

Twisted bialgebras, cofreeness and cointeraction

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Abstract

We study twisted bialgebras and double twisted bialgebras, that is to say bialgebras in the category of linear species, or in the category of species in the category of coalgebras. We define the notion of cofree twisted coalgebra and generalize Hoffman's quasi-shuffle product, obtaining in particular a twisted bialgebra of set compositions *Comp*. Given a special character, this twisted bialgebra satisfies a terminal property, generalizing the one of the Hopf algebra of quasisymmetric functions proved by Aguiar, Bergeron and Sottile.

We give *Comp* a second coproduct, making it a double twisted bialgebra, and prove that it is a terminal object in the category of double twisted bialgebras. Actions of characters on morphisms allow to obtain every twisted bialgebra morphisms from a connected double twisted bialgebra to *Comp*.

These results are applied to examples based on graphs and on finite topologies, obtaining species versions of the chromatic symmetric series and chromatic polynomials, or of the Ehrhart polynomials. Moreover, through actions of monoids of characters, we obtain a twisted bialgebraic interpretation of the duality principle.

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Introduction

Species, introduced by Joyal [20] in the 80's, are functors from the category of finite sets with bijective maps to the category of vector spaces. They form a monoidal category and a natural idea is to define and construct algebras, coalgebras and bialgebras in this category: this is for example explored by Aguiar and Mahajan in [2, 3]. In the present paper, following [28], these objects will be qualified of *twisted*. A bunch of functors from species to vector spaces send twisted algebras, coalgebras or bialgebras to algebras, coalgebras or bialgebras: we shall use here the full Fock functor \mathcal{K} and the bosonic Fock functor $\widehat{\mathcal{K}}$, in the terminology of [2]. Our aim in this paper is to mimic several known results on bialgebras involving cofreeness, shuffles and (co)interactions in the frame of twisted bialgebras.

We shall start in the first section with reminder on species, and examples of twisted algebras, coalgebras and bialgebras, that is to say algebras, coalgebras and bialgebras in the category of species. A simple example is given by the species **Com**: for any finite set A , $\mathbf{Com}[A] = \mathbb{K}$; the product and the coproduct of **Com** are given by:

$$m_{A,B}(1 \otimes 1) = 1 \in \mathbf{Com}[A \sqcup B],$$

$$\Delta_{A,B}(1) = 1 \otimes 1 \in \mathbf{Com}[A] \otimes \mathbf{Com}[B].$$

Two more complex examples will be particularly studied in the sequel: firstly, the twisted bialgebra of graphs $\mathcal{G}r'$, which product is given by the disjoint union of graphs, and which coproduct

by extraction of subgraphs. For example, if A , B and C are finite sets:

$$\begin{aligned}\Delta_{A,B \sqcup C}(\mathfrak{!}_A^C) &= \bullet_A \otimes \mathfrak{!}_B^C, & \Delta_{B \sqcup C,A}(\mathfrak{!}_A^C) &= \mathfrak{!}_B^C \otimes \bullet_A, \\ \Delta_{B,A \sqcup C}(\mathfrak{!}_A^C) &= \bullet_B \otimes \bullet_{A \cdot C}, & \Delta_{A \sqcup C,B}(\mathfrak{!}_A^C) &= \bullet_{A \cdot C} \otimes \bullet_B, \\ \Delta_{C,A \sqcup B}(\mathfrak{!}_A^C) &= \bullet_C \otimes \mathfrak{!}_A^B, & \Delta_{A \sqcup B,C}(\mathfrak{!}_A^C) &= \mathfrak{!}_A^B \otimes \bullet_C.\end{aligned}$$

This twisted bialgebra is commutative and cocommutative. Secondly, the twisted bialgebra of finite topologies, or, by Alexandroff's theorem [4], of quasi-posets, which product is given by the disjoint union of finite topologies, and which coproduct is given by the extraction of open subsets. For example, if A , B and C are nonempty finite sets:

$$\begin{aligned}\Delta_{A,B \sqcup C}(\mathfrak{!}_A^C) &= \bullet_A \otimes \mathfrak{!}_B^C, & \Delta_{B \sqcup C,A}(\mathfrak{!}_A^C) &= 0, \\ \Delta_{B,A \sqcup C}(\mathfrak{!}_A^C) &= 0, & \Delta_{A \sqcup C,B}(\mathfrak{!}_A^C) &= 0, \\ \Delta_{C,A \sqcup B}(\mathfrak{!}_A^C) &= 0, & \Delta_{A \sqcup B,C}(\mathfrak{!}_A^C) &= \mathfrak{!}_A^B \otimes \bullet_C.\end{aligned}$$

We here represent finite topologies or quasi-posets by their Hasse graphs. This twisted bialgebra is commutative and not cocommutative.

We deal with the cofree twisted coalgebra $co\mathcal{T}(\mathcal{P})$ (co)generated by a species \mathcal{P} (Definition 2.1) in the second section. It satisfies a universal property, justifying the adjective "cofree" (Theorem 2.2). Two particular examples will be used: the first one, $Comp$, based on set compositions, is generated by the augmentation ideal of \mathbf{Com} ; the second one, Ord , based on total orders, is generated by a species concentrated in degree 1. The universal property is used to extend Hoffman's quasi-shuffle product to the twisted case: if \mathcal{P} is a twisted algebra, then the cofree twisted coalgebra $co\mathcal{T}(\mathcal{P}_+)$ of its augmentation ideal \mathcal{P}_+ inherits a product \boxplus making it a twisted bialgebra; when the product of \mathcal{P}_+ is trivial, one obtains the simpler shuffle product \sqcup . This is applied to $Comp$ and to Ord . For example, in $Comp$, if A , B , C and D are nonempty finite sets:

$$\begin{aligned}(A) \boxplus (B) &= (A, B) + (B, A) + (A \sqcup B), \\ (A, B) \boxplus (C) &= (A, B, C) + (A, C, B) + (C, A, B) + (A \sqcup B, C) + (A, B \sqcup C), \\ (A, B) \boxplus (C, D) &= (A, B, C, D) + (A, C, B, D) + (C, A, B, D) \\ &\quad + (A, C, D, B) + (C, A, D, B) + (C, D, A, B) \\ &\quad + (A, B \sqcup C, D) + (A \sqcup C, B, D) + (A \sqcup C, D, B) \\ &\quad + (A, C, B \sqcup D) + (C, A, B \sqcup D) + (C, A \sqcup D, B) \\ &\quad + (A \sqcup C, B \sqcup D),\end{aligned}$$

whereas:

$$\begin{aligned}(A) \sqcup (B) &= (A, B) + (B, A), \\ (A, B) \sqcup (C) &= (A, B, C) + (A, C, B) + (C, A, B), \\ (A, B) \sqcup (C, D) &= (A, B, C, D) + (A, C, B, D) + (C, A, B, D) \\ &\quad + (A, C, D, B) + (C, A, D, B) + (C, D, A, B).\end{aligned}$$

These shuffle bialgebras are, in some sense, universal for commutative cofree twisted coalgebras, see Theorem 3.7.

We extend in the third section the notion of characters to twisted bialgebras (Definition 3.3). As in the classical case, the characters form a monoid $\text{Char}(\mathcal{P})$ for the convolution product $*$

associated to Δ . If \mathcal{P} is a connected twisted bialgebra, that is to say if $\mathcal{P}[\emptyset]$ is one-dimensional, this monoid has a richer structure: for any $q \in \mathbb{K}$, there is a map $\lambda \longrightarrow \lambda^q$ such that:

$$\forall q, q' \in \mathbb{K}, \forall \lambda \in \text{Char}(\mathcal{P}), \quad \lambda^q * \lambda^{q'} = \lambda^{q+q'}, \quad (\lambda^q)^{q'} = \lambda^{qq'}.$$

Taking $q = -1$, one obtains an inverse of λ : in this case, $\text{Char}(\mathcal{P})$ is a group. Moreover, the twisted bialgebra Comp inherits a special character ε' making it a terminal object in a suitable category of connected twisted bialgebras with a character (Theorem 3.6): this is the twisted version of Aguiar, Bergeron and Sottile's theorem on combinatorial Hopf algebras [1].

All these constructions are based on the Cauchy tensor product of species. We turn to the Hadamard tensor product of species in the fourth section. This allows to define a second kind of bialgebras in the category of species (Definition 4.1). We shall be especially interested in species with both sort of structures and with a nice cointeraction between them, mimicking the notion of bialgebras in cointeraction we now recall.

A pair of bialgebras in cointeraction is a pair (A, m_A, Δ) and (B, m_B, δ) of bialgebras with a (right) coaction $\rho : A \longrightarrow A \otimes B$ of B on A such that the product, coproduct, counit and unit of A are morphisms of right comodules. In other words:

- $\rho(1_A) = 1_A \otimes 1_B$ and for any $a, b \in A$, $\rho(ab) = \rho(a)\rho(b)$: ρ is an algebra morphism.
- $(\Delta \otimes \text{Id}) \circ \rho = m_{1,3,24} \circ (\rho \otimes \rho) \circ \Delta$, where

$$m_{1,3,24} : \begin{cases} A \otimes B \otimes A \otimes B & \longrightarrow & A \otimes A \otimes b \\ a_1 \otimes b_1 \otimes a_2 \otimes b_2 & \longrightarrow & a_1 \otimes a_1 \otimes b_1 b_2. \end{cases}$$

- For any $a \in A$, $(\varepsilon_A \otimes \text{Id}) \circ \rho(a) = \varepsilon_A(a)1_B$.

For example, the polynomial algebra $\mathbb{K}[X]$ is given two multiplicative coproducts by:

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \delta(X) = X \otimes X.$$

Then $(\mathbb{K}[X], m, \Delta)$ and $(\mathbb{K}[X], m, \delta)$ are in cointeraction with the coaction $\rho = \delta$. In numerous examples, $(A, m) = (B, m)$ and $\rho = \delta$; see [8, 12, 13, 24] for combinatorial examples based on families of graphs, oriented graphs or posets, and applications of the cointeraction.

This notion is extended to species, obtaining double twisted bialgebras (Definition 4.3). For example, the species **Com** is a double twisted bialgebra: it has a second coproduct δ , such that for any finite set A :

$$\delta_A : \begin{cases} \mathbf{Com}[A] & \longrightarrow & \mathbf{Com}[A] \otimes \mathbf{Com}[A] \\ 1 & \longrightarrow & 1 \otimes 1. \end{cases}$$

The set of characters of a double twisted bialgebra \mathcal{P} holds a second convolution product \star associated to δ , making it a monoid. Moreover:

$$\forall \lambda, \mu, \nu \in \text{Char}(\mathcal{P}), \quad (\lambda * \mu) \star \nu = (\lambda \star \nu) * (\mu \star \nu).$$

In other words, $(\text{Char}(\mathcal{P}), \star)$ acts on $(\text{Char}(\mathcal{P}), *)$ by monoid endomorphisms. For any twisted bialgebra \mathcal{Q} , this monoid also acts on the set $\text{Mor}_B(\mathcal{P}, \mathcal{Q})$ of twisted bialgebra morphisms from \mathcal{P} to \mathcal{Q} (Proposition 4.8), giving a right action:

$$\begin{cases} \text{Mor}_B(\mathcal{P}, \mathcal{Q}) \times \text{Char}(\mathcal{P}) & \longrightarrow & \text{Mor}_B(\mathcal{P}) \\ (\phi, \lambda) & \longrightarrow & \phi \leftarrow \lambda. \end{cases}$$

In particular, we obtain an injective morphism of monoids:

$$\chi_{\mathcal{P}} : \begin{cases} (\text{Char}(\mathcal{P}), \star) & \longrightarrow & (\text{Mor}_B(\mathcal{P}, \mathcal{P}), \circ) \\ \lambda & \longrightarrow & \text{Id}_{\mathcal{P}} \leftarrow \lambda. \end{cases}$$

We prove that if \mathcal{P} is a commutative double twisted bialgebra, then the twisted cofree coalgebra $co\mathcal{T}(\mathcal{P}_+)$ is a double twisted bialgebra, with the quasi-shuffle product \boxplus , the deconcatenation coproduct Δ and another coproduct δ defined by an extraction-contraction process (Proposition 4.5). This is directly applied to $Comp$ in Corollary 4.6. For example, if A, B and C are nonempty finite subsets:

$$\begin{aligned}\delta(A) &= (A) \otimes (A), \\ \delta(A, B) &= (A, B) \otimes ((A) \boxplus (B)) + (A \sqcup B) \otimes (A, B), \\ \delta(A, B, C) &= (A, B, C) \otimes ((A) \boxplus (B) \boxplus (C)) + (A, B \sqcup C) \otimes ((A) \boxplus (B, C)) \\ &\quad + (A \sqcup B, C) \otimes ((A, B) \boxplus (C)) + (A \sqcup B \sqcup C) \otimes (A, B, C).\end{aligned}$$

We then prove a universal property for this twisted double bialgebra in Theorem 4.11: if \mathcal{P} is a twisted double bialgebra satisfying a condition of connectivity, then there exists a unique morphism ϕ of twisted double bialgebra from \mathcal{P} to $Comp$. Moreover, all the morphisms of twisted bialgebras from \mathcal{P} to $Comp$ can be deduced from ϕ by Theorem 3.6, using the action \leftarrow of the monoid $\text{Char}(\mathcal{P})$.

Graphs and finite topologies are given a structure of double twisted bialgebras in the next two sections. For example, in $\mathcal{G}r'$, If A, B, C are nonempty finite sets:

$$\begin{aligned}\delta(\bullet_A) &= \bullet_A \otimes \bullet_A, \\ \delta(\downarrow_A^B) &= \downarrow_A^B \otimes \bullet_{A \cdot B} + \bullet_{A \sqcup B} \otimes \downarrow_A^B, \\ \delta(\downarrow_A^{B \cdot C}) &= \downarrow_A^{B \cdot C} \otimes \bullet_{A \cdot B \cdot C} + \downarrow_{A \sqcup B}^C \otimes \downarrow_A^B \cdot C + \downarrow_{A \sqcup C}^B \otimes \downarrow_A^C \cdot B + \bullet_{A \sqcup B \sqcup C} \otimes \downarrow_A^{B \cdot C}.\end{aligned}$$

In $\mathcal{T}op$:

$$\begin{aligned}\delta(\bullet_A) &= \bullet_A \otimes \bullet_A, \\ \delta(\downarrow_A^B) &= \downarrow_A^B \otimes \bullet_{A \cdot B} + \bullet_{A \sqcup B} \otimes \downarrow_A^B, \\ \delta(\downarrow_A^{B \cdot C}) &= \downarrow_A^{B \cdot C} \otimes \bullet_{A \cdot B \cdot C} + \downarrow_{A \sqcup B}^C \otimes \downarrow_A^B \cdot C + \downarrow_{A \sqcup C}^B \otimes \downarrow_A^C \cdot B + \bullet_{A \sqcup B \sqcup C} \otimes \downarrow_A^{B \cdot C}, \\ \delta(\downarrow_A^{\frac{C}{B}}) &= \downarrow_A^{\frac{C}{B}} \otimes \bullet_{A \cdot B \cdot C} + \downarrow_{A \sqcup B}^C \otimes \downarrow_A^B \cdot C + \downarrow_{A \sqcup C}^B \otimes \bullet_A \downarrow_B^C + \bullet_{A \sqcup B \sqcup C} \otimes \downarrow_A^{\frac{C}{B}}.\end{aligned}$$

The unique double twisted bialgebra morphism $\phi_{chr} : \mathcal{G}r' \longrightarrow Comp$ is related to the noncommutative chromatic formal series of [16] (Proposition 5.3). For example:

$$\begin{aligned}\phi_{chr}(\bullet_A) &= (A), \\ \phi_{chr}(\downarrow_A^B) &= (A, B) + (B, A), \\ \phi_{chr}(\downarrow_A^{\frac{C}{B}}) &= (A, B, C) + (A, C, B) + (B, A, C) + (B, C, A) + (C, A, B) + (C, B, A) \\ &\quad + (A \sqcup C, B) + (B, A \sqcup C).\end{aligned}$$

The unique double twisted bialgebra morphism $\phi_{ehr} : \mathcal{G}r' \longrightarrow Comp$ is related to linear extensions and to the Ehrhart quasisymmetric function (Theorem 6.9). For example:

$$\begin{aligned}\phi_{ehr}(\bullet_A) &= (A), \\ \phi_{ehr}(\downarrow_A^B) &= (A, B), \\ \phi_{ehr}(\downarrow_A^{B \cdot C}) &= (A, B, C) + (A, C, B) + (A, B \sqcup C), \\ \phi_{ehr}(\downarrow_A^{\frac{C}{B}}) &= (A, B, C).\end{aligned}$$

We also prove in this section that $\mathcal{T}op$ is a cofree twisted coalgebra, so is isomorphic to a quasi-shuffle twisted bialgebra (Theorem 6.10).

The seventh section is devoted to the study of homogeneity. We consider now graded twisted bialgebras, that is to say twisted bialgebra \mathcal{P} admitting a decomposition

$$\mathcal{P} = \bigoplus_{n \geq 0} \mathcal{P}_n,$$

compatible with the product and coproduct of \mathcal{P} (Definition 7.1). For example, $\mathcal{G}r'$ admits a graduation, given by the number of vertices of the graphs, and $\mathcal{T}op$ admits a graduation, given by the number of classes of the quasi-posets, or equivalently the number of vertices of the Hasse graphs. If \mathcal{P} is graded, it admits a family $(\iota_q)_{q \in \mathbb{K}}$ of bialgebra endomorphisms, such that for any $q, q' \in \mathbb{K}$:

$$\iota_q \circ \iota_{q'} = \iota_{qq'}.$$

Conversely, we define in Proposition 7.2 such a family $(\theta_q)_{q \in \mathbb{K}}$ of endomorphisms of $\mathcal{C}omp$, given by:

$$\theta_q = \text{Id}_{\mathcal{C}omp} \leftarrow \varepsilon'^q.$$

These morphisms imply a graduation of $\mathcal{C}omp$. For example:

$$\mathcal{C}omp_1[A] = \text{Vect}((A)),$$

$$\mathcal{C}omp_2[A] = \text{Vect} \left((A_1, A_2) + \frac{1}{2}(A_1 \sqcup A_2), (A_1, A_2) \in \mathbf{Comp}[A] \right),$$

$$\mathcal{C}omp_3[A] = \text{Vect} \left((A_1, A_2, A_3) + \frac{1}{2}(A_1 \sqcup A_2, A_3) + \frac{1}{2}(A_1, A_2 \sqcup A_3) + \frac{1}{6}(A_1 \sqcup A_2 \sqcup A_3), (A_1, A_2, A_3) \in \mathbf{Comp}[A] \right).$$

For any graded connected twisted bialgebra \mathcal{P} , we prove that the set of homogeneous morphisms from \mathcal{P} to $\mathcal{C}omp$ is in one-to-one correspondence with the set of morphisms from \mathcal{P}_1 to \mathbf{Com} (Corollary 7.4). As applications, we describe all homogeneous twisted bialgebra morphisms from $\mathcal{G}r'$ or from $\mathcal{T}op$ to $\mathcal{C}omp$ (Proposition 7.5 and 7.13). For graphs, we shall especially consider a one-parameter family of them, denoted by $(\phi_q)_{q \in \mathbb{K}}$. For example:

$$\begin{aligned} \phi_q(\bullet_A) &= q(A), \\ \phi_q(\bullet_A^B) &= q^2(A, B) + q^2(B, A) + q^2(A \sqcup B) \\ \phi_q(\bullet_A^{B \star C}) &= q^3(A, B, C) + q^3(A, C, B) + q^3(B, A, C) + q^3(B, C, A) + q^3(C, A, B) + q^3(C, B, A) \\ &\quad + q^3(A \sqcup B, C) + q^3(A \sqcup C, B) + q^3(B \sqcup C, A) \\ &\quad + q^3(A, B \sqcup C) + q^3(B, A \sqcup C) + q^3(C, A \sqcup B) + q^3(A \sqcup B \sqcup C). \end{aligned}$$

Playing with the graduation, we deform the morphism ϕ_{chr} , putting:

$$\forall q \in \mathbb{K}, \quad \phi_{chr_q} = \theta_q^{-1} \circ \phi_{chr} \circ \iota_q.$$

We prove the existence of a character λ_{chr_q} such that:

$$\phi_{chr_q} = \phi_1 \leftarrow \lambda_{chr_q}.$$

This rather complicated character is invertible for the \star product, and for any graph G , $\lambda_{chr_q}^{-1}(G)$ is a power of q (Corollary 7.10). We finally obtain in Corollary 7.10 a diagram of twisted bialgebras morphisms

$$\begin{array}{ccc} \mathcal{G}r' & \xrightarrow{\Gamma} & \mathcal{G}r' \\ & \searrow \phi_{chr_{-1}} & \swarrow \phi_{chr_1} \\ & \mathcal{C}omp & \end{array}$$

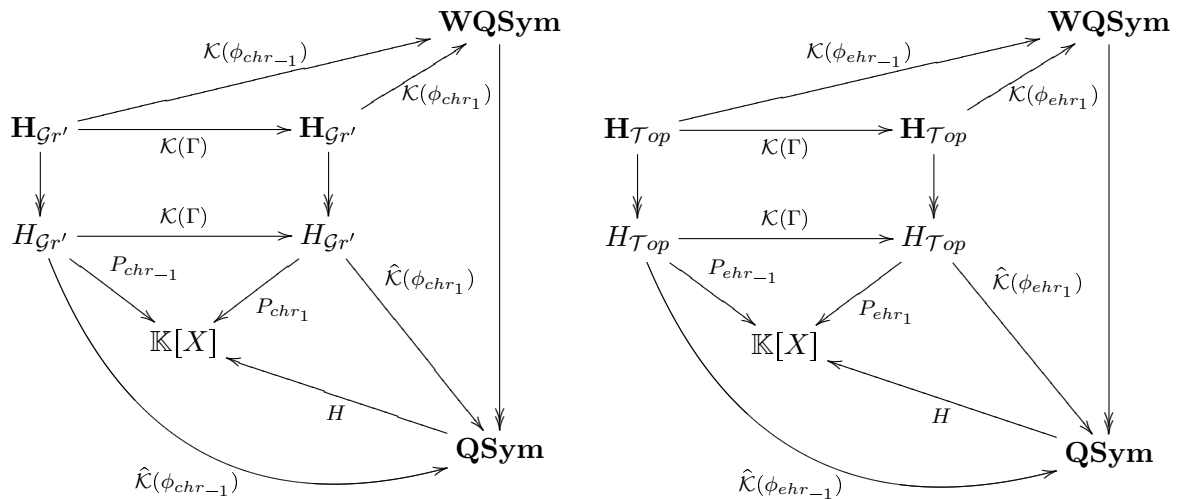
where Γ is a twisted bialgebra automorphism related to a character defined by acyclic orientations of graphs. Similar results are obtained for finite topologies.

We apply the Fock functors to these constructions in the eighth section. Both of them send twisted bialgebras to bialgebras, and the bosonic one sends double twisted bialgebras to bialgebras in cointeraction (Theorem 8.2). Note that this second point generally does not work for the full Fock functor by a lack of commutativity. We obtain:

- For **Com**: both Fock functors give $\mathbb{K}[X]$, with its usual product and coproducts.
- For **Ord**: the full Fock functor gives Malvenuto and Reutenauer's Hopf algebra **FQSym** of permutations [9, 23], whereas the bosonic Fock functor gives again the polynomial algebra $\mathbb{K}[X]$.
- For **Comp**: the full Fock functor gives the Hopf algebra of packed words (or of set compositions) **WQSym** [26, 27, 25] with its internal coproduct; the bosonic Fock functor give the Hopf algebra of quasi-symmetric functions **QSym**.
- For the species of graphs $\mathcal{G}r'$, we obtain the Hopf algebras $\mathbf{H}_{\mathcal{G}r'}$ and $H_{\mathcal{G}r'}$ described in [12]. The induced morphism ϕ is Stanley's chromatic symmetric function [30].
- For the species of finite topologies $\mathcal{T}op$, we obtain the Hopf algebras $\mathbf{H}_{\mathcal{T}op}$ and $H_{\mathcal{T}op}$ described in [13]. The induced morphism ϕ is the (strict) Ehrhart quasisymmetric function.

We also prove in Theorem 8.8 that the image of a cofree twisted bialgebra by the full Fock functor is free and cofree (getting back a known result on **FQSym** and **WQSym**).

The last section is devoted to applications of these results on graphs and finite topologies. We also prove that there exists a unique morphism $H : \mathbf{QSym} \rightarrow \mathbb{K}[X]$ of double bialgebra, which we use to recover the chromatic polynomial P_{chr} and the (strict) Ehrhart polynomial P_{ehr} . We finally obtain the following morphisms of bialgebra morphisms:



Notations 0.1. • Let \mathbb{K} be a commutative field of characteristic zero. All the objects of this text (vector spaces, algebras, coalgebras, bialgebras...) will be taken over \mathbb{K} .

- For any $n \in \mathbb{N}$, we denote by \underline{n} the set $\{1, \dots, n\}$. In particular, $\underline{0} = \emptyset$.
- For any $n \in \mathbb{N}$, we denote by \mathfrak{S}_n the n -th symmetric group.
- If A is a finite set, we denote by $\sharp A$ its cardinality.

1 Bialgebras in the category of species

1.1 Reminders and examples

We refer to [6, 20] for a more complete exposition of species. We denote by **Set** the category of finite sets with bijections and by **Vect** the category of vector spaces over \mathbb{K} with linear maps. A linear species is a functor from **Set** to **Vect**. In other words, a species \mathcal{P} is given by:

- For any finite set A , a vector space $\mathcal{P}[A]$.
- For any bijection $\sigma : A \longrightarrow B$ between two finite sets, a linear map $\mathcal{P}[\sigma] : \mathcal{P}[A] \longrightarrow \mathcal{P}[B]$.

The following properties are satisfied:

- For any finite set A , $\mathcal{P}[\text{Id}_A] = \text{Id}_{\mathcal{P}[A]}$.
- For any finite sets, A, B, C and any bijections $\sigma : A \longrightarrow B$ and $\tau : B \longrightarrow C$,

$$\mathcal{P}[\tau \circ \sigma] = \mathcal{P}[\tau] \circ \mathcal{P}[\sigma].$$

If \mathcal{P} and \mathcal{Q} are two species, a morphism of species $f : \mathcal{P} \longrightarrow \mathcal{Q}$ is a natural transformation from \mathcal{P} to \mathcal{Q} , that is, for any finite set A , $f[A]$ is a linear map from $\mathcal{P}[A]$ to $\mathcal{Q}[A]$, such that for any bijection $\sigma : A \longrightarrow B$ between two finite sets, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}[A] & \xrightarrow{\mathcal{P}[\sigma]} & \mathcal{P}[B] \\ f[A] \downarrow & & \downarrow f[B] \\ \mathcal{Q}[A] & \xrightarrow{\mathcal{Q}[\sigma]} & \mathcal{Q}[B] \end{array}$$

Example 1.1. 1. We define a species **Com** by the following:

- For any finite set A , $\mathbf{Com}[A] = \mathbb{K}$.
 - For any bijection $\sigma : A \longrightarrow B$, $\mathbf{Com}[\sigma] = \text{Id}_{\mathbb{K}}$.
2. For any finite set A , let us denote by $\mathbf{Comp}[A]$ the set of set compositions of A , that is to say finite sequences (A_1, \dots, A_k) of nonempty subsets of A , such that $A_1 \sqcup \dots \sqcup A_k = A$. Let $\mathit{Comp}[A]$ be the space generated by $\mathbf{Comp}[A]$. If $\sigma : A \longrightarrow B$ is a bijection:

$$\mathit{Comp}[\sigma] : \begin{cases} \mathit{Comp}[A] & \longrightarrow & \mathit{Comp}[B] \\ (A_1, \dots, A_k) \in \mathbf{Comp}[A] & \longrightarrow & (\sigma(A_1), \dots, \sigma(A_k)) \in \mathbf{Comp}[B]. \end{cases}$$

For example, if a, b are two different elements of a set A :

$$\begin{aligned} \mathit{Comp}[\emptyset] &= \text{Vect}(\emptyset), \\ \mathit{Comp}[\{a\}] &= \text{Vect}(\{a\}), \\ \mathit{Comp}[\{a, b\}] &= \text{Vect}(\{\{a\}, \{b\}\}, \{\{b\}, \{a\}\}, \{\{a, b\}\}) \end{aligned}$$

3. We define the species of graphs \mathcal{Gr} by the following:

- For any finite set A , $\mathcal{Gr}[A]$ is the vector space generated by graphs G which vertex set $V(G)$ is equal to A .
- For any bijection $\sigma : A \longrightarrow B$, $\mathcal{Gr}[\sigma]$ sends the graph $G = (A, E(G))$, where $E(G)$ is the set of edges of A , to the graph $\mathcal{Gr}[\sigma](G) = G' = (B, E(G'))$, with:

$$E(G') = \{\{\sigma(a), \sigma(b)\}, \{a, b\} \in E(G)\}.$$

For example, if a, b are two different elements of a set A :

$$\mathcal{Gr}[\emptyset] = \text{Vect}(\emptyset), \quad \mathcal{Gr}[\{a\}] = \text{Vect}(\bullet_a), \quad \mathcal{Gr}[\{a, b\}] = \text{Vect}(\bullet_a \bullet_b, \uparrow_a^b).$$

4. We define the species of posets $\mathcal{P}os$ by the following:

- For any finite set A , $\mathcal{P}os[A]$ is the vector space generated by posets P which vertex set is A , that is to say pairs $P = (A, \leq_P)$, such that \leq_P is a partial order on A .
- For any bijection $\sigma : A \rightarrow B$, $\mathcal{P}os[\sigma]$ sends the poset $P = (A, \leq_P)$, to the poset $\mathcal{P}os[\sigma](P) = (B, \leq'_P)$, with:

$$\forall a, b \in B, \quad a \leq'_P b \iff \sigma^{-1}(a) \leq_P \sigma^{-1}(b).$$

Posets will be represented by their Hasse graphs. For example, if a, b are two different elements of a set A :

$$\begin{aligned} \mathcal{P}os[\emptyset] &= \text{Vect}(\emptyset), \\ \mathcal{P}os[\{a\}] &= \text{Vect}(\bullet_a), \\ \mathcal{P}os[\{a, b\}] &= \text{Vect}(\bullet_a \bullet_b, \uparrow_a^b, \uparrow_b^a). \end{aligned}$$

Notations 1.1. If \mathcal{P} and \mathcal{Q} are species, we define the composition of the species \mathcal{P} and \mathcal{Q} by:

$$\mathcal{P} \circ \mathcal{Q}[A] = \bigoplus_{I \text{ partition of } A} \mathcal{P}[I] \otimes \left(\bigotimes_{X \in I} \mathcal{Q}[X] \right).$$

We shall especially consider the case where $\mathcal{Q} = \mathbf{Com}$:

$$\mathcal{P} \circ \mathbf{Com}[A] = \bigoplus_{I \text{ partition of } A} \mathcal{P}[I].$$

Example 1.2. 1. We put $\mathcal{G}r' = \mathcal{G}r \circ \mathbf{Com}$. For any finite set A , $\mathcal{G}r'[A]$ is the vector space generated by graphs which set of vertices is a partition of A . For example, if $A = \{a, b\}$:

$$\mathcal{G}r'[A] = \text{Vect}(\bullet_{\{a\}} \bullet_{\{b\}}, \uparrow_{\{a\}}^{\{b\}}, \bullet_{\{a, b\}}).$$

If $\sigma : A \rightarrow B$ is a bijection, the map $\mathcal{G}r'[\sigma] : \mathcal{G}r'[A] \rightarrow \mathcal{G}r'[B]$ is given by the action of σ on the elements of A .

2. The species $\mathcal{T}op$ of finite topologies is given by the following:

- For any finite set A , $\mathcal{T}op[A]$ is the vector space generated by the set of topologies on A , that is to say subsets T of the set of subsets of A with the following properties:
 - (a) \emptyset and A belong to T .
 - (b) If $X, Y \in T$, then $X \cup Y \in T$ and $X \cap Y \in T$.
- If $\sigma : A \rightarrow B$ and T is a topology on A , then:

$$\mathcal{T}op[\sigma](T) = \{\sigma(X), X \in T\}$$

A quasi-poset is a pair (A, \leq_P) , where A is a set and \leq_P a quasi-order on A , that is to say a transitive and reflexive relation on A . By Alexandroff's theorem [4, 31], for any finite set A , there is a bijection between topologies on A and quasi-posets $P = (A, \leq_P)$. This bijection associates to a quasi-order \leq on A the set $\mathcal{T}op(\leq)$ of its open sets: $X \subseteq A$ is \leq -open if:

$$\forall a, b \in A, \quad a \leq b \text{ and } a \in X \implies b \in X.$$

In the sequel, we shall identify in this way finite topologies and quasi-posets. If $T = (A, \leq_T)$ is a quasi-poset, one defines an equivalence on A by:

$$a \sim_T b \text{ if } a \leq_T b \text{ and } b \leq_T a.$$

The quotient space A/\sim_T is given a partial order \preceq_T by:

$$X \preceq_T Y \text{ if } \forall a \in X, \forall b \in Y, a \leq_T b.$$

Therefore, $(A/\sim_T, \preceq_T)$ is a poset. This implies that the species $\mathcal{T}op$ and $\mathcal{P}os \circ \mathbf{Com}$ are isomorphic. For example, if $A = \{a, b\}$, we represent finite topologies on A by the Hasse graph of \preceq_T , the vertices being decorated by the corresponding equivalence class of \sim_T :

$$\mathcal{T}op[A] = \text{Vect}(\bullet_{\{a\}} \bullet_{\{b\}}, \uparrow_{\{a\}}^{\{b\}}, \uparrow_{\{b\}}^{\{a\}}, \bullet_{\{a, b\}}).$$

The open sets of these topologies are given in the following table:

	\emptyset	$\{a\}$	$\{b\}$	$\{a, b\}$
$\bullet_{\{a\}} \bullet_{\{b\}}$	\times	\times	\times	\times
$\uparrow_{\{a\}}^{\{b\}}$	\times		\times	\times
$\uparrow_{\{b\}}^{\{a\}}$	\times	\times		\times
$\bullet_{\{a, b\}}$	\times			\times

Notations 1.2. 1. If G is a graph, the number of its vertices is denoted by $\text{deg}(G)$, and the number of its connected components is denoted by $\text{cc}(G)$.

2. If T is a finite topology on a set A , we denote by $CL(T)$ the quotient space A/\sim_T and by $\text{cl}(T)$ its cardinal (which is also the number of vertices of the Hasse graph of T), and we denote by $\text{cc}(T)$ then number of its connected components (which is also the number of connected components of the Hasse graph of T).

1.2 Algebras in the category of species

Let \mathcal{P}, \mathcal{Q} be two species. The Cauchy tensor product of species $\mathcal{P} \otimes \mathcal{Q}$ is defined as this:

- For any finite set A :

$$\mathcal{P} \otimes \mathcal{Q}[A] = \bigoplus_{I \subseteq A} \mathcal{P}[I] \otimes \mathcal{Q}[A \setminus I].$$

- For any bijection $\sigma : A \rightarrow B$ between two finite sets:

$$\mathcal{P} \otimes \mathcal{Q}[\sigma] : \begin{cases} \mathcal{P} \otimes \mathcal{Q}[A] & \rightarrow \mathcal{P} \otimes \mathcal{Q}[A] \\ x \otimes y \in \mathcal{P}[I] \otimes \mathcal{Q}[A \setminus I] & \rightarrow \underbrace{\mathcal{P}[\sigma|_I](x) \otimes \mathcal{Q}[\sigma|_{A \setminus I}](y)}_{\in \mathcal{P}[\sigma(I)] \otimes \mathcal{Q}[\sigma(A \setminus I)]} \end{cases}$$

For any species \mathcal{P}, \mathcal{Q} and \mathcal{R} , $\mathcal{P} \otimes (\mathcal{Q} \otimes \mathcal{R}) = (\mathcal{P} \otimes \mathcal{Q}) \otimes \mathcal{R}$. The unit is the species \mathcal{I} :

$$\mathcal{I}[A] = \begin{cases} \mathbb{K} & \text{if } A = \emptyset, \\ (0) & \text{otherwise.} \end{cases}$$

For any species \mathcal{P} , $\mathcal{P} \otimes \mathcal{I} = \mathcal{I} \otimes \mathcal{P} = \mathcal{P}$.

If $f : \mathcal{P} \rightarrow \mathcal{P}'$ and $g : \mathcal{Q} \rightarrow \mathcal{Q}'$ are morphisms of species, then $f \otimes g : \mathcal{P} \otimes \mathcal{Q} \rightarrow \mathcal{P}' \otimes \mathcal{Q}'$ is a morphism of species:

$$f \otimes g[A] : \begin{cases} \mathcal{P} \otimes \mathcal{Q}[A] & \rightarrow \mathcal{P}' \otimes \mathcal{Q}'[A] \\ x \otimes y \in \mathcal{P}[I] \otimes \mathcal{Q}[A \setminus I] & \rightarrow \underbrace{f[I](x) \otimes g[A \setminus I](y)}_{\in \mathcal{P}'[I] \otimes \mathcal{Q}'[A \setminus I]} \end{cases}$$

For any species \mathcal{P}, \mathcal{Q} , we define a species isomorphism $c_{\mathcal{P}, \mathcal{Q}} : \mathcal{P} \otimes \mathcal{Q} \rightarrow \mathcal{Q} \otimes \mathcal{P}$ by:

$$c_{\mathcal{P}, \mathcal{Q}}[A] : \begin{cases} \mathcal{P} \otimes \mathcal{Q}[A] & \rightarrow \mathcal{Q} \otimes \mathcal{P}[A] \\ x \otimes y \in \mathcal{P}[I] \otimes \mathcal{Q}[A \setminus I] & \rightarrow \underbrace{y \otimes x}_{\in \mathcal{Q}[A \setminus I] \otimes \mathcal{P}[I]} \end{cases}$$

Definition 1.1. An algebra in the category of species, or twisted algebra, is a pair (\mathcal{P}, m) where \mathcal{P} is a species and $m : \mathcal{P} \otimes \mathcal{P} \rightarrow \mathcal{P}$ is a morphism of species such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}^{\otimes 3} & \xrightarrow{m \otimes \text{Id}_{\mathcal{P}}} & \mathcal{P}^{\otimes 2} \\ \text{Id}_{\mathcal{P}} \otimes m \downarrow & & \downarrow m \\ \mathcal{P}^{\otimes 2} & \xrightarrow{m} & \mathcal{P} \end{array}$$

Moreover, there exists a morphism of species $\iota : \mathcal{I} \rightarrow \mathcal{P}$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{I} \otimes \mathcal{P} & \xrightarrow{\iota \otimes \text{Id}_{\mathcal{P}}} & \mathcal{P}^{\otimes 2} & \xleftarrow{\text{Id}_{\mathcal{P}} \otimes \iota} & \mathcal{P} \otimes \mathcal{I} \\ & \searrow \text{Id} & \downarrow m & \swarrow \text{Id} & \\ & & \mathcal{P} & & \end{array}$$

We shall say that (\mathcal{P}, m) is commutative if $m \circ c_{\mathcal{P}, \mathcal{P}} = m$.

In other words, a twisted algebra \mathcal{P} is given, for any finite sets A and B , a map $m_{A,B} : \mathcal{P}[A] \otimes \mathcal{P}[B] \rightarrow \mathcal{P}[A \sqcup B]$ such that:

1. For any bijections $\sigma : A \rightarrow A'$ and $\tau : B \rightarrow B'$ between finite sets, then:

$$m_{A',B'} \circ (\mathcal{P}[\sigma] \otimes \mathcal{P}[\tau]) = \mathcal{P}[\sigma \sqcup \tau] \circ m_{A,B} : \mathcal{P}[A] \otimes \mathcal{P}[B] \rightarrow \mathcal{P}[A' \sqcup B'].$$

2. For any finite sets A, B, C :

$$m_{A \sqcup B, C} \circ (m_{A,B} \otimes \text{Id}_{\mathcal{P}[C]}) = m_{A, B \sqcup C} \circ (\text{Id}_{\mathcal{P}[A]} \otimes m_{B,C}).$$

3. There exists an element $1_{\mathcal{P}} \in \mathcal{P}[\emptyset]$ such that for any finite set A , and $x \in \mathcal{P}[A]$:

$$m_{\emptyset, A}(1_{\mathcal{P}} \otimes x) = m_{A, \emptyset}(x \otimes 1_{\mathcal{P}}) = x.$$

Moreover, \mathcal{P} is commutative if, and only if, for any finite set A, B , in $\mathcal{P}[A \sqcup B]$:

$$\forall x \in \mathcal{P}[A], \forall y \in \mathcal{P}[B], \quad m_{A,B}(x \otimes y) = m_{B,A}(y \otimes x).$$

Example 1.3. The following examples can be found in [3]:

1. The product of \mathbb{K} induces an algebra structure on **Com**: for any finite sets A and B , for any $\lambda \in \mathbb{K} = \mathbf{Com}[A]$, $\mu \in \mathbb{K} = \mathbf{Com}[B]$,

$$m_{A,B}(\lambda \otimes \mu) = \lambda \mu \in \mathbb{K} = \mathbf{Com}[A \sqcup B].$$

The unit of **Com** is the unit 1 of $\mathbb{K} = \mathbf{Com}[\emptyset]$.

2. The species *Comp* is an algebra, with the concatenation product:

$$m_{A,B}((A_1, \dots, A_k) \otimes (B_1, \dots, B_l)) = (A_1, \dots, A_k, B_1, \dots, B_l).$$

Its unit is the empty composition \emptyset .

3. The species of graphs $\mathcal{G}r'$ is an algebra with the product given by disjoint union of graphs. The unit is the empty graph. For example, if A, B, C, D and E are finite sets, then:

$$m_{A \sqcup B \sqcup C, D \sqcup E}({}^B \vee_A^C \otimes \downarrow_B^E) = {}^B \vee_A^C \downarrow_B^E.$$

4. The species *Top* is a twisted algebra with the disjoint union product.

Notations 1.3. Let \mathcal{P} be a twisted algebra. If $(A_1, \dots, A_k) \in \mathbf{Comp}[A]$, we inductively define m_{A_1, \dots, A_k} from $\mathcal{P}[A_1] \otimes \dots \otimes \mathcal{P}[A_k]$ into $\mathcal{P}[A]$ by:

$$m_A = \text{Id}_{\mathcal{P}[A]},$$

$$m_{A_1, \dots, A_k} = m_{A_1 \sqcup A_2, A_3, \dots, A_k} \circ (m_{A_1, A_2} \otimes \text{Id}_{\mathcal{P}[A_3]} \otimes \dots \otimes \text{Id}_{\mathcal{P}[A_k]}).$$

By associativity, if $(A_1, \dots, A_{k+l}) \in \mathbf{Comp}[A]$:

$$m_{A_1 \sqcup \dots \sqcup A_k, A_{k+1} \sqcup \dots \sqcup A_{k+l}} \circ (m_{A_1, \dots, A_k} \otimes m_{A_{k+1}, \dots, A_{k+l}}) = m_{A_1, \dots, A_{k+l}}.$$

1.3 Coalgebras and bialgebras in the category of species

Definition 1.2. A coalgebra in the category of species, or twisted coalgebra, is a pair (\mathcal{P}, Δ) where \mathcal{P} is a species and $\Delta : \mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{P}$ is a morphism of species such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\Delta} & \mathcal{P}^{\otimes 2} \\ \Delta \downarrow & & \downarrow \text{Id}_{\mathcal{P}} \otimes \Delta \\ \mathcal{P}^{\otimes 2} & \xrightarrow{\Delta \otimes \text{Id}_{\mathcal{P}}} & \mathcal{P}^{\otimes 3} \end{array}$$

Moreover, there exists a morphism of species $\varepsilon : \mathcal{P} \rightarrow \mathcal{I}$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{I} \otimes \mathcal{P} & \xrightarrow{\text{Id}} & \mathcal{P} & \xleftarrow{\text{Id}} & \mathcal{P} \otimes \mathcal{I} \\ & \searrow \varepsilon \otimes \text{Id}_{\mathcal{P}} & \downarrow \Delta & \nearrow \text{Id}_{\mathcal{P}} \otimes \varepsilon & \\ & & \mathcal{P}^{\otimes 2} & & \end{array}$$

We shall say that (\mathcal{P}, Δ) is cocommutative if $c_{\mathcal{P}, \mathcal{P}} \circ \Delta = \Delta$. We shall say that the coalgebra (\mathcal{P}, Δ) is connected if $\mathcal{P}[\emptyset]$ is one-dimensional.

In other words, a twisted coalgebra \mathcal{P} is given, for any finite sets A and B , a map $\Delta_{A,B} : \mathcal{P}[A \sqcup B] \rightarrow \mathcal{P}[A] \otimes \mathcal{P}[B]$ such that:

1. For any bijections $\sigma : A \rightarrow A'$ and $\tau : B \rightarrow B'$ between finite sets, then:

$$(\mathcal{P}[\sigma] \otimes \mathcal{P}[\tau]) \circ \Delta_{A,B} = \Delta_{A',B'} \circ \mathcal{P}[\sigma \sqcup \tau] : \mathcal{P}[A \sqcup B] \rightarrow \mathcal{P}[A'] \otimes \mathcal{P}[B'].$$

2. For any finite sets A, B, C :

$$(\Delta_{A,B} \otimes \text{Id}_{\mathcal{P}[C]}) \circ \Delta_{A \sqcup B, C} = (\text{Id}_{\mathcal{P}[A]} \otimes \Delta_{B,C}) \circ \Delta_{A, B \sqcup C}.$$

3. There exists a linear map $\varepsilon_{\mathcal{P}} : \mathcal{P}[\emptyset] \rightarrow \mathbb{K}$ such that for any finite set A :

$$(\varepsilon_{\mathcal{P}} \otimes \text{Id}_{\mathcal{P}[A]}) \circ \Delta_{\emptyset, A} = (\text{Id}_{\mathcal{P}[A]} \otimes \varepsilon_{\mathcal{P}}) \circ \Delta_{A, \emptyset} = \text{Id}_{\mathcal{P}[A]}.$$

Example 1.4. 1. The species **Com** is given a coalgebra structure by the following: for any finite sets A, B ,

$$\forall \lambda \in \mathbb{K} = \mathbf{Com}[A \sqcup B], \quad \Delta_{A,B}(\lambda) = \lambda(1 \otimes 1).$$

The counit is the identity of \mathbb{K} .

2. The species **Comp** is given a coalgebra structure by:

$$\Delta_{A,B}(A_1, \dots, A_k) = \begin{cases} (A_1, \dots, A_p) \otimes (A_{p+1}, \dots, A_k) & \text{if there exists a } p \text{ (necessarily unique)} \\ & \text{such that } A_1 \sqcup \dots \sqcup A_p = A, \\ 0 & \text{otherwise.} \end{cases}$$

The counit is given by $\varepsilon[\emptyset] = 1$.

Remark 1.1. 1. If $(\mathcal{P}, m_{\mathcal{P}})$ and $(\mathcal{Q}, m_{\mathcal{Q}})$ are two algebras in the category of species, then $\mathcal{P} \otimes \mathcal{Q}$ is too. Its product is given by:

$$m_{\mathcal{P} \otimes \mathcal{Q}} = (m_{\mathcal{P}} \otimes m_{\mathcal{Q}}) \circ (\text{Id}_{\mathcal{P}} \otimes c_{\mathcal{P}, \mathcal{Q}} \otimes \text{Id}_{\mathcal{Q}}).$$

Its unit is $1_{\mathcal{P}} \otimes 1_{\mathcal{Q}}$.

2. Dually, if $(\mathcal{P}, \Delta_{\mathcal{P}})$ and $(\mathcal{Q}, \Delta_{\mathcal{Q}})$ are two coalgebras in the category of species, then $\mathcal{P} \otimes \mathcal{Q}$ is too. Its coproduct is given by:

$$\Delta_{\mathcal{P} \otimes \mathcal{Q}} = (\text{Id}_{\mathcal{P}} \otimes c_{\mathcal{P}, \mathcal{Q}} \otimes \text{Id}_{\mathcal{Q}}) \circ (\Delta_{\mathcal{P}} \otimes \Delta_{\mathcal{Q}}).$$

Its counit is $\varepsilon_{\mathcal{P}} \otimes \varepsilon_{\mathcal{Q}}$.

Lemma 1.3. *Let us assume that the twisted coalgebra (\mathcal{P}, Δ) is connected. There exists a unique $1_{\mathcal{P}} \in \mathcal{P}[\emptyset]$ such that $\varepsilon(1_{\mathcal{P}}) = 1$. Moreover, for any finite set A , for any $x \in \mathcal{P}[A]$:*

$$\Delta_{A, \emptyset}(x) = x \otimes 1_{\mathcal{P}}, \quad \Delta_{\emptyset, A}(x) = 1_{\mathcal{P}} \otimes x.$$

Proof. For any finite set A , $(\varepsilon \otimes \text{Id}) \circ \Delta_{\emptyset, A} = \text{Id}_{\mathcal{P}[A]}$, so $\varepsilon \neq 0$. As $\mathcal{P}[\emptyset]$ is one dimensional, $\varepsilon : \mathcal{P}[\emptyset] \rightarrow \mathbb{K}$ is bijective. Let $1_{\mathcal{P}}$ be the unique element of $\mathcal{P}[\emptyset]$ such that $\varepsilon(1_{\mathcal{P}}) = 1$. For any $x \in \mathcal{P}[A]$, there exists $y \in \mathcal{P}[A]$ such that $\Delta_{\emptyset, A}(x) = 1_{\mathcal{P}} \otimes y$. Then:

$$(\varepsilon \otimes \text{Id}) \circ \Delta_{\emptyset, A}(x) = x = y,$$

so $\Delta_{\emptyset, A}(x) = 1_{\mathcal{P}} \otimes x$. Similarly, $\Delta_{A, \emptyset}(x) = x \otimes 1_{\mathcal{P}}$. □

Notations 1.4. Let \mathcal{P} be a twisted coalgebra. If $A_1 \sqcup \dots \sqcup A_k = A$, we inductively define Δ_{A_1, \dots, A_k} by:

$$\begin{aligned} \Delta_A &= \text{Id}_{\mathcal{P}[A]}, \\ \Delta_{A_1, \dots, A_{k+1}} &= (\Delta_{A_1, \dots, A_k} \otimes \text{Id}_{\mathcal{P}[A_{k+1}]}) \circ \Delta_{A_1 \sqcup \dots \sqcup A_k, A_{k+1}} \text{ if } k \geq 1. \end{aligned}$$

By coassociativity, for any finite sets A_1, \dots, A_{k+l} :

$$(\Delta_{A_1, \dots, A_k} \otimes \Delta_{A_{k+1}, \dots, A_{k+l}}) \circ \Delta_{A_1 \sqcup \dots \sqcup A_k, A_{k+1} \sqcup \dots \sqcup A_{k+l}} = \Delta_{A_1, \dots, A_{k+l}}.$$

As for "classical" bialgebras:

Definition 1.4. *Let \mathcal{P} be a species, both a twisted algebra $(\mathcal{P}, m_{\mathcal{P}})$ and a twisted coalgebra $(\mathcal{P}, \Delta_{\mathcal{P}})$. The following conditions are equivalent:*

1. $\varepsilon_{\mathcal{P}} : \mathcal{P} \rightarrow \mathcal{I}$ and $\Delta_{\mathcal{P}} : \mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{P}$ are algebra morphisms.
2. $\iota : \mathcal{I} \rightarrow \mathcal{P}$ and $m : \mathcal{P} \otimes \mathcal{P} \rightarrow \mathcal{P}$ are coalgebra morphisms.

If this holds, we shall say that $(\mathcal{P}, m_{\mathcal{P}}, \Delta_{\mathcal{P}})$ is a bialgebra in the category of species or a twisted bialgebra [2, 28].

The compatibility between the product and coproduct can also be written in this way: if A is a finite set and $A = I \sqcup J = I' \sqcup J'$,

$$\begin{aligned} \Delta_{I, J} \circ m_{I', J'} &= (m_{I' \cap I, J' \cap I} \otimes m_{I' \cap J, J' \cap J}) \circ (\text{Id}_{\mathcal{P}[I \cap I]} \otimes c_{\mathcal{P}[I' \cap J], \mathcal{P}[J' \cap I]} \otimes \text{Id}_{\mathcal{P}[J' \cap J]}) \\ &\quad \circ (\Delta_{I' \cap I, I' \cap J} \otimes \Delta_{J' \cap I, J' \cap J}). \end{aligned}$$

Example 1.5. The species **Com**, with the product and coproduct defined below, is a twisted bialgebra.

Remark 1.2. If \mathcal{P} is a connected twisted bialgebra, that is to say if it is connected as a coalgebra, the elements $1_{\mathcal{P}}$ defined in Lemma 1.3 is necessarily the unit of the algebra \mathcal{P} .

2 Cofreeness

2.1 Cofree coalgebras

Definition 2.1. Let \mathcal{P} be a species such that $\mathcal{P}[\emptyset] = (0)$. The cofree coalgebra (co)generated by \mathcal{P} is the species:

$$\text{co}\mathcal{T}(\mathcal{P}) = \bigoplus_{n=0}^{\infty} \mathcal{P}^{\otimes n},$$

that is to say, for any finite set A :

$$\text{co}\mathcal{T}(\mathcal{P})[A] = \bigoplus_{(A_1, \dots, A_k) \in \mathbf{Comp}[A]} \mathcal{P}[A_1] \otimes \dots \otimes \mathcal{P}[A_k].$$

For any $x_i \in \mathcal{P}[A_i]$, $i \in \underline{k}$, we denote by $x_1 \dots x_k$ the tensor product of the elements x_1, \dots, x_k in $\mathcal{P}[A_1] \otimes \dots \otimes \mathcal{P}[A_k] \subseteq \text{co}\mathcal{T}(\mathcal{P})[A_1 \sqcup \dots \sqcup A_k]$. This species is given a coproduct making it a twisted coalgebra: for any finite sets A, B , $(C_1, \dots, C_k) \in \mathbf{Comp}[A \sqcup B]$, $x_i \in \mathcal{P}[A_i]$,

$$\Delta_{A,B}(x_1 \dots x_k) = \begin{cases} x_1 \dots x_i \otimes x_{i+1} \dots x_k & \text{if there exists an } i \text{ (necessarily unique)} \\ & \text{such that } A_1 \sqcup \dots \sqcup A_i = A, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\pi : \text{co}\mathcal{T}(\mathcal{P}) \rightarrow \mathcal{P}$ the canonical projection vanishing on $\mathcal{P}^{\otimes k}$ if $k \neq 1$.

More generally, we shall say that a twisted coalgebra is cofree if it is isomorphic to a coalgebra $\text{co}\mathcal{T}(\mathcal{P})$.

Theorem 2.2. Let \mathcal{P} be a species such that $\mathcal{P}[\emptyset] = (0)$ and let \mathcal{C} be a connected twisted coalgebra. For any species morphism $\phi : \mathcal{C} \rightarrow \mathcal{P}$, there exists a unique coalgebra morphism $\Phi : \mathcal{C} \rightarrow \text{co}\mathcal{T}(\mathcal{P})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Phi} & \text{co}\mathcal{T}(\mathcal{P}) \\ & \searrow \phi & \downarrow \pi \\ & & \mathcal{P} \end{array}$$

Moreover, for any finite set A :

$$\Phi[A] = \sum_{(A_1, \dots, A_k) \in \mathbf{Comp}[A]} (\phi[A_1] \otimes \dots \otimes \phi[A_k]) \circ \Delta_{A_1, \dots, A_k}. \quad (1)$$

Proof. Existence. Let Φ be the species morphism defined by (1). For any finite set A :

$$\pi[A] \circ \Phi[A] = \phi[A] \circ \Delta_A = \phi[A],$$

so $\pi \circ \Phi = \phi$. Let A and B be two finite sets. Let us prove that:

$$\Delta_{A,B} \circ \Phi[A \sqcup B] = (\Phi[A] \otimes \Phi[B]) \circ \Delta_{A,B}.$$

If $A = B = \emptyset$, it is enough to apply this two maps on $1_{\mathcal{C}}$:

$$\Delta_{\emptyset, \emptyset} \circ \Phi[\emptyset \sqcup \emptyset](1_{\mathcal{C}}) = 1 \otimes 1 = (\Phi[\emptyset] \otimes \Phi[\emptyset]) \circ \Delta_{\emptyset, \emptyset}(1_{\mathcal{C}}).$$

If $A = \emptyset$ and $B \neq \emptyset$, for any $x \in \mathcal{C}(A)$:

$$\Delta_{\emptyset, B} \circ \Phi[B](x) = 1 \otimes \Phi[B](x) = (\Phi[\emptyset] \otimes \Phi[B]) \circ \Delta_{\emptyset, B}(x).$$

A similar computation works if $B = \emptyset$ and $A \neq \emptyset$. Let us assume now that $A, B \neq \emptyset$.

$$\begin{aligned}
\Delta_{A,B} \circ \Phi[A \sqcup B] &= \Delta_{A,B} \circ \left(\sum_{(A_1, \dots, A_k) \in \mathbf{Comp}[A \sqcup B]} \phi^{\otimes k} \circ \Delta_{A_1, \dots, A_k} \right) \\
&= \Delta_{A,B} \circ \left(\sum_{\substack{(A_1, \dots, A_p) \in \mathbf{Comp}[A], \\ (B_1, \dots, B_q) \in \mathbf{Comp}[B]}} \phi^{\otimes(p+q)} \circ \Delta_{A_1, \dots, A_p, B_1, \dots, B_q} \right) \\
&= \sum_{\substack{(A_1, \dots, A_p) \in \mathbf{Comp}[A], \\ (B_1, \dots, B_q) \in \mathbf{Comp}[B]}} \phi^{\otimes p} \circ \Delta_{A_1, \dots, A_p} \otimes \phi^{\otimes q} \circ \Delta_{B_1, \dots, B_q} \\
&= (\Phi[A] \otimes \Phi[B]) \circ \Delta_{A,B}.
\end{aligned}$$

So Φ is a coalgebra morphism.

Unicity. By definition of Δ , for any nonempty set A , in $co\mathcal{T}(\mathcal{P})[A]$:

$$\bigcap_{(I,J) \in \mathbf{Comp}[A]} \ker(\Delta_{I,J}) = \mathcal{P}[A].$$

Let Φ and Φ' be two coalgebra morphisms satisfying the required conditions. Let us prove that $\Phi[A] = \Phi'[A]$ by induction on $\sharp A$. If $A = \emptyset$, as 1 is the unique nonzero element of $co\mathcal{T}(\mathcal{P})[\emptyset]$ such that $\Delta_{\emptyset, \emptyset}(1) = 1 \otimes 1$:

$$\Phi[\emptyset](1_C) = \Phi'[\emptyset](1_C) = 1.$$

As $\mathcal{C}[\emptyset] = \mathbb{K}1_C$, $\Phi[\emptyset] = \Phi'[\emptyset]$. Let us assume the result at all ranks $< \sharp A$. For any $(I, J) \in \mathbf{Comp}[A]$:

$$\Delta_{I,J} \circ \Phi[A] = (\Phi[I] \otimes \Phi[J]) \circ \Delta_{I,J} = (\Phi'[I] \otimes \Phi'[J]) \circ \Delta_{I,J} = \Delta_{I,J} \circ \Phi'[A].$$

Hence, $\Phi[A] - \Phi'[A]$ takes its values in $\mathcal{P}[A]$, so:

$$\Phi[A] - \Phi'[A] = \pi[A] \circ \Phi[A] - \pi[A] \circ \Phi'[A] = \phi[A] - \phi[A] = 0.$$

Therefore, $\Phi = \Phi'$. □

Example 2.1. Let us consider the species \mathcal{P} defined by:

$$\mathcal{P}[A] = \begin{cases} \mathbb{K} & \text{if } \sharp A = 1, \\ (0) & \text{otherwise.} \end{cases}$$

The cofree coalgebra $co\mathcal{T}(\mathcal{P})$ is denoted by $\mathcal{O}rd$. For any finite set A , noticing that compositions of A which parts have cardinality 1 are in bijection with bijections from A to $\underline{\sharp A}$:

$$\mathcal{O}rd(A) = \bigoplus_{\substack{f: A \rightarrow \underline{\sharp A}, \\ \text{bijection}}} \mathbb{K}^{\otimes \sharp A}.$$

Hence, $\mathcal{O}rd(A)$ has a basis indexed by bijections from A to $\underline{\sharp A}$, or equivalently a basis indexed by total orders on A . If $A = I \sqcup J$ and \leq is a total order on A :

$$\Delta_{I,J}(\leq) = \begin{cases} \leq|_I \otimes \leq|_J & \text{if for any } x \in I, y \in J, x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

2.2 Quasishuffle products

Notations 2.1. 1. Let n_1, \dots, n_k be integers. Let us denote by $\text{QSh}(n_1, \dots, n_k)$ the set of (n_1, \dots, n_k) -quasi-shuffles, that is to say surjective map $\sigma : \underline{n_1 + \dots + n_k} \longrightarrow \underline{\max(\sigma)}$, such that for all $i \in \underline{k}$, $\sigma|_{\{n_1 + \dots + n_{i-1} + 1, \dots, n_1 + \dots + n_i\}}$ is non decreasing. Bijective quasi-shuffles are called shuffles; the set of (n_1, \dots, n_k) -shuffles is denoted by $\text{Sh}(n_1, \dots, n_k)$.

2. Let \mathcal{P} be a species. We consider the species \mathcal{P}_+ defined by:

$$\mathcal{P}_+[A] = \begin{cases} (0) & \text{if } A = \emptyset, \\ \mathcal{P}[A] & \text{if } A \neq \emptyset. \end{cases}$$

Note that if \mathcal{P} is a twisted algebra, then \mathcal{P}_+ is stable under the product of \mathcal{P} .

3. Let \mathcal{P} be a twisted algebra. Let $(A_1, \dots, A_k) \in \mathbf{Comp}[A]$, where A is a finite set, and let $\sigma : \underline{k} \longrightarrow \underline{\max(\sigma)}$ be a surjection. For any $x_i \in \mathcal{P}[A_i]$, $i \in \underline{k}$, we put:

$$\sigma \rightarrow x_1 \dots x_k = \left(\prod_{i, \sigma(i)=1}^m x_i \right) \dots \left(\prod_{i, \sigma(i)=\max(\sigma)}^m x_i \right),$$

where each product is taken in \mathcal{P} . This is an element of $\mathcal{P}^{\otimes \max(\sigma)}[A]$.

The construction of quasi-shuffle products [19, 11] extends to the category of species:

Theorem 2.3. *Let \mathcal{P} be a twisted algebra. The cofree coalgebra $\text{co}\mathcal{T}(\mathcal{P}_+)$ is given an associative product \boxplus : if, for any $i \in \underline{k+l}$, $x_i \in \mathcal{P}[A_i]$, then*

$$x_1 \dots x_k \boxplus x_{k+1} \dots x_{k+l} = \sum_{\sigma \in \text{QSh}(k,l)} \sigma \rightarrow (x_1 \dots x_{k+l}).$$

Then $(\text{co}\mathcal{T}(\mathcal{P}_+), \boxplus, \Delta)$ is a twisted bialgebra. Moreover, \boxplus is commutative if, and only if, m is commutative.

Proof. Let \mathcal{C} be the coalgebra $\text{co}\mathcal{T}(\mathcal{P}_+) \otimes \text{co}\mathcal{T}(\mathcal{P}_+)$. Then $\mathcal{C}[\emptyset] = \mathbb{K}1 \otimes 1$: \mathcal{C} is connected. We shall identify the unit 1 of \mathbb{K} (also unit of $\text{co}\mathcal{T}(\mathcal{P}_+)$) and the unit of \mathcal{P} . We consider the species morphism $\phi : \mathcal{C} \longrightarrow \mathcal{P}_+$ defined by:

$$\phi|_{\mathcal{P}^{\otimes k} \otimes \mathcal{P}^{\otimes l}} = \begin{cases} m & \text{if } (k, l) \in \{(0, 1), (1, 0), (1, 1)\}, \\ 0 & \text{otherwise.} \end{cases}$$

There exists a unique coalgebra $\boxplus : \mathcal{C} \longrightarrow \text{co}\mathcal{T}(\mathcal{P}_+)$ such that $\pi \circ \boxplus = \phi$. Let us prove that \boxplus is associative. Let us consider the coalgebra morphisms $\boxplus \circ (\boxplus \otimes \text{Id})$ and $\boxplus \circ (\text{Id} \otimes \boxplus)$ from $\text{co}\mathcal{T}(\mathcal{P}_+)^{\otimes 3}$ to $\text{co}\mathcal{T}(\mathcal{P}_+)$. For any $k, l, p \geq 0$:

$$\begin{aligned} \pi \circ \boxplus \circ (\boxplus \otimes \text{Id})|_{\mathcal{P}^{\otimes k} \otimes \mathcal{P}^{\otimes l} \otimes \mathcal{P}^{\otimes p}} &= \begin{cases} m \circ (m \otimes \text{Id}) & \text{if } k, l, p \leq 1 \text{ and } (k, l, p) \neq (0, 0, 0), \\ 0 & \text{otherwise;} \end{cases} \\ \pi \circ \boxplus \circ (\text{Id} \otimes \boxplus)|_{\mathcal{P}^{\otimes k} \otimes \mathcal{P}^{\otimes l} \otimes \mathcal{P}^{\otimes p}} &= \begin{cases} m \circ (\text{Id} \otimes m) & \text{if } k, l, p \leq 1 \text{ and } (k, l, p) \neq (0, 0, 0), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

As m is associative, $\pi \circ \boxplus \circ (\boxplus \otimes \text{Id}) = \pi \circ \boxplus \circ (\text{Id} \otimes \boxplus)$. By unicity in Theorem 2.2, $\boxplus \circ (\boxplus \otimes \text{Id}) = \boxplus \circ (\text{Id} \otimes \boxplus)$. So $(\text{co}\mathcal{T}(\mathcal{P}_+), \boxplus, \Delta)$ is a twisted bialgebra.

Let $x_i \in \mathcal{P}[A_i]$ for all i . We put $A = A_1 \sqcup \dots \sqcup A_{k+l}$. By Theorem 2.2:

$$\begin{aligned}
& \boxplus (x_1 \dots x_k \otimes x_{k+1} \dots x_{k+l}) \\
&= \sum_{(B_1, \dots, B_n) \in \mathbf{Comp}[A]} \phi^{\otimes n} \circ \Delta_{A_1, \dots, A_n} (x_1 \dots x_k \otimes x_{k+1} \dots x_{k+l}) \\
&= \sum_{\substack{1=i_0 \leq i_1 \leq \dots \leq i_p = k, \\ 1=j_0 \leq j_1 \leq \dots \leq j_p = l}} \phi (x_1 \dots x_{i_1} \otimes x_{k+1} \dots x_{k+j_1}) \dots \phi (x_{i_{p-1}+1} \dots x_{i_p} \otimes x_{k+j_{p-1}+1} \dots x_{k+j_p}) \\
&= \sum_{\substack{1=i_0 \leq i_1 \leq \dots \leq i_p = k, \\ 1=j_0 \leq j_1 \leq \dots \leq j_p = l, \\ \forall s, i_{s+1} - i_s \leq 1, \\ \forall t, j_{t+1} - j_t \leq 1}} \phi (x_1 \dots x_{i_1} \otimes x_{k+1} \dots x_{k+j_1}) \dots \phi (x_{i_{p-1}+1} \dots x_{i_p} \otimes x_{k+j_{p-1}+1} \dots x_{k+j_p}) \\
&= \sum_{\sigma \in \text{QSh}(k, l)} \sigma \rightarrow (x_1 \dots x_{k+l}).
\end{aligned}$$

Let us assume that \boxplus is commutative. For any finite set A and B :

$$\begin{aligned}
& \boxplus_{B, A} \circ c_{\text{co}\mathcal{T}(\mathcal{P}_+)[A], \text{co}\mathcal{T}(\mathcal{P}_+)[B]} = \boxplus_{A, B}, \\
& \pi[A \sqcup B] \circ \boxplus_{B, A} \circ c_{\mathcal{P}[A], \mathcal{P}[B]} = (\pi[A \sqcup B] \circ \boxplus_{A, B})|_{\mathcal{P}[A] \otimes \mathcal{P}[B]}, \\
& m_{B, A} \circ c_{\mathcal{P}[A], \mathcal{P}[B]} = m_{A, B}.
\end{aligned}$$

So m is commutative.

Let us assume that m is commutative. Then \boxplus and $\boxplus \circ c$ are both coalgebra morphisms from $\text{co}\mathcal{T}(\mathcal{P}_+)^{\otimes 2}$ to $\text{co}\mathcal{T}(\mathcal{P}_+)$ and, as m is commutative, $\pi \circ \boxplus \circ c = \pi \circ m = \phi$. By unicity in Theorem 2.2, $\boxplus = \boxplus \circ c$. \square

Example 2.2. 1. Let us consider the bialgebra \mathbf{Com} . The cofree coalgebra $\text{co}\mathcal{T}(\mathbf{Com}_+)$ is a twisted bialgebra. For any finite set A :

$$\text{co}\mathcal{T}(\mathbf{Com}_+)[A] = \bigoplus_{(A_1, \dots, A_k) \in \mathbf{Comp}[A]} \mathbf{Com}[A_1] \otimes \dots \otimes \mathbf{Com}[A_k].$$

The element $1 \otimes \dots \otimes 1$ of $\mathbf{Com}[A_1] \otimes \dots \otimes \mathbf{Com}[A_k]$ will be identified with $(A_1, \dots, A_k) \in \mathbf{Comp}[A]$: this defines a species isomorphism between $\text{co}\mathcal{T}(\mathbf{Com}_+)$ and \mathbf{Comp} . The product \boxplus induced on \mathbf{Comp} is given by:

$$(A_1, \dots, A_k) \boxplus (B_1, \dots, B_k) = \sum_{\sigma \in \text{QSh}(k, l)} \left(\bigcup_{i \in \sigma^{-1}(1)} A_i \right) \dots \left(\bigcup_{i \in \sigma^{-1}(\max(\sigma))} A_i \right).$$

The coproduct is given by:

$$\Delta_{A, B}(A_1, \dots, A_k) = \begin{cases} (A_1, \dots, A_p) \otimes (A_{p+1}, \dots, A_k) & \text{if there exists } p \text{ (necessarily unique)} \\ & \text{such that } A_1 \sqcup \dots \sqcup A_p = A, \\ 0 & \text{otherwise.} \end{cases}$$

The counit is given by $\varepsilon_{\mathbf{Comp}}[\emptyset] = 1$. For example, if A, B, C, D are nonempty finite sets:

$$\begin{aligned}
(A) \boxplus (B) &= (A, B) + (B, A) + (A \sqcup B), \\
(A, B) \boxplus (C) &= (A, B, C) + (A, C, B) + (C, A, B) + (A \sqcup B, C) + (A, B \sqcup C), \\
(A, B) \boxplus (C, D) &= (A, B, C, D) + (A, C, B, D) + (C, A, B, D) \\
&\quad + (A, C, D, B) + (C, A, D, B) + (C, D, A, B) \\
&\quad + (A, B \sqcup C, D) + (A \sqcup C, B, D) + (A \sqcup C, D, B) \\
&\quad + (A, C, B \sqcup D) + (C, A, B \sqcup D) + (C, A \sqcup D, B) \\
&\quad + (A \sqcup C, B \sqcup D).
\end{aligned}$$

2. Let \mathcal{P} be any species such that $\mathcal{P}[\emptyset] = (0)$. We give it a trivial product $m = 0$. If for all $i \in \underline{k+l}$, $x_i \in \mathcal{P}[A_i]$ and $\sigma \in \text{QSh}(k, l)$, then $\sigma \rightarrow (x_1 \dots x_{k+l}) = 0$ if σ is not bijective, that is to say if σ is not a shuffle. Hence, the product induced by m on $\text{co}\mathcal{T}(\mathcal{P})$ is the shuffle product:

$$\sqcup(x_1 \dots x_k \otimes x_{k+1} \dots x_{k+l}) = \sum_{\sigma \in \text{Sh}(k, l)} \sigma \rightarrow (x_1 \dots x_{k+l}).$$

We obtain the shuffle product of [3]. In particular, this holds for Ord . If A and B are finite sets and \leq_A, \leq_B are total orders on respectively A and B , for any $\sigma \in \text{QSh}(\sharp A, \sharp B)$, $\sigma \rightarrow (\leq_A \leq_B)$ is nonzero if, and only if, σ is a bijective, that is to say if, and only if, σ is a shuffle. Consequently:

$$\leq_A \sqcup \leq_B = \sum_{\substack{\leq \text{ total order on } A \sqcup B, \\ \leq|_A = \leq_A, \leq|_B = \leq_B}} \leq.$$

Proposition 2.4. *Let (\mathcal{P}, m) be a twisted algebra. The map $\varepsilon' : \text{co}\mathcal{T}(\mathcal{P}_+) \rightarrow \mathcal{P}$ defined by $\varepsilon'[\emptyset](1) = 1_{\mathcal{P}}$ and $\varepsilon'[A] = \pi[A]$ if $A \neq \emptyset$ is an algebra morphism.*

Proof. As 1 is the unit of \sqcup , it is enough to prove that:

$$\varepsilon'[A \sqcup B] \circ \sqcup_{A, B} = m_{A, B} \circ (\varepsilon'[A] \otimes \varepsilon'[B]),$$

for any nonempty sets A and B . Let $(A_1, \dots, A_k) \in \mathbf{Comp}[A]$, $(B_1, \dots, B_l) \in \mathbf{Comp}[B]$, $x_i \in \mathcal{P}[A_i]$, $x_{k+i} \in \mathcal{P}[B_i]$. By construction of \sqcup :

$$\begin{aligned} \varepsilon'[A \sqcup B] \circ \sqcup_{A, B}(x_1 \dots x_k \otimes x_{k+1} \dots x_{k+l}) &= \begin{cases} m(x_1 \otimes x_2) & \text{if } k = l = 1, \\ 0 & \text{otherwise;} \end{cases} \\ &= m_{A, B} \circ (\varepsilon'[A] \otimes \varepsilon'[B])(x_1 \dots x_k \otimes x_{k+1} \dots x_{k+l}). \end{aligned}$$

So ε' is a twisted algebra morphism. \square

As Comp is isomorphic to $\text{co}\mathcal{T}(\mathbf{Com}_+)$:

Corollary 2.5. *The following map is an algebra morphism from (Comp, \sqcup) to (\mathbf{Com}, m) :*

$$\varepsilon' : \begin{cases} \text{Comp}[A] & \longrightarrow \mathbf{Com}[A] \\ (A_1, \dots, A_k) & \longrightarrow \delta_{k, 1}. \end{cases}$$

Theorem 2.6. *Let \mathcal{P} be a twisted algebra and let \mathcal{B} be a connected twisted bialgebra. For any species morphism $\phi : \mathcal{B} \rightarrow \mathcal{P}$, there exists a unique twisted coalgebra morphism Φ from \mathcal{B} to $(\text{co}\mathcal{T}(\mathcal{P}_+), \sqcup, \Delta)$ such that $\varepsilon' \circ \Phi = \phi$. For any finite set A :*

$$\Phi[A] = \sum_{(A_1, \dots, A_k) \in \mathbf{Comp}[A]} \phi^{\otimes k} \circ \Delta_{A_1, \dots, A_k}.$$

Moreover, Φ is a twisted bialgebra morphism if, and only if, ϕ is twisted algebra morphism.

Proof. Let ϕ' defined by:

$$\phi'[A] = \begin{cases} 0 & \text{if } A = \emptyset, \\ \phi[A] & \text{if } A \neq \emptyset. \end{cases}$$

For any coalgebra morphism $\Phi : \mathcal{B} \rightarrow \text{co}\mathcal{T}(\mathcal{P}_+)$, $\Phi(1_{\mathcal{B}}) = 1$. As $\mathcal{B}[\emptyset] = \mathbb{K}1_C$, $\varepsilon' \circ \Phi = \phi$ if, and only if, $\pi \circ \Phi = \phi'$. By Theorem 2.2, such a morphism exists and is unique.

Let us assume that Φ is a bialgebra morphism. By composition, $\phi = \varepsilon' \circ \Phi$ is an algebra morphism. Let us assume that ϕ is an algebra morphism. Let us consider the coalgebra morphisms $\Phi \circ m$ and $\boxplus \circ (\Phi \otimes \Phi)$, both from $\mathcal{B} \otimes \mathcal{B}$ to $\text{co}\mathcal{T}(\mathcal{P}_+)$.

$$\begin{aligned}\varepsilon' \circ \Phi \circ m &= \phi \circ m, \\ \varepsilon' \circ \boxplus \circ (\Phi \otimes \Phi) &= m_{\mathcal{P}} \circ (\varepsilon' \otimes \varepsilon') \circ (\Phi \otimes \Phi) \\ &= m_{\mathcal{P}} \circ (\phi \otimes \phi) \\ &= \phi \circ m.\end{aligned}$$

Consequently, $\varepsilon' \circ \Phi \circ m = \varepsilon' \circ \boxplus \circ (\Phi \otimes \Phi)$, so $\pi \circ \Phi \circ m = \pi \circ \boxplus \circ (\Phi \otimes \Phi)$. By unicity in Theorem 2.2, $\Phi \circ m = \boxplus \circ (\Phi \otimes \Phi)$, so Φ is a bialgebra morphism. \square

2.3 A criterion of cofreeness

Let us give here a twisted version of Loday and Ronco's rigidity theorem for bialgebras [21].

Theorem 2.7. *Let (\mathcal{P}, Δ) be a connected twisted coalgebra. We denote by 1 the unique nonzero element of $\mathcal{P}[\emptyset]$ such that $\Delta_{\emptyset, \emptyset}(1) = 1 \otimes 1$. The following propositions are equivalent:*

1. (\mathcal{P}, Δ) is isomorphic to a cofree coalgebra $\text{co}\mathcal{T}(\mathcal{Q})$.
2. There exists a structure of twisted algebra (\mathcal{P}, m) on \mathcal{P} , of unit 1 , such that for any finite sets I, J, I' and J' such that $I \sqcup J = I' \sqcup J'$, for any $x \in \mathcal{P}[I']$, $y \in \mathcal{P}[J']$:

$$\Delta_{I, J} \circ m_{I', J'}(x \otimes y) = \begin{cases} (m_{I', I \setminus I'} \otimes \text{Id})(x \otimes \Delta_{I \setminus I', J}(y)) & \text{if } I' \subseteq I, \\ (\text{Id} \otimes m_{J \setminus J', J'})(\Delta_{I, J \setminus J'}(x) \otimes y) & \text{if } J' \subseteq J, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. $1 \implies 2$. Up to an isomorphism, we assume that the coalgebras (\mathcal{P}, Δ) and $\text{co}\mathcal{T}(\mathcal{Q})$ are equal. Let us denote by m the concatenation product of \mathcal{P} . Let us consider $x_i \in \mathcal{Q}[A_i]$ for all $i \in \underline{k+l}$. Putting $I' = A_1 \sqcup \dots \sqcup A_k$ and $J' = A_{k+1} \sqcup \dots \sqcup A_{k+l}$, if $I \sqcup J = I' \sqcup J'$:

$$\begin{aligned}\Delta_{I, J} \circ m_{I', J'}(x_1 \dots x_k \otimes x_{k+1} \dots x_{k+l}) \\ &= \Delta_{I, J}(x_1 \dots x_{k+l}) \\ &= \begin{cases} x_1 \dots x_j \otimes x_{j+1} \dots x_{k+l} & \text{if there exists } j \text{ such that } A_1 \sqcup \dots \sqcup A_j = I, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Three cases are possible.

- Such a j exists and $j \leq k$. Then $J' \subseteq J$ and:

$$\begin{aligned}\Delta_{I, J} \circ m_{I', J'}(x_1 \dots x_k \otimes x_{k+1} \dots x_{k+l}) \\ &= x_1 \dots x_j \otimes x_{j+1} \dots x_k \cdot x_{k+1} \dots x_{k+l} \\ &= (\text{Id} \otimes m_{J \setminus J', J'})(\Delta_{I, J \setminus J'}(x_1 \dots x_k) \otimes x_{k+1} \dots x_{k+l}).\end{aligned}$$

- Such a j exists and $j \geq k$. Then $I' \subseteq I$ and:

$$\begin{aligned}\Delta_{I, J} \circ m_{I', J'}(x_1 \dots x_k \otimes x_{k+1} \dots x_{k+l}) \\ &= x_1 \dots x_k \cdot x_{k+1} \dots x_l \otimes x_{j+1} \dots x_{k+l} \\ &= (m_{I', I \setminus I'} \otimes \text{Id})(x_1 \dots x_k \otimes \Delta_{I \setminus I', J}(x_{k+1} \dots x_{k+l}))\end{aligned}$$

- No such j exists. Then $I' \not\subseteq I$, $J' \not\subseteq J$ and:

$$\Delta_{I, J} \circ m_{I', J'}(x_1 \dots x_k \otimes x_{k+1} \dots x_{k+l}) = 0.$$

2 \implies 1. Let \mathcal{Q} be the subspecies of \mathcal{P} defined by:

$$\mathcal{Q}[A] = \{x \in \mathcal{P}[A], \forall (I, J) \in \mathbf{Comp}(A), \Delta_{I,J}(x) = 0\}.$$

We shall consider the following morphism:

$$\Phi : \begin{cases} \text{co}\mathcal{T}(\mathcal{Q}) & \longrightarrow \mathcal{P} \\ x_1 \dots x_k \in \mathcal{Q}[A_1] \otimes \dots \otimes \mathcal{Q}[A_k] & \longrightarrow m_{A_1, \dots, A_k}(x_1 \otimes \dots \otimes x_k). \end{cases}$$

Let us first prove that Φ is a coalgebra morphism. Let $x_1 \dots x_k \in \mathcal{Q}[A_1] \otimes \dots \otimes \mathcal{Q}[A_k]$.

- If $I = A_1 \sqcup \dots \sqcup A_i$ for a certain i :

$$\begin{aligned} \Delta_{I,J} \circ \Phi(x_1 \dots x_k) &= \Delta_{I,J}(x_1 \cdot \dots \cdot x_k) \\ &= \Delta_{I,J}((x_1 \cdot \dots \cdot x_i) \cdot (x_{i+1} \cdot \dots \cdot x_k)) \\ &= (m_{I \emptyset} \otimes \text{Id})(x_1 \cdot \dots \cdot x_i \otimes \Delta_{\emptyset, J}(x_{i+1} \cdot \dots \cdot x_k)) \\ &= (m_{I \emptyset} \otimes \text{Id})(x_1 \cdot \dots \cdot x_i \otimes 1 \otimes x_{i+1} \cdot \dots \cdot x_k) \\ &= x_1 \cdot \dots \cdot x_i \otimes x_{i+1} \cdot \dots \cdot x_k \\ &= (\Phi \otimes \Phi) \circ \Delta_{I,J}(x_1 \dots x_k). \end{aligned}$$

- Otherwise, let i be the greatest integer such that $A_1 \sqcup \dots \sqcup A_i \subseteq I$. Then $A_{i+1} \cap I \neq \emptyset$.

$$\begin{aligned} \Delta_{I,J} \circ \Phi(x_1 \dots x_k) &= \Delta_{I,J}(x_1 \cdot \dots \cdot x_k) \\ &= (x_1 \cdot \dots \cdot x_i \otimes 1) \cdot \Delta_{I \setminus A_1 \sqcup \dots \sqcup A_i, J}(x_{i+1} \cdot \dots \cdot x_k). \end{aligned}$$

If $I \not\subseteq A_1 \sqcup \dots \sqcup A_{i+1}$, this is zero. Otherwise, $I \cap A_{i+1} \neq \emptyset$. Then:

$$\begin{aligned} \Delta_{I,J} \circ \Phi(x_1 \dots x_k) &= \Delta_{I,J}(x_1 \cdot \dots \cdot x_k) \\ &= (x_1 \cdot \dots \cdot x_i \otimes 1) \cdot \Delta_{I \setminus A_1 \sqcup \dots \sqcup A_i, J}(x_{i+1} \cdot \dots \cdot x_k) \\ &= (x_1 \cdot \dots \cdot x_i \otimes 1) \cdot \Delta_{A_{i+1} \cap I, A_{i+1} \cap J}(x_{i+1}) \cdot (1 \otimes x_{i+2} \cdot \dots \cdot x_k). \end{aligned}$$

By definition of \mathcal{Q} , $\Delta_{A_{i+1} \cap I, A_{i+1} \cap J}(x_{i+1}) = 0$. Finally:

$$\Delta_{I,J} \circ \Phi(x_1 \dots x_k) = 0 = (\Phi \otimes \Phi) \circ \Delta_{I,J}(x_1 \dots x_k).$$

Hence, Φ is a coalgebra morphism.

Let us now prove that Φ is injective. Let $x \in \text{co}\mathcal{T}(\mathcal{Q})[A]$, nonzero, such that $\Phi[A](x) = 0$. We assume that $\sharp A$ is minimal. If $A = I \sqcup J$, with $I \neq \emptyset$ and $J \neq \emptyset$:

$$(\Phi[I] \otimes \Phi[J]) \circ \Delta_{I,J}(x) = \Delta_{I,J} \circ \Phi[A](x) = 0.$$

By hypothesis on A , $\Phi[I]$ and $\Phi[J]$ are injective, so $\Delta_{I,J}(x) = 0$, and $x \in \mathcal{Q}[A]$. Then $\Phi[A](x) = x \neq 0$: this is a contradiction. Therefore, Φ is injective.

Let us finally prove that Φ is surjective. As $\mathcal{P}[\emptyset] = \mathbb{K}1$, $\mathcal{P}[\emptyset] = \Phi[\emptyset](\text{co}\mathcal{T}(\mathcal{P})[\emptyset])$. Let A be a nonempty set. For any $x \in \mathcal{P}[A]$, we put:

$$k(x) = \max\{l, \exists (I_1, \dots, I_l) \in \mathbf{Comp}(A), \Delta_{I_1, \dots, I_l}(x) \neq 0\}.$$

Obviously, $k(x)$ exists and $k(x) \leq \sharp A$. Let us prove that $x \in \Phi[A](\text{co}\mathcal{T}(\mathcal{Q})[A])$ by induction on $k(x)$. If $k(x) = 1$, then $x \in \mathcal{Q}[A]$ and $x = \Phi[A](x)$. If $k = k(x) \geq 2$, we put, for any $I = (I_1, \dots, I_k) \in \mathbf{Comp}(A)$:

$$\Delta_{I_1, \dots, I_k}(x) = x_1^{(I)} \otimes \dots \otimes x_k^{(I)}.$$

By maximality of k , $x_i^{(I)} \in \mathcal{Q}$ for any I, i . We consider:

$$y = \sum_{I=(I_1, \dots, I_k) \in \mathbf{Comp}(A)} x_1^{(I)} \cdot \dots \cdot x_k^{(I)}.$$

Then, if $l > k$ and $(I_1, \dots, I_l) \in \mathbf{Comp}(A)$, $\Delta_{I_1, \dots, I_l}(y) = 0$. If $I = (I_1, \dots, I_k) \in \mathbf{Comp}(A)$, $\Delta_{I_1, \dots, I_l}(y) = x_1^{(I)} \otimes \dots \otimes x_k^{(I)} = \Delta_{I_1, \dots, I_k}(x)$. Hence, $k(x - y) < k$, so $x - y \in \Phi[A](\text{co}\mathcal{T}(\mathcal{Q})[A])$. As $y \in \Phi[A](\text{co}\mathcal{T}(\mathcal{Q})[A])$, $x \in \Phi[A](\text{co}\mathcal{T}(\mathcal{Q})[A])$. \square

3 Convolution and characters

3.1 Convolution

As for "usual" algebras and coalgebras:

Proposition 3.1. *Let $(\mathcal{A}, m_{\mathcal{A}})$ be an algebra and $(\mathcal{C}, \Delta_{\mathcal{C}})$ be a coalgebra, both in the category of species. The space $\text{End}(\mathcal{C}, \mathcal{A})$ of species morphisms from \mathcal{C} to \mathcal{A} is a monoid for the convolution product $*$:*

$$\forall f, g \in \text{End}(\mathcal{C}, \mathcal{A}), \quad f * g = m_{\mathcal{A}} \circ (f \otimes g) \circ \Delta_{\mathcal{C}}.$$

The unit is the morphism $u = \iota_{\mathcal{A}} \circ \varepsilon_{\mathcal{C}}$.

Lemma 3.2. *Let \mathcal{A} be a commutative twisted algebra and \mathcal{B} be a twisted bialgebra. We denote by $\text{Char}(\mathcal{B}, \mathcal{A})$ the set of algebra morphisms from \mathcal{B} to \mathcal{A} .*

1. $\text{Char}(\mathcal{B}, \mathcal{A})$ is a submonoid of $(\text{End}(\mathcal{B}, \mathcal{A}), *)$.
2. Let $f \in \text{Char}(\mathcal{B}, \mathcal{A})$, with an inverse $g \in \text{End}(\mathcal{B}, \mathcal{A})$. Then $g \in \text{Char}(\mathcal{B}, \mathcal{A})$.

Proof. 1. For any $f, g \in \text{Char}(\mathcal{B}, \mathcal{A})$:

$$f * g(1_{\mathcal{B}}) = f(1_{\mathcal{B}})g(1_{\mathcal{B}}) = 1_{\mathcal{A}}1_{\mathcal{A}} = 1_{\mathcal{A}}.$$

Moreover:

$$\begin{aligned} m_{\mathcal{A}} \circ (f * g \otimes f * g) &= m_{\mathcal{A}} \circ (m_{\mathcal{A}} \otimes m_{\mathcal{A}}) \circ (f \otimes g \otimes f \otimes g) \circ (\Delta_{\mathcal{B}} \otimes \Delta_{\mathcal{B}}), \\ f * g \circ m_{\mathcal{B}} &= m_{\mathcal{A}} \circ (f \otimes g) \circ (\Delta_{\mathcal{B}} \circ m_{\mathcal{B}}) \\ &= m_{\mathcal{A}} \circ (f \otimes g) \circ (m_{\mathcal{B}} \otimes m_{\mathcal{B}}) \circ (\text{Id}_{\mathcal{B}} \otimes c_{\mathcal{B}, \mathcal{B}} \otimes \text{Id}_{\mathcal{B}}) \circ (\Delta_{\mathcal{B}} \otimes \Delta_{\mathcal{B}}) \\ &= m_{\mathcal{A}} \circ (m_{\mathcal{A}} \otimes m_{\mathcal{A}}) \circ (f \otimes f \otimes g \otimes g) \circ (\text{Id}_{\mathcal{B}} \otimes c_{\mathcal{B}, \mathcal{B}} \otimes \text{Id}_{\mathcal{B}}) \circ (\Delta_{\mathcal{B}} \otimes \Delta_{\mathcal{B}}) \\ &= m_{\mathcal{A}} \circ (m_{\mathcal{A}} \otimes m_{\mathcal{A}}) \circ (\text{Id}_{\mathcal{A}} \otimes c_{\mathcal{A}, \mathcal{A}} \otimes \text{Id}_{\mathcal{A}}) \circ (f \otimes g \otimes f \otimes g) \circ (\Delta_{\mathcal{B}} \otimes \Delta_{\mathcal{B}}). \end{aligned}$$

As \mathcal{A} is commutative, $m_{\mathcal{A}} \circ (f * g \otimes f * g) = f * g \circ m_{\mathcal{B}}$. So $f * g \in \text{Char}(\mathcal{B}, \mathcal{A})$.

2. As \mathcal{B} is a bialgebra, $\mathcal{B} \otimes \mathcal{B}$ is a bialgebra. Let us consider the maps $F = f \circ m_{\mathcal{B}} = m_{\mathcal{A}} \circ (f \otimes f)$, $G = g \circ m_{\mathcal{B}}$ and $H = m_{\mathcal{A}} \circ (g \otimes g)$, all three in $\text{End}(\mathcal{B} \otimes \mathcal{B}, \mathcal{A})$.

$$\begin{aligned} F * G &= m_{\mathcal{A}} \circ (f \otimes g) \circ (m_{\mathcal{B}} \otimes m_{\mathcal{B}}) \circ (\text{Id}_{\mathcal{B}} \otimes c_{\mathcal{B}, \mathcal{B}} \otimes \text{Id}_{\mathcal{B}}) \circ (\Delta_{\mathcal{B}} \otimes \Delta_{\mathcal{B}}) \\ &= m_{\mathcal{A}} \circ (f \otimes g) \circ \Delta_{\mathcal{B}} \circ m_{\mathcal{B}} \\ &= f * g \circ m_{\mathcal{B}} \\ &= \iota_{\mathcal{A}} \circ \varepsilon_{\mathcal{B}} \circ m_{\mathcal{B}} \\ &= \iota_{\mathcal{A}} \circ m_{\mathcal{I}} \circ (\varepsilon_{\mathcal{B}} \otimes \varepsilon_{\mathcal{B}}) \\ &= \iota_{\mathcal{A}} \circ \varepsilon_{\mathcal{B} \otimes \mathcal{B}}. \end{aligned}$$

So G is a right inverse of F .

$$\begin{aligned}
H * F &= m_{\mathcal{A}} \circ (m_{\mathcal{A}} \otimes m_{\mathcal{A}}) \circ (f \otimes f \otimes g \otimes g) \circ (\text{Id}_{\mathcal{B}} \otimes c_{\mathcal{B}, \mathcal{B}} \otimes \text{Id}_{\mathcal{B}}) \circ (\Delta_{\mathcal{B}} \otimes \Delta_{\mathcal{B}}) \\
&= m_{\mathcal{A}} \circ (m_{\mathcal{A}} \otimes m_{\mathcal{A}}) \circ (\text{Id}_{\mathcal{A}} \otimes c_{\mathcal{A}, \mathcal{A}} \otimes \text{Id}_{\mathcal{A}}) \circ (f \otimes g \otimes f \otimes g) \circ (\Delta_{\mathcal{B}} \otimes \Delta_{\mathcal{B}}) \\
&= m_{\mathcal{A}} \circ (m_{\mathcal{A}} \otimes m_{\mathcal{A}}) \circ (f \otimes g \otimes f \otimes g) \circ (\Delta_{\mathcal{B}} \otimes \Delta_{\mathcal{B}}) \\
&= m_{\mathcal{A}} \circ (f * g \otimes f * g) \\
&= m_{\mathcal{A}} \circ (\iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}}) \circ (\varepsilon_{\mathcal{B}} \otimes \varepsilon_{\mathcal{B}}) \\
&= \iota_{\mathcal{A}} \circ m_{\mathcal{I}} \circ (\varepsilon_{\mathcal{B}} \otimes \varepsilon_{\mathcal{B}}) \\
&= \iota_{\mathcal{A}} \circ \varepsilon_{\mathcal{B} \otimes \mathcal{B}}.
\end{aligned}$$

So H is a left inverse of F . Therefore, $H = H * (F * G) = (H * F) * G = G$, so $g \circ m_{\mathcal{B}} = m_{\mathcal{A}} \circ (g \otimes g)$. Finally:

$$f * g(1_{\mathcal{B}}) = 1_{\mathcal{A}} = f(1_{\mathcal{B}})g(1_{\mathcal{B}}) = g(1_{\mathcal{B}}),$$

Hence, $g(1_{\mathcal{B}}) = 1_{\mathcal{A}}$. □

3.2 Characters

By Lemma 3.2:

Definition 3.3. Let \mathcal{P} be a twisted algebra. A character on \mathcal{P} is an algebra morphism from \mathcal{P} to \mathbf{Com} . The set of characters on \mathcal{P} is denoted by $\text{Char}(\mathcal{P})$ instead of $\text{Char}(\mathcal{P}, \mathbf{Com})$. Then $\text{Char}(\mathcal{P})$ is given a monoid structure in the following way:

$$\forall \alpha, \beta \in \text{Char}(\mathcal{P}), \quad \alpha * \beta = m_{\mathbf{Com}} \circ (\alpha \otimes \beta) \circ \Delta.$$

The unit of this monoid is the counit $\varepsilon : \mathcal{P} \rightarrow \mathcal{I} \rightarrow \mathbf{Com}$, where the second arrow is the canonical injection from \mathcal{I} into \mathbf{Com} .

Remark 3.1. We proved in Corollary 2.5 that ε' is a character of \mathcal{Comp} .

Let \mathcal{P} be a twisted bialgebra and let $\lambda : \mathcal{P} \rightarrow \mathbf{Com}$ be a morphism such that $\lambda[\emptyset] = 0$. Then, for any $k \in \mathbb{N}$,

$$\lambda^{*k}[A] = \sum_{(I_1, \dots, I_k) \in \mathbf{Comp}[A]} \lambda^{\otimes k} \circ \Delta_{I_1, \dots, I_k}(x).$$

If $k > \sharp A$, this is zero: λ is locally nilpotent for the convolution product $*$. We obtain an algebra morphism:

$$\left\{ \begin{array}{l} \mathbb{K}[[X]] \longrightarrow \text{End}(\mathcal{P}, \mathcal{Comp}) \\ f(X) = \sum_{n=0}^{\infty} a_n X^n \longrightarrow f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^{*n}. \end{array} \right.$$

If \mathcal{P} is connected, then for any character $\lambda \in \text{Char}(\mathcal{P})$, $(\lambda - \varepsilon)[\emptyset] = 0$, so $f(\lambda - \varepsilon)$ exists for any $f \in \mathbb{K}[[X]]$. In particular, for any $q \in \mathbb{K}$, we put:

$$\lambda^q = (1 + (\lambda - \varepsilon))^q = \sum_{n=0}^{\infty} H_n(q) (\lambda - \varepsilon)^{*n},$$

where for any $n \geq 0$, H_n is the n -th Hilbert polynomial:

$$H_n(X) = \frac{X(X-1)\dots(X-n+1)}{n!}.$$

For any $q, q' \in \mathbb{K}$, $(1 + X)^q (1 + X)^{q'} = (1 + X)^{q+q'}$, so:

$$\lambda^q * \lambda^{q'} = \lambda^{q+q'}.$$

Moreover, if $f_q = (1 + X)^q - 1$, $(1 + f_q(X))^{q'} = (1 + X)^{qq'}$, so:

$$(\lambda^q)^{q'} = \lambda^{qq'}.$$

Proposition 3.4. *Let \mathcal{P} be a connected twisted bialgebra and let $\lambda \in \text{Char}(\mathcal{P})$. For any $q \in \mathbb{K}$, $\lambda^q \in \text{Char}(\mathcal{P})$.*

Proof. We define a species morphism:

$$M : \begin{cases} L(\mathcal{P}, \mathbf{Com}) & \longrightarrow L(\mathcal{P} \otimes \mathcal{P}, \mathbf{Com}) \\ \lambda & \longrightarrow \lambda \circ m. \end{cases}$$

Let $\lambda, \mu \in L(\mathcal{P}, \mathbf{Com})$. For any $x \in \mathcal{P}[A]$, $y \in \mathcal{P}[B]$:

$$\begin{aligned} M(\lambda) * M(\mu)(x \otimes y) &= M(\lambda) \otimes M(\mu)(x^{(1)} \otimes y^{(1)} \otimes x^{(2)} \otimes y^{(2)}) \\ &= \lambda(x^{(1)}y^{(1)})\mu(x^{(2)}y^{(2)}) \\ &= \lambda * \mu(xy) \\ &= M(\lambda * \mu)(x \otimes y). \end{aligned}$$

So M is an algebra morphism. Moreover, if $\lambda \in \text{Char}(\mathcal{P})$, then $M(\lambda) = \lambda \otimes \lambda$, so:

$$\begin{aligned} M(\ln(1 + (\lambda - \varepsilon))) &= \ln(1 + \lambda \otimes \lambda - \varepsilon \otimes \varepsilon) \\ &= \ln(1 + \lambda - \varepsilon) \otimes \varepsilon + \varepsilon \otimes \ln(1 + \lambda - \varepsilon), \\ M(\lambda^q) &= \exp(q \ln(1 + \lambda - \varepsilon) \otimes \varepsilon + \varepsilon \otimes q \ln(1 + \lambda - \varepsilon)) \\ &= \exp(q \ln(1 + \lambda - \varepsilon)) \otimes \exp(q \ln(1 + \lambda - \varepsilon)) \\ &= \lambda^q \otimes \lambda^q. \end{aligned}$$

Hence, if $x \in \mathcal{P}[A]$, $y \in \mathcal{P}[B]$, $\lambda^q(xy) = \lambda^q(x)\lambda^q(y)$, so $\lambda^q \in \text{Char}(\mathcal{P})$. \square

Corollary 3.5. *Let (\mathcal{P}, m) be a connected twisted bialgebra. Then $(\text{Char}(\mathcal{P}), *)$ is a group.*

Proof. For any $\lambda \in \text{Char}(\mathcal{P})$, $\lambda^{-1} * \lambda = \lambda * \lambda^{-1} = \lambda_0 = \varepsilon$, so λ^{-1} is indeed an inverse of λ for the convolution. \square

Let us now give a species version of Aguiar, Bergeron and Sottile's Theorem on the Hopf algebra of quasisymmetric functions [1]:

Theorem 3.6. *Let \mathcal{B} be a connected twisted bialgebra. Let $\alpha : \mathcal{B} \longrightarrow \mathbf{Com}$ be a species morphism. There exists a unique coalgebra morphism $\phi : \mathcal{B} \longrightarrow \text{Comp}$ such that $\varepsilon' \circ \phi = \alpha$. For any $x \in \mathcal{B}[A]$, with $A \neq \emptyset$:*

$$\phi(x) = \sum_{(A_1, \dots, A_k) \in \mathbf{Comp}[A]} \alpha^{\otimes k} \circ \Delta_{A_1, \dots, A_k}(x)(A_1, \dots, A_k).$$

Moreover, ϕ is a bialgebra morphism if, and only if, ϕ is a character.

Proof. Immediate corollary of Theorem 2.6, as $\text{Comp} = \text{co}\mathcal{T}(\mathbf{Com}_+)$. \square

3.3 Subalgebras of shuffle twisted bialgebras

Theorem 3.7. *Let \mathcal{P} be a twisted, connected and commutative bialgebra.*

1. *There exists a species \mathcal{Q} and an injective morphism of twisted bialgebras from \mathcal{P} to the shuffle bialgebra $(\text{co}\mathcal{T}(\mathcal{Q}), \sqcup, \Delta)$.*
2. *There exists a species \mathcal{Q} and an isomorphism of twisted bialgebras from \mathcal{P} to the shuffle bialgebra $(\text{co}\mathcal{T}(\mathcal{Q}), \sqcup, \Delta)$ if, and only if, \mathcal{P} is cofree.*

Proof. First step. Let \mathcal{P} and \mathcal{Q} be species. Then the set of linear morphisms from \mathcal{P} to \mathcal{Q} is a species, defined by:

1. $L(\mathcal{P}, \mathcal{Q})[A]$ is the space of linear maps from $\mathcal{P}[A]$ to $\mathcal{Q}[A]$.

2. If $\sigma : A \rightarrow B$ is a bijection and $f \in L(\mathcal{P}, \mathcal{Q})[A]$, $L(\mathcal{P}, \mathcal{Q})[\sigma](f) = \mathcal{Q}[\sigma] \circ f \circ \mathcal{P}[\sigma^{-1}]$.

If \mathcal{P} is a unitary twisted bialgebra, $m : \mathcal{P} \otimes \mathcal{P} \rightarrow \mathcal{P}$ is a surjective map of species. Let us denote by $\bar{m} : (\mathcal{P} \otimes \mathcal{P})/\ker(m) \rightarrow \mathcal{P}$ the canonically associated isomorphism of species. If \mathcal{P} is commutative, then m is a twisted algebra morphism: indeed, if $x \in \mathcal{P}[A]$, $y \in \mathcal{P}[B]$, $x' \in \mathcal{P}[A']$, $y' \in \mathcal{P}[B']$:

$$m_{A \sqcup A', B \sqcup B'}((x \otimes y) \otimes (x' \otimes y')) = xx'yy' = xyx'y' = m_{A,B}(x \otimes y)m_{A',B'}(x' \otimes y').$$

Therefore, $(\mathcal{P} \otimes \mathcal{P})/\ker(m)$ becomes an algebra and \bar{m} is an algebra isomorphism.

If now \mathcal{P} is a commutative twisted bialgebra, we shall consider the morphism of species defined by:

$$M : \begin{cases} L(\mathcal{P}, \mathcal{P}) & \longrightarrow L(\mathcal{P} \otimes \mathcal{P}, (\mathcal{P} \otimes \mathcal{P})/\ker(m)) \\ f & \longrightarrow \bar{m}^{-1} \circ f \circ m. \end{cases}$$

In other terms, for any $f : \mathcal{P}[A] \rightarrow \mathcal{P}[A]$, if $A = I \sqcup J$:

$$f \circ m_{I,J} = \bar{m}_{I,J} \circ M[A](f).$$

As $\mathcal{P} \otimes \mathcal{P}$ is a twisted coalgebra and $(\mathcal{P} \otimes \mathcal{P})/\ker(m)$ is a twisted algebra, the species $L(\mathcal{P} \otimes \mathcal{P}, (\mathcal{P} \otimes \mathcal{P})/\ker(m))$ is given a convolution product $*$. Let $f, g \in L(\mathcal{P}, \mathcal{P})[A]$. We put $F = M[A](f)$ and $G = M[A](g)$. Then, if $A = I \sqcup J$, $x \in \mathcal{P}[I]$, $y \in \mathcal{P}[J]$:

$$\begin{aligned} (f * g) \circ m_{I,J}(x \otimes y) &= \pi \circ m \circ (f \otimes g) \circ (m \otimes m) \circ (\text{Id} \otimes c \otimes \text{Id}) \circ (\Delta \otimes \Delta)(x \otimes y) \\ &= m \circ (\bar{m} \otimes \bar{m}) \circ (F \otimes G) \circ (\text{Id} \otimes c \otimes \text{Id}) \circ (\Delta \otimes \Delta)(x \otimes y). \end{aligned}$$

Let us denote by π the canonical projection from $\mathcal{P} \otimes \mathcal{P}$ onto $(\mathcal{P} \otimes \mathcal{P})/\ker(m)$. We shall write:

$$F = \pi \circ \left(\sum F'_j \otimes F''_j \right), \quad G = \pi \circ \left(\sum G'_k \otimes G''_k \right).$$

where F'_j, F''_j, G'_k, G''_k are linear endomorphisms of certain $\mathcal{P}[B]$, with $B \subseteq A$. Then, with Sweedler's notation:

$$\begin{aligned} (f * g) \circ m_{I,J}(x \otimes y) &= f(x^{(1)}y^{(1)})g(x^{(2)}y^{(2)}) \\ &= \sum F'_j(x^{(1)})F''_j(y^{(1)})G'_k(x^{(2)})G''_k(y^{(2)}) \\ &= \sum F'_j(x^{(1)})G'_k(x^{(2)})F''_j(y^{(1)})G''_k(y^{(2)}) \\ &= \sum F'_j * G'_k(x)F''_j * G''_k(y) \\ &= F * G(xy). \end{aligned}$$

Hence, $M(f * g) = M(f) * M(g)$: M is compatible with $*$. Moreover, for any $x \in \mathcal{P}[A]$, $y \in \mathcal{P}[B]$:

$$\text{Id} \circ m(x \otimes y) = xy = \bar{m} \circ \pi(x \otimes y).$$

Hence, $M(\text{Id}) = \pi$.

Second step. Let us denote by ρ the canonical projection from \mathcal{P} to \mathcal{P}_+ . As \mathcal{P} is connected, $\text{Id} = \nu + \rho$, where ν is the unit of the convolution product of $L(\mathcal{P}, \mathcal{P})$. For any finite set A , by connectivity of \mathcal{P} :

$$\rho^{*k}[A] = \sum_{(A_1, \dots, A_k) \in \mathbf{Comp}(A)} m_{A_1, \dots, A_k} \circ \Delta_{A_1, \dots, A_k}.$$

In particular, if $k > \sharp A$, $\rho^{*k}[A] = 0$. We can then consider the morphism defined by:

$$f = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \rho^{*k} = \ln(1 + \rho) = \ln(\text{Id}).$$

As M is compatible with $*$:

$$M(f) = \ln(1 + M(\rho)) = \ln(1 + (\pi - \nu)) = \ln(\pi) = \pi \circ \ln(\text{Id} \otimes \text{Id}).$$

Moreover, $\text{Id} \otimes \text{Id} = (\text{Id} \otimes \nu) * (\nu \otimes \text{Id})$, by property of the \ln formal series:

$$M(f) = \pi \circ (\ln(\text{Id} \otimes \nu) + \ln(\nu \otimes \text{Id})) = \pi \circ (\ln(\text{Id}) \otimes \nu + \nu \otimes \ln(\text{Id})) = \pi \circ (f \otimes 1 + 1 \otimes f).$$

In other words, if $x \in \mathcal{P}[A]$, $y \in \mathcal{P}[B]$, with $A, B \neq \emptyset$:

$$f(xy) = f(x)\varepsilon(y) + \varepsilon(x)f(y) = 0.$$

Let \mathcal{Q} be the subspecies of \mathcal{P} defined by:

$$\mathcal{Q}[A] = \{x \in \mathcal{P}[A], \forall (I, J) \in \mathbf{Comp}[A], \Delta_{I,J}(x) = 0\}.$$

If $x \in \mathcal{Q}[A]$:

$$f(x) = \sum_{(A_1, \dots, A_k) \in \mathbf{Comp}(A)} \frac{(-1)^{k+1}}{k} m_{A_1, \dots, A_k} \circ \Delta_{A_1, \dots, A_k}(x) = x + 0.$$

As a consequence, for any finite set A :

$$\mathcal{Q}[A] \cap m(\mathcal{P}_+ \otimes \mathcal{P}_+)[A] = (0).$$

Third step. For any $n \geq 1$, $m(\mathcal{P}_+ \otimes \mathcal{P}_+)[\underline{n}]$ is a \mathfrak{S}_n -submodule of $\mathcal{P}[\underline{n}]$. By semisimplicity, there exists a \mathfrak{S}_n -submodule R_n of $\mathcal{P}[\underline{n}]$ such that:

$$\mathcal{P}[\underline{n}] = \mathcal{Q}[\underline{n}] \oplus R_n, \quad m(\mathcal{P}_+ \otimes \mathcal{P}_+)[\underline{n}] \subseteq R_n.$$

For any finite set A , of cardinality n , let σ be a bijection from \underline{n} to A . We put $\mathcal{R}[A] = \mathcal{P}[\sigma](R_n)$. This does not depend on the choice of σ : indeed, if τ is another bijection, then $\tau^{-1} \circ \sigma \in \mathfrak{S}_n$, so $\mathcal{P}[\tau]^{-1} \circ \mathcal{P}[\sigma](R_n) = \mathcal{P}[\tau^{-1} \circ \sigma](V_n) = V_n$, and finally $\mathcal{P}[\tau](R_n) = \mathcal{P}[\sigma](R_n)$. We define in this way a subspecies \mathcal{R} of \mathcal{P} such that:

$$\mathcal{P} = \mathcal{Q} \oplus \mathcal{R}, \quad m(\mathcal{P}_+ \otimes \mathcal{P}_+) \subseteq \mathcal{R}.$$

Let $\varpi : \mathcal{P} \rightarrow \mathcal{Q}$ be the canonical projection. By universal property, there exists a unique coalgebra morphism $\Phi : \mathcal{P} \rightarrow \text{co}\mathcal{T}(\mathcal{Q})$, such that $\pi \circ \phi = \varpi$, where $\pi : \text{co}\mathcal{T}(\mathcal{Q}) \rightarrow \mathcal{Q}$ is the canonical projection.

Let us assume that Φ is not injective. Let $x \in \mathcal{P}[A]$, nonzero, such that $\Phi[A](x) = 0$. We assume that the cardinality of A is minimal. If $(I, J) \in \mathbf{Comp}(A)$:

$$(\Phi[I] \otimes \Phi[J]) \circ \Delta_{I,J}(x) = \Delta_{I,J} \circ \Phi[A](x) = 0.$$

By minimality of $\sharp A$, $\Phi[I]$ and $\Phi[J]$ are injective, so $\Delta_{I,J}(x) = 0$. Hence, $x \in \mathcal{Q}$, so $\Phi[A](x) = \pi \circ \Phi[A](x) = \varpi(x) = x$: this is a contradiction. Therefore, Φ is injective.

Let us consider $\Phi_1 = \sqcup \circ (\Phi \otimes \Phi)$ and $\Phi_2 = \Phi \circ m$. Both are coalgebra morphisms from $\mathcal{P} \otimes \mathcal{P}$ to $\text{co}\mathcal{T}(\mathcal{Q})$. Moreover, if $x \in \mathcal{P}[A]$, $y \in \mathcal{Q}[B]$:

$$\begin{aligned} \pi \circ \Phi_1[A \sqcup B](x \otimes y) &= \pi[A \sqcup B](\Phi[A](x) \sqcup \Phi[B](y)) \\ &= \begin{cases} \varepsilon(x)\pi \circ \Phi[B](y) & \text{if } A = \emptyset, \\ \varepsilon(y)\pi \circ \Phi[A](x) & \text{if } B = \emptyset, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \varepsilon(x)\varpi[B](y) & \text{if } A = \emptyset, \\ \varepsilon(y)\varpi[A](x) & \text{if } B = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand, as $\varpi \circ m(\mathcal{P}_+ \otimes \mathcal{P}_+) = (0)$:

$$\begin{aligned} \pi \circ \Phi_2[A \sqcup B](x \otimes y) &= \varpi \circ m(x \otimes y) \\ &= \begin{cases} \varepsilon(x)\varpi[B](y) & \text{if } A = \emptyset, \\ \varepsilon(y)\varpi[A](x) & \text{if } B = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

So $\pi \circ \Phi_1 = \pi \circ \Phi_2$: by unicity in the universal property, $\Phi_1 = \Phi_2$, so Φ is a bialgebra morphism.

Last step. Let us assume that \mathcal{P} is cofree, and let us prove that Φ is surjective. Let $k \geq 0$, $x_i \in \mathcal{Q}[A_i]$ for all i and $A = A_1 \sqcup \dots \sqcup A_k$. Let us prove that $x = x_1 \dots x_k \in \Phi[A](\mathcal{P}[A])$ by induction on k . This is obvious if $k \leq 1$. If $k \geq 2$, as \mathcal{P} is cofree, there exists an element $y \in \mathcal{P}$ such that:

$$\Delta_{A_1, \dots, A_k}(y) = x_1 \otimes \dots \otimes x_k.$$

Then:

$$\Delta_{A_1, \dots, A_k} \circ \Phi[A](y) = \Phi[A_i](x_1) \otimes \dots \otimes \Phi[A_k](x_k) = x_1 \otimes \dots \otimes x_k.$$

Hence, the induction hypothesis can be applied to $x - \Phi[A](y)$, so $x \in \Phi[A](\mathcal{P}[A])$. \square

4 The Hadamard product of species

4.1 Double twisted bialgebras

Definition 4.1 (Hadamard product of species). *1. Let \mathcal{P} and \mathcal{Q} be two species.*

- For any finite set A , we put $\mathcal{P} \boxtimes \mathcal{Q}[A] = \mathcal{P}[A] \otimes \mathcal{Q}[A]$.
- For any bijection $\sigma : A \rightarrow B$, we put $\mathcal{P} \boxtimes \mathcal{Q}[\sigma] = \mathcal{P}[\sigma] \otimes \mathcal{Q}[\sigma]$.

Then $\mathcal{P} \boxtimes \mathcal{Q}$ is a species.

- 2. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_1, \mathcal{Q}_2$ be species, and $\phi_1 : \mathcal{P}_1 \rightarrow \mathcal{Q}_1, \phi_2 : \mathcal{P}_2 \rightarrow \mathcal{Q}_2$ be species morphisms. We define a morphism $\phi_1 \boxtimes \phi_2$ of species from $\mathcal{P}_1 \boxtimes \mathcal{P}_2$ to $\mathcal{Q}_1 \boxtimes \mathcal{Q}_2$ by:*

$$\phi_1 \boxtimes \phi_2[A] = \phi_1[A] \otimes \phi_2[A] : \mathcal{P}_1 \boxtimes \mathcal{P}_2[A] \rightarrow \mathcal{Q}_1 \boxtimes \mathcal{Q}_2[A].$$

- 3. Let \mathcal{P} and \mathcal{Q} be two species. The following defines a morphism of species from $\mathcal{P} \boxtimes \mathcal{Q}$ to $\mathcal{Q} \boxtimes \mathcal{P}$:*

$$\tau_{\mathcal{P}, \mathcal{Q}} : \begin{cases} \mathcal{P} \boxtimes \mathcal{Q}[A] & \rightarrow \mathcal{Q} \boxtimes \mathcal{P}[A] \\ x \otimes y & \rightarrow y \otimes x. \end{cases}$$

The species **Com** is the identity for this tensor product: for any species \mathcal{P} ,

$$\mathcal{P} \boxtimes \mathbf{Com} = \mathbf{Com} \boxtimes \mathcal{P} = \mathcal{P}.$$

Definition 4.2. A twisted bialgebra of the second kind is a family (\mathcal{P}, m, δ) where (\mathcal{P}, m) is a twisted algebra and $\delta : \mathcal{P} \longrightarrow \mathcal{P} \boxtimes \mathcal{P}$ is a morphism of species such that:

1. The following diagram commutes:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\delta} & \mathcal{P} \boxtimes 2 \\ \delta \downarrow & & \downarrow \text{Id}_{\mathcal{P}} \boxtimes \delta \\ \mathcal{P} \boxtimes 2 & \xrightarrow{\delta \boxtimes \text{Id}_{\mathcal{P}}} & \mathcal{P} \boxtimes 3 \end{array}$$

2. There exists a morphism of species $\varepsilon' : \mathcal{P} \longrightarrow \mathbf{Com}$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathbf{Com} \boxtimes \mathcal{P} & \xrightarrow{\text{Id}} & \mathcal{P} & \xleftarrow{\text{Id}} & \mathcal{P} \boxtimes \mathbf{Com} \\ & \searrow \varepsilon' \boxtimes \text{Id}_{\mathcal{P}} & \downarrow \delta & \nearrow \text{Id}_{\mathcal{P}} \boxtimes \varepsilon' & \\ & & \mathcal{P} \boxtimes 2 & & \end{array}$$

3. δ is an algebra morphism from \mathcal{P} to $\mathcal{P} \boxtimes \mathcal{P}$, that is to say, for any finite sets A, B :

$$\delta_{A \sqcup B} \circ m_{A,B} = (m_{A,B} \otimes m_{A,B}) \circ (\text{Id}_{\mathcal{P}[A]} \otimes c_{A,B} \otimes \text{Id}_{\mathcal{P}[B]}) \circ (\delta_A \otimes \delta_B),$$

$$\text{and } \delta_{\emptyset}(1_{\mathcal{P}}) = 1_{\mathcal{P}} \otimes 1_{\mathcal{P}}.$$

In other words, for any finite set A , there exists a coproduct $\delta_A : \mathcal{P}[A] \longrightarrow \mathcal{P}[A] \otimes \mathcal{P}[A]$, making $\mathcal{P}[A]$ a coalgebra of counit ε'_A . If $\sigma : A \longrightarrow B$ is a bijection, $\mathcal{P}[\sigma]$ is a coalgebra isomorphism from $(\mathcal{P}[A], \delta_A)$ to $(\mathcal{P}[B], \delta_B)$. In other terms, these objects are algebras in the category of species in the category of coalgebras, that is to say the category of functors from the category of finite sets with bijections to the category of coalgebras.

Notations 4.1. If \mathcal{P} is a twisted bialgebra of the second kind, we adopt Sweedler's notation for its coproduct: if A in a finite set and $x \in \mathcal{P}[A]$,

$$\begin{aligned} \delta_A(x) &= x' \otimes x'', \\ (\delta_A \otimes \text{Id}_{\mathcal{P}[A]}) \circ \delta_A(x) &= (\text{Id}_{\mathcal{P}[A]} \otimes \delta_A) \circ \delta_A(x) = x' \otimes x'' \otimes x'''. \end{aligned}$$

Definition 4.3. A double twisted bialgebra is a family $(\mathcal{P}, m, \Delta, \delta)$ such that:

1. (\mathcal{P}, m, Δ) is a twisted bialgebra. Its counit is denoted by ε .
2. (\mathcal{P}, m, δ) is a twisted bialgebra of the second kind. Its counit is denoted by ε' .
3. Δ is a right comodule morphism, that is, for any finite sets A, B :

$$(\Delta_{A,B} \otimes \text{Id}_{\mathcal{P}[A \sqcup B]}) \circ \delta_{A \sqcup B} = m_{1,3,24} \circ (\delta_A \otimes \delta_B) \circ \Delta_{A,B},$$

where:

$$m_{1,3,24} : \begin{cases} \mathcal{P}[A] \otimes \mathcal{P}[A] \otimes \mathcal{P}[B] \otimes \mathcal{P}[B] & \longrightarrow & \mathcal{P}[A] \otimes \mathcal{P}[B] \otimes \mathcal{P}[A \sqcup B] \\ x \otimes y \otimes z \otimes t & \longrightarrow & x \otimes z \otimes m_{A,B}(y \otimes t). \end{cases}$$

4. The counit $\varepsilon : \mathcal{P} \longrightarrow \mathcal{I}$ is a right comodule morphism, that is, for any $x \in \mathcal{P}[\emptyset]$:

$$(\varepsilon \otimes \text{Id}) \circ \delta_{\emptyset}(x) = \varepsilon(x) 1_{\mathcal{P}} \otimes 1_{\mathcal{P}}.$$

Example 4.1. The species \mathbf{Com} is a double bialgebra with the coproduct defined on $\mathbf{Com}[A] = \mathbb{K}$ by:

$$\delta_A(1) = 1 \otimes 1.$$

Remark 4.1. Let (A, m, Δ, δ) be double twisted bialgebra. For any finite sets A, B , $\mathcal{P}[A] \otimes \mathcal{P}[B]$ is a right $(\mathcal{P}[A \sqcup B], \delta_{A \sqcup B})$ -comodule with the coaction:

$$\rho_{A,B} = \underbrace{(\text{Id}^{\otimes 2} \otimes m_{A,B}) \circ (\text{Id}_{\mathcal{P}[A]} \otimes \tau_{A,B} \otimes \text{Id}_{\mathcal{P}[B]})}_{=m_{1,3,24}} \circ (\delta_A \otimes \delta_B).$$

More generally, if A_1, \dots, A_k are finite sets, then $\mathcal{P}[A_1] \otimes \dots \otimes \mathcal{P}[A_k]$ is a right $\mathcal{P}[A_1 \sqcup \dots \sqcup A_k]$ -comodule, with the coaction defined by:

$$\rho_{A_1, \dots, A_k} = m_{1,3, \dots, 2n-1, 24 \dots 2n} \circ (\delta_{A_1} \otimes \dots \otimes \delta_{A_n}).$$

For any finite sets A_1, \dots, A_k , $m_{A_1, \dots, A_k} : \mathcal{P}[A_1] \otimes \dots \otimes \mathcal{P}[A_k] \rightarrow \mathcal{P}[A_1 \sqcup \dots \sqcup A_k]$ and $\Delta_{A_1, \dots, A_k} : \mathcal{P}[A_1 \sqcup \dots \sqcup A_k] \rightarrow \mathcal{P}[A_1] \otimes \dots \otimes \mathcal{P}[A_k]$ are right comodule morphisms.

Proposition 4.4. *Let $(\mathcal{P}, m, \Delta, \delta)$ be a connected twisted double bialgebra. Then m is commutative.*

Proof. Let A and B be finite sets, $x \in \mathcal{P}[A]$, $y \in \mathcal{P}[B]$. As $\mathcal{P}[\emptyset] = \mathbb{K}1_{\mathcal{P}}$:

$$\Delta_{\emptyset, A}(x) = 1_{\mathcal{P}} \otimes x, \quad \Delta_{B, \emptyset}(y) = y \otimes 1_{\mathcal{P}}.$$

By the compatibility between the product m and the coproduct Δ :

$$\Delta_{B,A}(xy) = (m_{B,B} \otimes m_{A,A}) \circ (\text{Id} \otimes c_{\mathcal{P}[B], \mathcal{P}[A]} \otimes \text{Id})(\Delta_{\emptyset, A}(x) \otimes \Delta_{B, \emptyset}(y)) = y \otimes x.$$

Then:

$$\begin{aligned} m_{1,3,24} \circ (\delta_B \otimes \delta_A) \circ \Delta_{B,A}(xy) &= m_{1,3,24} \circ (\delta_B \otimes \delta_A)(y \otimes x) \\ &= y' \otimes x' \otimes y'' x''. \end{aligned}$$

Applying $\varepsilon'[B] \otimes \varepsilon'[A] \otimes \text{Id}$:

$$(\varepsilon'[B] \otimes \varepsilon'[A] \otimes \text{Id}) \circ m_{1,3,24} \circ (\delta_B \otimes \delta_A) \circ \Delta_{B,A}(xy) = yx.$$

Moreover:

$$\begin{aligned} m_{1,3,24} \circ (\delta_B \otimes \delta_A) \circ \Delta_{B,A}(xy) &= (\Delta_{B,A} \otimes \text{Id}) \circ \delta_{A \sqcup B}(xy) \\ &= \Delta_{B,A}((xy)') \otimes (xy)'' \\ &= \Delta_{B,A}(x'y') \otimes x''y'' \\ &= y' \otimes x' \otimes x''y''. \end{aligned}$$

Applying $\varepsilon'[B] \otimes \varepsilon'[A] \otimes \text{Id}$:

$$(\varepsilon'[B] \otimes \varepsilon'[A] \otimes \text{Id}) \circ m_{1,3,24} \circ (\delta_B \otimes \delta_A) \circ \Delta_{B,A}(xy) = xy.$$

Hence, $xy = yx$: m is commutative. □

Proposition 4.5. *Let $(\mathcal{P}, \cdot, \delta_{\mathcal{P}})$ be a connected commutative twisted bialgebra of the second kind. For any $k \geq 0$, we consider the set \mathbf{cont}_k of pairs (σ, τ) , where:*

1. $\sigma : \underline{k} \rightarrow \underline{\max(\sigma)}$ is a non decreasing surjection.
2. $\tau : \underline{k} \rightarrow \underline{\max(\tau)}$ is a surjection.
3. For any $i, j \in \underline{k}$, if $i < j$ and $\sigma(i) = \sigma(j)$, then $\tau(i) < \tau(j)$.

There exists a unique coproduct $\delta : \text{co}\mathcal{T}(\mathcal{P}_+) \rightarrow \text{co}\mathcal{T}(\mathcal{P}_+) \otimes \text{co}\mathcal{T}(\mathcal{P}_+)$ such that:

1. $(\text{co}\mathcal{T}(\mathcal{P}_+), \boxplus, \Delta, \delta)$ is a double twisted bialgebra.

2. $\delta|_{\mathcal{P}} = \delta_{\mathcal{P}}$.

3. For any $(A_1, \dots, A_k) \in \mathbf{Comp}[A]$, $x_i \in \mathcal{P}[A_i]$:

$$(\varepsilon' \otimes \text{Id}) \circ \delta(x_1 \dots x_k) = \varepsilon'(x'_1 \dots x'_k) x''_1 \dots x''_k.$$

Moreover, for any $(A_1, \dots, A_k) \in \mathbf{Comp}[A]$, $x_i \in \mathcal{P}[A_i]$:

$$\delta_A(x_1 \dots x_k) = \sum_{(\sigma, \tau) \in \mathbf{cont}_k} \sigma \rightarrow x'_1 \dots x'_k \otimes \tau \rightarrow x''_1 \dots x''_k. \quad (2)$$

The counit is given by:

$$\epsilon'_{|\mathcal{P}^{\otimes k}} = \begin{cases} \varepsilon'_{\mathcal{P}} & \text{if } k = 1, \\ 0 & \text{if } k \geq 2. \end{cases}$$

Moreover, $\varepsilon' : (\text{co}\mathcal{T}(\mathcal{P}_+), m, \delta) \rightarrow (\mathcal{P}, m_{\mathcal{P}}, \delta_{\mathcal{P}})$ is a morphism of bialgebras of the second kind.

Proof. Existence. Let us consider the coproduct defined by (2). If $x \in \mathcal{P}[A]$, as $\mathbf{cont}_1 = \{(\text{Id}, \text{Id})\}$:

$$\delta_A(x) = x' \otimes x'' = \delta_{\mathcal{P}}(x).$$

We denote by $1_k : \underline{k} \rightarrow \underline{1}$ the constant map. For any $(A_1, \dots, A_k) \in \mathbf{Comp}[A]$, $x_i \in \mathcal{P}[A_i]$:

$$\begin{aligned} (\varepsilon' \otimes \text{Id}) \circ \delta_A(x_1 \dots x_k) &= \sum_{(1_k, \tau) \in \mathbf{cont}_k} \varepsilon'_{\mathcal{P}}(1_k \rightarrow x'_1 \dots x''_k) \tau \rightarrow x''_1 \dots x''_k \\ &= \varepsilon'_{\mathcal{P}}(1_k \rightarrow x'_1 \dots x''_k) x''_1 \dots x''_k \\ &= \varepsilon'_{\mathcal{P}}(x'_1 \dots x'_k) x''_1 \dots x''_k \\ &= \varepsilon'_{\mathcal{P}}(x'_1) \dots \varepsilon'_{\mathcal{P}}(x'_k) x''_1 \dots x''_k \\ &= x_1 \dots x_k. \end{aligned}$$

Similarly, $(\text{Id} \otimes \varepsilon') \circ \delta_A = \text{Id}_{\text{co}\mathcal{T}(\mathcal{P})[A]}$, so ε' is a counit for δ .

For any $k, l \geq 0$, we denote by $\mathbf{cont}'_{k,l}$ the set of pairs (α, β) such that:

1. $\alpha : \underline{k+l} \rightarrow \underline{m}$ is a surjection, non decreasing on \underline{k} and on $\underline{k+l} \setminus \underline{k}$.
2. $\beta : \underline{k+l} \rightarrow \underline{n}$ is a surjection.
3. For any $i, j \in \underline{k+l}$, if $i < j$ and $\alpha(i) = \alpha(j)$, then $\beta(i) < \beta(j)$.

Let $a = a_1 \dots a_k \in \text{co}\mathcal{T}(\mathcal{P})[A]$ and $b = b_1 \dots b_l \in \text{co}\mathcal{T}(\mathcal{P})[B]$. We put:

$$a' \otimes a'' = a'_1 \dots a'_k \otimes a''_1 \dots a''_k, \quad b' \otimes b'' = b'_1 \dots b'_l \otimes b''_1 \dots b''_l.$$

$$\begin{aligned} \delta_{A \sqcup B} \circ m_{A,B}(a \otimes b) &= \sum_{\substack{\alpha \in \text{QSh}(k,l), \\ (\sigma, \tau) \in \mathbf{cont}_{\max(\alpha)}}} (\sigma \circ \alpha) \rightarrow (a'b') \otimes (\tau \circ \alpha) \rightarrow (a''b'') \\ &= \sum_{(\alpha, \beta) \in \mathbf{cont}'_{k,l}} \alpha \rightarrow (a'b') \otimes \beta \rightarrow (a''b''); \end{aligned}$$

$$\begin{aligned} m_{13,24} \circ (\delta_A \otimes \delta_B)(a \otimes b) &= \sum_{\substack{(\sigma', \tau') \in \mathbf{cont}_k, \\ (\sigma'', \tau'') \in \mathbf{cont}_l, \\ \alpha \in \text{QSh}(\max(\sigma'), \max(\sigma'')), \\ \beta \in \text{QSh}(\max(\tau'), \max(\tau''))}} (\alpha \circ (\sigma' \otimes \sigma'')) \rightarrow (a'b') \otimes (\beta \circ (\tau' \otimes \tau'')) \rightarrow (a''b'') \\ &= \sum_{(\alpha, \beta) \in \mathbf{cont}'_{k,l}} \alpha \rightarrow (a'b') \otimes \beta \rightarrow (a''b''). \end{aligned}$$

So δ is multiplicative.

Let us prove that Δ is a right $(\text{co}\mathcal{T}(\mathcal{P}_+), m, \delta)$ -comodule. For any $k \geq p \geq 0$, let us denote by $\mathbf{cont}''_{p,k}$ the set of triples (α, β, γ) such that:

1. $\alpha : \underline{p} \longrightarrow \underline{l}$ and $\beta : \underline{k} \setminus \underline{p} \longrightarrow \underline{m}$ are non decreasing bijections.
2. $\gamma : \underline{k} \longrightarrow \underline{n}$ is a surjection.
3. For any $i, j \in \underline{p}$, if $i < j$ and $\alpha(i) = \alpha(j)$, then $\gamma(i) < \gamma(j)$.
4. For any $i, j \in \underline{k} \setminus \underline{p}$, if $i < j$ and $\beta(i) = \beta(j)$, then $\gamma(i) < \gamma(j)$.

Let $a = x_1 \dots x_k \in \text{co}\mathcal{T}(\mathcal{P}_+)[A \sqcup B]$. If there exists a (unique) $p \leq k$ such that $x_1 \dots x_i \in \text{co}\mathcal{T}(\mathcal{P}_+)[A]$, we put $a' \otimes a'' \otimes a''' = x'_1 \dots x'_p \otimes x'_{p+1} \dots x'_k \otimes x''_1 \dots x''_k$. Then:

$$\begin{aligned} (\Delta_{A,B} \otimes \text{Id}) \circ \delta_{A \sqcup B}(a) &= \sum_{\substack{(\sigma, \tau) \in \mathbf{cont}''_k, \\ \sigma(p) < \sigma(p+1)}} \sigma|_{\underline{p}} \rightarrow a' \otimes \sigma|_{\underline{k} \setminus \underline{p}} \rightarrow a'' \otimes \tau \rightarrow a''' \\ &= \sum_{(\alpha, \beta, \gamma) \in \mathbf{cont}''_{p,k}} \alpha \rightarrow a' \otimes \beta \rightarrow a'' \otimes \gamma \rightarrow a'''; \\ m_{1,3,24} \circ (\delta_A \otimes \delta_B) \circ \Delta_{A,B}(a) &= \sum_{\substack{(\sigma', \tau') \in \mathbf{cont}''_p, \\ (\sigma'', \tau'') \in \mathbf{cont}''_{k-p}, \\ \alpha \in \text{QSh}(\max(\tau'), \max(\tau''))}} \sigma' \rightarrow a' \otimes \sigma'' \rightarrow a'' \otimes (\alpha \circ (\tau' \cdot \tau'')) \rightarrow a''' \\ &= \sum_{(\alpha, \beta, \gamma) \in \mathbf{cont}''_{p,k}} \alpha \rightarrow a' \otimes \beta \rightarrow a'' \otimes \gamma \rightarrow a'''. \end{aligned}$$

Otherwise, both are equal to 0. So Δ is a right comodule morphism.

Let us prove that δ is coassociative. We work on $\text{co}\mathcal{T}(\mathcal{P}_+)[A]$ and proceed by induction on $\sharp A$. If $A = \emptyset$, then:

$$(\delta_{\emptyset} \otimes \text{Id}) \circ \delta_{\emptyset}(1)(\text{Id} \otimes \delta_{\emptyset}) \circ \delta_{\emptyset}(1) = 1 \otimes 1 \otimes 1.$$

Let us assume the result at all ranks $< \sharp A$. Let $(I, J) \in \mathbf{Comp}[A]$. For any $x \in \text{co}\mathcal{T}(\mathcal{P}_+)[A]$, putting $\Delta_{I,J}(x) = x_I \otimes x_J$:

$$\begin{aligned} &(\Delta_{I,J} \otimes \text{Id} \otimes \text{Id}) \circ (\delta_A \otimes \text{Id}) \circ \delta_A(x) \\ &= (m_{1,3,24} \otimes \text{Id}) \circ (\delta_I \otimes \delta_J \otimes \text{Id}) \circ (\Delta_{I,J} \otimes \text{Id}) \circ \delta_A(x) \\ &= (m_{1,3,24} \otimes \text{Id}) \circ (\delta_I \otimes \delta_J \otimes \text{Id}) \circ m_{1,3,24} \circ (\delta_I \otimes \delta_J) \circ \Delta_{I,J}(x) \\ &= (x_I)' \otimes (x_J)' \otimes (x_I)''(x_J)'' \otimes (x_I)'''(x_J)''', \end{aligned}$$

$$\begin{aligned} &\Delta_{I,J} \otimes \text{Id} \otimes \text{Id} \circ (\text{Id} \otimes \delta_A) \circ \delta_A(x) \\ &= (\text{Id} \otimes \text{Id} \otimes \delta_A) \circ m_{1,3,24} \circ (\delta_I \otimes \delta_J) \circ \Delta_{I,J}(x) \\ &= (x_I)' \otimes (x_J)' \otimes \delta_A((x_I)''(x_J)'') \\ &= (x_I)' \otimes (x_J)' \otimes (x_I)''(x_J)'' \otimes (x_I)'''(x_J)'''. \end{aligned}$$

Hence, $(\delta_A \otimes \text{Id}) \circ \delta_A(x) - (\text{Id} \otimes \delta_A) \circ \delta_A(x)$ belongs to:

$$\bigcap_{(I,J) \in \mathbf{Comp}[A]} \ker(\Delta_{I,J}) \otimes \text{co}\mathcal{T}(\mathcal{P}_+)[A] = \mathcal{P}[A] \otimes \text{co}\mathcal{T}(\mathcal{P}_+)[A].$$

If $x \in \mathcal{P}^{\otimes k}(A)$:

$$\begin{aligned} (\pi \otimes \text{Id} \otimes \text{Id}) \circ (\delta_A \otimes \text{Id}) \circ \delta_A(x) &= \sum_{(\sigma, \tau) \in \mathbf{cont}_k} x' \otimes \sigma \rightarrow x'' \otimes \tau \rightarrow x''' \\ &= (\text{Id} \otimes \delta_A) \circ (\pi \otimes \text{Id}) \circ \delta_A(x) \\ &= (\pi \otimes \text{Id} \otimes \text{Id}) \circ (\text{Id} \otimes \delta_A) \circ \delta_A(x). \end{aligned}$$

Consequently, δ_A is coassociative.

Unicity. Let δ' be another coproduct with the same properties. Let us show that $\delta_A = \delta'_A$ by induction on $\sharp A$. This is obvious if $A = \emptyset$. Otherwise, if $(I, J) \in \mathbf{Comp}[A]$:

$$\begin{aligned} (\Delta_{I,J} \otimes \text{Id}) \circ \delta_A &= m_{1,3,24} \circ (\delta_I \otimes \delta_J) \circ \Delta_{I,J} \\ &= m_{1,3,24} \circ (\delta'_I \otimes \delta'_J) \circ \Delta_{I,J} \\ &= (\Delta_{I,J} \otimes \text{Id}) \circ \delta'_A. \end{aligned}$$

Hence, $\delta_A - \delta'_A$ takes its values in $\mathcal{P}[A] \otimes \text{co}\mathcal{T}(\mathcal{P}_+)[A]$. As $(\varepsilon' \otimes \text{Id}) \circ \delta_A = (\varepsilon' \otimes \text{Id}) \circ \delta'_A$, $\delta_A = \delta'_A$.

We already proved that ε' is an algebra morphism. Let $x \in \mathcal{P}_+[A]^{\otimes k}$, with $k \geq 1$. Then:

$$\begin{aligned} \delta_A \circ \varepsilon'(x) &= \begin{cases} 0 & \text{if } k \geq 2, \\ \delta_{\mathcal{P}}(x) & \text{if } k = 1; \end{cases} \\ (\varepsilon' \otimes \varepsilon') \circ \delta_A(x) &= \begin{cases} \varepsilon(1_k \longrightarrow x') \otimes \varepsilon(1_k \longrightarrow x'') & \text{if } (1_k, 1_k) \in \mathbf{cont}'_1, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 0 & \text{if } k \geq 2, \\ \delta_{\mathcal{P}}(x) & \text{if } k = 1. \end{cases} \end{aligned}$$

Therefore, $\delta_A \circ \varepsilon' = (\varepsilon' \otimes \varepsilon') \circ \delta_A$. □

Example 4.2. We denote by \cdot the product of \mathcal{P} and we use Sweedler's notation for the coproduct of \mathcal{P} . If $x \in \mathcal{P}[A]$, $y \in \mathcal{P}[B]$, $z \in \mathcal{P}[C]$:

$$\begin{aligned} \delta(x) &= x' \otimes x'', \\ \delta(xy) &= x'y' \otimes x'' \boxplus y'' + x' \cdot y' \otimes x''y'', \\ &= x'y' \otimes (x''y'' + y''x'' + x'' \cdot y'') + x' \cdot y' \otimes x''y'', \\ \delta(xyz) &= x'y'z' \otimes x'' \boxplus y'' \boxplus z'' + x'(y' \cdot z') \otimes x'' \boxplus y''z'' \\ &+ (x' \cdot y')z' \otimes x''y'' \boxplus z'' + x' \cdot y' \cdot z' \otimes x''y''z''. \end{aligned}$$

Corollary 4.6. *The bialgebra $(\text{Comp}, \boxplus, \Delta)$ is made a double bialgebra with the coproduct given by the following: if $(A_1, \dots, A_k) \in \mathbf{Comp}[A]$,*

$$\begin{aligned} &\delta_A(A_1, \dots, A_k) \\ &= \sum_{(\sigma, \tau) \in \mathbf{cont}_k} \sigma \rightarrow (A_1, \dots, A_k) \otimes \tau \rightarrow (A_1, \dots, A_k) \\ &= \sum_{1 \leq i_1 < \dots < i_p < k} (A_1 \sqcup \dots \sqcup A_{i_1}, \dots, A_{i_p+1} \sqcup \dots \sqcup A_k) \otimes (A_1, \dots, A_{i_1}) \boxplus \dots \boxplus (A_{i_p+1}, \dots, A_k). \end{aligned}$$

The counit is given by $\varepsilon'(A_1, \dots, A_k) = \delta_{k,1}$.

Proof. This comes from $\text{Comp} = \text{co}\mathcal{T}(\mathbf{Com}_+)$. Note that in this case, $\varepsilon' = \varepsilon$. □

Example 4.3. In $Comp$, if A, B and C are finite sets:

$$\begin{aligned}\delta(A) &= (A) \otimes (A), \\ \delta(A, B) &= (A, B) \otimes ((A) \boxplus (B)) + (A \sqcup B) \otimes (A, B), \\ \delta(A, B, C) &= (A, B, C) \otimes ((A) \boxplus (B) \boxplus (C)) + (A, B \sqcup C) \otimes ((A) \boxplus (B, C)) \\ &\quad + (A \sqcup B, C) \otimes ((A, B) \boxplus (C)) + (A \sqcup B \sqcup C) \otimes (A, B, C).\end{aligned}$$

4.2 Characters of a double bialgebra

Proposition 4.7. *Let $\mathcal{P} = (\mathcal{P}, m, \Delta, \delta)$ be a double twisted bialgebra. The set of characters $\text{Char}(\mathcal{P})$ has a second convolution product \star , making it a monoid:*

$$\forall f, g \in \text{Char}(\mathcal{P}), \quad f \star g = m_{\text{Com}} \circ (f \otimes g) \circ \delta.$$

Its unit is ε' . For any $f, g, h \in \text{Char}(\mathcal{P})$:

$$(f \star g) \star h = (f \star h) \star (g \star h).$$

We denote by $M_{\mathcal{P}}$ the monoid $(\text{Char}(\mathcal{P}), \star)$.

Proof. As δ is an algebra morphism, if $f, g \in \text{Char}(\mathcal{P})$, then $f \star g \in \text{Char}(\mathcal{P})$.

$$\begin{aligned}(f \star g) \star h &= (f \otimes g \otimes h) \circ (\Delta \otimes \text{Id}_{\mathcal{P}}) \circ \delta \\ &= (f \otimes g \otimes h) \circ m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta \\ &= (f \otimes h \otimes g \otimes h) \circ (\delta \otimes \delta) \circ \Delta \\ &= (f \star h) \star (g \star h).\end{aligned}$$

The associativity of \star comes from the coassociativity of δ . □

Proposition 4.8. *Let $\mathcal{P} = (\mathcal{P}, m, \Delta, \delta)$ be a double twisted bialgebra. Then $M_{\mathcal{P}}$ acts on the space $\text{Mor}(\mathcal{P}, \mathcal{Q})$ of species morphisms from \mathcal{P} to \mathcal{Q} by:*

$$\begin{cases} \text{Mor}(\mathcal{P}, \mathcal{Q}) \times M_{\mathcal{P}} & \longrightarrow \text{Mor}(\mathcal{P}, \mathcal{Q}) \\ (\phi, f) & \longrightarrow \phi \leftarrow f = (\phi \otimes f) \circ \delta. \end{cases}$$

Moreover:

1. *Let \mathcal{Q} be a twisted algebra. We denote by $\text{Mor}_A(\mathcal{P}, \mathcal{Q})$ the set of algebra morphisms from \mathcal{P} to \mathcal{Q} . Then $\text{Mor}_A(\mathcal{P}, \mathcal{Q})$ is a $M_{\mathcal{P}}$ -submodule of $\text{Mor}(\mathcal{P}, \mathcal{Q})$.*
2. *Let \mathcal{Q} be a twisted coalgebra. We denote by $\text{Mor}_C(\mathcal{P}, \mathcal{Q})$ the set of coalgebra morphisms from \mathcal{P} to \mathcal{Q} . Then $\text{Mor}_C(\mathcal{P}, \mathcal{Q})$ is a $M_{\mathcal{P}}$ -submodule of $\text{Mor}(\mathcal{P}, \mathcal{Q})$.*
3. *Let \mathcal{Q} be a twisted bialgebra. We denote by $\text{Mor}_B(\mathcal{P}, \mathcal{Q})$ the set of bialgebra morphisms from \mathcal{P} to \mathcal{Q} . Then $\text{Mor}_B(\mathcal{P}, \mathcal{Q})$ is a $M_{\mathcal{P}}$ -submodule of $\text{Mor}(\mathcal{P}, \mathcal{Q})$.*

Proof. Let $\phi \in \text{Mor}(\mathcal{P}, \mathcal{Q})$, $f, g \in M_{\mathcal{P}}$.

$$(\phi \leftarrow f) \leftarrow g = (\phi \otimes f \otimes g) \circ (\delta \otimes \text{Id}_{\mathcal{P}}) \circ \delta = (\phi \otimes f \otimes g) \circ (\text{Id}_{\mathcal{P}} \otimes \delta) \circ \delta = \phi \leftarrow (f \star g).$$

So \leftarrow is indeed an action.

1. Let $\phi \in \text{Mor}_A(\mathcal{P}, \mathcal{Q})$ and $f \in M_{\mathcal{P}}$. As ϕ, f and δ are algebra morphisms, $\phi \leftarrow f$ is an algebra morphism, so belong to $\text{Mor}_A(\mathcal{P}, \mathcal{Q})$.

2. Let $\phi \in \text{Mor}_C(\mathcal{P}, \mathcal{Q})$ and $f \in M_{\mathcal{P}}$.

$$\begin{aligned}
\Delta \circ (\phi \leftarrow f) &= \Delta \circ (\phi \otimes f) \circ \delta \\
&= (\phi \otimes \phi \otimes f) \circ (\Delta \otimes \text{Id}_{\mathcal{P}}) \circ \delta \\
&= (\phi \otimes \phi \otimes f) \circ m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta \\
&= (\phi \otimes f \otimes \phi \otimes f) \circ (\delta \otimes \delta) \circ \Delta \\
&= (\phi \leftarrow f \otimes \phi \leftarrow f) \circ \Delta.
\end{aligned}$$

So $\phi \leftarrow f \in \text{Mor}_C(\mathcal{P}, \mathcal{Q})$. The fact that \leftarrow is an action is proved as in the first point.

3. This comes from $\text{Mor}_B(\mathcal{P}, \mathcal{Q}) = \text{Mor}_A(\mathcal{P}, \mathcal{Q}) \cap \text{Mor}_C(\mathcal{P}, \mathcal{Q})$. \square

Lemma 4.9. *Let \mathcal{P} be a double twisted bialgebra, \mathcal{Q} and \mathcal{R} be twisted bialgebras. For any species morphisms $\phi : \mathcal{P} \rightarrow \mathcal{Q}$ and $\psi : \mathcal{Q} \rightarrow \mathcal{R}$, for any $f \in M_{\mathcal{P}}$:*

$$(\psi \circ \phi) \leftarrow f = \psi \circ (\phi \leftarrow f).$$

Proof. For any finite set A , for any $x \in \mathcal{P}[A]$:

$$\begin{aligned}
(\psi \circ \phi) \leftarrow f(x) &= (\psi \circ \phi \otimes f) \circ \delta(x) \\
&= \psi \circ \phi(x') f(x'') \\
&= \psi(\phi(x') f(x'')) \\
&= \psi(\phi \leftarrow f(x)) \\
&= \psi \circ (\phi \leftarrow f)(x),
\end{aligned}$$

where we put $\delta_A(x) = x' \otimes x''$. \square

Proposition 4.10. *Let \mathcal{P} be a double twisted bialgebra. The following map is an injective morphism of monoids:*

$$\chi_{\mathcal{P}} : \begin{cases} (M_{\mathcal{P}}, \star) & \longrightarrow (\text{Mor}_B(\mathcal{P}, \mathcal{P}), \circ) \\ f & \longrightarrow \text{Id}_{\mathcal{P}} \leftarrow f. \end{cases}$$

Proof. Let $f, g \in M_{\mathcal{P}}$. By Lemma 4.9:

$$(\text{Id}_{\mathcal{P}} \leftarrow f) \circ (\text{Id}_{\mathcal{P}} \leftarrow g) = ((\text{Id}_{\mathcal{P}} \leftarrow f) \circ \text{Id}_{\mathcal{P}}) \leftarrow g = (\text{Id}_{\mathcal{P}} \leftarrow f) \leftarrow g = \text{Id} \leftarrow (f \star g).$$

So $\chi_{\mathcal{P}}(f) \circ \chi_{\mathcal{P}}(g) = \chi_{\mathcal{P}}(f \star g)$. Let us consider the map:

$$\chi'_{\mathcal{P}} : \begin{cases} \text{Mor}_B(\mathcal{P}, \mathcal{P}) & \longrightarrow M_{\mathcal{P}} \\ \phi & \longrightarrow \varepsilon' \circ \phi. \end{cases}$$

For any $f \in M_{\mathcal{P}}$:

$$\chi'_{\mathcal{P}} \circ \chi_{\mathcal{P}}(f) = \varepsilon' \circ (\text{Id}_{\mathcal{P}} \otimes f) \circ \delta = (\varepsilon' \otimes f) \circ \delta = \varepsilon' \star f = f,$$

so $\chi'_{\mathcal{P}} \circ \chi_{\mathcal{P}} = \text{Id}_{M_{\mathcal{P}}}$. Consequently, $\chi_{\mathcal{P}}$ is injective (and $\chi'_{\mathcal{P}}$ is surjective). \square

4.3 The terminal property of *Comp*

Theorem 4.11. *Let $\mathcal{P} = (\mathcal{P}, m, \Delta, \delta)$ be a connected double twisted bialgebra. There exists a unique morphism ϕ of double bialgebras from \mathcal{P} to *Comp*. Moreover, the following maps are bijections, inverse one from the other:*

$$\begin{aligned}
\varsigma : \begin{cases} \text{Char}(\mathcal{P}) & \longrightarrow \text{Mor}_B(\mathcal{P}, \text{Comp}) \\ f & \longrightarrow \phi \leftarrow f, \end{cases} \\
\varsigma' : \begin{cases} \text{Mor}_B(\mathcal{P}, \text{Comp}) & \longrightarrow \text{Char}(\mathcal{P}) \\ \phi & \longrightarrow \varepsilon' \circ \phi. \end{cases}
\end{aligned}$$

Proof. Unicity of ϕ . If ϕ is such a morphism, then $\varepsilon' \circ \phi = \varepsilon'_{\mathcal{P}}$. By Theorem 3.6, ϕ is unique.

Existence of ϕ . Let ϕ be the unique morphism from (\mathcal{P}, m, Δ) to (Comp, m, Δ) , such that $\varepsilon' \circ \phi = \varepsilon'_{\mathcal{P}}$. Let us prove that it is compatible with δ . Let A be a finite set and $x \in \mathcal{P}[A]$. We proceed by induction on $\sharp A$. If $A = \emptyset$, we can assume that $x = 1_{\mathcal{P}}$. Then:

$$\delta_{\emptyset} \circ \phi(1_{\mathcal{P}}) = \delta_{\emptyset}(1_{\text{Comp}}) = 1_{\text{Comp}} \otimes 1_{\text{Comp}} = (\phi \otimes \phi) \circ \delta(1_{\mathcal{P}}).$$

Let us now assume that $A \neq \emptyset$. Let $\emptyset \subsetneq I \subsetneq A$ and $J = A \setminus I$. Then:

$$\begin{aligned} (\Delta_{I,J} \otimes \text{Id}_{\text{Comp}(A)}) \circ \delta_A \circ \phi(x) &= m_{1,3,24} \circ (\delta_I \otimes \delta_J) \circ \Delta_{I,J} \circ \phi(x) \\ &= m_{1,3,24} \circ (\delta_I \otimes \delta_J) \circ (\phi \otimes \phi) \circ \Delta_{I,J}(x) \\ &= m_{1,3,24} \circ (\phi \otimes \phi \otimes \phi \otimes \phi) \circ (\delta_I \otimes \delta_J) \circ (\phi \otimes \phi) \circ \Delta_{I,J}(x) \\ &= (\phi \otimes \phi \otimes \phi) \circ m_{1,3,24} \circ (\delta_I \otimes \delta_J) \circ \Delta_{I,J}(x) \\ &= (\phi \otimes \phi \otimes \phi) \circ (\Delta_{I,J} \otimes \text{Id}_{\mathcal{P}[A]}) \circ \delta_A(x) \\ &= (\Delta_{I,J} \otimes \text{Id}_{\text{Comp}(A)}) \circ (\phi \otimes \phi) \circ \delta_A(x). \end{aligned}$$

We put $y = (\phi \otimes \phi) \circ \delta_A(x) - \delta_A \circ \phi(x)$. Then:

$$y \in \bigcap_{(I,J) \in \mathbf{Comp}(A)} \ker(\Delta_{I,J} \otimes \text{Id}_{\text{Comp}(A)}) = (A) \otimes \text{Comp}(A).$$

We put $y = (A) \otimes z$. Then:

$$\begin{aligned} z &= (\varepsilon' \otimes \text{Id}_{\text{Comp}(A)})(y) \\ &= (\varepsilon' \circ \phi \otimes \phi) \circ \delta_A(x) - (\varepsilon' \otimes \text{Id}_{\text{Comp}(A)}) \circ \delta_A \circ \phi(x) \\ &= (\text{Id} \circ \phi) \circ (\varepsilon' \otimes \text{Id}) \circ \delta_A(x) - \phi(x) \\ &= \phi(x) - \phi(x) \\ &= 0. \end{aligned}$$

Therefore, $y = 0$, so $(\phi \otimes \phi) \circ \delta_A(x) = \delta_A \circ \phi(x)$.

Bijectivity of ς and ς' . Let $f \in \text{Char}(\mathcal{P})$.

$$\varepsilon' \circ (\phi \leftarrow f) = \varepsilon' \circ (\phi \otimes f) \circ \delta = (\varepsilon' \circ \phi) \otimes f \circ \delta = (\varepsilon'_{\mathcal{P}} \otimes f) \circ \delta = f.$$

Hence, $\Upsilon' \circ \Upsilon = \text{Id}_{\text{Char}(\mathcal{P})}$. By Theorem 2.6, Υ' is bijective, so its inverse is Υ , which is bijective too. \square

5 The example of graphs

5.1 Double twisted bialgebra of graphs

Let us now give the species $\mathcal{G}r'$ a structure of double bialgebra.

Notations 5.1. Let G be a graph in $\mathcal{G}r'[A]$.

1. The relation \sim_G is the equivalence whose classes are the vertices of G .
2. Let $I \subseteq A$. We assume that I is a union of vertices of A . The graph $G|_I \in \mathcal{G}r'[I]$ is defined by:
 - $V(G|_I) = \{x \in V(G), x \subseteq I\}$.
 - $E(G|_I) = \{\{x, y\} \in E(G), x \cup y \subseteq I\}$.

3. Let \sim be an equivalence on A . We shall write $\sim \triangleleft G$ if the following hold:

- For any $x, y \in A$, if $x \sim_G y$, then $x \sim y$.
- For any class I of \sim , $G|_I$ is a connected graph.

If so, we define two graphs $G|_{\sim}$ and G/\sim :

- (a) If C_1, \dots, C_k are the classes of \sim , then $G|_{\sim} = G|_{C_1} \dots G|_{C_k}$.
- (b) $V(G/\sim)$ is the set of classes of \sim .
- (c) Two classes C and C' of \sim are related by an edge in G/\sim if there exist vertices x, x' of G , related by an edge in G , such that $x \subseteq C$ and $x' \subseteq C'$.

Proposition 5.1. *The species $\mathcal{G}r'$ is a double twisted bialgebra, with the following coproducts:*

1. If G is a graph in $\mathcal{G}r'[I \sqcup J]$:

$$\Delta_{I,J}(G) = \begin{cases} G|_I \otimes G|_J & \text{if } I \text{ is a union of vertices of } G, \\ 0 & \text{otherwise.} \end{cases}$$

2. If G is a graph in $\mathcal{G}r'[A]$:

$$\delta_A(G) = \sum_{\sim \triangleleft G} G/\sim \otimes G|_{\sim}.$$

The counit of δ is given by:

$$\varepsilon'_A(G) = \begin{cases} 1 & \text{if } G \text{ has no edge,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $G \in \mathcal{G}r'[I \sqcup J \sqcup K]$. Then:

$$\begin{aligned} & (\Delta_{I,J} \otimes \text{Id}_{\mathcal{G}r'(K)}) \circ \Delta_{I \sqcup J, K}(G) \\ &= \begin{cases} G|_I \otimes G|_J \otimes G|_K & \text{if } I, J, K \text{ are union of vertices of } G, \\ 0 & \text{otherwise.} \end{cases} \\ &= (\text{Id}_{\mathcal{G}r'[I]} \otimes \Delta_{J,K}) \circ \Delta_{I, J \sqcup K}(G) \end{aligned}$$

Hence, Δ is coassociative.

Let G, H be graphs in respectively $\mathcal{G}r'[A]$ and $\mathcal{G}r'[B]$. If $A \sqcup B = I \sqcup J$, then:

$$\begin{aligned} \Delta_{I,J}(GH) &= \begin{cases} (GH)|_I \otimes (GH)|_J & \text{if } I \text{ is a union of vertices of } G \text{ and } H, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} G|_{A \cap I} H|_{B \cap I} \otimes G|_{A \cap J} H|_{B \cap J} & \text{if } I \text{ is a union of vertices of } G \text{ and } H, \\ 0 & \text{otherwise} \end{cases} \\ &= \Delta_{A \cap I, A \cap J}(G) \Delta_{A \cap J, B \cap J}(H). \end{aligned}$$

Hence, $(\mathcal{G}r', m, \Delta)$ is a twisted bialgebra.

Let A be a graph in $\mathcal{G}r'[A]$. We consider the set X of pairs (\sim, \sim') of equivalences on A such that:

1. For any $x, y \in A$, if $x \sim_G y$, then $x \sim y$; if $x \sim y$, then $x \sim' y$.
2. The equivalence classes of \sim and \sim' are connected graphs.

Then:

$$\begin{aligned}
(\delta_A \otimes \text{Id}_{\mathcal{G}r'[A]}) \circ \delta_A(G) &= \sum_{\substack{\sim \triangleleft G, \\ \sim' \triangleleft G/\sim}} (G/\sim)/\sim' \otimes (G/\sim)|\sim' \otimes G|\sim \\
&= \sum_{(\sim, \sim') \in X} (G/\sim)/\sim' \otimes (G/\sim)|\sim' \otimes G|\sim \\
&= \sum_{(\sim, \sim') \in X} G/\sim' \otimes (G|\sim')/\sim \otimes (G|\sim')|\sim \\
&= \sum_{\substack{\sim' \triangleleft G, \\ \sim \triangleleft G|\sim'}} G/\sim' \otimes (G|\sim')/\sim \otimes (G|\sim')|\sim \\
&= (\text{Id}_{\mathcal{G}r'[A]} \otimes \delta_A) \circ \delta_A(G).
\end{aligned}$$

So δ is coassociative. Let \sim'_G be the equivalence whose classes are the connected classes of G . Then $\sim_G, \sim'_G \triangleleft G$ and:

$$\begin{aligned}
(\varepsilon' \otimes \text{Id}_{\mathcal{G}r'[A]}) \circ \delta_A(G) &= \varepsilon'(G/\sim'_G)G|\sim'_G + 0 = G, \\
(\text{Id}_{\mathcal{G}r'[A]} \otimes \varepsilon') \circ \delta_A(G) &= \varepsilon'(G|\sim_G)G/\sim_G + 0 = G.
\end{aligned}$$

So ε' is indeed the counit of δ .

Let G, H be graphs in respectively $\mathcal{G}r'[A]$ and $\mathcal{G}r'[B]$. As the connected components of GH are the connected components of G and the connected components of H , for any equivalence \sim on $A \sqcup B$, $\sim \triangleleft GH$, if, and only if, $\sim = \sim' \sqcup \sim''$, with $\sim' \triangleleft G$ and $\sim'' \triangleleft H$. Consequently:

$$\begin{aligned}
\delta_{A \sqcup B}(GH) &= \sum_{\substack{\sim' \triangleleft G, \\ \sim'' \triangleleft H}} (GH)/\sim' \sqcup \sim'' \otimes (GH)|\sim' \sqcup \sim'' \\
&= \sum_{\substack{\sim' \triangleleft G, \\ \sim'' \triangleleft H}} (G/\sim')(H/\sim'') \otimes (G|\sim')(H|\sim'') \\
&= \delta_{A \sqcup B}(G)\delta_{A \sqcup B}(H).
\end{aligned}$$

Let G be a graph in $\mathcal{G}r'[A \sqcup B]$. If A is not a union of vertices of G , then:

$$(\Delta_{A,B} \otimes \text{Id}_{\mathcal{G}r'[A]}) \circ \delta_{A \sqcup B}(G) = m_{1,3,24} \circ (\delta_A \otimes \delta_B) \circ \Delta_{A,B}(G) = 0.$$

Otherwise:

$$\begin{aligned}
(\Delta_{A,B} \otimes \text{Id}_{\mathcal{G}r(A)}) \circ \delta_{A \sqcup B}(G) &= \sum_{\substack{\sim \triangleleft G, \\ A \text{ and } B \text{ are union of classes of } \sim}} (G/\sim)|_A \otimes (G/\sim)|_B \otimes G/\sim \\
&= \sum_{\substack{\sim' \triangleleft G|_A, \\ \sim'' \triangleleft G|_B}} (G|_A)/\sim' \otimes (G|_B)/\sim'' \otimes (G|_A)|\sim' (G|_B)|\sim'' \\
&= m_{1,3,24} \circ (\delta_A \otimes \delta_B) \circ \Delta_{A,B}(G).
\end{aligned}$$

Consequently, $\mathcal{G}r'$ is a double twisted bialgebra. □

Example 5.1. If A, B, C are nonempty finite sets:

$$\begin{aligned}
\Delta(\cdot_A) &= \cdot_A \otimes 1 + 1 \otimes \cdot_A, \\
\Delta(\mathfrak{!}_A^B) &= \mathfrak{!}_A^B \otimes 1 + \cdot_A \otimes \cdot_B + \cdot_B \otimes \cdot_A + 1 \otimes \mathfrak{!}_A^B, \\
\Delta({}^B\mathfrak{V}_A^C) &= {}^B\mathfrak{V}_A^C \otimes 1 + \mathfrak{!}_A^B \otimes \cdot_C + \mathfrak{!}_A^C \otimes \cdot_B + \cdot_B \otimes \cdot_C \\
&\quad + \cdot_C \otimes \mathfrak{!}_A^B + \cdot_B \otimes \mathfrak{!}_A^C + \cdot_A \otimes \cdot_{B \cdot C} + 1 \otimes {}^B\mathfrak{V}_A^C; \\
\delta(\cdot_A) &= \cdot_A \otimes \cdot_A, \\
\delta(\mathfrak{!}_A^B) &= \mathfrak{!}_A^B \otimes \cdot_{A \cdot B} + \cdot_{A \sqcup B} \otimes \mathfrak{!}_A^B, \\
\delta({}^B\mathfrak{V}_A^C) &= {}^B\mathfrak{V}_A^C \otimes \cdot_{A \cdot B \cdot C} + \mathfrak{!}_{A \sqcup B}^C \otimes \mathfrak{!}_A^B \cdot C + \mathfrak{!}_{A \sqcup C}^B \otimes \mathfrak{!}_A^C \cdot B + \cdot_{A \sqcup B \sqcup C} \otimes {}^B\mathfrak{V}_A^C.
\end{aligned}$$

5.2 Morphism to *Comp*

We now look for the unique morphism $\phi : \mathcal{G}r' \longrightarrow \mathit{Comp}$ of Theorem 4.11.

Definition 5.2. Let G be a graph in $\mathcal{G}r'[A]$. A valid packed coloration of G is a surjective map $c : A \longrightarrow [\max(c)]$, such that:

1. For any $x, y \in A$, if $x \sim_G y$, then $c(x) = c(y)$.
2. For any $x, y \in A$, if there is an edge in G between the \sim_G -equivalence classes of x and y , then $c(x) \neq c(y)$.

The set of valid colorations of G is denoted by $VC(G)$.

Proposition 5.3. The unique morphism of double bialgebras from $\mathcal{G}r'$ to *Comp* of Theorem 3.6 is given on any graph G by:

$$\phi(G) = \sum_{c \in VC(G)} (c^{-1}(1), \dots, c^{-1}(\max(c))). \quad (3)$$

Proof. By Theorem 3.6:

$$\begin{aligned}
\phi(G) &= \sum_{(A_1, \dots, A_k)} \varepsilon'^{\otimes k} \circ \Delta_{A_1, \dots, A_k}(G)(A_1, \dots, A_k) \\
&= \sum_{(A_1, \dots, A_k)} \varepsilon'(G|_{A_1}) \dots \varepsilon'(G|_{A_k})(A_1, \dots, A_k).
\end{aligned}$$

By definition of ε' , $\varepsilon'(G|_{A_1}) \dots \varepsilon'(G|_{A_k}) = 1$ if the map c sending any $a \in A_i$ to i for any i is a valid coloration of G , and 0 otherwise, which implies (3). \square

Example 5.2. If A, B and C are finite sets:

$$\begin{aligned}
\phi(\cdot_A) &= (A), \\
\phi(\mathfrak{!}_A^B) &= (A, B) + (B, A), \\
\phi({}^B\mathfrak{V}_A^C) &= (A, B, C) + (A, C, B) + (B, A, C) + (B, C, A) + (C, A, B) + (C, B, A) \\
&\quad + (A, B \sqcup C) + (B \sqcup C, A).
\end{aligned}$$

6 The example of finite topologies

6.1 Double twisted bialgebra of finite topologies

Definition 6.1. Let $T = (A, \leq)$ be a finite topology and let \sim be an equivalence on A .

1. We define a second quasi-order $\leq_{T|\sim}$ on A by the relation:

$$\forall x, y \in A, \quad x \leq_{T|\sim} y \text{ if } (x \leq y \text{ and } x \sim y).$$

2. We define a third quasi-order $\leq_{T/\sim}$ on A as the transitive closure of the relation R defined by:

$$\forall x, y \in A, \quad xRy \text{ if } (x \leq y \text{ or } x \sim y).$$

In other words, $x \leq_{T/\sim} y$ if there exist $x_1, y_1, \dots, x_n, y_n \in A$, such that:

$$x = x_1 \sim y_1 \leq_T \dots \leq_T x_k \sim y_k = y.$$

3. We shall say that \sim is T -compatible and we shall write $\sim \triangleleft T$ if the two following conditions are satisfied:

- The restriction of T to any equivalence class of \sim is connected.
- The equivalences $\sim_{T/\sim}$ and \sim are equal. In other words:

$$\forall x, y \in A, \quad (x \leq_{T/\sim} y \text{ and } y \leq_{T/\sim} x) \implies x \sim y;$$

note that the converse assertion trivially holds.

The set of T -compatible equivalences is denoted by $\text{CE}(T)$.

Lemma 6.2. Let T be a finite topology on a set A and let \sim be an equivalence on A .

1. The open sets of T/\sim are the \sim -saturated open sets of T .
2. The open sets of $T|\sim$ are the sets of the form

$$(O_1 \cap X_1) \cup \dots \cup (O_n \cap X_n),$$

where O_1, \dots, O_n are open sets of T and X_1, \dots, X_n are equivalence classes of \sim .

3. $\sim \in \text{CE}(T)$ if, and only if, the following hold:

- (a) For any equivalence class X of \sim , $T|_X$ is connected.
- (b) For any equivalence class X of \sim , there exist an \sim -saturated open set O and a \sim -saturated closed set C such that $X = O \cap C$.

Proof. 1. Let O be an open set of T/\sim . If $x \in O$ and $y \in A$, such that $x \leq_T y$ or $x \sim y$, then $x \leq_{T/\sim} y$, so $y \in O$: O is an \sim -saturated open set of T . Conversely, let O be a \sim -saturated open set of T . Let $x \in O$ and $x \leq_{T/\sim} y$. There exist $x_1, y_1, \dots, x_n, y_n \in A$, such that:

$$x = x_1 \sim y_1 \leq_T \dots \leq_T x_k \sim y_k = y.$$

As O is open and saturated, for any i , $x_i, y_i \in O$, so $y \in O$: O is an open set of T/\sim .

2. Let O be an open set of $T|\sim$. For any equivalence class X of \sim , we put:

$$O_X = \{y \in A, \exists x \in O \cap X, x \leq_T y\}.$$

Then O_X is obviously an open set of T containing $O \cap X$. If $y \in O_X \cap X$, there exists $x \in O_X$, such that $x \leq_T y$. As O is open, $y \in O$, so $y \in O \cap X$. We finally obtain that $O_X \cap X = O \cap X$, and:

$$O = \bigcup_{X \text{ class of } \sim} (O_X \cap X).$$

Conversely, let us assume that O has the form

$$O = (O_1 \cap X_1) \cup \dots \cup (O_n \cap X_n).$$

Let $x \in O$ and $x \leq_{T/\sim} y$. Then $x \leq_T y$ and $x \sim y$. There exists i such that $x \in O_i \cap X_i$. As O_i is open, $y \in O_i$. As X_i is an equivalence class of \sim , $y \in X_i$, so $y \in O_i \cap X_i \subseteq O$.

3. For any $X \subseteq A$, we put:

$$O(X) = \{y \in A, \exists x \in X, x \leq_{T/\sim} y\}, \quad C(X) = \{y \in A, \exists x \in X, y \leq_{T/\sim} x\}.$$

Then $O(X)$ is an open set of T/\sim , so is a \sim -saturated open set of T ; $C(X)$ is a closed set of T/\sim , so is a \sim -saturated closed set of T . Moreover, $X \subseteq O(X) \cap C(X)$.

\Leftarrow . Let us assume that $\sim \in \text{CE}(T)$ and let X be an equivalence class of \sim . By hypothesis, $T|_X$ is connected. Let $y \in O(X) \cap C(X)$. There exists $x, x' \in X$, such that $x \leq_{T/\sim} y \leq_{T/\sim} x'$. As $x \sim x'$, $x \sim_{T/\sim} x'$ so $x \sim_{T/\sim} y \sim_{T/\sim} x'$. Moreover, $\sim \in \text{CE}(T)$, so $x \sim y \sim x'$, and $y \in X$. We proved that $X = O(X) \cap C(X)$.

\Rightarrow . Let O' be a \sim -saturated open set and C' be a \sim -saturated open set such that $X = O' \cap C'$. Let $y \in O(X)$. There exists $x \in X$, such that $x \leq_{T/\sim} y$. As O' is a open set of T/\sim by the first point, $y \in O'$, so $O(X) \subset O'$. Similarly, $C(X) \subset C'$. Hence:

$$X \subseteq O(X) \cap C(X) \cap O' \cap C' = X.$$

We proved that $X = O(X) \cap C(X)$.

Let $x, y \in A$, such that $x \sim_{T/\sim} y$. Let us denote by X the equivalence class of x . As $x \leq_{T/\sim} y$, $y \in O(X)$; as $y \leq_{T/\sim} x$, $y \in C(X)$. So $y \in X$ and $x \sim y$. \square

Lemma 6.3. *Let T be a finite topology on a set A and \sim, \sim' be equivalences on A such that:*

$$\forall x, y \in A, \quad x \sim y \implies x \sim' y.$$

Then:

$$(T| \sim')| \sim = T| \sim, \quad (T/\sim)/ \sim' = T/\sim'.$$

If moreover $\sim' \in \text{CE}(T)$, then:

$$(T| \sim')/ \sim = (T/\sim)| \sim'.$$

Proof. Let $x, y \in A$.

$$\begin{aligned} x \leq_{(T\sim')| \sim} y &\iff x \leq_{T|\sim'} y \text{ and } x \sim y \\ &\iff x \leq_T y \text{ and } x \sim' y \text{ and } x \sim y \\ &\iff x \leq_T y \text{ and } x \sim y \\ &\iff x \leq_{T|\sim} y. \end{aligned}$$

So $(T| \sim')| \sim = T| \sim$.

By definition, $\leq_{(T/\sim)/ \sim'}$ is the transitive closure of the relation given by:

$$x \leq_T y \text{ or } x \sim y \text{ or } x \sim' y,$$

or equivalently:

$$x \leq_T y \text{ or } x \sim' y.$$

Therefore, $(T/\sim)/\sim' = T/\sim'$.

Let $x, y \in A$. If $x \leq_{(T/\sim)'} y$, there exist $x_1, y_1, \dots, x_k, y_k \in A$ such that:

$$x = x_1 \sim y_1 \leq_{T|\sim'} x_2 \sim y_2 \leq_{T|\sim'} \dots \leq_{T|\sim'} x_k \sim y_k = y.$$

Consequently:

$$x = x_1 \sim y_1 \leq_T x_2 \sim y_2 \leq_T \dots \leq_T x_k \sim y_k = y,$$

so $x \leq_{T/\sim} y$. Moreover,

$$x = x_1 \sim' y_1 \sim' \dots \sim' x_k \sim' y_k = y,$$

so $x \sim' y$ and $x \leq_{(T/\sim)'} y$. Let us assume that $\sim' \in \text{CE}(T)$. If $x \leq_{(T/\sim)'} y$, then $x \leq_{T/\sim} y$ and $x \sim' y$. There exist $x_1, y_1, \dots, x_k, y_k \in A$, such that:

$$x = x_1 \sim y_1 \leq_T x_2 \sim y_2 \leq_T \dots \leq_T x_k \sim y_k = y.$$

Consequently:

$$x = x_1 \leq_{T/\sim'} y_1 \leq_{T/\sim'} \dots \leq_{T/\sim'} x_k \leq_{T/\sim'} y_k \leq_{T/\sim'} y \leq_{T/\sim'} x,$$

so all the elements of A appearing here are $\sim_{T/\sim'}$ -equivalent. As $\sim' \in \text{CE}(T)$, they are \sim' -equivalent. Hence:

$$x = x_1 \sim y_1 \leq_{T|\sim'} x_2 \sim y_2 \leq_{T|\sim'} \dots \leq_{T|\sim'} x_k \sim y_k = y.$$

So $x \leq_{(T|\sim)'} y$. Finally, $(T|\sim')/\sim = (T/\sim)|\sim'$. □

Proposition 6.4. *Let T be a finite topology on a set A and $\sim \in \text{CE}(T)$.*

1. *The Hasse graph of $T|\sim$ is obtained from the Hasse graph of T by deleting the edges of the graph of T between non \sim -equivalent vertices.*
2. *The Hasse graph of T/\sim is obtained from the Hasse graph of T by:*
 - *Contracting any equivalence class of \sim to a single vertex.*
 - *Deleting the superfluous edges created in this process.*

Proof. First, if $x \sim_T y$, then $x \sim_{T/\sim} y$, so $x \sim y$ as $\sim \in \text{CE}(T)$. Hence, the sentence describing the construction of the Hasse graph of $T|\sim$ makes sense.

1. Obviously, $\sim_{T|\sim} = \sim_T$, so the vertices of the Hasse graph of $T|\sim$ are the vertices of the Hasse graph of T , that is to say the classes of \sim_T . For any $x \in A$, we denote by $cl_T(x)$ its equivalence class for \sim_T .

Let us prove that the edges of the Hasse graph of $T|\sim$ are of the form $(cl_T(x), cl_T(y))$, with $x \sim y$. Firstly, if $(cl_T(x), cl_T(y))$ is an edge of the Hasse graph of $T|\sim$, then $x \leq_{T|\sim} y$, so $x \leq_T y$ and $x \sim y$. If $x \leq_T z \leq_T y$, then $x \leq_{T/\sim} z \leq_{T/\sim} y \leq_{T/\sim} x$, so $x \sim_{T/\sim} z \sim_{T/\sim} y$. As $\sim \in \text{CE}(T)$, $x \sim z \sim y$, so $x \leq_{T|\sim} z \leq_{T|\sim} y$. Consequently, $x \sim_T z$ or $y \sim_T z$. Secondly, if $(cl_T(x), cl_T(y))$ is an edge of the Hasse graph of T and $x \sim y$, then $x \leq_{T|\sim} y$. If $x \leq_{T/\sim} z \leq_{T|\sim} y$, then $x \leq_T z \leq_T y$, so $x \sim_T y$ or $y \sim_T z$.

2. As $\sim \in \text{CE}(T)$, the vertices of the Hasse graph of T/\sim are the classes of \sim . Let us prove that for any edge $(cl_\sim(x), cl_\sim(y))$ of the Hasse graph of T/\sim , there exist $x' \sim x$ and $y' \sim y$ such that $(cl_T(x'), cl_T(y'))$ is an edge of the Hasse graph of T : this will prove the second point. As $x \leq_T y$, there exist $x_1, y_1, \dots, x_k, y_k \in A$ such that:

$$x = x_1 \sim y_1 \leq_T \dots \leq_T x_k \sim y_k = y.$$

Consequently:

$$x = x_1 \leq_{T/\sim} y_1 \leq_{T/\sim} \dots \leq_{T/\sim} x_k \leq_{T/\sim} y_k = y.$$

As $(cl_{\sim}(x), cl_{\sim}(y))$ is an edge, there exists i such that:

$$x = x_1 \sim y_1 \sim \dots \sim y_i \leq_T x_{i+1} \sim \dots \sim x_k \sim y_k = y.$$

We consider the set:

$$X = \{x' \in A, x' \sim x, \exists y' \in A, y' \sim y \text{ and } x' \leq_T y'\}.$$

We proved that X is nonempty: let $x' \in X$, maximal for \leq_T . We now consider:

$$Y = \{y' \in A, y' \sim y \text{ and } x' \leq_T y'\}.$$

By definition of x' , this is nonempty: let $y \in Y'$, minimal for \leq_T . Then $x' \leq_T y'$, $x' \sim x$ and $y' \sim y$. Let us assume that $x' \leq_T z \leq_T y$. Then $x' \leq_{T/\sim} z \leq_{T/\sim} y$, so $x' \sim z$ or $z \sim y'$.

- If $x' \sim z$, by maximality of x' , $x' \sim_T z$.
- If $z \sim y'$, by minimality of y' , $z \sim_T y'$.

So $(cl_T(x'), cl_T(y'))$ is an edge of the Hasse graph of T . □

Corollary 6.5. *Let T be a finite topology on a set A and let $\sim \in \text{CE}(T)$.*

1. *Let $X \subseteq A$, \sim -saturated. Then $T|_X$ is connected if, and only if, $(T/\sim)|_X$ is connected.*
2. *Let $X \subseteq A$, included in a class of \sim . Then $T|_X$ is connected if, and only if, $(T|\sim)|_X$ is connected.*

Proof. 1. \Leftarrow . Let $x, y \in X$. There exists a path from $cl_T(x)$ to $cl_T(y)$ in the Hasse graph of T , with all its vertices in X . By construction of the Hasse graph of T/\sim , there exists a path from $cl_{\sim}(x)$ to $cl_{\sim}(y)$ in the Hasse graph of T/\sim , with all its vertices in X . So X is T/\sim -connected.

\Rightarrow . Let $x, y \in X$. There exists a path from $cl_{\sim}(x)$ to $cl_{\sim}(y)$ in the Hasse graph of T/\sim , with all its internal vertices $x = x_0, x_1, \dots, x_k = y$ be classes of elements of x . By construction of the Hasse graph of T/\sim , for any i there exist x'_i, x''_i with an edge between $cl_T(x'_i)$ and $cl_T(x''_i)$ in the Hasse graph of T , $x'_i \sim x_i$ and $x''_i \sim x_{i+1}$. As the classes of \sim are connected, there exists a path in the Hasse graph of T between $cl_T(x'_i)$ and $cl_T(x_i)$, with all its vertices being equivalent to x'_i . As X is \sim -saturated, all these vertices belong to X . Similarly, there exists a path in the Hasse graph of T between $cl_T(x''_i)$ and $cl_T(x_{i+1})$, with all its vertices in X . We finally obtain a path in the Hasse graph of T between $cl_T(x)$ and $cl_T(y)$, so X is T -connected.

2. \Leftarrow . There exists a path from $cl_T(x)$ to $cl_T(y)$ in the Hasse graph of T , with all its vertices in X . As X is included in a single class of \sim , all the edges between them belong to the Hasse graph of $T|\sim$, so X is $T|\sim$ -connected.

\Rightarrow . Immediate, as the Hasse graph of $(T|\sim)|_X$ is a subgraph of the Hasse graph of $T|_X$. □

Theorem 6.6. *The species $\mathcal{T}op$ is a double algebra with the following coproducts:*

1. *For any quasi-poset $T \in \mathcal{T}op[A \sqcup B]$,*

$$\Delta_{A,B}(T) = \begin{cases} T|_A \otimes T|_B & \text{if } B \text{ is an open set of } T, \\ 0 & \text{otherwise.} \end{cases}$$

2. *For any quasi-poset $T \in \mathcal{T}op[A]$,*

$$\delta_A(T) = \sum_{\sim \in \text{CE}(T)} T/\sim \otimes T|\sim.$$

The counit ε' of δ is given by:

$$\varepsilon'_A(T) = \begin{cases} 1 & \text{if } \leq_T \text{ is an equivalence,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let T be a quasi-poset on a set E , and let $E = A \sqcup B \sqcup C$.

$$\begin{aligned}
& (\Delta_{A,B} \otimes \text{Id}) \circ \Delta_{A \sqcup B, C}(T) \\
&= \begin{cases} T|_A \otimes T|_B \otimes T|_C & \text{if } C \text{ is an open set of } T \text{ and } B \text{ is an open set of } T|_{A \sqcup B}, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} T|_A \otimes T|_B \otimes T|_C & \text{if } B \sqcup C \text{ and } C \text{ are open sets of } T, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} T|_A \otimes T|_B \otimes T|_C & \text{if } B \sqcup C \text{ is an open set of } T \text{ and } C \text{ is an open set of } T|_{B \sqcup C}, \\ 0 & \text{otherwise,} \end{cases} \\
&= (\text{Id} \otimes \Delta_{B,C}) \circ \Delta_{A, B \sqcup C}(T).
\end{aligned}$$

So Δ is coassociative.

Let T and T' be a finite topology on A and A' , respectively. Let B and B' be finite sets such that $A \sqcup A' = B \sqcup B'$.

$$\begin{aligned}
\Delta_{B, B'}(TT') &= \begin{cases} (TT')|_B \otimes (TT')|_{B'} & \text{if } B' \text{ is an open set of } TT', \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} T|_{A \cap B} T'|_{A' \cap B} \otimes T|_{A \cap B'} T'|_{A' \cap B'} & \text{if } A \cap B' \text{ and } A' \cap B \text{ are open sets of } T \text{ and } T', \\ 0 & \text{otherwise,} \end{cases} \\
&= \Delta_{A \cap B, A \cap B'}(T) \Delta_{A' \cap B, A' \cap B'}(T').
\end{aligned}$$

So $(\mathcal{Top}, m, \Delta)$ is a twisted bialgebra.

Let $T \in \mathcal{Top}[A]$ and $T' \in \mathcal{Top}[A']$. If $\sim \in \text{CE}(TT')$, as its classes are connected, \sim can be uniquely written as $\sim = \sim' \sqcup \sim''$, with $\sim' \in \text{CE}(T)$ and $\sim'' \in \text{CE}(T')$. Conversely, if $\sim' \in \text{CE}(T)$ and $\sim'' \in \text{CE}(T')$, then $\sim = \sim' \sqcup \sim'' \in \text{CE}(TT')$. Hence:

$$\begin{aligned}
\delta_{A \sqcup A'}(TT') &= \sum_{\sim' \in \text{CE}(T), \sim'' \in \text{CE}(T')} (TT')/(\sim' \sqcup \sim'') \otimes (TT')|(\sim' \sqcup \sim'') \\
&= \sum_{\sim' \in \text{CE}(T), \sim'' \in \text{CE}(T')} (T/\sim')(T'/\sim'') \otimes (T|\sim')(T'|\sim'') \\
&= \delta_A(T) \delta_{A'}(T').
\end{aligned}$$

Let $T \in \mathcal{Top}[A]$. Let $\sim_0 = \sim_T$ and \sim_1 be the equivalence which classes are the connected components of T . Then both belong to $\text{CE}(T)$. For any $\sim \in \text{CE}(T)$:

$$\begin{aligned}
\leq_{T/\sim} & \text{ is an equivalence} \iff \sim = \sim_1, \\
\leq_{T|\sim} & \text{ is an equivalence} \iff \sim = \sim_0.
\end{aligned}$$

Consequently:

$$(\varepsilon' \otimes \text{Id}) \circ \delta(A) = T|\sim_1 + 0 = T, \quad (\text{Id} \otimes \varepsilon') \circ \delta_A(T) = T/\sim_0 = T.$$

So ε' is the counit of δ .

Let $T \in \mathcal{Top}[A]$.

$$\begin{aligned}
(\delta_A \otimes \text{Id}) \circ \delta(T) &= \sum_{\sim \in \text{CE}(T), \sim' \in \text{CE}(T/\sim)} (T/\sim)/\sim' \otimes (T/\sim)|\sim' \otimes T|\sim, \\
(\text{Id} \otimes \delta_A) \circ \delta_A(T) &= \sum_{\sim' \in \text{CE}(T), \sim \in \text{CE}(T|\sim')} T/\sim' \otimes (T|\sim')/\sim \otimes (T|\sim')|\sim.
\end{aligned}$$

We consider the two sets:

$$\mathcal{X} = \{(\sim, \sim'), \sim \in \text{CE}(T), \sim' \in \text{CE}(T/\sim)\}, \quad \mathcal{Y} = \{(\sim, \sim'), \sim' \in \text{CE}(T), \sim \in \text{CE}(T|\sim')\}.$$

Let $(\sim, \sim') \in \mathcal{X}$. If $x \sim y$, $x \sim_{T/\sim} y$ and $x \sim_{(T/\sim)/\sim'} y$. As $\sim' \in \text{CE}(T/\sim)$, $x \sim' y$. Let $(\sim, \sim') \in \mathcal{Y}$. If $x \sim y$, then x and y are in the same connected component of $T|\sim'$ as the classes of \sim are $T|\sim'$ -connected. Hence, $x \sim' y$. By Lemma 6.3:

$$\begin{aligned} (\delta_A \otimes \text{Id}) \circ \delta(T) &= \sum_{(\sim, \sim') \in \mathcal{X}} T/\sim' \otimes (T/\sim)|\sim' \otimes T|\sim, \\ (\text{Id} \otimes \delta_A) \circ \delta_A(T) &= \sum_{(\sim, \sim') \in \mathcal{Y}} T/\sim' \otimes (T/\sim)|\sim' \otimes T|\sim. \end{aligned}$$

In order to prove the coassociativity of δ , it is enough to prove that $\mathcal{X} = \mathcal{Y}$.

Let $(\sim, \sim') \in \mathcal{X}$. Let X' be an equivalence class of \sim' . Then it is \sim' -saturated, so also \sim -saturated. By Corollary 6.5, as X' is T/\sim -connected, it is also T -connected. Let X be an equivalence class of \sim . Then it is included in a single class of \sim' . By Corollary 6.5, as X is T -connected, it is also $T|\sim'$ -connected.

By Lemma 6.3, $\sim_{T/\sim'} = \sim_{(T/\sim)/\sim'} = \sim'$ as $\sim' \in \text{CE}(T/\sim)$. So $\sim' \in \text{CE}(T)$. Consequently, $\sim_{(T|\sim')/\sim} = \sim_{(T/\sim)|\sim'}$. For any $x, y \in A$, as $\sim \in \text{CE}(T)$:

$$\begin{aligned} x \sim_{(T/\sim)|\sim'} y &\iff x \sim' y \text{ and } x \sim_{T/\sim} y \\ &\iff x \sim' y \text{ and } x \sim y \\ &\iff x \sim y. \end{aligned}$$

We obtain that $\sim \in \text{CE}(T|\sim')$. Therefore, $\mathcal{X} \subseteq \mathcal{Y}$.

Let $(\sim, \sim') \in \mathcal{Y}$. Let X be a class of \sim . Then X is $T|\sim'$ -connected. By Corollary 6.5, it is T -connected. Let X' be a class of \sim' . Then it is \sim -saturated and T -connected. By Corollary 6.5, it is T/\sim -connected.

By Lemma 6.3, $\sim_{(T/\sim)/\sim'} = \sim_{T/\sim'} = \sim'$ as $\sim' \in \text{CE}(T)$, so $\sim' \in \text{CE}(T/\sim)$. Let $x, y \in A$. If $x \sim_{T/\sim} y$, as $\sim \in \text{CE}(T|\sim')$, x and y are in the same connected component of $T|\sim'$, that is to say in the same class of \sim' as $\sim' \in \text{CE}(T)$, so $x \sim' y$. Consequently, $x \sim_{(T/\sim)|\sim'} y$, so $x \sim_{(T|\sim')/\sim} y$ and $x \sim y$, as $\sim \in \text{CE}(T|\sim')$. Hence, $\sim \in \text{CE}(T)$. We proved that $\mathcal{Y} \subseteq \mathcal{X}$.

Let $T \in \mathcal{Top}[A]$ and $A = I \sqcup J$.

$$\begin{aligned} (\Delta_{I,J} \otimes \text{Id}) \circ \delta_A(T) &= \sum_{\substack{\sim \in \text{CE}(T), \\ J \in O(T/\sim)}} (T/\sim)|_I \otimes (T/\sim)|_J \otimes T|\sim \\ &= \sum_{\substack{\sim \in \text{CE}(T), \\ J \in O(T), \sim\text{-saturated}}} (T/\sim)|_I \otimes (T/\sim)|_J \otimes T|\sim \\ &= \begin{cases} \sum_{\substack{\sim' \in \text{CE}(T|_I), \\ \sim'' \in \text{CE}(T|_J)}} (T|_I)/\sim' \otimes (T|_J)/\sim'' \otimes T|(\sim' \sqcup \sim'') & \text{if } J \in O(T), \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \sum_{\substack{\sim' \in \text{CE}(T|_I), \\ \sim'' \in \text{CE}(T|_J)}} (T|_I)/\sim' \otimes (T|_J)/\sim'' \otimes (T|_I)|\sim' (T|_J)|\sim'' & \text{if } J \in O(T), \\ 0 & \text{otherwise,} \end{cases} \\ &= m_{1,3,24} \circ (\delta_I \otimes \delta_J) \circ \Delta_{I,J}(T). \end{aligned}$$

So T is indeed a double twisted bialgebra. □

Example 6.1. If A, B, C are finite sets:

$$\begin{aligned}
\Delta(\cdot_A) &= \cdot_A \otimes 1 + 1 \otimes \cdot_A, \\
\Delta(\mathfrak{!}_A^B) &= \mathfrak{!}_A^B \otimes 1 + \cdot_A \otimes \cdot_B + 1 \otimes \mathfrak{!}_A^B, \\
\Delta(\mathfrak{!}_A^{B \vee C}) &= \mathfrak{!}_A^{B \vee C} \otimes 1 + \mathfrak{!}_A^B \otimes \cdot_C + \mathfrak{!}_A^C \otimes \cdot_B + \cdot_A \otimes \cdot_{B \cdot C} + 1 \otimes \mathfrak{!}_A^{B \vee C}, \\
\Delta(\mathfrak{!}_A^C) &= \mathfrak{!}_A^C \otimes 1 + \mathfrak{!}_A^B \otimes \cdot_C + \cdot_A \otimes \mathfrak{!}_B^C + 1 \otimes \mathfrak{!}_A^C; \\
\\
\delta(\cdot_A) &= \cdot_A \otimes \cdot_A, \\
\delta(\mathfrak{!}_A^B) &= \mathfrak{!}_A^B \otimes \cdot_{A \cdot B} + \cdot_{A \sqcup B} \otimes \mathfrak{!}_A^B, \\
\delta(\mathfrak{!}_A^{B \vee C}) &= \mathfrak{!}_A^{B \vee C} \otimes \cdot_{A \cdot B \cdot C} + \mathfrak{!}_{A \sqcup B}^C \otimes \mathfrak{!}_A^B \cdot C + \mathfrak{!}_{A \sqcup C}^B \otimes \mathfrak{!}_A^C \cdot B + \cdot_{A \sqcup B \sqcup C} \otimes \mathfrak{!}_A^{B \vee C}, \\
\delta(\mathfrak{!}_A^C) &= \mathfrak{!}_A^C \otimes \cdot_{A \cdot B \cdot C} + \mathfrak{!}_{A \sqcup B}^C \otimes \mathfrak{!}_A^B \cdot C + \mathfrak{!}_A^{B \sqcup C} \otimes \cdot_A \mathfrak{!}_B^C + \cdot_{A \sqcup B \sqcup C} \otimes \mathfrak{!}_A^C.
\end{aligned}$$

6.2 Morphism to \mathcal{Comp}

Let us describe the unique morphism $\phi : \mathcal{Top} \longrightarrow \mathcal{Comp}$ of double twisted bialgebra.

Definition 6.7. Let $T = (A, \leq_T)$ be a quasi-poset.

1. We denote by $L'(T)$ the set of surjections $f : A \longrightarrow \underline{\max(f)}$ such that:
 - For any $a, b \in A$, if $a \leq_T b$, then $f(a) \leq f(b)$.
2. We denote by $L(T)$ the set of surjections $f : A \longrightarrow \underline{\max(f)}$ such that:
 - For any $a, b \in A$, if $a \leq_T b$, then $f(a) \leq f(b)$.
 - For any $a, b \in A$, if $a \leq_T b$ and $f(a) = f(b)$, then $a \sim_T b$.

Proposition 6.8. Let $\lambda \in \text{Char}(\mathcal{Top})$. The unique twisted bialgebra morphism $\psi : \mathcal{Top} \longrightarrow \mathcal{Comp}$ such that $\varepsilon' \circ \psi = \lambda$ (Theorem 3.6) is given by the following: for any quasi-poset $T = (A, \leq_T)$,

$$\psi(T) = \sum_{f \in L'(T)} \lambda(T|_{f^{-1}(1)}) \dots \lambda(T|_{f^{-1}(\max(f))})(f^{-1}(1), \dots, f^{-1}(\max(f))).$$

Proof. We denote by $S(T)$ the set of sequences (A_1, \dots, A_k) of subsets of A such that:

1. For any $i \in \underline{k}$, A_i is nonempty and $A_1 \sqcup \dots \sqcup A_k = A$.
2. For any $i \in \underline{k}$, $A_i \sqcup \dots \sqcup A_k$ is an open set of T .

By Theorem 3.6:

$$\psi(T) = \sum_{(A_1, \dots, A_k) \in S(T)} \lambda(T|_{A_1} \dots T|_{A_k})(A_1, \dots, A_k).$$

The following map is a bijection:

$$\begin{cases} L'(T) & \longrightarrow S(T) \\ f & \longrightarrow (f^{-1}(1), \dots, f^{-1}(\max(f))), \end{cases}$$

which implies the result. □

In order to obtain the morphism ϕ of Theorem 4.11, we consider $\lambda = \varepsilon'$. Firstly, observe that $L(T) \subseteq L'(T)$. Moreover, for any $f \in L'(T)$, the following assertions are equivalent:

1. $f \in L(T)$.

2. $\forall i \in \underline{\max(f)}, \leq_{T|_{f^{-1}(i)}}$ is an equivalence.
3. $\forall i \in \underline{\max(f)}, \varepsilon'(T|_{f^{-1}(i)}) = 1$.
4. $\forall i \in \underline{\max(f)}, \varepsilon'(T|_{f^{-1}(i)}) \neq 0$.

We obtain:

Theorem 6.9. *The unique double bialgebra morphism from \mathcal{Top} to \mathcal{Comp} of Theorem 4.11 is given by the following: for any quasi-poset T ,*

$$\phi(T) = \sum_{f \in L(T)} (f^{-1}(1), \dots, f^{-1}(\max(f))).$$

Example 6.2. If A, B and C are finite sets:

$$\begin{aligned} \phi(\bullet_A) &= (A), \\ \phi(\mathfrak{!}_A^B) &= (A, B), \\ \phi({}^B\mathfrak{V}_A^C) &= (A, B, C) + (A, C, B) + (A, B \sqcup C), \\ \phi(\mathfrak{!}_A^C) &= (A, B, C). \end{aligned}$$

6.3 Structure of \mathcal{Top}

Theorem 6.10. *The bialgebra \mathcal{Top} is isomorphic to a shuffle twisted bialgebra.*

Proof. By theorem 3.7, it is enough to prove that \mathcal{Top} is cofree. We shall apply Theorem 2.7. Let us consider the product \otimes (joint product) defined on \mathcal{Top} by the following: if $T = (A, \leq_T)$ and $S = (B, \leq_S)$ are quasi-posets, then, for any $x, y \in A \sqcup B$,

$$\begin{aligned} x \leq_{S \otimes T} y &\text{ if } (x, y) \in A^2 \text{ and } x \leq_S y \\ &\text{ or } (x, y) \in B^2 \text{ and } x \leq_T y \\ &\text{ or } (x, y) \in A \times B. \end{aligned}$$

The open sets of $S \otimes T$ are:

- The open sets of T ;
- The sets $O \sqcup B$, where O is an open set of S .

Hence, if $A \sqcup B = A' \sqcup B'$:

1. If B' is an open set of T , then $B' \subseteq B$ and:

$$\Delta_{A', B'}(S \otimes T) = S \otimes T|_{B \setminus B'} \otimes T|_{B'} = (S \otimes 1) \otimes \Delta_{B \setminus B', B'}(T).$$

2. If A' is a closed set of S , then $A' \subseteq A$ and:

$$\Delta_{A', B'}(S \otimes T) = S|_{A'} \otimes S|_{A \setminus A'} \otimes T = \Delta_{A', A \setminus A'}(S) \otimes (1 \otimes T).$$

3. Otherwise, $\Delta_{A', B'}(S \otimes T) = 0$; if $A' \subseteq A$, $\Delta_{A', A \setminus A'}(S) = 0$; if $B' \subseteq B$, $\Delta_{B \setminus B', B'}(T) = 0$.

Therefore, the product \otimes satisfies the properties of Theorem 2.7. □

7 Graduation and homogeneous morphisms

7.1 Definition

Definition 7.1. Let \mathcal{P} be a twisted bialgebra. We shall say that \mathcal{P} is graded if there exists a family of subspecies $(\mathcal{P}_n)_{n \geq 0}$ such that

$$\mathcal{P} = \bigoplus_{n \geq 0} \mathcal{P}_n,$$

with:

$$\begin{aligned} \forall k, l \in \mathbb{N}, & \quad m(\mathcal{P}_k \otimes \mathcal{P}_l) \subseteq \mathcal{P}_{k+l}, \\ \forall n \in \mathbb{N}, & \quad \Delta(\mathcal{P}_n) \subseteq \bigoplus_{k+l=n} \mathcal{P}_k \otimes \mathcal{P}_l. \end{aligned}$$

We also assume that for any finite set A :

$$\mathcal{P}_0[A] = \begin{cases} \mathcal{P}[A] & \text{if } A = \emptyset, \\ (0) & \text{otherwise.} \end{cases}$$

If \mathcal{P} and \mathcal{Q} are two graded twisted bialgebras and if $\phi : \mathcal{P} \rightarrow \mathcal{Q}$ is a morphism, we shall say that ϕ is homogeneous if for any $n \geq 0$, $\phi(\mathcal{P}_n) \subseteq \mathcal{Q}_n$.

Example 7.1. 1. $\mathcal{G}r'$ is graded: for any $n \in \mathbb{N}$, for any finite set A , $\mathcal{G}r'_n[A]$ is the vector space of graphs in $\mathcal{G}r'[A]$ with n vertices. For example, if A is a finite set:

$$\begin{aligned} \mathcal{G}r'_1[A] &= \text{Vect}(\cdot_A), \\ \mathcal{G}r'_2[A] &= \bigoplus_{(B,C) \in \mathbf{Comp}[A]} \text{Vect}(\mathbf{!}_B^{\mathcal{G}}, \cdot_{B \cdot C}). \end{aligned}$$

2. $\mathcal{T}op$ is graded: for any $n \in \mathbb{N}$, for any finite set A , $\mathcal{T}op_n[A]$ is the vector space of finite topologies on A with n equivalence classes. For example, if A is a finite set:

$$\begin{aligned} \mathcal{T}op_1[A] &= \text{Vect}(\cdot_A), \\ \mathcal{T}op_2[A] &= \bigoplus_{(B,C) \in \mathbf{Comp}[A]} \text{Vect}(\mathbf{!}_B^{\mathcal{G}}, \mathbf{!}_C^{\mathcal{E}}, \cdot_{B \cdot C}). \end{aligned}$$

3. Let \mathcal{P} be a species with $\mathcal{P}[\emptyset] = (0)$. The shuffle twisted bialgebra $(\text{coT}(\mathcal{P}), \sqcup, \Delta)$ is graded by the length: for any $n \in \mathbb{N}$, $\text{coT}(\mathcal{P})_n = \mathcal{P}^{\otimes n}$.

If \mathcal{P} is graded, for any $q \in \mathbb{K}$, we obtain a twisted bialgebra endomorphism:

$$\iota_q : \begin{cases} \mathcal{P} & \rightarrow \mathcal{P} \\ x \in \mathcal{P}_n[A] & \rightarrow q^n x. \end{cases}$$

Remark 7.1. For any $q, q' \in \mathbb{K}$, $\iota_q \circ \iota_{q'} = \iota_{qq'}$, and $\iota_1 = \text{Id}_{\mathcal{P}}$. Moreover, if $\phi : \mathcal{P} \rightarrow \mathcal{Q}$ is a morphism between two graded twisted bialgebras, it is homogeneous if, and only if, for any $q \in \mathbb{K}$, $\iota_q \circ \phi = \phi \circ \iota_q$. Conversely, if \mathcal{P} is a twisted bialgebra and $(\iota_q)_{q \in \mathbb{K}}$ is a family of twisted bialgebra endomorphisms such that:

- For any $q, q' \in \mathbb{K}$, $\iota_q \circ \iota_{q'} = \iota_{qq'}$.
- $\iota_1 = \text{Id}_{\mathcal{P}}$.

For any $n \in \mathbb{N}$, any finite set A , we put:

$$\mathcal{P}_n[A] = \{x \in A, \forall q \in \mathbb{K}, \iota_q(x) = q^n x\}.$$

Then the direct sum \mathcal{P}' of the subspecies \mathcal{P}_n is a graded twisted bialgebra.

7.2 Graduation of quasi-shuffles bialgebras

Proposition 7.2. *Let \mathcal{Q} be a commutative twisted bialgebra of the second type and let $\mathcal{P} = (\text{co}T(\mathcal{Q}), \boxplus, \Delta, \delta)$ be the associated double twisted bialgebra. We denote by $\epsilon' : \mathcal{P} \rightarrow \mathbf{Com}$ the counit of δ .*

1. For any $q \in \mathbb{K}$, we put $\lambda_q = \epsilon'^q$. Then:

$$\lambda_q : \begin{cases} \mathcal{P} & \longrightarrow \mathbf{Com} \\ x_1 \dots x_n & \longrightarrow H_n(q) \epsilon'(x_1) \dots \epsilon'(x_n). \end{cases}$$

2. For any $q \in \mathbb{K}$, we put $\theta_q = \text{Id}_{\mathcal{P}} \leftarrow \lambda_q$. Then θ_q is a twisted bialgebra endomorphism and, for any $q, q' \in \mathbb{K}$, $\theta_q \circ \theta_{q'} = \theta_{qq'}$.

3. For any $n \in \mathbb{N}$, for any finite set A , we put:

$$\mathcal{P}_n[A] = \{x \in \mathcal{P}[A], \forall q \in \mathbb{K}, \theta_q(x) = q^n x\}.$$

This gives $(\mathcal{P}, \boxplus, \Delta)$ a graduation.

Proof. First, note that for any $q, q' \in \mathbb{K}$, $\lambda_q * \lambda_{q'} = \lambda_{q+q'}$ and $\lambda_q \star \lambda_{q'} = \lambda_{qq'}$.

1. Let $x_1 \dots x_n \in \mathcal{P}[A]$. Then:

$$\begin{aligned} \lambda_q(x_1 \dots x_n) &= \sum_{k=1}^{\infty} H_k(q) \sum_{1 \leq i_1 < \dots < i_k < n} \epsilon'^{\otimes k}(x_1 \dots x_{i_1} \otimes \dots \otimes x_{i_k+1} \dots x_n) \\ &= H_n(q) \epsilon'^{\otimes n}(x_1 \otimes \dots \otimes x_n) + 0 \\ &= H_n(q) \epsilon'(x_1) \dots \epsilon'(x_n). \end{aligned}$$

2. For any $q, q' \in \mathbb{K}$:

$$\theta_q \circ \theta_{q'} = (\text{Id} \leftarrow \lambda_q) \circ (\text{Id} \leftarrow \lambda_{q'}) = \text{Id} \leftarrow (\lambda_q \star \lambda_{q'}) = \text{Id} \leftarrow \lambda_{qq'} = \lambda_{qq'}.$$

3. For any $x_1 \dots x_n \in \mathcal{P}[A]$:

$$\begin{aligned} \theta_q(x_1 \dots x_n) &= x'_1 \dots x'_n \lambda_q(x''_1 \boxplus \dots \boxplus x''_n) + \text{a span of tensors of length } < n \\ &= x'_1 \dots x'_n \lambda_q(x''_1) \dots \lambda_q(x''_n) + \text{a span of tensors of length } < n \\ &= x'_1 \dots x'_n q \epsilon'(x''_1) \dots q \epsilon'(x''_n) + \text{a span of tensors of length } < n \\ &= q^n x_1 \dots x_n + \text{a span of tensors of length } < n. \end{aligned}$$

Hence, for any $q \in \mathbb{K}$, $\theta_q[A]$ is diagonalisable and its eigenvalues are powers of q . For any integer n , we put:

$$\mathcal{P}_n[A] = \{x \in \mathcal{P}[A], \theta_2(x) = 2^n x\}.$$

Then:

$$\mathcal{P} = \bigoplus_{n \in \mathbb{N}} \mathcal{P}_n.$$

Moreover, for any n , the restriction of the canonical projection π_n on $\mathcal{P}^{\otimes n}$ to \mathcal{P}_n is a bijection $\varpi_n : \mathcal{P}_n \rightarrow \mathcal{P}^{\otimes n}$. For any $q \in \mathbb{K}$, $\theta_q \circ \theta_2 = \theta_2 \circ \theta_q = \theta_{2q}$, so for any n , $\theta_q(\mathcal{P}_n) \subseteq \mathcal{P}_n$. Moreover, if $x \in \mathcal{P}_n[A]$,

$$\varpi_n \circ \theta_q(x) = q^n \varpi_n(x) = \varpi_n(q^n x).$$

As ϖ_n is a bijection, $\theta_q(x) = q^n x$. We obtain:

$$\mathcal{P}_n[A] = \{x \in \mathcal{P}[A], \forall q \in \mathbb{K}, \theta_q(x) = q^n x\}.$$

If $x \in \mathcal{P}_k[A]$ and $y \in \mathcal{P}_l[B]$:

$$\theta_2(x \boxplus y) = \theta_2(x) \boxplus \theta_2(y) = (2^k x) \boxplus (2^l y) = 2^{k+l}(x \boxplus y),$$

so $\boxplus(\mathcal{P}_k \otimes \mathcal{P}_l) \subseteq \mathcal{P}_{k+l}$. If $x \in \mathcal{P}_n[A]$ and $A = A_1 \sqcup A_2$:

$$(\theta_2 \otimes \theta_2) \circ \Delta_{A_1, A_2}(x) = \Delta_{A_1, A_2} \circ \theta_2(x) = 2^n \Delta_{A_1, A_2}(x),$$

so:

$$\Delta_{A_1, A_2}(x) \in \bigoplus_{k+l=n} \mathcal{P}_k[A_1] \otimes \mathcal{P}_l[A_2].$$

Hence, \mathcal{P} is a graded twisted bialgebra. □

Example 7.2. We obtain:

$$\begin{aligned} \theta_q(x_1) &= qx_1, \\ \theta_q(x_1 x_2) &= q^2 x_1 x_2 + \frac{q(q-1)}{2}(x_1 \cdot x_2), \\ \theta_q(x_1 x_2 x_3) &= q^3 x_1 x_2 x_3 + \frac{q^2(q-1)}{2}(x_1 \cdot x_2)x_3 \\ &\quad + \frac{q^2(q-1)}{2}x_1(x_2 \cdot x_3) + \frac{q(q-1)(q-2)}{6}(x_1 \cdot x_2 \cdot x_3), \\ \theta_q(x_1 x_2 x_3 x_4) &= q^4 x_1 x_2 x_3 x_4 \\ &\quad + \frac{q^2(q-1)}{2}(x_1 \cdot x_2)x_3 x_4 + \frac{q^2(q-1)}{2}x_1(x_2 \cdot x_3)x_4 \\ &\quad + \frac{q^2(q-1)}{2}x_1 x_2(x_3 \cdot x_4) + \frac{q^2(q-1)^2}{4}(x_1 \cdot x_2)(x_3 \cdot x_4) \\ &\quad + \frac{q^2(q-1)(q-2)}{6}(x_1 \cdot x_2 \cdot x_3)x_4 + \frac{q^2(q-1)(q-2)}{6}x_1(x_2 \cdot x_3 \cdot x_4) \\ &\quad + \frac{q(q-1)(q-2)(q-3)}{24}(x_1 \cdot x_2 \cdot x_3 \cdot x_4). \end{aligned}$$

We shall give in Proposition 7.3 another description of the graduation of \mathcal{P} .

7.3 Homogeneous morphisms to quasi-shuffle bialgebras

Proposition 7.3. *Let \mathcal{Q} be a commutative twisted bialgebra of the second type and let $\mathcal{P} = \text{coT}(\mathcal{Q})$. Let $\varrho : (\mathcal{P}, \sqcup, \Delta) \rightarrow (\mathcal{Q}, \boxplus, \Delta)$ defined by:*

$$\varrho((x_1 \dots x_n)) = \sum_{1 \leq i_1 < \dots < i_k < n} \frac{1}{i_1!(i_2 - i_1)! \dots (n - i_k)!} (x_1 \dots x_{i_1}) \dots (x_{i_k+1} \dots x_n).$$

This is a homogeneous twisted bialgebra morphism.

Proof. Let $\lambda : \mathcal{P} \rightarrow \mathcal{Q}$, defined by

$$\lambda(x_1 \dots x_n) = \frac{x_1 \dots x_n}{n!}.$$

This is a twisted algebra morphism from (\mathcal{P}, \sqcup) to (\mathcal{Q}, \cdot) : for any $x_1 \dots x_k$ and $x_{k+1} \dots x_{k+l} \in \mathcal{P}$, as \mathcal{Q} is commutative:

$$\begin{aligned} \lambda(x_1 \dots x_k \sqcup x_{k+1} \dots x_{k+l}) &= \sum_{\sigma \in \text{Sh}(k, l)} \frac{1}{(k+l)!} x_{\sigma^{-1}(1)} \dots x_{\sigma^{-1}(k+l)} \\ &= \frac{1}{(k+l)!} \sum_{\sigma \in \text{Sh}(k, l)} x_1 \dots x_{k+l} \\ &= \frac{1}{k!} l! x_1 \dots x_{k+l} \\ &= \lambda(x_1 \dots x_k) \lambda(x_{k+1} \dots x_{k+l}). \end{aligned}$$

Moreover, ϱ is the unique twisted bialgebra morphism such that $\epsilon' \circ \varrho = \lambda$ of Theorem 2.6.

Let $x_1 \dots x_n \in \mathcal{Q}$. For any $q \in \mathbb{K}$:

$$\pi \circ \varrho \circ \iota_q(x_1 \dots x_n) = \frac{q^n}{n!} x_1 \cdot \dots \cdot x_n.$$

Moreover, if $q \in \mathbb{N}$:

$$\begin{aligned} & \pi \circ \theta_q \circ \rho(x_1 \dots x_n) \\ &= \sum_{1 \leq i_1 < \dots < i_k \leq n} \frac{1}{i_1!(i_2 - i_1)! \dots (n - i_k)!} \pi \circ \theta_q((x_1 \dots x_{i_1}) \dots (x_{i_k+1} \dots x_n)) \\ &= \sum_{1 \leq i_1 < \dots < i_k \leq n} \frac{1}{i_1!(i_2 - i_1)! \dots (n - i_k)!} H_{k+1}(q) x_1 \cdot \dots \cdot x_n \\ &= \sum_{l=1}^n \sum_{a_1 + \dots + a_l = n} \frac{1}{a_1! \dots a_l!} H_l(q) x_1 \cdot \dots \cdot x_n \\ &= \frac{1}{n!} \sum_{k=1}^n \sum_{a_1 + \dots + a_k = n} \frac{(a_1 + \dots + a_k)!}{a_1! \dots a_k!} \binom{q}{k} x_1 \cdot \dots \cdot x_n \\ &= \frac{1}{n!} \sum_{k=1}^{\min(n,q)} |\{f : \underline{n} \longrightarrow \underline{q}, |f(\underline{n})| = k\}| x_1 \cdot \dots \cdot x_n \\ &= \frac{1}{n!} |\{f : \underline{n} \longrightarrow \underline{q}\}| x_1 \cdot \dots \cdot x_n \\ &= \frac{q^n}{n!} x_1 \cdot \dots \cdot x_n \\ &= \pi \circ \rho \circ \iota_q(x_1 \dots x_n). \end{aligned}$$

As $\theta_q \circ \rho$ and $\rho \circ \iota_q$ are both coalgebra morphisms, by unicity in the universal property of $coT(\mathcal{P})$, $\theta_q \circ \rho = \rho \circ \iota_q$ for any $q \in \mathbb{N}$. Hence, for any $n \geq 2$, for any finite set A , we have:

$$\begin{aligned} coT(\mathcal{P})_n[A] &= \{x \in \mathcal{P}[A], \theta_2(x) = 2^n x\}, \\ \mathcal{P}^{\otimes n}[A] &= \{x \in \mathcal{P}[A], \iota_2(x) = 2^n x\}, \end{aligned}$$

we obtain that $\rho(\mathcal{P}^{\otimes n}) \subseteq coT(\mathcal{P})_n$. So ρ is homogeneous. □

Consequently, in the particular case of *Comp*, for any $n \in \mathbb{N}$, for any finite set A :

$$Comp_n[A] = \text{Vect}(\varrho((A_1, \dots, A_n)), (A_1, \dots, A_n) \in \mathbf{Comp}[A]).$$

For example:

$$\begin{aligned} Comp_1[A] &= \text{Vect}((A)), \\ Comp_2[A] &= \text{Vect} \left((A_1, A_2) + \frac{1}{2}(A_1 \sqcup A_2), (A_1, A_2) \in \mathbf{Comp}[A] \right), \\ Comp_3[A] &= \text{Vect} \left((A_1, A_2, A_3) + \frac{1}{2}(A_1 \sqcup A_2, A_3) + \frac{1}{2}(A_1, A_2 \sqcup A_3) + \frac{1}{6}(A_1 \sqcup A_2 \sqcup A_3), \right. \\ & \quad \left. (A_1, A_2, A_3) \in \mathbf{Comp}[A] \right). \end{aligned}$$

Corollary 7.4. *Let \mathcal{P} be a graded, connected twisted bialgebra. We denote by $\text{Mor}_B^0(\mathcal{P}, \mathbf{Comp})$ the set of homogeneous twisted bialgebra morphisms from \mathcal{P} to *Comp* and by $L(\mathcal{P}_1, \mathbf{Com})$ the set of species morphisms from \mathcal{P}_1 to **Com**. The following map is a bijection:*

$$\begin{cases} \text{Mor}_B^0(\mathcal{P}, \mathbf{Comp}) & \longrightarrow L(\mathcal{P}_1, \mathbf{Com}) \\ \phi & \longrightarrow \epsilon' \circ \phi|_{\mathcal{P}_1}. \end{cases}$$

Proof. Injectivity. and let $\phi, \phi' : \mathcal{P} \longrightarrow \mathbf{Comp}$ be two homogeneous twisted bialgebra morphism, such that $\varepsilon' \circ \phi|_{\mathcal{P}_1} = \varepsilon' \circ \phi'|_{\mathcal{P}_1}$. We put $\lambda = \varepsilon' \circ \phi$ and $\lambda' = \varepsilon' \circ \phi'$. Let us prove that $\phi = \phi'$. By Theorem 4.11, it is enough to prove that $\lambda = \lambda'$: let us prove that $\lambda|_{\mathcal{P}_n} = \lambda'|_{\mathcal{P}_n}$ by induction on n . This is obvious if $n = 0$ or 1 . If $n \geq 2$, then for any $q \in \mathbb{K}$, if $x \in \mathcal{P}_n[A]$, by the induction hypothesis:

$$\begin{aligned} q^n \lambda(x) &= \lambda \circ \iota_q(x) \\ &= \lambda^q(x) \\ &= H_1(q)\lambda(x) + \sum_{k=2}^{\infty} \sum_{(A_1, \dots, A_k) \in \mathbf{Comp}[A]} (\lambda - \varepsilon)^{\otimes k} \circ \Delta_{A_1, \dots, A_k}(x). \end{aligned}$$

Hence, by the induction hypothesis:

$$\begin{aligned} (q^n - q)\lambda(x) &= \sum_{k=2}^{\infty} \sum_{(A_1, \dots, A_k) \in \mathbf{Comp}[A]} (\lambda - \varepsilon)^{\otimes k} \circ \Delta_{A_1, \dots, A_k}(x) \\ &= \sum_{k=2}^{\infty} \sum_{(A_1, \dots, A_k) \in \mathbf{Comp}[A]} (\lambda' - \varepsilon)^{\otimes k} \circ \Delta_{A_1, \dots, A_k}(x) \\ &= (q^n - q)\lambda'(x). \end{aligned}$$

Choosing $q = 2$, as $n \geq 2$, we obtain $\lambda(x) = \lambda'(x)$.

Surjectivity. Let $\lambda_1 : \mathcal{P}_1 \longrightarrow \mathbf{Com}$ be a species morphism. We define $\lambda_0 : \mathcal{P}_0 \longrightarrow \mathbf{Com}$ by $\lambda_0(1_{\mathcal{P}}) = 1$. Let us define $\lambda_n : \mathcal{P}_n \longrightarrow \mathbf{Com}$ for $n \geq 2$ by induction on n . If $x \in \mathcal{P}_n[A]$, we put:

$$\begin{aligned} \lambda_n(x) &= \frac{1}{2^n - 2} \sum_{k=2}^n \sum_{(A_1, \dots, A_k) \in \mathbf{Comp}[A]} H_k(2)(\lambda_{\#A_1} \otimes \dots \otimes \lambda_{\#A_k}) \circ \Delta_{A_1, \dots, A_k}(x) \\ &= \frac{1}{2^n - 2} \sum_{(A_1, A_2) \in \mathbf{Comp}[A]} (\lambda_{\#A_1} \otimes \lambda_{\#A_2}) \circ \Delta_{A_1, A_2}(x), \end{aligned}$$

which we shortly write as:

$$\lambda_n(x) = \frac{1}{2^n - 2} \lambda(x^{(1)})\lambda(x^{(2)}).$$

We define in this way a map $\lambda : \mathcal{P} \longrightarrow \mathbf{Comp}$, such that $\lambda^2 = \lambda \circ \iota_2$. Let us prove that λ is a character: let $x \in \mathcal{P}_k[A]$, $y \in \mathcal{P}_l[B]$, and let us prove that $\lambda(xy) = \lambda(x)\lambda(y)$ by induction on $k+l = n$. If $k = 0$ or $l = 0$, we can assume that $x = 1_{\mathcal{P}}$ or $y = 1_{\mathcal{P}}$ and the result is obvious. We now assume that $k, l \geq 1$. There is nothing to prove if $n = 0$ or 1 . Otherwise, by the induction hypothesis:

$$\begin{aligned} \lambda(xy) &= \frac{1}{2^{k+l} - 2} \left(\begin{array}{c} 2\lambda(x)\lambda(y) + \lambda(x^{(1)}y)\lambda(x^{(2)}) \\ + \lambda(x^{(1)})\lambda(x^{(2)}y) + \lambda(xy^{(1)})\lambda(y^{(2)}) \\ + \lambda(y^{(1)})\lambda(xy^{(2)}) + \lambda(x^{(1)}y^{(1)})\lambda(x^{(2)}y^{(2)}) \end{array} \right) \\ &= \frac{1}{2^{k+l} - 2} \left(\begin{array}{c} 2\lambda(x)\lambda(y) + 2\lambda(x^{(1)})\lambda(x^{(2)})\lambda(y) \\ + 2\lambda(x)\lambda(y^{(1)})\lambda(y^{(2)}) + \lambda(x^{(1)})\lambda(y^{(1)})\lambda(x^{(2)})\lambda(y^{(2)}) \end{array} \right) \\ &= \frac{\lambda(x)\lambda(y)}{2^{k+l} - 2} (2 + 2(2^k - 2) + 2(2^l - 2) + (2^k - 2)(2^l - 2)) \\ &= \lambda(x)\lambda(y). \end{aligned}$$

So λ is a character, $\lambda|_{\mathcal{P}_1} = \lambda_1$ and $\lambda^2 = \lambda \circ \iota_2$. Let $\phi : \mathcal{P} \longrightarrow \mathbf{Comp}$ be the unique twisted bialgebra morphism such that $\varepsilon' \circ \phi = \lambda$.

Let $q \in \mathbb{K}$. For any $x \in \mathcal{P}[A]$:

$$\begin{aligned}
\lambda_q \circ \phi(x) &= \sum_{k \geq 0} \sum_{(A_1, \dots, A_k) \in \mathbf{Comp}[A]} \lambda^{\otimes k} \circ \Delta_{A_1, \dots, A_k}(x) \lambda_q(A_1, \dots, A_k) \\
&= \sum_{k \geq 0} H_k(q) \sum_{(A_1, \dots, A_k) \in \mathbf{Comp}[A]} \lambda^{\otimes k} \circ \Delta_{A_1, \dots, A_k}(x) \\
&= \sum_{k \geq 0} H_k(q) (\lambda - \varepsilon)^{*k}(x) \\
&= \lambda^q(x).
\end{aligned}$$

So $\lambda_q \circ \phi = \lambda^q$. For any finite set A :

$$\begin{aligned}
\mathbf{Comp}_n[A] &= \{x \in \mathbf{Comp}[A], \theta_2(x) = 2^n x\}, \\
\mathcal{P}_n[A] &= \{x \in \mathcal{P}_n[A], \iota_2(x) = 2^n x\}.
\end{aligned}$$

As the characters $\lambda \circ \iota_2 = \varepsilon' \circ \phi \circ \iota_2$ and $\lambda^2 = \lambda_2 \circ \phi = \varepsilon' \circ \theta_2 \circ \phi$ are equal, by Theorem 4.11 $\phi \circ \iota_2 = \theta_2 \circ \phi$. Consequently, if $x \in \mathcal{P}_n[A]$:

$$\theta_2 \circ \phi(x) = \phi \circ \iota_2(x) = 2^n \phi(x),$$

so $\phi(x) \in \mathbf{Comp}_n[A]$: ϕ is homogeneous. Moreover, $\varepsilon' \circ \phi|_{\mathcal{P}_1} = \lambda_1$. \square

7.4 The example of graphs

Proposition 7.5. 1. Let $u = (u_k)_{k \geq 1}$ be a sequence of scalars. The following map is a homogeneous morphism of twisted bialgebras:

$$\phi_u : \begin{cases} \mathcal{G}r' & \longrightarrow \mathbf{Comp} \\ G \in \mathcal{G}r'[A] & \longrightarrow \left(\prod_{I \in V(G)} u_{\#I} \right) \prod_{I \in V(G)}^{\boxplus} (I). \end{cases}$$

All homogeneous morphisms from $\mathcal{G}r'$ to \mathbf{Comp} are obtained in this way.

2. Let $\lambda_0 : \mathcal{G}r' \longrightarrow \mathbf{Comp}$ be defined by $\lambda_0(G) = 1$ for any graph G . Then λ_0 is a character of $\mathcal{G}r'$ and for any $q \in \mathbb{K}$:

$$\lambda_0^q(G) = q^{\deg(G)} G.$$

Proof. For any finite set A , $\mathcal{G}r'_1[A]$ is one-dimensional, generated by \bullet_A . Hence, the species morphisms from $\mathcal{G}r'_1$ to \mathbf{Com} are given by:

$$\mu_u(\bullet_A) = u_{\#A},$$

where u is a sequence of scalars. By Corollary 7.4, there exists a unique homogeneous morphism of twisted bialgebra $\phi_u : \mathcal{G}r' \longrightarrow \mathbf{Comp}$ such that $(\varepsilon' \circ \phi_u)|_{\mathcal{G}r'_1} = \mu_u$. Let us consider the map:

$$\iota_u : \begin{cases} \mathcal{G}r' & \longrightarrow \mathcal{G}r' \\ G & \longrightarrow \left(\prod_{I \in V(G)} u_{\#I} \right) G. \end{cases}$$

This is obviously a homogeneous endomorphism of twisted bialgebras, and for any sequences u, v , then $\phi_u \circ \iota_v$ is a homogeneous morphism and for any finite set A of cardinal n :

$$\varepsilon' \circ \phi_u \circ \iota_v(\bullet_A) = u_n v_n,$$

so, putting $uv = (u_n v_n)_{n \geq 1}$, $\phi_{uv} = \phi_u \circ \iota_v$. It is now enough to describe $\phi_{\mathbf{1}}$, with $\mathbf{1}_n = 1$ for any n . Let us put $\lambda_0 = \varepsilon' \circ \phi_{\mathbf{1}}$, and let us prove that $\lambda_0(G) = 1$ for any graph G . We proceed by induction on $\deg(G) = n$. If $n = 1$, then $G = \bullet_A$ and $\lambda_0(G) = \mathbf{1}_n = 1$. Otherwise, as $\lambda_0^2 = \lambda_0 \circ \iota_2$:

$$\begin{aligned} \lambda_0(G) &= \frac{1}{2^n - 2} \sum_{(A,B) \in \mathbf{Comp}[V(G)]} \lambda(G|_A) \lambda(G|_B) \\ &= \frac{1}{2^n - 2} \sum_{(A,B) \in \mathbf{Comp}[V(G)]} 1 \\ &= \frac{1}{2^n - 2} \sum_{k=1}^{n-1} \binom{n}{k} \\ &= 1. \end{aligned}$$

So $\lambda_0(G) = 1$ for any G . As $\phi_{\mathbf{1}}$ is homogeneous, $\lambda_0^q = \lambda_0 \circ \iota_q$ for any $q \in \mathbb{K}$. \square

Example 7.3. If A, B and C are finite sets:

$$\begin{aligned} \phi_{\mathbf{1}}(\bullet_A) &= (A), \\ \phi_{\mathbf{1}}(\mathbf{1}_A^B) &= (A) \boxplus (B) \\ &= (A, B) + (B, A) + (A \sqcup B), \\ \phi_{\mathbf{1}}(\mathbf{1}_A^{B \times C}) &= (A) \boxplus (B) \boxplus (C) \\ &= (A, B, C) + (A, C, B) + (B, A, C) + (B, C, A) + (C, A, B) + (C, B, A) \\ &\quad + (A \sqcup B, C) + (A \sqcup C, B) + (B \sqcup C, A) \\ &\quad + (A, B \sqcup C) + (B, A \sqcup C) + (C, A \sqcup B) + (A \sqcup B \sqcup C). \end{aligned}$$

7.5 Application: acyclic orientations

Let $q \in \mathbb{K}$. We shall consider the homogeneous morphism defined in Proposition 7.5, with $u_n = q$ for any n :

$$\phi_q : \begin{cases} \mathcal{G}r' & \longrightarrow \mathit{Comp} \\ G \in \mathcal{G}r'[A] & \longrightarrow q^{\deg(G)} \prod_{I \in V(G)} (I). \end{cases}$$

Then:

$$\phi_q = \phi \leftarrow \lambda_q,$$

where $\lambda_q = \varepsilon' \circ \phi_q$ is the character given by:

$$\lambda_q(G) = q^{\deg(G)}.$$

If $q \neq 0$, we put:

$$\phi_{chr_q} = \theta_{q^{-1}} \circ \phi \circ \iota_q.$$

This is a twisted bialgebra morphism from $\mathcal{G}r'$ to Comp . In particular, $\phi_{chr_1} = \phi$.

Example 7.4. If A, B and C are finite sets:

$$\begin{aligned} \phi_{chr_q}(\bullet_A) &= (A), \\ \phi_{chr_q}(\mathbf{1}_A^B) &= (A, B) + (B, A) + (1 - q)(A \sqcup B), \\ \phi_{chr_q}(\mathbf{1}_A^{B \times C}) &= (A, B, C) + (A, C, B) + (B, A, C) + (B, C, A) + (C, A, B) + (C, B, A) \\ &\quad + (1 - q)(A \sqcup B, C) + (1 - q)(A \sqcup C, B) + (B \sqcup C, A) \\ &\quad + (1 - q)(C, A \sqcup B) + (1 - q)(B, A \sqcup C) + (A, B \sqcup C) + 2(1 - q)(A \sqcup B \sqcup C). \end{aligned}$$

Lemma 7.6. *Let $\lambda \in \text{Char}(\mathcal{G}r')$. Then λ is invertible for \star if, and only if, for any finite set A , $\lambda(\cdot_A) \neq 0$.*

Proof. \implies . Let μ be the inverse of λ . For any finite set A , $\lambda \star \mu(\cdot_A) = \lambda(\cdot_A)\mu(\cdot_A) = \varepsilon(\cdot_A) = 1$, so $\lambda(\cdot_A) \neq 0$.

\impliedby . We define inductively scalars $\mu_1(G)$ and $\mu_2(G)$ for any graph G by induction on the number k of edges of G . If $k = 0$, then $G = \cdot_{A_1} \dots \cdot_{A_n}$, and we put:

$$\mu_1(G) = \mu_2(G) = \frac{1}{\lambda(\cdot_{A_1}) \dots \lambda(\cdot_{A_n})}.$$

Otherwise, let C_1, \dots, C_l be the connected components of G , and let A_1, \dots, A_k be the decorations of its vertices. Then:

$$\delta(G) = G \otimes \cdot_{A_1} \dots \cdot_{A_k} + \cdot_{C_1} \dots \cdot_{C_l} \otimes G + \text{sum of terms } G' \otimes G'', \text{ with } \#E(G'), \#E(G'') < \#E(G).$$

We then put:

$$\begin{aligned} \mu_1(G) &= -\frac{1}{\lambda(\cdot_{C_1}) \dots \lambda(\cdot_{C_l})} (\mu_1(\cdot_{A_1} \dots \cdot_{A_k})\lambda(G) + \mu_1(G')\lambda(G'')), \\ \mu_2(G) &= -\frac{1}{\lambda(\cdot_{A_1}) \dots \lambda(\cdot_{A_k})} (\lambda(G)\mu_2(\cdot_{C_1} \dots \cdot_{C_l}) + \lambda(G')\mu_2(G'')). \end{aligned}$$

By construction, $\mu_1 \star \lambda = \lambda \star \mu_2 = \varepsilon'$. Hence, λ is invertible, and its inverse is $\mu_1 = \mu_2$. \square

Consequently, if $q \neq 0$, λ_q is invertible. We denote by λ_{chr} the inverse of λ_1 and for any $q \neq 0$, $\lambda_{chr_q} = \lambda_{chr} \star (\varepsilon' \circ \phi_{chr_q})$. Then:

$$\phi_{chr_q} = \phi \leftarrow (\varepsilon' \circ \phi_{chr_q}) = (\phi_1 \leftarrow \lambda_{chr}) \leftarrow (\varepsilon' \circ \phi_{chr_q}) = \phi_1 \leftarrow \lambda_{chr}.$$

In particular, $\lambda_{chr_q} = \lambda_{chr}$. Consequently:

Corollary 7.7. *For any graph $G \in \mathcal{G}r'[A]$:*

$$\phi_{chr_q}(G) = \sum_{\sim \triangleleft G} \lambda_{chr_q}(G | \sim) C_{\sim}^{(1)} \boxplus \dots \boxplus C_{\sim}^{(\text{cl}(\sim))},$$

where $C_{\sim}^{(1)}, \dots, C_{\sim}^{(\text{cl}(\sim))}$ are the equivalence classes of \sim .

Proposition 7.8 (Extraction-contraction principle). *Let G be a graph and e be an edge of G . We denote by $G \setminus e$ the graph obtained in deleting the edge e and by G/e the graph obtained in contracting the edge e , the two extremities of e being identified with their union. Then, for any graph G , for any edge e of G :*

$$\phi_{chr_1}(G) = \phi_{chr_1}(G \setminus e) - \phi_{chr_1}(G/e).$$

For any $q \neq 0$:

$$\phi_{chr_q}(G) = \phi_{chr_q}(G \setminus e) - q\phi_{chr_q}(G/e).$$

In particular, $\lambda_{ao}(G) = \varepsilon' \circ \phi_{chr_{-1}}(G)$ is the number of acyclic orientations of G .

Proof. We denote by X and Y the set of valid colorations of $G \setminus e$ is the set of colorations c of G such that the conditions for c to be valid is satisfied, except maybe for pairs of elements $(x, y) \in X \times Y$. The set of valid colorations of G/e is the set of colorations c of G such that the conditions for c to be valid is satisfied, except for any pair of elements $(x, y) \in X \times Y$. Hence,

$VC(G \setminus e) = VC(G) \sqcup VC(G/e)$, which gives $\phi_{chr_1}(G \setminus e) = \phi_{chr_1}(G) - \phi_{chr_1}(G/e)$.

If $q \neq 0$:

$$\begin{aligned}\phi_{chr_q}(G) &= q^{\deg(G)} \theta_{q^{-1}} \circ \phi_{chr_1}(G) \\ &= q^{\deg(G)} \theta_{q^{-1}} (\phi_{chr_1}(G \setminus e) - \phi_{chr_1}(G/e)) \\ &= \theta_{q^{-1}} \left(q^{\deg(G \setminus e)} \phi_{chr_1}(G \setminus e) - q^{\deg(G/e)+1} \phi_{chr_1}(G/e) \right) \\ &= \phi_{chr_q}(G \setminus e) - q \phi_{chr_q}(G/e).\end{aligned}$$

We denote by $\lambda_{ao}(G)$ the number of acyclic orientations of G and we put $\mu(G) = \varepsilon' \circ \phi_{chr_{-1}}(G)$. Let us show that $\mu(G) = \lambda_{ao}(G)$ by induction on the number k of edges of G . If $k = 0$, then $\mu(G) = 1$ and the result is obvious. If $k \geq 1$, let e be an edge of G . Let O be an orientation of G and let O' be the orientation of G obtained by changing the orientation of e in G . Both O and O' give the same orientation of $G \setminus e$, which is acyclic if, and only if, O or O' is acyclic. Both O and O' induce an orientation of G/e , which is acyclic if, and only if, both O and O' are acyclic. Consequently, $\lambda_{ao}(G) = \lambda_{ao}(G \setminus e) + \lambda_{ao}(G/e) = \mu(G \setminus e) + \mu(G/e) = \mu(G)$. \square

Corollary 7.9. *Let $q \neq 0$. Moreover for any graph G :*

$$\lambda_{chr_q}(G) = q^{\deg(G) - \text{cc}(G)} \lambda_{chr_1}(G).$$

Moreover, the character λ_{chr_q} is invertible and for any graph G , $\lambda_{chr_q}^{*-1}(G) = q^{\deg(G) - \text{cc}(G)}$.

Proof. As $\phi_{chr_q} = \phi_1 \leftarrow \lambda_{chr_q} = \theta_{q^{-1}} \circ \phi_{chr_1} \circ \iota_q$:

$$\begin{aligned}\phi_{chr_1} \circ \iota_q &= \theta_q \circ (\phi_1 \leftarrow \lambda_{chr_q}) \\ &= (\theta_q \circ \phi_1) \leftarrow \lambda_{chr_q}, \\ \varepsilon' \circ \iota_q &= \varepsilon' \circ \phi_{chr_1} \circ \iota_q \\ &= (\varepsilon' \circ \theta_q \circ \phi_1) \star \lambda_{chr_q}.\end{aligned}$$

For any graph G , if $V(G) = \{A_1, \dots, A_k\}$:

$$\begin{aligned}\varepsilon' \circ \theta_q \circ \phi_1(G) &= \varepsilon' \circ \theta_q((A_1) \boxplus \dots \boxplus (A_k)) \\ &= \varepsilon' \circ \theta_q((A_1)) \dots \varepsilon' \circ \theta_q((A_k)) \\ &= q^k \\ &= q^{\deg(G)}.\end{aligned}$$

Therefore:

$$\begin{aligned}\varepsilon' \circ \iota_q(G) &= q^{\deg(G)} \varepsilon'(G) \\ &= \sum_{\sim \triangleleft G} q^{\deg(G/\sim)} \lambda_{chr_q}(G | \sim) \\ &= \sum_{\sim \triangleleft G} q^{\text{cc}(G/\sim)} \lambda_{chr_q}(G | \sim).\end{aligned}$$

So:

$$\varepsilon'(G) = \sum_{\sim \triangleleft G} q^{\text{cc}(G/\sim) - \deg(G/\sim)} \lambda_{chr_q}(G | \sim).$$

Hence, the inverse of the character defined by $\lambda(G) = 1$ for any graph G is given by:

$$\lambda^{*-1}(G) = q^{\text{cc}(G) - \deg(G)} \lambda_{chr_q}(G).$$

If $q = 1$, $\lambda^{\star-1}(G) = \lambda_{chr_1}(G)$ and, if $q \neq 0$:

$$\lambda_{chr_q}(G) = q^{\deg(G) - \text{cc}(G)} \lambda_{chr_1}(G).$$

Moreover:

$$\begin{aligned} \varepsilon'(G) &= \sum_{\sim \triangleleft G} q^{\text{cc}(G|\sim) - \deg(G|\sim)} \lambda_{chr_q}(G|\sim) \\ &= q^{\text{cc}(G) - \deg(G)} \sum_{\sim \triangleleft G} q^{\deg(G/\sim) - \text{cc}(G/\sim)} \lambda_{chr_q}(G|\sim). \end{aligned}$$

If $\varepsilon'(G) \neq 0$, then $\text{cc}(G) = \deg(G)$, so:

$$\sum_{\sim \triangleleft G} q^{\deg(G/\sim) - \text{cc}(G/\sim)} \lambda_{chr_q}(G|\sim) = \varepsilon'(G).$$

Therefore, $\lambda_{chr_q}^{\star-1}(G) = q^{\deg(G) - \text{cc}(G)}$. □

Corollary 7.10. 1. The character λ_{ao} is invertible and, for any graph G :

$$\lambda_{ao}^{\star-1}(G) = (-1)^{\deg(G) + \text{cc}(G)} \lambda_{ao}(G).$$

2. The following maps are twisted bialgebra automorphisms, inverse one from the other:

$$\begin{aligned} \Gamma : \begin{cases} \mathcal{G}r' & \longrightarrow \mathcal{G}r' \\ G \in \mathcal{G}r'[A] & \longrightarrow \sum_{\sim \triangleleft G} \lambda_{ao}(G|\sim)G/\sim, \end{cases} \\ \Gamma' : \begin{cases} \mathcal{G}r' & \longrightarrow \mathcal{G}r' \\ G \in \mathcal{G}r'[A] & \longrightarrow \sum_{\sim \triangleleft G} (-1)^{\text{cl}(\sim) + \deg(G)} \lambda_{ao}(G|\sim)G/\sim. \end{cases} \end{aligned}$$

Moreover, $\phi_{chr_{-1}} = \phi_{chr_1} \circ \Gamma$ and $\phi_{chr_1} = \phi_{chr_{-1}} \circ \Gamma'$.

Proof. 1. By definition of λ_{ao} :

$$\lambda_{ao} = \varepsilon' \circ \phi_{chr_{-1}} = \varepsilon' \circ (\phi_1 \leftarrow \lambda_{chr_{-1}}) = \lambda_{chr_1}^{\star-1} \star \lambda_{chr_{-1}}.$$

So $\lambda_{ao}^{\star-1} = \lambda_{chr_{-1}}^{\star-1} \star \lambda_{chr_1}$, and for any graph G :

$$\begin{aligned} \lambda_{ao}^{\star-1}(G) &= \sum_{\sim \triangleleft G} (-1)^{\text{cl}(\sim) - \text{cc}(G)} \lambda_{chr_1}(G|\sim) \\ &= (-1)^{\deg(G) + \text{cc}(G)} \sum_{\sim \triangleleft G} (-1)^{\deg(G|\sim) + \text{cc}(G|\sim)} \lambda_{chr_1}(G|\sim) \\ &= (-1)^{\deg(G) + \text{cc}(G)} \sum_{\sim \triangleleft G} \lambda_{chr_{-1}}(G|\sim) \\ &= (-1)^{\deg(G) + \text{cc}(G)} \lambda_{chr_1}^{\star-1} \star \lambda_{chr_{-1}}(G) \\ &= (-1)^{\deg(G) + \text{cc}(G)} \lambda_{ao}(G). \end{aligned}$$

2. We obtain:

$$\phi_{chr_1} \circ \Gamma = \phi_{chr_1} \circ (\text{Id} \leftarrow \lambda_{ao}) = \phi_{chr_1} \leftarrow \lambda_{ao} = \phi_{chr_1} \leftarrow (\lambda_{chr_1}^{\star-1} \star \lambda_{chr_{-1}}) = \phi_1 \leftarrow \lambda_{chr_{-1}} = \phi_{chr_{-1}}.$$

Note that $\Gamma = \text{Id} \leftarrow \lambda_{ao}$ and $\Gamma' = \text{Id} \leftarrow \lambda_{ao}^{\star-1}$, so they are indeed twisted bialgebra morphisms, inverse one from each other. □

To summarize, we obtain twisted bialgebra morphisms ϕ_q and ϕ_{chr_q} from $\mathcal{G}r'$ to $Comp$, such that the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{G}r' & \xrightarrow{\iota_q} & \mathcal{G}r' \\ \phi_{chr_{q'}} \downarrow & & \downarrow \phi_{chr_{\frac{q'}{q}}} \\ Comp & \xrightarrow{\theta_q} & Comp \end{array} \qquad \begin{array}{ccc} \mathcal{G}r' & \xrightarrow{\iota_q} & \mathcal{G}r' \\ \phi_{q'} \downarrow & & \downarrow \phi_{q'} \\ Comp & \xrightarrow{\theta_q} & Comp \end{array}$$

We also have a twisted bialgebra automorphism Γ of $\mathcal{G}r'$ making the following diagram commutating:

$$\begin{array}{ccc} \mathcal{G}r' & \xrightarrow{\Gamma} & \mathcal{G}r' \\ & \searrow \phi_{chr_{-1}} & \swarrow \phi_{chr_1} \\ & & Comp \end{array}$$

They are related to several characters of $\mathcal{G}r'$:

$$\phi_{chr_q} = \phi_1 \leftarrow \lambda_{chr_q}, \qquad \Gamma = \text{Id} \leftarrow \lambda_{ao}.$$

For any graph G :

$$\lambda_{chr_q}(G) = q^{\deg(G) - \text{cc}(G)} \lambda_{chr_1}(G), \quad \lambda_{chr_q}^{*-1}(G) = q^{\deg(G) - \text{cc}(G)}, \quad \lambda_{ao}^{*-1}(G) = (-1)^{\deg(G) + \text{cc}(G)} \lambda_{ao}(G).$$

Remark 7.2. Taking the limit when q goes to 0, one gives a meaning to ϕ_{chr_0} and to λ_{chr_0} . For any graph G :

$$\begin{aligned} \lambda_{chr_0}(G) &= \begin{cases} 1 & \text{if } \deg(G) = \text{cc}(G), \\ 0 & \text{otherwise} \end{cases} \\ &= \varepsilon'(G). \end{aligned}$$

$$\text{So } \phi_{chr_0} = \phi_1 \leftarrow \lambda_{chr_0} = \phi_1 \leftarrow \varepsilon' = \phi_1.$$

7.6 The example of finite topologies

Definition 7.11. For any quasi-poset $T = (A, \leq_T)$, we denote by $\mathbf{HO}(T)$ the set of heap-orders on T , that is to say surjections $f : A \rightarrow \underline{\max}(f)$ such that:

1. For any $a, b \in A$, if $a \leq_T b$, then $f(a) \leq f(b)$.
2. For any $a, b \in A$, $f(a) = f(b)$ if, and only if, $a \sim_T b$.

Note that for any $f \in \mathbf{HO}(T)$, $\max(f) = \text{cl}(T)$. We denote by $\text{ho}(T)$ the cardinality of $\mathbf{HO}(T)$.

Lemma 7.12. Let T be a finite topology on a finite set A . Then:

$$\sum_{\substack{O \text{ non trivial} \\ \text{open set of } T}} \frac{\text{cl}(T)!}{\text{cl}(T|_O)! \text{cl}(T|_{A \setminus O})!} \text{ho}(T|_{A \setminus O}) \text{ho}(T|_O) = (2^{\text{cl}(T)} - 2) \text{ho}(T).$$

Proof. We consider the two following sets:

1. X is the set of pairs (h, k) , where $h \in \mathbf{HO}(T)$ and $1 \leq k \leq \text{cl}(T)$.
2. Y is the set of triples (O, h', h'') , where O is a non trivial open set of T , $h' \in \mathbf{HO}(T|_{A \setminus O})$ and $h'' \in \mathbf{HO}(T|_O)$.

For any (h, k) in X , we put:

- $O = h^{-1}(\{k+1, \dots, \text{cl}(T)\})$. As $h \in \mathbf{HO}(T)$, this is an open set of T , and as $1 \leq k < \text{cl}(T)$, it is non trivial.
- $h' = h|_{A \setminus O}$ and $h'' = h|_O - k$. As $h \in \mathbf{HO}(T)$, $(O, h', h'') \in B$.

We define in this way a map $\vartheta : X \longrightarrow Y$.

If $(O, h, h') \in Y$, we put:

1. $k = \text{cl}(A \setminus O) = \max(h')$.
2. $h : A \longrightarrow \underline{\text{cl}(T)}$ defined by:

$$h(x) = \begin{cases} h'(x) & \text{if } x \notin O, \\ h''(x) + k & \text{if } x \in O. \end{cases}$$

It is not difficult to show that $h \in \mathbf{HO}(T)$.

We define in this way a map $\vartheta' : Y \longrightarrow X$, and it is immediate that $\vartheta \circ \vartheta' = \text{Id}_Y$ and $\vartheta' \circ \vartheta = \text{Id}_X$. So ϑ and ϑ' are bijections. We obtain:

$$\begin{aligned} & \sum_{\substack{O \text{ non trivial} \\ \text{open set of } T}} \frac{\text{cl}(T)!}{\text{cl}(T|_O)! \text{cl}(T|_{A \setminus O})!} \text{ho}(T|_{A \setminus O}) \text{ho}(T|_O) \\ &= \sum_{(O, h', h'') \in Y} \frac{\text{cl}(T)!}{\text{cl}(T|_O)! \text{cl}(T|_{A \setminus O})!} \\ &= \sum_{(h, k) \in X} \frac{n!}{k!(n-k)!} \\ &= \sum_{h \in \mathbf{HO}(T)} \sum_{k=1}^{n-1} \frac{n!}{k!(n-k)!} \\ &= \sum_{h \in \mathbf{HO}(T)} (2^n - 2) \\ &= (2^n - 2) \text{ho}(T), \end{aligned}$$

where $n = \text{cl}(T)$. □

Proposition 7.13. 1. Let $u = (u_k)_{k \geq 1}$ be a sequence of scalars. The following map is a homogeneous morphism of twisted bialgebras:

$$\phi_u : \begin{cases} \mathcal{Top} & \longrightarrow \mathcal{Comp} \\ T \in \mathcal{Top}[A] & \longrightarrow \left(\prod_{I \in \mathcal{CL}(T)} u_{\#I} \right) \phi \leftarrow \lambda_{\text{ho}}(T). \end{cases}$$

All homogeneous morphisms from \mathcal{Top} to \mathcal{Comp} are obtained in this way.

2. We put:

$$\lambda_{\text{ho}}(T) = \frac{\text{ho}(T)}{\text{cl}(T)!}.$$

Then λ_{ho} is a character of \mathcal{Top} . Moreover, for any $q \in \mathbb{K}$, $\lambda_{\text{ho}} \circ \iota_q = \lambda_{\text{ho}}^q$.

Proof. For any finite set A , $\mathcal{Top}_1[A]$ is one-dimensional, generated by \bullet_A . Hence, the species morphisms from \mathcal{Top}_1 to **Com** are given by:

$$\mu_u(\bullet_A) = u_{\#A},$$

where u is a sequence of scalars. Corollary 7.4 implies that there exists a unique homogeneous morphism of twisted bialgebra $\phi_u : \mathcal{Top} \rightarrow \mathcal{Comp}$ such that $(\varepsilon' \circ \phi_u)|_{\mathcal{G}_{r_1}} = \mu_u$.

Let us consider the map:

$$\iota_u : \begin{cases} \mathcal{Top} & \rightarrow \mathcal{Top} \\ T & \rightarrow \left(\prod_{I \in CL(T)} u_{\#I} \right) T. \end{cases}$$

This is obviously a homogeneous endomorphism of twisted bialgebras, and for any sequences u, v , then $\phi_u \circ \iota_v$ is a homogeneous morphism and for any finite set A of cardinal n :

$$\varepsilon' \circ \phi_u \circ \iota_v(\bullet_A) = u_n v_n.$$

It is now enough to describe $\phi_{\mathbf{1}}$, with $\mathbf{1}_n = 1$ for any n . Let us put $\lambda_{ho} = \varepsilon' \circ \phi_{\mathbf{1}}$, and let us prove that for any finite topology T on a finite set A :

$$\lambda_{ho}(T) = \frac{\text{ho}(T)}{\text{cl}(T)!}.$$

We proceed by induction on $\text{cl}(T) = n$. If $n = 1$, then $T = \bullet_A$ and $\lambda_{ho}(T) = \mathbf{1}_n = 1$. Otherwise, as $\lambda_{ho}^2 = \lambda_{ho} \circ \iota_2$, by the preceding Lemma:

$$\begin{aligned} \lambda_{ho}(T) &= \frac{1}{2^n - 2} \sum_{O \text{ non trivial open set of } T} \lambda_{ho}(T|_{V(T) \setminus O}) \lambda_{ho}(T|_O) \\ &= \frac{1}{2^n - 2} \sum_{O \text{ non trivial open set of } T} \frac{\text{cl}(T)!}{\text{cl}(T|_O)! \text{cl}(T|_{A \setminus O})!} \text{ho}(T|_O) \text{ho}(T|_{A \setminus O}) \\ &= \frac{\text{ho}(T)}{n!}. \end{aligned}$$

As $\phi_{\mathbf{1}}$ is homogeneous, $\lambda_{ho}^q = \lambda_{ho} \circ \iota_q$ for any $q \in \mathbb{K}$. □

Example 7.5. If A, B and C are finite sets:

$$\begin{aligned} \phi_{\mathbf{1}}(\bullet_A) &= (A), \\ \phi_{\mathbf{1}}(\mathfrak{!}_A^B) &= (A, B) + \frac{1}{2}(A \sqcup B), \\ \phi_{\mathbf{1}}(\mathfrak{!}_A^{B \vee C}) &= (A, B, C) + (A, C, B) + (A, B \sqcup C) \\ &\quad + \frac{1}{2}(A \sqcup B, C) + \frac{1}{2}(A \sqcup C, B) + \frac{1}{3}(A \sqcup B \sqcup C), \\ \phi_{\mathbf{1}}(\mathfrak{!}_A^{\overset{C}{B}}) &= (A, B, C) + \frac{1}{2}(A, B \sqcup C) + \frac{1}{2}(A \sqcup B, C) + \frac{1}{6}(A \sqcup B \sqcup C). \end{aligned}$$

7.7 Application: duality principle

Lemma 7.14. *Let λ be a character on \mathcal{Top} . It is invertible for \star if, and only if, for any finite set A , $\lambda(\bullet_A) \neq 0$.*

Proof. Similar to the proof of Lemma 7.6. □

Let $q \in \mathbb{K}$. We denote by ϕ_q the homogeneous morphism associated to the sequence defined by $u_n = q$ for any n in Proposition 7.13:

$$\phi_q : \begin{cases} \mathcal{T}op & \longrightarrow \mathcal{C}omp \\ T \in \mathcal{T}op[A] & \longrightarrow q^{\text{cl}(T)} \phi \leftarrow \lambda_{ho}(T). \end{cases}$$

The associated character $\lambda_q = \varepsilon' \circ \phi_q$ is defined by:

$$\lambda_q(T) = q^{\text{cl}(T)} \frac{\text{ho}(T)}{\text{cl}(T)!}.$$

If $q \neq 0$, for any finite set A , $\lambda_q(\bullet_A) = q \neq 0$, so λ_q is invertible.

For any $q \in \mathbb{K}$, nonzero, we put $\phi_{ehr_q} = \theta_{q^{-1}} \circ \phi \circ \iota_q : \mathcal{T}op \longrightarrow \mathcal{C}omp$. These are twisted bialgebra morphisms, and $\phi_{ehr_1} = \phi$. We put:

$$\lambda_{ehr_q} = \lambda_1^{*-1} \star (\varepsilon' \circ \phi_{ehr_q}),$$

which implies that for any $q \in \mathbb{K}$:

$$\phi_{ehr_q} = \phi_1 \leftarrow \lambda_{ehr_q}.$$

Example 7.6. If A, B and C are finite sets:

$$\begin{aligned} \phi_{ehr_q}(\bullet_A) &= (A), \\ \phi_{ehr_q}(\downarrow_A^B) &= (A, B) + \frac{1-q}{2}(A \sqcup B), \\ \phi_{ehr_q}(\overset{B}{\bullet} \downarrow_A^C) &= (A, B, C) + (A, C, B) + (A, B \sqcup C) \\ &\quad + \frac{1-q}{2}(A \sqcup B, C) + \frac{1-q}{2}(A \sqcup C, B) + \frac{(1-q)(2-q)}{6}(A \sqcup B \sqcup C), \\ \phi_{ehr_q}(\downarrow_A^{\overset{C}{\bullet}}) &= (A, B, C) + \frac{1-q}{2}(A, B \sqcup C) + \frac{1-q}{2}(A \sqcup B, C) + \frac{(1-q)(2-q)}{6}(A \sqcup B \sqcup C). \end{aligned}$$

Proposition 7.15. *For any finite topology T , $\varepsilon' \circ \phi_{ehr_{-1}}(T) = 1$ and:*

$$\phi_{ehr_{-1}}(T) = \sum_{f \in L'(T)} (\#f^{-1}(1), \dots, \#f^{-1}(\max(f))).$$

Proof. For any finite topology T :

$$\begin{aligned} \varepsilon' \circ \phi_{ehr_{-1}}(T) &= (-1)^{\text{cl}(T)} \varepsilon' \circ \theta_{-1} \circ \phi_{ehr_1}(T) \\ &= \sum_{f \in L(T)} (-1)^{\text{cl}(T) + \max(f)}. \end{aligned}$$

Let us prove that this is equal to 1 for any quasi-poset T . We proceed by induction on $n = \text{cl}(T)$. This is obvious if $n = 0$. If $n \geq 1$, let us denote by $\min(T)$ the set of minimal classes (for \leq_T) of T . For any linear extension f of T , $f^{-1}(T)$ is a subset of $\min(T)$; we obtain a bijection:

$$\left\{ \begin{array}{l} \bigsqcup_{\emptyset \neq I \subseteq \min(T)} L(T|_{V(T) \setminus I}) \longrightarrow L(T) \\ f \in L(T|_{V(T) \setminus I}) \longrightarrow \tilde{f} : \begin{cases} V(T) & \longrightarrow \mathbb{N} \\ x & \longrightarrow \begin{cases} 1 & \text{if } x \in I, \\ f(x) + 1 & \text{otherwise.} \end{cases} \end{cases} \end{array} \right.$$

Hence, using the induction hypothesis on the quasi-posets $T_{|V(T)\setminus I}$:

$$\begin{aligned}
\sum_{f \in L(T)} (-1)^{\text{cl}(T) + \max(f)} &= \sum_{\emptyset \subsetneq I \subseteq \min(T)} \sum_{f \in L(T_{|V(T)\setminus I})} (-1)^{\#I + \text{cl}(T_{|V(T)\setminus I}) + 1 + \max(f)} \\
&= - \sum_{\emptyset \subsetneq I \subseteq \min(T)} (-1)^{\#I} \\
&= - \sum_{I \subseteq \min(T)} (-1)^{\#I} + 1 \\
&= 1.
\end{aligned}$$

Hence, $\varepsilon' \circ \phi_{ehr_{-1}}(T) = 1$ for any T . The formula for $\phi_{ehr_{-1}}(T)$ comes from Proposition 6.8. \square

Proposition 7.16. *For any quasi-poset T , for any $q \in \mathbb{K}$, nonzero:*

$$\lambda_{ehr_q}(T) = q^{\text{cl}(T) - \text{cc}(T)} \lambda_{ehr_1}(T), \quad \lambda_{ehr_q}^{\star-1}(T) = q^{\text{cl}(T) - \text{cc}(T)} \frac{\text{ho}(T)}{\text{cl}(T)!}.$$

Proof. As $\phi_{ehr_q} = \phi_1 \leftarrow \lambda_{ehr_q} = \theta_{q^{-1}} \circ \phi_{ehr_1} \circ \iota_q$:

$$\begin{aligned}
\theta_q \circ \phi_{ehr_q} &= \phi_{ehr_1} \circ \iota_q & \varepsilon' \circ \iota_q &= \varepsilon' \circ \phi_{ehr_1} \circ \iota_q \\
&= (\theta_q \circ \phi_1) \leftarrow \lambda_{ehr_q}, & &= \varepsilon' \circ \theta_q \circ \phi_{ehr_q} \\
& & &= (\varepsilon' \circ \theta_q \circ \phi_1) \star \lambda_{ehr_q}.
\end{aligned}$$

Let T be a finite topology. Then:

$$\varepsilon' \circ \theta_q \circ \phi_1(T) = \frac{q^{\text{cl}(T)} \text{ho}(T)}{\text{cl}(T)!}.$$

Hence:

$$\begin{aligned}
q^{\text{cl}(T)} \varepsilon'(T) &= \sum_{\sim \triangleleft T} \frac{q^{\text{cl}(\sim)} \text{ho}(T/\sim)}{\text{cl}(\sim)} \lambda_{ehr_q}(T|\sim), \\
\varepsilon'(T) &= \sum_{\sim \triangleleft T} \frac{\text{ho}(T/\sim)}{\text{cl}(T/\sim)!} q^{\text{cc}(T|\sim) - \text{cl}(T|\sim)} \lambda_{ehr_q}(T|\sim) \\
&= \sum_{\sim \triangleleft T} \lambda_{ho}(T/\sim) q^{\text{cc}(T|\sim) - \text{cl}(T|\sim)} \lambda_{ehr_q}(T|\sim).
\end{aligned}$$

Hence, the inverse of the character λ_{ho} is given by:

$$\lambda_{ho}^{\star-1}(T) = q^{\text{cc}(T) - \text{cl}(T)} \lambda_{ehr_q}(T).$$

Consequently, for $q = 1$, $\lambda_{ho}^{\star-1} = \lambda_{ehr_1}$ and, for any $q \neq 0$:

$$\lambda_{ehr_q}(T) = q^{\text{cl}(T) - \text{cc}(T)} \lambda_{ehr_1}(T).$$

Moreover:

$$\varepsilon'(T) = q^{\text{cc}(T) - \text{cl}(T)} \sum_{\sim \triangleleft T} \lambda_{ho}(T/\sim) q^{\text{cl}(T/\sim) - \text{cc}(T/\sim)} \lambda_{ehr_q}(T|\sim).$$

If $\varepsilon'(T) \neq 0$, then \leq_T is an equivalence and $\text{cl}(T) = \text{cc}(T)$, so:

$$\varepsilon'(T) = \sum_{\sim \triangleleft T} \lambda_{ho}(T/\sim) q^{\text{cl}(T/\sim) - \text{cc}(T/\sim)} \lambda_{ehr_q}(T|\sim).$$

Hence, the inverse of λ_{ehr_q} is given by $\lambda_{ehr_q}^{\star-1}(T) = q^{\text{cl}(T) - \text{cc}(T)} \lambda_{ho}(T)$. \square

Corollary 7.17. *The following maps are twisted bialgebra automorphisms, inverse one from each other:*

$$\Gamma : \begin{cases} \mathcal{Top} & \longrightarrow \mathcal{Top} \\ T & \longrightarrow \sum_{\sim \triangleleft T} T / \sim, \end{cases}$$

$$\Gamma' : \begin{cases} \mathcal{Top} & \longrightarrow \mathcal{Top} \\ T & \longrightarrow \sum_{\sim \triangleleft T} (-1)^{\text{cl}(\sim) + \text{cl}(T)} T / \sim. \end{cases}$$

Moreover, $\phi_{ehr_{-1}} = \phi_{ehr_1} \circ \Gamma$ and $\phi_{ehr_1} = \phi_{ehr_{-1}} \circ \Gamma'$.

Proof. Let λ be the character defined by $\lambda(T) = 1$ for any finite topology T . It is an invertible character and, as $\Gamma = \text{Id} \leftarrow \lambda$, Γ is a twisted bialgebra automorphism. By Proposition 7.15, $\lambda = \varepsilon' \circ \phi_{ehr_{-1}}$, so:

$$\lambda = \varepsilon' \circ (\phi_1 \leftarrow \lambda_{ehr_{-1}}) = \varepsilon' \circ (\phi_{ehr_1} \leftarrow (\lambda_{ehr_1}^{*-1} \star \lambda_{ehr_{-1}})) = \lambda_{ehr_1}^{*-1} \star \lambda_{ehr_{-1}}.$$

Hence:

$$\phi_{ehr_1} \circ \Gamma = \phi_{ehr_1} \leftarrow \lambda = \phi_1 \leftarrow (\lambda_{ehr_1} \star \lambda) = \phi_1 \leftarrow (\lambda_{ehr_{-1}}) = \phi_{ehr_{-1}}.$$

Moreover, $\Gamma^{-1} = \text{Id} \leftarrow \lambda^{*-1}$. It remains to compute $\lambda^{*-1} = \lambda_{ehr_{-1}}^{*-1} \star \lambda_{ehr_1}$. Let T be a finite topology:

$$\begin{aligned} \lambda_{ehr_{-1}}^{*-1} \star \lambda_{ehr_1}(T) &= \sum_{\sim \triangleleft T} (-1)^{\text{cl}(T/\sim) + \text{cc}(T/\sim)} \lambda_{ehr_1}^{*-1}(T/\sim) \lambda_{ehr_1}(T|\sim) \\ &= \sum_{\sim \triangleleft T} (-1)^{\text{cc}(T|\sim) + \text{cc}(T)} \lambda_{ehr_1}^{*-1}(T/\sim) \lambda_{ehr_1}(T|\sim) \\ &= (-1)^{\text{cl}(T) + \text{cc}(T)} \sum_{\sim \triangleleft T} (-1)^{\text{cc}(T|\sim) + \text{cl}(T|\sim)} \lambda_{ehr_1}^{*-1}(T/\sim) \lambda_{ehr_1}(T|\sim) \\ &= (-1)^{\text{cl}(T) + \text{cc}(T)} \sum_{\sim \triangleleft T} \lambda_{ehr_1}^{*-1}(T/\sim) \lambda_{ehr_1}(T|\sim) \\ &= (-1)^{\text{cl}(T) + \text{cc}(T)} \lambda_{ehr_1}^{*-1} \star \lambda_{ehr_{-1}}(T) \\ &= (-1)^{\text{cl}(T) + \text{cc}(T)} \lambda(T) \\ &= (-1)^{\text{cl}(T) + \text{cc}(T)}. \end{aligned}$$

As $\Gamma' = \text{Id} \leftarrow \lambda^{*-1}$, $\Gamma' = \Gamma^{-1}$. □

To summarize, we obtain twisted bialgebra morphisms ϕ_q and ϕ_{ehr_q} from \mathcal{Top} to \mathcal{Comp} , such that the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{Top} & \xrightarrow{\iota_q} & \mathcal{Top} \\ \phi_{ehr_{q'}} \downarrow & & \downarrow \phi_{ehr_{q'/q}} \\ \mathcal{Comp} & \xrightarrow{\theta_q} & \mathcal{Comp} \end{array} \qquad \begin{array}{ccc} \mathcal{Top} & \xrightarrow{\iota_q} & \mathcal{Top} \\ \phi_{q'} \downarrow & & \downarrow \phi_{q'} \\ \mathcal{Comp} & \xrightarrow{\theta_q} & \mathcal{Comp} \end{array}$$

We also have a twisted bialgebra automorphism Γ of \mathcal{Top} making the following diagram commutating:

$$\begin{array}{ccc} \mathcal{Top} & \xrightarrow{\Gamma} & \mathcal{Top} \\ & \searrow \phi_{ehr_{-1}} & \swarrow \phi_{ehr_1} \\ & \mathcal{Comp} & \end{array}$$

They are related to several characters of $\mathcal{T}op$:

$$\phi_{ehr_q} = \phi_1 \leftarrow \lambda_{ehr_q}, \quad \Gamma = \text{Id} \leftarrow \lambda.$$

For any finite toplogy T :

$$\begin{aligned} \lambda_{ehr_q}(G) &= q^{\text{cl}(T) - \text{cc}(T)} \lambda_{ehr_1}(T), & \lambda_{ehr_q}^{\star-1}(T) &= q^{\text{cl}(T) - \text{cc}(T)} \frac{\text{ho}(T)}{\text{cl}(T)!}, \\ \lambda(T) &= 1, & \lambda^{\star-1}(T) &= (-1)^{\text{cl}(T) + \text{cc}(T)}. \end{aligned}$$

Remark 7.3. Taking the limit when q goes to 0, one gives a meaning to ϕ_{ehr_0} and to λ_{ehr_0} . For any finite topology T :

$$\begin{aligned} \lambda_{ehr_0}(T) &= \begin{cases} 1 & \text{if } \text{cl}(T) = \text{cc}(T), \\ 0 & \text{otherwise} \end{cases} \\ &= \varepsilon'(T). \end{aligned}$$

So $\phi_{ehr_0} = \phi_1 \leftarrow \lambda_{ehr_0} = \phi_1 \leftarrow \varepsilon' = \phi_1$.

8 Fock functors

8.1 Definition

Let us now use the Fock functors of [2].

Notations 8.1. Let V be a left \mathfrak{S}_n -module. The space of coinvariants of V is:

$$\text{Coinv}(V) = \frac{V}{\text{Vect}(x - \sigma.x, x \in V, \sigma \in \mathfrak{S}_n)}.$$

In particular, if \mathcal{P} is a species, for any $n \geq 0$, $\mathcal{P}[\underline{n}]$ is a left \mathfrak{S}_n -module.

Definition 8.1. 1. (*Full Fock functor*). Let \mathcal{P} be a species. We put:

$$\mathcal{K}(\mathcal{P}) = \bigoplus_{n=0}^{\infty} \mathcal{P}[\underline{n}].$$

If $\phi : \mathcal{P} \rightarrow \mathcal{Q}$ is a species morphism, we put:

$$\mathcal{K}(\phi) : \begin{cases} \mathcal{K}(\mathcal{P}) & \rightarrow \mathcal{K}(\mathcal{Q}) \\ x \in \mathcal{P}[\underline{n}] & \rightarrow \phi[\underline{n}](x) \in \mathcal{Q}[\underline{n}]. \end{cases}$$

This defines a functor from the category of species to the category of graded vector spaces.

2. (*Bosonic Fock functor*). Let \mathcal{P} be a species. We put:

$$\widehat{\mathcal{K}}(\mathcal{P}) = \bigoplus_{n=0}^{\infty} \text{Coinv}(\mathcal{P}[\underline{n}]).$$

If $\phi : \mathcal{P} \rightarrow \mathcal{Q}$ is a morphism of species, we put:

$$\widehat{\mathcal{K}}(\phi) : \begin{cases} \widehat{\mathcal{K}}(\mathcal{P}) & \rightarrow \widehat{\mathcal{K}}(\mathcal{Q}) \\ \bar{x} \in \text{Coinv}(\mathcal{P}[\underline{n}]) & \rightarrow \frac{\phi[\underline{n}](x)}{\#} \in \text{Coinv}(\mathcal{Q}[\underline{n}]). \end{cases}$$

This defines a functor from the category of species to the category of graded vector spaces.

Remark 8.1. 1. $\widehat{\mathcal{K}}(\phi)$ is well-defined: if $x \in \mathcal{P}[\underline{n}]$ and $\sigma \in \mathfrak{S}_n$, $\mathcal{K}(\phi)(x - \sigma.x) = \mathcal{K}(\phi)(x) - \sigma.\mathcal{K}(\phi)(x)$. We have a commutative diagram:

$$\begin{array}{ccc} \mathcal{K}(\mathcal{P}) & \xrightarrow{\mathcal{K}(\phi)} & \mathcal{K}(\mathcal{Q}) \\ \downarrow & & \downarrow \\ \widehat{\mathcal{K}}(\mathcal{P}) & \xrightarrow{\widehat{\mathcal{K}}(\phi)} & \widehat{\mathcal{K}}(\mathcal{Q}) \end{array}$$

where the vertical arrows are the canonical surjections.

2. For all $n \geq 0$, the n -th homogeneous component of $\mathcal{K}(\mathcal{P})$ inherits a structure of left \mathfrak{S}_n -module. This induces a trivial left action of \mathfrak{S}_n on the n -th homogeneous component of $\widehat{\mathcal{K}}(\mathcal{P})$.

8.2 Fock functors applied to twisted (double) bialgebras

Notations 8.2. Let $m, n \in \mathbb{N}$. We denote by $\sigma_{m,n} : \underline{m} \sqcup \underline{n} \longrightarrow \underline{m+n}$ the bijection defined by:

$$\begin{array}{ll} \forall i \in \underline{m}, & \sigma_{m,n}(i) = i, \\ \forall j \in \underline{n}, & \sigma_{m,n}(j) = j + m. \end{array}$$

For any $I \subseteq \underline{n}$, we denote by σ_I the unique increasing bijection from I to $\#I$.

Theorem 8.2. 1. Let \mathcal{P} be a twisted algebra. Then $\mathcal{K}(\mathcal{P})$ is a graded algebra, with the product defined by:

$$\forall x \in \mathcal{P}[\underline{m}], \forall y \in \mathcal{P}[\underline{n}], \quad x \cdot y = \mathcal{P}[\sigma_{m,n}] \circ m_{\mathcal{P}}(x \otimes y).$$

The unit is $1_{\mathcal{P}} \in \mathcal{P}[\emptyset]$.

2. Let \mathcal{P} be a twisted coalgebra. Then $\mathcal{K}(\mathcal{P})$ is a graded coalgebra, with the coproduct defined by:

$$\forall x \in \mathcal{P}[\underline{n}], \quad \Delta(x) = \sum_{I \subseteq \underline{n}} (\mathcal{P}[\sigma_I] \otimes \mathcal{P}[\sigma_{\underline{n} \setminus I}]) \circ \Delta_{I, \underline{n} \setminus I}(x).$$

We shall put $\Delta^{(I, \underline{n} \setminus I)} = (\sigma_I \otimes \sigma_{\underline{n} \setminus I}) \circ \Delta_{I, \underline{n} \setminus I}$ for any $I \subseteq \underline{n}$.

3. If \mathcal{P} is a twisted bialgebra, then $\mathcal{K}(\mathcal{P})$ is a graded bialgebra.
4. If \mathcal{P} is a double twisted bialgebra, then $\mathcal{K}(\mathcal{P})$ inherits a second coproduct δ , defined by:

$$\forall x \in \mathcal{P}[\underline{n}], \quad \delta(x) = \delta_{\underline{n}}(x).$$

The triple $(\mathcal{K}(\mathcal{P}), m, \delta)$ is a bialgebra. Moreover, for any $I \subseteq \underline{n}$:

$$(\Delta^{(I, \underline{n} \setminus I)} \otimes \text{Id}_{\mathcal{P}[\underline{n}]}) \circ \delta = (\text{Id} \otimes \text{Id} \otimes \mathcal{P}[\alpha_{I, \underline{n} \setminus I}]) \circ m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta^{(I, \underline{n} \setminus I)},$$

where $\alpha_{i, \underline{n} \setminus I}$ is an element of \mathfrak{S}_n .

Proof. 1. Let $x \in \mathcal{P}[\underline{m}]$, $y \in \mathcal{P}[\underline{n}]$ and $z \in \mathcal{P}[\underline{p}]$. We consider the bijection $\sigma_{m,n,p} : \underline{m} \sqcup \underline{n} \sqcup \underline{p} \longrightarrow \underline{m+n+p}$ defined by:

$$\sigma_{m,n,p}(i) = \begin{cases} i & \text{if } i \in \underline{m}, \\ i + m & \text{if } i \in \underline{n}, \\ i + m + n & \text{if } i \in \underline{p}. \end{cases}$$

Then:

$$\begin{aligned}
(x \cdot y) \cdot z &= \mathcal{P}[\sigma_{m+n,p}] \circ (\mathcal{P}[\sigma_{m,n}] \otimes \text{Id}) \circ m \circ (m \otimes \text{Id})(x \otimes y \otimes z) \\
&= \mathcal{P}[\sigma_{m,n,p}] \circ m \circ (m \otimes \text{Id})(x \otimes y \otimes z) \\
&= \mathcal{P}[\sigma_{m,n,p}] \circ m \circ (\text{Id} \otimes m)(x \otimes y \otimes z) \\
&= \mathcal{P}[\sigma_{m,n+p}] \circ (\text{Id} \otimes \mathcal{P}[\sigma_{n,p}]) \circ m \circ (\text{Id} \otimes m)(x \otimes y \otimes z) \\
&= x \cdot (y \cdot z).
\end{aligned}$$

2. Let $x \in \mathcal{P}[\underline{n}]$.

$$\begin{aligned}
(\Delta \otimes \text{Id}) \circ \Delta(x) &= \sum_{I \sqcup J \sqcup K = \underline{n}} (\mathcal{P}[\sigma_I] \otimes \mathcal{P}[\sigma_J] \otimes \text{Id}) \circ (\Delta_{I,J} \otimes \text{Id}) \circ (\text{Id} \otimes \mathcal{P}[\sigma_K]) \circ \Delta_{I \sqcup J, K}(x) \\
&= \sum_{I \sqcup J \sqcup K = \underline{n}} (\mathcal{P}[\sigma_I] \otimes \mathcal{P}[\sigma_J] \otimes \mathcal{P}[\sigma_K]) \circ \Delta_{I,J,K}(x) \\
&= \sum_{I \sqcup J \sqcup K = \underline{n}} (\text{Id} \otimes \mathcal{P}[\sigma_J] \otimes \mathcal{P}[\sigma_K]) \circ (\text{Id} \otimes \Delta_{J,K}) \circ (\mathcal{P}[\sigma_I] \otimes \text{Id}) \circ \Delta_{I, J \sqcup K}(x) \\
&= (\text{Id} \otimes \Delta) \circ \Delta(x).
\end{aligned}$$

3. Let $x \in \mathcal{P}[\underline{m}]$ and $y \in \mathcal{P}[\underline{n}]$.

$$\begin{aligned}
\Delta(x \cdot y) &= (\mathcal{P} \otimes \mathcal{P})[\sigma_{m,n}] \circ \Delta \circ m(x \otimes y) \\
&= (\mathcal{P} \otimes \mathcal{P})[\sigma_{m,n}] \circ (m \otimes m) \circ (\text{Id} \otimes c_{\mathcal{P}, \mathcal{P}} \otimes \text{Id}) \circ (\Delta \otimes \Delta)(x \otimes y) \\
&= \sum_{\substack{I \sqcup J = \underline{m}, \\ I' \sqcup J' = \underline{n}}} (\mathcal{P} \otimes \mathcal{P})[\sigma_{m,n}] \circ (m \otimes m) \circ (\text{Id} \otimes c_{\mathcal{P}, \mathcal{P}} \otimes \text{Id}) \\
&\quad \circ (\mathcal{P}[\sigma_I] \otimes \mathcal{P}[\sigma_J] \otimes \mathcal{P}[\sigma_{I'}] \otimes \mathcal{P}[\sigma_{J'}]) \circ (\Delta_{I,J} \otimes \Delta_{I',J'})(x \otimes y) \\
&= \sum_{\substack{I \sqcup J = \underline{m}, \\ I' \sqcup J' = \underline{n}}} (\mathcal{P} \otimes \mathcal{P})[\sigma_{m,n}] \circ (m \otimes m) \circ (\mathcal{P}[\sigma_I] \otimes \mathcal{P}[\sigma_{I'}] \otimes \mathcal{P}[\sigma_J] \otimes \mathcal{P}[\sigma_{J'}]) \\
&\quad \circ (\text{Id} \otimes c_{\mathcal{P}, \mathcal{P}} \otimes \text{Id}) \circ (\Delta_{I,J} \otimes \Delta_{I',J'})(x \otimes y) \\
&= \sum_{\substack{I \sqcup J = \underline{m}, \\ I' \sqcup J' = \underline{n}}} \Delta^{(I,J)}(x) \cdot \Delta^{(I',J')}(y) \\
&= \Delta(x) \cdot \Delta(y).
\end{aligned}$$

4. Let $x \in \mathcal{P}[\underline{m}]$ and $y \in \mathcal{P}[\underline{n}]$.

$$\begin{aligned}
\delta(x \cdot y) &= \delta \circ \mathcal{P}[\sigma_{m,n}] \circ m(x \otimes y) \\
&= (\mathcal{P}[\sigma_{m,n}] \otimes \mathcal{P}[\sigma_{m,n}]) \circ (m \otimes m) \circ (\text{Id} \otimes c_{\mathcal{P}, \mathcal{P}} \otimes \text{Id}) \circ (\delta \otimes \delta)(x \otimes y) \\
&= \delta(x) \cdot \delta(y).
\end{aligned}$$

If $x \in \mathcal{P}[\underline{m}]$ and $\underline{m} = I \sqcup J$:

$$\begin{aligned}
(\Delta^{(I,J)} \otimes \text{Id}) \circ \delta(x) &= (\mathcal{P}[\sigma_I] \otimes \mathcal{P}[\sigma_J] \otimes \text{Id}) \circ m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta_{I,J}(x), \\
m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta^{(I, \underline{m} \setminus I)} &= m_{1,3,24} \circ (\mathcal{P}[\sigma_I] \otimes \mathcal{P}[\sigma_I] \otimes \mathcal{P}[\sigma_J] \otimes \mathcal{P}[\sigma_J]) \circ (\delta \otimes \delta) \circ \Delta_{I,J}(x) \\
&= (\mathcal{P}[\sigma_I] \otimes \mathcal{P}[\sigma_J] \otimes \mathcal{P}[\sigma_I \sqcup \sigma_J]) \circ m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta_{I,J}(x) \\
&= (\text{Id} \otimes \text{Id} \otimes \mathcal{P}[\sigma_{I,J}]) \circ (\Delta^{(I,J)} \otimes \text{Id}) \circ \delta(x),
\end{aligned}$$

with $\sigma_{I,J} = \sigma_I \sqcup \sigma_J$. This permutation is nondecreasing on I and J and sends I to $\underline{\#}I$, so is the inverse of a shuffle. \square

Corollary 8.3. *Let \mathcal{P} be a twisted algebra [respectively coalgebra, bialgebra]. Then $\widehat{\mathcal{K}}(\mathcal{P})$ is an algebra [respectively a coalgebra, a bialgebra], quotient of $\mathcal{K}(\mathcal{P})$. If \mathcal{P} is a double twisted bialgebra, then $\widehat{\mathcal{K}}(\mathcal{P})$ is a bialgebra in cointeraction in the sense of [12, 13]:*

$$(\Delta \otimes \text{Id}) \circ \delta = m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta.$$

Proof. Let I be the subspace of $\mathcal{K}(\mathcal{P})$ generated by the elements $x - \mathcal{P}[\sigma](x)$, where $x \in \mathcal{P}[\underline{n}]$, $\sigma \in \mathfrak{S}_n$, $n \geq 0$. Then $\widehat{\mathcal{K}}(\mathcal{P}) = \mathcal{K}(\mathcal{P})/I$.

If \mathcal{P} is an algebra, let us prove that I is an ideal of $\mathcal{K}(\mathcal{P})$. For any $x \in \mathcal{P}[\underline{m}]$, $\sigma \in \mathfrak{S}_m$, $y \in \mathcal{P}[\underline{n}]$:

$$\begin{aligned} (x - \mathcal{P}[\sigma](x)) \cdot y &= \mathcal{P}[\sigma_{m,n}] \circ m(x \otimes y - \mathcal{P}[\sigma](x) \times y) \\ &= \mathcal{P}[\sigma_{m,n}] \circ m(x \otimes y) - \mathcal{P}[\sigma_{m,n}] \circ \mathcal{P}[\text{Id}_m \sqcup \sigma] \circ m(x \otimes y) \\ &= X - \mathcal{P}[\tau](X), \end{aligned}$$

with:

$$X = \mathcal{P}[\sigma_{m,n}] \circ m(x \otimes y), \quad \tau = \sigma_{m,n} \circ (\text{Id}_m \sqcup \sigma) \circ \sigma_{m,n}^{-1}.$$

So $(x - \mathcal{P}[\sigma](x)) \cdot y \in I$. Similarly, $y \cdot (x - \mathcal{P}[\sigma](x)) \in I$.

If \mathcal{P} is a coalgebra, let us prove that I is a coideal of $\mathcal{K}(\mathcal{P})$. For any $x \in \mathcal{P}[\underline{m}]$, $\tau \in \mathfrak{S}_m$:

$$\begin{aligned} &\Delta(\mathcal{P}[\tau](x)) \\ &= \sum_{I \sqcup J = \underline{m}} (\mathcal{P}[\sigma_{\tau(I)}] \otimes \mathcal{P}[\sigma_{\tau(J)}]) \circ (\mathcal{P} \otimes \mathcal{P}[\tau]) \circ \Delta_{I,J}(x) \\ &= \sum_{I \sqcup J = \underline{m}} (\mathcal{P}[\sigma_{\tau(I)}] \otimes \mathcal{P}[\sigma_{\tau(J)}]) \circ (\mathcal{P} \otimes \mathcal{P}[\tau]) \circ (\mathcal{P}[\sigma_I^{-1}] \otimes \mathcal{P}[\sigma_J^{-1}]) \circ (\mathcal{P}[\sigma_I] \otimes \mathcal{P}[\sigma_J]) \circ \Delta_{I,J}(x) \\ &= \sum_{I \sqcup J = \underline{m}} \mathcal{P} \otimes \mathcal{P}[\tau_{I,J}] \circ (\mathcal{P}[\sigma_I] \otimes \mathcal{P}[\sigma_J]) \circ \Delta_{I,J}(x), \end{aligned}$$

where $\tau_{I,J}$ is the permutation defined by:

- If $i \leq \#I$, $\tau_{I,J}(i) = \sigma_{\tau(I)} \circ \tau \circ \sigma_I^{-1}(i)$.
- If $i > \#I$, $\tau_{I,J}(i) = \sigma_{\tau(J)} \circ \tau \circ \sigma_J^{-1}(i)$.

Hence, putting $k = \#I$, $\tau_{I,J}(\underline{k}) = \underline{k}$. There exists $\tau'_{I,J} \in \mathfrak{S}_k$ and $\tau''_{I,J} \in \mathfrak{S}_{m-k}$ such that $\tau_{I,J} = \tau'_{I,J} \sqcup \tau''_{I,J}$. Hence, putting $(\mathcal{P}[\sigma_I] \otimes \mathcal{P}[\sigma_J]) \circ \Delta_{I,J}(x) = x'_{I,J} \otimes x''_{I,J}$:

$$\begin{aligned} \Delta(x - \mathcal{P}[\tau](x)) &= \sum_{I \sqcup J = \underline{m}} x'_{I,J} \otimes x''_{I,J} - (\mathcal{P}[\tau'_{I,J}] \otimes \mathcal{P}[\tau''_{I,J}])(x'_{I,J} \otimes x''_{I,J}) \\ &= \sum_{I \sqcup J = \underline{m}} (x'_{I,J} - \mathcal{P}[\tau'_{I,J}](x'_{I,J})) \otimes x''_{I,J} + \mathcal{P}[\tau'_{I,J}](x'_{I,J}) \otimes (x''_{I,J} - \mathcal{P}[\tau''_{I,J}](x''_{I,J})) \\ &\in I \otimes \mathcal{K}(\mathcal{P}) + \mathcal{K}(\mathcal{P}) \otimes I. \end{aligned}$$

So I is a coideal for Δ .

Let us assume that \mathcal{P} is a double bialgebra. By the preceding points, $(\widehat{\mathcal{K}}(\mathcal{P}), m, \Delta)$ is a bialgebra. For any $x \in \mathcal{P}[\underline{m}]$, $\tau \in \mathfrak{S}_m$, putting $\delta(x) = x' \otimes x''$:

$$\begin{aligned} \delta(x - \mathcal{P}[\tau](x)) &= \delta(x) - (\mathcal{P}[\tau] \otimes \mathcal{P}[\tau]) \circ \delta(x) \\ &= (x' - \mathcal{P}[\tau](x')) \otimes x'' + \mathcal{P}[\tau](x') \otimes (x'' - \mathcal{P}[\tau](x'')) \\ &\in I \otimes \mathcal{K}(\mathcal{P}) + \mathcal{K}(\mathcal{P}) \otimes I. \end{aligned}$$

So I is a coideal for δ , and $\widehat{\mathcal{K}}(\mathcal{P})$ inherits a second coproduct δ . By Theorem 8.2, as the action of the symmetric groups of $\widehat{\mathcal{K}}(\mathcal{P})$ are trivial:

$$(\Delta \otimes \text{Id}) \circ \delta = m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta.$$

So $(\widehat{\mathcal{K}}(\mathcal{P}), m, \Delta)$ is a bialgebra in the category of right $(\widehat{\mathcal{K}}(\mathcal{P}), m, \Delta)$ -comodules. \square

Remark 8.2. Let \mathcal{P} be a double twisted bialgebra. Denoting by π the canonical projection from $\mathcal{K}(\mathcal{P})$ to $\widehat{\mathcal{K}}(\mathcal{P})$, we can define a right coaction of $\widehat{\mathcal{K}}(\mathcal{P})$ on $\mathcal{K}(\mathcal{P})$ by:

$$\rho = (\text{Id} \otimes \pi) \circ \delta : \mathcal{K}(\mathcal{P}) \longrightarrow \mathcal{K}(\mathcal{P}) \otimes \widehat{\mathcal{K}}(\mathcal{P}).$$

Then $(\mathcal{K}(\mathcal{P}), m, \Delta)$ is a bialgebra in the category of right $(\widehat{\mathcal{K}}(\mathcal{P}), m, \delta)$ -comodules:

$$(\Delta \otimes \text{Id}) \circ \rho = m_{1,3,24} \circ (\rho \otimes \rho) \circ \Delta.$$

Example 8.1. Let us apply the Fock functors to **Com**. As the action of \mathfrak{S}_n over **Com** $[\underline{n}]$ is trivial, $\mathcal{K}(\mathbf{Com}) = \widehat{\mathcal{K}}(\mathbf{Com})$. If we denote by e_n the unit of $\mathbb{K} = \mathbf{Com}[\underline{n}]$ for all $n \geq 0$, then a basis of $\mathcal{K}(\mathbf{Com})$ is $(e_n)_{n \geq 0}$. For any $k, l \geq 0$, $e_k \cdot e_l = e_{k+l}$, so putting $X = e_1$, for any $n \geq 0$, $e_n = X^n$ and $\mathcal{K}(\mathbf{Com}) = \mathbb{K}[X]$. For any $n \geq 0$:

$$\Delta(X^n) = \sum_{k=0}^n \binom{n}{k} X^k \otimes X^{n-k}, \quad \delta(X^n) = X^n \otimes X^n.$$

So $\mathcal{K}(\mathbf{Com})$ is the algebra $\mathbb{K}[X]$ with its usual coproducts Δ and δ , as considered in [12, 13].

Example 8.2. Let us apply the Fock functors to **Ord**. For any total order \leq on \underline{n} , there exists a unique $\sigma \in \mathfrak{S}_n$, such that:

$$\forall i, j \in \underline{n}, \quad i \leq j \iff \sigma(i) \leq \sigma(j).$$

Hence, $\mathcal{K}(\mathbf{Ord})$ has a basis indexed by permutations. For any $\sigma \in \mathfrak{S}_n$:

$$\Delta(\sigma) = \sum_{i=0}^n \sigma_i^{(1)} \otimes \sigma_i^{(2)},$$

where, if σ is represented by the word $\sigma(1) \dots \sigma(n)$:

- $\sigma_i^{(1)}$ is obtained by keeping only the letters $1, \dots, i$, in the same order.
- $\sigma_i^{(2)}$ is obtained by keeping only the letters $i+1, \dots, n$, in the same order, and by subtracting i to any of these letters.

If $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_l$:

$$\sigma \boxplus \tau = \sum_{\alpha \in \text{Sh}(k,l)} \alpha \circ (\sigma \otimes \tau).$$

In other words, $\mathcal{K}(\mathbf{Ord})$ is the Hopf algebra of permutations **FQSym** of [9, 22]. Taking the coinvariants, $\widehat{\mathcal{K}}(\mathbf{Ord})$ has a basis $(e_n)_{n \geq 0}$, where for all n e_n is the canonical projection of any permutation $\sigma \in \mathfrak{S}_n$. For any $k, l, n \geq 0$:

$$\Delta(e_n) = \sum_{i+j=n} e_i \otimes e_j, \quad e_k \boxplus e_l = \binom{k+l}{k} e_{k+l}.$$

Therefore, $\widehat{\mathcal{K}}(\mathbf{Ord})$ is isomorphic to the Hopf algebra $\mathbb{K}[X]$, through the morphism:

$$\begin{cases} \widehat{\mathcal{K}}(\mathbf{Ord}) & \longrightarrow \mathbb{K}[X] \\ e_n & \longrightarrow n!X^n. \end{cases}$$

Example 8.3. Let us apply the Fock functors to \mathcal{Comp} .

1. The bialgebra $\mathcal{K}(\mathcal{Comp})$ has for basis the set of set compositions of \underline{n} , with $n \geq 0$, or, equivalently, the set of packed words, that is to say surjective maps from \underline{n} to \underline{k} , for any $k \geq 1$. Its product is given by quasishuffles: if u, v are packed words of respective length k and l ,

$$u \boxplus v = \sum_{\sigma \in \text{QSh}(k,l)} \sigma \circ (u \sqcup v[\max(u)]),$$

where $v[\max(u)] = (v_1 + \max(u)) \dots + (v_l + \max(u))$ if $v = v_1 \dots v_l$. For example:

$$\begin{aligned} (11) \boxplus (11) &= (1122) + (2211) + (1111), \\ (12) \boxplus (11) &= (1233) + (1322) + (2311) + (1211) + (1222). \end{aligned}$$

The coproduct Δ is given by extractions of letters: if u is a packed word of length n ,

$$\Delta(u) = \sum_{k=0}^n u_{\underline{k}} \otimes u_{\underline{n} \setminus \underline{k}},$$

where for any $I \subseteq \underline{n}$, u_I is obtained by the following steps:

- (a) Keep only the letters of u which belong to I : one obtains a word $u'_1 \dots u'_p$.
- (b) Let $\sigma : I \rightarrow \underline{k}$ be the unique increasing bijection between I and set \underline{k} , where $k = |I|$. Then $u_I = f(u'_1) \dots f(u'_p)$.

For example:

$$\Delta((1123)) = 1 \otimes (1123) + (11) \otimes (12) + (112) \otimes (1) + (1123) \otimes 1.$$

The second coproduct δ is given by:

$$\delta(u) = \sum_{(\sigma, \tau) \in \mathbf{cont}_{\max(u)}} \sigma \circ u \otimes \tau \circ u.$$

For example:

$$\begin{aligned} \delta((11)) &= (11) \otimes (11), \\ \delta((12)) &= (12) \otimes ((12) + (21) + (11)) + (11) \otimes (12). \end{aligned}$$

In other words, $\mathcal{K}(\mathcal{Comp})$ is the bialgebra **WQSym** [26, 27, 25].

2. The bialgebra $\widehat{\mathcal{K}}(\mathcal{Comp})$ has for basis the set of compositions, that is to say finite sequence of positive integers. The product is given by quasishuffles, the coproduct Δ by deconcatenation and the coproduct δ by the action of the elements of \mathbf{cont}_k . For example, if

$a, b, c, d \geq 1$:

$$\begin{aligned}
(a) \boxplus (b) &= (a, b) + (b, a) + (a + b), \\
(a, b) \boxplus (c) &= (a, b, c) + (a, c, b) + (c, a, b) + (a + b, c) + (a, b + c), \\
(a, b) \boxplus (c, d) &= (a, b, c, d) + (a, c, b, d) + (c, a, b, d) \\
&\quad + (a, c, d, b) + (c, a, d, b) + (c, d, a, b) \\
&\quad + (a, b + c, d) + (a + c, b, d) + (a + c, d, b) \\
&\quad + (a, c, b + d) + (c, a, b + d) + (c, a + d, b) \\
&\quad + (a + c, b + d);
\end{aligned}$$

$$\begin{aligned}
\Delta(a, b, c, d) &= 1 \otimes (a, b, c, d) + (a) \otimes (b, c, d) + (a, b) \otimes (c, d) \\
&\quad + (a, b, c) \otimes (d) + (a, b, c, d) \otimes 1;
\end{aligned}$$

$$\begin{aligned}
\delta(a) &= (a) \otimes (a), \\
\delta(a, b) &= (a, b) \otimes ((a) \boxplus (b)) + (a + b) \otimes (a, b), \\
\delta(a, b, c) &= (a, b, c) \otimes ((a) \boxplus (b) \boxplus (c)) + (a, b + c) \otimes ((a) \boxplus (b, c)) \\
&\quad + (a + b, c) \otimes ((a, b) \boxplus (c)) + (a + b + c) \otimes (a, b, c).
\end{aligned}$$

This is the Hopf algebra of quasisymmetric functions **QSym**; see for example [17] for more details.

8.3 The terminal property of $\mathbb{K}[X]$

We observed that $\mathbb{K}[X]$, with its two coproducts Δ and δ , is a bialgebra in cointeraction.

Proposition 8.4. *Let (A, Δ) be a graded and connected coalgebra. We denote by 1_A the unique element of A_0 such that $\varepsilon(1_A) = 1$. Let $\lambda : A \rightarrow \mathbb{K}$ be any linear form such that $\lambda(1_A) = 1$.*

1. *There exists a unique coalgebra morphism $\psi : (A, \Delta) \rightarrow (\mathbb{K}[X], \Delta)$, such that $\varepsilon' \circ \psi = \lambda$.*
2. *Let us assume that (A, m, Δ) is a bialgebra. Then $\psi : (A, m, \Delta) \rightarrow (\mathbb{K}[X], m, \Delta)$ is a bialgebra morphism if, and only if, λ is a character of A .*
3. *Let us assume that (A, m, Δ, δ) is a bialgebra in cointeraction. Then $\psi : (A, m, \Delta, \delta) \rightarrow (\mathbb{K}[X], m, \Delta, \delta)$ is a morphism of bialgebras in cointeraction if, and only if, λ is the counit ε' of (A, m, δ) .*

Proof. 1. We define $\psi(a)$ for any $a \in A_n$ by induction on n . If $n = 0$, it is defined by $\psi(1_A) = 1$. Otherwise, we put $\Delta(a) = a \otimes 1_A + 1_A \otimes 1 + \tilde{\Delta}(a)$, and by the connectivity condition:

$$\tilde{\Delta}(a) \in \sum_{i=1}^{n-1} A_i \otimes A_{n-i}.$$

By the induction hypothesis, $X = (\psi \otimes \psi) \circ \tilde{\Delta}(a)$ is well-defined. Moreover:

$$\begin{aligned}
(\tilde{\Delta} \otimes \text{Id})(X) &= ((\tilde{\Delta} \circ \psi) \otimes \psi) \circ \tilde{\Delta}(a) \\
&= (\psi \otimes \psi \otimes \psi) \circ (\tilde{\Delta} \otimes \text{Id}) \circ \tilde{\Delta}(a) \\
&= (\psi \otimes \psi \otimes \psi) \circ (\\
&\quad \text{id} \otimes \tilde{\Delta}) \circ \tilde{\Delta}(a) \\
&= (\text{Id} \otimes \tilde{\Delta}) \circ (\psi \otimes \psi)(a) \\
&= (\text{Id} \otimes \tilde{\Delta})(X).
\end{aligned}$$

So $X \in \ker(\tilde{\Delta} \otimes \text{Id} - \text{Id} \otimes \tilde{\Delta})$. It is not difficult to prove that in $\mathbb{K}[X]$:

$$\ker(\tilde{\Delta} \otimes \text{Id} - \text{Id} \otimes \tilde{\Delta}) = \text{Im}(\tilde{\Delta}).$$

Therefore, there exists a linear map $\psi' : A_n \longrightarrow \mathbb{K}[X]$ such that for any $a \in A_n$, $(\psi \otimes \psi) \circ \tilde{\Delta}(a) = \tilde{\Delta} \circ \psi'(a)$. We then put:

$$\psi(a) = \psi'(a) - \varepsilon' \circ \psi'(a)X + \lambda(a)X.$$

As $X \in \ker(\tilde{\Delta})$, for any $a \in A_n$, $(\psi \otimes \psi) \circ \tilde{\Delta}(a) = \tilde{\Delta} \circ \psi(a)$ and $\varepsilon' \circ \psi(a) = \lambda(a)$.

Unicity. Let ψ and ψ' be two such morphisms. Let us prove that $\psi|_{A_n} = \psi'|_{A_n}$ by induction on n . This is obvious for $n = 0$, as ψ and ψ' both sends the group-like element 1_A to the unique group-like element 1 of $(\mathbb{K}[X], \Delta)$. Let us assume the result at all ranks $< n$. For any $a \in A_n$:

$$\tilde{\Delta} \circ \psi(a) = (\psi \otimes \psi) \circ \tilde{\Delta}(a) = (\psi' \otimes \psi') \circ \tilde{\Delta}(a) = \tilde{\Delta} \circ \psi'(a),$$

so $\psi(a) - \psi'(a) \in \ker(\tilde{\Delta}) = \text{Vect}(X)$: let us put $\psi(a) - \psi'(a) = \mu X$, with $\mu \in \mathbb{K}$. Then:

$$\mu = \varepsilon'(\psi(a) - \psi'(a)) = \lambda(a) - \lambda(a) = 0,$$

so $\psi|_{A_n} = \psi'|_{A_n}$.

2. \implies . Then $\lambda = \varepsilon' \circ \psi$ is an algebra morphism by composition. \impliedby . Let us consider the two linear maps $\psi_1, \psi_2 : A \otimes A \longrightarrow \mathbb{K}[X]$ defined by $\psi_1 = m \circ (\psi \otimes \psi)$ and $\psi_2 = \psi \circ m$. As m is a coalgebra morphism, ψ_1 and ψ_2 are coalgebra morphisms. Moreover, for any $a, b \in A$:

$$\begin{aligned} \varepsilon' \circ \psi_1(a \otimes b) &= \varepsilon'(\psi(a)\psi(b)) \\ &= \varepsilon' \circ \psi(a)\varepsilon' \circ \psi(b) \\ &= \lambda(a)\lambda(b), \\ \varepsilon' \circ \psi_2(a \otimes b) &= \varepsilon' \circ \psi(ab) \\ &= \lambda(ab). \end{aligned}$$

As λ is a character, $\varepsilon' \circ \psi_1$ and $\varepsilon' \circ \psi_2$ are equal. The coalgebra $A \otimes A$ being connected, by unicity in the first point, $\psi_1 = \psi_2$, so ψ is an algebra morphism.

3. \implies . If ψ is a morphism of bialgebras in cointeration, then $\varepsilon' \circ \psi = \varepsilon'$. \impliedby . The counit ε' is a character of A , so ψ is compatible with m and Δ . Let us prove that for any $a \in A$, for any $k, l \geq 1$,

$$\delta \circ \psi(a)(k, l) = (\psi \otimes \psi) \circ \delta(a)(k, l).$$

Note that $\delta \circ \psi(a)(k, l) = \psi(a)(kl)$ by definition of the coproduct δ of $\mathbb{K}[X]$. If $k = 1$:

$$\begin{aligned} \delta \circ \psi(a)(1, l) &= (\varepsilon' \otimes \text{Id}) \circ \delta \circ \psi(a)(l) \\ &= \psi(a)(l), \\ (\psi \otimes \psi) \circ \delta(a)(1, l) &= (\varepsilon' \circ \psi \otimes \psi) \circ \delta(a)(l) \\ &= (\varepsilon' \otimes \psi) \circ \delta(a)(l) \\ &= \psi(a)(l). \end{aligned}$$

Let us assume the result at rank k . As ψ is compatible with m and Δ , and by the induction

hypothesis:

$$\begin{aligned}
\psi(a)((k+1)l) &= \psi(a)(kl+1l) \\
&= \Delta \circ \psi(a)(kl, 1l) \\
&= (\delta \otimes \delta) \circ \Delta \circ \psi(a)(k, l, 1, l) \\
&= m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta \circ \psi(a)(k, 1, l) \\
&= (\psi \otimes \psi \otimes \psi) \circ m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta(a)(k, 1, l) \\
&= (\psi \otimes \psi \otimes \psi)(\Delta \otimes \text{Id}) \circ \delta(a)(k, 1, l) \\
&= (\psi \otimes \psi) \circ \delta(a)(k+1, l).
\end{aligned}$$

Hence, for any $\delta \circ \psi = (\psi \otimes \psi) \circ \delta$. □

Let us apply this to **QSym**:

Proposition 8.5. *There exists a unique morphism $H : \mathbf{QSym} \rightarrow \mathbb{K}[X]$ such that:*

1. $H : (\mathbf{QSym}, \boxplus, \Delta) \rightarrow (\mathbb{K}[X], m, \Delta)$ is a bialgebra morphism.
2. $H : (\mathbf{QSym}, \boxplus, \delta) \rightarrow (\mathbb{K}[X], m, \delta)$ is a bialgebra morphism.

For any composition (a_1, \dots, a_n) :

$$H(a_1, \dots, a_n) = H_n(X) = \frac{X(X-1)\dots(X-n+1)}{n!}.$$

Proof. Let us first prove that $H : (\mathbf{QSym}, \Delta) \rightarrow (\mathbb{K}[X], \Delta)$ is a coalgebra morphism. For any composition (a_1, \dots, a_n) , for any $k, l \geq 0$:

$$\begin{aligned}
\Delta \circ H((a_1, \dots, a_n))(k, l) &= H(a_1, \dots, a_n)(k+l) \\
&= H_n(k+l) \\
&= \binom{k+l}{n}, \\
(H \otimes H) \circ \Delta((a_1, \dots, a_n))(k, l) &= \sum_{i=0}^n H_i(k)H_{n-i}(l) \\
&= \sum_{i=0}^n \binom{k}{i} \binom{l}{n-i} \\
&= \binom{k+l}{n},
\end{aligned}$$

so $\Delta \circ H = (H \otimes H) \circ \Delta$. For any composition (a_1, \dots, a_n) :

$$\varepsilon' \circ H((a_1, \dots, a_n)) = H_n(1) = \delta_{n,1} + \delta_{n,0} = \varepsilon'((a_1, \dots, a_n)),$$

so H is a morphism of bialgebras in cointeraction. □

Here is an application:

Proposition 8.6. *Let $q \in \mathbb{K}$. The following maps are bialgebra endomorphisms:*

$$\begin{aligned}
\mathcal{K}(\theta_q) : & \begin{cases} \mathbf{WQSym} & \longrightarrow & \mathbf{WQSym} \\ u & \longrightarrow & \sum_{\substack{f: \max(u) \rightarrow k, \\ \text{surjective, non decreasing}}} H_{\sharp f^{-1}(1)}(q) \dots H_{\sharp f^{-1}(\max(f))}(q) f \circ u, \end{cases} \\
\widehat{\mathcal{K}}(\theta_q) : & \begin{cases} \mathbf{QSym} & \longrightarrow & \mathbf{QSym} \\ (a_1, \dots, a_n) & \longrightarrow & \sum_{1 \leq i_1 < \dots < i_k < n} H_{i_1}(q) H_{i_2 - i_1}(q) \dots H_{n - i_k}(q) \\ & & (a_1 + \dots + a_{i_1}, \dots, a_{i_1 + \dots + i_k + 1} + \dots + a_n), \end{cases} \\
\Theta_q : & \begin{cases} \mathbb{K}[X] & \longrightarrow & \mathbb{K}[X] \\ P(X) & \longrightarrow & P(qX). \end{cases}
\end{aligned}$$

Moreover, the following diagram commutes:

$$\begin{array}{ccccc}
\mathbf{WQSym} & \twoheadrightarrow & \mathbf{QSym} & \xrightarrow{H} & \mathbb{K}[X] \\
\mathcal{K}(\theta_q) \downarrow & & \widehat{\mathcal{K}}(\theta_q) \downarrow & & \downarrow \Theta_q \\
\mathbf{WQSym} & \twoheadrightarrow & \mathbf{QSym} & \xrightarrow{H} & \mathbb{K}[X]
\end{array}$$

Proof. The formulas for $\mathcal{K}(\theta_q)$ and $\widehat{\mathcal{K}}(\theta_q)$ are directly obtained, and it is obvious that Θ_q is a Hopf algebra endomorphism. It remains to prove that $H \circ \widehat{\mathcal{K}}(\theta_q) = \Theta_q \circ H$. Let (a_1, \dots, a_n) be a composition.

$$\begin{aligned}
\varepsilon' \circ H \circ \widehat{\mathcal{K}}(\theta_q)(a_1, \dots, a_n) &= \varepsilon' \circ \widehat{\mathcal{K}}(\theta_q)(a_1, \dots, a_n) \\
&= H_n(q), \\
\varepsilon' \circ \Theta_q \circ H(a_1, \dots, a_n) &= \varepsilon' \circ \Theta_q(H_n(X)) \\
&= H_n(q),
\end{aligned}$$

so $\varepsilon' \circ H \circ \widehat{\mathcal{K}}(\theta_q) = \varepsilon' \circ \Theta_q \circ H$. By Proposition 8.4, $H \circ \widehat{\mathcal{K}}(\theta_q) = \Theta_q \circ H$. \square

Remark 8.3. If $q = -1$, $H_k(-1) = (-1)^k$ for any k , so:

$$\begin{aligned}
\mathcal{K}(\theta_{-1}) : & \begin{cases} \mathbf{WQSym} \longrightarrow \mathbf{WQSym} \\ u \longrightarrow (-1)^{\max(u)} \sum_{\substack{f: \max(u) \rightarrow k, \\ \text{surjective, non decreasing}}} f \circ u, \end{cases} \\
\widehat{\mathcal{K}}(\theta_{-1}) : & \begin{cases} \mathbf{QSym} \longrightarrow \mathbf{QSym} \\ (a_1, \dots, a_n) \longrightarrow (-1)^n \sum_{1 \leq i_1 < \dots < i_k < n} (a_1 + \dots + a_{i_1}, \dots, a_{i_1 + \dots + i_k + 1} + \dots + a_n). \end{cases}
\end{aligned}$$

Example 8.4. If $a, b, c \geq 1$, we put:

$$\begin{aligned}
(1^a) &= (\underbrace{1, \dots, 1}_a), \\
(1^a 2^b) &= (\underbrace{1, \dots, 1}_a, \underbrace{2, \dots, 2}_b), \\
(1^a 2^b 3^c) &= (\underbrace{1, \dots, 1}_a, \underbrace{2, \dots, 2}_b, \underbrace{3, \dots, 3}_c).
\end{aligned}$$

Then:

$$\begin{aligned}
\mathcal{K}(\theta_q)((1^a)) &= q(1^a), \\
\mathcal{K}(\theta_q)((1^a 2^b)) &= q^2(1^a 2^b) + \frac{q(q-1)}{2}(1^{a+b}), \\
\mathcal{K}(\theta_q)((1^a 2^b 3^c)) &= q^3(1^a 2^b 3^c) + \frac{q^2(q-1)}{2}(1^a 2^b 1^c) + \frac{q^2(q-1)}{2}(1^a 2^{b+c}) + \frac{q(q-1)(q-2)}{6}(1^{a+b+c}), \\
\widehat{\mathcal{K}}(\theta_q)((a)) &= q(a), \\
\widehat{\mathcal{K}}(\theta_q)((a, b)) &= q^2(a, b) + \frac{q(q-1)}{2}(a+b), \\
\widehat{\mathcal{K}}(\theta_q)((a, b, c)) &= q^3(a, b, c) + \frac{q^2(q-1)}{2}(a+b, c) + \frac{q^2(q-1)}{2}(a, b+c) + \frac{q(q-1)(q-2)}{6}(a+b+c).
\end{aligned}$$

8.4 Fock functors and characters

The monoids of characters of a twisted bialgebra and of its images by the Fock functor are related:

Proposition 8.7. *Let \mathcal{P} be a twisted bialgebra.*

1. *We denote by $\text{Char}(\mathcal{K}(\mathcal{P}))$ the monoid of characters of the bialgebra $\mathcal{K}(\mathcal{P})$. The following map is an injective map of monoids:*

$$\rho : \begin{cases} \text{Char}(\mathcal{P}) & \longrightarrow & \text{Char}(\mathcal{K}(\mathcal{P})) \\ f & \longrightarrow & \rho(f) = \bigoplus_{n \geq 0} f[\underline{n}]. \end{cases}$$

2. *ρ induces a monoid isomorphism between $\text{Char}(\mathcal{P})$ and the monoid of characters $\text{Char}(\widehat{\mathcal{K}}(\mathcal{P}))$ of the bialgebra $\widehat{\mathcal{K}}(\mathcal{P})$.*

Proof. 1. *First step.* Let $\sigma : A \longrightarrow B$ be a bijection between two finite sets. As f is a species morphism, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{P}[A] & \xrightarrow{\mathcal{P}[\sigma]} & \mathcal{P}[B] \\ f[A] \downarrow & & \downarrow f[B] \\ \mathbf{Com}[A] & \xrightarrow{\mathbf{Com}[\sigma]} & \mathbf{Com}[B] \end{array}$$

As $\mathbf{Com}[\sigma] = \text{Id}_{\mathbb{K}}$, $f[B] \circ \mathcal{P}[\sigma] = f[A]$.

Second step. Let us prove that ρ is well-defined. If $f \in \text{Char}(\mathcal{P})$, then if $x \in \mathcal{P}[\underline{m}]$ and $y \in \mathcal{P}[\underline{n}]$, by the first step:

$$\begin{aligned} \rho(f)(xy) &= f[\underline{m+n}] \circ \mathcal{P}[\sigma_{m,n}] \circ m(x \otimes y) \\ &= f[\underline{m} \sqcup \underline{n}] \circ m(x \otimes y) \\ &= f[\underline{m}](x) f[\underline{n}](y) \\ &= \rho(f)(x) \rho(f)(y). \end{aligned}$$

So $\rho(f)$ is indeed a character of $\mathcal{K}(\mathcal{P})$.

Let us now prove that ρ is a monoid morphism. Let $f, g \in \text{Char}(\mathcal{P})$. For any $x \in \mathcal{P}[\underline{n}]$, by the first step:

$$\begin{aligned} \rho(f) * \rho(g)(x) &= \sum_{I \subseteq \underline{n}} (f[\underline{\#}I] \otimes g[\underline{n} - \underline{\#}I]) \circ (\mathcal{P}[\sigma_I] \otimes \mathcal{P}[\sigma_{\underline{n} \setminus I}]) \circ \Delta_{I, \underline{n} \setminus I}(x) \\ &= \sum_{I \subseteq \underline{n}} (f[I] \otimes g[\underline{n} \setminus I]) \circ \Delta_{I, \underline{n} \setminus I}(x) \\ &= (f * g)[\underline{n}](x) \\ &= \rho(f * g)(x). \end{aligned}$$

So $\rho(f) * \rho(g) = \rho(f * g)$.

Let us now prove that ρ is injective. If $\rho(f) = \rho(g)$, then for any $n \geq 0$, $f[\underline{n}] = g[\underline{n}]$. For any finite set A , let us denote by n the cardinality of A and let us choose $\sigma : A \longrightarrow \underline{n}$ be a bijection. By the first step:

$$f[A] = f[\underline{n}] \circ \mathcal{P}[\sigma] = g[\underline{n}] \circ \mathcal{P}[\sigma] = g[A].$$

Hence, $f = g$.

2. Let $f \in \text{Char}(\mathcal{P})$. For any $x \in \mathcal{P}[\underline{n}]$, $\sigma \in \mathfrak{S}_n$:

$$\begin{aligned}\rho(f)(x - \mathcal{P}[\sigma](x)) &= f[\underline{n}](x) - f[\underline{n}](\mathcal{P}[\sigma](x)) \\ &= f[\underline{n}](x) - \mathcal{P}[\sigma](f[\underline{n}](x)) \\ &= f[\underline{n}](x) - f[\underline{n}](x) \\ &= 0.\end{aligned}$$

Hence, $\rho(f)$ induces a character $\bar{\rho}(f)$ of $\widehat{\mathcal{K}}(\mathcal{P})$ defined by:

$$\forall x \in \mathcal{P}[\underline{n}], \quad \bar{\rho}(f)(\bar{x}) = \rho(f)(x) = f[\underline{n}](x).$$

This defines a monoid morphism $\bar{\rho}$ from $\text{Char}(\mathcal{P})$ to $\text{Char}(\widehat{\mathcal{K}}(\mathcal{P}))$. It is injective: indeed, if $\bar{\rho}(f) = \bar{\rho}(g)$, then $\rho(f) = \rho(g)$, so $f = g$. Let us prove it is surjective. Let us consider $g \in \text{Char}(\widehat{\mathcal{K}}(\mathcal{P}))$. Let A be a finite set. We define $f[A] : \mathcal{P}[A] \rightarrow \mathbb{K}$ by:

$$\forall x \in \mathcal{P}[A], \quad f[A](x) = g(\overline{\mathcal{P}[\sigma](x)}),$$

where $\sigma : A \rightarrow \underline{n}$ is any bijection. This does not depend on the choice of σ : if τ is another bijection, then, as $\sigma \circ \tau^{-1} \in \mathfrak{S}_n$:

$$g(\overline{\mathcal{P}[\tau](x)}) = g(\overline{\mathcal{P}[\sigma \circ \tau^{-1]} \circ \mathcal{P}[\tau](x)}) = g(\overline{\mathcal{P}[\sigma](x)}).$$

Hence, $f[A]$ is well-defined. If $\tau : A \rightarrow B$ is a bijection, for any $x \in \mathcal{P}[A]$, choosing a bijection $\sigma : A \rightarrow \underline{n}$:

$$\begin{aligned}f[B](\mathcal{P}[\tau](x)) &= g(\overline{\mathcal{P}[\sigma \circ \tau^{-1]} \circ \mathcal{P}[\tau](x)}) \\ &= g(\overline{\mathcal{P}[\sigma](x)}) \\ &= \mathbf{Com}[\tau] \circ f[A](x).\end{aligned}$$

So f is a species morphism. Let $x \in \mathcal{P}[A]$, $y \in \mathcal{P}[B]$. Choosing bijections $\sigma : A \rightarrow \underline{m}$ and $\tau : B \rightarrow \underline{n}$:

$$\begin{aligned}f(xy) &= g(\overline{\mathcal{P}[\sigma \sqcup \tau](xy)}) \\ &= g(\overline{\mathcal{P}[\sigma](x) \mathcal{P}[\tau](y)}) \\ &= g(\overline{\mathcal{P}[\sigma](x)})g(\overline{\mathcal{P}[\tau](y)}) \\ &= f(x)f(y).\end{aligned}$$

So $f \in \text{Char}(\mathcal{P})$. If $x \in \mathcal{P}[\underline{n}]$:

$$\bar{\rho}(f)(\bar{x}) = f[\underline{n}](x) = g(\overline{\mathcal{P}[\text{Id}_{\underline{n}}](x)}) = g(\bar{x}).$$

Therefore, $\bar{\rho}(f) = g$. □

Remark 8.4. It may happen that ρ is not surjective. Let us consider for example the twisted algebra of posets \mathcal{Pos} . We define a character g on $\mathcal{K}(\mathcal{Pos})$ by the following: if $P = (\underline{n}, \leq_P)$ is a poset,

$$g(P) = \begin{cases} 1 & \text{if } \forall i, j \in \underline{n}, i \leq_P j \implies i \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

Let us assume that there exists a character $f \in \text{Char}(\mathcal{Pos})$ such that $\rho(f) = g$. Then $f[\underline{2}](\uparrow_1^2) = 1$ and $f[\underline{2}](\uparrow_2^1) = 0$. Moreover, by the first step of the preceding proof, for $\sigma = (12)$:

$$f[\underline{2}](\uparrow_2^1) = f[\underline{2}] \circ \mathcal{Pos}[\sigma](\uparrow_1^2) = f[\underline{2}](\uparrow_1^2).$$

This is a contradiction. So ρ is not surjective.

8.5 Hopf algebraic structure from the first Fock functor

Theorem 8.8. 1. Let \mathcal{P} be a connected twisted bialgebra. We define another coproduct \blacktriangle on $\mathcal{K}(\mathcal{P})$ by the following:

$$\forall x \in \mathcal{P}[\underline{n}], \quad \blacktriangle(x) = \sum_{k=0}^n (\mathcal{P}[\sigma_{\underline{k}}] \otimes \mathcal{P}[\sigma_{\underline{n} \setminus \underline{k}}]) \circ \Delta_{\underline{k}, \underline{n} \setminus \underline{k}}(x).$$

Then $(\mathcal{K}(\mathcal{P}), m, \blacktriangle)$ is an infinitesimal bialgebra, in the sense of Loday and Ronco [21].

2. If \mathcal{P} is a connected twisted bialgebra, then $\mathcal{K}(\mathcal{P})$ is freely generated as an algebra by the space of primitive elements of \blacktriangle .

3. If \mathcal{P} is a cofree twisted coalgebra, then $\mathcal{K}(\mathcal{P})$ is a cofree coalgebra.

Proof. Let $x \in \mathcal{P}[\underline{n}]$. Then:

$$(\blacktriangle \otimes \text{Id}) \circ \blacktriangle(x) = (\text{Id} \otimes \blacktriangle) \circ \blacktriangle(x) = \sum_{0 \leq k \leq l \leq n} (\mathcal{P}[\sigma_{\underline{k}}] \otimes \mathcal{P}[\sigma_{\underline{l} \setminus \underline{k}}] \otimes \mathcal{P}[\sigma_{\underline{n} \setminus \underline{l}}]) \circ \Delta_{\underline{k}, \underline{l} \setminus \underline{k}, \underline{n} \setminus \underline{l}}(x).$$

So \blacktriangle is coassociative. Let $x \in \mathcal{P}[\underline{m}]$ and $y \in \mathcal{P}[\underline{n}]$.

$$\begin{aligned} \blacktriangle(xy) &= \sum_{0 \leq k \leq m+n} (\mathcal{P}[\sigma_{\underline{k}}] \otimes \mathcal{P}[\sigma_{\underline{m+n} \setminus \underline{k}}]) \circ \Delta_{\underline{k}, \underline{m+n} \setminus \underline{k}} \circ \mathcal{P}[\sigma_{\underline{m}, \underline{n}}] \circ m(x \otimes y) \\ &= \sum_{0 \leq k \leq m} (\mathcal{P}[\sigma_{\underline{k}}] \otimes \mathcal{P}[\sigma_{\underline{m+n} \setminus \underline{k}}]) \circ \Delta_{\underline{k}, \underline{m+n} \setminus \underline{k}} \circ \mathcal{P}[\sigma_{\underline{m}, \underline{n}}] \circ m(x \otimes y) \\ &\quad + \sum_{m \leq k \leq m+n} (\mathcal{P}[\sigma_{\underline{k}}] \otimes \mathcal{P}[\sigma_{\underline{m+n} \setminus \underline{k}}]) \circ \Delta_{\underline{k}, \underline{m+n} \setminus \underline{k}} \circ \mathcal{P}[\sigma_{\underline{m}, \underline{n}}] \circ m(x \otimes y) \\ &\quad - (\mathcal{P}[\sigma_{\underline{m}}] \otimes \mathcal{P}[\sigma_{\underline{m+n} \setminus \underline{m}}]) \circ \Delta_{\underline{m}, \underline{m+n} \setminus \underline{m}} \circ \mathcal{P}[\sigma_{\underline{m}, \underline{n}}] \circ m(x \otimes y) \\ &= \sum_{0 \leq k \leq m} (\mathcal{P}[\sigma_{\underline{k}}] \otimes \mathcal{P}[\sigma_{\underline{m+n} \setminus \underline{k}}]) \circ (\Delta_{\underline{k}, \underline{m} \setminus \underline{k}}(x) \Delta_{\emptyset, \underline{n}}(y)) \\ &\quad + \sum_{m \leq k \leq m+n} (\mathcal{P}[\sigma_{\underline{k}}] \otimes \mathcal{P}[\sigma_{\underline{m+n} \setminus \underline{k}}]) (\Delta_{\underline{m}, \emptyset}(x) \Delta_{\underline{k-m}, \underline{m+n-k} \setminus \underline{k-m}}(y)) \\ &\quad - (\mathcal{P}[\sigma_{\underline{m}}] \otimes \mathcal{P}[\sigma_{\underline{m+n} \setminus \underline{m}}]) (\Delta_{\underline{m}, \emptyset}(x) \Delta_{\emptyset, \underline{n}}(y)) \\ &= \sum_{0 \leq k \leq m} (\mathcal{P}[\sigma_{\underline{k}}] \otimes \mathcal{P}[\sigma_{\underline{m+n} \setminus \underline{k}}]) (\Delta_{\underline{k}, \underline{m} \setminus \underline{k}}(x) 1 \otimes y) \\ &\quad + \sum_{m \leq k \leq m+n} (\mathcal{P}[\sigma_{\underline{k}}] \otimes \mathcal{P}[\sigma_{\underline{m+n} \setminus \underline{k}}]) (x \otimes 1 \Delta_{\underline{k-m}, \underline{m+n-k} \setminus \underline{k-m}}(y)) \\ &\quad - (\mathcal{P}[\sigma_{\underline{m}}] \otimes \mathcal{P}[\sigma_{\underline{m+n} \setminus \underline{m}}]) (x \otimes y) \\ &= \blacktriangle(x)(1 \otimes y) + (x \otimes 1)\blacktriangle(y) - x \otimes y. \end{aligned}$$

Hence, $\mathcal{K}(\mathcal{P})$ is an infinitesimal bialgebra.

2. This is a direct consequence of the first point, by Loday and Ronco's rigidity theorem [21].

3. We assume that $\mathcal{P} = \text{co}\mathcal{T}(\mathcal{Q})$. We define another product $*$ on $\mathcal{K}(\mathcal{P})$ by:

$$\forall x = q_1 \dots q_k, y = q_{k+1} \dots q_{k+l} \in \mathcal{P}, \quad x * y = p_1 \dots p_k q_{k+1}^+ \dots q_{k+l}^+,$$

where we use the following notation: if $q_i \in \mathcal{Q}[I]$, then $q_i^+ = \mathcal{Q}[(\sigma_{m,n})_{|I}](q_i)$. This product is

associative. Moreover, by definition of the coproduct of $\text{co}\mathcal{T}(\mathcal{Q})$:

$$\begin{aligned}\Delta(x * y) &= \sum_{i=0}^k \mathcal{P}[\sigma'_i](p_1 \dots p_i) \otimes \mathcal{P}[\sigma''_i](p_{i+1} \dots p_k q_1^+ \dots q_i^+) \\ &\quad + \sum_{i=k}^l \mathcal{P}[\sigma'_i](p_1 \dots p_k q_1^+ \dots q_i^+) \otimes \mathcal{P}[\sigma''_i](q_{i+1}^+ \dots q_l^+) \\ &\quad - q_1 \dots q_k \otimes q_{k+1} \dots q_{k+l} \\ &= \Delta(x) * (1 \otimes y) + (x \otimes 1) * \Delta(y) - x \otimes y,\end{aligned}$$

where σ'_i, σ''_i are suitable bijections. Hence, $(\mathcal{K}(\mathcal{P}), *, \Delta)$ is an infinitesimal bialgebra. By Loday and Ronco's rigidity theorem, $(\mathcal{K}(\mathcal{P}), \Delta)$ is cofree. \square

Remark 8.5. We obtain again the well-known result that **FQSym** and **WQSym** are both free and cofree bialgebras.

9 Applications

9.1 Hopf algebras and bialgebras of graphs

We put $\mathbf{H}_{\mathcal{G}r'}$ and $H_{\mathcal{G}r'} = \widehat{\mathcal{K}}(\mathcal{G}r')$. The Hopf algebra $\mathbf{H}_{\mathcal{G}r'}$ has for basis the set of graphs whose vertices are a set composition of \underline{n} , for a given n . Its product is given by shifted concatenation, for example:

$$\begin{matrix} \{3\} \\ \bullet \\ \{1, 2\} \end{matrix} \bullet \begin{matrix} \{4\} \\ \bullet \\ \{1, 3\} \end{matrix} = \begin{matrix} \{3\} \\ \bullet \\ \{1, 2\} \end{matrix} \bullet \begin{matrix} \{4\} \\ \bullet \\ \{5, 7\} \end{matrix}.$$

Its coproduct is given by decompositions of the set of vertices into two parts with standardization, for example:

$$\begin{aligned}\Delta(\bullet_{\{1\}}) &= \bullet_{\{1\}} \otimes 1 + 1 \otimes \bullet_{\{1\}}, \\ \Delta(\bullet_{\{1\}^{\{2, 3\}}}) &= \bullet_{\{1\}^{\{2, 3\}}} \otimes 1 + \bullet_{\{1\}} \otimes \bullet_{\{1, 2\}} + \bullet_{\{1, 2\}} \otimes \bullet_{\{1\}} + 1 \otimes \bullet_{\{1\}^{\{2, 3\}}}, \\ \Delta(\begin{matrix} \{2\} \\ \bullet \\ \{1, 3\} \end{matrix} \bullet \begin{matrix} \{4\} \\ \bullet \\ \{1, 3\} \end{matrix}) &= \begin{matrix} \{2\} \\ \bullet \\ \{1, 3\} \end{matrix} \bullet \begin{matrix} \{4\} \\ \bullet \\ \{1, 3\} \end{matrix} \otimes 1 + \begin{matrix} \{2\} \\ \bullet \\ \{1, 3\} \end{matrix} \otimes \bullet_{\{1\}} + \begin{matrix} \{3\} \\ \bullet \\ \{1, 2\} \end{matrix} \otimes \bullet_{\{1\}} + \bullet_{\{1\}} \bullet_{\{2\}} \otimes \bullet_{\{1, 2\}} \\ &\quad + \bullet_{\{1\}} \otimes \begin{matrix} \{2\} \\ \bullet \\ \{1, 3\} \end{matrix} + \bullet_{\{1\}} \otimes \begin{matrix} \{3\} \\ \bullet \\ \{1, 2\} \end{matrix} + \bullet_{\{1, 2\}} \otimes \bullet_{\{1\}} \bullet_{\{2\}} + 1 \otimes \begin{matrix} \{2\} \\ \bullet \\ \{1, 3\} \end{matrix} \bullet \end{aligned}$$

The second coproduct is given by a contraction-extraction process; for example:

$$\begin{aligned}\delta(\bullet_A) &= \bullet_A \otimes \bullet_A, \\ \delta(\bullet_A^B) &= \bullet_A^B \otimes \bullet_{A \bullet B} + \bullet_{A \sqcup B} \otimes \bullet_A^B, \\ \delta(\begin{matrix} B \\ \bullet \\ A \end{matrix} \bullet \begin{matrix} C \\ \bullet \\ A \end{matrix}) &= \begin{matrix} B \\ \bullet \\ A \end{matrix} \bullet \begin{matrix} C \\ \bullet \\ A \end{matrix} \otimes \bullet_{A \bullet B \bullet C} + \begin{matrix} C \\ \bullet \\ A \sqcup B \end{matrix} \otimes \begin{matrix} B \\ \bullet \\ A \end{matrix} \bullet \begin{matrix} C \\ \bullet \\ A \end{matrix} + \begin{matrix} B \\ \bullet \\ A \sqcup C \end{matrix} \otimes \begin{matrix} C \\ \bullet \\ A \end{matrix} \bullet \begin{matrix} B \\ \bullet \\ A \end{matrix} + \bullet_{A \sqcup B \sqcup C} \otimes \begin{matrix} B \\ \bullet \\ A \end{matrix} \bullet \begin{matrix} C \\ \bullet \\ A \end{matrix},\end{aligned}$$

where in the first computation, $(A) = \underline{n}$; in the second one, $(A, B) \in \mathbf{Comp}[\underline{n}]$ and, in the last one, $(A, B, C) \in \mathbf{Comp}[\underline{n}]$. This is a variant of the noncommutative Hopf algebra of graphs of [12].

The Hopf algebra $H_{\mathcal{G}r'}$ has for basis the set of decorated graphs, that is to say graphs with a map from their set of vertices to the set of positive integers. The product is given by concatenation, for example, if a, b, c, d and $e \geq 1$:

$$\begin{matrix} c \\ \bullet \\ a \end{matrix} \bullet \begin{matrix} b \\ \bullet \\ d \end{matrix} \bullet \begin{matrix} e \\ \bullet \\ d \end{matrix} = \begin{matrix} b \\ \bullet \\ a \end{matrix} \bullet \begin{matrix} c \\ \bullet \\ d \end{matrix} \bullet \begin{matrix} e \\ \bullet \\ d \end{matrix}.$$

The first product is given by partitions of the vertices into two subsets and the second one by an extraction-contraction process. For example, if $a, b, c \geq 1$:

$$\begin{aligned}\Delta(\bullet_a) &= \bullet_a \otimes 1 + 1 \otimes \bullet_a, \\ \Delta(\mathfrak{!}_a^b) &= \mathfrak{!}_a^b \otimes 1 + \bullet_a \otimes \bullet_b + \bullet_b \otimes \bullet_a + 1 \otimes \mathfrak{!}_a^b, \\ \Delta({}^b\mathfrak{V}_a^c) &= {}^b\mathfrak{V}_a^c \otimes 1 + \mathfrak{!}_a^b \otimes \bullet_c + \mathfrak{!}_a^c \otimes \bullet_b + \bullet_b \bullet_c \otimes \bullet_a \\ &\quad + \bullet_c \otimes \mathfrak{!}_a^b + \bullet_b \otimes \mathfrak{!}_a^c + \bullet_a \otimes \bullet_b \bullet_c + 1 \otimes {}^b\mathfrak{V}_a^c;\end{aligned}$$

$$\begin{aligned}\delta(\bullet_a) &= \bullet_a \otimes \bullet_a, \\ \delta(\mathfrak{!}_a^b) &= \mathfrak{!}_a^b \otimes \bullet_a \bullet_b + \bullet_{a+b} \otimes \mathfrak{!}_a^b, \\ \delta({}^b\mathfrak{V}_a^c) &= {}^b\mathfrak{V}_a^c \otimes \bullet_a \bullet_b \bullet_c + \mathfrak{!}_{a+b}^c \otimes \mathfrak{!}_a^b \bullet_c + \mathfrak{!}_{a+c}^b \otimes \mathfrak{!}_a^c \bullet_b + \bullet_{a+b+c} \otimes {}^b\mathfrak{V}_a^c.\end{aligned}$$

This is a decorated variant of the commutative Hopf algebra of graphs of [12, 18].

9.2 Morphisms from graphs to \mathbf{WQSym} , \mathbf{QSym} and $\mathbb{K}[X]$

The Fock functors applied to the morphism ϕ_{chr_q} gives a commutative diagram of morphisms compatible with the product and both coproducts:

$$\begin{array}{ccccc}\mathbf{H}_{\mathcal{G}_{r'}} & \longrightarrow & H_{\mathcal{G}_{r'}} & & \\ \mathcal{K}(\phi_{chr_q}) \downarrow & & \hat{\mathcal{K}}(\phi_{chr_q}) \downarrow & \searrow P_{chr_q} & \\ \mathbf{WQSym} & \longrightarrow & \mathbf{QSym} & \xrightarrow{H} & \mathbb{K}[X]\end{array}$$

where $P_{chr_q} = H \circ \hat{\mathcal{K}}(\phi_{chr_q})$. For any graph in $\mathbf{H}_{\mathcal{G}_{r'}}$ of degree n :

$$\begin{aligned}\mathcal{K}(\phi_{chr_1})(G) &= \sum_{c \in VC(G)} c(1) \dots c(n), \\ \hat{\mathcal{K}}(\phi_{chr_1})(G) &= \sum_{c \in VC(G)} (\#c^{-1}(1), \dots, \#c^{-1}(\max(c))).\end{aligned}$$

For example:

$$\begin{aligned}\mathcal{K}(\phi_{chr_1})(\bullet_{\{1\}}) &= (1), \\ \mathcal{K}(\phi_{chr_1})(\mathfrak{!}_{\{1,3\}}^2) &= (121) + (212), \\ \mathcal{K}(\phi_{chr_1})({}^{\{3\}}\mathfrak{V}_{\{1,4\}}^{\{2\}}) &= (1231) + (1321) + (2132) + (2312) + (3123) + (3213) + (1221) + (2112).\end{aligned}$$

This is a colored version of the noncommutative chromatic symmetric function of [16]. The morphism $\hat{\mathcal{K}}(\phi_{chr_1})$ is a colored version of Stanley's chromatic symmetric series [30, 29], as it is explained in [12]. For example, if $a, b, c \geq 1$:

$$\begin{aligned}\hat{\mathcal{K}}(\phi_{chr_1})(\bullet_a) &= (a), \\ \hat{\mathcal{K}}(\phi_{chr_1})(\mathfrak{!}_a^b) &= (a, b) + (b, a), \\ \hat{\mathcal{K}}(\phi_{chr_1})({}^b\mathfrak{V}_a^c) &= (a, b, c) + (a, c, b) + (b, a, c) + (b, c, a) + (c, a, b) + (c, b, a) + (a, b + c) + (b + c, a).\end{aligned}$$

Reformulating in terms of formal series in a infinite set $\{X_1, X_2, \dots\}$ of indeterminates, with the help of the polynomial realization of \mathbf{QSym} :

$$\begin{aligned}\hat{\mathcal{K}}(\phi_{chr_1})(\bullet_a) &= \sum_i X_i^a, \\ \hat{\mathcal{K}}(\phi_{chr_1})(\mathfrak{!}_a^b) &= \sum_{i \neq j} X_i^a X_j^b, \\ \hat{\mathcal{K}}(\phi_{chr_1})({}^b\mathfrak{V}_a^c) &= \sum_{i \neq j, i \neq k, j \neq k} X_i^a X_j^b X_k^c + \sum_{i \neq j} X_i^a X_j^{b+c}.\end{aligned}$$

Composing $\widehat{\mathcal{K}}(\phi_{chr_1})$ with H , we obtain the chromatic polynomial P_{chr_1} [7]. For example:

$$\begin{aligned} P_{chr_1}(\bullet_a) &= H_1(X) = X, \\ P_{chr_1}(\downarrow_a^b) &= 2H_2(X) = X(X-1) \\ P_{chr_1}({}^b\downarrow_a^c) &= 6H_3(X) + 2H_2(X) = X(X-1)^2. \end{aligned}$$

Proposition 9.1. *If $q \neq 0$, for any graph $G \in H_{\mathcal{G}r'}$:*

$$P_{chr_q}(G)(X) = q^{\deg(G)} P_{chr_1}(G) \left(\frac{X}{q} \right).$$

Proof. Indeed:

$$\begin{aligned} P_{chr_q}(G)(X) &= H \circ \widehat{\mathcal{K}}(\theta_{q^{-1}}) \circ \widehat{\mathcal{K}}(\phi_{chr_1}) \circ \widehat{\mathcal{K}}(\iota_q)(G)(X) \\ &= q^{\deg(G)} H \circ \widehat{\mathcal{K}}(\theta_{q^{-1}}) \circ \widehat{\mathcal{K}}(\phi_{chr_1})(G)(X) \\ &= q^{\deg(G)} \Theta_{q^{-1}} \circ H \circ \widehat{\mathcal{K}}(\phi_{chr_1})(G)(X) \\ &= q^{\deg(G)} \Theta_{q^{-1}} \circ P_{chr_1}(G)(X) \\ &= q^{\deg(G)} P_{chr_1}(G) \left(\frac{X}{q} \right). \end{aligned}$$

□

Proposition 9.2. *Let $q \in \mathbb{K}$.*

1. *For any graph $G \in \mathcal{G}r'[\underline{n}]$:*

$$\mathcal{K}(\phi_q)(G) = q^{\deg(G)} \sum_{w \in W(G)} w,$$

where w is the set of packed words of length n such that for any $i \in \underline{n}$, if i and j are in the same vertex of G , then $w(i) = w(j)$.

2. *For any decorated graph G , denoting by d_1, \dots, d_k the decoration of its vertices:*

$$\widehat{\mathcal{K}}(\phi_q)(G) = q^{\deg(G)} (d_1) \boxplus \dots \boxplus (d_k).$$

3. *For any decorated graph G :*

$$H \circ \widehat{\mathcal{K}}(\phi_q)(G) = q^{\deg(G)} X^{\deg(G)}.$$

Proof. The two first points are immediate. If G is a decorated graph:

$$\begin{aligned} H \circ \widehat{\mathcal{K}}(\phi_q)(G) &= q^{\deg(G)} H((d_1) \boxplus \dots \boxplus (d_k)) \\ &= q^{\deg(G)} H((a_1)) \dots H((a_k)) \\ &= q^{\deg(G)} X^k \\ &= q^{\deg(G)} X^{\deg(G)}. \end{aligned}$$

□

As $\phi_{chr_1} = \phi_1 \leftarrow \lambda_{chr_1}$:

Corollary 9.3. *For any graph $G \in H_{\mathcal{G}r'}$:*

$$\begin{aligned} \widehat{\mathcal{K}}(\phi_{chr_1})(G) &= \sum_{\sim \triangleleft G} \lambda_{chr_1}(G | \sim) \prod_{C \text{ equivalence classe of } \sim}^{\boxplus} (C), \\ P_{chr_1}(G) &= \sum_{\sim \triangleleft G} \lambda_{chr_1}(G | \sim) X^{\text{cl}(\sim)}. \end{aligned}$$

Applying the Fock functor to Γ :

Proposition 9.4. *The following maps are Hopf algebra automorphisms:*

$$\mathcal{K}(\Gamma) : \begin{cases} \mathbf{H}_{G_{r'}} & \longrightarrow \mathbf{H}_{G_{r'}} \\ G & \longrightarrow \sum_{\sim \triangleleft G} \#\{\text{acyclic orientations of } G \mid \sim\} G / \sim, \end{cases}$$

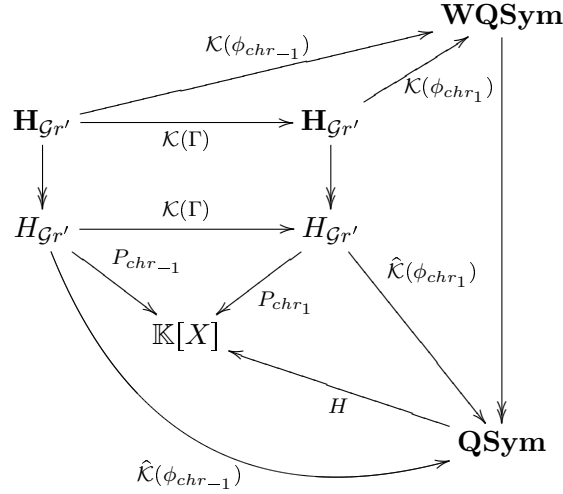
$$\widehat{\mathcal{K}}(\Gamma) : \begin{cases} H_{G_{r'}} & \longrightarrow H_{G_{r'}} \\ G & \longrightarrow \sum_{\sim \triangleleft G} \#\{\text{acyclic orientations of } G \mid \sim\} G / \sim. \end{cases}$$

Their inverse are given by:

$$\mathcal{K}(\Gamma)^{-1} : \begin{cases} \mathbf{H}_{G_{r'}} & \longrightarrow \mathbf{H}_{G_{r'}} \\ G & \longrightarrow \sum_{\sim \triangleleft G} (-1)^{\deg(G) + \text{cl}(\sim)} \#\{\text{acyclic orientations of } G \mid \sim\} G / \sim, \end{cases}$$

$$\widehat{\mathcal{K}}(\Gamma)^{-1} : \begin{cases} H_{G_{r'}} & \longrightarrow H_{G_{r'}} \\ G & \longrightarrow \sum_{\sim \triangleleft G} (-1)^{\deg(G) + \text{cl}(\sim)} \#\{\text{acyclic orientations of } G \mid \sim\} G / \sim. \end{cases}$$

Moreover, the following diagram commutes:



9.3 Hopf algebras and bialgebras on finite topologies

We put $\mathbf{H}_{\mathcal{T}op} = \mathcal{K}(\mathcal{T}op)$ and $H_{\mathcal{T}op} = \widehat{\mathcal{K}}(\mathcal{T}op)$. The Hopf algebra $\mathbf{H}_{\mathcal{T}op}$ has for basis the set of finite topologies on a set \underline{n} . Its product is given by shifted concatenation. For example:

$$\begin{matrix} \{3\} \\ \mathbf{V}_{\{1,2\}}^{\{4\}} \end{matrix} \cdot \begin{matrix} \{2\} \\ \mathbf{V}_{\{1,3\}}^{\{3\}} \end{matrix} = \begin{matrix} \{3\} \\ \mathbf{V}_{\{1,2\}}^{\{4\}} \end{matrix} \begin{matrix} \{6\} \\ \mathbf{V}_{\{5,7\}}^{\{7\}} \end{matrix}.$$

Its coproduct is given by partitions of the underlying sets in an open and a closed set, for example:

$$\Delta(\bullet_{\{1\}}) = \bullet_{\{1\}} \otimes 1 + 1 \otimes \bullet_{\{1\}},$$

$$\Delta(\begin{matrix} \{2\} \\ \mathbf{V}_{\{1\}}^{\{3\}} \end{matrix}) = \begin{matrix} \{2\} \\ \mathbf{V}_{\{1\}}^{\{3\}} \end{matrix} \otimes 1 + \bullet_{\{1\}} \otimes \bullet_{\{1,2\}} + 1 \otimes \begin{matrix} \{2,3\} \\ \mathbf{V}_{\{1\}}^{\{3\}} \end{matrix},$$

$$\Delta(\begin{matrix} \{2\} \\ \mathbf{V}_{\{1,3\}}^{\{4\}} \end{matrix}) = \begin{matrix} \{2\} \\ \mathbf{V}_{\{1,3\}}^{\{4\}} \end{matrix} \otimes 1 + \begin{matrix} \{2\} \\ \mathbf{V}_{\{1,3\}}^{\{4\}} \end{matrix} \otimes \bullet_{\{1\}} + \begin{matrix} \{3\} \\ \mathbf{V}_{\{1,2\}}^{\{4\}} \end{matrix} \otimes \bullet_{\{1\}} + \bullet_{\{1,2\}} \otimes \bullet_{\{1\} \bullet \{2\}} + 1 \otimes \begin{matrix} \{2\} \\ \mathbf{V}_{\{1,3\}}^{\{4\}} \end{matrix},$$

$$\Delta(\begin{matrix} \{2\} \\ \mathbf{V}_{\{1,3\}}^{\{4\}} \end{matrix}) = \begin{matrix} \{2\} \\ \mathbf{V}_{\{1,3\}}^{\{4\}} \end{matrix} \otimes 1 + \begin{matrix} \{3\} \\ \mathbf{V}_{\{1,2\}}^{\{4\}} \end{matrix} \otimes \bullet_{\{1\}} + \bullet_{\{1,2\}} \otimes \begin{matrix} \{1\} \\ \mathbf{V}_{\{2\}}^{\{2\}} \end{matrix} + 1 \otimes \begin{matrix} \{2\} \\ \mathbf{V}_{\{1,3\}}^{\{4\}} \end{matrix}.$$

The second coproduct is given by a contraction-extraction process; for example:

$$\delta(\bullet_A) = \bullet_A \otimes \bullet_A,$$

$$\delta(\begin{matrix} B \\ \mathbf{V}_A^B \end{matrix}) = \begin{matrix} B \\ \mathbf{V}_A^B \end{matrix} \otimes \bullet_{A \bullet B} + \bullet_{A \sqcup B} \otimes \begin{matrix} B \\ \mathbf{V}_A^B \end{matrix},$$

$$\delta(\begin{matrix} B \\ \mathbf{V}_A^C \end{matrix}) = \begin{matrix} B \\ \mathbf{V}_A^C \end{matrix} \otimes \bullet_{A \bullet B \bullet C} + \begin{matrix} C \\ \mathbf{V}_{A \sqcup B}^C \end{matrix} \otimes \begin{matrix} B \\ \mathbf{V}_A^C \end{matrix} + \begin{matrix} B \\ \mathbf{V}_{A \sqcup C}^C \end{matrix} \otimes \begin{matrix} C \\ \mathbf{V}_A^C \end{matrix} + \bullet_{A \sqcup B \sqcup C} \otimes \begin{matrix} B \\ \mathbf{V}_A^C \end{matrix},$$

$$\delta(\begin{matrix} C \\ \mathbf{V}_A^B \end{matrix}) = \begin{matrix} C \\ \mathbf{V}_A^B \end{matrix} \otimes \bullet_{A \bullet B \bullet C} + \begin{matrix} C \\ \mathbf{V}_{A \sqcup B}^C \end{matrix} \otimes \begin{matrix} B \\ \mathbf{V}_A^C \end{matrix} + \begin{matrix} B \\ \mathbf{V}_{A \sqcup C}^C \end{matrix} \otimes \bullet_{A \bullet B} + \bullet_{A \sqcup B \sqcup C} \otimes \begin{matrix} C \\ \mathbf{V}_A^B \end{matrix}.$$

where in the first computation, $A = \underline{n}$; in the second one, $(A, B) \in \mathbf{Comp}[\underline{n}]$ and, in the last one, $(A, B, C) \in \mathbf{Comp}[\underline{n}]$. This is the Hopf algebra of finite topologies of [10, 14, 15, 13].

The Hopf algebra $H_{\mathcal{T}op}$ has for basis the set of finite topologies, up to homeomorphism. We represent such objects by Hasse graphs of the underlying quasi-order, where the cardinality of the equivalence classes of the quasi-order are represented on the graph. The product is given by disjoint union, for example, if a, b, c, d and $e \geq 1$:

$${}^c \mathbf{V}_a^b \cdot \mathfrak{!}_d^e = {}^b \mathbf{V}_a^c \mathfrak{!}_d^e.$$

The first product is given by partitions into a closed set and an open set, and the second one by an extraction-contraction process. For example, if $a, b, c \geq 1$:

$$\begin{aligned} \Delta(\cdot_a) &= \cdot_a \otimes 1 + 1 \otimes \cdot_a, \\ \Delta(\mathfrak{!}_a^b) &= \mathfrak{!}_a^b \otimes 1 + \cdot_a \otimes \cdot_b + 1 \otimes \mathfrak{!}_a^b, \\ \Delta({}^b \mathbf{V}_a^c) &= {}^b \mathbf{V}_a^c \otimes 1 + \mathfrak{!}_a^c \otimes \cdot_c + \mathfrak{!}_a^c \otimes \cdot_b + \cdot_c \otimes \mathfrak{!}_a^b + 1 \otimes {}^b \mathbf{V}_a^c, \\ \Delta(\mathfrak{!}_a^c) &= \mathfrak{!}_a^c \otimes 1 + \mathfrak{!}_a^b \otimes \cdot_c + \cdot_a \otimes \mathfrak{!}_b^c + 1 \otimes \mathfrak{!}_a^c; \\ \\ \delta(\cdot_a) &= \cdot_a \otimes \cdot_a, \\ \delta(\mathfrak{!}_a^b) &= \mathfrak{!}_a^b \otimes \cdot_a \cdot_b + \cdot_a \cdot_b \otimes \mathfrak{!}_a^b, \\ \delta({}^b \mathbf{V}_a^c) &= {}^b \mathbf{V}_a^c \otimes \cdot_a \cdot_b \cdot_c + \mathfrak{!}_{a+b}^c \otimes \mathfrak{!}_a^b \cdot_c + \mathfrak{!}_{a+c}^b \otimes \mathfrak{!}_a^c \cdot_b + \cdot_a \cdot_b \cdot_c \otimes {}^b \mathbf{V}_a^c, \\ \delta(\mathfrak{!}_a^c) &= \mathfrak{!}_a^c \otimes \cdot_a \cdot_b \cdot_c + \mathfrak{!}_{a+b}^c \otimes \mathfrak{!}_a^b \cdot_c + \mathfrak{!}_{a+c}^b \otimes \cdot_a \mathfrak{!}_b^c + \cdot_a \cdot_b \cdot_c \otimes \mathfrak{!}_a^c. \end{aligned}$$

As $\mathcal{T}op$ is a cofree twisted bialgebra, by Theorem 8.8:

Proposition 9.5. $\mathbf{H}_{\mathcal{T}op}$ is a free and cofree bialgebra.

9.4 Morphisms from finite topologies to \mathbf{WQSym} , \mathbf{QSym} or $\mathbb{K}[X]$

The Fock functors applied to the morphism ϕ_{ehr_q} gives a commutative diagram of morphisms compatible with the product and both coproducts:

$$\begin{array}{ccccc} \mathbf{H}_{\mathcal{T}op} & \longrightarrow & H_{\mathcal{T}op} & & \\ \mathcal{K}(\phi_{ehr_q}) \downarrow & & \hat{\mathcal{K}}(\phi_{ehr_q}) \downarrow & \searrow P_{ehr_q} & \\ \mathbf{WQSym} & \longrightarrow & \mathbf{QSym} & \xrightarrow{H} & \mathbb{K}[X] \end{array}$$

where $P_{ehr_q} = H \circ \hat{\mathcal{K}}(\phi_q)$. For any finite topology T in $\mathbf{H}_{\mathcal{T}op}$ of degree n :

$$\mathcal{K}(\phi_{ehr_1})(T) = \sum_{c \in L(T)} c(1) \dots c(n), \quad \mathcal{K}(\phi_{ehr_{-1}})(T) = \sum_{c \in L'(T)} c(1) \dots c(n).$$

For example:

$$\begin{aligned} \mathcal{K}(\phi_{ehr_1})(\cdot_{\{1\}}) &= (1), \\ \mathcal{K}(\phi_{ehr_1})(\mathfrak{!}_{\{1,3\}}^2) &= (121), \\ \mathcal{K}(\phi_{ehr_1})({}^{\{3\}} \mathbf{V}_{\{1,4\}}^{\{2\}}) &= (1231) + (1321) + (1221), \\ \mathcal{K}(\phi_{ehr_1})(\mathfrak{!}_{\{1,3\}}^{\{2\}}) &= (1312); \\ \mathcal{K}(\phi_{ehr_{-1}})(\cdot_{\{1\}}) &= (1), \\ \mathcal{K}(\phi_{ehr_{-1}})(\mathfrak{!}_{\{1,3\}}^2) &= (121) + (111), \\ \mathcal{K}(\phi_{ehr_{-1}})({}^{\{3\}} \mathbf{V}_{\{1,4\}}^{\{2\}}) &= (1231) + (1321) + (1221) + (1211) + (1121) + (1111), \\ \mathcal{K}(\phi_{ehr_{-1}})(\mathfrak{!}_{\{1,3\}}^{\{2\}}) &= (1312) + (1211) + (1221) + (1111). \end{aligned}$$

The morphism $\mathcal{K}(\phi_{ehr_1})$ is the noncommutative version of the strict quasisymmetric Ehrhart functions [5], as it is explained in [13], whereas $\widehat{\mathcal{K}}(\phi_{ehr_1})$ is the strict quasisymmetric Ehrhart function. For example, if $a, b, c \geq 1$:

$$\begin{aligned}\widehat{\mathcal{K}}(\phi_{ehr_1})(\bullet_a) &= (a), \\ \widehat{\mathcal{K}}(\phi_{ehr_1})(\uparrow_a^b) &= (a, b), \\ \widehat{\mathcal{K}}(\phi_{ehr_1})(\mathbf{V}_a^c) &= (a, b, c) + (a, c, b) + (a, b + c), \\ \widehat{\mathcal{K}}(\phi_{ehr_1})(\uparrow_a^c) &= (a, b, c); \\ \\ \widehat{\mathcal{K}}(\phi_{ehr_{-1}})(\bullet_a) &= (a), \\ \widehat{\mathcal{K}}(\phi_{ehr_{-1}})(\uparrow_a^b) &= (a, b) + (a + b), \\ \widehat{\mathcal{K}}(\phi_{ehr_{-1}})(\mathbf{V}_a^c) &= (a, b, c) + (a, c, b) + (a, b + c) + (a + b, c) + (a + c, b) + (a + b + c), \\ \widehat{\mathcal{K}}(\phi_{ehr_{-1}})(\uparrow_a^c) &= (a, b, c) + (a + b, c) + (a, b + c) + (a + b + c).\end{aligned}$$

Reformulating in terms of formal series in a infinite set $\{X_1, X_2, \dots\}$ of indeterminates:

$$\begin{aligned}\widehat{\mathcal{K}}(\phi_{ehr_1})(\bullet_a) &= \sum_i X_i^a, \\ \widehat{\mathcal{K}}(\phi_{ehr_1})(\uparrow_a^b) &= \sum_{i < j} X_i^a X_j^b, \\ \widehat{\mathcal{K}}(\phi_{ehr_1})(\mathbf{V}_a^c) &= \sum_{i < j < k} X_i^a X_j^b X_k^c + \sum_{i < j < k} X_i^a X_j^c X_k^b + \sum_{i < j} X_i^a X_j^{b+c}, \\ \widehat{\mathcal{K}}(\phi_{ehr_1})(\uparrow_a^c) &= \sum_{i < j < k} X_i^a X_j^b X_k^c.\end{aligned}$$

Composing $\widehat{\mathcal{K}}(\phi_{ehr_1})$ with H , we obtain the strict Ehrhart polynomial P_{ehr_1} . For example:

$$\begin{aligned}P_{ehr_1}(\bullet_a) &= H_1(X) = X, \\ P_{ehr_1}(\uparrow_a^b) &= H_2(X) = \frac{X(X-1)}{2}, \\ P_{ehr_1}(\mathbf{V}_a^c) &= 2H_3(X) + H_2(X) = \frac{X(X-1)(2X-1)}{6}, \\ P_{ehr_1}(\uparrow_a^c) &= H_3(X) = \frac{X(X-1)(X-2)}{6}.\end{aligned}$$

Composing $\widehat{\mathcal{K}}(\phi_{ehr_{-1}})$ with H , we obtain the Ehrhart polynomial $P_{ehr_{-1}}$. For example:

$$\begin{aligned}P_{ehr_{-1}}(\bullet_a) &= H_1(X) = X, \\ P_{ehr_{-1}}(\uparrow_a^b) &= H_2(X) + H_1(X) = \frac{X(X+1)}{2}, \\ P_{ehr_{-1}}(\mathbf{V}_a^c) &= 2H_3(X) + 3H_2(X) + H_1(X) = \frac{X(X+1)(2X+1)}{6}, \\ P_{ehr_{-1}}(\uparrow_a^c) &= H_3(X) + 2H_2(X) + H_1(X) = \frac{X(X+1)(X+2)}{6}.\end{aligned}$$

Proposition 9.6 (Duality principle). *For any finite topology $T \in H_{\mathcal{T}op}$:*

$$P_{ehr_q}(T)(X) = q^{\text{cl}(T)} P_{ehr_1}(T) \left(\frac{X}{q} \right).$$

Proof. Indeed:

$$\begin{aligned}
P_{ehr_q}(T)(X) &= H \circ \widehat{\mathcal{K}}(\theta_{q^{-1}}) \circ \widehat{\mathcal{K}}(\phi_{ehr_1}) \circ \widehat{\mathcal{K}}(\iota_q)(T)(X) \\
&= q^{\text{cl}(T)} H \circ \widehat{\mathcal{K}}(\theta_{q^{-1}}) \circ \widehat{\mathcal{K}}(\phi_{ehr_1})(T)(X) \\
&= q^{\text{cl}(T)} \Theta_{q^{-1}} \circ H \circ \widehat{\mathcal{K}}(\phi_{ehr_1})(T)(X) \\
&= q^{\text{cl}(T)} \Theta_{q^{-1}} \circ P_{ehr_1}(T)(X) \\
&= q^{\text{cl}(T)} P_{ehr}(T) \left(\frac{X}{q} \right). \quad \square
\end{aligned}$$

Proposition 9.7. *Let $q \in \mathbb{K}$. For any finite topology T :*

$$H \circ \widehat{\mathcal{K}}(\phi_q)(T) = \lambda_{ho}(T) X^{\text{cl}(T)}.$$

Proof. In order to lighten the proof, we write $F_q = H \circ \widehat{\mathcal{K}}(\phi_q)$. For any $q' \in \mathbb{K}$, $\phi_q \circ \iota_{q'} = \theta_{q'} \circ \phi_q$, so:

$$\begin{aligned}
q'^{\text{cl}(T)} F_q(T) &= F_q \circ \widehat{\mathcal{K}}(\iota_{q'})(T) \\
&= H \circ \widehat{\mathcal{K}}(\theta_{q'}) \circ \widehat{\mathcal{K}}(\phi_q)(T) \\
&= \Theta_{q'} \circ H \circ \widehat{\mathcal{K}}(\phi_q)(T) \\
&= F_q(T)(q'X),
\end{aligned}$$

so $F_q(T)$ is homogeneous of degree $\text{cl}(T)$. We put $F_q(T) = \mu(T) X^{\text{cl}(T)}$. Then:

$$\begin{aligned}
\mu(T) &= \varepsilon'(F_q(T)) \\
&= \varepsilon' \circ H \circ \widehat{\mathcal{K}}(\phi_q)(T) \\
&= \varepsilon' \circ \widehat{\mathcal{K}}(\phi_q)(T) \\
&= \varepsilon' \circ \phi_q(T) \\
&= q^{\text{cl}(T)} \lambda_{ho}(T).
\end{aligned}$$

So $F_q(T) = q^{\text{cl}(T)} \lambda_{ho}(T)$. □

As $\phi_{ehr_1} = \phi_1 \leftarrow \lambda_{ehr_1}$:

Corollary 9.8. *For any finite topology $T \in H_{\mathcal{T}op}$:*

$$P_{ehr_1}(T) = \sum_{\sim \triangleleft G} \lambda_{ehr_1}(G | \sim) \lambda_{ho}(T / \sim) X^{\text{cl}(\sim)}.$$

Applying the Fock functor to Γ :

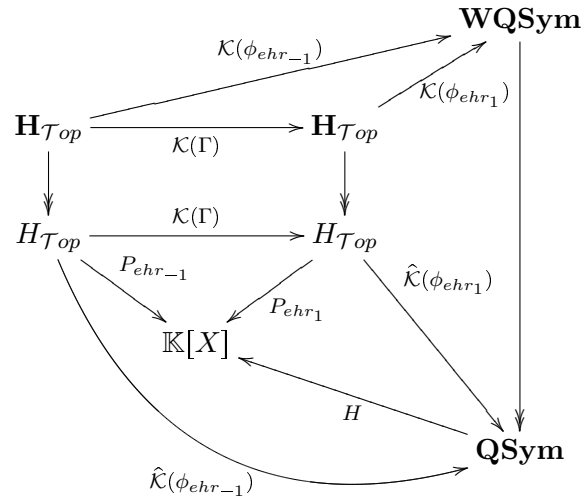
Proposition 9.9. *The following maps are Hopf algebra automorphisms:*

$$\begin{aligned}
\mathcal{K}(\Gamma) : & \begin{cases} \mathbf{H}_{\mathcal{T}op} & \longrightarrow & \mathbf{H}_{\mathcal{T}op} \\ T & \longrightarrow & \sum_{\sim \triangleleft T} T / \sim, \end{cases} \\
\widehat{\mathcal{K}}(\Gamma) : & \begin{cases} H_{\mathcal{T}op} & \longrightarrow & H_{\mathcal{T}op} \\ G & \longrightarrow & \sum_{\sim \triangleleft T} T / \sim. \end{cases}
\end{aligned}$$

Their inverse are given by:

$$\begin{aligned}
\mathcal{K}(\Gamma)^{-1} : & \begin{cases} \mathbf{H}_{\mathcal{T}op} & \longrightarrow & \mathbf{H}_{\mathcal{T}op} \\ T & \longrightarrow & \sum_{\sim \triangleleft T} (-1)^{\text{cl}(T) + \text{cl}(\sim)} T / \sim, \end{cases} \\
\widehat{\mathcal{K}}(\Gamma)^{-1} : & \begin{cases} H_{\mathcal{T}op} & \longrightarrow & H_{\mathcal{T}op} \\ T & \longrightarrow & \sum_{\sim \triangleleft T} (-1)^{\text{cl}(T) + \text{cl}(\sim)} T / \sim. \end{cases}
\end{aligned}$$

Moreover, the following diagram commutes:



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