Com-PreLie algebra, such that the dimension of $Prim(A)$) is at least 2. Then $f_A = 0$, and there exists a map $F : A \longrightarrow A$, such that:*

1. *For all $x, y \in A_+$, $x \bullet y = F(x \otimes y')y'' + F(x \otimes 1)y$, with Sweedler's notation $\Delta(y) = y \otimes 1 + 1 \otimes y + y' \otimes y''$.*

2. *For all $x_1, x_2 \in A$, $F(x_1 x_2 \otimes y) = F(x_1 \otimes y)x_2 + x_1 F(x_2 \otimes y)$.*

3. *$F(Prim(A) \otimes A) \subseteq \mathbb{K}$.*

Proof. We assume that $A = S(V)$ as a bialgebra, with its usual product and coproduct Δ, and that $dim(V) \geq 2$. Let $x, y \in V$. Then:

$$\Delta(x \bullet y) = x \bullet y \otimes 1 + 1 \otimes x \bullet y + f_A(x) \otimes y.$$

As A is cocommutative, for all $x, y \in V$, $f_A(x)$ and y are colinear. As $dim(V) \geq 2$, necessarily $f = 0$.

We now construct linear maps $F_i : V \otimes S^i(V) \longrightarrow \mathbb{K}$, such that for all $k \geq 0$, putting:

$$F^{(k)} = \bigoplus_{i=0}^{k} F_i : \bigoplus_{i=0}^{k} V \otimes S^i(V) \longrightarrow \mathbb{K},$$

for all $x \in V$, $y \in S^k(V)$:

$$x \bullet y = F^{(k)}(x \otimes y') \otimes y'' + F^{(k)}(x \otimes 1)y.$$

We proceed by induction on k. Let us first construct $F^{(0)}$. Let $x, y \in V$.

$$\Delta(x \bullet y^2) = 1 \otimes x \bullet y^2 + x \bullet y^2 \otimes 1 + 2x \bullet y \otimes y.$$

As Δ is cocommutative, $x \bullet y$ and y are colinear, so there exists a linear map $g : V \longrightarrow \mathbb{K}$ such that $x \bullet y = g(x)y$. We the take $F^{(0)}(x \otimes 1) = g(x)$. For all $x, y \in V$, $x \bullet y = F(x \otimes 1)y$, so the result holds for $k = 0$.

Let us assume that $F^{(0)}, \ldots, F^{(k-2)}$ are constructed for $k \geq 2$. Let $x, y_1, \ldots, y_k \in V$. For all $I \subseteq [k] = \{1, \ldots, k\}$, we put $y_I = \prod_{i \in I} y_i$. Then:

$$\tilde{\Delta}(y_1 \ldots y_k) = \sum_{I \sqcup J = [k], I, J \neq \emptyset} y_I \otimes y_J,$$

and:

$$\Delta(x \bullet y_1 \ldots y_k) = 1 \otimes x \bullet y_1 \ldots y_k + x \bullet y_1 \ldots y_k \otimes 1 + \sum_{[k] = I \sqcup J, J \neq \emptyset} x \bullet y_I \otimes y_J$$

$$= 1 \otimes x \bullet y_1 \ldots y_k + x \bullet y_1 \ldots y_k \otimes 1 + \sum_{I \sqcup J \sqcup K = [k], J, K \neq \emptyset} F^{(k-2)}(x \otimes y_I) \otimes y_J \otimes y_K.$$

We put:

$$P(x, y_1 \ldots y_k) = x \bullet y_1 \ldots y_k - \sum_{I \sqcup J = [k], |J| \geq 2} F^{(k-2)}(x \otimes y_I)y_J.$$

The preceding computation show that $P(x, y_1 \ldots, y_k)$ is primitive, so belongs to V. Let $y_{k+1} \in V$.

$$\tilde{\Delta}(x \bullet y_1 \ldots y_{k+1}) = \sum_{I \sqcup J \sqcup K = [k+1], K \neq \emptyset, |J| \geq 2} \underbrace{F^{(k-2)}(x \otimes y_I)y_J}_{\in S_{\geq 2}(V)} \otimes y_K$$

$$+ P(x, y_1 \ldots y_k) \otimes y_{k+1} + \sum_{i=1}^{k} P(x, y_1 \ldots y_{i-1}y_{i+1} \ldots y_{k+1}) \otimes y_i.$$

By cocommutativity, considering the projection on $V \otimes V$, we deduce that $P(x, y_1 \ldots y_k) \in Vect(y_1, \ldots, y_k, y_{k+1})$ for all nonzero $y_{k+1} \in V$. In particular, for $y_1 = y_{k+1}$, $P(x \otimes y_1 \ldots y_k) \in Vect(y_1, \ldots, y_k)$. By multilinearity, there exists $F'_1, \ldots, F'_k \in (V \otimes S_{k-1}(V))^*$, such that for all $x, y_1, \ldots, y_k \in V$:

$$P(x, y_1 \ldots y_k) = F'_1(x \otimes y_2 \ldots y_k)y_1 + \ldots + F'_k(x \otimes y_1 \ldots y_{k-1})y_k.$$

By symmetry in $y_1, \ldots, y_k$, $F'_1 = \ldots = F'_k = F_{k-1}$. Then:

$$x \bullet y_1 \ldots y_k = \sum_{I \sqcup J = [k], |j| \geq 2} F^{(k-2)}(x \otimes y_I) y_J + \sum_{I \sqcup J = [k], |J| = 1} F_{k-1}(x \otimes y_I) y_J$$

$$= \sum_{I \sqcup J = [k], |j| \geq 1} F^{(k-1)}(x \otimes y_I) y_J$$

$$= F^{(k-1)}(x \otimes (y_1 \ldots y_k)')(y_1 \ldots y_k)'' + F(x \otimes 1) y_1 \ldots y_k.$$

We finally defined a map $F : V \otimes S(V) \longrightarrow K$, such that for all $x \in V$, $b \in S_+(V)$,

$$x \bullet b = F(x \otimes b') b'' + F(x \otimes 1) b.$$

We extend F in a map from $S(V) \otimes S(V) \longrightarrow S(V)$ by $F(1 \otimes b) = 0$ and, for all $x_1, \ldots, x_k \in V$:

$$F(x_1 \ldots x_k \otimes b) = \sum_{i=1}^{k} x_1 \ldots x_{i-1} F(x_1 \otimes b) x_{i+1} \ldots x_k.$$

This map F satisfies points 2 and 3. We consider:

$$B = \{a \in A \mid \forall b \in S_+(V), a \bullet b = F(a \otimes b') b'' + F(a \otimes 1) b\}.$$

As $1 \bullet b = 0$ for all $b \in S(V)$, $1 \in B$. By construction of F, $V \subseteq B$. Let $a_1, a_2 \in B$. For any $b \in S_+(V)$:

$$a_1 a_2 \bullet b = (a_1 \bullet b) a_2 + a_1 (a_2 \bullet b)$$
$$= F(a_1 \otimes b') a_2 b'' + a_1 F(a_2 \bullet b') b'' + F(a_1 \otimes 1) a_2 b + a_1 F(a_2 \otimes 1) b$$
$$= F(a_1 a_2 \otimes b') b'' + F(a_1 a_2 \otimes 1) b.$$

So $a_1 a_2 \in B$. We obtain that B is a subalgebra of $S(V)$ containing V, so is equal to $S(V)$: F satisfies the first point. $\qquad \square$

Remark 4.   1. In this case, for all primitive element v, the 1-cocycle of the coalgebra A defined by $L(x) = a \bullet x$ is the coboundary associated to the linear form sending x to $-F(a \otimes x)$.

   2. In particular, the preLie product of two elements x, y of $Prim(A)$ si given by:

$$x \bullet y = F(x \otimes 1) y.$$

Lemma 12. *With the preceding hypothesis, let us assume that $F(x \otimes 1) = 0$ for all $x \in Prim(A)$. Then $\bullet = 0$.*

Proof. We assume that $A = S(V)$ as a bialgebra. Note that for all $a, b \in S_+(V)$:

$$\tilde{\Delta}(a \bullet b) = a \bullet b' \otimes b'' + a' \bullet b' \otimes a'' b'' + a' \bullet b \otimes a'' + a' \otimes a'' \bullet b.$$

Let us prove the following assertion by induction on N: for all $k < N$, for all $x, y_1, \ldots, y_k \in V$, $x \bullet y_1 \ldots y_k = 0$. By hypothesis, this is true for $N = 1$. Let us assume the result at a certain rank $N \geq 2$. Let us choose $x, y_1, \ldots, y_N \in V$. Then, by the condition on N:

$$\tilde{\Delta}(x \bullet y_1 \ldots y_N) = 0 + 0 + 0 + 0 = 0.$$

So $x \bullet y_1 \ldots y_N$ is primitive.

Up to a factorization, we can write any $x \bullet y_1 \ldots y_N$ as a linear span of terms of the form $z_1 \bullet z_1^{\beta_1} \ldots z_n^{\beta_n}$, with $z_1, \ldots, z_n$ linearly independent, $\beta_1, \ldots, \beta_n \in \mathbb{N}$, with $\beta_1 + \ldots + \beta_n = N$. If $n = 1$, as $dim(V) \geq 2$ we can choose any z_2 linearly independent with z_1 and take $\beta_2 = 0$. It is

now enough to consider $z_1 \bullet z_1^{\beta_1} \ldots z_n^{\beta_n}$, with $n \geq 2$, $z_1, \ldots, z_n$ linearly independent, $\beta_1, \ldots, \beta_n \in \mathbb{N}$, $\beta_1 + \ldots + \beta_n = N$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$, such that $\alpha_1 + \ldots + \alpha_n = N + 1$.

$$\tilde{\boldsymbol{\Delta}}(z_1 \bullet z_1^{\alpha_1} \ldots z_n^{\alpha_n}) = \sum_{i=1}^{n} \alpha_i z_1 \bullet z_1^{\alpha_1} \ldots z_i^{\alpha_i-1} \ldots z_n^{\alpha_n} \otimes z_i,$$

$$\tilde{\boldsymbol{\Delta}}\left(\frac{z_1^2}{2} \bullet z_1^{\alpha_1} \ldots z_n^{\alpha_n}\right) = \sum_{i=1}^{n} \alpha_i (z_1 \bullet z_1^{\alpha_1} \ldots z_i^{\alpha_i-1} \ldots z_n^{\alpha_n}) z_1 \otimes z_i$$
$$+ \sum_{i=1}^{n} \alpha_i z_1 \bullet z_1^{\alpha_1} \ldots z_i^{\alpha_i-1} \ldots z_n^{\alpha_n} \otimes z_1 z_i$$
$$+ z_1 \bullet z_1^{\alpha_1} \ldots z_n^{\alpha_n} \otimes z_1 + z_1 \otimes z_1 \bullet z_1^{\alpha_1} \ldots z_n^{\alpha_n},$$

$$(\tilde{\boldsymbol{\Delta}} \otimes Id) \circ \tilde{\boldsymbol{\Delta}}\left(\frac{z_1^2}{2} \bullet z_1^{\alpha_1} \ldots z_n^{\alpha_n}\right) = \sum_{i=1}^{n} \alpha_i z_1 \bullet z_1^{\alpha_1} \ldots z_i^{\alpha_i-1} \ldots z_n^{\alpha_n} \otimes z_1 \otimes z_i$$
$$+ \sum_{i=1}^{n} \alpha_i z_1 \otimes z_1 \bullet z_1^{\alpha_1} \ldots z_i^{\alpha_i-1} \ldots z_n^{\alpha_n} \otimes z_i$$
$$+ \sum_{i} \alpha_i z_1 \bullet z_1^{\alpha_1} \ldots z_i^{\alpha_i-1} \ldots z_n^{\alpha_n} \otimes z_i \otimes z_1.$$

The cocommutativity implies that for all $1 \leq i \leq n$, $\alpha_i z_1 \bullet z_1^{\alpha_1} \ldots z_i^{\alpha_i-1} \ldots z_n^{\alpha_n}$ and z_i are colinear. We first choose $\alpha_1 = \beta_1 + 1$, $\alpha_i = \beta_i$ for all $i \geq 2$, and we obtain for $i = 1$ that $z_1 \bullet z_1^{\beta_1} \ldots z_n^{\beta_n} \in Vect(z_1)$. We then choose $\alpha_n = \beta_n + 1$ and $\alpha_i = \beta_i$ for all $i \leq n - 1$, and we obtain for $i = n$ that $z_1 \bullet z_1^{\beta_1} \ldots z_n^{\beta_n} \in Vect(z_n)$. Finally, as $n \geq 2$, $z_1 \bullet z_1^{\beta_1} \ldots z_n^{\beta_n} \in Vect(z_1) \cap Vec(z_2) = (0)$; the hypothesis is true at trank N.

We proved that for all $x \in V$, for all $b \in S(V)$, $x \bullet b = 0$. By the derivation property of $\bullet$, as V generates $S(V)$, for all $a, b \in S(V)$, $a \bullet b = 0$. $\square$

Lemma 13. *Under the preceding hypothesis, Let us assume that $F(Prim(A) \otimes \mathbb{K}) \neq (0)$. Then A is isomorphic to a certain $S(Prim(A), f, \lambda)$, with $f(x) = F(x \otimes 1)$ for all $x \in V$.*

Proof. We assume that $A = S(V)$ as a bialgebra. Let $a, b, c \in S_+(V)$. Then:

$$\tilde{\boldsymbol{\Delta}}([a,b]) = a' \otimes a'' \bullet b + a \bullet b' \otimes b'' + a' \bullet b \otimes a''$$
$$- b' \otimes b'' \bullet a - b \bullet a' \otimes a'' - b' \otimes a \otimes b'' + [a', b'] \otimes a''b'',$$

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where $[-,-]$ is the Lie bracket associated to $\bullet$. Hence:

$$
\begin{aligned}
(a \bullet b) \bullet c &= F(a \otimes 1)b \bullet c + F(a \otimes b')b'' \bullet c \\
&= F(a \otimes 1)F(b \otimes 1)c + F(a \otimes 1)F(b \otimes c')c'' \\
&\quad + F(a \otimes b')F(b'' \otimes 1)c + F(a \otimes b')F(b'' \otimes c')c'',
\end{aligned}
$$

$$
\begin{aligned}
(a \bullet c) \bullet b &= F(a \otimes 1)F(c \otimes 1)b + F(a \otimes 1)F(c \otimes b')b'' \\
&\quad + F(a \otimes c')F(c'' \otimes 1)b + F(a \otimes c')F(c'' \otimes b')b'',
\end{aligned}
$$

$$
\begin{aligned}
a \bullet [b,c] &= F(a \otimes 1)F(b \otimes 1)c + F(a \otimes 1)F(b \otimes c')c'' - F(a \otimes 1)F(c \otimes 1)b \\
&\quad - F(a \otimes 1)F(c \otimes b')b'' + F(a \otimes b')F(b'' \otimes 1)c + F(a \otimes b')F(b'' \otimes c')c'' \\
&\quad - F(a \otimes c')F(c'' \otimes 1)b - F(a \otimes c')F(c'' \otimes b')b'' + F(a \otimes F(b \otimes 1)c')c'' \\
&\quad + F(a \otimes F(b \otimes c')c'')c''' - F(a \otimes F(c \otimes 1)b')b'' - F(a \otimes F(c \otimes b')b'')b''' \\
&\quad + F(a \otimes F(b' \otimes 1)c)b'' + F(a \otimes F(b' \otimes c')c'')b'' - F(a \otimes F(c' \otimes 1)b)c'' \\
&\quad + F(a \otimes F(c' \otimes b')b'')c'' + F(a \otimes F(b' \otimes 1)c')b''c'' + F(a \otimes F(b' \otimes c')c'')b''c''' \\
&\quad - F(a \otimes F(c' \otimes 1)b')b''c''' - F(a \otimes F(c' \otimes b')b'')b'''c''.
\end{aligned}
$$

The preLie identity implies that:

$$
\begin{aligned}
0 &= F(a \otimes F(b \otimes 1)c')c'' + F(a \otimes F(b \otimes c')c'')c''' - F(a \otimes F(c \otimes 1)b')b'' \\
&\quad - F(a \otimes F(c \otimes b')b'')b''' + F(a \otimes F(b' \otimes 1)c)b'' + F(a \otimes F(b' \otimes c')c'')b'' \\
&\quad - F(a \otimes F(c' \otimes 1)b)c'' + F(a \otimes F(c' \otimes b')b'')c'' + F(a \otimes F(b' \otimes 1)c')b''c'' \\
&\quad + F(a \otimes F(b' \otimes c')c'')b''c''' - F(a \otimes F(c' \otimes 1)b')b''c''' - F(a \otimes F(c' \otimes b')b'')b'''c''.
\end{aligned}
$$

For $a = x \in V$, $b = y \in V$, as $F(V \otimes S(V)) \subset \mathbb{K}$, this simplifies to:

$$
F(x \otimes c')F(y \otimes 1)c'' + F(y \otimes c')F(x \otimes c'')c''' = F(x \otimes F(c' \otimes 1)y)c''. \tag{1}
$$

Let $x_1, \ldots, x_k \in V$, linearly independent, $\alpha_1, \ldots, \alpha_k \in \mathbb{N}$, with $\alpha_1 + \ldots + \alpha_N \geq 1$. We take $c = x_1^{\alpha_1+1} \ldots x_k^{\alpha_k}$ and $d = x_1^{\alpha_1} \ldots x_k^{\alpha_k}$. The coefficient of x_1 in (1), seen as an equality between two polynomials in $x_1, \ldots, x_k$, gives:

$$
(\alpha_1 + 1)(F(x \otimes d)F(y \otimes 1) + F(y \otimes d')F(x \otimes d'')) = (\alpha_1 + 1)F(x \otimes F(d \otimes 1)y).
$$

Hence, for all $x, y \in V$, for all $c \in S_+(V)$:

$$
F(x \otimes c)F(y \otimes 1) + F(y \otimes c')F(x \otimes c'') = F(x \otimes F(c \otimes 1)y). \tag{2}
$$

We put $f(x) = F(x \otimes 1)$ for all $x \in V$. If $z_1, \ldots, z_k \in Ker(g)$, then:

$$
F(z_1 \ldots z_k \otimes 1) = \sum_{i=1}^{k} z_1 \ldots g(z_i) \ldots z_k = 0.
$$

Consequently, if $c \in S_+(Ker(f)) \subseteq S_+(V)$, (2) gives:

$$
F(x \otimes c')F(y \otimes 1) + F(y \otimes c')F(x \otimes c'') = 0.
$$

Let us choose y such that $F(y \otimes 1) = 0$. An easy induction on the length of c proves that for all $c \in S_+(Ker(g))$, $F(x \otimes c) = 0$ for all $x \in V$. So there exists linear forms $g_k \in V^*$, such that for all $x, y_1, \ldots, y_k \in V$:

$$
F(x \otimes y_1 \ldots y_k) = g_k(x)f(y_1) \ldots f(y_k).
$$

In particular, $h_0 = f$. The preLie product is then given by:

$$x \bullet y_1 \ldots y_k = \sum_{i=1}^{k-1} g_i(x) \sum_{1 \le j_1 < \ldots < j_i \le k} y_1 \ldots f(y_{j_1}) \ldots f(y_{j_i}) \ldots y_k.$$

Let $x, y, z_1, \ldots, z_k \in V$.

$$x \bullet (y \bullet z_1 \ldots z_k) = x \bullet \sum_{i=0}^{k-1} g_i(y) \sum_{j_1, \ldots, j_i} z_i \ldots f(z_{j-1}) \ldots f(z_{j_i}) \ldots z_l$$

$$= \sum_{i=0}^{k-1} g_{l-i-1}(x) g_i(x) \binom{l-1}{i} \sum_{j=1}^{k} f(z_1) \ldots f(z_{j-1}) z_j f(z_{j+1}) \ldots f(z_k) + S_{\geq 2}(V),$$

$$(x \bullet y) \bullet z_1 \ldots z_k = f(x) y \bullet z_1 \ldots z_k$$

$$= f(x) g_{k-1}(y) \sum_{j=1}^{k} f(z_1) \ldots f(z_{j-1}) z_j f(z_{j+1}) \ldots f(z_k) + S_{\geq 2}(V),$$

$$x \bullet (z_1 \ldots z_k \bullet y) = \sum_{i=1}^{k} f(y_i) x \bullet z_1 \ldots z_{i-1} z_{i+1} \ldots z_k y$$

$$= k g_{k-1}(x) f(z_1) \ldots f(z_k) y$$

$$+ (k-1) f(x) g_{k-1}(y) \sum_{j=1}^{k} f(z_1) \ldots f(z_{j-1}) z_j f(z_{j+1}) \ldots f(z_k) + S_{\geq 2}(V),$$

$$(x \bullet z_1 \ldots z_k) \bullet y = \sum_{i=0}^{k-1} g_i(x) \sum_{j_1, \ldots, j_i} z_1 \ldots f(z_{j_1}) \ldots f(z_{j_i}) \ldots z_k \bullet y$$

$$= k f(x) g_{k-1}(y) \sum_{j=1}^{k} f(z_1) \ldots f(z_{j-1}) z_j f(z_{j+1}) \ldots f(z_k) + S_{\geq 2}(V).$$

Let us choose $z_1 = \ldots = z_k$ such that $f(z) = 1$. Then:

$$\sum_{j=1}^{k} f(z_1) \ldots f(z_{j-1}) z_j f(z_{j+1}) \ldots f(z_k) = kz \neq 0.$$

The preLie identity implies:

$$f(x) g_{k-1}(y) + (k-1) g_{k-1}(x) f(y) - \sum_{i=0}^{k-1} g_i(y) g_{k-i-1}(x) \binom{k-1}{i} = 0,$$

so, for all $l \geq 1$:

$$l g_l(x) f(y) = \sum_{i=1}^{l} g_i(y) g_{l_i}(x) \binom{l}{i}. \tag{3}$$

Let us choose x such that $f(x) = 1$. Let us consider $y \in Ker(f)$, let us prove that $g_i(y) = 0$ for all $i \geq 0$. As $g_0 = 0$, this is obvious for $i = 0$. Let us assume the result at all rank $< l$, with $l \geq 1$. Then (3) gives:

$$0 = \sum_{i=1}^{l-1} g_i(y) g_{l_i}(x) \binom{l}{i} + g_l(y) f(x) = g_l(y).$$

Consequently, for all $l \geq 1$, there exists a scalar λ_l such that $g_l = \lambda f$. Equation (3, for x, y such that $f(x) = f(y) = 1$, gives, for all $l \geq 1$:

$$l \lambda l = \sum_{i=1}^{l} \lambda_i \lambda_{l-i} \binom{l}{i} = \sum_{i=1}^{l-1} \lambda_i \lambda_{l-i} \binom{l}{i} + \lambda_l,$$

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so, for all $l \geq 2$:

$$\lambda_l = \frac{1}{l-1} \sum_{i=1}^{l-1} \lambda_i \lambda_{l-i} \binom{l}{i}.$$

An easy induction then proves that $\lambda_l = l! \lambda_1^l$ for all $l \geq 1$. Putting $\lambda_1 = \lambda$, for all $x, x_1, \ldots, x_n \in V$:

$$x \bullet x_1 \ldots x_k = \sum_{I \subsetneq \{1,\ldots,k\}} |I|! \lambda^{|I|} f(x) \prod_{i \in I} f(x_i) \prod_{i \notin I} x_i.$$

This is the preLie product of $S(V, f, \lambda)$. $\qquad\qquad\square$

4.2 Second case

We now assume that V is one-dimensional. Then $S(V)$ and $\mathbb{K}[X]$ are isomorphic as bialgebras. Let us describe all the preLie products on $\mathbb{K}[X]$ making it a Com-PreLie bialgebra.

Proposition 14. *Let $\lambda, \mu \in \mathbb{K}$. We define:*

$$X^k \bullet X^l = \lambda k l! \sum_{i=k}^{k+l-1} \frac{\mu^{k+l-i-1}}{(i-k+1)!} X^i.$$

Then $(\mathbb{K}[X], m, \prec, \Delta)$ is a Zinbiel-PreLie algebra, denoted by $\mathfrak{g}'(\lambda, \mu)$.

Proof. This is obvious if $\lambda = 0$. Let us assume that $\lambda \neq 0$. We consider a one-dimensional vector space V, with basis (x), $f \in V^*$ defined by $f(x) = \lambda$, in $S(V, f, \mu/\lambda)$, for any $k, l \geq 0$:

$$x^k \bullet x^l = k x^{k-1} x \bullet x^l$$

$$= \sum_{j=0}^{l-1} \binom{l}{j} j! k \frac{\mu^j}{\lambda^j} \lambda^{j+1} x^{k+l-j-1}$$

$$= \lambda k l! \sum_{j=0}^{l-1} \frac{\mu^j}{(l-j)!} x^{k \circ l - j - 1}$$

$$= \lambda k l! \sum_{i=k}^{k+l-1} \frac{\mu^{k+l-i-1}}{(i-k+1)!} x^i.$$

Using the Hopf algebra morphim from $S(V)$ to $\mathbb{K}[X]$ sending x to X, we obtain that $(\mathbb{K}[X], m, \bullet, \Delta)$ is a Com-PreLie bialgebra. Let us now prove the Leibniz rule for the $\prec$ product. If $k, l \geq 1$ and $m \geq 0$:

$$(X^k \bullet X^m) \prec X^l + X^k \prec (X^l \bullet X^m)$$

$$= \lambda m! \left(\sum_{i=k}^{k+m-1} \frac{\mu^{k+m-i-1}}{(i-k+1)!} \frac{ki}{i+l} X^{i+l} + \sum_{i=l}^{l+m-1} \frac{\mu^{l+m-i-1}}{(i-l+1)!} \frac{kl}{k+i} X^{k+1} \right)$$

$$= \lambda m! \sum_{i=k+l}^{k+l+m-1} \frac{\mu^{k+l+m-i-1}}{(i-k-l+1)!} \frac{k(i-l)+kl}{i} X^i$$

$$= \lambda m! k \sum_{i=k+l}^{k+l+m-1} \frac{\mu^{k+l+m-i-1}}{(i-k-l+1)!} X^i$$

$$= \frac{k}{k+l} \lambda m! (k+l) \sum_{i=k+l}^{k+l+m-1} \frac{\mu^{k+l+m-i-1}}{(i-k-l+1)!} X^i$$

$$= \frac{k}{k+l} X^{k+l} \bullet X^m$$

$$= (X^k \prec X^l) \bullet X^m.$$

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So $(\mathbb{K}[X], \prec, \bullet, \Delta)$ is a Zinbiel-PreLie bialgebra. $\qquad\qquad\square$

Proposition 15. *Let $\bullet$ a preLie product on $\mathbb{K}[X]$ such that $(\mathbb{K}[X], m, \bullet, \Delta)$ is a Com-PreLie bialgebra. Then $(\mathbb{K}[X], m, \bullet, \Delta) = \mathfrak{g}^{(1)}(1, \lambda, 1)$ for a certain $\lambda \in \mathbb{K}$, or $\mathfrak{g}'(\lambda, \mu)$ for a certain $(\lambda, \mu) \in \mathbb{K}^2$.*

Proof. Let $\pi : \mathbb{K}[X] \longrightarrow \mathbb{K}[X]$ be the canonical projection on $Vect(X)$:

$$
\pi : \begin{cases} \mathbb{K}[X] & \longrightarrow \mathbb{K}[X] \\ X^k & \longrightarrow \delta_{k,1} X. \end{cases}
$$

For all $k \geq 0$, we put $\pi(X \bullet X^k) = \lambda_k X$.

We shall use the map $\varpi = m \circ (\pi \otimes Id) \circ \Delta$. For all $k \geq 0$:

$$
\varpi(X^k) = m \circ (\pi \otimes Id) \left(\sum_{i=0}^{k} \binom{k}{i} X^i \otimes X^{k-i} \right) = m(kX \otimes X^{k-1}) = kX^k.
$$

First step. We fix $l \geq 0$. For all $P, Q \in \mathbb{K}[X]$, $\varepsilon(P \bullet Q) = 0$; hence, we can write $X \bullet X^l = \sum_{i=1}^{\infty} a_i X^i$.

$$
\begin{aligned}
\varpi(X \bullet X^l) &= \sum_{i=1}^{\infty} i a_i X^i \\
&= m \circ (\pi \otimes Id) \circ \Delta(X \bullet X^l) \\
&= m \circ (\pi \otimes Id) \left(1 \otimes X \bullet X^l + \sum_{i=0}^{l} \binom{l}{i} X \bullet X^i \otimes X^{l-i} \right) \\
&= m \left(\sum_{i=0}^{l} \binom{l}{i} \lambda_i X \otimes X^{l-i} \right) \\
&= \sum_{i=0}^{l} \binom{l}{i} \lambda_i X^{l-i+1} \\
&= \sum_{j=1}^{l+1} \binom{l}{l-j+1} \lambda_{l-j+1} X^j.
\end{aligned}
$$

Hence:

$$
X \bullet X^l = \sum_{j=1}^{l+1} \binom{l}{l-j+1} \frac{\lambda_{l-j+1}}{j} X^j.
$$

By the Leibniz axiom, for all $k \geq 0$, $X^k \bullet X^l = kX^{k-1}(X \bullet X^l)$, so for all $k, l \geq 0$:

$$
X^k \bullet X^l = \sum_{j=1}^{l+1} k \binom{l}{l-j+1} \frac{\lambda_{l-j+1}}{j} X^{j+k-1}.
$$

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Second step. In particular, for all $k \geq 0$, $X^k \bullet 1 = k\lambda_0 X^k$, and $X \bullet X = \frac{\lambda_0}{2} X^2 + \lambda_1 X$. Hence:

$$X \bullet (X \bullet 1) - (X \bullet X) \bullet 1 = \frac{\lambda_0^2}{2} X^2 + \lambda_0 \lambda_1 X - \frac{\lambda_0}{2} X^2 \bullet 1 - \lambda_1 X \bullet 1$$

$$= \frac{\lambda_0^2}{2} X^2 + \lambda_0 \lambda_1 X - \lambda_0^2 X^2 - \lambda_0 \lambda_1 X$$

$$= -\frac{\lambda_0^2}{2} X^2;$$

$$X \bullet (1 \bullet X) - (X \bullet 1) \bullet X = 0 - \lambda_0 X \bullet X$$

$$= -\frac{\lambda_0^2}{2} X^2 - \lambda_0 \lambda_1 X.$$

By the preLie identity, $\lambda_0 \lambda_1 = 0$. We shall now study three subcases:

1. $\lambda_0 \neq 0$; **2.** $\lambda_0 = 0$, $\lambda_1 = 0$; **3.** $\lambda_0 = 0$, $\lambda_1 \neq 0$.

Third step. First subcase: $\lambda_0 \neq 0$. Let us prove that $\lambda_k = 0$ for all $k \geq 1$ by induction on k. It is given by the second step if $k = 1$. Let us assume that $\lambda_1 = \ldots = \lambda_{k-1} = 0$. Then $X \bullet X^k = \frac{\lambda_0}{k+1} X^{k+1} + \lambda_k X$, and:

$$X \bullet (X^k \bullet 1) - (X \bullet X^k) \bullet 1 = k\lambda_0 \left(\frac{\lambda_0}{k+1} X^{k+1} + \lambda_k X \right) - \left(\frac{\lambda_0}{k+1} X^{k+1} + \lambda_k X \right) \bullet 1$$

$$= \frac{k}{k+1} \lambda_0^2 X^{k+1} + \lambda_0 \lambda_k X - \lambda_0^2 X^{k+1} - \lambda_0 \lambda_k X$$

$$= \frac{-1}{k+1} \lambda_0^2 X^{k+1};$$

$$X \bullet (1 \bullet X^k) - (X \bullet 1) \bullet X^k = 0 - \lambda_0 \left(\frac{\lambda_0}{k+1} X^{k+1} + \lambda_k X \right)$$

$$= \frac{-1}{k+1} \lambda_0^2 X^{k+1} - \lambda_0 \lambda_k X.$$

By the preLie identity, $\lambda_0 \lambda_k = 0$. As $\lambda_0 \neq 0$, $\lambda_k = 0$.

Finally, we obtain, by the first step, $X^k \bullet X^l = \lambda_0 \frac{k}{l+1} X^{k+l}$ for all $k, l \geq 0$. So this is the preLie product of $\mathbb{K}[X]_{\lambda_0}$.

Fourth step. Second subcase: $\lambda_0 = \lambda_1 = 0$. Let us prove that $\lambda_k = 0$ for all $k \geq 0$. It is obvious if $k = 0, 1$. Let us assume that $\lambda_0 = \ldots = \lambda_{k-1} = 0$, with $k \geq 2$. Then $X^i \bullet X^j = 0$ for all $j < k$, $i \geq 0$. Hence:

$$X \bullet (X^{k+1} \bullet X^{k-1}) = (X \bullet X^{k+1}) \bullet X^{k-1} = (X \bullet X^{k-1}) \bullet X^{k+1} = 0.$$

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By the preLie identity, $X \bullet (X^{k-1} \bullet X^{k+1}) = 0$. Moreover:

$$X \bullet (X^{k-1} \bullet X^{k+1}) = X \bullet \left(\sum_{j=1}^{k-2} \binom{k+1}{k+2-j}(k-1)\frac{\lambda_{k+2-j}}{j} X^{k+2-j} \right)$$

$$= X \bullet \left((k-1)\lambda_{k+1}X^{k-1} + (k+1)(k-1)\frac{\lambda_k}{2}X^k \right)$$

$$= 0 + \frac{(k-1)(k+1)}{2}\lambda_k X \bullet X^k$$

$$= \frac{(k-1)(k+1)}{2}\lambda_k \left(\sum_{j=1}^{k-1} \binom{k}{k+1-j}\frac{\lambda_{k+1-j}}{j} X^j \right)$$

$$= \frac{(k-1)(k+1)}{2}\lambda_k^2 X + 0.$$

Hence, $\lambda_k = 0$.

We finally obtain by the first step $X^k \bullet X^l = 0$ for all $k, l \geq 0$. So this is the trivial preLie product of $\mathbb{K}[X]_0 = \mathbb{K}[X]_{0,0}$.

Fifth step. Last subcase: $\lambda_0 = 0$, $\lambda_1 \neq 0$. Let us prove that $\lambda_k = \frac{k!}{2^{k-1}} \frac{\lambda_2^{k-1}}{\lambda_1^{k-2}}$ for all $k \geq 1$. It is obvious if $k = 1$ or $k = 2$. Let us assume the result at all rank $< k$, with $k \geq 2$.

$$\pi((X \bullet X) \bullet X^k) = \pi(\lambda_1 X \bullet X^k)$$
$$= \lambda_1 \lambda_k X;$$

$$\pi(X \bullet (X \bullet X^k)) = \pi \left(\sum_{j=1}^{k} \binom{k}{k+1-j}\frac{\lambda_{k+1-j}}{j} X \bullet X^j \right)$$

$$= \sum_{j=1}^{k} \binom{k}{k+1-j}\frac{\lambda_{k+1-j}\lambda_j}{j} X$$

$$= \left(\lambda_k \lambda_1 + \sum_{j=2}^{k-1} \frac{1}{j}\binom{k}{k+1-j}\frac{(k+1-j)!j!}{2^{k-j+j-1}}\frac{\lambda_2^{k-j+j-1}}{\lambda_1^{k-j-1+j-2}} + \frac{k}{k}\lambda_1\lambda_k \right) X$$

$$= \left(2\lambda_1\lambda_k + \sum_{j=2}^{k-1} \frac{k!}{2^{k-1}}\frac{\lambda_2^{k-1}}{\lambda_1^{k-3}} \right) X$$

$$= \left(2\lambda_1\lambda_k + (k-2)\frac{k!}{2^{k-1}}\frac{\lambda_2^{k-1}}{\lambda_1^{k-3}} \right) X;$$

$$\pi((X \bullet X^k) \bullet X) = \sum_{j=1}^{k} \binom{k}{k+1-j}\frac{\lambda_{k+1-j}}{j}\pi(X^j \bullet X)$$

$$= \sum_{j=1}^{k} \binom{k}{k+1-j}\frac{\lambda_{k+1-j}}{j}\pi(j\lambda_1 X^j)$$

$$= \lambda_1 \lambda_k X + 0;$$

$$\pi(X \bullet (X^k \bullet X)) = k\lambda_1 \pi(X \bullet X^k)$$
$$= k\lambda_1\lambda_k X.$$

By the preLie identity:

$$\lambda_1 \lambda_k - 2\lambda_1 \lambda_k - (k-2)\frac{k!}{2^{k-1}}\frac{\lambda_2^{k-1}}{\lambda_1^{k-3}} = \lambda_1 \lambda_k - k\lambda_1 \lambda_k,$$

which gives, as $\lambda_1 \neq 0$ and $k \geq 3$, $\lambda_k = \dfrac{k!}{2^{k-1}}\dfrac{\lambda_2^{k-1}}{\lambda_1^{k-2}}$. Finally, the first step gives, for all $k, l \geq 0$, with $\lambda = \lambda_1$ and $\mu = \frac{\lambda_2}{2\lambda_1}$:

$$
\begin{aligned}
X^k \bullet X^l &= \sum_{j=1}^{k+1} k\binom{l}{l+1-j}\frac{\lambda_{l+1-j}}{j} X^{j+k-1} \\
&= \sum_{j=1}^{k} k\frac{l!(l+1-j)!}{(l+1-j)!(j-1)!j2^{l-j}}\frac{\lambda_2^{l-j}}{\lambda_1^{l-1-j}} X^{j+k-1} \\
&= \lambda k l! \sum_{j=1}^{k}\frac{\mu^{l-j}}{j!} X^{j+k-1} \\
&= \lambda k l! \sum_{i=k}^{k+l-1}\frac{\mu^{k+l-i-1}}{(i-k+1)!} X^{i}.
\end{aligned}
$$

So this is the preLie product of $\mathfrak{g}'(\lambda, \mu)$. $\qquad\square$

As $\mathfrak{g}'(\lambda, \mu)$ is a special case of $S(V, f, \lambda)$, this ends the proof of Theorem 10. Let us describe the underlying Lie algebra of $\mathfrak{g}'(\lambda, \mu)$, in a similar way as Proposition 9. If $\lambda = 0$, it is abelian. Otherwise:

Proposition 16. *Let $\mathfrak{g} = \mathfrak{g}'(\lambda, \mu)$, with $\lambda \neq 0$. As a Lie algebra:*

$$\mathfrak{g} \approx (\mathfrak{g}_{FdB} \oplus_{deg} \mathbb{K}) \oplus \mathbb{K}.$$

Proof. Let us denote by $\mathfrak{g}_+$ the augmentation ideal of $\mathfrak{g}$. Obviously, $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathbb{K}$. For any $k \geq 1$, we put $Y_k = \dfrac{X^k}{\lambda}$. Then, for any $k, l \geq 1$:

$$
\begin{aligned}
Y_k \bullet Y_k &= kY_{k+l-1} + \text{terms of smaller degree}, \\
[Y_k, Y_l] &= (k-l)Y_{k+l-1} + \text{terms of smaller degree}.
\end{aligned}
$$

We put $f_1 = Y_1$. For any $n \geq 1$, we put $V_n = Vect(Y_1, \ldots, Y_n)$. The matrix of the endomorphism of V_n sending x to $[x, f_1]$ is triangular, with diagonal $(0, 1, \ldots, n-1)$. Consequently, it is diagonalizable, and for any $n \geq 1$, there exists a unique f_n of the form $f_n = Y_n + a_{n-1}Y_{n-1} + \ldots + a_1 Y_1$, such that $[f_n, f_1] = (n-1)f_n$. We obtain a basis $(f_n)_{n \geq 1}$ of $\mathfrak{g}_+$. For any $k, l \geq 1$:

$$
\begin{aligned}
[[f_k, f_l], f_1] &= [[f_k, f_1], f_l] + [f_k, [f_l, f_1]] \\
&= (k-1+l-1)[f_k, f_l],
\end{aligned}
$$

so there exists a scalar $\lambda(k, l)$, such that $[f_k, f_l] = \lambda(k, l)f_{k+l-1}$. Moreover:

$$
\begin{aligned}
[f_k, f_l] &= [Y_k, Y_l] + \text{terms of smaller degree} \\
&= (k-l)Y_{k+l-1} + \text{terms of smaller degree}.
\end{aligned}
$$

So $[f_k, f_l] = (k-l)f_{k+l-1}$. Let $\mathfrak{g}_1 = Vect(f_k, k \geq 2)$ and $\mathfrak{g}_2 = Vect(f_1)$. Both are Lie subalgebras of $\mathfrak{g}_+$ and, as a vector space, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. We put $e_k = f_{k+1}$ for any $k \geq 1$. Then, for any $k, l \geq 1$:

$$[e_k, e_l] = [f_{k+1}, f_{l+1}] = (k-1-l+1)f_{k+l+1} = (k-l)e_{k+l},$$

so $\mathfrak{g}_1$ is isomorphic to $\mathfrak{g}_{FdB}$. For any $k \geq 1$:

$$[e_k, f_1] = (k + 1 - 1)f_{k+1} = ke_k,$$

so $\mathfrak{g}_+ \approx \mathfrak{g}_{FdB} \oplus_{deg} \mathbb{K}$. $\qquad\square$

Corollary 17. *Let $\mathfrak{g} = S(V, f, \lambda)$ be a Com-PreLie algebra of Theorem 5. If f is nonzero, then, as a Lie algebra:*

$$\mathfrak{g} \approx ((\mathfrak{g}_{FdB} \oplus_{deg} \mathbb{K}) \oplus \mathbb{K}) \otimes S(Ker(f)),$$

where the Lie bracket is given in the following way: for any $x, x' \in (\mathfrak{g}_{FdB} \oplus_{deg} \mathbb{K}) \oplus \mathbb{K}$ and $y, y' \in S(Ker(f))$,

$$[x \otimes y, x' \otimes y'] = [x, x'] \otimes yy'.$$

Proof. Let $x_1 \in V$, such that $f(x_1) = 1$. Then, as an algebra, $\mathfrak{g} \equiv \mathbb{K}[x_1] \otimes S(Ker(f))$. Moreover, $\mathbb{K}[x_1]$ is a Com-Prelie bialgebra, isomorphic to $\mathfrak{g}'(1, \lambda)$. Moreover, if $k, l \in \mathbb{N}$ and $y, z \in S(Ker(f))$, by definition of the preLie product:

$$[x_1^k y, x_1^l z] = ((k - l)x_1^{k+l-1} + \text{terms of smaller degree})yz.$$

The end of the proof is similar to to the one of Proposition 16. $\qquad\square$

5 Com-PreLie structures on group Hopf algebras

Let G be an abelian group. The group algebra $\mathbb{K}G$ is a Hopf algebra, which product is given by the bilinear extension of the product of G and:

$$\forall g \in G, \qquad\qquad \Delta(g) = g \otimes g.$$

Lemma 18. *Let G be a group. For any $g, h \in G$, we put:*

$$P_{g,h} = \{x \in \mathbb{K}G \mid \Delta(x) = g \otimes x + x \otimes h\}.$$

Then $P_{g,h} = Vect(g - h)$.

Proof. Firstly:

$$\Delta(g - h) = g \otimes g - h \otimes h = g \otimes (g - h) + (g - h) \otimes h.$$

Let $x = \sum_{k \in G} a_k k \in P_{g,h}$. Then:

$$\Delta(x) = \sum_{k \in G} a_k k \otimes k = \sum_{k \in G} a_k(g \otimes k + k \otimes h).$$

Identifying in the basis $(k \otimes l)_{k,l \in G}$ of $\mathbb{K}G^{\otimes 2}$, we obtain that:

- If $k \notin \{g, h\}$, $a_k = 0$.

- If $g \neq h$, identifying the coefficients of $g \otimes h$, we obtain $0 = a_g + a_h$. In this case, $x = a_g(g - h)$.

- If $g = h$, identifying the coefficients of $g \otimes g$, we obtain $a_g = 2a_g$, so $a_g = 0$. In this case, $x = 0 = 0(g - g)$.

So $P_{g,h} = Vect(g - h)$. $\qquad\square$

5.1 General case

Theorem 19. *Let G be an abelian group and $\bullet$ be a bilinear product on $\mathbb{K}G$. Then $(\mathbb{K}G, m, \bullet, \Delta)$ is a Com-PreLie bialgebra if, and only if, there exists a family $(\lambda(g,h))_{g,h \in G}$ of scalars such that:*

- *For all $g \in G$, $\lambda(g, 1) = 0$.*

- *For all $g, h, k \in G$:*

$$\lambda(gh, k) = \lambda(g, k) + \lambda(h, k). \tag{4}$$

- *For all $g, h, k \in G$, such that $hk \neq 1$:*

$$\lambda(g, hk)\lambda(h, k) - \lambda(g, h)\lambda(h, k) = \lambda(g, hk)\lambda(k, h) - \lambda(g, k)\lambda(k, h). \tag{5}$$

- *For all $g, h \in G$:*

$$g \bullet h = \lambda(g, h)(g - gh). \tag{6}$$

Proof. Let us assume that $\bullet$ makes $\mathbb{K}G$ a Com-PreLie bialgebra. For any $g, h \in G$:

$$\Delta(g \bullet h) = g \otimes g \bullet h + g \bullet h \otimes gh.$$

By Lemma 18, there exists a scalar $\lambda(g, h)$ such that $g \bullet h = \lambda(g, h)(g - gh)$.

Let $\bullet$ be a product defined on $\mathbb{K}G$ by $g \bullet h = \lambda(g, h)(g - gh)$, for any $g, h \in G$. For any $g \in G$, $g \bullet h = 0$: we can assume that $\lambda(g, 1) = 0$. For any $g, h \in G$:

$$\Delta(g \bullet h) = g \otimes g \bullet h + g \bullet h \otimes gh,$$

so $\bullet$ satisfies the compatibility with Δ. Moreover:

> $\bullet$ satisfies the Leibniz identity
> $\iff \forall g, h, k \in G, \ (gh) \bullet k = (g \bullet k)h + g(h \bullet k)$
> $\iff \forall g, h, k \in G, \ \lambda(gh, k)(gh - ghk) = \lambda(g, k)(gh - ghk) + \lambda(h, k)(gh - ghk)$
> $\iff \forall g, h, k \in G, \ \lambda(gh, k)(gh - ghk) = (\lambda(g, k) + \lambda(h, k))(gh - ghk)$
> $\iff \forall g, h \in G, k \in G \setminus \{1\}, \ \lambda(gh, k) = \lambda(g, k) + \lambda(h, k).$

As $\lambda(g, 1) = 0$ for any $g \in G$, this identity is trivially satisfies if $k = 1$. Hence:

> $\bullet$ satisfies the Leibniz identity $\iff \forall g, h, k \in G, \ \lambda(gh, k) = \lambda(g, k) + \lambda(h, k).$

We now assume that $\bullet$ satisfies the Leibniz identity. Let $g, h, k \in G$.

$$
\begin{aligned}
(g \bullet h) \bullet k - g \bullet (h \bullet k) &= (\lambda(g, h)\lambda(g, k) - \lambda(g, h)\lambda(h, k) + \lambda(g, hk\lambda(h, k))g \\
&\quad - \lambda(g, h)\lambda(g, k)(gh + gk) + (\lambda(g, h)\lambda(g, k) \\
&\quad + \lambda(g, h)\lambda(h, k) - \lambda(g, hk)\lambda(h, k))ghk.
\end{aligned}
$$

Hence:

> $\bullet$ is preLie
> $\iff \forall g, h, k \in G, \ (g \bullet h) \bullet k - g \bullet (h \bullet k) = (g \bullet k) \bullet h - g \bullet (k \bullet h)$
> $\iff \forall g, h, k \in G, \ (\lambda(g, hk)\lambda(h, k) - \lambda(g, h)\lambda(h, k))(g - ghk)$
> $\qquad = (\lambda(g, hk)\lambda(k, h) - \lambda(g, k)\lambda(k, h))(g - ghk)$
> $\iff \forall g, h, k \in G$ with $hk \neq 1, \ \lambda(g, hk)\lambda(h, k) - \lambda(g, h)\lambda(h, k)$
> $\qquad = \lambda(g, hk)\lambda(k, h) - \lambda(g, k)\lambda(k, h).$

Consequently, $\bullet$ makes $\mathbb{K}G$ a Com-PreLie bialgebra if, and only if, the four conditions of Theorem 19 are satisfied. $\qquad \square$

Remark 5.   1. For any $h \in G$, $\lambda(-, h) : G \longrightarrow (\mathbb{K}, +)$ is a group morphism.

2. If $g \in G$ is an element of finite order n, then for any $h \in G$:

$$\lambda(g^n, h) = n\lambda(g, h) = \lambda(1, h) = 0.$$

As the characteristic of $\mathbb{K}$ is zero, $\lambda(g, h) = 0$. Consequently, if any element of G is of finite order, then the only product making $\mathbb{K}G$ a Com-PreLie bialgebra is 0.

Proposition 20. *Let G be an abelian group. Let $\lambda : G \longrightarrow (\mathbb{K}, +)$ be a group morphism and let $g_0 \in G$. We define a product $\bullet$ on $\mathbb{K}G$ by:*

$$\forall g, h \in G, \qquad\qquad g \bullet h = \lambda(g)\delta_{h,g_0}g(1 - g_0).$$

Then $(\mathbb{K}G, m, \bullet, \Delta)$ is a Com-PreLie bialgebra.

Proof. First, note that if $g_0 = 1$, then $\bullet = 0$. We now assume that $g_0 \neq 1$.

Here, $\lambda(g, h) = \lambda(g)\delta_{h,g_0}$ for any $g, h \in G$. Let us prove that the conditions of Theorem 19 are satisfied. let $g, h, k \in G$. Then:

$$\lambda(g, k) + \lambda(h, k) = \delta_{k,g_0}(\lambda(g) + \lambda(h)) = \delta_{k,g_0}\lambda(gh) = \lambda(gh, k).$$

As $\lambda(1) = 0$ and $g_0 \neq 1$:

$$\lambda(g, hk)\lambda(h, k) - \lambda(g, h)\lambda(h, k) = \lambda(g)\lambda(h)\delta_{k,g_0}(\delta_{hk,g_0} - \delta_{h,g_0})$$

$$= \begin{cases} -\lambda(g)\lambda(h) \text{ if } h = k = g_0, \\ \lambda(g)\lambda(h) \text{ if } h = 1 \text{ and } k = g_0, \\ 0 \text{ otherwise}; \end{cases}$$

$$= \begin{cases} -\lambda(g)\lambda(h) \text{ if } h = k = g_0, \\ 0 \text{ otherwise}; \end{cases}$$

$$= \lambda(g, hk)\lambda(k, h) - \lambda(g, k)\lambda(k, h).$$

So $(\mathbb{K}G, m, \bullet, \Delta)$ is indeed a Com-PreLie bialgebra. $\qquad\square$

5.2   Examples on $\mathbb{Z}$

Our aim here is the classification of all Com-PreLie bialgebra structures on $\mathbb{K}\mathbb{Z}$. In order to avoid confusion between the sum of $\mathbb{Z}$ and the sum of $\mathbb{K}\mathbb{Z}$, we identify $\mathbb{K}\mathbb{Z}$ with the Laurent polynomial ring $\mathbb{K}[X, X^{-1}]$, with the coproduct defined by $\Delta(X) = X \otimes X$.

Theorem 21. *There are three families of products $\bullet$ on $\mathbb{K}[X, X^{-1}]$ making it a Com-PreLie bialgebra:*

1. $\bullet = 0$.

2. *There exists $k_0 \in \mathbb{Z}$, nonzero, $a \in \mathbb{K}$, nonzero, such that:*

$$\forall k, l \in \mathbb{Z}, \qquad\qquad X^k \bullet X^l = a\delta_{l,k_0}k(X^k - X^{k+l}).$$

3. *There exists $\alpha, \beta \in \mathbb{K} \setminus \{0\}$, such that for all $n \neq -1$, $n\alpha - (n-1)\beta \neq 0$, there exists $N \geq 1$ such that:*

$$\forall k, l \in \mathbb{Z}, \qquad X^k \bullet X^l = \begin{cases} \dfrac{\alpha\beta}{\left(\frac{l}{N} - 1\right)\alpha - \left(\frac{l}{N} - 2\right)\beta} k(X^k - X^{k+l}) \text{ if } N \mid l, \\ 0 \text{ otherwise.} \end{cases}$$

Proof. We shall use Theorem 19. Let $\bullet$ be a product on $\mathbb{K}[X, X^{-1}]$, making it a Com-PreLie bialgebra. Then:

$$\forall k, l \in \mathbb{Z}, \qquad\qquad X^k \bullet X^l = \lambda(k,l)(X^k - X^{k+l}).$$

Moreover, for any $l \in \mathbb{Z}$, $\lambda(-, l) : (\mathbb{Z}, +) \longrightarrow (\mathbb{K}, +)$ is a group morphism so there exists a scalar a_l such that for any $k \in \mathbb{Z}$, $\lambda(k, l) = a_l k$. The conditions of Theorem 19 become:

- $a_0 = 0$.

- For all $h, k \in \mathbb{Z}$, such that $h + k \neq 0$:

$$
\begin{aligned}
& a_{h+k} a_k h - a_k a_h h = a_{h+k} a_h k - a_k a_h k \\
\Longleftrightarrow\ & a_{h+k}(a_k h - a_h k) = a_h a_k (h - k).
\end{aligned}
\tag{7}
$$

Let us assume that there exists $n \geq 1$, such that $a_n \neq 0$. Let $N = \min\{n \geq 1, a_n \neq 0\}$. Let us prove that for all $n \geq 1$, if $a_n \neq 0$, then $N \mid n$. Let us write $n = Nq + r$, with $0 \leq r \leq N - 1$. We proceed by induction on q. By definition of N, $a_0 = a_1 = \ldots = a_{N-1} = 0$, so this proves the result for $q = 0$. Let us assume the result at rank $q - 1$. If $r \neq 0$, then by the induction hypothesis, $a_{(q-1)N+r} = 0$. Hence, by (7) for $h = (q-1)N + r$ and $k = N$:

$$a_{qN+r} a_N ((q-1)N + r) = 0.$$

As $a_N \neq 0$, $a_{qN+r} = 0$.

Let us put $b_n = a_{Nn}$ for all $n \geq 1$. Then $b_1 \neq 0$ and, for all $h, k \geq 1$:

$$b_{h+k}(b_k h - b_h k) = b_h b_k (h - k).\tag{8}$$

Let us assume that $b_2 = 0$. We prove that $b_n = 0$ for all $n \geq 2$ by induction on n. This is obvious if $n = 2$. If $b_n = 0$, with $n \geq 2$, by (8), with $h = n$ and $k = 1$, $b_{n+1} b_1 n = 0$. As $b_1 \neq 0$, $b_{n+1} = 0$.

Let us assume that $b_2 \neq 0$. Let us show that for any $n \geq 1$, $(n-1)b_1 \neq (n-2)b_2$ and:

$$b_n = \frac{b_1 b_2}{(n-1)b_1 - (n-2)b_2}.$$

This is obvious if $n = 1$ or $n = 2$, as $b_1, b_2 \neq 1$. Let us assume the result at rank n, $n \geq 2$. By (8) with $h = n$, $k = 1$:

$$
\begin{aligned}
b_{n+1}\left(b_1 n - \frac{b_1 b_2}{(n-1)b_1 - (n-2)b_2}\right) &= b_1 \frac{b_1 b_2}{(n-1)b_1 - (n-2)b_2}(n-1) \\
b_{n+1}(n(n-1)b_1 - (n-1)^2 b_2) &= b_1 b_2 (n-1) \\
b_{n+1}(n b_1 - (n-1)b_2) &= b_1 b_2.
\end{aligned}
$$

As $b_1, b_2 \neq 0$, $n b_1 - (n-1)b_2 \neq 0$, so:

$$b_{n+1} = \frac{b_1 b_2}{n b_1 - (n-1)b_2}.$$

We proved that there are three possibilities for $(a_n)_{n \geq 1}$:

1. For all $n \geq 1$, $a_n = 0$.

2. There exists a unique $N \geq 1$ such that $a_N \neq 0$.

3. There exists $N \geq 1$, $\alpha, \beta \neq 0$ such that for all $n \geq 0$, $n\alpha - (n-1)\beta \neq 0$ and:

$$a_n = \begin{cases} \dfrac{\alpha\beta}{\left(\frac{n}{N} - 1\right)\alpha - \left(\frac{n}{N} - 2\right)\beta} & \text{if } N \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, there are three possibilities for $(a_{-n})_{n \geq 1}$:

1.' For all $n \geq 1$, $a_{-n} = 0$.

2.' There exists a unique $N' \geq 1$ such that $a_{-N'} \neq 0$.

3.' There exists $N' \geq 1$, $\alpha', \beta' \neq 0$ such that for all $n \geq 0$, $n\alpha' - (n-1)\beta' \neq 0$ and:

$$a_{-n} = \begin{cases} \dfrac{\alpha'\beta'}{\left(\frac{n}{N'} - 1\right)\alpha' - \left(\frac{n}{N'} - 2\right)\beta'} & \text{if } N' \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

If (2. or 3.) and (2.' or 3.') is satisfied, let us assume that $N \neq N'$. For example, we assume that $N' > N$. By (7) with $h = -N'$ and $k = N$, then:

$$a_{N-N'}(-N'a_N - Na_{-N'}) = a_{-N'}a_N(-N' - N) \neq 0.$$

So $a_{N-N'} \neq 0$: this contradicts the definition of $N' = \min\{n \geq 1, a_{-n} \neq 0\}$. So $N = N'$.

Let us assume that ((1'. or 2') and 3.) is satisfied. By (7) for $h = 3N$ and $k = -2N$:

$$a_N(2Na_{3N}) = 0.$$

We obtain $a_N = 0$, so $\alpha = 0$: contradiction. So ((1'. or 2') and 3.) is impossible. Similarly, (1. or 2) and 3'.) is impossible. If (2. and 2.') is satisfied, by (7) for $h = N$ and $k = -2N$:

$$a_{-N}(2Na_N) = 0.$$

As a_{-N} and a_N are both nonzero, this is a contradiction. So (2. and 2.') is impossible.

It remains the following cases:

1. If (1. and 1.') is satisfied, then $\bullet = 0$.

2. If (1. and 2.') or (1.' and 2.) is satisfied, this is the second case of Theorem 21.

3. If (3. and 3.') is satisfied, by (7) with $h = 2N$ and $k = -N$, $h = 3N$ and $k = -2N$, we obtain:

$$\alpha' = \frac{-\alpha\beta}{2\alpha - 3\beta}, \qquad\qquad \beta' = \frac{-\alpha\beta}{3\alpha - 4\beta}.$$

Hence, if $n \geq 1$:

$$a_{-nN} = \frac{\alpha\beta}{(-n-1)\alpha - (-n-2)\beta}.$$

This is the third case of Theorem 21.

It remains to prove that this three cases give indeed Com-PreLie bialgebras. It is obvious in the first case. The second case is Proposition 20. The third case is left to the reader. $\qquad\square$

Remark 6. In the third case, if $-\alpha + 2\beta = 0$, the formula for $X^k \bullet X^0$ is not well-defined: by convention, $X^k \bullet 1 = 0$.

6 Examples of non connected Com-PreLie bialgebras

Let H be a commutative and cocommutative Hopf algebra. If $\mathbb{K}$ is an algebraically closed field of characteristic zero, denoting by G the group of group-like elements of H and by V the space of its primitive elements, then H is isomorphic to $\mathbb{K}G \otimes S(V)$, which we shortly denote as $\mathbb{K}G \cdot S(V)$. We now look for products on $\mathbb{K}G \otimes S(V)$, making it a Com-PreLie bialgebra.

6.1 Several lemmas on $\mathbb{K}G \cdot S(V)$

Lemma 22. *A 1-cocycle of $S(V)$ is a linear map $\phi : S(V) \longrightarrow S(V)$ such that for all $x \in S(V)$:*

$$\Delta \circ \phi(x) = 1 \otimes \phi(x) + (\phi \otimes Id) \circ \Delta(x).$$

1. If $\dim(V) \geq 2$, for any 1-cocycle ϕ of $S(V)$, there exists $\lambda \in \mathbb{K}$ and $F : S^+(V) \longrightarrow \mathbb{K}$ such that:

$$\phi(1) = 0,$$
$$\forall x \in S^+(V), \qquad \phi(x) = \lambda x + (F \otimes Id) \circ \tilde{\Delta}(x). \qquad (9)$$

2. If $\dim(V) = 1$, let (X) be a basis of V. There exists scalars a, λ and a map $F : S^+(V) \longrightarrow \mathbb{K}$ such that:

$$\phi(1) = aX,$$
$$\forall n \geq 1, \qquad \phi(X^n) = a\frac{X^{n+1}}{n+1} + \lambda X^n + \sum_{i=1}^{n-1}\binom{n}{i}F(X^i)X^{n-i}.$$

Proof. First step. Let $\lambda \in \mathbb{K}$ and $F : S^+(V) \longrightarrow \mathbb{K}$ be a map. We consider the map ϕ defined on $S(V)$ by (9). Let us prove that it is a 1-cocycle. If $x = 1$:

$$\Delta \circ \phi(1) = 0 = 1 \otimes \phi(1) + \phi(1) \otimes 1.$$

If $x \in S^+(V)$, then:

$$1 \otimes \phi(x) + (\phi \otimes Id) \circ \Delta(x) = 1 \otimes \phi(x) + \phi(x) \otimes 1 + \lambda x' \otimes x'' + F(x')x'' \otimes x'''$$
$$= \Delta(\lambda x + F(x')x'')$$
$$= \Delta \circ \phi(x).$$

Second step. Let ϕ be a 1-cocycle and $k \geq 2$ such that $\phi_{|S^l(V)} = 0$ if $l < k$. We fix a basis $(e_i)_{i \in I}$ of V. Let $x = \prod e_i^{\alpha_i} \in S^k(V)$, with $\sum \alpha_i = k$. Then:

$$\Delta \circ \phi(x) = 1 \otimes \phi(x) + \phi(x) \otimes 1 + \phi(1) \otimes x + \phi(x') \otimes x'' = \phi(x) \otimes 1 + 1 \otimes \phi(x),$$

so $\phi(x) \in V$. Let us take $j \in J$, we shall consider the linear form defined on $S^{k-1}(V)$ by:

$$\forall x = \prod_{i \in I} e_i^{\beta_i} \in S^{k-1}(V), \qquad F_j(x) = \frac{1}{\beta_j + 1}e_j^* \circ \phi(xe_j).$$

Let ϕ_j be the 1-cocycle defined in the first section with $\lambda = 0$ and $F = F_j$. Then:

$$\Delta \circ \phi(xe_j) = 1 \otimes \phi(xe_j) + \phi(xe_j) \otimes 1 + \sum_{i \in I} \alpha_i \phi\left(\frac{xe_j}{e_i}\right) \otimes e_i + \phi(x) \otimes e_j.$$

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As $S(V)$ is cocommutative:

$$(e_j^* \otimes Id) \circ \tilde{\Delta} \circ \phi(xe_j) = \sum_{i \in I} \alpha_i e_j^* \circ \phi \left(\frac{xe_j}{e_i} \right) e_i + e_j^* \circ \phi(x)e_j$$

$$= (Id \otimes e_j^*) \circ \tilde{\Delta} \circ \phi(xe_j) = (\alpha_j + 1)\phi(x).$$

Hence:

$$\phi(x) = \sum_{i \neq j} \frac{\alpha_i}{\alpha_j + 1} e_j^* \circ \phi \left(\frac{xe_j}{e_i} \right) e_i + e_j^* \circ \phi(x)e_j$$

$$= \sum_{i \in I} \alpha_i F_j \left(\frac{xe_i}{e_j} \right) e_i$$

$$= \phi_j(x).$$

We proved that if ϕ is a 1-cocycle such that $\phi_{|S^l(V)} = 0$ for any $l < k$, with $k \geq 2$, then there exists $F : S^{k-1}(V) \longrightarrow \mathbb{K}$ such that for any $x \in S^k(V)$:

$$\phi(x) = F(x') \otimes x''.$$

Third step. Let ϕ be a 1-cocyle such that $\phi(1) = 0$ and $\phi_{|V} = 0$. Let us construct $F_k : S^k(V) \longrightarrow \mathbb{K}$ by induction on k, $k \geq 0$, such that for any $x \in S^{k+1}(V)$:

$$\phi(x) = ((F_1 \oplus \ldots \oplus F_k) \otimes Id) \circ \tilde{\Delta}(x).$$

If $k = 0$, there is nothing to construct, as $\phi_{|V} = 0$. Let us assume $F_1, \ldots F_{k-1}$ constructed, with $k \geq 1$. Let ψ be the 1-cocycle associated in the first step to $\lambda = 0$ and $F_1 \oplus \ldots \oplus F_{k-1}$. For any $x \in S^l(V)$, with $l \leq k$, $\phi(x) = \psi(x)$ by the induction hypothesis. By the second step applied to $\phi - \psi$, there exists F_k such that for any $x \in S^{k+1}(V)$:

$$\phi(x) - \psi(x) = F_k(x')x''.$$

Hence, for any $x \in S^{k+1}(V)$:

$$\phi(x) = (F_1 \oplus \ldots \oplus F_k)(x')x''.$$

We proved that for any 1-cocycle ϕ such that $\phi(1) = 0$ and $\phi_{|V} = 0$, there exists $F : S^+(V) \longrightarrow \mathbb{K}$ such that for any $x \in S^+(V)$, then $\phi(x) = F(x')x''$.

Fourth step. Let us consider a 1-cocycle such that $\phi(1) = 0$. Let $x \in V$.

$$\Delta \circ \phi(x) = 1 \otimes \phi(x) + \phi(x) \otimes 1,$$

so $\phi(x) \in V$. If V is 1-dimensional, there exists λ such that $\phi_{|V} = \lambda Id_V$. If $dim(V) \geq 2$, for any $x \in V$:

$$\Delta \circ \phi(x^2) = 1 \otimes \phi(x^2) + \phi(x^2) \otimes 1 + 2\phi(x) \otimes x.$$

As $S(V)$ is cocommutative, $\phi(x)$ and x are colinear for any $x \in V$. Hence, there exists $\lambda \in \mathbb{K}$ such that $\phi_{|V} = \lambda Id_V$. Consequently, if ψ is the 1-cocycle associated to λ and $F = 0$, then $\psi(1) = \phi(1) = 0$ and $\phi_{|V} = \psi_{|V} = \lambda Id_V$. By the third step applied to $\phi - \psi$, there exists F such that (9) holds.

Fifth step. Let φ is a 1-cocycle of $S(V)$, with $dim(V) \geq 2$. Then:

$$\Delta \circ \phi(1) = \phi(1) \otimes 1 + 1 \otimes \phi(1),$$

so $\phi(1) \in V$. Let $x \in V$.

$$\Delta \circ \phi(x) = 1 \otimes \phi(x) + \phi(x) \otimes 1 + \phi(1) \otimes x.$$

As Δ is cocommutative, $\phi(1)$ and x are colinear, for any $x \in V$. As $dim(V) \geq 2$, $\phi(1) = 0$. Combined with the fourth step, we obtain point 1.

Last step. We prove the first assertion of point 2. Let us consider the map $\psi : S(V) \longrightarrow S(V)$, defined by:

$$\psi(X^n) = \frac{X^{n+1}}{n+1}.$$

It is a 1-cocycle of $S(V)$, with $V = Vect(X)$. If ϕ is 1-cocycle of $S(V)$, then $\phi(1)$ is a primitive element of $S(V)$, so belong to $Vect(X)$. Hence, there exists $a \in \mathbb{K}$ such that $\phi(1) = \lambda X$. Then $\phi - a\psi$ is a 1-cocycle of $S(V)$, vanishing on 1. We conclude with the fourth step. $\square$

Lemma 23. *In $\mathbb{K}G \cdot S(V)$:*

1. *Let $g, h \in G$, with $g \neq h$.*

$$\{x \in \mathbb{K}G \cdot S(V) \mid \Delta(x) = g \otimes x + x \otimes h\} = Vect(g - h).$$

2. *Let $g \in G$.*

$$\{x \in \mathbb{K}G \cdot S(V) \mid \Delta(x) - g \otimes x - x \otimes g \in (\mathbb{K}G \cdot S^+(G))^{\otimes 2}\} = gS^+(V)$$

Proof. For both points, the inclusion $\supseteq$ is trivial. Note that:

$$\mathbb{K}G \cdot S(V) = \mathbb{K}G \oplus \mathbb{K}G \cdot S^+(V) = \bigoplus_{k \in G} \mathbb{K}k \oplus \mathbb{K}G \cdot S^+(V).$$

For any $k \in G$, we denote by ϖ_k the canonical projection on $\mathbb{K}k$ in this direct sum.

1. Let $x \in \mathbb{K}G \cdot S(V)$, such that $\Delta(x) = g \otimes x + x \otimes h$. We put $\varpi_g(x) = \alpha g$. By cocommutativity:

$$(\varpi_g \otimes Id) \circ \Delta(x) = g \otimes x + \alpha g \otimes h = (Id \otimes \varpi) \circ \Delta(x) = \alpha g \otimes g + 0.$$

Hence, $x + \alpha h = \alpha g$, so $x = \alpha(g - h)$.

2. We put:

$$x = \sum_{h \in G} h x_h,$$

where $x_h \in S(V)$ for any $h \in G$. For any $h \in G$, let us put $\varpi_h(x) = \alpha_h h$.

$$\Delta(x) = \sum_{h \in G} h x_h^{(1)} \otimes h x_h^{(2)},$$
$$(\varpi_h \otimes Id) \circ \Delta(x) = \delta_{g,h} g \otimes x + \alpha_h h \otimes g = h \otimes x_h.$$

For $h \neq g$, we obtain $h x_h = \alpha_h g$, so $x_h = 0$ and $\alpha_h = 0$. Hence, $x = g x_g \in gS(V)$. For $h = g$, we obtain $g \otimes g x_g = g \otimes x + \alpha_g g \otimes g$, so $\alpha_g = 0$, and $x \in gS^+(V)$. $\square$

6.2 PreLie products on $\mathbb{K}G \cdot S(V)$

Proposition 24. *Let $\bullet$ be a product on $\mathbb{K}G \cdot S(V)$, making it a Com-PreLie bialgebra.*

1. *for $g \in G \setminus \{1\}$, for any $v \in S(V)$, for any $w \in S(V)$, $v \bullet gw = 0$.*

2. *$S(V)$ is a preLie subalgebra of $\mathbb{K}G \cdot S(V)$.*

3. *If $\dim(V) \geq 2$ or if the preLie product is nonzero on $S(V)$, then $\mathbb{K}G$ is a preLie subalgebra of $\mathbb{K}G \cdot S(V)$.*

Proof. 1. By the Leibniz identity, it is enough to prove it for $v \in V$. For any $k \geq 0$, let us prove by induction that for any $w \in S^l(V)$, with $l < k$, $v \bullet gw = 0$. It is trivial if $k = 0$. Let us assume the result at rank k, $k \geq 0$. Let $x \in S^k(V)$.

$$\Delta(v \bullet gw) = 1 \otimes v \bullet gw + v \bullet gw^{(1)} \otimes gw^{(2)}$$

$$= \begin{cases} 1 \otimes v \bullet gw + v \bullet gw \otimes g \text{ if } k = 0, \\ 1 \otimes v \bullet gw + v \bullet gw \otimes g + \underbrace{v \bullet g \otimes gw + v \bullet gw' \otimes w''}_{=0 \text{ by induction hypothesis}} \text{ if } k \geq 1. \end{cases}$$

By Lemma 23, there exists a scalar $\lambda(v \otimes w)$ such that $v \bullet gw = \lambda(v \otimes w)(g - 1)$.

Let $(e_i)_{i \in I}$ be a basis of V and $w = \prod e_i^{\alpha_i} \in S^k(V)$, with $\sum \alpha_i = k$. For any $i \in I$:

$$\Delta(v \bullet gwe_i) = 1 \otimes v \bullet gwe_i + v \bullet gwe_i \otimes g + \sum_{j \neq i} \alpha_j \lambda\left(v \otimes \frac{we_i}{e_j}\right)(g - 1) \otimes ge_j$$

$$+ (\alpha_i + 1)\lambda(v \otimes w)(g - 1) \otimes ge_i.$$

Let ϖ be the projector on $\mathbb{K}G \cdot S^+(V)$ which vanishes on $\mathbb{K}G$. By cocommutativity:

$$(\varpi \otimes Id) \circ \Delta(v \bullet gwe_i) = \varpi(g \otimes we_i) \otimes g$$

$$= (Id \otimes \varpi) \circ \Delta(v \bullet gwe_i) = \sum_{j \neq i} \alpha_j \lambda\left(v \otimes \frac{we_i}{e_j}\right)(g - 1) \otimes ge_j \quad + (\alpha_i + 1)\lambda(v \otimes w)(g - 1) \otimes ge_i.$$

Applying ϖ_g, as $\varpi_g \circ \varpi = 0$, we obtain:

$$0 = (\varpi_g \otimes Id) \circ (\varpi \otimes Id) \circ \Delta(v \bullet gwe_i)$$

$$= \sum_{j \neq i} \alpha_j \lambda\left(v \otimes \frac{we_i}{e_j}\right) g \otimes ge_j + (\alpha_i + 1)\lambda(v \otimes w)g \otimes ge_i.$$

Hence, $\lambda(v \otimes w) = 0$, so $v \bullet gw = 0$.

2. By the Leibniz identity, it is enough to prove that $v \bullet w \in S^+(V)$ for any $v \in V$, $w \in S^k(V)$, $k \geq 0$, by induction on k. If $k = 0$, then:

$$\Delta(v \bullet 1) = v \bullet 1 \otimes 1 + 1 \otimes v \bullet 1.$$

By Lemma 23, $v \bullet 1 \in S^+(V)$. Let us assume the result at all ranks $< k$, with $k \geq 1$. Then:

$$\Delta(v \bullet w) = v \bullet w \otimes 1 + 1 \otimes v \bullet w + \underbrace{v \bullet w' \otimes w''}_{\in S^+(V)^{\otimes 2} \text{ by the induction hypothesis}}.$$

By Lemma 23, $v \bullet w \in S^+(V)$.

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3. Let $g, h \in G$.

$$\Delta(g \bullet h) = g \otimes g \bullet h + g \bullet h \otimes gh.$$

If $h \neq 1$, by Lemma 23, $g \bullet h \in Prim(g - gh) \subseteq \mathbb{K}G$. If $h = 1$, then $g^{-1}(g \bullet 1)$ is a primitive element of $\mathbb{K}G \cdot S(V)$, so belong to V. Let us put $g \bullet 1 = gx_g$, with $x_g \in V$.

Let us assume that $dim(V) \geq 2$. For any $y \in V$:

$$\Delta(g \bullet y) = g \otimes g \bullet y + g \bullet y \otimes g + gx_g \otimes gy.$$

By cocommutativity, x_g and y are colinear, for any $y \in V$. As $dim(V) \geq 2$, necessarily $x_g = 0$, so $g \bullet 1 = 0 \in \mathbb{K}G$.

Let us assume that V is 1-dimensional, and that the restriction of $\bullet$ to $S(V)$ is nonzero. Up to an isomorphism, we replace $S(V)$ by $\mathbb{K}[X]$. By Proposition 15, there are two possibilities.

1. There exist $\lambda, \mu \in \mathbb{K}$, such that for any $k, l \in \mathbb{N}$:

$$X^k \bullet X^l = \lambda kl! \sum_{i=1}^{k+l-1} \frac{\mu^{k+l-i-1}}{(i-k+1)!} X^i.$$

2. There exists $\lambda \in \mathbb{K}$, such that for any $k, l \in \mathbb{N}$:

$$X^k \bullet X^l = \lambda \frac{k}{l+1} X^{k+l}.$$

As $\bullet$ is nonzero, in both cases $\lambda \neq 0$. For any $g \in G$, $g^{-1}(g \bullet 1) \in V = Vect(X)$: we put $g \bullet 1 = \alpha(g)gX$. If $g = 1$, as $1 \bullet 1 = 0$, $\alpha(g) = 0$. Let us assume that $g \neq 1$. Then, by the first point:

$$(X \bullet g) \bullet 1 = 0;$$

$$X \bullet (g \bullet 1) = \alpha(g)X \bullet X = \begin{cases} \alpha(g)\lambda X \text{ (first case)}, \\ \alpha(g)\dfrac{\lambda}{2}X^2 \text{ (second case)}; \end{cases}$$

$$(X \bullet 1) \bullet g = 0,$$

$$X \bullet (1 \bullet g) = 0.$$

By the preLie identity, as $\lambda \neq 0$, $\alpha(g) = 0$. So $g \bullet 1 = 0 \in \mathbb{K}G$. $\qquad\square$

Proposition 25. *Let $\bullet$ be a product on $\mathbb{K}G \cdot S(V)$, making it a Com-PreLie bialgebra. We assume that $\mathbb{K}G$ is a sub-preLie algebra of $\mathbb{K}G \cdot S(V)$.*

1. *For any $g \in G$, there exist $\lambda(g) \in \mathbb{K}$ and $F_g : S^+(V) \longrightarrow \mathbb{K}$ such that for any $v \in S^+(V)$:*

$$g \bullet v = \lambda(g)gv + F_g(v')gv''.$$

   Moreover, for any $g, h \in G$:

$$\lambda(gh) = \lambda(g) + \lambda(h), \qquad\qquad F_{gh} = F_g + F_h.$$

2. *Recall that $S(V)$ is a sub-preLie algebra of $\mathbb{K}G \cdot S(V)$; if it is equal to $S(V, f, \lambda)$, with $f \neq 0$, then for any $g \in G$, there exists $\mu(g) \in \mathbb{K}$ such that for any $x_1, \ldots, x_n \in V$:*

$$F_g(x_1 \ldots x_n) = \mu(g)n!\lambda^{n-1}f(x_1) \ldots f(x_n).$$

   Moreover, for any $g, h \in G$:

$$\mu(gh) = \mu(g) + \mu(h).$$

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3. If $(S(V), \bullet) = S(V, f, \lambda)$, with $f \neq 0$, then for any $g, h \in G \setminus \{1\}$, for any $v \in S^+(V)$:

$$g \bullet hv = -\lambda(g, h) ghv.$$

Proof. 1. Let us consider the map:

$$\phi_g : \begin{cases} S(V) & \longrightarrow & \mathbb{K}G \cdot S(V) \\ x & \longrightarrow & g^{-1}(g \bullet x). \end{cases}$$

As $\mathbb{K}G$ is a sub-preLie algebra, by Theorem 19, $g \bullet 1 = \lambda(g, 1)(g - g) = 0$, so $\phi_g(1) = 0$. Moreover, for any $v \in S(V)$:

$$\Delta \circ \phi(v) = (g^{-1} \otimes g^{-1})(g \otimes g \bullet v + g \bullet v^{(1)} \otimes gv^{(2)})$$
$$= 1 \otimes \phi_g(v) + \phi_g(v^{(1)}) \otimes v^{(2)}.$$

By Lemma 23, $\phi_g(v) \in S(V)$, and ϕ_g is a 1-cocycle of $S(V)$. Lemma 22 gives the existence of $\lambda(g)$ and F_g.

If $g, h \in G$, for any $v \in S^+(V)$:

$$gh \bullet v = \lambda(gh) gv + F_{gh}(v') gv''$$
$$= (g \bullet v)h + g(h \bullet v)$$
$$= (\lambda(g) + \lambda(h)) ghv + (F_g(v') + F_h(v')) ghv''.$$

Hence, $\lambda(gh) = \lambda(g) + \lambda(h)$ and $F_{gh} = F_g + F_h$.

2. Let $x, y \in S^+(V)$.

$$(g \bullet x) \bullet y = \lambda(g)^2 gxy + \lambda(g) F_g(y') gxy'' + \lambda(g) F_g(x') gx''y + F_g(x') F_g(y') gx''y''$$
$$+ \lambda(g) g(x \bullet y) + F_g(x') g(x'' \bullet y),$$
$$g \bullet (x \bullet y) = \lambda(g) g(x \bullet y) + F_g(x' \bullet y) gx'' + F_g(x \bullet y') gy'' + F_g(x' \bullet y') gx''y'' + F_g(x') g(x'' \bullet y).$$

The preLie identity implies that for any $x, y \in S^+(V)$:

$$F_g(x' \bullet y) x'' + F_g(x \bullet y') y'' + F_g(x' \bullet y') x'' y'' \tag{10}$$
$$= F_g(y' \bullet x) y'' + F_g(y \bullet x') x'' + F_g(y' \bullet x') x'' y''$$

Let $(e_i)_{i \in I}$ be a basis of V, such that there exists $i_0 \in I$, with $f(e_i) = \delta_{i, i_0}$ for any $i \in I$. By (10) for $x = e_{i_0}$, primitive element, for any $y \in S^+(V)$:

$$F_g(e_{i_0} \bullet y') y'' = F_g(y' \bullet e_{i_0}) y''.$$

Applying $e_{i_0}^*$ on both sides:

$$F_g\left(e_{i_0} \bullet \frac{\partial y}{\partial e_{i_0}}\right) = F_g\left(\frac{\partial y}{\partial e_{i_0}} \bullet e_{i_0}\right).$$

By surjectivity of $\frac{\partial}{\partial e_{i_0}}$, for any $z \in S^+(V)$:

$$F_g(z \bullet e_{i_0}) = F_g(z \bullet e_{i_0}).$$

Let us consider $z = \prod e_i^{\alpha_i}$, with $\sum \alpha_i = n \geq 1$. Then:

$$z \bullet e_{i_0} = \alpha_{i_0} z,$$

$$e_{i_0} \bullet z = \begin{cases} \displaystyle\sum_{k=0}^{\alpha_{i_0}} \frac{n!}{(n-k)!} \lambda^k e_{i_0}^{\alpha_{i_0}-k} \prod_{i \neq i_0} e_i^{\alpha_i} & \text{if } n \neq \alpha_{i_0}, \\[3mm] \displaystyle\sum_{k=0}^{\alpha_{i_0}-1} \frac{n!}{(n-k)!} \lambda^k e_{i_0}^{\alpha_{i_0}-k} \prod_{i \neq i_0} e_i^{\alpha_i} & \text{if } n = \alpha_{i_0}. \end{cases}$$

Noticing that $F_g(1) = 0$, we obtain that:

$$F_g\left((1 - \alpha_{i_0})z + \sum_{k=1}^{\alpha_{i_0}} \frac{n!}{(n-k)!}\lambda^k e_{i_0}^{\alpha_{i_0}-k}\prod_{i \neq i_0} e_i^{\alpha_i}\right) = 0. \tag{11}$$

We put $\mu(g) = F_g(e_{i_0})$. Let us prove that for any $n \geq 1$:

$$F_g(e_{i_0}^n) = n!\lambda^{n-1}\mu(g) = n!\lambda^{n-1}\mu(g)f(e_{i_0})^n.$$

This is obvious if $n = 1$. If the result is true at all ranks $1 \leq k < n$, we obtain from (11):

$$0 = (1-n)F_g(e_{i_0}^n) + \sum_{k=1}^{n-1} \frac{n!}{(n-k)!}\lambda^k(n-k)!\lambda^{n-k-1}\mu(g)$$
$$= (1-n)F_g(e_{i_0}^n) + (n-1)n!\lambda^{n-1}\mu(g).$$

This implies the result at rank n.

Let us consider $j \in I \setminus \{i_0\}$. By (10), for $x = e_j$, primitive, for any $y \in S^+(V)$:

$$F_g(y' \bullet e_j)y'' = 0.$$

Applying e_j^*, for any $y \in S^+(V)$:

$$F_g\left(\frac{\partial y}{\partial e_j} \bullet e_j\right) = 0.$$

By surjectivity of $\frac{\partial}{\partial e_j}$, for any $z \in S^+(V)$:

$$F_g(z \bullet e_j) = 0.$$

If $z = \prod e_i^{\alpha_i}$, then:

$$z \bullet e_j = \alpha_{i_0} e_j e_{i_0}^{\alpha_{i_0}-1}\prod_{i \in I \setminus \{i_0\}} e_i^{\alpha_i}.$$

Consequently, if $y = \prod e_i^{\alpha_i} \in S^+(V)$, with $j \neq i_0$ such that $\alpha_j \neq 0$, denoting $n = \sum \alpha_i$:

$$F_g(y) = 0 = \mu(g)n!\lambda^{n-1}\prod_{i \in I} f(e_i)^{\alpha_i},$$

as $f(e_j) = 0$ and $\alpha_j \geq 1$. Finally, for any $y = \prod e_i^{\alpha_i} \in S^+(V)$, denoting $n = \sum \alpha_i$:

$$F_g(y) = \mu(g)n!\lambda^{n-1}\prod_{i \in I} f(e_i)^{\alpha_i}.$$

As $F_{gh} = F_g + F_h$, $\mu(gh) = F_{gh}(e_{i_0}) = F_g(e_{i_0}) + F_h(e_{i_0}) = \mu(g) + \mu(h)$.

3. For any $v \in S^+(V)$, we put $\varpi_g(g \bullet hv) = \alpha(v)g$, where ϖ_g is defined in the proof of Lemma 23. Then:

$$\Delta(g \bullet hv) = g \bullet hv \otimes gh + g \bullet h \otimes ghv + g \bullet hv' \otimes ghv'' + g \otimes g \bullet hv$$
$$= g \bullet hv \otimes gh + \lambda(g, h)(g - gh) \otimes ghv + g \bullet hv' \otimes ghv'' + g \otimes g \bullet hv.$$

By cocommutativity:

$$(\varpi_g \otimes Id) \circ \Delta(g \bullet hv) = \alpha(v)g \otimes gh + \lambda(g,h)g \otimes ghv + \alpha(v')g \otimes ghv'' + g \otimes g \bullet hv$$
$$= (Id \otimes \varpi_g) \circ \Delta(g \bullet hv) = \alpha(v)g \otimes g.$$

This gives:

$$g \bullet hv = -\lambda(g,h)ghv + \alpha(v)(g - gh) - \alpha(v')ghv''.$$

It remains to prove that the linear form α is zero. Let $x, y \in S^+(V)$.

$(g \bullet hx) \bullet y \in \mathbb{K}G \cdots S^+(V),$

$g \bullet (hx \bullet y) = \left(\lambda(h)\alpha(xy) + \lambda(h)\alpha(xy'')F_g(y') + \alpha(x \bullet y)\right)(g - gh) + \text{terms in } \mathbb{K}G \cdots S^+(V),$

$(g \bullet y) \bullet hx \in \mathbb{K}G \cdots S^+(V),$

$g \bullet (y \bullet hx) \in \mathbb{K}G \cdots S^+(V).$

By the preLie identity, for any $x, y \in S^+(V)$:

$$\alpha\left(\lambda(h)xy + F_h(y')xy'' + x \bullet y\right) = 0. \tag{12}$$

First subcase. We assume that $\lambda(h) = 0$. For $x \in S^+(V)$ and $y \in V$, we obtain that:

$$\alpha(S^+(V) \bullet V) = (0).$$

Let $y = \prod e_i^{\alpha_i} \in S^+(V)$, with $\sum \alpha_i = n \geq 1$. If $\alpha_{i_0} \geq 1$, then:

$$y \bullet e_{i_0} \alpha_{i_0} y,$$

so $y \in S^+(V) \bullet V$. Otherwise, there exists $j \neq i_0$, such that $\alpha_j \geq 1$.

$$\left(e_{i_0} e_j^{\alpha_j - 1} \prod_{i \neq i_0, j} e_i^{\alpha_i}\right) \bullet e_j = (\alpha_{i_0} + 1)y = y,$$

so $y \in S^+(V) \bullet V$. As a conclusion, $S^+(V) \bullet V = S^+(V)$, so $\alpha = 0$.

Second subcase. We assume that $\lambda(h) \neq 0$. Let us first prove that the ideal I generated by $Ker(f)$ is a subspace of $Ker(\alpha)$. Let $x_1, \ldots, x_n \in V$ and $y \in Ker(f)$, let us prove that $x_1 \ldots x_n y \in Ker(\alpha)$ by induction on n. If $n = 0$, by (12) with $x \in V$:

$$\alpha(\lambda(h)xy + x \bullet y) = \alpha(\lambda(h)xy + y \bullet x) = 0.$$

Hence, $x \bullet y - y \bullet x = f(x)y - f(y)x \in Ker(\alpha)$. Choosing g such that $f(x) = 1$ we obtain $y \in Ker(\alpha)$. Let us assume the result at rank $n - 1$, $n \geq 1$. The following element belongs to $Ker(\alpha)$ by (12), with $x = x_1 \ldots x_n$:

$$\lambda(h)x_1 \ldots x_n y + \sum_{i=1}^{n} f(x_i)x_1 \ldots x_{i-1}yx_{i+1} \ldots x_n.$$

Applying the induction hypothesis and $\lambda(h) \neq 0$, we obtain that $x_1 \ldots x_n y \in Ker(\alpha)$.

Consequently, there exists a family of scalars $(\beta(n))_{n \geq 1}$ such that for any $x_1, \ldots, x_n \in V$:

$$\alpha(x_1 \ldots x_n) = \beta(n)f(x_1) \ldots f(x_n);$$

with the notations of the proof of point 2, $\beta(n) = \alpha(e_{i_0}^n)$. By (12), with $x = e_{i_0}^n$ and $y = e_{i_0}$, for any $n \geq 1$:

$$\lambda(h)\beta(n+1) + n\beta(n) = 0,$$

which gives, for all $n \geq 1$:

$$\beta(n) = \frac{(-1)^{n-1}(n-1)!}{\lambda(h)^{n-1}}\beta(1).$$

Let us assume that $\beta(1) \neq 0$. By (12) with $x = e_{i_0}$ and $y = e_{i_0}^2$, we obtain:

$$\frac{\beta(1)}{\lambda(h)}(-2\mu(g) + 2\lambda\lambda(h) + 1) = 0.$$

Hence, $\mu(g) = \lambda\lambda(h) + \frac{1}{2}$. By (12) with $x = e_{i_0}$ and $y = e_{i_0}^3$, we obtain:

$$\frac{\beta(1)}{\lambda(h)^2} = 0.$$

This is a contradiction, so $\beta(1) = 0$ and, therefore, $\alpha = 0$. □

Theorem 26. *Let $\bullet$ be a product on $\mathbb{K}G \cdot S(V)$, making it a Com-PreLie bialgebra. We assume that the restriction of $\bullet$ to $S(V)$ is nonzero and that one of the following assertions holds:*

1. *$dim(V) \geq 2$.*

2. *$dim(V) = 1$ and $v \bullet 1 = 0$ for any $v \in V$.*

Then:

1. *There exist $f : V \longrightarrow \mathbb{K}$, nonzero, and $\lambda \in \mathbb{K}$ such that the Com-PreLie Hopf subalgebra $(S(V), m, \bullet, \Delta)$ is equal to $S(V, f, \lambda)$.*

2. *There exist a family of scalars $(\lambda(g, h))_{g,h \in G}$ satisfying the following conditions:*

   - *For all $g \in G$, $\lambda(g, 1) = 0$;*
   - *For all $g, h, k \in G$, $\lambda(gh, k) = \lambda(g, k) + \lambda(h, k)$;*
   - *For all $g, h, k \in G$, such that $hk \neq 1$:*

   $$\lambda(g, hk)\lambda(h, k) - \lambda(g, h)\lambda(h, k) = \lambda(g, hk)\lambda(k, h) - \lambda(g, k)\lambda(k, h);$$

   - *For all $g \in G$, $\lambda(g, g) = 0$;*

   such that for all $g, h \in G$, $x \in S^+(V)$:

   $$g \bullet h = \lambda(g, h)(g - gh), \qquad\qquad g \bullet hx = -\lambda(g, h)ghx.$$

3. *For any $x \in S(V)$, $g \in G \setminus \{1\}$, $y \in S(V)$, $x \bullet gy = 0$.*

4. *There exist group morphisms $\lambda, \mu : G \longrightarrow (\mathbb{K}, +)$ such that for any $g \in G$, $x_1, \ldots, x_n \in V$, $n \geq 1$:*

   $$g \bullet x_1 \ldots x_n = \lambda(g)gx_1 \ldots x_n + \mu(g)g \sum_{\emptyset \subsetneq I \subsetneq [n]} |I|! \lambda^{|I|-1} \prod_{i \in I} f(x_i) \prod_{i \notin I} x_i.$$

   Conversely, if one define a product $\bullet$ on $\mathbb{K}G \cdot S(V)$ with the point 1.-4. and, for any $g \in G$, $x \in S(V)$, $y \in \mathbb{K}G \cdot S(V)$:

   $$gx \bullet y = (g \bullet y)x + g(x \bullet y),$$

   then $(\mathbb{K}G \cdot S(V), m, \bullet, \Delta)$ is a Com-PreLie bialgebra.

Proof. Let $\bullet$ be a product on $\mathbb{K}G \cdot S(V)$, making it a Com-PreLie bialgebra. By Proposition 24-2, we obtain that $S(V)$ is a Com-PreLie subalgebra of $\mathbb{K}G \cdot S(V)$. Theorem 10 and the hypotheses on $S(V)$ imply that $(S(V), \bullet) = S(V, f, \lambda)$ for well-chosen f and λ, which is point 1. By Proposition 24-3, $\mathbb{K}G$ is a Com-PreLie subalgebra of $\mathbb{K}G \cdot S(V)$; Theorem 19 gives the existence of the scalars $\lambda(g, h)$ satisfying the first four points, such that for any $g, h \in G$, $g \bullet h = \lambda(g, h)(g - gh)$. By Proposition 25-3, for any $g, h \in G \setminus \{1\}$, $v \in S^+(V)$, $g \bullet hv = -\lambda(g, h)ghv$, which gives point 2. Point 3 comes from Proposition 24, together with the Leibniz identity, and Point 4 from Proposition 25-2.

Conversely, given $\lambda \in \mathbb{K}$, $f : V \longrightarrow \mathbb{K}$, $\mu, \lambda : G \longrightarrow (\mathbb{K}, +)$ be group morphisms, and scalars $(\lambda(g, h))_{g,h \in G}$ satisfying the two first conditions of point 2, points 1-4 define a product $\bullet$ on $\mathbb{K}G \cdot S(V)$ satisfying the Leibniz identity and the compatibility with the coproduct. It remains to prove the preLie identity. Because of the Leibniz identity, it is enough to prove it in the following cases:

1. $x \in V$, $y \in S(V)$, $z \in S(V)$. This comes from Theorem 2.

2. $x \in G \setminus \{1\}$, $y, z \in G$. By Theorem 19, this holds if, and only if, the third first conditions of point 2 are satisfied. We now assume that these conditions hold.

3. $x \in V$, $y \in G$. This is immediate, as all terms in the preLie identity are zero.

4. $x \in V$, $y \in S(V)$, $z \in G \setminus \{1\} \cdots S^+(V)$. This is immediate, as all terms in the preLie identity are zero.

5. $x \in V$, $y, z \in G \setminus \{1\} \cdots S^+(V)$. We put $y = gv$ and $z = hw$. If $gh \neq 1$, all terms in the reLie identity are zero. Otherwise:

$$x \bullet (gv \bullet hw) - (x \bullet gv) \bullet hw = \lambda(g, h)x \bullet vw,$$
$$x \bullet (hw \bullet gv) - (x \bullet hw) \bullet gvw = \lambda(h, g)x \bullet vw.$$

As $\lambda(h^{-1}, h) = -\lambda(h, h)$. For well-chosen x, v, w, $x \bullet vw \neq 0$, so the preLie identity holds in this case if, and only if, for any $h \in G$:

$$\lambda(h, h^{-1}) = -\lambda(h, h). \tag{13}$$

We now assume that this condition holds. Note that it is implies by the fourth condition of point 2.

6. $x \in G \setminus \{1\}$, $y \in S(V)$: this is proved by direct computations, separating the cases $z \in S(V)$, $z \in G$ and $z \in G \setminus \{1\} \cdots S^+(V)$.

7. $x, y \in G \setminus \{1\}$, $z \in G \setminus \{1\} \cdots S^+(V)$. We put $x = g$, $y = hv$, $z = kw$. If $hk \neq 1$, this is proved by direct computations. If $hk = 1$, the preLie identity is satisfied if, and only if:

$$\lambda(h, k)(\lambda(g, h) - \lambda(g, k)) = 0.$$

In particular, if $g = h$, by (13):

$$\lambda(h, k)(\lambda(g, h) - \lambda(g, k)) = -\lambda(h, h)(\lambda(h, h) + \lambda(h, h)) = 2\lambda(h, h)^2 = 0,$$

so $\lambda(h, h) = 0$: this is the fourth condition of point 2.

8. $x \in G \setminus \{1\}$, $y, z \in G \setminus \{1\} \cdots S^+(V)$. If the conditions of point 2 hold, a direct computation shows that the preLie identity is satisfied in this case.

Finally, if $\bullet$ gives $\mathbb{K}G \cdot S(V)$ a Com-PreLie bialgebra structure, then points 1-4 are satisfied, and conversely. $\qed$

Let us now consider the coefficients $\lambda(g, h)$ satisfying the conditions of point 2 if $G = \mathbb{Z}$.

Proposition 27. *Let $(\lambda(g, h))_{g,h\in\mathbb{Z}}$ be coefficients satisfying the conditions of point 2 of Theorem 26, with $G = \mathbb{Z}$. Then for any $g, h \in \mathbb{Z}$, $\lambda(g, h) = 0$.*

Proof. We use Theorem 21. In the second case, there exists $k_0 \in \mathbb{Z}$, nonzero, $a \in \mathbb{K}$, nonzero, such that for all $k, l \in \mathbb{Z}$:

$$\lambda(k, l) = a\delta_{l,k_0}k.$$

Then $\lambda(k_0, k_0) = ak_0 = 0$, so $k_0 = 0$: this is a contradiction. In the third case, there exist nonzero scalars α, β and $N \geq 1$ such that for any $k, l \in \mathbb{Z}$:

$$\lambda(k, l) = \begin{cases} \dfrac{\alpha\beta}{\frac{l}{N}(\alpha - \beta) + 2\beta - \alpha} & \text{if } N \mid l, \\ 0 \text{ otherwise.} \end{cases}$$

Then:

$$\lambda(N, N) = \alpha = 0.$$

This is a contradiction. Consequently, $\lambda(k, l) = 0$ for any $k, l \in \mathbb{Z}$. $\qquad\square$

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