

ProPs of graphs and generalised traces

Pierre J. Clavier^{1,2}, Loïc Foissy³, Sylvie Paycha^{1,4}

¹ *Universität Potsdam, Institut für Mathematik,
Campus II - Golm, Haus 9
Karl-Liebknecht-Straße 24-25
D-14476 Potsdam, Germany*

² *Technische Universität, Institut für Mathematik,
Str. des 17. Juni 136,
10587 Berlin, Germany*

³ *Univ. Littoral Côte d'Opale, UR 2597
LMPA, Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville
F-62100 Calais, France*

⁴ *On leave from the University of Clermont-Auvergne, Clermont-Ferrand, France*

emails: clavier@math.uni-potsdam.de, foissy@univ-littoral.fr, paycha@math.uni-potsdam.de

Abstract

We assign generalised convolutions (resp. traces) to graphs whose edges are decorated by smooth kernels (resp. smoothing operators) on a closed manifold. To do so, we introduce the concept of TraPs (Traces and Permutations), which roughly correspond to ProPs (Products and Permutations) without vertical concatenation and equipped with families of generalised partial traces. They can be equipped with a ProP structure in deriving vertical concatenation from the partial traces and we relate TraPs to wheeled ProPs first introduced by Merkulov. We further build their free object and give precise proofs of universal properties of ProPs and TraPs.

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Introduction

State of the art

ProPs (Products and Permutations)¹ provide an algebraic structure that allows to deal with an arbitrary number of inputs and outputs. As such they generalise many other algebraic structures such as operads, which have one output and multiple inputs. ProPs appeared in [Lan65] and later in the book [BV73] in the context of cartesian categories. Operads stemmed from this work in [May72], although their origin can also be traced back to the earlier work [BV68]².

¹The traditional notation for ProP is PROP or and more recently prop. We choose to use ProP as an acronym with capital letters for the first letter of the words and use the same convention for the related concept of TraP.

²We thank B. Vallette for his enlightening comments on these historical aspects.

An important asset of ProPs over operads is that they encompass algebraic structures such as bialgebras and Hopf algebras that lie outside the realm of operads or co-operads. We refer the reader to [Pir01] for the study of bialgebras in the ProPs framework and [Mar08] for other classical examples of ProPs.

Our two central examples of ProPs are the ProP \mathbf{Hom}_V of homomorphisms of a finite dimensional vector space V which we generalise to the ProP \mathbf{Hom}_V^c of continuous homomorphisms of the nuclear Fréchet space V , and the ProP \mathbf{Gr}^\cup of graphs³. In the context of deformation quantisation, the complex of oriented graphs whether directed or wheeled, plays an important role in the construction of a free ProP generated by a $\mathfrak{S} \times \mathfrak{S}^{op}$ -module (see e.g [Mer04, Paragraph 2.1.3]). However, to our knowledge, the ProP of oriented graphs, briefly mentioned in [Ion07b], has not yet found concrete applications in the perturbative approach to quantum field theory. Filling this gap is a long term goal we have in mind.

ProPs and oriented graphs

In space-time variables, a Feynman rule is expected to assign to a graph G with k incoming and l outgoing edges, a correlation function (it is actually a distribution) K_G in $k + l$ variables. Our long term goal is to interpret the correlation function associated with the composition $G \circ G'$ of two graphs as a generalised convolution $K_G \star K_{G'}$ of the correlation functions K_G and $K_{G'}$ associated with G and G' , aiming to derive the existence and the properties of the map $G \mapsto K_G$ from a universal property of the ProP structure on graphs.

ProPs entail two operations, called horizontal and vertical concatenations, which are the natural operations implemented on oriented graphs. With the goal we have in mind, ProPs are therefore natural structures to consider. We provide a precise formulation of the well-known fact that oriented graphs can be equipped with a ProP structure as well as a complete proof (see Theorem 1.3.3) of this statement. We also give a similar statement for (resp. planar) vertex decorated graphs in Theorem 4.1.2 (resp. Theorem 4.2.3). The horizontal concatenation of this ProP is the natural concatenation of graphs and the vertical concatenation is the composition, which to a graph G with k incoming and l outgoing edges, and a graph G' with l incoming and m outgoing edges, assigns a graph $G \circ G'$ with k incoming and m outgoing edges. Roughly speaking, $G \circ G'$ is obtained by gluing together the outgoing edges of G and the incoming edges of G' according to their indexation.

In Theorem 3.2.1, we show that the ProP of oriented graphs is the free ProP generated by what we call indecomposable graphs (see Definition 3.1.1). We provide a planar version of this result in Theorem 3.4.3. These universal properties are generalised to decorated graphs in Theorem 4.1.4. Such universal properties were stated without detailed proofs in previous work, see e.g. [Mar08, Proposition 57] and [Val03, Val04].

We make use of the universal property of oriented graphs when decorating the corresponding ProP \mathbf{Gr}^\cup with another ProP whose structure is compatible with that of the one on graphs (see Subsection 4.3). In particular, we show in Theorem 4.3.1 that $\Gamma^\dagger(X)$ is the free ProP generated by the $\mathfrak{S} \times \mathfrak{S}^{op}$ -module X . The decorating set X will eventually be a ProP of smooth kernels. Along the way, we use Theorem 4.3.1 in Corollaries 4.4.2, 4.4.3 to build algebra over ProPs; see Definition 4.4.1. The same constructions and universal properties hold for edge-decorated oriented graphs, i.e. Feynman graphs (see Remark 3.3.1).

We have chosen to work with the ProP \mathbf{Gr}^\cup which comprises loops, although the latter play a passive role in the presentation of a ProP. Yet, they will be relevant in the presentation of TraPs that come later in Section 5.3. Introducing them right at the beginning unifies the presentation, since otherwise two similar constructions over two different sets of graphs would have been necessary.

³We use Merkulov's notations.

Correlation functions and generalised convolutions

By means of blow-up methods, generalised convolutions of Green functions were built on a closed Riemannian manifold in [DZ17], with the goal of renormalising multiple loop amplitudes for Euclidean QFT on Riemannian manifolds. We hope to be able to simplify the intricate analytic aspects of the renormalisation procedure for multiple loop amplitudes, by adopting an algebraic point of view on correlation functions using ProPs. There were earlier attempts to describe QFT theories in terms of ProPs (see e.g. [Ion07a, Ion07b]), yet to our knowledge, none with the focus we are putting on generalised convolutions to describe correlation functions.

Our goal is to use the ProP (actually TraP) structure of graphs decorated by distribution (e.g. Green) kernels to build the resulting convolutions as generalised convolutions of kernels associated with the decorated graph. The expected singularities of the resulting correlation functions are immediate obstacles in defining such generalised convolutions. In this paper, we focus on the smooth setup, for which the correlation functions are smooth. Our goal in the smooth case is to provide an adequate algebraic and analytic framework in which we carry out this construction for correlation functions emanating from graphs decorated with smooth kernels.

A smooth kernel K on a closed manifold M gives rise to a smoothing operator

$$\mathcal{D}'(M) \ni u \longmapsto \left(L_K(u) : x \mapsto \int_M K(x, y) u(y) dy \right) \in C^\infty(M),$$

which maps the space $\mathcal{D}'(M)$ of distributions on M to the space $C^\infty(M)$ of smooth functions on M . So, in generalising the convolution of smooth kernels, we generalise the composition of smoothing operators.

Graphs with oriented cycles and TraPs

One challenge present both in the smooth and non-smooth case is the treatment of oriented cycles. A first step is the study of the sub-ProPs of (decorated and non decorated) graphs without oriented cycles carried out in Subsection 3.3. These structures are then used in Section 4. Yet in order to tackle Feynman graphs, we need graphs that can contain oriented cycles.

TraPs (see Definition 5.1.1) provide a natural structure to take into account oriented cycles in the graph. It indeed provides a framework to host (partial) traces on graphs that generalise the ordinary trace $\text{Tr}(L_K) = \int_M K(x, x) dx$. The TraP structure, which we relate in Section 7.1 to Merkulov's notion of wheeled ProPs (see Corollary 7.1.4), encompasses families of generalised traces. In Definition 1.3.1, we introduce the set of \mathbf{Gr}° of graphs which includes graphs with oriented loops. Proposition 5.3.1 shows that \mathbf{Gr}° can be equipped with a TraP structure and Theorem 5.4.1 shows that this TraP is free. This result is then generalised by Theorem 5.4.2 which describes free TraP. An appendix is dedicated to the precise definition of the trace on \mathbf{Gr}° . Paragraph 5.4 provides a description of a free TraP generated by a given set.

We have postponed the detailed proofs of two main results Theorem 3.2.1 and Theorem 5.4.1 to the appendix, so as not to burden the bulk of the paper with technicalities. A sketch of the proof is given straight after the statement so that the reader can nevertheless have an idea of the proof.

Alongside the ProP of graphs, another guiding example throughout the paper is the ProP of homomorphisms, which we investigate in the infinite dimensional setup. In Theorem 2.2.5, we introduce the ProP \mathbf{Hom}_V^c of *continuous morphisms* for a topological Fréchet nuclear space V , which generalises the well-known ProP \mathbf{Hom}_V (see e.g. the classical monograph [Mar08]) of morphisms on a finite dimensional vector space (see Definition 1.2.1).

In Proposition 7.2.1 we define the TraP $(\mathbf{Hom}_V^c(k, l))_{k, l \geq 0}$ corresponding to the ProP \mathbf{Hom}_V^c of continuous morphisms on an infinite dimensional Fréchet nuclear space V . In the finite dimensional case it reduces to the TraP $(\mathbf{Hom}_V(k, l))_{k, l \geq 0}$.

Functorial properties: TraPs versus wheeled ProPs

Much in the same way as we build the functor (see Proposition 4.2.5)

$$\Gamma^\uparrow : \mathbf{Mod}_\mathfrak{S} \longrightarrow \mathbf{ProP}$$

from the category $\mathbf{Mod}_\mathfrak{S}$ (Definition 4.2.1) of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules to the category \mathbf{ProP} (Definition 1.1.3) of ProPs, which to a $\mathfrak{S} \times \mathfrak{S}^{op}$ -module P assigns a graph-ProP $\Gamma^\uparrow(P)$ whose vertices are decorated by P , following Merkulov's approach, we build a functor

$$\Gamma^\circ : \mathbf{Mod}_\mathfrak{S} \longrightarrow \mathbf{TraP}$$

which takes $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules to TraPs (Proposition 6.1.1). Combining them with forgetful functors from \mathbf{ProP} or \mathbf{TraP} to $\mathbf{Mod}_\mathfrak{S}$, we can view Γ^\uparrow as an endofunctor of $\mathbf{Mod}_\mathfrak{S}$ or of \mathbf{ProP} , and Γ° as an endofunctor of $\mathbf{Mod}_\mathfrak{S}$ or of \mathbf{TraP} .

In Paragraph 6.2, we provide a detailed description of Merkulov's construction of the monad structure of Γ° on the category $\mathbf{Mod}_\mathfrak{S}$ (Proposition 6.2.2), by means of which (wheeled) ProPs are defined. Our Definition 5.1.1 of TraPs corresponds to unital wheeled ProPs. Using the construction of free TraPs of Section 5.4, in Corollary 7.1.4 we establish an isomorphism between the categories of wheeled TraPs on the one hand and of TraPs on the other hand.

Our constructions have some similarity with those underlying traced monoidal categories introduced in [JSV96], yet the framework and the axioms in the two approaches differ.

TraPs viewed as ProPs: the trace and the composition

It follows from the identification between TraPs and wheeled ProPs mentioned above, that a TraP is a ProP. In Proposition 7.2.1, we provide a detailed description of the ProP structure on TraPs as a result of the fact that both the trace and composition of morphisms (see Lemma 5.2.2) can be expressed in terms of a dual pairing. Let us illustrate this fact in the finite dimensional setup.

Given a finite dimensional vector space V over a commutative field \mathbb{K} , both the composition and the trace on the algebra of morphisms $\mathbf{Hom}(V) \simeq V^* \otimes V$ involve the dual pairing

$$V^* \times V \ni (v^*, w) \mapsto v^*(w) \in \mathbb{K},$$

between the algebraic dual V^* and the space V .

Extending this to the infinite dimensional setup requires the use of a completed tensor product $\widehat{\otimes}$ in order to have an isomorphism

$$\mathbf{Hom}_V^c(k, l) \simeq (V')^{\widehat{\otimes} k} \widehat{\otimes} V^{\widehat{\otimes} l},$$

where $\mathbf{Hom}_V^c(k, l)$ stands for the algebra of continuous morphisms from $V^{\widehat{\otimes} l}$ to $V^{\widehat{\otimes} k}$ (see Definition 2.2.4) and V' for the topological dual of a topological space V . This holds in the framework of Fréchet nuclear spaces which form a monoidal category under the completed tensor product $E \widehat{\otimes} F$ (Lemma 2.1.4). On Fréchet nuclear spaces, the composition can indeed be described as a dual pairing (see Lemma 5.2.2) so it comes as no surprise that (see Proposition 7.2.2) for a Fréchet nuclear space V , the ProP built from the TraP $(\mathbf{Hom}_V^c(k, l))_{k, l \geq 0}$ is isomorphic, as a ProP, to the ProP \mathbf{Hom}_V^c . In the finite dimensional setting, this induces an isomorphism of ProPs between TraP $(\mathbf{Hom}_V(k, l))_{k, l \geq 0}$ and \mathbf{Hom}_V .

In practice, the partial trace maps $t_{i,j}$ arising in the definition of a TraP might not be defined on every operator. To circumvent this difficulty, in Paragraph 7.3, we introduce the notion of quasi-TraP, which we embed in a complete TraP.

Openings

As announced in the abstract, by means of a (quasi-) TraP structure, we were able to build generalised convolutions (resp. traces) associated with graphs decorated with smooth kernels (see Remark 8.1.1). We expect this algebraic approach to enable us to tackle non smooth kernels and thus to describe correlation functions as generalised convolutions of distribution kernels associated with graphs. At this stage these are open questions we hope to address in future work.

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Notation

1. Any vector space in this text is taken over \mathbb{K} , chosen to be the field \mathbb{R} or the field \mathbb{C} .
2. For any $k \in \mathbb{N}_0 = \mathbb{Z}_{\geq 0}$, we denote by $[k]$ the set $\{1, \dots, k\}$. In particular, $[0] = \emptyset$.

1 Two guiding examples of ProPs

We define ProPs, the first main protagonists of the paper, and two ProPs which we shall use as a driving thread throughout the paper.

1.1 Definition

Following [Val03, Mar08], a ProP is a symmetric strict monoidal category, whose objects are identified with $(\mathbb{N}_0)^2$ and such that the tensor product of two objects is identified with the sum of integers on each copy of \mathbb{N}_0 . Here is a more detailed description.

Definition 1.1.1. *A **ProP** is a family $P = (P(k, l))_{k, l \in \mathbb{N}_0}$ of vector spaces such that:*

1. *P is a $\mathfrak{S} \times \mathfrak{S}^{op}$ -module, that is to say, for any $(k, l) \in \mathbb{N}_0^2$, $P(k, l)$ is a $\mathfrak{S}_l \times \mathfrak{S}_k^{op}$ -module. In other words, there exist maps*

$$\left\{ \begin{array}{l} \mathfrak{S}_l \times P(k, l) \longrightarrow P(k, l) \\ (\sigma, p) \longrightarrow \sigma \cdot p, \end{array} \right. \quad \left\{ \begin{array}{l} P(k, l) \times \mathfrak{S}_k \longrightarrow P(k, l) \\ (p, \tau) \longrightarrow p \cdot \tau, \end{array} \right.$$

such that for any $(k, l) \in \mathbb{N}_0^2$, for any $(\sigma, \sigma', \tau, \tau') \in \mathfrak{S}_l^2 \times \mathfrak{S}_k^2$, for any $p \in P(k, l)$,

$$\begin{aligned} \text{Id}_{[l]} \cdot p &= p \cdot \text{Id}_{[k]} = p, \\ \sigma \cdot (\sigma' \cdot p) &= (\sigma \sigma') \cdot p, & \sigma \cdot (p \cdot \tau) &= (\sigma \cdot p) \cdot \tau, & (p \cdot \tau) \cdot \tau' &= p \cdot (\tau \tau'). \end{aligned}$$

2. *For any $(k, l, k', l') \in \mathbb{N}_0^4$, there exists a product $*$ from $P(k, l) \otimes P(k', l')$ to $P(k + k', l + l')$ such that:*

(a) *For any $(k, l, k', l', k'', l'') \in \mathbb{N}_0^6$, for any $(p, p', p'') \in P(k, l) \times P(k', l') \times P(k'', l'')$,*

$$p * (p' * p'') = (p * p') * p''.$$

(b) There exists $I_0 \in P(0,0)$, such that for any $(k,l) \in \mathbb{N}_0^2$, for any $p \in P(k,l)$,

$$p * I_0 = I_0 * p = p.$$

This product $*$ is called the **horizontal concatenation**.

3. For any $(k,l,m) \in \mathbb{N}_0^3$, there exists a product \circ from $P(l,m) \otimes P(k,l)$ to $P(k,m)$ such that:

(a) For any $(k,l,m,n) \in \mathbb{N}_0^4$, for any $(p,q,r) \in P(m,n) \times P(l,m) \times P(k,l)$,

$$p \circ (q \circ r) = (p \circ q) \circ r.$$

(b) There exists $I_1 \in P(1,1)$, such that for any $(k,l) \in \mathbb{N}_0^2$, for any $p \in P(k,l)$,

$$p \circ I_k = I_l \circ p = p,$$

where we put $I_n = I_1^{*n}$ for any $n \in \mathbb{N}_0$, with the convention $I_1^{*0} = I_0$.

This product \circ is called the **vertical concatenation**.

4. The vertical and horizontal concatenations are compatible: for any $(k,k',l,l',m,m') \in \mathbb{N}_0^6$, for any $(p,p',q,q') \in P(l,m) \times P(l',m') \times P(k,l) \times P(k',l')$,

$$(p * p') \circ (q * q') = (p \circ q) * (p' \circ q').$$

5. The vertical concatenation and the action of $\mathfrak{S} \times \mathfrak{S}^{op}$ are compatible: for any $(k,l,m) \in \mathbb{N}_0^3$, for any $(p,q) \in P(l,m) \times P(k,l)$, for any $(\sigma,\tau,\nu) \in \mathfrak{S}_m \times \mathfrak{S}_l \times \mathfrak{S}_k$,

$$\sigma \cdot (p \circ q) = (\sigma \cdot p) \circ q, \quad (p \circ q) \cdot \nu = p \circ (q \cdot \nu), \quad (p \cdot \tau) \circ q = p \circ (\tau \cdot q).$$

6. The horizontal concatenation and the action of $\mathfrak{S} \times \mathfrak{S}^{op}$ are compatible:

(a) For any $(k,k',l,l') \in \mathbb{N}_0^4$, for any $(p,p') \in P(k,l) \times P(k',l')$, for any $(\sigma,\sigma',\tau,\tau') \in \mathfrak{S}_l \times \mathfrak{S}_{l'} \times \mathfrak{S}_k \times \mathfrak{S}_{k'}$,

$$(\sigma \cdot p) * (\sigma' \cdot p') = (\sigma \otimes \sigma') \cdot (p * p'), \quad (p \cdot \tau) * (p' \cdot \tau') = (p * p') \cdot (\tau \otimes \tau'),$$

where for any $\alpha \in \mathfrak{S}_m$, $\beta \in \mathfrak{S}_n$, $\alpha \otimes \beta \in \mathfrak{S}_{m+n}$ is defined by:

$$\alpha \otimes \beta(i) = \begin{cases} \alpha(i) & \text{if } i \leq m, \\ \beta(i-m) + m & \text{if } i > m. \end{cases}$$

(b) (Commutativity of the horizontal concatenation). For any $(k,k',l,l') \in \mathbb{N}_0^4$, for any $(p,p') \in P(k,l) \times P(k',l')$,

$$c_{l,l'} \cdot (p * p') = (p' * p) \cdot c_{k,k'}, \tag{1}$$

where for any $(m,n) \in \mathbb{N}_0^2$, $c_{m,n} \in \mathfrak{S}_{m+n}$ is defined by:

$$c_{m,n}(i) = \begin{cases} i + n & \text{if } i \leq m, \\ i - m & \text{if } i > m. \end{cases} \tag{2}$$

Remark 1.1.1. 1. Note that $c_{k,0} = \text{Id}_{[k]} = c_{0,k}$.

2. In particular, $(P(0,0), *)$ is a unitary associative and commutative algebra, whose unit is I_0 , which, consequently is unique.

3. Similarly, $(P(1, 1), \circ)$ is a unitary associative non commutative algebra, whose unit is I_1 which, consequently is unique.
4. For any $\sigma \in \mathfrak{S}_k$, as a consequence of the compatibility between the vertical concatenation and the action of $\mathfrak{S} \times \mathfrak{S}^{op}$ and the definition of $I_k \in P(k, k)$:

$$I_k \cdot \sigma = (I_k \cdot \sigma) \circ I_k = I_k \circ (\sigma \cdot I_k) = \sigma \cdot I_k.$$

Hence, $I_k \cdot \sigma = \sigma \cdot I_k$.

5. By the commutativity axiom, if $p \in P(k, l)$ and $p_0 \in P(0, 0)$, by the first item of this Remark, it follows from (1) that $p * p_0 = p_0 * p$. So the elements of $P(0, 0)$ are central for the horizontal concatenation. If $q \in P(l, m)$, by the compatibility between the two concatenations:

$$\begin{aligned} (p * p_0) \circ q &= (p * p_0) \circ (q * I_0) \\ &= (p \circ q) * (p_0 \circ I_0) \\ &= (p \circ q) * p_0. \end{aligned}$$

Similarly, $p \circ (q * p_0) = (p \circ q) * p_0$.

We adapt the definition of morphisms of ProPs of [Val03] in our non categorical language.

Definition 1.1.2. Let $P = (P(k, l))_{k, l \geq 0}$ and $Q = (Q(k, l))_{k, l \geq 0}$ be two ProPs. A **morphism of ProPs** is a family $\phi = (\phi_{k, l})_{k, l \geq 0}$ of linear maps $\phi_{k, l} : P(k, l) \rightarrow Q(k, l)$ which form a morphism for the horizontal concatenation, the vertical concatenation and the actions of the symmetric groups. More precisely, for any $(k, l, m, n) \in \mathbb{N}_0^4$:

- $\forall (p, q) \in P(l, m) \times P(k, l), \phi_{k, m}(p \circ q) = \phi_{l, m}(p) \circ \phi_{k, l}(q),$
- $\forall (p, q) \in P(k, l) \times P(n, m), \phi_{k+n, l+m}(p * q) = \phi_{k, l}(p) * \phi_{n, m}(q),$
- $\forall (\sigma, p) \in \mathfrak{S}_l \times P(k, l), \phi_{k, l}(\sigma \cdot p) = \sigma \cdot \phi_{k, l}(p),$
- $\forall (p, \tau) \in P(k, l) \times \mathfrak{S}_k, \phi_{k, l}(p \cdot \tau) = \phi_{k, l}(p) \cdot \tau.$

By abuse of notation, we shall write $\phi(p)$ instead of $\phi_{k, l}(p)$ for $p \in P(k, l)$.

In particular, ProPs form a category.

Definition 1.1.3. Let **ProP** be the category with objects given by $P = (P(k, l))_{(k, l) \in \mathbb{N}_0^2}$ and the morphisms of which are morphisms $\phi : P \rightarrow Q$ of ProPs given by families $(\phi_{k, l})_{(k, l) \in \mathbb{N}_0^2}$. Here, for any $(k, l) \in \mathbb{N}_0^2$, $\phi_{k, l} : P(k, l) \rightarrow Q(k, l)$ is a morphism of $\mathfrak{S}_l \otimes \mathfrak{S}_k^{op}$ -modules, compatible with the vertical and horizontal concatenations, which sends the units I_0 and I_1 of P to the corresponding units of Q . More explicitly, we have that

- For any $(k, l, k', l') \in \mathbb{N}_0^4$, for any $(p, p') \in P(k, l) \times P(k', l')$, $\phi_{k+k', l+l'}(p * p') = \phi_{k, l}(p) * \phi_{k', l'}(p')$.
- For any $(k, l, m) \in \mathbb{N}_0^3$, for any $(p, p') \in P(l, m) \times P(k, l)$, $\phi_{k, m}(p \circ p') = \phi_{l, m}(p) \circ \phi_{k, l}(p')$.
- $\phi_{0, 0}(I_0) = J_0$ and $\phi_{1, 1}(I_1) = J_1$, where I_0, I_1 are the units of P and J_0, J_1 are the units of Q .

Let $P = (P(k, l))_{k, l \geq 0}$ be a ProP and, for any $k, l \geq 0$, $Q(k, l)$ be a subspace of $P(k, l)$. We shall say that $Q = (Q(k, l))_{k, l \geq 0}$ is a sub-ProP of P if it is stable under the horizontal and vertical compositions, under the action of the symmetric groups and if it contains the units I_0 and I_1 . More precisely:

- For any $(k, l, m) \in \mathbb{N}_0^3$, $Q(l, m) \circ Q(k, l) \subseteq Q(k, m)$.
- For any $(k, l, k', l') \in \mathbb{N}_0^4$, $Q(k, l) * Q(k', l') \subseteq Q(k + k', l + l')$.
- For any $(k, l) \in \mathbb{N}_0^2$, for any $(\sigma, \tau) \in \mathfrak{S}_l \times \mathfrak{S}_k$, $\sigma.Q(k, l).\tau \subseteq Q(k, l)$.
- $I_0 \in Q(0, 0)$ and $I_1 \in Q(1, 1)$.

Let P be a ProP.

- If Q is a sub-ProP of P , then Q is also a ProP, and the canonical injection from P to Q is a ProP morphism.
- If $(Q_i)_{i \in I}$ is a family of sub-ProPs of P , then $\bigcap_{i \in I} Q_i$ is also a sub-ProP of P .

This leads to the following

Definition-Proposition 1.1.4. *Let P be a ProP. If for any $k, l \geq 0$, $R(k, l)$ is a subspace of $P(k, l)$, then there exists a smallest sub-ProP of P containing $R = (R(k, l))_{k, l \geq 0}$*

$$\langle R \rangle := \bigcap_{\substack{Q \text{ sub-ProP} \\ \text{of } P \text{ containing } R}} Q.$$

Remark 1.1.2. Since Q contains I_1 , by $*$ -stability, Q contains $\underbrace{I_1 * \dots * I_1}_{k \text{ times}} = I_k$ and as a consequence of stability under the action of the symmetry groups, Q further contains $\sigma.I_k.\tau$.

1.2 The ProP of linear morphisms: \mathbf{Hom}_V

We recall a classical example of ProP.

Definition-Proposition 1.2.1. *Given a finite dimensional \mathbb{K} -vector space V , the ProP \mathbf{Hom}_V is defined in the following way:*

1. For any $k, l \in \mathbb{N}_0$,

$$\mathbf{Hom}_V(k, l) := \mathbf{Hom}(V^{\otimes k}, V^{\otimes l}).$$

2. For any $\sigma \in \mathfrak{S}_n$, let θ_σ be the endomorphism of $V^{\otimes n}$ defined by

$$\theta_\sigma(v_1 \otimes \dots \otimes v_n) := v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(n)}.$$

This defines a left action of \mathfrak{S}_n on $V^{\otimes n}$. For any $(k, l) \in \mathbb{N}_0^2$, for any $f \in \mathbf{Hom}_V(k, l)$, for any $(\sigma, \tau) \in \mathfrak{S}_l \times \mathfrak{S}_k$, we set:

$$\sigma \cdot f := \theta_\sigma \circ f, \quad f \cdot \tau := f \circ \theta_\tau.$$

3. The horizontal concatenation is the tensor product of maps and $I_0 : \mathbb{K} \longrightarrow \mathbb{K}$ is the identity map $I_0 := \text{Id}_{\mathbb{K}}$.
4. The vertical concatenation is the usual composition of maps and $I_1 : V \longrightarrow V$ is the identity map $I_1 := \text{Id}_V$.

Remark 1.2.1. This ProP is mentioned in [Val03] and [Mar08], but without an explicit proof of its ProP structure. We add such a proof here for completeness and in preparation for the infinite dimensional case, which will be similar in spirit.

Remark 1.2.2. Following our convention for a ProP $P = (P(k, l))_{k, l \in \mathbb{N}_0}$, where an element in $P(k, l)$ has “ k entries and l exits”, for the ProP \mathbf{Hom}_V , an element $f \in \mathbf{Hom}_V(k, l)$ has “ k entries and l exits”.

Proof. 1. The maps θ_σ turns \mathbf{Hom}_V into a $\mathfrak{S}_l \times \mathfrak{S}_k^{op}$ -module by associativity of the composition product.

2. The horizontal concatenation is associative as a result of the associativity of the tensor product \otimes , and we trivially have that \otimes maps $\mathbf{Hom}_V(k, l) \otimes \mathbf{Hom}_V(k', l')$ to $\mathbf{Hom}_V(k + k', l + l')$. Furthermore, if $(k, l) \in \mathbb{N}_0^2$ and $f \in \mathbf{Hom}_V(k, l)$, for any $v \in V^{\otimes k}$, we have

$$(I_0 \otimes f)(v) = (I_0 \otimes f)(1.v) := I_0(1) \otimes f(v) = 1_{\mathbb{K}} \otimes f(v) = f(v)$$

3. The vertical concatenation is associative as the consequence of the associativity of the composition product. We furthermore have $I_n := I_1^{\otimes n} = \text{Id}_V^{\otimes n} = \text{Id}_{V^{\otimes n}}$ where the last identity follows from the definition of the tensor product of maps.

4. For any $f \in \mathbf{Hom}_V(l, m)$, $f' \in \mathbf{Hom}_V(l', m')$, $g \in \mathbf{Hom}_V(k, l)$, $g' \in \mathbf{Hom}_V(k', l')$, $v \in V^{\otimes k}$ and $v' \in V^{\otimes k'}$ we have

$$\begin{aligned} (f \otimes f') \circ (g \otimes g')(v \otimes v') &= (f \otimes f')(g(v) \otimes g'(v')) \\ &= (f \circ g)(v) \otimes (f' \circ g')(v') \\ &= [(f \circ g) \otimes (f' \circ g)](v \otimes v'). \end{aligned}$$

Thus, the horizontal and vertical concatenation are compatible.

5. The vertical concatenation and the action of $\mathfrak{S} \times \mathfrak{S}^{op}$ are compatible by associativity of the composition product.

6. For any $f \in \mathbf{Hom}_V(k, l)$, $f' \in \mathbf{Hom}_V(k', l')$, $\sigma \in \mathfrak{S}_l$, $\sigma' \in \mathfrak{S}_{l'}$, $v \in V^{\otimes k}$, $v' \in V^{\otimes k'}$ we have

$$\begin{aligned} (\sigma.f) \otimes (\sigma'.f')(v \otimes v') &= (\theta_\sigma \circ f) \otimes (\theta_{\sigma'} \circ f')(v \otimes v') \\ &= \theta_\sigma(f(v)) \otimes \theta_{\sigma'}(f'(v')) \\ &= (\theta_\sigma \otimes \theta_{\sigma'})[f(v) \otimes f'(v')] \\ &= (\sigma \otimes \sigma').(f \otimes f')(v \otimes v'). \end{aligned}$$

Similarly, we have $(f.\tau) \otimes (f'.\tau') = (f \otimes f').(\tau \otimes \tau')$ and $c_{l,l'} \cdot (f * f') = (f * f') \cdot c_{k,k'}$, therefore the horizontal action of $\mathfrak{S} \times \mathfrak{S}^{op}$ are compatible. \square

Remark 1.2.3. Let V be an n -dimensional \mathbb{K} -vector space equipped with a basis (e_1, \dots, e_n) , and let (e^1, \dots, e^n) be the dual basis. We write $v_j = \sum_{k_j=1}^n b_j^{k_j} e_{k_j}$ the elements of V and $v_i^* = \sum_{k_i=1}^n a_i^{k_i} e^{k_i}$ the elements of V^* . Then an element $f = v_1^* \otimes \dots \otimes v_k^* \otimes v_1 \otimes \dots \otimes v_l \in \mathbf{Hom}_V(k, l)$ reads

$$f = \sum_{\vec{I}, \vec{J}} a_{\vec{J}}^{\vec{I}} e^{\vec{J}} \otimes e_{\vec{I}},$$

where, for two finite sequences $\vec{I} = (i_1, \dots, i_k)$, $\vec{J} = (j_1, \dots, j_l)$ of k and l elements of $[n]$, we have set

$$e_{\vec{I}} := e_{i_1} \otimes \dots \otimes e_{i_k}; \quad e^{\vec{J}} := e^{j_1} \otimes \dots \otimes e^{j_l}$$

and the $a_{\vec{J}}^{\vec{I}} \in \mathbb{K}$ are coefficients built from sums of products of the coefficients $a_{k_i}^i$ and $b_j^{k_j}$. In particular, an element of this ProP is completely determined by this collection of numbers $a_{\vec{J}}^{\vec{I}}$. We can therefore view f as a map from pairs of subsets I, J of $[n]$ with k and l elements respectively into \mathbb{K} .

It follows that for any n -dimensional vector space V , \mathbf{Hom}_V is isomorphic as a ProP to the set of maps from pairs of finite sequences of elements of $[n]$ to \mathbb{K} :

$$\mathbf{Hom}_V \simeq (\{a : \text{Seq}_k([n]) \times \text{Seq}_l([n]) \longrightarrow \mathbb{K}\})_{k,l \geq 0}.$$

1.3 The ProP of graphs: Gr^\cup

Definition 1.3.1. A **graph** is a family $G = (V(G), E(G), I(G), O(G), IO(G), L(G), s, t, \alpha, \beta)$, where:

1. $V(G)$ (set of vertices), $E(G)$ (set of internal edges), $I(G)$ (set of input edges), $O(G)$ (set of output edges), $IO(G)$ (set of input-output edges) and $L(G)$ (set of loops, that is to say edges with no endpoints) are finite (maybe empty) sets.
2. $s : E(G) \sqcup O(G) \longrightarrow V(G)$ is a map (source map).
3. $t : E(G) \sqcup I(G) \longrightarrow V(G)$ is a map (target map).
4. $\alpha : I(G) \sqcup IO(G) \longrightarrow [i(G)]$ is a bijection, with $i(G) = |I(G)| + |IO(G)|$ (indexation of the input edges).
5. $\beta : O(G) \sqcup IO(G) \longrightarrow [o(G)]$ is a bijection, with $o(G) = |O(G)| + |IO(G)|$ (indexation of the output edges).

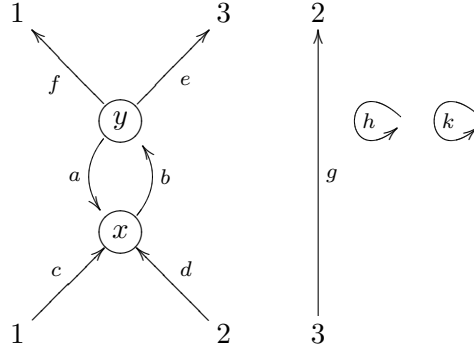
Example 1.3.1. Here is a graph G :

$$V(G) = \{x, y\}, \quad E(G) = \{a, b\}, \quad I(G) = \{c, d\}, \quad O(G) = \{e, f\}, \quad IO(G) = \{g\}, \quad L(G) = \{h, k\},$$

and:

$$s : \begin{cases} a \mapsto y \\ b \mapsto x \\ e \mapsto y \\ f \mapsto y, \end{cases} \quad t : \begin{cases} a \mapsto x \\ b \mapsto y \\ c \mapsto x \\ d \mapsto x, \end{cases} \quad \alpha : \begin{cases} c \mapsto 1 \\ d \mapsto 2 \\ g \mapsto 3, \end{cases} \quad \beta : \begin{cases} e \mapsto 3 \\ f \mapsto 1 \\ g \mapsto 2. \end{cases}$$

This is graphically represented as follows:



Note that this graph contains two loops, represented by $\begin{array}{c} \curvearrowright \\ h \end{array}$ and $\begin{array}{c} \curvearrowright \\ k \end{array}$.

Remark 1.3.1. As explained in the introduction, although loops play a passive role in the presentation of a ProP, their role will be essential in the presentation of TraPs, see Section 5.3.

Definition 1.3.2. Let G and G' be two graphs. An (**resp. iso-**)**morphism** of graphs from G to G' is a family of (**resp. bijections**) maps $f = (f_V, f_E, f_I, f_O, f_{IO}, f_L)$ with:

$$\begin{aligned} f_V : V(G) &\longrightarrow V(G'), & f_E : E(G) &\longrightarrow E(G'), & f_I : I(G) &\longrightarrow I(G'), \\ f_O : O(G) &\longrightarrow O(G'), & f_{IO} : IO(G) &\longrightarrow IO(G'), & f_L : L(G) &\longrightarrow L(G'), \end{aligned}$$

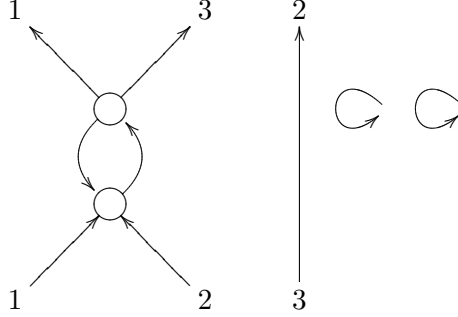
such that:

$$\begin{aligned} s' \circ f_E &= f_V \circ s|_{E(G)}, & s' \circ f_O &= f_V \circ s|_{O(G)}, \\ t' \circ f_E &= f_V \circ t|_{E(G)}, & t' \circ f_I &= f_V \circ t|_{I(G)}, \\ \alpha' \circ f_I &= \alpha|_{I(G)}, & \alpha' \circ f_{IO} &= \alpha|_{IO(G)}, \\ \beta' \circ f_O &= \beta|_{O(G)}, & \beta' \circ f_{IO} &= \beta|_{IO(G)}. \end{aligned}$$

For any $k, l \in \mathbb{N}_0$, we denote by $\mathbf{Gr}^\cup(k, l)$ the space generated by the isoclasses of graphs G such that $i(G) = k$ and $o(G) = l$, i.e. $\mathbf{Gr}^\cup(k, l)$ is the quotient space of graphs with k input edges and l output edges by the equivalence relation given by isomorphism.

In what follows, we shall write *graphs* for *isoclasses of graphs*.

Example 1.3.2. The isomorphism class of the graph of Example 1.3.1 is represented by:



We now want to equip the set \mathbf{Gr}^\cup of isoclasses of graphs with a ProP structure.

- We first define the **horizontal concatenation**. If G and G' are two disjoint graphs, we define a graph $G * G'$ in the following way:

$$\begin{aligned} V(G * G') &= V(G) \sqcup V(G'), & E(G * G') &= E(G) \sqcup E(G'), & L(G * G') &= L(G) \sqcup L(G'), \\ I(G * G') &= I(G) \sqcup I(G'), & O(G * G') &= O(G) \sqcup O(G'), & IO(G * G') &= IO(G) \sqcup IO(G'). \end{aligned}$$

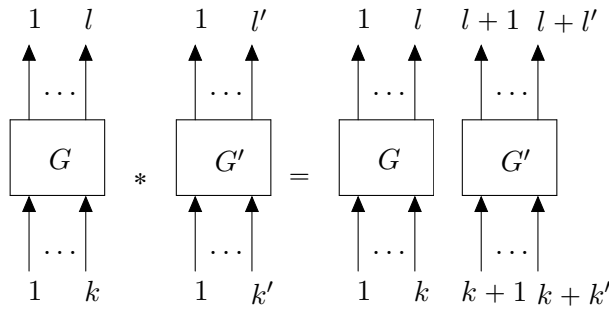
The source and target maps are given by:

$$\begin{aligned} s''_{|E(G) \sqcup O(G)} &= s, & s''_{|E(G') \sqcup O(G')} &= s', \\ t''_{|E(G) \sqcup I(G)} &= t, & t''_{|E(G') \sqcup I(G')} &= t'. \end{aligned}$$

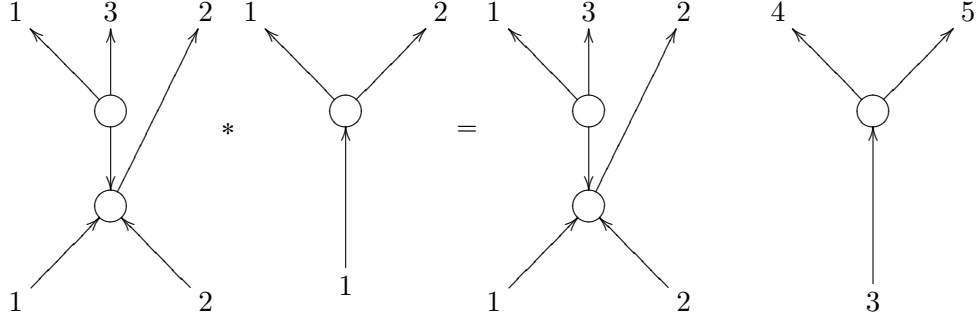
The indexations of the input and output edges are given by:

$$\begin{aligned} \alpha''_{|I(G) \sqcup IO(G)} &= \alpha, & \alpha''_{|I(G') \sqcup IO(G')} &= i(G) + \alpha', \\ \beta''_{|O(G) \sqcup IO(G)} &= \beta, & \beta''_{|O(G') \sqcup IO(G')} &= o(G) + \beta' \end{aligned}$$

with an obvious abuse of notation in the definition of the second column. Notice that this product is not commutative in the usual sense for $G * G'$ and $G' * G$ might differ by the indexation of their input and output edges. However, it is commutative in the sense of Axiom 6.(b) of ProPs. Roughly speaking, $G * G'$ is the disjoint union of G and G' , the input and output edges of G' being indexed after the input and output edges of G .



Example 1.3.3. Here is an example of horizontal concatenation :



This product of graphs induces a product $*$: $\mathbf{Gr}^{\circlearrowleft}(k, l) \otimes \mathbf{Gr}^{\circlearrowleft}(k', l') \longrightarrow \mathbf{Gr}^{\circlearrowleft}(k+k', l+l')$.
If G, G' and G'' are three graphs, clearly

$$G * (G' * G'') = (G * G') * G''.$$

Hence, the product $*$ is associative. Its unit I_0 is the unique graph such that $V(I_0) = E(I_0) = I(I_0) = O(I_0) = \emptyset$.

- We now define the **vertical concatenation**. Let G and G' be disjoint graphs such that $o(G) = i(G')$. We define a graph $G'' = G' \circ G$ in the following way:

$$\begin{aligned} V(G'') &= V(G) \sqcup V(G'), \\ E(G'') &= E(G) \sqcup E(G') \sqcup \{(f, e) \in O(G) \times I(G') : \beta(f) = \alpha'(e)\}, \\ I(G'') &= I(G) \sqcup \{(f, e) \in IO(G) \times I(G') : \beta(f) = \alpha'(e)\}, \\ O(G'') &= O(G) \sqcup \{(f, e) \in O(G) \times IO(G') : \beta(f) = \alpha'(e)\}, \\ IO(G'') &= \{(f, e) \in IO(G) \times IO(G') : \beta(f) = \alpha'(e)\}, \\ L(G'') &= L(G) \sqcup L(G'). \end{aligned}$$

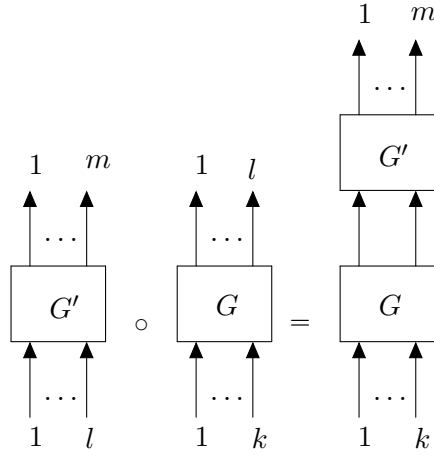
Its **source** and **target** maps are given by:

$$\begin{aligned} s''_{|E(G)} &= s_{|E(G)}, & s''_{|E(G')} &= s'_{|E(G')}, & s''_{|O(G')} &= s'_{|O(G')}, & s''((f, e)) &= s(f), \\ t''_{|E(G)} &= t_{|E(G)}, & t''_{|E(G')} &= s'_{|E(G')}, & t''_{|I(G)} &= s_{|I(G)}, & s''((f, e)) &= t'(e). \end{aligned}$$

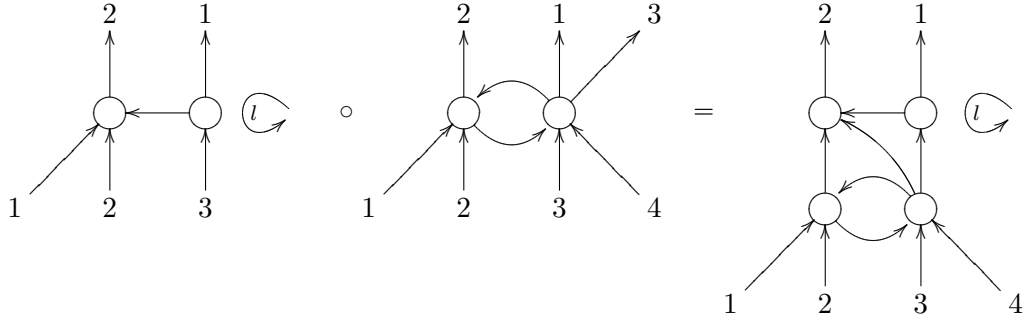
The indexations of its input and output edges are given by:

$$\begin{aligned} \alpha''_{|I(G)} &= \alpha_{|I(G)}, & \alpha''((f, e)) &= \alpha(f), \\ \beta''_{|O(G')} &= \beta'_{|O(G')}, & \beta''((f, e)) &= \beta'(e). \end{aligned}$$

Roughly speaking, $G' \circ G$ is obtained by gluing together the outgoing edges of G and the incoming edges of G' according to their indexation.



Example 1.3.4. Here is an example of vertical concatenation :



Theorem 1.3.3. *The family $\mathbf{Gr}^\cup = (\mathbf{Gr}^\cup_{k,l})_{k,l \in \mathbb{N}_0}$, equipped with this $\mathfrak{S} \times \mathfrak{S}^{op}$ -action and these horizontal and vertical concatenations, is a Prop.*

Proof. • We check the **associativity** of \circ . Let G , G' and G'' be three graphs with $o(G) = i(G')$ and $o(G') = i(G'')$. The graphs $(G'' \circ G') \circ G$ and $G'' \circ (G' \circ G)$ may be different, but both are isomorphic to the graph H defined by:

$$\begin{aligned}
V(H) &= V(G) \sqcup V(G') \sqcup V(G''), \\
E(H) &= E(G) \sqcup E(G') \sqcup E(G'') \\
&\sqcup \{(f, e) \in O(G) \times I(G') : \beta(f) = \alpha'(e)\} \sqcup \{(f, e) \in O(G') \times I(G'') : \beta'(f) = \alpha''(e)\} \\
&\sqcup \{(f, f', e) \in O(G) \times IO(G') \times I(G'') : \beta(f) = \alpha'(f'), \beta'(f') = \alpha''(e)\}, \\
I(H) &= I(G) \sqcup \{(f, e) \in IO(G) \times I(G') : \beta(f) = \alpha'(e)\} \\
&\sqcup \{(f, f', e) \in IO(G) \times IO(G') \times I(G'') : \beta(f) = \alpha'(f'), \beta'(f') = \alpha''(e)\}, \\
O(H) &= O(G'') \sqcup \{(f, e) \in O(G') \times IO(G'') : \beta'(f) = \alpha''(e)\} \\
&\sqcup \{(f, f', e) \in O(G) \times IO(G') \times IO(G'') : \beta(f) = \alpha'(f'), \beta'(f') = \alpha''(e)\}, \\
IO(H) &= \{(f, f', e) \in IO(G) \times IO(G') \times IO(G'') : \beta(f) = \alpha'(f'), \beta'(f') = \alpha''(e)\}, \\
L(H) &= L(G) \sqcup L(G') \sqcup L(G''),
\end{aligned}$$

with immediate source, target and indexation maps. So \circ induces an associative product $\circ : \mathbf{Gr}^\cup(l, m) \otimes \mathbf{Gr}^\cup(k, l) \longrightarrow \mathbf{Gr}^\cup(k, m)$.

- Let I_1 be the graph such that

$$V(I_1) = E(I_1) = I(I_1) = O(I_1) = L(I_1) = \emptyset, \quad IO(I_1) = [1].$$

We show that I_1 is the **unit** for \circ : The indexation maps are both the identity of $[1]$. For any integer $n \in \mathbb{N}_0$, I_1^{*n} is isomorphic to the graph I_n such that

$$V(I_n) = E(I_n) = I(I_n) = O(I_n) = L(I_n) = \emptyset, \quad IO(I_n) = [n],$$

the indexation maps being both the identity of $[n]$. If G is a graph and $k = i(G)$, then $H = G \circ I_k$ is the graph such that:

$$\begin{aligned} V(H) &= V(G), & I(H) &= \{(\alpha(e), e) : e \in I(G)\}, \\ E(H) &= E(G), & IO(H) &= \{(\alpha(e), e) : e \in IO(G)\}, \\ O(H) &= O(G), & L(H) &= L(G), \end{aligned}$$

with immediate source, target and indexation maps. This graph H is isomorphic to G , via the isomorphism given by:

$$\begin{aligned} f_V &= \text{Id}_{V(G)}, & f_I((\alpha(e), e)) &= e, \\ f_E &= \text{Id}_{E(G)}, & f_{IO}((\alpha(e), e)) &= e, \\ f_O &= \text{Id}_{O(G)}, \\ f_L &= \text{Id}_{L(G)}. \end{aligned}$$

Similarly, $I_l \circ G$ and G are isomorphic. Hence, I_1 is the unit of \circ in \mathbf{Gr}° .

- We check the **compatibility** of the horizontal and vertical concatenations. Let G, G', H and H' be graphs such that $o(G) = i(H)$ and $o(G') = i(H')$. The graphs $(H * H') \circ (G * G')$ and $(H \circ G) * (H' \circ G')$ are both equal to the graph K , such that:

$$\begin{aligned} V(K) &= V(G) \sqcup V(G') \sqcup V(H) \sqcup V(H'), \\ E(K) &= E(G) \sqcup E(G') \sqcup E(H) \sqcup E(H') \\ &\quad \sqcup \{(f, e) \in O(G) \times I(H) : \beta(f) = \alpha'(e)\} \\ &\quad \sqcup \{(f, e) \in O(G') \times I(H') : \beta(f) = \alpha'(e)\}, \\ I(K) &= I(G) \sqcup I(G') \sqcup \{(f, e) \in IO(G) \times I(H) : \beta(f) = \alpha'(e)\} \\ &\quad \sqcup \{(f, e) \in IO(G') \times I(H') : \beta(f) = \alpha'(e)\}, \\ O(K) &= O(H) \sqcup O(H') \sqcup \{(f, e) \in O(G) \times IO(H) : \beta(f) = \alpha'(e)\} \\ &\quad \sqcup \{(f, e) \in O(G') \times IO(H') : \beta(f) = \alpha'(e)\}, \\ IO(K) &= \sqcup \{(f, e) \in IO(G) \times IO(H) : \beta(f) = \alpha'(e)\} \\ &\quad \sqcup \{(f, e) \in IO(G') \times IO(H') : \beta(f) = \alpha'(e)\}, \\ L(K) &= L(G) \sqcup L(G') \sqcup L(H) \sqcup L(H'), \end{aligned}$$

with obvious source, target and indexation maps. Hence, the vertical and the horizontal concatenations are compatible.

- We check the **module** structure of \mathbf{Gr}° over the symmetric group. Let G be a graph, $\sigma \in \mathfrak{S}_{o(G)}$ and $\tau \in \mathfrak{S}_{i(G)}$. We set:

$$\begin{aligned} \sigma \cdot G &= (V(G), E(G), I(G), O(G), IO(G), L(G), s, t, \alpha, \sigma \circ \beta), \\ G \cdot \tau &= (V(G), E(G), I(G), O(G), IO(G), L(G), s, t, \tau^{-1} \circ \alpha, \beta). \end{aligned} \quad (3)$$

This induces a structure of $\mathfrak{S} \times \mathfrak{S}^{op}$ -module over \mathbf{Gr}° .

Let us prove the compatibility of this action with the vertical concatenation. Let G and G' be two graphs such that $o(G) = i(G')$, and let $\sigma \in \mathfrak{S}_{o(G')}$, $\tau \in \mathfrak{S}_{o(G)}$, $\nu \in \mathfrak{S}_{i(G)}$. Clearly, the graphs $\sigma \cdot (G' \circ G)$ and $(\sigma \cdot G') \circ G$ are equal; the graphs $(G' \circ G) \cdot \nu$ and $G' \circ (G \cdot \nu)$

are equal. Let us compare the graphs $H = (G' \cdot \tau) \circ G$ and $H' = G' \circ (\tau \cdot G)$. Their set of vertices coincide. Moreover:

$$\begin{aligned} E(H) &= E(G) \sqcup E(G') \sqcup \{(f, e) \in O(G) \times I(G') : \beta(f) = \tau^{-1} \circ \alpha'(e)\}, \\ E(H') &= E(G) \sqcup E(G') \sqcup \{(f, e) \in O(G) \times I(G') : \tau \circ \beta(f) = \alpha'(e)\}, \end{aligned}$$

so $E(H) = E(H')$. Similarly, $I(H) = I(H')$, $O(H) = O(H')$, $IO(H) = IO(H')$ and $L(H) = L(H')$. Moreover, the source, target and indexation maps are the same for H and H' , so $H = H'$.

- We finally prove the **compatibility** of the $\mathfrak{S} \times \mathfrak{S}^{op}$ -action with the horizontal composition. Let G and G' be two graphs, $\sigma \in \mathfrak{S}_{o(G)}$ and $\sigma' \in \mathfrak{S}_{o(G')}$. We put $H = (\sigma \cdot G) * (\sigma' \cdot G')$ and $H' = (\sigma \otimes \sigma') \cdot (G * G')$. They have the same set of vertices, whether internal, input, output and input-output edges, and the source, target and indexation of output edges maps for H and H' coincide. Both indexations of the set of output edges are given by:

$$\sigma''(e) = \begin{cases} \sigma \circ \beta(e) & \text{if } e \in O(G) \sqcup IO(G), \\ o(G) + \sigma' \circ \beta'(e) & \text{if } e \in O(G') \sqcup IO(G'). \end{cases}$$

So $H = H'$.

Let G and G' be graphs. We set $H = c_{o(G), o(G')} \cdot (G * G')$ and $H' = (G' * G) \cdot c_{i(G), i(G')}$, where $c_{m,n} \in \mathfrak{S}_{m+n}$ was defined in (2). They have the same sets of vertices, internal, input, output and input-output edges, and the same source and target maps. The indexations maps are given by:

$$\begin{aligned} \alpha_H(e) &= \begin{cases} \alpha(e) + i(G') & \text{if } e \in I(G) \sqcup IO(G), \\ \alpha'(e) & \text{if } e \in I(G') \sqcup IO(G'), \end{cases} \\ \beta_H(e) &= \begin{cases} \beta(e) & \text{if } e \in O(G) \sqcup IO(G), \\ \beta'(e) + o(G) & \text{if } e \in O(G') \sqcup IO(G'), \end{cases} \\ \alpha_{H'}(e) &= \begin{cases} \alpha'(e) & \text{if } e \in I(G') \sqcup IO(G'), \\ \alpha(e) + i(G') & \text{if } e \in I(G) \sqcup IO(G), \end{cases} \\ \beta_{H'}(e) &= \begin{cases} \beta'(e) + o(G) & \text{if } e \in O(G') \sqcup IO(G'), \\ \beta(e) & \text{if } e \in O(G) \sqcup IO(G), \end{cases} \end{aligned}$$

so $H = H'$. □

2 The ProP of continuous morphisms: \mathbf{Hom}_V^c

We now generalise the ProP \mathbf{Hom}_V of Subsection 1.2 to a ProP \mathbf{Hom}_V^c (the superscript "c" for continuous) for a topological vector space V .

We work in the context of nuclear Fréchet spaces. One could relax these conditions (for example Fréchet could be replaced by barreled) yet the nuclear setup is comfortable to work in and general enough for our purposes. We refer the reader to [Tre67] for the more general cases.

2.1 Fréchet nuclear spaces

Nuclear spaces were defined in the seminal work [Gro54]. Most of the results stated here can be found in [Gro52, Gro54]. We also refer to the more recent presentation [Tre67].

We recall that

- A topological vector space is **Fréchet** if it is Hausdorff, has its topology induced by a family of semi-norms and is complete with respect to this family of semi-norms.
- A topological vector space is called **reflexive** if $E'' = (E')' = E$, where E' is the topological dual of E .

In the following E and F are two topological vector spaces and $\mathbf{Hom}^c(E, F)$ is the set of continuous linear maps from E to F .

Remark 2.1.1. When E and F are finite dimensional, we have $\mathbf{Hom}^c(E, F) = \mathbf{Hom}(E, F)$.

In order to build the **Hom** ProP in the infinite dimensional case, we need Grothendieck's completion of the tensor product, a notion we recall here in the setup of locally convex topological \mathbb{K} -vector spaces.

Let E and F be two vector spaces. Recall that there exists a vector space $E \otimes F$, and a bilinear map $\phi : E \times F \longrightarrow E \otimes F$ such that for any vector space V and bilinear map $f : E \times F \longrightarrow V$, there is a unique linear map $\tilde{f} : E \otimes F \rightarrow V$ satisfying $f = \tilde{f} \circ \phi$. The space $E \otimes F$ is unique modulo isomorphism and is called the **tensor product** of E and F .

Given two topological vector spaces, E and F one can a priori equip $E \otimes F$ with several topologies, among which the ϵ -**topology** and the **projective topology** whose construction are recalled in Appendix A. We denote by $E \otimes_\epsilon F$ (resp. $E \otimes_\pi F$) the space $E \otimes F$ endowed with the ϵ -topology (resp. the projective topology) and by $E \widehat{\otimes}_\epsilon F$ (resp. $E \widehat{\otimes}_\pi F$) of $E \otimes_\epsilon F$ (resp. $E \otimes_\pi F$) their completion with respect to the ϵ -topology (resp. projective topology). These two spaces differ in general but coincide for nuclear spaces.

Definition 2.1.1. [Gro54] *A locally convex topological vector space E is **nuclear** if, and only if, for any locally convex topological vector space F ,*

$$E \widehat{\otimes}_\epsilon F = E \widehat{\otimes}_\pi F =: E \widehat{\otimes} F$$

*holds, in which case $E \widehat{\otimes} F$ is called the **completed tensor product** of E and F .*

There are other equivalent definitions of nuclearity, see for example [GV64, HS08].

Given a locally convex topological vector space E , its topological dual E' can be endowed with various topologies. An important one for our applications will be the **strong topology**, generated by the family of semi-norms of E' defined, on any $f \in E'$: $\|f\|_B := \sup_{x \in B} |f(x)|$ for any bounded set B of E . The topological dual E' endowed with this topology is called the **strong dual**.

For Fréchet spaces, nuclearity is preserved under strong duality.

Proposition 2.1.2. • [Tre67, Proposition 50.6] *A Fréchet space is nuclear if and only if its strong dual is nuclear.*

- [Tre67, Proposition 36.5] *A Fréchet nuclear space is reflexive.*

Many spaces relevant to renormalisation issues are Fréchet and nuclear. We list here some examples.

Example 2.1.1. Any finite dimensional vector space can be equipped with a norm and for any of these norms, they are trivially Banach, hence Fréchet and nuclear. If E and F are finite dimensional vector spaces we have $\mathbf{Hom}^c(E, F) = \mathbf{Hom}(E, F) \simeq E^* \otimes F$, where $\mathbf{Hom}(E, F)$ stands for the space of F -valued linear maps on E and where the dual E^* is the **algebraic dual**.

Example 2.1.2. Let U be an open subset of \mathbb{R}^n . Take $E = C^\infty(U) =: \mathcal{E}(U)$. The topological dual is the space $E' = \mathcal{E}'(U)$ of distributions on U with compact support.

Then E is Fréchet ([Tre67], pp. 86-89), and E' is nuclear ([Tre67], Corollary p. 530). By Proposition 2.1.2, E is also nuclear.

Remark 2.1.2. Note that the dual E' of a Fréchet space E is never a Fréchet space (for any of the natural topologies on E'), unless E is actually a Banach space (see for example [Köt69]). In particular, $\mathcal{E}'(U)$ is generally not Fréchet.

We now sum up various results of [Tre67] of importance for later purposes.

Theorem 2.1.3. [Tre67, Equations (50.17)–(50.19)] *Let E and F be two Fréchet spaces, with E nuclear. The following isomorphisms of topological vector spaces hold.*

$$E' \widehat{\otimes} F \simeq \mathbf{Hom}^c(E, F) \quad (4)$$

$$E \widehat{\otimes} F \simeq \mathbf{Hom}^c(E', F) \quad (5)$$

$$E' \widehat{\otimes} F' \simeq (E \widehat{\otimes} F)' \simeq \mathcal{B}^c(E \times F, \mathbb{K}). \quad (6)$$

with $\mathcal{B}^c(E \times F, \mathbb{K})$ the set of continuous bilinear maps $K : E \times F \rightarrow \mathbb{K}$. Here the duals are endowed with the strong dual topology, $\mathbf{Hom}^c(E, F)$ with the strong topology and $\mathcal{B}^c(E \times F, \mathbb{K})$ with the topology of uniform convergence on products of bounded sets.

We also need the stability of Fréchet nuclear spaces under completed tensor products, for which we need the following lemma.

Lemma 2.1.4. *The completed tensor product $E \widehat{\otimes} F$ of two Fréchet nuclear spaces is a Fréchet nuclear space.*

Proof. If E and F are two nuclear spaces then $E \widehat{\otimes} F$ is a nuclear space ([Tre67, Equation (50.9)]). It is moreover complete since $E \widehat{\otimes} F$ is obtained by completion. \square

Proposition 2.1.5. *Let V be a Fréchet nuclear space. Then*

$$(V^{\widehat{\otimes} k})' \simeq (V')^{\widehat{\otimes} k} \quad (7)$$

holds for any $k \geq 1$, where the duals are endowed with their strong topologies.

Proof. Let V be a Fréchet nuclear space. The case $k = 1$ is trivial. Then Equation (7) with $k = 2$ holds by Equation (6) with $E = F = V$. The cases $k \geq 2$ are proved by induction, using $E = V^{\widehat{\otimes} k-1}$ and $F = V$. The induction holds by Lemma 2.1.4. \square

2.2 A ProP for Fréchet nuclear spaces

We start by recalling the definition of distributions over a finite dimensional smooth manifold X . We quote [Hö89, Definition 6.3.3].

Definition 2.2.1. *To every coordinate system $\kappa : U_k \subset X \rightarrow V_k \subset \mathbb{R}^n$ we associate a distribution $u_k \in \mathcal{D}'(V_k)$ such that*

$$u_{k'} = (\kappa \circ \kappa'^{-1})^* u_k$$

in $\kappa'(U_k \cap U_{k'})$; with $(\kappa \circ \kappa'^{-1})^ u_k$ the pullback of u_k by $\kappa \circ \kappa'^{-1}$ whose existence and uniqueness is given by [Hö89, Theorem 6.1.2]. Then the system u_k of distributions is called a distribution on X . The set of distributions on X is written $\mathcal{D}'(X)$. Similarly we define $\mathcal{E}'(X)$, the set of distributions with compact support.*

Proposition 2.2.2. *$\mathcal{E}(X)$ is a Fréchet nuclear space.*

It is a classical result of functional analysis that the space of functions over a smooth manifold is Fréchet (see for example [vdBC13, Exercise 2.3.2]). The fact that the same space is nuclear is a folklore result, often stated without proof nor references. A proof was recently given in [BDLGR17, p. 4].

It then follows from Proposition 2.1.2, that the space $\mathcal{E}'(X)$ is also nuclear.

Remark 2.2.1. (Compare with Remark 2.1.2). Note that the space $\mathcal{E}'(X)$ is *not* Fréchet since the dual of a Fréchet space F is Fréchet if and only if F is Banach (see for example [Köt69]) which is not the case of $\mathcal{E}(X)$.

One further useful result is

Proposition 2.2.3. *Let X and Y be two finite dimensional smooth manifolds. Then*

$$\mathbf{Hom}^c(\mathcal{E}'(X), \mathcal{E}(Y)) \simeq \mathcal{E}(X) \widehat{\otimes} \mathcal{E}(Y) \simeq \mathcal{E}(X \times Y)$$

holds.

The second isomorphism [Gro52, Chap. 5, p. 105] can be proved using a version of the Schwartz kernel theorem for smoothing operators [vdBC13, Theorem 2.4.5] by means of the identification $\mathbf{Hom}^c(\mathcal{E}'(X), \mathcal{E}(Y)) \simeq \mathcal{E}(X \times Y)$. The result then follows from (5) applied to $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ which are Fréchet nuclear spaces.

Definition 2.2.4. *Let V be a Fréchet nuclear space. For any $k, l \in \mathbb{N}_0$, we set*

$$\mathbf{Hom}_V^c(k, l) = \mathbf{Hom}^c(V^{\widehat{\otimes} k}, V^{\widehat{\otimes} l}) \simeq (V')^{\widehat{\otimes} k} \widehat{\otimes} V^{\widehat{\otimes} l},$$

where, as before V' stands for the strong topological dual. Furthermore we set $\mathbf{Hom}_V^c := (\mathbf{Hom}_V^c(k, l))_{k, l \geq 0}$.

For any $\sigma \in \mathfrak{S}_n$, let θ_σ be the endomorphism of $V^{\widehat{\otimes} n}$ defined by

$$\theta_\sigma(v_1 \otimes \dots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(n)}.$$

It extends to a continuous linear map $\overline{\theta}_\sigma$ on the closure $V^{\widehat{\otimes} n}$. For any $f \in \mathbf{Hom}_V^c(k, l)$, $\sigma \in \mathfrak{S}_l$, $\tau \in \mathfrak{S}_k$, we set:

$$\sigma \cdot f = \overline{\theta}_\sigma \circ f, \quad f \cdot \tau = f \circ \overline{\theta}_\tau.$$

In the above definition, the superscript ‘‘c’’ stands for continuous. The family \mathbf{Hom}_V^c carries a ProP structure.

Theorem 2.2.5. *Let V be a Fréchet nuclear space. \mathbf{Hom}_V^c , with the action of $\mathfrak{S} \times \mathfrak{S}^{\text{op}}$ described above, is a ProP. Its horizontal concatenation is the usual (topological) tensor product of maps with $I_0 : \mathbb{K} \rightarrow \mathbb{K}$ is the constant map $I_0(x) := 1_{\mathbb{K}}$ and its vertical concatenation is the usual composition of maps and $I_1 : V \rightarrow V$ is the identity map.*

Proof. The proof is exactly the same as the proof of Definition-Proposition 1.2.1. \square

Example 2.2.1. For a finite dimensional vector space V the classical ProP \mathbf{Hom}_V of Proposition-Definition 1.2.1 coincides with the the ProP \mathbf{Hom}_V^c .

Example 2.2.2. Let U be an open of \mathbb{R}^n . From Example 2.1.2 and Equation (7) the family $(\mathcal{K}_U(k, l))_{k, l \geq 0}$, with $\mathcal{K}_U(k, l) = (\mathcal{E}'(U))^{\widehat{\otimes} k} \widehat{\otimes} (\mathcal{E}(U))^{\widehat{\otimes} l}$ defines a ProP.

Example 2.2.3. Let X be a smooth finite dimensional manifold. From Proposition 2.2.2 and Equation (7) the family $(\mathcal{K}_X(k, l))_{k, l \geq 0}$, with $\mathcal{K}_X(k, l) = (\mathcal{E}'(X))^{\widehat{\otimes} k} \widehat{\otimes} \mathcal{E}(X)^{\widehat{\otimes} l}$ defines a ProP.

3 Freeness of the ProP \mathbf{Gr}^\cup of graphs

The goal of this section is to build free ProPs generated by indecomposable graphs (see Definition 3.1.1 below). A free ProP was already described by Hackney and Robertson in [HR12]. Their construction is on the category of ‘‘megagraphs’’, which are special types of graphs with decorations on their vertices and edges. Their work is categorical and not very adapted for the applications we have in mind, which require a more explicit description of the structures at hand. This is why we carry out the proof of the freeness of the ProP introduced in subsection 1.3. The complete proof of the main theorem (Theorem 3.2.1) is postponed to Appendix C.1.

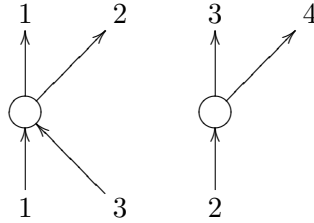
3.1 Indecomposable graphs

Definition 3.1.1. We call a graph G *indecomposable* if the five following conditions hold:

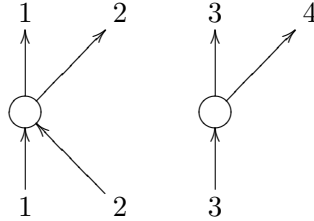
1. $V(G) \neq \emptyset$.
2. $IO(G) = \emptyset$.
3. $L(G) = \emptyset$ or G is reduced to a single loop.
4. If G' and G'' are two graphs such that $G = G' \circ G''$, then $V(G') = \emptyset$ or $V(G'') = \emptyset$.
5. If G' and G'' are two graphs and σ, τ are two permutations such that $G = \sigma \cdot (G' * G'') \cdot \tau$, then $V(G') = \emptyset$ or $V(G'') = \emptyset$.

For any $k, l \in \mathbb{N}_0$, the subspace of $\mathbf{Gr}^\cup(k, l)$ generated by isoclasses of indecomposable graphs G with $i(G) = k$ and $o(G) = l$ is denoted by $\mathbf{Gr}_{\text{ind}}^\cup(k, l)$.

Remark 3.1.1. 1. The permutations in the fifth item of the definition of indecomposable graphs play an important role: without them, one would allow for non connected graphs to be indecomposable, which can well happen when the indexations of the inputs and outputs of the various connected components do not match. For example, the graph



would be indecomposable. Permuting inputs we obtain



which is decomposable. The same requirement does not arise for the vertical concatenation since one can write $\sigma \cdot (P \circ Q) \cdot \tau = (\sigma \cdot P) \circ (Q \cdot \tau) = P' \circ Q'$.

2. There is one special indecomposable graph \mathcal{O} , formed by a unique loop. The other indecomposable graphs have no loop.

Proposition 3.1.2. Let G be a graph, $\sigma \in \mathfrak{S}_{o(G)}$ and $\tau \in \mathfrak{S}_{i(G)}$. Then G is indecomposable if, and only if, $\sigma \cdot G \cdot \tau$ is indecomposable.

Proof. Let us assume that $H = \sigma \cdot G \cdot \tau$ is indecomposable. Then $V(G) = V(H) \neq \emptyset$ and $IO(G) = IO(H) = \emptyset$. Let us assume that $G = G' \circ G''$. Then:

$$H = \sigma \cdot (G' \circ G'') \cdot \tau = (\sigma \cdot G') \circ (G'' \cdot \tau).$$

As H is indecomposable, $V(G') = V(\sigma \cdot G') = \emptyset$ or $V(G'') = V(G'' \cdot \tau) = \emptyset$. Let us assume that $G = \sigma' \cdot (G' * G'') \cdot \tau'$. Then:

$$H = \sigma \cdot (\sigma' \cdot (G' * G'') \cdot \tau') \cdot \tau = ((\sigma\sigma') \cdot G') * (G'' \cdot (\tau\tau')).$$

As H is indecomposable, $V(G') = V((\sigma\sigma') \cdot G') = \emptyset$ or $V(G'') = V(G'' \cdot (\tau\tau')) = \emptyset$.

Conversely, if G is indecomposable, then $G = \sigma^{-1} \cdot H \cdot \tau^{-1}$ is indecomposable, so H is indecomposable. \square

Notations 3.1.1. Let G be a graph.

1. Let $J \subseteq V(G)$. We define (non uniquely due to the non uniqueness of the maps α' and β') the graph $G_{|J}$ by:

$$\begin{aligned} V(G_{|J}) &= J, \\ E(G_{|J}) &= \{e \in E(G) : s(e) \in J, t(e) \in J\}, \\ I(G_{|J}) &= \{e \in I(G) : t(e) \in J\} \sqcup \{e \in E(G) : s(e) \notin J, t(e) \in J\}, \\ O(G_{|J}) &= \{e \in O(G) : s(e) \in J\} \sqcup \{e \in E(G) : s(e) \in J, t(e) \notin J\}, \\ IO(G_{|J}) &= IO(G), \\ L(G_{|J}) &= \emptyset. \end{aligned}$$

The source and target maps are defined by:

$$\begin{aligned} \forall e \in E(G_{|J}) \sqcup O(G_{|J}), & \quad s_{G_{|J}}(e) = s(e), \\ \forall e \in E(G_{|J}) \sqcup I(G_{|J}), & \quad t_{G_{|J}}(e) = t(e), \end{aligned}$$

The indexation of the input edges is any indexation map α' such that:

$$\forall e, e' \in (I(G) \sqcup IO(G)) \cap (I(G_{|J}) \sqcup IO(G_{|J})), \quad \alpha'(e) < \alpha'(e') \iff \alpha(e) < \alpha(e').$$

The indexation of the output edges is any indexation map β' such that:

$$\forall f, f' \in (O(G) \sqcup IO(G)) \cap (O(G_{|J}) \sqcup IO(G_{|J})), \quad \beta'(f) < \beta'(f') \iff \beta(f) < \beta(f').$$

2. We denote by \tilde{G} the graph defined by:

$$\begin{aligned} V(\tilde{G}) &= V(G), & E(\tilde{G}) &= E(G), & L(\tilde{G}) &= \emptyset, \\ I(\tilde{G}) &= I(G), & O(\tilde{G}) &= O(G), & IO(\tilde{G}) &= \emptyset, \\ \tilde{s} &= s, & \tilde{t} &= t. \end{aligned}$$

The indexation of the input edges is the unique indexation map $\tilde{\alpha}$ such that:

$$\forall e, e' \in I(G), \quad \tilde{\alpha}(e) < \tilde{\alpha}(e') \iff \alpha(e) < \alpha(e').$$

The indexation of the output edges is the unique indexation map $\tilde{\beta}$ such that:

$$\forall f, f'' \in O(G), \quad \tilde{\beta}(f) < \tilde{\beta}(f') \iff \beta(f) < \beta(f').$$

Roughly speaking, \tilde{G} is obtained from G by deletion of all the input-output edges and all the loops.

Definition 3.1.3. *Let G be a graph.*

1. A **path** in G is a sequence $p = (e_1, \dots, e_k)$ of internal edges of G such that for any $i \in [k-1]$, $t(e_i) = s(e_{i+1})$. The source of p is $s(e_1)$ and its target is $t(e_k)$, and we shall say that p is a path from $s(e_1)$ to $t(e_k)$ of length k . By convention, for any $x \in V(G)$, there exists a unique path from x to x of length 0.
2. We shall say that a path p is a **cycle** if its source and its target are equal and if its length is nonzero.

Remark 3.1.2. A cycle of length one is to be distinguished from a loop.

We consider oriented-pathwise connected components of graphs.

Lemma 3.1.4. *Let G be a graph such that $V(G) \neq \emptyset$. We denote by $\mathcal{O}(G)$ the set of nonempty subsets I of $V(G)$ such that for any $x \in I$, for any $y \in V(G)$, if there exists a path in G from x to y , then $y \in I$. Then:*

1. *If $I, J \in \mathcal{O}(G)$, either $I \cap J = \emptyset$ or $I \cap J \in \mathcal{O}(G)$.*
2. *For any $x \in V(G)$, there exists a unique element $\langle x \rangle \in \mathcal{O}(G)$ which contains x and is minimal for the inclusion. Moreover:*

$$\langle x \rangle = \{y \in V(G) : \text{there exists a path in } G \text{ from } x \text{ to } y\}.$$

Notice that, if G_x is the connected component of G that contains x , then $\langle x \rangle \subseteq G_x$, but we do not necessarily have an equality, as the edges are *oriented*.

Proof. 1. If $I \cap J \neq \emptyset$, let $x \in I \cap J$ and $y \in V(G)$ such that there exists a path in G from x to y . As $I, J \in \mathcal{O}(G)$, $y \in I \cap J$, so $I \cap J \in \mathcal{O}(G)$.

2. Note that $V(G) \in \mathcal{O}(G)$. Let $x \in V(G)$; by the first item, the following element of $\mathcal{O}(G)$ is the minimal (for the inclusion) element of $\mathcal{O}(G)$ that contains x :

$$\langle x \rangle = \bigcap_{I \in \mathcal{O}(G), x \in I} I.$$

On the one hand, a set I in $\mathcal{O}(G)$ contains x if and only if any path emanating from x ends at an element of I . So it contains all the ending vertices of such paths and hence the set

$$I_x := \{y \in V(G) : \text{there exists a path in } G \text{ from } x \text{ to } y\}.$$

Thus, $I_x \subseteq \langle x \rangle$. On the other hand, let $y \in I$ and $z \in V(G)$, such that there exists a path from y to z in G . As there exists a path from x to y in G , there exists a path from x to z , so $z \in I_x$. Hence, I_x lies in $\mathcal{O}(G)$ which in turn contains x , so $\langle x \rangle \subseteq I_x$. \square

Proposition 3.1.5. *Let G be a graph such that $V(G) \neq \emptyset$. We denote by J_1, \dots, J_k the minimal elements (for the inclusion) of the set $\mathcal{O}(G)$ of nonempty subsets I of $V(G)$ stable under paths as in Lemma 3.1.4, and we set $G_i = \tilde{G}_{|J_i}$ for any $i \in [k]$. Then G_1, \dots, G_k are indecomposable graphs with no loop and there exists a graph G_0 with no loop, integers p, ℓ and a permutation γ such that:*

$$G \approx (\gamma \cdot (G_1 * \dots * G_k * I_p) \circ G_0) * \mathcal{O}^{*\ell},$$

where, as before \mathcal{O} is the indecomposable graph formed by a unique loop. Such a decomposition will be called *minimal*.

Proof. By definition, $V(G_i) = J_i \neq \emptyset$ and $IO(G_i) = \emptyset$ for any i . Let us assume that $G_i = G' \circ G''$. If $V(G') \neq \emptyset$, then clearly $V(G') \in \mathcal{O}(G_i)$ and, as $J_i \in \mathcal{O}(G)$, we deduce that $V(G') \in \mathcal{O}(G)$. As J_i is minimal in $\mathcal{O}(G)$, $V(G') = J_i = V(G_i)$, so $V(G'') = \emptyset$. Similarly, if $G_i = \sigma \cdot (G' * G'') \cdot \tau$, then $V(G') = \emptyset$ or $V(G'') = \emptyset$: we proved that G_i is indecomposable.

Let us assume that $I = V(G_i) \cap V(G_j) \neq \emptyset$. Then $I \in \mathcal{O}(G)$ and, by minimality of J_i and J_j , $J_i = J_j = I$, so the J_i are disjoint.

Let us set $K := V(G) \setminus (J_1 \cup \dots \cup J_k)$ and $G' := G|_K$. As J_1, \dots, J_k lie in $\mathcal{O}(G)$, there is no internal edge of G from a vertex of G_i to a vertex of G' , and any outgoing edge of G' is either glued in G to an incoming edge of G_i or is an outgoing edge of G . Hence, there exists permutations γ, σ and τ , and three integers $p := |IO(G)|$, $q := |\{e \in I(G) : t(e) \in J_1 \cup \dots \cup J_k\}|$ and $\ell := |L(G)|$ such that:

$$G = \gamma \cdot (G_1 * \dots * G_k * I_p) \circ (\sigma \cdot (I_q * G') \cdot \tau) * \mathcal{O}^{*\ell}.$$

We conclude in taking $G_0 = \sigma \cdot (I_q * G') \cdot \tau$. \square

Note that this decomposition is not unique: it depends on the indexation of the minimal elements of $\mathcal{O}(G)$ and of the choice of the indexation of their input and output edges. Importantly, it depends only on that.

Proposition 3.1.6. *Let G be a graph such that $V(G) \neq \emptyset$ and $IO(G) = \emptyset$. The graph G is indecomposable if, and only if, $L(G) = \emptyset$ and for any $x, y \in V(G)$, there exists a path from x to y in G .*

Proof. First notice that if $|V(G)| = 1$ the result trivially holds. In the following, we therefore assume that $|V(G)| \geq 2$.

Let $G = \gamma \cdot (G_1 * \dots * G_k * I_p) \circ G_0 * \mathcal{O}^{*\ell}$ a minimal decomposition of G .

\implies Note that $V(G_1) \neq \emptyset$. As G is indecomposable, necessarily $\ell = 0$, $V(G_0) = \emptyset$, and there exists a permutation $\tau \in \mathfrak{S}_p$ such that $G_0 = I_q \cdot \tau$. Therefore, $G = \gamma \cdot (G_1 * \dots * G_k * I_p) \cdot \tau$. As G is indecomposable, $k = 1$ and $V(G) = V(G_1) = J_1$. Hence, $V(G_1)$ is both a minimal and the maximal element of $\mathcal{O}(G)$, which is consequently reduced to the singleton $\{V(G)\}$. Therefore, for any $x \in V(G)$, $\langle x \rangle = V(G)$, so for any $y \in V(G)$, there exists a path from x to y in G .

\impliedby Firstly, note that $L(G) = \emptyset \implies \ell = 0$. If $k \geq 2$, there is no path in G from any vertex of G_1 to any vertex of G_2 , so $k = 1$. Thus, $V(G_0) = \emptyset$ and there exists a permutation τ such that $G_0 = I_p \cdot \tau$. We obtain that

$$G = \gamma \cdot (G_1 * I_p) \cdot \tau.$$

As $IO(G) = \emptyset$, we obtain that $p = 0$, so $G = \gamma \cdot G \cdot \tau$ is indecomposable. \square

Remark 3.1.3. Another way to formulate the above Proposition is to say that a graph G is indecomposable if, and only if, one (and only one) of the following conditions holds:

- $G = \mathcal{O}$.
- G has no loop, is connected and for any of its vertices x , a cycle of strictly positive length goes through x .

3.2 Freeness of \mathbf{Gr}^\cup

We now state and give a sketch of the proof of one of the main results of this section, namely the freeness of the ProP \mathbf{Gr}^\cup . To our knowledge, this result is new.

Theorem 3.2.1. *Let P be a ProP and $\phi : \mathbf{Gr}_{\text{ind}}^\cup \longrightarrow P$ be a morphism of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules. There exists a unique ProP morphism $\Phi : \mathbf{Gr}^\cup \longrightarrow P$ such that $\Phi|_{\mathbf{Gr}_{\text{ind}}^\cup} = \phi$. In other words, \mathbf{Gr}^\cup is the free ProP generated by $\mathbf{Gr}_{\text{ind}}^\cup$.*

Proof. We provide here a sketch of the proof, and refer the reader to Appendix C.1 for a full proof. We define $\Phi(G)$ for any graph G by induction on its number n of vertices. If $n = 0$, there exists a permutation $\sigma \in \mathfrak{S}_k$ such that $G = \sigma \cdot I_k$. We set

$$\Phi(G) = \sigma \cdot I_k.$$

If $n > 0$ and G is indecomposable, we set $\Phi(G) = \phi(G)$. Otherwise, let

$$G = \gamma \cdot (G_1 * \dots * G_k * I_p) \circ G_0 * \mathcal{O}^{*\ell}$$

be a minimal decomposition of G . As $V(G_1) \neq \emptyset$, $|V(G_0)| < n$, we set:

$$\Phi(G) = \gamma \cdot (\phi(G_1) * \dots * \phi(G_k) * I_p) \circ \Phi(G_0) * \phi(\mathcal{O})^{*\ell}.$$

One can prove that this does not depend on the choice of the minimal decomposition of G with the help of the ProP axioms applied to P . Using minimal decompositions of vertical or horizontal concatenations of graphs, one can show that Φ is compatible with both concatenations. \square

3.3 Cycleless graphs

Definition 3.3.1. For any $k, l \in \mathbb{N}_0$, we denote by $\mathbf{Gr}^\uparrow(k, l)$ the subspace of $\mathbf{Gr}^\circ(k, l)$ generated by the graphs which do not contain any cycle nor any loop. Note that \mathbf{Gr}^\uparrow is a $\mathfrak{S} \times \mathfrak{S}^{op}$ -sub-module of \mathbf{Gr}° .

As before, we write $\mathbf{Gr}_{\text{ind}}^\uparrow$ for the set of indecomposable cycleless and loopless graphs in \mathbf{Gr}^\uparrow .

A simple yet important observation is the following.

Proposition 3.3.2. \mathbf{Gr}^\uparrow is a sub-ProP of \mathbf{Gr}° .

Proof. First, notice that I_0 and I_1 are in \mathbf{Gr}^\uparrow . Let us check the stability of \mathbf{Gr}^\uparrow under the horizontal and vertical concatenation.

Let G_1, G_2 be two graphs without cycle. By construction, there is no edge e of $G_1 * G_2$ such that $s(e) \in V(G_1)$ and $t(e) \in V(G_2)$, or such that $s(e) \in V(G_2)$ and $t(e) \in V(G_1)$. So a cycle in $G_1 * G_2$ is a cycle in G_1 or G_2 . Thus \mathbf{Gr}^\uparrow is stable by horizontal concatenation.

Similarly, let G_1, G_2 be two graphs without cycle such that $G_1 \circ G_2$ is defined. Then using the same argument, a cycle of $G_1 \circ G_2$ must either be a cycle of G_1 , a cycle of G_2 (both being contradictions) or contain an edge e such that $s(e) \in V(G_1)$ and $t(e) \in V(G_2)$. This contradicts the definition of \circ for graphs. \square

In this particular example, we recover the description of a free ProP in terms of oriented graphs [Val03, Val09]:

Proposition 3.3.3. For any $k, l \in \mathbb{N}_0$, we denote by $G_{k,l}$ the graph such that:

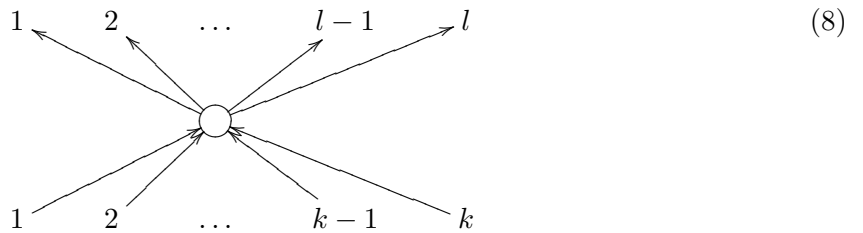
$$\begin{aligned} V(G_{k,l}) &= \{\star\}, & I(G_{k,l}) &= [k], & IO(G_{k,l}) &= \emptyset, \\ E(G_{k,l}) &= \emptyset, & O(G_{k,l}) &= [l] & L(G_{k,l}) &= \emptyset. \end{aligned}$$

For any $i \in [k]$, for any $j \in [l]$:

$$\begin{aligned} \alpha(i) &= i, & \beta(j) &= j, \\ t(i) &= \star, & s(j) &= \star. \end{aligned}$$

These graphs generate a trivial $\mathfrak{S} \times \mathfrak{S}^{op}$ -module $\mathbf{Gr}_{\text{ind}}^\uparrow$, and \mathbf{Gr}^\uparrow is the free ProP generated by $\mathbf{Gr}_{\text{ind}}^\uparrow$.

Graphically, $G_{k,l}$ is represented as follows:



Proof. For any permutations $\sigma \in \mathfrak{S}_l$, $\tau \in \mathfrak{S}_k$, $\sigma \cdot G_{k,l} \cdot \tau$ is isomorphic to $G_{k,l}$, through the isomorphism defined by:

$$f_V = \text{Id}_{\{\star\}}, \quad f_I = \tau^{-1}, \quad f_O = \sigma.$$

so indeed these graphs generate a trivial $\mathfrak{S} \times \mathfrak{S}^{op}$ -module.

Since sub-graphs of a graph without cycle, are without cycle, the following is an easy consequence of Proposition 3.1.5.

Lemma 3.3.4. *Let G be a graph without cycle and without loop, such that $V(G) \neq \emptyset$, then a minimal decomposition*

$$G \approx \gamma \cdot (G_1 * \dots * G_k * I_p) \circ G_0$$

yields a decomposition without cycles. Note that, as G has no loop, $\ell = 0$.

Let G be an indecomposable graph without cycle nor loop and let us assume that $V(G) \geq 2$. Let $x \neq y$ in $V(G)$. As G is indecomposable, by Proposition 3.1.6, there exists a path (e_1, \dots, e_k) from x to y in G and a path (f_1, \dots, f_l) from y to x in G . Hence, there exists a cycle $(e_1, \dots, e_k, f_1, \dots, f_l)$ in G : this is a contradiction. We obtain that $V(G)$ is reduced to a single element. As $IO(G) = L(G) = \emptyset$, $G = G_{i(G), o(G)}$. This gives:

$$\mathbf{Gr}^\uparrow \cap \mathbf{Gr}_{\text{ind}}^\circ = \mathbf{Gr}_{\text{ind}}^\uparrow.$$

As \mathbf{Gr}° is the free ProP generated by $\mathbf{Gr}_{\text{ind}}^\circ$, for any $\mathfrak{S} \times \mathfrak{S}^{op}$ -sub-module P of $\mathbf{Gr}_{\text{ind}}^\circ$, the sub-ProP of \mathbf{Gr}° generated by P is freely generated by P . This holds in particular for $\mathbf{Gr}_{\text{ind}}^\uparrow$. It remains to prove that the sub-ProP $\langle \mathbf{Gr}_{\text{ind}}^\uparrow \rangle$ generated by $\mathbf{Gr}_{\text{ind}}^\uparrow$ is \mathbf{Gr}^\uparrow .

Clearly, if G and G' are graphs without cycles, then $G * G'$ and $G \circ G'$ are without cycles, so \mathbf{Gr}^\uparrow is a sub-ProP of \mathbf{Gr}° , which contains $\mathbf{Gr}_{\text{ind}}^\uparrow$. Consequently, $\langle \mathbf{Gr}_{\text{ind}}^\uparrow \rangle \subseteq \mathbf{Gr}^\uparrow$. Conversely, let G be a graph without cycle and let us prove that $G \in \langle \mathbf{Gr}_{\text{ind}}^\uparrow \rangle$ by induction on $n = |V(G)|$. If $n = 0$, then $G = \sigma \cdot I_k$ for a certain permutation $\sigma \in \mathfrak{S}_k$, so G belongs to $\langle \mathbf{Gr}_{\text{ind}}^\uparrow \rangle$. Otherwise, let us consider a minimal decomposition of G in \mathbf{Gr}^c (see Lemma 3.3.4):

$$G = \gamma \cdot (G_1 * \dots * G_k * I_p) \circ G_0.$$

Since G_1, \dots, G_k are indecomposable, they lie in $\mathbf{Gr}_{\text{ind}}^\uparrow$. Since $k \geq 1$ and $V(G_i) \neq \emptyset$ we have $G_0 \in \langle \mathbf{Gr}_{\text{ind}}^\uparrow \rangle$ by the induction hypothesis, so $G \in \langle \mathbf{Gr}_{\text{ind}}^\uparrow \rangle$. \square

Remark 3.3.1. One can also work with graphs with various types of edges: each edge e (internal, input, output or input-output) of the graph under consideration has a type $\text{type}(e)$, chosen in a fixed set of types T . The horizontal composition of two graphs G and G' exists in any case, whereas their vertical concatenation exists if, and only if, for any $i \in [o(G)]$, the type of the output edge $\beta^{-1}(i)$ of G and the type of the input edge $\alpha^{-1}(i)$ of G' are the same. One obtains a T -coloured ProP, and one can prove similarly a freeness result. Restricting to typed graphs without cycles, we obtain a free T -coloured ProP generated by graphs with only one vertex (and no input-output edge).

3.4 Planar graphs and free ProPs

We recall from Definition 1.3.1 that $s : E(G) \sqcup O(G) \longrightarrow V(G)$ stands for the source map and $t : E(G) \sqcup I(G) \longrightarrow V(G)$ stands for the target map.

Definition 3.4.1. *Let G be a graph and $v \in E(G)$ be a vertex of G . We put:*

$$\begin{aligned} I(v) &= \{e \in I(G) \sqcup E(G), t(e) = v\}, \\ O(v) &= \{e \in O(G) \sqcup E(G), s(e) = v\}. \end{aligned}$$

We also set $i(v) = |I(v)|$ and $o(v) = |O(v)|$.

The number $i(v)$ (resp. $o(v)$) counts the number of input (resp. output) edges and ingoing (resp. outgoing) arrows at the vertex v .

Definition 3.4.2. *A **planar graph** is a graph G such that, for any vertex $v \in V(G)$, $I(v)$ and $O(v)$ are totally ordered. The set of planar graphs is denoted by \mathbf{PGr}° and the set of planar graphs with no cycle and no loop is denoted by \mathbf{PGr}^\uparrow . The set of planar graphs G (resp. of planar graphs G with no cycle and no loop) with $|I(G)| + |IO(G)| = k$ and $|O(G)| + |IO(G)| = l$ is denoted by $\mathbf{PGr}^\circ(k, l)$ (resp. by $\mathbf{PGr}^\uparrow(k, l)$).*

Graphically, we shall represent the orders on the incoming and outgoing edges of a vertex by drawing the vertices by boxes, the incoming and outgoing edges of any vertex being ordered from left to right. For example, we distinguish the two following situations:



Remark 3.4.1. This notion of planarity is not the usual one used in graph theory, as we authorise crossings of edges.

Since a planar graph is a graph, the horizontal and vertical concatenation of planar graphs are defined by the concatenations of the underlying graphs, which preserve the orders around each of the vertices. It is a simple exercise to check that $\mathbf{PGr}^{\circlearrowleft}$ is still a $\mathfrak{S} \times \mathfrak{S}^{op}$ -module and we left it to the reader. Hence, $\mathbf{PGr}^{\circlearrowleft}$ inherits a ProP structure from $\mathbf{Gr}^{\circlearrowleft}$. As before, \mathbf{PGr}^{\uparrow} is a sub-ProP of $\mathbf{PGr}^{\circlearrowleft}$.

We shall say that a planar graph is indecomposable if the underlying graph is indecomposable. The set $\mathbf{PGr}_{\text{ind}}^{\circlearrowleft}$ of indecomposable planar graphs forms a $\mathfrak{S} \times \mathfrak{S}^{op}$ -module. We then obtain a minimal decomposition of planar graphs similar to the one of Proposition 3.1.5. For any $k, l \in \mathbb{N}$, we denote by $PG_{k,l}$ the planar graph obtained from $G_{k,l}$ by ordering the sets $[k]$ and $[l]$ of incoming and outgoing edges of the unique vertex \star by their usual orders. We obtain the planar counterpart of Theorem 3.2.1.

Theorem 3.4.3. 1. Let P be a ProP and $\phi : \mathbf{PGr}_{\text{ind}}^{\circlearrowleft} \longrightarrow P$ be a morphism of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules. There exists a unique ProP morphism $\Phi : \mathbf{Gr}^{\circlearrowleft} \longrightarrow P$ such that $\Phi|_{\mathbf{PGr}_{\text{ind}}^{\circlearrowleft}} = \phi$. In other words, $\mathbf{PGr}^{\circlearrowleft}$ is the free ProP generated by $\mathbf{PGr}_{\text{ind}}^{\circlearrowleft}$.

2. The planar graphs $PG_{k,l}$ generate a free $\mathfrak{S} \times \mathfrak{S}^{op}$ -module $\mathbf{PGr}_{\text{ind}}^{\uparrow}$, and \mathbf{PGr}^{\uparrow} is the free ProP generated by $\mathbf{PGr}_{\text{ind}}^{\uparrow}$.

4 Graphs decorated by ProPs and endofunctors of ProPs

This section is motivated by Feynman graphs, in which case the decorations are distribution kernels. Since we expect to be able to equip the later with a ProP structure, we study here graphs decorated by ProPs. The results of Section 3 then allow us to build a endofunctor Γ^{\uparrow} on the category of ProPs.

4.1 The ProP $\mathbf{Gr}^{\circlearrowleft}(X)$ of decorated graphs as a free ProP

Throughout this paragraph, $X = (X_{k,l})_{k,l \geq 0}$ is a family of sets.

Definition 4.1.1. A graph decorated by X (or X -decorated graph, or simply decorated graph) is a couple (G, d_G) with G a graph as in Definition 1.3.1 and $d_G : V(G) \longrightarrow \bigsqcup_{k,l \in \mathbb{N}_0} X_{k,l}$ a map,

such that for any vertex $v \in V(G)$, $d_G(v) \in X_{i(v), o(v)}$. We denote by $\mathbf{Gr}^{\circlearrowleft}(X)$ (resp. $\mathbf{Gr}^{\uparrow}(X)$) the set of graphs (resp. the set of cycleless graphs) decorated by X . We define similarly X -decorated planar graphs and we denote by $\mathbf{PGr}^{\circlearrowleft}(X)$ (resp. $\mathbf{PGr}^{\uparrow}(X)$) the set of planar graphs (resp. the set of cycleless planar graphs) decorated by X .

Most of the results on graphs naturally generalise to X -decorated graphs. In particular, we have the horizontal (resp. vertical) concatenation of graphs, denoted by $*$ (resp. \circ):

$$(G, d_G) * (G', d_{G'}) = (G * G', d_{G * G'}), \quad (G, d_G) \circ (G', d_{G'}) = (G \circ G', d_{G \circ G'}).$$

The set of vertices of $G * G'$ and $G' \circ G'$ both being the disjoint union $V(G) \sqcup V(G')$ of the vertices of G and G' , we define $d_{G * G'} = d_{G \circ G'}$ on the set $V(G) \cup V(G')$ by $d_{G * G'}|_{V(G)} := d_G$, $d_{G * G'}|_{V(G')} := d_{G'}$.

Furthermore, the actions on the left and on the right of the permutation group on \mathbf{Gr}^\cup extend to actions on $\mathbf{Gr}^\cup(X)$ since the aforementioned actions leave the set of vertices of a graph invariant. Here are the decorated and cycleless versions of Theorem 1.3.3, which to our knowledge is new:

Theorem 4.1.2. *The families $\mathbf{Gr}^\cup(X)$ and $\mathbf{PGr}^\cup(X)$, equipped the above $\mathfrak{S} \times \mathfrak{S}^{op}$ -action and the above horizontal and vertical concatenations, are ProPs. The family $\mathbf{Gr}^\uparrow(X)$ is a sub-ProP of $\mathbf{Gr}^\cup(X)$ and the family $\mathbf{PGr}^\uparrow(X)$ is a sub-ProP of $\mathbf{PGr}^\cup(X)$.*

Proposition 3.1.5 generalises to the case of decorated graphs.

Proposition 4.1.3. *Let (G, d_G) be an X -decorated graph such that $V(G) \neq \emptyset$. We denote by J_1, \dots, J_k the minimal elements (for the inclusion) of $\mathcal{O}(G)$. As before, we set $G_i = \tilde{G}|_{J_i}$ and $d_i = d_G|_{J_i}$ for any $i \in [k]$. Then there exists an X -decorated graph (G_0, d_0) with no loop, integers p, ℓ and a permutation γ such that:*

$$(G, d_G) \approx \gamma \cdot ((G_1, d_1) * \dots * (G_k, d_k) * I_p) \circ (G_0, d_0) * \mathcal{O}^{*\ell}.$$

As in the non decorated case, we call such a decomposition minimal.

Proof. By Proposition 3.1.5, G admits a minimal decomposition

$$G \approx \gamma \cdot (G_1 * \dots * G_k * I_p) \circ G_0 * \mathcal{O}^{*\ell}.$$

We observe that we can identify the vertices of G_0 with those of $G \setminus (G_1 \sqcup \dots \sqcup G_k)$. We can therefore set $d_0 := d_G|_{V(G) \setminus (V(G_1) \sqcup \dots \sqcup V(G_k))}$. The result then follows from the definition of the actions of the permutation group on $\mathbf{Gr}^\cup(X)$ using the horizontal and vertical concatenations. \square

As in the non decorated case, we denote by $\mathbf{Gr}_{\text{ind}}^\cup(X)$ (resp. $\mathbf{Gr}_{\text{ind}}^\uparrow(X)$) the indecomposable graphs (resp. the cycleless indecomposable graphs) decorated by X . Notice that the graphs (G_i, d_i) ; for $i \in [k]$, are indecomposable. We define $\mathbf{PGr}_{\text{ind}}^\cup(X)$ and $\mathbf{PGr}_{\text{ind}}^\uparrow(X)$ similarly.

The key result of this paragraph is the decorated version of the universal property (Theorem 3.2.1).

Theorem 4.1.4. 1. *Let P be a ProP and $\phi : \mathbf{Gr}_{\text{ind}}^\cup(X) \longrightarrow P$ be a morphism of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules. There exists a unique ProP morphism $\Phi : \mathbf{Gr}^\cup(X) \longrightarrow P$ such that $\Phi|_{\mathbf{Gr}_{\text{ind}}^\cup(X)} = \phi$. In other words, $\mathbf{Gr}^\cup(X)$ is the free ProP generated by the $\mathfrak{S} \times \mathfrak{S}^{op}$ -module $\mathbf{Gr}_{\text{ind}}^\cup(X)$.*

Furthermore, $\mathbf{Gr}^\uparrow(X)$ is the free ProP generated by the $\mathfrak{S} \times \mathfrak{S}^{op}$ -module $\mathbf{Gr}_{\text{ind}}^\uparrow(X)$, which is isomorphic to the trivial $\mathfrak{S} \times \mathfrak{S}^{op}$ -module generated by X .

2. *Let P be a ProP and $\phi : \mathbf{PGr}_{\text{ind}}^\cup(X) \longrightarrow P$ be a morphism of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules. There exists a unique ProP morphism $\Phi : \mathbf{PGr}^\cup(X) \longrightarrow P$ such that $\Phi|_{\mathbf{PGr}_{\text{ind}}^\cup(X)} = \phi$. In other words, $\mathbf{PGr}^\cup(X)$ is the free ProP generated by the $\mathfrak{S} \times \mathfrak{S}^{op}$ -module $\mathbf{PGr}_{\text{ind}}^\cup(X)$.*

Furthermore, $\mathbf{PGr}^\uparrow(X)$ is the free ProP generated by the $\mathfrak{S} \times \mathfrak{S}^{op}$ -module $\mathbf{PGr}_{\text{ind}}^\uparrow(X)$ generated by X , which is isomorphic to the free $\mathfrak{S} \times \mathfrak{S}^{op}$ -module generated by X .

Remark 4.1.1. 1. This result generalises Theorem 3.2.1 and Proposition 3.3.3. However, it is *not* a direct consequence of these previous results. Given $G \in \mathbf{Gr}^\cup$, $d_G, d'_G : V(G) \longrightarrow X$ two decoration maps of G , we a priori have $\Phi(G, d_G) \neq \Phi(G, d'_G)$.

2. The $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules $\mathbf{Gr}_{\text{ind}}^{\uparrow}(X)$ and $\mathbf{PGr}_{\text{ind}}^{\uparrow}(X)$ differ from one another in so far as for any $k, l \in \mathbb{N}_0$, $\mathbf{Gr}_{\text{ind}}^{\uparrow}(k, l)$ is a trivial $\mathfrak{S}_l \otimes \mathfrak{S}_k^{op}$ -module, whereas $\mathbf{PGr}_{\text{ind}}^{\uparrow}(k, l)$ is a free $\mathfrak{S}_l \otimes \mathfrak{S}_k^{op}$ -module. They are both generated by $X_{k,l}$.

Proof. The proofs of Theorem 3.2.1 and Proposition 3.3.3 can be reproduced in extenso in the decorated setup, simply replacing graphs by decorated graphs and using the decorated version of the minimal decomposition and will therefore not reproduce it here. Let us however notice that

- the transformation of type A arising in the proof of Theorem 3.2.1 only concerns indexation of edges. As such, it easily generalises to decorated graphs.
- the transformation of type B arising in the proof of Theorem 3.2.1 exchanges two subgraphs of G . It therefore extends to the decorated case as a transformation exchanging two decorated graphs. The rest of the proof of Theorem 3.2.1 remains unchanged.
- the cycleless indecomposable graphs are still in the decorated case the graphs with exactly one vertex, since the decorations play no role in the definition of indecomposable.
- the rest of the proof of 3.3.3 also generalises in a straightforward manner to the decorated case. \square

4.2 An endofunctor of the category of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules

We now assume that the family $X = (X_{k,l})_{k,l \in \mathbb{N}_0}$ is a $\mathfrak{S} \times \mathfrak{S}^{op}$ -module. We define another $\mathfrak{S} \times \mathfrak{S}^{op}$ -module $\Gamma^{\cup}(X)$ on graphs, taking into account this module structure.

Let $G \in \mathbf{PGr}^{\cup}$. As G is a planar graph, the sets $I(v)$ and $O(v)$ are canonically identified with $[i(v)]$ and $[o(v)]$ thanks to their total orders.

For any vertex $v \in V(G)$, there is a natural action of $\mathfrak{S}_{o(v)} \times \mathfrak{S}_{i(v)}^{op}$, obtained by acting on the total orders of $O(v)$ and $I(v)$. The graph obtained from G by the action of (σ, τ) on the vertex v is denoted by

$$\sigma \cdot_v G \cdot_v \tau.$$

For example:

$$(12) \cdot_v \begin{array}{c} \boxed{w} \\ \uparrow \quad \uparrow \\ \boxed{v} \end{array} = \begin{array}{c} \boxed{w} \\ \uparrow \quad \uparrow \\ \boxed{v} \end{array} \cdot_w (12) = \begin{array}{c} \boxed{w} \\ \swarrow \quad \searrow \\ \boxed{v} \end{array}.$$

Let $G \in \mathbf{PGr}^{\cup}(k, l)$ and X a $\mathfrak{S} \times \mathfrak{S}^{op}$ -module. We define

$$G(X) = \bigotimes_{v \in X(G)} X(i(v), o(v)).$$

The elements of $G(X)$ will be written as linear spans of tensors

$$\bigotimes_{v \in V(G)} x_v.$$

In other words, we decorate any vertex of G by an element of X , with respect to the number of incoming and outgoing edges of v , and we take these decorations to be linear in each vertex.

Let

$$\mathbf{PGr}^{\cup}(X)(k, l) := \bigoplus_{G \in \mathbf{PGr}^{\cup}(k, l)} \mathbb{C}G \otimes G(X),$$

whose elements are linear spans of tensors

$$G \otimes \left(\bigotimes_{v \in V(G)} x_v \right).$$

Remark 4.2.1. Graphically, this element of $\mathbf{PGr}^\cup(X)(k, l)$ is represented by the planar graph G where each vertex $v \in V(G)$ is decorated by x_v .

$\mathbf{PGr}^\cup(X)(k, l)$ is a $\mathfrak{S}_l \times \mathfrak{S}_k^{op}$ -module, by the action on the indexation of the incoming and outgoing edges of the graphs.

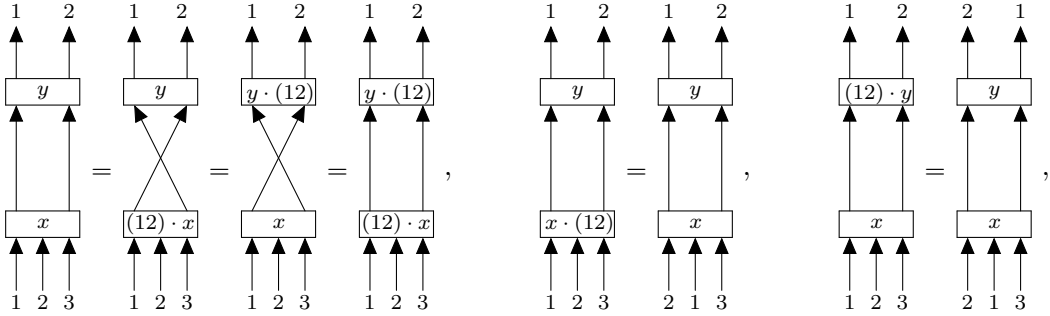
Let $I(k, l)$ be the $\mathfrak{S}_l \times \mathfrak{S}_k^{op}$ -submodule of $\mathbf{PGr}^\cup(X)(k, l)$ generated by elements of the form

$$\sigma \cdot_{v_0} G \cdot_{v_0} \tau \otimes \left(\bigotimes_{v \in V(G)} x_v \right) - G \otimes \left(\left(\bigotimes_{v \in V(G) \setminus \{v_0\}} x_v \right) \otimes \sigma \cdot x_{v_0} \cdot \tau \right), \quad (9)$$

where $G \in \mathbf{PGr}^\cup(k, l)$, $v_0 \in V(G)$, $\sigma \in \mathfrak{S}_{o(v_0)}$ and $\tau \in \mathfrak{S}_{i(v_0)}$. We further define

$$\Gamma^\cup(X)(k, l) := \frac{\mathbf{PGr}^\cup(X)(k, l)}{I(k, l)}.$$

Here is the type of relations we obtain graphically:



where $x \in X_{3,2}$ and $y \in X_{2,2}$.

Example 4.2.1. Let us assume that X is a trivial $\mathfrak{S} \times \mathfrak{S}^{op}$ -module: for any $k, l \in \mathbb{N}_0$, for any $x \in X_{k,l}$, for any $(\sigma, \tau) \in \mathfrak{S}_l \times \mathfrak{S}_l$, $\sigma \cdot x \cdot \tau = x$. The relations defining $\Gamma^\cup(X)$ which boil down to

$$\sigma \cdot_{v_0} G \cdot_{v_0} \tau \otimes \left(\bigotimes_{v \in V(G)} x_v \right) - G \otimes \left(\bigotimes_{v \in V(G)} x_v \right), \quad (10)$$

amount to the identification of two planar X -decorated graphs with the same underlying X -decorated graph. In this case we recover the $\mathfrak{S} \times \mathfrak{S}^{op}$ -module $\mathbf{Gr}^\cup(X)$.

More generally, according to the relations defining $\Gamma^\cup(X)$, if for any graph G , we choose a planar graph \bar{G} , the underlying graph of which is some $G \in \mathbf{Gr}^\cup(X)(k, l)$, then the set of graphs $\bar{G}(X)(k, l)$ is a basis of $\Gamma^\cup(X)(k, l)$. As there is no canonical way to choose the graphs \bar{G} , we prefer to consider $\Gamma^\cup(X)$ as a quotient of $\mathbf{PGr}^\cup(X)$.

Alongside the category \mathbf{ProP} introduced in Definition 1.1.3, we now introduce a second category.

Definition 4.2.1. Let $\mathbf{Mod}_{\mathfrak{S}}$ denote the category of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules: its objects are families $P = (P(k, l))_{(k, l) \in \mathbb{N}_0^2}$, where for any $(k, l) \in \mathbb{N}_0^2$, $P(k, l)$ is a $\mathfrak{S}_l \otimes \mathfrak{S}_k^{op}$ -module; a morphism $\phi : P \rightarrow Q$ is a family $(\phi_{k, l})_{(k, l) \in \mathbb{N}_0^2}$, where for any $(k, l) \in \mathbb{N}_0^2$, $\phi_{k, l} : P(k, l) \rightarrow Q(k, l)$ is a morphism of $\mathfrak{S}_l \otimes \mathfrak{S}_k^{op}$ -modules.

To a $\mathfrak{S} \times \mathfrak{S}^{op}$ -module X in $\mathbf{Mod}_{\mathfrak{S}}$ we have assigned another $\mathfrak{S} \times \mathfrak{S}^{op}$ -module $\Gamma^\cup(P)$ in $\mathbf{Mod}_{\mathfrak{S}}$. One easily checks that a morphism $\varphi : P \rightarrow Q$ of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules induces a morphism of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules

$$\Gamma^\cup(\varphi) : \Gamma^\cup(P) \rightarrow \Gamma^\cup(Q)$$

defined by pull-back on the decorations of the vertices of the graphs:

$$\Gamma^\cup(\varphi)(G, d_G) := (G, \varphi \circ d_G). \quad (11)$$

In summary, we have proven the following

Proposition 4.2.2. *The map $\Gamma^\cup : \mathbf{Mod}_{\mathfrak{S}} \longrightarrow \mathbf{Mod}_{\mathfrak{S}}$ defines an endofunctor of the category $\mathbf{Mod}_{\mathfrak{S}}$.*

Moreover, for any $\mathfrak{S} \times \mathfrak{S}^{op}$ -module, $\Gamma^\cup(X)$ is a ProP:

Theorem 4.2.3. *Let X be a $\mathfrak{S} \times \mathfrak{S}^{op}$ -module. The vertical and horizontal concatenations of the ProP $\mathbf{PGr}^\cup(X)$ induce a ProP structure on $\Gamma^\cup(X)$.*

Proof. We have to prove that if $P \in I$ and $H \in \mathbf{PGr}^\cup(X)$, then $P * H$, $H * P$, $H \circ P$ and $P \circ H$ (if these vertical concatenations are possible) belong to I . We can restrict ourselves to the case $P = G - G'$, where G is a X -decorated planar graph and G' is obtained from G by the action of two permutations on a vertex v of G . It is then immediate that $G' * H$ is obtained from $G * H$ by the action of two permutations on a vertex v of $G * H$, so that $G * H - G' * H = P * H \in I$. Similarly, $H * P \in I$, and $H \circ P$ and $P \circ H$ belong to I if these vertical concatenations are possible. So Γ^\cup inherits a structure of ProP from the structure of \mathbf{PGr}^\cup . \square

Definition 4.2.4. *For any $\mathfrak{S} \times \mathfrak{S}^{op}$ -module X , the ProP $\Gamma^\uparrow(X)$ is defined by*

$$\Gamma^\uparrow(X) = \frac{\mathbf{PGr}^\uparrow(X)}{I \cap \mathbf{PGr}^\uparrow(X)}.$$

As $\mathbf{PGr}^\uparrow(X)$ is a sub-ProP of $\mathbf{PGr}^\cup(X)$, $\Gamma^\uparrow(X)$ is a sub-ProP of $\Gamma^\cup(X)$.

Example 4.2.2. As in Example 4.2.1, if X is a trivial $\mathfrak{S} \times \mathfrak{S}^{op}$ module, we recover the ProP $\mathbf{Gr}^\uparrow(X)$.

We have seen (Theorem 4.2.3) that for any $\mathfrak{S} \times \mathfrak{S}^{op}$ -module, $\Gamma^\cup(X)$ has a ProP structure, and that the same holds for $\Gamma^\uparrow(X)$. We can then lift $\Gamma^\cup(X)$ and $\Gamma^\uparrow(X)$ to functors between these categories:

Proposition 4.2.5. *The maps $\Gamma^\cup : \mathbf{Mod}_{\mathfrak{S}} \longrightarrow \mathbf{ProP}$ and $\Gamma^\uparrow : \mathbf{Mod}_{\mathfrak{S}} \longrightarrow \mathbf{ProP}$ define functors from the category $\mathbf{Mod}_{\mathfrak{S}}$ to the category \mathbf{ProP} of ProPs.*

Proof. Let X and Y be two $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules and $\varphi : X \longrightarrow Y$ a morphism of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules and let $\mathbf{PGr}^\cup(\varphi) : \mathbf{PGr}^\cup(X) \longrightarrow \mathbf{PGr}^\cup(Y)$ be its pullback defined by

$$\mathbf{PGr}^\cup(\varphi)(G, d_G) := (G, \varphi \circ d_G) \tag{12}$$

for any $G \in \mathbf{PGr}^\cup(X)$. As the structure of ProP of $\Gamma^\cup(X)$ is combinatorially given by disjoint union and grafting of graphs, $\mathbf{PGr}^\cup(\varphi)$ clearly defines a morphism of ProPs from $\mathbf{PGr}^\cup(X)$ to $\mathbf{PGr}^\cup(Y)$. As φ is a morphism of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules, it follows that $\mathbf{PGr}^\cup(\varphi)$ sends the ideal defining $\Gamma^\cup(X)$ to the ideal defining $\Gamma^\cup(Y)$, hence it induces a morphism $\Gamma^\cup(\varphi) : \Gamma^\cup(X) \longrightarrow \Gamma^\cup(Y)$ of ProPs. A similar proof holds for Γ^\uparrow . \square

4.3 The ProP $\Gamma^\uparrow(P)$ of graphs decorated by another ProP

The ProPs $\Gamma^\uparrow(X)$ satisfy a universal property:

Theorem 4.3.1. *Let X be a $\mathfrak{S} \times \mathfrak{S}^{op}$ -module, P a ProP and $\varphi : X \longrightarrow P$ a morphism of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules. There exists a unique morphism of ProPs $\Phi : \Gamma^\uparrow(X) \longrightarrow P$, extending φ so that the following diagramme commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & P \\ \downarrow \iota & \nearrow \Phi & \\ \Gamma^\uparrow(X) & & \end{array}$$

where $\iota : X \hookrightarrow \Gamma^\uparrow(X)$ is the map that sends an element x of X to the X -decorated graph $G(x) = (G_{k,l}, d)$ with d sending the unique vertex of $G_{k,l}$ to x .

In other words, $\Gamma^\uparrow(X)$ is the free ProP generated by the $\mathfrak{S} \times \mathfrak{S}^{op}$ -module X .

Proof. Uniqueness. As a quotient of $\mathbf{PGr}_{\text{ind}}^{\uparrow}(X)$, the ProP $\Gamma^{\uparrow}(X)$ is generated by graphs with only one vertex v , decorated by an element of X respecting $i(v)$ and $o(v)$. Hence, such a morphism Φ is unique.

Existence. As $\mathbf{PGr}_{\text{ind}}^{\uparrow}(X)$ is the free ProP generated by the space X , for any linear map $\varphi : X \rightarrow P$, there exists a unique morphism of TraPs $\bar{\Phi} : \mathbf{PGr}_{\text{ind}}^{\uparrow}(X) \rightarrow P$, extending φ . If φ is a morphism of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules, that any element of the form (9) belongs to the kernel of $\bar{\Phi}$, thanks to the compatibility of the concatenation products of P and the action of the symmetric groups. so $\bar{\Phi}$ induces a morphism $\Phi : \Gamma^{\uparrow}(X) \rightarrow P$. \square

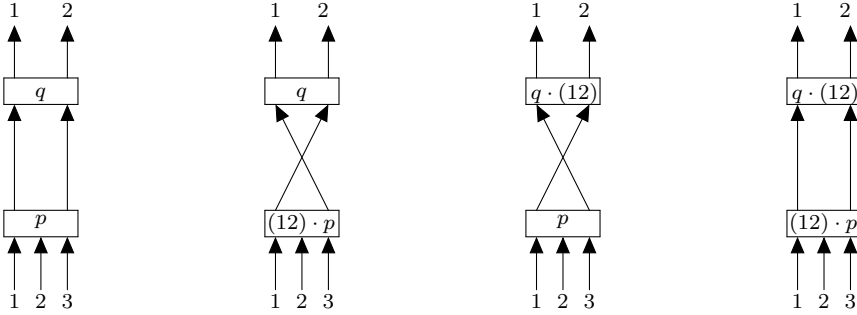
Corollary 4.3.2. *Given a ProP P , there is a canonical morphism of ProPs*

$$\alpha_P : \Gamma^{\uparrow}(P) \rightarrow P$$

induced by the decoration.

Proof. This is a straightforward consequence of Theorem 4.3.1, with $\varphi = \text{Id}_P$. \square

Example 4.3.1. Let $p \in P(3, 2)$ and $q \in P(2, 2)$. The four following graphs (which are equal in $\Gamma^{\uparrow}(P)$)



are respectively sent to

$$q \circ p, \quad (q \cdot (12)) \circ ((12) \cdot p), \quad (q \cdot (12)) \circ ((12) \cdot p), \quad (q \cdot (12)) \circ ((12) \cdot p),$$

which are equal in P . The two following graphs (which are equal in $\Gamma^{\uparrow}(P)$)



are respectively sent to

$$q \circ (p \cdot (12)), \quad (q \circ p) \cdot (12),$$

coincide in P . The two following graphs (which are equal in $\Gamma^\uparrow(P)$)



are respectively sent to

$$((12) \cdot q) \circ p,$$

$$(12) \cdot (q \circ p),$$

which are equal in P .

Recall from Definition 1.1.1 that a ProP is a $\mathfrak{S} \times \mathfrak{S}^{op}$ -module. Composing with the forgetful functor $F : \mathbf{ProP} \rightarrow \mathbf{Mod}_{\mathfrak{S}}$ endofunctors $\Gamma^\cup \circ F$ and $\Gamma^\uparrow \circ F$ of the category \mathbf{ProP} , which we denote also by Γ^\cup and Γ^\uparrow with a slight abuse of notations.

Proposition 4.3.3. *The maps α_P defined in Corollary 4.3.2 give a natural transformation from the identity endofunctor of \mathbf{ProP} to the endofunctor Γ^\uparrow , that is to say, for any morphism $\varphi : P \rightarrow Q$ of ProPs, the following diagram commutes:*

$$\begin{array}{ccc} \Gamma^\uparrow(P) & \xrightarrow{\Gamma^\uparrow(\varphi)} & \Gamma^\uparrow(Q) \\ \alpha_P \downarrow & & \downarrow \alpha_Q \\ P & \xrightarrow{\varphi} & Q \end{array}$$

Proof. Since $\Gamma^\uparrow(\varphi)$, α_P , α_Q and φ are morphisms of ProPs, $\alpha_Q \circ \Gamma^\uparrow(\varphi)$ and $\varphi \circ \alpha_P$ are morphisms of ProPs. As $\Gamma^\uparrow(P)$ is generated by classes of graphs with only one vertex, it is enough to prove that $\alpha_Q \circ \Gamma^\uparrow(\varphi)$ and $\varphi \circ \alpha_P$ coincide on such graphs. Let us consider the planar graph $G_p = PG_{k,l}$, with its unique vertex decorated by $p \in P(k, l)$. Then, if \overline{G}_p is the class of G_p in $\Gamma^\uparrow(X)$:

$$\alpha_Q \circ \Gamma^\uparrow(\varphi)(\overline{G}_p) = \alpha_Q(\overline{G}_{\varphi(p)}) = \varphi(p) = \varphi \circ \alpha_P(\overline{G}_p).$$

So $\alpha_Q \circ \Gamma^\uparrow(\varphi) = \varphi \circ \alpha_P$. □

4.4 The case of \mathbf{Hom}_V^c

Specialising the results of the previous Subsection to $Q := \mathbf{Hom}_V^c$ for some Fréchet nuclear topological vector space V leads us to algebras over ProPs, see e.g. [Mar08].

Definition 4.4.1. *A Fréchet nuclear topological vector space V is an algebra over a ProP P or a P -algebra if there is a representation*

$$\varphi : P \rightarrow \mathbf{Hom}_V^c,$$

of the ProP P on the vector space V i.e. if φ is a morphism of ProPs.

Remark 4.4.1. In the literature of ProPs, the \mathbf{Hom}_V ProP consists of the algebraic counterpart of our \mathbf{Hom}_V^c .

Remark 4.4.2. Algebras over ProPs arise in Segal's axiomatic approach to conformal field theory (CFT) [ABM⁺01], by which a CFT is viewed as an algebra over the Segal ProP. A CFT is viewed as an algebra over the Segal ProP in [Ion07a], where the author claims that Feynman rules of a given QFT, may be presented functorially as an algebra over the corresponding Feynman ProP.

Applying Corollary 4.3.2 to $P = \mathbf{Hom}_V^c$ and $\varphi = \text{Id}|_{\mathbf{Hom}_V^c}$ yields:

Corollary 4.4.2. *A topological vector space V has a canonical algebra structure over $\Gamma^\uparrow(\mathbf{Hom}_V^c)$ given by the canonical morphism of ProP*

$$\alpha_V : \Gamma^\uparrow(\mathbf{Hom}_V^c) \longrightarrow \mathbf{Hom}_V^c.$$

Proposition 4.3.3 applied to $Q = \mathbf{Hom}_V^c$ yields the following statement.

Corollary 4.4.3. *Let P be a ProP and V an algebra over P given by a ProP-morphism $\varphi : P \longrightarrow \mathbf{Hom}_V^c$. Then V also canonically has the structure of an algebra over $\Gamma^\uparrow(P)$ given by the map $\alpha_V \circ \Gamma^\uparrow(\varphi) = \varphi \circ \alpha_P$.*

5 Traces and Permutations (TraPs)

This section is dedicated to TraPs, the other main protagonist of the paper. As for ProP, the main objects of interests in the category of TraP will be the TraP of graphs \mathbf{Gr}^\cup together with its variants and the TraP \mathbf{Hom}_V^c of continuous morphisms on a Fréchet nuclear space V .

5.1 The category of TraPs

Definition 5.1.1. *A TraP is a family $(P(k, l))_{k, l \geq 0}$ of vector spaces, equipped with the following structures:*

1. For any $k, l \in \mathbb{N}_0$, $P(k, l)$ is a $\mathfrak{S}_l \otimes \mathfrak{S}_k^{op}$ -module.
2. For any $k, l, k', l' \in \mathbb{N}_0$, there is a map

$$* : \begin{cases} P(k, l) \otimes P(k', l') & \longrightarrow P(k + k', l + l') \\ p \otimes p' & \longrightarrow p * p', \end{cases}$$

called the **horizontal concatenation**, such that:

(a) (Associativity). For any $(k, l, k', l', k'', l'') \in \mathbb{N}_0^6$, for any $(p, p', p'') \in P(k, l) \times P(k', l') \times P(k'', l'')$,

$$(p * p') * p'' = p * (p' * p'').$$

(b) (Unity). There exists $I_0 \in P(0, 0)$ such that for any $(k, l) \in \mathbb{N}_0^2$, for any $p \in P(k, l)$,

$$I_0 * p = p * I_0 = p.$$

(c) (Compatibility with the symmetric actions). For any $(k, l, k', l') \in \mathbb{N}_0^4$, for any $(p, p') \in P(k, l) \times P(k', l')$, for any $(\sigma, \tau, \sigma', \tau') \in \mathfrak{S}_l \times \mathfrak{S}_k \times \mathfrak{S}_{l'} \times \mathfrak{S}_{k'}$,

$$(\sigma \cdot p \cdot \tau) * (\sigma' \cdot p' \cdot \tau') = (\sigma \otimes \sigma') \cdot (p * p') \cdot (\tau \otimes \tau').$$

(d) (Commutativity). For any $(k, k', l') \in \mathbb{N}_0^4$, For any $p \in P(k, l)$, $p' \in P(k', l')$,

$$c_{l, l'} \cdot (p * p') = (p' * p) \cdot c_{k, k'},$$

where $c_{k, k'}$ and $c_{l, l'}$ are defined by (2).

3. For any $k, l \geq 1$, for any $i \in [k]$, $j \in [l]$, there is a map

$$t_{i,j} : \begin{cases} P(k, l) & \longrightarrow & P(k-1, l-1) \\ p & \longrightarrow & t_{i,j}(p), \end{cases} \quad (13)$$

called the **partial trace map**, such that:

(a) (Commutativity). For any $k, l \geq 2$, for any $i \in [k]$, $j \in [l]$, $i' \in [k-1]$, $j' \in [l-1]$,

$$t_{i',j'} \circ t_{i,j} = \begin{cases} t_{i-1,j-1} \circ t_{i',j'} & \text{if } i' < i, j' < j, \\ t_{i,j-1} \circ t_{i'+1,j'} & \text{if } i' \geq i, j' < j, \\ t_{i-1,j} \circ t_{i',j'+1} & \text{if } i' < i, j' \geq j, \\ t_{i,j} \circ t_{i'+1,j'+1} & \text{if } i' \geq i, j' \geq j. \end{cases}$$

(b) (Compatibility with the symmetric actions). For any $k, l \geq 1$, for any $i \in [k]$, $j \in [l]$, $\sigma \in \mathfrak{S}_l$, $\tau \in \mathfrak{S}_k$, for any $p \in P(k, l)$,

$$t_{i,j}(\sigma \cdot p \cdot \tau) = \sigma_j \cdot (t_{\tau(i), \sigma^{-1}(j)}(p)) \cdot \tau_i,$$

with the following notation: if $\alpha \in \mathfrak{S}_n$ and $p \in [n]$, then $\alpha_p \in \mathfrak{S}_{n-1}$ is defined by

$$\alpha_p(k) = \begin{cases} \alpha(k) & \text{if } k < \alpha^{-1}(p) \text{ and } \alpha(k) < p, \\ \alpha(k) - 1 & \text{if } k < \alpha^{-1}(p) \text{ and } \alpha(k) > p, \\ \alpha(k+1) & \text{if } k \geq \alpha^{-1}(p) \text{ and } \alpha(k) < p, \\ \alpha(k+1) - 1 & \text{if } k \geq \alpha^{-1}(p) \text{ and } \alpha(k) > p. \end{cases} \quad (14)$$

In other words, if we represent α by a word $\alpha_1 \dots \alpha_n$, then α_p is represented by the word obtained by suppression of the letter p in $\alpha_1 \dots \alpha_n$ and subtraction of 1 to all the letters $> p$.

(c) (Compatibility with the horizontal concatenation). For any $k, l, k', l' \geq 1$, for any $i \in [k+l]$, $j \in [k'+l']$, for any $p \in P(k, l)$, $p' \in P(k', l')$:

$$t_{i,j}(p * p') = \begin{cases} t_{i,j}(p) * p' & \text{if } i \leq k, j \leq l, \\ p * t_{i-k, j-l}(p') & \text{if } i > k, j > l. \end{cases}$$

(d) (Unit). There exists $I \in P(1, 1)$ such that for any $k, l \geq 1$, for any $i \in [k+1]$, $j \in [l+1]$, for any $p \in P(k, l)$:

$$\begin{aligned} t_{1,j}(I * p) &= (1, 2, \dots, j-1) \cdot p \text{ if } j \geq 2, \\ t_{i,1}(I * p) &= p \cdot (1, 2, \dots, i-1)^{-1} \text{ if } i \geq 2, \\ t_{k+1,j}(p * I) &= (j, j+1, \dots, l)^{-1} \cdot p \text{ if } j \leq l, \\ t_{i,l+1}(p * I) &= p \cdot (i, i+1, \dots, k) \text{ if } i \leq k. \end{aligned}$$

Remark 5.1.1. 1. We do not require that $t_{1,1}(I) = I_0$, hence the terminology partial trace map.

2. By commutativity of $*$, for any $p \in P(0, 0)$, for any $(k, l) \in \mathbb{N}_0^2$, for any $p' \in P(k, l)$:

$$p * p' = p' * p,$$

since $c_{0,k} = \text{Id}_{[k]}$.

Remark 5.1.2. Our notion of TraP is an axiomatised version of Merkulov's notion of **wheeled ProPs** introduced in [Mer06]. The link between TraPs and wheeled ProPs will be made in Section 7.1, Corollary 7.1.3.

Our approach mainly differs from Merkulov's categorical approach in that it comprises units. Units of wheeled ProPs are mentioned in [Mer10b, Remark 2.3.1] but their axioms are not explicitly written down in the literature. Our axiomatic approach is tailored to address analytic issues regarding products of singularities. This axiomatic approach allows us to give a simple definition of quasi-TraPs in Section 7.3, a notion that seem absent in previous works on wheeled ProPs. However, the categorical approach seems better suited for classification problems, e.g. regarding the solutions of the master equation in the BV formalism [MMS09, Mer10b].

Lemma 5.1.2. *Let $P = (P(k, l))_{k, l \in \mathbb{N}_0}$ be a $\mathfrak{S} \otimes \mathfrak{S}^{op}$ -module, equipped with a horizontal concatenation $*$ satisfying axioms 2. (a)-(d), and with maps $t_{i, j}$ satisfying axioms 3. (a)-(b).*

1. We assume that for any $k, l, k', l' \geq 1$, for any $p \in P(k, l)$, $p' \in P(k', l')$,

$$t_{1,1}(p * p') = t_{1,1}(p) * p'.$$

Then axiom 3.(c) is satisfied.

2. We assume for for any $k, l \geq 1$, for any $p \in P(k, l)$,

$$t_{1,2}(I * p) = p.$$

Then axiom 3.(d) is satisfied.

Proof. 1. Let $p \in P(k, l)$ and $p' \in P(k', l')$. Let us take $i \in [k + l]$, $j \in [k' + l']$, consider the transpositions $\sigma = (1, j)$ and $\tau = (1, i)$, with the convention $(1, 1) = \text{Id}$. If $i \leq k$ and $j \leq l$, then:

$$\begin{aligned} t_{i,j}(p * p') &= t_{i,j}(\sigma^2 \cdot (p * p') \cdot \tau^2) \\ &= \sigma_j \cdot t_{1,1}(\sigma \cdot (p * p') \cdot \tau) \cdot \tau_i \\ &= \sigma_j \cdot (t_{1,1}((\sigma \cdot p \cdot \tau) * p') \cdot \tau_i) \\ &= \sigma_j \cdot (t_{1,1}(\sigma \cdot p \cdot \tau) * p') \cdot \tau_i \\ &= (\sigma_j \cdot (t_{1,1}(\sigma \cdot p \cdot \tau) \cdot \tau_i) * p') \\ &= t_{i,j}(p) * p'. \end{aligned}$$

If $i > k$ and $j > l$, using $c_{m,n}^{-1} = c_{n,m}$:

$$\begin{aligned} t_{i,j}(p * p') &= t_{i,j}(c_{l',l} \cdot (p' * p) \cdot c_{k,k'}) \\ &= (c_{l',l})_j \cdot t_{i-k,j-l}(p' * p) \cdot (c_{k,k'})_i \\ &= c_{l'-1,l} \cdot (t_{i-k,j-l}(p') * p) \cdot c_{k,k'-1} \\ &= p * t_{i-k,j-l}(p'). \end{aligned}$$

2. Let us take $j \geq 2$.

$$\begin{aligned} t_{1,j}(I * p) &= t_{1,j}((2, j)^2 \cdot (I * p)) \\ &= (2, \dots, j-1) \cdot t_{1,2}((2, j) \cdot (I * p)) \\ &= (2, \dots, j-1) \cdot t_{1,2}(I * (1, j-1) \cdot p) \\ &= (2, \dots, j-1) \cdot ((1, j-1) \cdot p) \\ &= (2, \dots, j-1)(1, j-1) \cdot p \\ &= (1, \dots, j-1) \cdot p. \end{aligned}$$

The three other relations are proved in the same way. □

Definition 5.1.3. Let $P = (P(k, l))_{k, l \geq 0}$ and $Q = (Q(k, l))_{k, l \geq 0}$ be two TraPs with partial trace maps $(t_{i, j}^P)_{i, j \geq 0}$ and $(t_{i, j}^Q)_{i, j \geq 0}$ respectively. A **morphism of TraPs** is a family $\phi = (\phi_{k, l})_{k, l \geq 0}$ of linear maps $\phi_{k, l} : P(k, l) \rightarrow Q(k, l)$ which are morphism for the horizontal concatenation, the actions of the symmetric groups and the partial trace maps. More precisely, for any $(k, l, m, n) \in \mathbb{N}_0^4$:

- $\forall (p, q) \in P(k, l) \times P(n, m), \phi_{k+n, l+m}(p * q) = \phi_{k, l}(p) * \phi_{n, m}(q),$
- $\forall (\sigma, p) \in \mathfrak{S}_l \times P(k, l), \phi_{k, l}(\sigma.p) = \sigma.\phi_{k, l}(p),$
- $\forall (p, \tau) \in P(k, l) \times \mathfrak{S}_k, \phi_{k, l}(p.\tau) = \phi_{k, l}(p).\tau.$
- $\forall (p, i, j) \in P(k, l) \times [k] \times [l], \phi_{k-1, l-1}(t_{i, j}^P(p)) = t_{i, j}^Q(\phi_{k, l}(p)).$

With a slight abuse of notations, we write $\phi(p)$ instead of $\phi_{k, l}(p)$ for $p \in P(k, l)$. In particular, TraPs form a category, which we denote by **TraP**.

Remark 5.1.3. The abuse of notation $t_{i, j}$ is legitimate since a full notation such as $t_{i, j}^{k, l}$ is not necessary in practice. Indeed the indices k and l in $t_{i, j}(p)$ are entirely determined by p to which $t_{i, j}$ is applied.

More so, $t_{i, j}$ does not strongly depend on k and l : indeed, let $f : P(k, l) \rightarrow P(k+1, l+1)$ be the map that sends p to $p * I$ (for the TraP of linear morphisms, this is the tensorisation by Id), then for $i \in [k]$ and $j \in [l]$, we have

$$t_{i, j} \circ f(p) = f \circ t_{i, j}(p),$$

which is the axiom 3.(c).

Lemma 5.1.4. Let P and Q be two TraPs and $\phi : P \rightarrow Q$ be a map. We assume that:

1. For any $(k, l) \in \mathbb{N}_0^2$, for any $(\sigma, \tau) \in \mathfrak{S}_l \times \mathfrak{S}_k$, for any $x \in P(k, l)$,

$$\phi(\sigma \cdot x \cdot \tau) = \sigma \cdot \phi(x) \cdot \tau.$$

2. For any $k, l \geq 1$, for any $x \in P(k, l)$,

$$t_{1, 1} \circ \phi(x) = \phi \circ t_{1, 1}(x).$$

Then for any $k, l \geq 1$, for any $(i, j) \in [k] \times [l]$, for any $x \in P(k, l)$,

$$t_{i, j} \circ \phi(x) = \phi \circ t_{i, j}(x).$$

Proof. If $i \in [k]$, $j \in [l]$, and $x \in P(k, l)$:

$$\begin{aligned} \phi \circ t_{i, j}(x) &= \phi \circ t_{i, j}((1, j)^2 \cdot x \cdot (1, i)^2) \\ &= \phi((1, j) \cdot t_{1, 1}((1, j) \cdot x \cdot (1, i)) \cdot (1, i)) \\ &= (1, j) \cdot \phi \circ t_{1, 1}((1, j) \cdot x \cdot (1, i)) \cdot (1, i) \\ &= (1, j) \cdot t_{1, 1} \circ \phi((1, j) \cdot x \cdot (1, i)) \cdot (1, i) \\ &= t_{i, j}((1, j) \cdot \phi((1, j) \cdot x \cdot (1, i)) \cdot (1, i)) \\ &= t_{i, j} \circ \phi(x), \end{aligned}$$

with the convention $(1, k) = \text{Id}$ if $k = 1$. □

In particular, to show that a collection of linear maps between two TraPs preserving the horizontal concatenation and the actions of the symmetry group is a morphism of TraP, it is enough to check the properties of Lemma 5.1.4.

5.2 The TraP \mathbf{Hom}_V^c

We start with the TraP version of the ProP of linear morphisms of section 1.2.

Proposition 5.2.1. *Let V be a finite dimensional vector space and V^* its algebraic dual. Then for any $(k, l) \in \mathbb{N}_0^2$:*

$$\mathbf{Hom}_V(k, l) = \mathbf{Hom}(V^{\otimes k}, V^{\otimes l}) \simeq V^{*\otimes k} \otimes V^{\otimes l}.$$

$\mathfrak{S}_l \otimes \mathfrak{S}_k^{op}$ acts on the ProP \mathbf{Hom}_V as readily described in Proposition-Definition 1.2.1. We shall make some abuse of notation setting $f_1 \cdots f_k := f_1 \otimes \cdots \otimes f_k \in V^{*\otimes k}$ and $v_1 \cdots v_l := v_1 \otimes \cdots \otimes v_l \in V^l$. We equip $V^{*\otimes k} \otimes V^{\otimes l}$ with a horizontal concatenation:

$$(f_1 \cdots f_k \otimes v_1 \cdots v_l) * (f'_1 \cdots f'_k \otimes v'_1 \cdots v'_l) = f_1 \cdots f_k f'_1 \cdots f'_k \otimes v_1 \cdots v_l v'_1 \cdots v'_l,$$

and partial trace maps:

$$t_{i,j}(f_1 \cdots f_k \otimes v_1 \cdots v_l) = f_i(v_j) f_1 \cdots f_{i-1} f_{i+1} \cdots f_k \otimes v_1 \cdots v_{j-1} v_{j+1} \cdots v_l$$

(with obvious abuses of notations). These make \mathbf{Hom}_V a TraP.

Proof. Properties 2.(a)-(d) are trivially satisfied, with $I_0 = 1 \in \mathbb{K} = V^{\otimes 0} \otimes V^{*\otimes 0}$. Property 3.(a) is direct. Let us prove Property 3. (b).

$$\begin{aligned} t_{i,j}(\sigma \cdot f_1 \cdots f_k \otimes v_1 \cdots v_l \cdot \tau) &= t_{i,j}(f_{\tau(1)} \cdots f_{\tau(k)} \otimes v_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(l)}) \\ &= f_{\tau(i)}(v_{\sigma^{-1}(j)}) f_{\tau(1)} \cdots f_{\tau(i-1)} f_{\tau(i+1)} \cdots f_{\tau(k)} \\ &\quad \otimes v_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(j-1)} v_{\sigma^{-1}(j+1)} \cdots v_{\sigma^{-1}(l)} \\ &= \sigma_j \cdot t_{\tau(i), \sigma^{-1}(j)}(f_1 \cdots f_k \otimes v_1 \cdots v_l) \cdot \tau_i. \end{aligned}$$

Property 3.(c) is straightforward. Let us prove property 3.(d) with the help of Lemma 5.1.2. Let us fix $(e_i)_{i \in I}$ a basis of V , then $(e_i^*)_{i \in I}$ is a basis of V^* and the identity map $I = \sum_{i \in I} e_i^* \otimes e_i$, acts as follows, $I(v) = \sum_{i \in I} e_i^*(v) e_i = v$ for all $v \in V$. Then:

$$\begin{aligned} t_{1,2}(I * f_1 \cdots f_k \otimes v_1 \cdots v_l) &= \sum_{i \in I} t_{1,2}(e_i^* f_1 \cdots f_k \otimes e_i v_1 \cdots v_l) \\ &= \sum_{i \in I} f_1 \cdots f_k \otimes e_i e_i^*(v_1) v_2 \cdots v_l \\ &= f_1 \cdots f_k \otimes I(v_1) v_2 \cdots v_l \\ &= f_1 \cdots f_k \otimes v_1 \cdots v_l. \end{aligned}$$

So \mathbf{Hom}_V is a TraP. □

Remark 5.2.1. In this example of TraP, $t_{1,1}(I) = \dim(V) = \dim(V)I_0$.

In order to generalise this construction to nuclear Fréchet spaces, we need to characterise the composition of linear morphisms of such spaces.

Lemma 5.2.2. *Let E_1, E_2 be two Fréchet nuclear spaces and E_3 a Fréchet space. Then the composition of continuous morphisms $L_1 : E_1 \longrightarrow E_2$, $L_2 : E_2 \longrightarrow E_3$ amounts to a dual pairing.*

Proof. Let E_1, E_2, E_3 be three topological spaces as in the statement. Then by (4) the identifications $\mathbf{Hom}^c(E_1, E_2) \simeq E_1' \hat{\otimes} E_2$ and $\mathbf{Hom}^c(E_2, E_3) \simeq E_2' \hat{\otimes} E_3$ hold. For $L_1 = \sum_{i,j} u_i^{1*} \otimes u_j^2 \in \mathbf{Hom}^c(E_1, E_2)$, $L_2 = \sum_{k,l} u_k^{2*} \otimes u_l^3 \in \mathbf{Hom}^c(E_2, E_3)$ and $u \in E_1$, we have

$$L_2 \circ L_1(u) = L_2 \left(\sum_{i,j} u_i^{1*}(u) u_j^2 \right) = \sum_{k,l} \sum_{i,j} u_i^{1*}(u) u_k^{2*}(u_j^2) u_l^3$$

so that

$$\mathbf{Hom}^c(E_1, E_3) \ni L_2 \circ L_1 = \sum_{i,l} \left(\sum_{k,j} u_k^{2*}(u_j^2) \right) u_i^{1*} \otimes u_l^3 \in E_1^* \hat{\otimes} E_3. \quad \square$$

Recall that, for a Fréchet nuclear space V , the ProP $(\mathbf{Hom}_V^c(k, l))_{k, l \geq 0}$ introduced in Subsection 2.2, Theorem 2.2.5, reads:

$$\mathbf{Hom}_V^c(k, l) \simeq (V')^{\widehat{\otimes} k} \widehat{\otimes} V^{\widehat{\otimes} l}.$$

Proposition 5.2.3. *Let V be a Fréchet nuclear space. The family $(\mathbf{Hom}_V^c(k, l))_{k, l \geq 0}$ equipped with the partial trace maps in the sense of (13) defined by*

$$\mathrm{tr}_{i,j} : \begin{cases} \mathbf{Hom}_V^c(k, l) & \longrightarrow \mathbf{Hom}_V^c(k-1, l-1) \\ (v_1^* \otimes \cdots \otimes v_k^*) \otimes (w_1 \otimes \cdots \otimes w_l) & \longmapsto \mathrm{tr}_{i,j}((v_1^* \otimes \cdots \otimes v_k^*) \otimes (w_1 \otimes \cdots \otimes w_l)) \end{cases}$$

with $\mathrm{tr}_{i,j}(v_1^* \otimes \cdots \otimes v_k^*) \otimes (w_1 \otimes \cdots \otimes w_l)$ defined as

$$v_i^*(w_j) (v_1^* \otimes \cdots \otimes \widehat{v_i^*} \otimes \cdots \otimes v_k^*) \otimes (w_1 \otimes \cdots \otimes \widehat{w_j} \otimes \cdots \otimes w_l)$$

for any $k, l \geq 1$, for any $i \in [k]$, $j \in [l]$, where $v_i^*(w_j)$ is the dual pairing, defines a TraP, with the topological tensor product as horizontal concatenation.

Proof. Commutativity follows from the commutativity of the field \mathbb{K} , compatibility with the symmetric actions and compatibility with the horizontal concatenation are shown as for the ProP \mathbf{Hom}_V^c . The unit is the identity map $I \in V^* \widehat{\otimes} V$. \square

Example 5.2.1. With the notations of Remark 1.2.3, $\mathrm{tr}_{i,j} \left(\sum_{\vec{I}, \vec{J}} a_{\vec{J}}^{\vec{I}} e^{\vec{J}} \otimes e_{\vec{I}} \right)$ is of the form $\sum_{\vec{I}_i, \vec{J}_j} b_{\vec{J}_j}^{\vec{I}_i} e^{\vec{J}_j} \otimes e_{\vec{I}_i}$, where $\vec{I}_i = (i_1, \dots, \hat{i}, \dots, i_k)$, $\vec{J}_j = (j_1, \dots, \hat{j}, \dots, j_l)$ and $b_{\vec{J}_j}^{\vec{I}_i}$ corresponds to the trace of the $n \times n$ matrix in the (i, j) entries of $a_{\vec{J}}^{\vec{I}}$ with the other indices frozen.

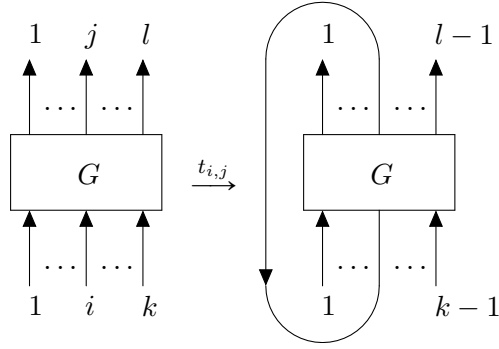
Example 5.2.2. Let U be an open of \mathbb{R}^n . Example 2.1.2 and Equation (7) imply that the family $(\mathcal{K}_U(k, l))_{k, l \geq 0}$, with $\mathcal{K}_U(k, l) = (\mathcal{E}'(U))^{\widehat{\otimes} k} \widehat{\otimes} \mathcal{E}(U)^{\widehat{\otimes} l}$ defines a TraP.

Example 5.2.3. Let X be a finite dimensional smooth manifold. Proposition 2.2.2 and Equation (7) imply that the family $(\mathcal{K}_X(k, l))_{k, l \geq 0}$, with $\mathcal{K}_X(k, l) = (\mathcal{E}'(X))^{\widehat{\otimes} k} \widehat{\otimes} \mathcal{E}(X)^{\widehat{\otimes} l}$ defines a TraP.

5.3 The TraP \mathbf{Gr}^\cup of graphs

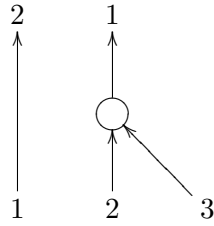
We now equip graphs and planar graphs with a TraP structure. We have already equipped \mathbf{Gr}^\cup and \mathbf{PGr}^\cup with a structure of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules and a horizontal concatenation, which we leave untouched. Let us now define partial trace maps. Let $G \in \mathbf{Gr}^\cup(k, l)$, $1 \leq i \leq k$ and $1 \leq j \leq l$. We set $e_i = \alpha_G^{-1}(i)$, $f_j = \beta_G^{-1}(j)$ and define $t_{i,j}(G)$ as the graph obtained by identifying the input of e_i with the output j of f_j . If $e_i \in I(G)$ and $f_j \in O(G)$, this creates an edge in $E(G)$. This case is illustrated in the figure below. Otherwise, we create an edge in $I(G)$, or $O(G)$ or $IO(G)$ or in $L(G)$. In all these cases, we then reindex increasingly the inputs and the outputs of the obtained graph.

Graphically:

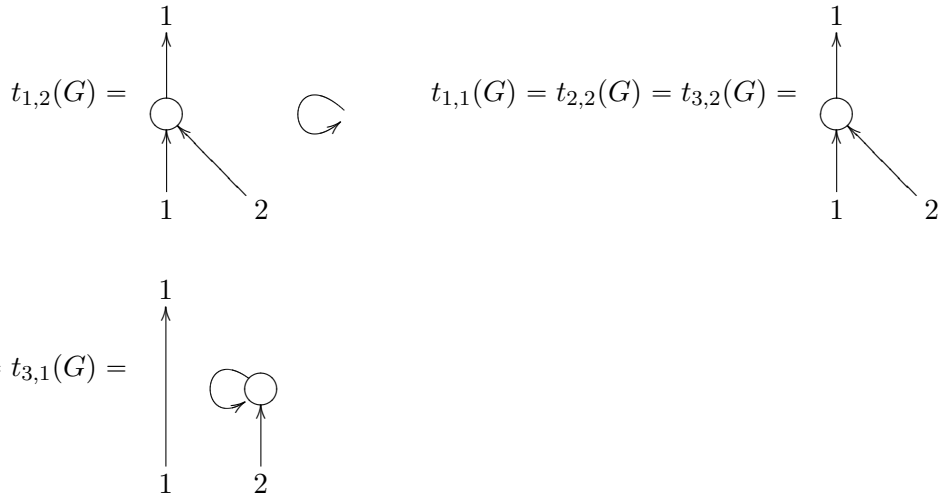


A more rigorous definition is given in the appendix. A similar definition can be given for planar graphs, by preserving the orders on incoming and outgoing edges of any vertex.

Example 5.3.1. Let G be the following graph:



Then:

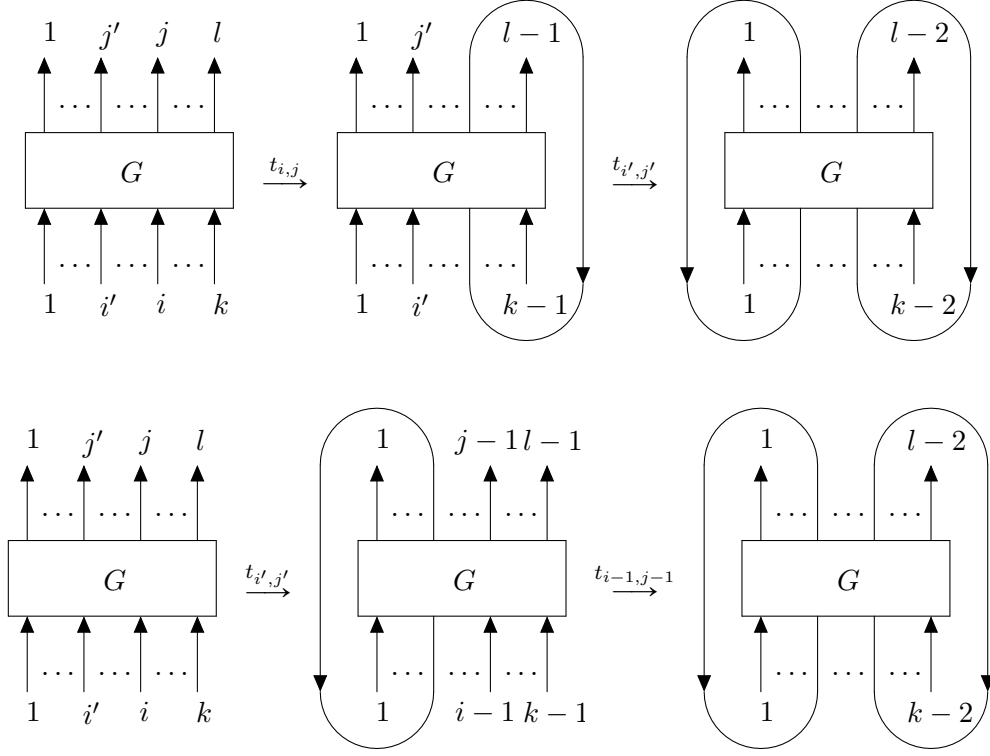


Note that $t_{1,2}$ creates a loop when applied on G .

Remark 5.3.1. In particular, $t_{1,1}(I)$ is the graph \mathcal{O} , which is essential for TraPs.

Proposition 5.3.1. \mathbf{Gr}° and \mathbf{PGr}° , with the usual horizontal concatenation and this partial trace map, are TraPs.

Proof. Properties 2.(a)-(d) are trivial. Let us give a graphical indication of the proof of Property 3.(a), when $i' < i$ and $j' < j$.



For Property 3.(b), let us consider $p = G$ a graph. As the input edge indexed by i in $\sigma \cdot G \cdot \tau$ is the input edge of G indexed by $\tau(i)$ and the output edge indexed by j in $\sigma \cdot G \cdot \tau$ is the output edge of G indexed by $\sigma^{-1}(j)$, $G_1 = t_{i,j}(\sigma \cdot G \cdot \tau)$ is the graph obtained by gluing together the input indexed by $\tau(j)$ and the output indexed by $\sigma^{-1}(j)$, reindexing the input according to σ_i and the output edges by τ_j , so $G_1 = \sigma_i \cdot t_{\tau(i), \sigma^{-1}(j)}(G) \cdot \tau_j$.

Let us prove Property 3.(c). By Lemma 5.1.2, it is enough to prove it for $(p, p') = (G, G')$ a pair of graphs and $(i, j) = (1, 1)$. In this case, e_i and f_j are both edges of G , so $t_{1,1}(G * G') = t_{1,1}(G) * G'$.

For Property 3.(d), let us consider the graph I such that

$$V(I) = E(I) = O(I) = I(I) = L(I) = \emptyset,$$

and $IO(I)$ being reduced to a single element. Then for any graph G with $|O(G)| \geq 1$,

$$t_{1,2}(I * G) = G.$$

By Lemma 5.1.2, Property 3.(d) is satisfied, so \mathbf{Gr}^\cup is a TraP. \square

5.4 Free TraPs

Theorem 5.4.1. *Let P be a TraP and, for any $k, l \in \mathbb{N}_0$, let $x_{k,l} \in P(k, l)$ such that:*

$$\forall \sigma \in \mathfrak{S}_l, \forall \tau \in \mathfrak{S}_k, \quad \sigma \cdot x_{k,l} \cdot \tau = x_{k,l}.$$

There exists a unique TraP morphism Φ from \mathbf{Gr}^\cup to P sending $G_{k,l}$ to $x_{k,l}$ for any $k, l \geq 0$.

Proof. We provide here a sketch of the proof, and refer the reader to the appendix for a full proof. We define $\Phi(G)$ for any graph $G \in \mathbf{Gr}^\cup(k, l)$ by induction on the number N of internal edges of G .

If $N = 0$, then G can be written as

$$G = \mathcal{O}^{*p} * \sigma \cdot (I^{*q} * G_{k_1, l_1} * \dots * G_{k_r, l_r}) \cdot \tau,$$

(recall that \mathcal{O} is the graph with no vertex, and only one edge belonging to $L(G)$) where $p, q, r \in \mathbb{N}_0$, $(k_i, k_i) \in \mathbb{N}_0^2$ for any i , and $\sigma \in \mathfrak{S}_{q+k_1+\dots+k_r}$, $\tau \in \mathfrak{S}_{q+l_1+\dots+l_r}$. We then put:

$$\Phi(G) = t_{1,1}(I)^{*p} * \sigma \cdot (I^{*q} * x_{k_1, l_1} * \dots * x_{k_r, l_r}) \cdot \tau.$$

We can prove that this does not depend on the choice of the decomposition of G , with the help of the TraP axioms applied to P and the invariance of the $x_{k,l}$. Let us assume now that $\Phi(G')$ is defined for any graph with $N - 1$ internal edges, for a given $N \geq 1$. Let G be a graph with N internal edges and let e be one of these edges. Let G_e be a graph obtained by cutting this edge in two, such that $G = t_{1,1}(G_e)$. We then set:

$$\Phi(G) = t_{1,1} \circ \Phi(G_e).$$

One can prove that this does not depend on the choice of e . It can then be shown that Φ defined as above is compatible with the partial trace maps. \square

The following TraP counterpart of Theorem 4.3.1 can be proved in a similar way as Theorem 5.4.1:

Theorem 5.4.2. *Let X be a $\mathfrak{S} \times \mathfrak{S}^{op}$ -module, P a TraP and $\varphi : X \rightarrow P$ be a morphism of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules. There exists a unique morphism of TraPs $\Phi : \mathbf{PGr}^\cup(X) \rightarrow P$, which extends φ so that the following diagramme commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & P \\ \downarrow \iota & \nearrow \Phi & \\ \mathbf{PGr}^\cup(X) & & \end{array}$$

where $\iota : X \hookrightarrow \mathbf{PGr}^\cup(X)$ is the map that sends an element x of X to the planar X -decorated graph $G(x) = (PG_{k,l}, d)$ with d sending the unique vertex of $PG_{k,l}$ to x .

In other words, $\mathbf{PGr}^\cup(X)$ is the free TroP generated by the $\mathfrak{S} \times \mathfrak{S}^{op}$ -module X .

Remark 5.4.1. The invariance condition of $x_{k,l}$ in Theorem 5.4.1 is replaced here with the planar condition on graphs. They play the same role, namely to allow us to show that the map Φ , defined inductively, is indeed well-defined.

6 The functor Γ^\cup applied on TraPs

6.1 The functor Γ^\cup as an endofunctor of TraP

Proposition 6.1.1. *Let X be a $\mathfrak{S} \times \mathfrak{S}^{op}$ -module. Then $\Gamma^\cup(X)$ is a TraP.*

Proof. Similarly to the proof of Proposition 5.3.1 concerning \mathbf{Gr}^\cup , we can prove that $\mathbf{PGr}^\cup(X)$ is a TraP.

If G and G' are two X -decorated planar graphs such that G' is obtained from G by the action of permutations on the incoming and outgoing edges of a vertex of G , then clearly, for any relevant i and j , $t_{i,j}(G')$ is obtained from G by the same operation. So $t_{i,j}(G - G') \in I$, and the partial trace maps of $\mathbf{PGr}^\cup(X)$ induce partial trace maps on $\Gamma^\cup(X)$. \square

Hence, Γ^\cup is a functor from the category $\mathbf{Mod}_{\mathfrak{S}}$ to the category \mathbf{TraP} . Combining with the forgetful functor $F : \mathbf{TraP} \rightarrow \mathbf{Mod}_{\mathfrak{S}}$, we obtain an endofunctor $\Gamma^\cup \circ F : \mathbf{TraP} \rightarrow \mathbf{TraP}$, which we denote by Γ^\cup , with a slight abuse of notations. As for ProPs (Corollary 4.3.2 and Proposition 4.3.3), we have the following statement.

Proposition 6.1.2. *Given a TraP P , there is a canonical morphism of TraPs*

$$\alpha_P : \Gamma^\cup(P) \longrightarrow P$$

induced by the decoration. These maps define a natural transformation from the endofunctor Γ^\cup to the identity endofunctor of \mathbf{TraP} , that is to say: for any morphism of TraP $\varphi : P \longrightarrow Q$, the following diagram commutes:

$$\begin{array}{ccc} \Gamma^\cup(P) & \xrightarrow{\Gamma^\cup(\varphi)} & \Gamma^\cup(Q) \\ \alpha_P \downarrow & & \downarrow \alpha_Q \\ P & \xrightarrow{\varphi} & Q \end{array}$$

Proof. Similar arguments as in the proofs of Corollary 4.3.2 and Proposition 4.3.3. \square

6.2 The endofunctor Γ^\cup as a monad

Let us now equip the endofunctor Γ^\cup with a **monad** structure, a terminology we borrow from [MMS09, Definition 2.13].

Definition 6.2.1. *A monad Γ (also called a triple) on a category \mathcal{C} is an associative and unital monoid (Γ, μ, ν) in the the unital monoid⁴ $\text{End}(\mathcal{C})$ of endofunctors of \mathcal{C} . This means that the multiplication $\mu : \Gamma \circ \Gamma \longrightarrow \Gamma$ and the unit morphism $\nu : Id_{\mathcal{C}} \longrightarrow \Gamma$ should satisfy the axioms given by commutativity of the diagrams below for any object P of the category \mathcal{C} .*

$$\begin{array}{ccc} \Gamma \circ \Gamma \circ \Gamma(P) & \xrightarrow{\Gamma(\mu_P)} & \Gamma \circ \Gamma(P) \\ \mu_{\Gamma(P)} \downarrow & & \downarrow \mu_P \\ \Gamma \circ \Gamma(P) & \xrightarrow{\mu_P} & \Gamma(P) \end{array} \qquad \begin{array}{ccc} \Gamma(P) & \xrightarrow{\Gamma(\nu_P)} \Gamma \circ \Gamma(P) & \xleftarrow{\nu_{\Gamma(P)}} \Gamma(P) \\ & \searrow Id_{\mathcal{C}} \quad \downarrow \mu_P \quad \swarrow Id_{\mathcal{C}} & \\ & & \Gamma(P) \end{array} \quad (15)$$

We want to define a transformation $\nu : Id_{\mathbf{Mod}_{\mathfrak{S}}} \longrightarrow \Gamma^\cup$, i.e. maps $\nu_P : P \longrightarrow \Gamma^\cup(P)$ for any $\mathfrak{S} \times \mathfrak{S}^{op}$ -module P . The morphism ν_P sends an element $p \in P(k, l)$ to the class of the graph $PG_{k,l}(p)$ with one vertex v decorated by p , and k incoming edges indexed from left to right by $1, \dots, k$, l outgoing edges indexed from left to right by $1, \dots, l$.

$$\nu_P(p) = \begin{array}{c} \begin{array}{c} 1 \dots l \\ \uparrow \quad \uparrow \\ \boxed{p} \\ \uparrow \quad \uparrow \\ 1 \dots k \end{array} \end{array} .$$

The morphism ν is a unit in $\text{End}(\mathbf{Mod}_{\mathfrak{S}})$ in the following sense: for any morphism $\phi : P \longrightarrow Q$, the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\nu_P} & \Gamma^\cup(P) \\ \phi \downarrow & & \downarrow \Gamma^\cup(\phi) \\ Q & \xrightarrow{\nu_Q} & \Gamma^\cup(Q) \end{array}$$

The multiplication is given by morphisms $\mu_P : \Gamma^\cup \circ \Gamma^\cup(P) \longrightarrow \Gamma^\cup(P)$ attached to $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules P . Elements of $\Gamma^\cup \circ \Gamma^\cup(P)$ are graphs G whose vertices v are decorated by graphs G_v ,

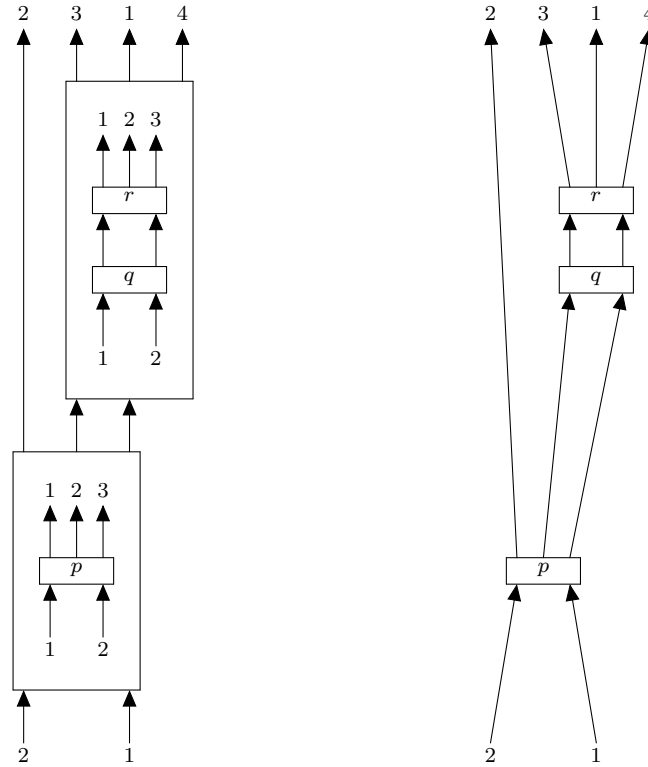
⁴The terminology monoid is used in this definition with the obvious abuse of vocabulary since Γ and $\text{End}(\mathcal{C})$ are not necessarily sets.

consistently with the number of incoming and outgoing edges. We denote by $\mu_P(G)$ the graph H such that

$$V(H) = \bigsqcup_{v \in V(G)} V(G_v),$$

whose edges are obtained by identifying, for any vertex v , the i -th incoming edges of v with the i -th incoming edge of G_v , and the j -th outgoing edge of v with the j -th outgoing edge of G_v .

To illustrate this graphically, here is an example in which μ_P sends the graph on the left to the graph on the right:



where $p \in P(2, 3)$, $q \in P(2, 2)$ and $r \in P(2, 3)$.

It is clear from the combinatorics that the relations corresponding to the diagrams (15) are satisfied. Hence:

Proposition 6.2.2. *The triple $\Gamma^\circ = (\Gamma^\circ, \mu, \nu)$ is a monad in the category $\mathbf{Mod}_\mathcal{E}$.*

7 TraPs versus ProPs

We have built the free TraPs by means of graphs discussed in Subsection 5.3. This, together with the functor Γ° of Sections 4 and 6 will allow us to show the equivalence of the categories of TraPs and wheeled ProPs.

7.1 TraPs are wheeled ProPs

The free TraP we previously built from a given TraP enables us to relate TraPs and Merkulov's notion of wheeled ProPs [MMS09]. We now build algebras on the monad Γ° . Let us first recall the notion of Γ -algebra (see e.g. [MMS09, Definition 2.1.4]).

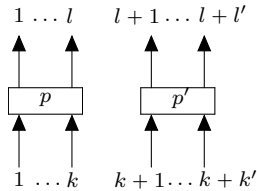
Definition 7.1.1. *Let \mathcal{C} be a category. An algebra over a monad $\Gamma \in \mathbf{End}(\mathcal{C})$ or a Γ -algebra is an object P of \mathcal{C} together with a structure morphism $\alpha : \Gamma(P) \rightarrow P$ such that the following*

diagrams commute:

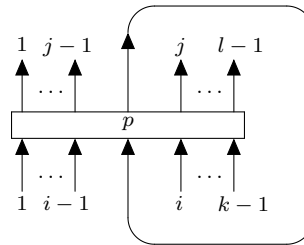
$$\begin{array}{ccc}
 \Gamma \circ \Gamma(P) & \xrightarrow{\Gamma(\alpha)} & \Gamma(P) \\
 \mu_P \downarrow & & \downarrow \alpha \\
 \Gamma(P) & \xrightarrow{\alpha} & P
 \end{array}
 \qquad
 \begin{array}{ccc}
 P & \xrightarrow{\nu_P} & \Gamma(P) \\
 Id \downarrow & \swarrow \alpha & \\
 P & &
 \end{array}
 \tag{16}$$

Proposition 7.1.2. Any Γ^\cup -algebra (P, α) defines a TraP defined as follows:

- For any $(p, p') \in P(k, l) \times P(k', l')$, $p * p'$ is obtained by applying α to the following graph:

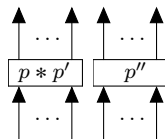


- For any $p \in P(k, l)$, for any $(i, j) \in [k] \times [l]$, $t_{i,j}(p)$ is obtained by the application of α to the following graph:

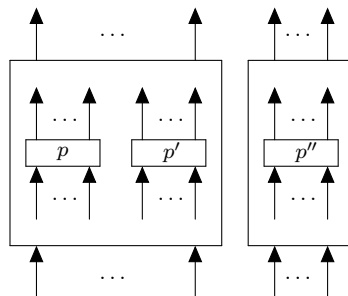


Proof. Let us prove some of the axioms of TraPs for P . The others can be proved in the same way and are left to the reader.

- (a). let $(p, p', p'') \in P(k, l) \times P(k', l') \times P(k'', l'')$. Then $(p * p') * p''$ is obtained by the application of α_P to the graph:



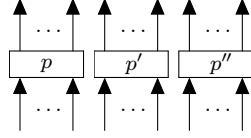
(For the sake of simplicity, we delete the indices of the input and output edges of this graph: they are always indexed from left to right). Hence, $(p * p') * p''$ is obtained by application of $\alpha \circ \Gamma^\cup(\alpha)$ to the graph:



Note that for the second connected component of this graph, this comes from:

$$\alpha \circ \Gamma^\cup(\alpha) \circ \Gamma^\cup(\nu_P)(p'') = \alpha \circ \Gamma^\cup(\alpha \circ \nu_P)(p'') = \alpha \circ \Gamma^\cup(Id_P)(p'') = \alpha(p'').$$

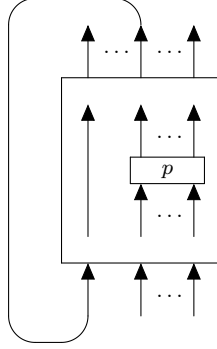
As $\alpha \circ \Gamma^\cup(\alpha) = \alpha \circ \mu_P$, $(p * p') * p''$ is obtained by application of α to the graph:



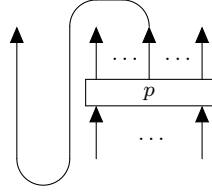
The same computation can be done for $p * (p' * p'')$, which gives the associativity of $*$.

2. (b). The unit is $I_0 = \alpha(\emptyset)$, where \emptyset is the graph with no vertex and no edge.

3. (d). The unit I is $\alpha(I_1)$, where I_1 is the graph with only one input-output edge. Let $p \in P(k, l)$ and $2 \leq j \leq l + 1$. Then $t_{1,j}(I * p)$ is obtained by application of $\alpha \circ \Gamma^\cup(\alpha)$ to the graph:



where the curved edge relate the first edge at the bottom to the j -th edge on the top. As $\alpha \circ \Gamma^\cup(\alpha) = \alpha \circ \mu_P$, $t_{1,j}(I * p)$ is obtained by application of α to the graph:



where the curved edge relate the first edge on the bottom to the j -th edge on the top (note that this edge is also the $(j - 1)$ -th outgoing the vertex decorated by p). As α is a $\mathfrak{S} \times \mathfrak{S}^{op}$ morphism, we obtain that this is $(1, \dots, j - 1) \cdot \alpha \circ \nu_P(p)$, that is to say $(1, \dots, j - 1) \cdot p$. \square

Proposition 7.1.3. *Any TraP is a Γ^\cup -algebra.*

Proof. Let P be a TraP. From Proposition 6.1.2, we obtain a unique TraP morphism $\alpha_P : \Gamma^\cup(P) \rightarrow P$, such that for any $(k, l) \in \mathbb{N}_0^2$, for any $p \in P(k, l)$, α_P sends the graph $\nu_P(p)$ to p . The map $\alpha_P \circ \Gamma^\cup(\alpha) : \Gamma^\cup \circ \Gamma^\cup(P) \rightarrow P$ is a TraP morphism, sending, for any graph $G \in \Gamma^\cup(P)$, $\nu_{\Gamma^\cup(P)}(G)$ to $\alpha(G)$. It is not difficult to see that $\mu_P : \Gamma^\cup \circ \Gamma^\cup(P) \rightarrow \Gamma^\cup(P)$ is a TraP morphism. Hence, $\alpha_P \circ \mu_P : \Gamma^\cup \circ \Gamma^\cup(P) \rightarrow P$ is a TraP morphism, sending, for any graph $G \in \Gamma^\cup(P)$, $\nu_{\Gamma^\cup(P)}(G)$ to $\alpha_P(G)$. As $\Gamma^\cup \circ \Gamma^\cup(P)$ is generated by the elements $\mu_P(G)$, both these morphisms coincide:

$$\alpha_P \circ \Gamma^\cup(\alpha) = \alpha \circ \mu_P.$$

For any $p \in P$, by construction of α_P , $\alpha_P \circ \nu_P(p) = p$, so:

$$\alpha_P \circ \nu_P = Id_P.$$

Therefore, P is a Γ^\cup -algebra. \square

Corollary 7.1.4. *The categories **TraP** of TraPs and $\Gamma^\cup - \mathbf{Alg}$ of Γ^\cup -algebras are isomorphic.*

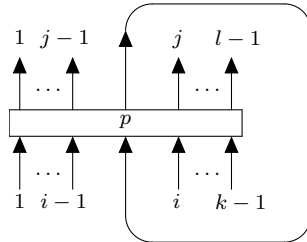
Proof. We defined in Propositions 7.1.2 and 7.1.3 two functors

$$\mathcal{F} : \mathbf{TraP} \longrightarrow \Gamma^\cup - \mathbf{Alg}, \quad \mathcal{G} : \Gamma^\cup - \mathbf{Alg} \longrightarrow \mathbf{TraP}.$$

Let P be a TraP and P' the TraP $\mathcal{G} \circ \mathcal{F}(P)$, with concatenation $*'$ and trace operators $t'_{i,j}$. We set $\mathcal{F}(P) := (P, \alpha_P)$: in other words, α_P is the TraP morphism of Proposition 6.1.2. For any $p, q \in P$:

$$p *' q = \alpha_P(\nu_P(p) * \nu_P(q)) = p * p',$$

where in the middle term $*$ is the concatenation in the TraP $\Gamma^\cup(P)$. Therefore, $* = *'$. If $p \in P(k, l)$, $(i, j) \in [k] \times [l]$, then $t'_{i,j}$ is obtained by the application of α_P to the graph:



which is $t_{i,j}(\nu_P(p))$, where here $t_{i,j}$ is the trace operator of $\Gamma^\cup(P)$. As α_P is a TraP morphism:

$$t'_{i,j}(p) = \alpha_P \circ t_{i,j} \circ \nu_P(p) = t_{i,j} \circ \alpha_P \circ \nu_P(p) = t_{i,j}(p),$$

so $P' = P$ and $\mathcal{G} \circ \mathcal{F}$ is the identity functor of \mathbf{TraP} .

Let now (P, α) be a Γ^\cup -algebra and let us consider (P', α') be the Γ^\cup -algebra $\mathcal{F} \circ \mathcal{G}(P)$. Both α and α' are TraP morphisms from $\Gamma^\cup(P)$ to $\mathcal{G}(P)$; for any $p \in P$,

$$\alpha \circ \nu_P(p) = \alpha' \circ \nu_P(p) = p.$$

As $\Gamma^\cup(P)$ is generated, as a TraP, by the elements $\nu_P(p)$, $\alpha = \alpha'$, so $\mathcal{F} \circ \mathcal{G}$ is the identity functor of $\Gamma^\cup - \mathbf{Alg}$. \square

Remark 7.1.1. Γ^\cup -algebras appear in the literature [Mer06, MMS09, Mer10b, Mer10a] under the name of unitary wheeled props; see [MMS09] for the description of the monad of graphs used for wheeled props, and [Mer10b, Mer10a] for applications of wheeled props.

We defined the structure of TraPs having their application to Feynman graphs in QFT in mind. Since our focus in this paper is on traces for which we need an explicit realisation of the structures under consideration, we choose to keep here the terminology TraP.

7.2 TraPs are ProPs

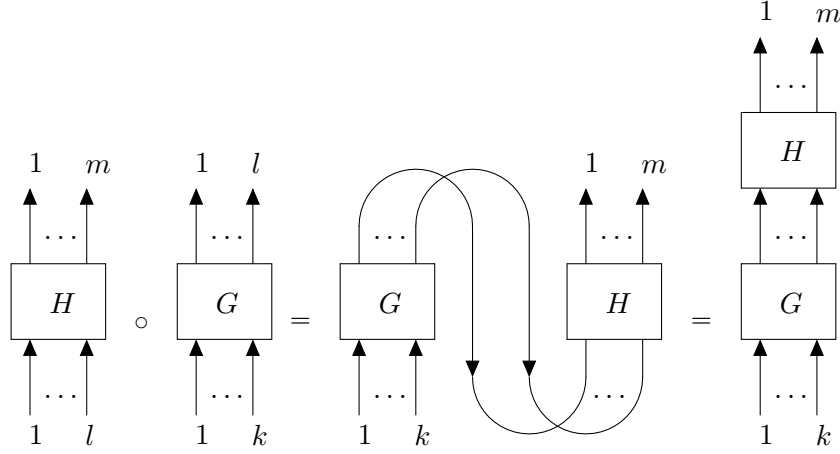
TraPs can be equipped with a ProP structure as a result of the fact that both the trace and composition of morphisms can be expressed in terms of a dual pairing. Corollary 7.1.4, yields an isomorphism between the categories of TraPs and wheeled ProPs. It is known that wheeled ProPs are ProPs, and we give here a detailed construction of the ProP structure on our TraPs, showing how the partial trace maps (referred to as contractions by Merkulov) of wheeled ProPs give rise to a vertical composition, and therefore to a ProP structure, a fact readily observed in [MMS09, Remarks 2.1.1].

Proposition 7.2.1. *Let P be a TraP. We define a vertical composition in the following way:*

$$\forall p \in P(k, l), \forall q \in P(l, m), \quad q \circ p = t_{k+1,1} \circ \dots \circ t_{k+l-1,l-1} \circ t_{k+l,l}(p * q).$$

Then P is a ProP.

Example 7.2.1. In the TraP of graphs \mathbf{Gr}^\cup :



Proof. It is enough to prove it for a free TraP $\mathbf{PGr}^\cup(X)$, as any TraP is the quotient of such an object. If $G \in \mathbf{PGr}^\cup(X)(k, l)$ and $H \in \mathbf{PGr}^\cup(X)(l, m)$ are two X -decorated planar graphs, then by definition of the partial trace maps, $G * H$ is the X -decorated planar graph obtained by grafting together the output edge i of G with the input edge j of H for any $i \in [k]$; this is precisely the vertical concatenation of graphs, adapted to X -decorated planar graphs. So it is indeed a ProP. \square

Example 7.2.2. 1. For graphs, we recover the composition defined in Section 1.3, extended to graphs.

2. For the \mathbf{Hom}_V TraP, for any $F = f_1 \dots f_k \otimes v_1 \dots v_l \in V^{*\otimes k} \otimes V^{\otimes l} \approx \mathbf{Hom}(V^{\otimes k}, V^{\otimes l})$ and $G = g_1 \dots g_l \otimes w_1 \dots w_n \in V^{*\otimes l} \otimes V^{\otimes m} \approx \mathbf{Hom}(V^{\otimes l}, V^{\otimes m})$:

$$F \circ G = g_1(v_1) \dots g_l(v_l) f_1 \dots f_k \otimes w_1 \dots w_n.$$

This is the composition of \mathbf{Hom}_V .

Applied to the TraP \mathbf{Hom}_V^c of Proposition 5.2.3, this method allows to recover the ProP \mathbf{Hom}_V^c of Theorem 2.2.5.

Proposition 7.2.2. *Let V be a Fréchet nuclear space. The ProP built from the TraP $(\mathbf{Hom}_V^c(k, l))_{k, l \geq 0}$ as in Proposition 7.2.1 is isomorphic, as a ProP, to the ProP \mathbf{Hom}_V^c of Theorem 2.2.5.*

Proof. It is enough to check that the composition of two homomorphisms will give the right object. Let $f = \mathbf{Hom}_V^c(k, l)$ and $g = \mathbf{Hom}_V^c(l, m)$. By Equation (4) we can write

$$f = \sum_{\alpha} ((v_1^{\alpha})^* \otimes \dots \otimes (v_l^{\alpha})^*) \otimes (w_1^{\alpha} \otimes \dots \otimes w_k^{\alpha}), \quad g = \sum_{\beta} ((u_1^{\beta})^* \otimes \dots \otimes (u_m^{\beta})^*) \otimes (r_1^{\beta} \otimes \dots \otimes r_l^{\beta}).$$

Then the definition of the composition product of Proposition 7.2.1 implies

$$f \circ g = \sum_{\alpha} \sum_{\beta} \left[\prod_{i=1}^l (v_i^{\alpha})^* (r_i^{\beta}) \right] \left((u_1^{\beta})^* \otimes \dots \otimes (u_m^{\beta})^* \right) \otimes (w_1^{\alpha} \otimes \dots \otimes w_k^{\alpha}).$$

Using Equation (7), we can apply Lemma 5.2.2 to the case $E_1 = V^{\widehat{\otimes} m}$, $E_2 = V^{\widehat{\otimes} l}$, $E_3 = V^{\widehat{\otimes} k}$. The result then follows from this lemma and the observation that

$$\prod_{i=1}^l (v_i^{\alpha})^* (r_i^{\beta}) = (v_1^{\alpha} \otimes \dots \otimes v_l^{\alpha})^* \otimes (r_1^{\beta} \otimes \dots \otimes r_l^{\beta})$$

for the duality pairing in E_2 . \square

We end this Subsection with a Corollary to Proposition 7.2.1.

Corollary 7.2.3. *Let P be a TraP. For any $p \in P(k, k)$, we set:*

$$\mathrm{Tr}(p) = t_{1,1} \circ \dots \circ t_{k,k}(p).$$

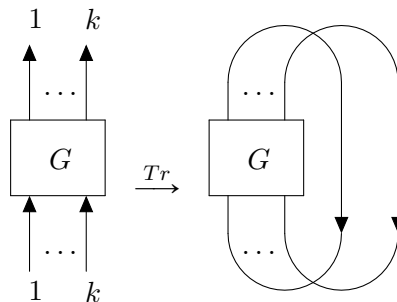
1. For any $(k, l) \in \mathbb{N}_0^2$, for any $(p, q) \in P(k, l) \times P(l, k)$,

$$\mathrm{Tr}(p \circ q) = \mathrm{Tr}(q \circ p).$$

2. For any $(k, l) \in \mathbb{N}_0^2$, for any $(p, q) \in P(k, k) \times P(l, l)$,

$$\mathrm{Tr}(p * q) = \mathrm{Tr}(p)\mathrm{Tr}(q).$$

Example 7.2.3. 1. In \mathbf{Gr}^\cup , for any graph $G \in \mathbf{Gr}^\cup(k, k)$, $\mathrm{Tr}(G)$ is obtained by gluing together the i -th output edge with the i -th output edge of G . In particular, $\mathcal{O} = \mathrm{Tr}(I)$. Graphically:



2. Let V be a finite dimensional vector space of dimension n . In the TraP \mathbf{Hom}_V introduced in Proposition 5.2.1 we obtain a trace for morphisms $F : V^{\otimes k} \mapsto V^{\otimes k}$. Specialising to the case $k = 1$, we recover the usual trace of linear endomorphisms: choose (e_1, \dots, e_n) a basis of V . Any morphism $f : V \mapsto V$ can be represented in this basis by $\sum_{i,j=1}^n a_{ij}^f e_i^* \otimes e_j$ for some complex numbers a_{ij}^f . Then $\mathrm{Tr}(f) = \sum_{i,j=1}^n a_{ij}^f e_i^*(e_j) = \sum_{i=1}^n a_{ii}^f$. $\mathrm{Tr}(f)$ lies in \mathbb{K} , is viewed here as an element of $\mathbf{Hom}_V(0, 0)$ via the identification of a constant λ in \mathbb{K} to a linear map $x \mapsto \lambda x$ on \mathbb{K} .

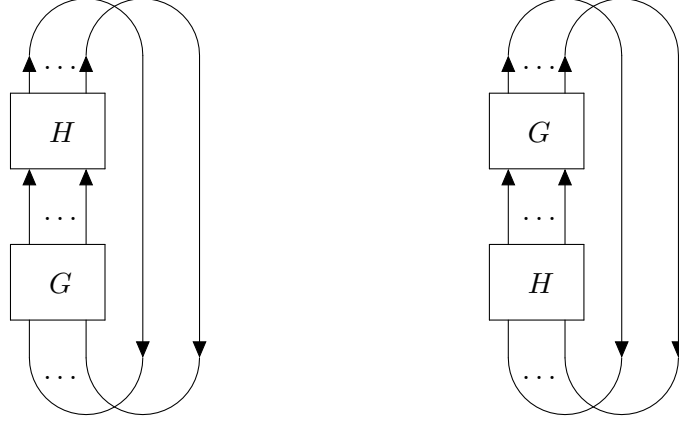
The vertical composition $f \circ g = t_{2,1}(f * g)$ of two morphisms f and g , defined according to Proposition 7.2.1 is indeed represented by the usual matrix product:

$$\sum_{i,j=1}^n \sum_{k,l=1}^n a_{ik}^f a_{lj}^g e_i^* \otimes e_k^* \otimes e_l \otimes e_j = \sum_{i,j=1}^n \left(\sum_{k=1}^n a_{ik}^f a_{kj}^g \right) e_i^* \otimes e_j,$$

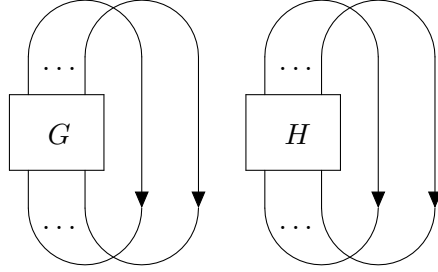
where $(a_{ij}^f)_{i,j}$, $(a_{ij}^g)_{i,j}$ are the matrix representations of f and g respectively.

Proof. Again, it is enough to prove the result for a free TraP $\mathbf{PGr}^\cup(X)$.

Let $G \in \mathbf{PGr}^\cup(X)(k, l)$ and $H \in \mathbf{PGr}^\cup(X)(l, k)$ be two graphs. Then $\mathrm{Tr}(H \circ G)$ is graphically represented by each of the graphs:



which are the same. So $\text{Tr}(H \circ G) = \text{Tr}(G \circ H)$. Moreover, the graph $\text{Tr}(G * H)$ is represented by the graph



which is also a graphical representation of $\text{Tr}(G) * \text{Tr}(H)$. So $\text{Tr}(G * H) = \text{Tr}(G) * \text{Tr}(H)$. \square

7.3 Quasi-TraPs

The partial trace maps $t_{i,j}$ arising in the definition of a TraP might not be defined on every operator. To circumvent this difficulty, we work with a $\mathfrak{S} \times \mathfrak{S}^{op}$ -module $(P(k,l))_{k,l \geq 0}$ with a horizontal concatenation \star , satisfying all the required axioms, and for any $k, l \geq 1$, for any $i \in [k]$, $j \in [l]$, a map $T_{i,j} : P(k,l) \rightarrow P(k-1, l-1)$ defined on a submodule of $P(k,l)$; we assume that it satisfies all the required axioms as soon as all the maps they imply are defined.

We can then embed such a quasi-TraP in a "complete" TraP: consider the TraP $\Gamma^{\cup}(P)$, and quotient it by the TraP ideal generated by the elements:

1. $\nu_P(p) * \nu_P(q) - \nu_P(p \star q)$, where $p, q \in P$.
2. $t_{i,j} \circ \nu_P(p) - \nu_P \circ T_{i,j}(p)$, where $p \in P$ such that $T_{i,j}(p)$ is defined.

We obtain in this way a TraP \overline{P} , with partial trace maps $t_{i,j}$ induced on the quotient by the partial trace maps of $\Gamma^{\cup}(P)$. It contains a $\mathfrak{S} \times \mathfrak{S}$ -module isomorphic to P and formed by graphs with only one vertex, which we identify with P itself. Then, if $T_{i,j}(p)$ is defined, $T_{i,j}(p) = t_{i,j}(p)$.

Example 7.3.1. Let $V = \mathbb{K}[X]$, $(X^n)_{n \geq 0}$ its canonical basis and $(\delta_n)_{n \geq 0}$ the dual basis. Let us denote by E^+ the subspace of $\mathbf{Hom}(V)$ generated by the endomorphisms of the form

$$f_{i,j} : \begin{cases} \mathbb{K}[X] & \longrightarrow \mathbb{K}[X] \\ X^k & \longrightarrow \delta_{i,k} X^j, \end{cases}$$

where $i, j \geq 0$ (i.e. $f_{i,j}(X^k) = X^j$ if $k = i$, and $f_{i,j}(X^k) = 0$ otherwise). This is the subspace of endomorphisms of V with a finite support when applied on monomials. Note that E^+ does

not contains Id_V : we put $E = E^+ \oplus \mathbb{K}\text{Id}_V$. For any $k, l \geq 0$, let $P(k, l)$ be the submodule of $\mathbf{Hom}(V^{\otimes k}, V^{\otimes k})$ generated by $E^{\otimes k}$ if $k = l$, and $\{0\}$ otherwise. This is stable under the horizontal concatenation of \mathbf{Hom}_V .

The elements of $P(k, k)$ are linear spans of terms:

$$\sigma \cdot (f_1 \otimes \dots \otimes f_k) \cdot \tau,$$

where $\sigma, \tau \in \mathfrak{S}_k$, and for any p , f_p is one of the $f_{i,j}$ or is Id_V . We define a partial trace map on P by putting $T_{1,1}(f_{i,j}) = \delta_{i,j}$; but $T_{1,1}(\text{Id}_V)$ is not defined. This is extended to P using the axioms of a TraP. For example:

$$\begin{aligned} T_{1,1}(f_{i,j} \otimes f_{k,l}) &= \delta_{i,j} f_{k,l}, & T_{1,1}(f_{i,j} \otimes \text{Id}_V) &= \delta_{i,j} \text{Id}_V, \\ T_{2,2}(f_{i,j} \otimes f_{k,l}) &= \delta_{k,l} f_{i,j}, & T_{2,2}(f_{i,j} \otimes \text{Id}_V) &\text{ is not defined,} \\ T_{1,2}(f_{i,j} \otimes f_{k,l}) &= \delta_{i,l} f_{k,j}, & T_{1,2}(f_{i,j} \otimes \text{Id}_V) &= f_{i,j}, \\ T_{2,1}(f_{i,j} \otimes f_{k,l}) &= \delta_{j,k} f_{i,l}, & T_{2,1}(f_{i,j} \otimes \text{Id}_V) &= f_{i,j}. \end{aligned}$$

Denoting by \mathcal{O} the graph with only one loop, we obtain that for any $k \geq 0$,

$$\overline{P}(k, k) = \mathbb{K}[\mathcal{O}] \otimes P(k, k),$$

and $t_{1,1}(\text{Id}_V) = \mathcal{O}$. Any $p \in P(k, k)$ is identified with $1 \otimes p \in \overline{P}(k, k)$. For example, in \overline{P} :

$$\begin{aligned} t_{1,1}(f_{i,j} \otimes f_{k,l}) &= \delta_{i,j} f_{k,l}, & t_{1,1}(f_{i,j} \otimes \text{Id}_V) &= \delta_{i,j} \text{Id}_V, \\ t_{2,2}(f_{i,j} \otimes f_{k,l}) &= \delta_{k,l} f_{i,j}, & t_{2,2}(f_{i,j} \otimes \text{Id}_V) &= f_{i,j} \otimes \mathcal{O}, \\ t_{1,2}(f_{i,j} \otimes f_{k,l}) &= \delta_{i,l} f_{k,j}, & t_{1,2}(f_{i,j} \otimes \text{Id}_V) &= f_{i,j}, \\ t_{2,1}(f_{i,j} \otimes f_{k,l}) &= \delta_{j,k} f_{i,l}, & t_{2,1}(f_{i,j} \otimes \text{Id}_V) &= f_{i,j}. \end{aligned}$$

Choosing for any $k \geq 1$ an element $f_k \in P(k, l)$, any graph G such that $L(G) = \emptyset$ is sent to an element of P by Φ .

8 The TraP $\overline{\mathcal{K}}_X^\infty$ of smoothing pseudo-differential operators

We apply our results on TraPs to tensor products of a class of Fréchet nuclear spaces introduced in Section 2, namely Fréchet spaces $\mathcal{E}(X)$ of smooth sections of X . Recall from Proposition 2.2.3 that such spaces are stable under tensor products and morphisms in $\text{Hom}^c(\mathcal{E}'(X), \mathcal{E}(Y))$ are determined by smoothing kernels in $\mathcal{E}(X \times Y)$.

8.1 Trace of smoothing pseudo-differential operators

Let X be a smooth finite dimensional closed manifold. Let us set $E = \mathcal{E}(X)$, and $F = \mathcal{E}'(X)$, which is not Fréchet, in which case Lemma 5.2.2 does not apply.

Instead, we restrict ourselves to smooth kernels which stabilise $\mathcal{E}(X)$. We set, for $(k, l) \neq (1, 1)$:

$$\mathcal{K}_X^\infty(k, l) := \mathcal{E}(X^k \times X^l) \simeq \mathcal{E}(X)^{\widehat{\otimes} k} \widehat{\otimes} \mathcal{E}(X)^{\widehat{\otimes} l},$$

where the identification holds by Proposition 2.2.3. For $(k, l) = (1, 1)$ we set

$$\mathcal{K}_X^\infty(1, 1) := \mathcal{E}(X \times X) \bigcup \{\delta\} \simeq \mathcal{E}(X) \widehat{\otimes} \mathcal{E}(X) \bigcup \{\delta\}$$

with δ the (singular) kernel of the identity operator on $\mathcal{E}(X)$. With the notations of Definition 5.1.1, we will have $I = \delta$.

For a closed Riemannian manifold X equipped with a volume measure μ , the canonical embedding $\mathcal{E}(X) \hookrightarrow \mathcal{E}'(X)$, $f \mapsto (\varphi \mapsto \int_X f(x) \varphi(x) d\mu(x))$ induces an embedding

$$\mathcal{K}_X^\infty(k, l) \hookrightarrow \mathcal{K}_X(k, l) \simeq \mathcal{E}'(X)^{\widehat{\otimes} k} \widehat{\otimes} \mathcal{E}(X)^{\widehat{\otimes} l}.$$

Proposition 8.1.1. *The family of topological vector spaces $(\mathcal{K}_X^\infty(k, l))_{k, l \geq 0}$ equipped with the partial traces*

$$t_{i,j} : \begin{cases} \mathcal{K}_X^\infty(k, l) & \longrightarrow \mathcal{K}_X^\infty(k-1, l-1) \\ K_1 \otimes K_2 & \longmapsto t_{i,j}(K_1 \otimes K_2) \end{cases}$$

with, for $K_1 \otimes K_2 \neq \delta$, $t_{i,j}(K_1 \otimes K_2)$ defined by

$$t_{i,j}(K_1 \otimes K_2)(x_1, \dots, x_{k-1}, y_1, \dots, y_{l-1}) := \int_X K_1(x_1, \dots, x_{i-1}, z, x_i \cdots, x_{k-1}) K_2(y_1, \dots, y_{j-1}, z, y_j \cdots, y_l) d\mu(z)$$

(with an obvious abuse of notation in the cases where i (or j) is equal to 1 or k (or to 1 or l)) defines a TraP, written \mathcal{K}_X^∞ .

Remark 8.1.1. Technically, \mathcal{K}_X^∞ is a quasi-TraP in the sense of Subsection 7.3 since $t_{1,1}(I) = t_{1,1}(\delta)$ is not defined. Following Subsection 7.3, this quasi-TraP can be completed to a full TraP $\overline{\mathcal{K}}_X^\infty$.

Proof. The unit $I_0 \in \mathcal{K}_X^\infty(0, 0) \simeq \mathbb{C} \otimes \mathbb{C}$ of the vertical concatenation $* = \otimes$ is the constant map defined by $f(x) = 1$. It is the unit of \otimes by bilinearity of the tensor product.

The unit $I \in \mathcal{K}_X^\infty(0, 0)$ is δ by definition of the action of Dirac's distribution on smooth kernels.

It suffices to show that $t_{i,j}(K_1 \otimes K_2)(x_1, \dots, x_{k-1}, y_1, \dots, y_{l-1})$ lies in $\mathcal{K}_X^\infty(k-1, l-1)$. The axioms of the TraP will then hold since they are in $\mathcal{K}_X(k, l)$ (Example 5.2.3).

The existence of the integral comes from the smoothness of K_1 and K_2 and the closedness of X . It is enough to show that the function $t_{i,j}(K_1 \otimes K_2) : X^{k-1} \times X^{l-1} \longrightarrow \mathbb{C}$ is smooth. Since K_1 and K_2 are smooth, the map

$$(x_1, \dots, x_{k-1}, y_1, \dots, y_l) \mapsto K_1(x_1, \dots, x_{i-1}, z, x_i \cdots, x_{k-1}) K_2(y_1, \dots, y_{j-1}, z, y_j \cdots, y_l)$$

is infinitely differentiable for any $z \in X$. Since X is compact, the partial derivatives

$$\partial_{\vec{x}}^{\vec{\alpha}} \partial_{\vec{y}}^{\vec{\beta}} K_1(x_1, \dots, x_{i-1}, z, x_i \cdots, x_{k-1}) K_2(y_1, \dots, y_{j-1}, z, y_j \cdots, y_l)$$

are bounded uniformly in z . We can therefore use the dominated convergence theorem to get that

$$\begin{aligned} & \int_X \partial_{\vec{x}}^{\vec{\alpha}} \partial_{\vec{y}}^{\vec{\beta}} K_1(x_1, \dots, x_{i-1}, z, x_i \cdots, x_{k-1}) K_2(y_1, \dots, y_{j-1}, z, y_j \cdots, y_l) d\mu(z) \\ &= \partial_{\vec{x}}^{\vec{\alpha}} \partial_{\vec{y}}^{\vec{\beta}} \int_X K_1(x_1, \dots, x_{i-1}, z, x_i \cdots, x_{k-1}) K_2(y_1, \dots, y_{j-1}, z, y_j \cdots, y_l) d\mu(z) \\ &= \partial_{\vec{x}}^{\vec{\alpha}} \partial_{\vec{y}}^{\vec{\beta}} t_{i,j}(K_1 \otimes K_2)(x_1, \dots, x_{k-1}, y_1, \dots, y_{l-1}). \end{aligned}$$

Therefore the map $t_{i,j}(K_1 \otimes K_2)(x_1, \dots, x_{k-1}, y_1, \dots, y_{l-1})$ is smooth. \square

In view of the fact that the trace of a smoothing pseudodifferential operator P with kernel K is

$$\text{Tr}(P) = \int_X K(x, x) d\mu(x)$$

Corollary 7.2.3 yields a generalised trace

$$\text{Tr} : \bigsqcup_{k \in \mathbb{N}_0} \mathcal{K}_X^\infty(k, k) \longrightarrow \mathbb{C}$$

on smoothing pseudo-differential operators on a closed smooth finite dimensional manifold. This trace is indeed cyclic for the horizontal and vertical composition products of $\overline{\mathcal{K}}_X^\infty$ in the sense of Corollary 7.2.3.

8.2 Generalised convolution of smoothing operators

Let X be a smooth finite dimensional closed Riemannian manifold. Set $X_{k,l} := \mathcal{K}_X^\infty(k, l)$. Recall from Proposition 6.1.2 that there exists a TraP map $\Phi : \Gamma^\cup(P) \longrightarrow X$, as X is a TraP.

Definition 8.2.1. *Let G be a graph decorated by $X = (\mathcal{K}_X^\infty(k, l))_{kl \in \mathbb{N}_0}$. The **generalised convolution** associated to G is the smoothing operator $\Phi(G) \in \overline{\mathcal{K}}_X^\infty$ given by the image of G under Φ .*

The name generalised convolution is justified by the following remark.

Remark 8.2.1. Let G be a ladder graph decorated by $X = (\mathcal{K}_X^\infty(k, l))_{kl \in \mathbb{N}_0}$ i.e., a graph such that $I(G) = O(G) = [1]$, $IO(G) = L(G) = \emptyset$, $V(G) = \{v_1, \dots, v_n\}$, $E(G) = \{e_1, \dots, e_{n-1}\}$ and the source and target maps defined by

$$\begin{aligned} \forall i \in [n-1], \quad s_G(1) &= v_n, & t_G(1) &= v_1, \\ s_G(e_i) &= v_i, & t_G(e_i) &= v_{i+1}. \end{aligned}$$

Here is a graphical representation of this graph:

$$1 \longrightarrow \textcircled{v_1} \longrightarrow \dots \longrightarrow \textcircled{v_n} \longrightarrow 1$$

Let O_i be the smoothing pseudo-differential operator defined by the kernel K_i that decorates the vertex v_i : $K_i := \text{dec}(v_i)$ for any $v_i \in [n]$. Then the generalised convolution associated to the graph G is the convolution of the kernels K_i, \dots, K_n , which is the kernel of the smoothing pseudo-differential operator $O_1 \circ \dots \circ O_n$.

The previous remark leads to the following statement.

Corollary 8.2.2. *The convolution of smoothing pseudo-differential operators is well-defined and associative.*

Proof. Well-definedness follows from the definition. The associativity follows from the fact that the vertical composition build from the TraP structure of graphs is associative, together with the fact that ϕ_{Id} is a morphism of TraP. \square

A Appendix: topologies on tensor products

Tensor products of topological spaces can be equipped with various topologies. A first possibility is the so-called ϵ -**topology**; [Tre67, Definition 43.1]. For two topological vector spaces E and F , one can show ([Tre67, Proposition 42.4]) the isomorphism of vector spaces $E \otimes F \simeq \mathcal{B}^c(E'_\sigma \times F'_\sigma, \mathbb{K})$ where $\mathcal{B}^c(E'_\sigma \times F'_\sigma, \mathbb{K})$ denotes the space of continuous bilinear maps from $E'_\sigma \times F'_\sigma$ to \mathbb{K} and E'_σ (resp. F'_σ) the topological dual of E (resp. F) for σ , the weak topology.

Recall that a bilinear map $f : E \times F \longrightarrow K$ is called separately continuous if, for any pair $(x, y) \in E \times F$, the maps $z \rightarrow f(x, z)$ and $z \rightarrow f(z, y)$ are continuous. We then clearly have that continuous bilinear maps build a linear subspace of the space $\mathcal{B}^{sc}(E \times F, \mathbb{K})$ of separately continuous bilinear maps.

The space $\mathcal{B}^{sc}(E \times F, \mathbb{K})$ can be equipped with the topology of uniform convergence on products of equicontinuous subsets of E'_σ with equicontinuous subsets of F'_σ . Recall that, for a topological space X and a topological vector space G , a set S of maps from X to G is said to be equicontinuous at $x_0 \in X$ if, for any $V \subseteq G$ neighbourhood of zero, there is some neighbourhood $V(x_0) \subseteq X$ of x_0 , such that

$$\forall f \in S, x \in V(x_0) \Rightarrow f(x) - f(x_0) \in V.$$

In our case, G is \mathbb{K} and X is E_σ (resp. F_σ). This topology induces a topology on the subspace $\mathcal{B}^c(E'_\sigma \times F'_\sigma, \mathbb{K})$ and thus on $E \otimes F$. We denote by $E \otimes_\epsilon F$ the topological vector space obtained by endowing $E \otimes F$ with this topology.

There is another topology on $E \otimes F$ called the **projective topology**; [Tre67, Definition 43.2]. The projective topology is defined as the strongest locally convex topology on $E \otimes F$ such that the canonical map $\phi : E \times F \longrightarrow E \otimes F$ is continuous. We write $E \otimes_{\pi} F$ the topological vector space obtained by endowing $E \otimes F$ with this topology.

The neighbourhoods of zero of the projective topology can be simply described in terms of neighbourhoods of zero in E and V . A convex subset S of $E \otimes F$ containing zero is a neighbourhood of zero if it exist a neighbourhood U (resp. V) of zero in E (resp. F) such that $U \otimes V := \{u \otimes v | u \in U \wedge v \in V\} \subseteq S$.

B Appendix: definition of the partial trace maps on \mathbf{Gr}^{\cup}

We give a rigorous definition of the partial trace maps on the space of graphs \mathbf{Gr}^{\cup} , which were only loosely defined in the bulk of the article.

Let $G \in \mathbf{Gr}^{\cup}(k, l)$ with $k, l \geq 1$, $i \in [k]$ and $j \in [l]$. We put $e_i = \alpha_G^{-1}(i)$ and $f_j = \beta_G^{-1}(j)$. We define the graph $G' = t_{i,j}(G)$ in the following way:

1. If $e_i \in I(G)$ and $f_j \in O(G)$, then:

$$\begin{aligned} V(G') &= V(G), & E(G') &= E(G) \sqcup \{(e_i, f_j)\}, \\ I(G') &= I(G) \setminus \{e_i\}, & O(G') &= O(G) \setminus \{f_j\}, \\ IO(G') &= IO(G), & L(G') &= L(G), \\ s_{G'}(e) &= \begin{cases} s_G(f_j) & \text{if } e = (e_i, f_j), \\ s_G(e) & \text{otherwise,} \end{cases} & t_{G'}(e) &= \begin{cases} t_G(e_i) & \text{if } e = (e_i, f_j), \\ t_G(e) & \text{otherwise,} \end{cases} \\ \alpha_{G'}(e) &= \begin{cases} \alpha_G(e) & \text{if } \alpha_G(e) < i, \\ \alpha_G(e) - 1 & \text{if } \alpha_G(e) \geq i, \end{cases} & \beta_{G'}(e) &= \begin{cases} \beta_G(e) & \text{if } \beta_G(e) < j, \\ \beta_G(e) - 1 & \text{if } \beta_G(e) \geq j. \end{cases} \end{aligned}$$

2. If $e_i \in IO(G)$ and $f_j \in O(G)$, then:

$$\begin{aligned} V(G') &= V(G), & E(G') &= E(G), \\ I(G') &= I(G), & O(G') &= O(G) \setminus \{f_j\} \sqcup \{(e_i, f_j)\}, \\ IO(G') &= IO(G) \setminus \{e_i\}, & L(G') &= L(G), \\ s_{G'}(e) &= \begin{cases} s_G(f_j) & \text{if } e = (e_i, f_j), \\ s_G(e) & \text{otherwise,} \end{cases} & t_{G'}(e) &= t_G(e), \\ \alpha_{G'}(e) &= \begin{cases} \alpha_G(e) & \text{if } \alpha_G(e) < i, \\ \alpha_G(e) - 1 & \text{if } \alpha_G(e) \geq i, \end{cases} & \beta_{G'}(e) &= \begin{cases} \beta_G(e_i) & \text{if } e = (e_i, f_j) \text{ and } \beta_G(e_i) < j, \\ \beta_G(e_i) - 1 & \text{if } e = (e_i, f_j) \text{ and } \beta_G(e_i) \geq j, \\ \beta_G(e) & \text{if } e \neq (e_i, f_j) \text{ and } \beta_G(e) < j, \\ \beta_G(e) - 1 & \text{if } e \neq (e_i, f_j) \text{ and } \beta_G(e) \geq j. \end{cases} \end{aligned}$$

3. If $e_i \in I(G)$ and $f_j \in IO(G)$, then:

$$\begin{aligned} V(G') &= V(G), & E(G') &= E(G), \\ I(G') &= I(G) \setminus \{e_i\} \sqcup \{(e_i, f_j)\}, & O(G') &= O(G), \\ IO(G') &= IO(G) \setminus \{f_j\}, & L(G') &= L(G), \\ s_{G'}(e) &= s_G(e), & t_{G'}(e) &= \begin{cases} t_G(e_i) & \text{if } e = (e_i, f_j), \\ t_G(e) & \text{otherwise,} \end{cases} \\ \alpha_{G'}(e) &= \begin{cases} \alpha_G(f_i) & \text{if } e = (e_i, f_j) \text{ and } \alpha_G(f_j) < i, \\ \alpha_G(f_i) - 1 & \text{if } e = (e_i, f_j) \text{ and } \alpha_G(f_j) \geq i, \\ \alpha_G(e) & \text{if } e \neq (e_i, f_j) \text{ and } \alpha_G(e) < i, \\ \alpha_G(e) - 1 & \text{if } e \neq (e_i, f_j) \text{ and } \alpha_G(e) \geq i, \end{cases} & \beta_{G'}(e) &= \begin{cases} \beta_G(e) & \text{if } \beta_G(e) < j, \\ \beta_G(e) - 1 & \text{if } \beta_G(e) \geq j. \end{cases} \end{aligned}$$

4. If $e_i \in IO(G)$, $f_j \in IO(G)$ and $e_i \neq f_j$, then:

$$\begin{aligned}
V(G') &= V(G), & E(G') &= E(G), \\
I(G') &= I(G), & O(G') &= O(G), \\
IO(G') &= \{(e_i, f_j)\} \sqcup IO(G) \setminus \{(e_i, f_j)\}, & L(G') &= L(G), \\
s_{G'}(e) &= s_G(e), & t_{G'}(e) &= t_G(e), \\
\alpha_{G'}(e) &= \begin{cases} \alpha_G(f_i) & \text{if } e = (e_i, f_j) \text{ and } \alpha_G(f_j) < i, \\ \alpha_G(f_i) - 1 & \text{if } e = (e_i, f_j) \text{ and } \alpha_G(f_j) \geq i, \\ \alpha_G(e) & \text{if } e \neq (e_i, f_j) \text{ and } \alpha_G(e) < i, \\ \beta_G(e) - 1 & \text{if } e \neq (e_i, f_j) \text{ and } \alpha_G(e) \geq i, \end{cases} \\
\beta_{G'}(e) &= \begin{cases} \beta_G(e_i) & \text{if } e = (e_i, f_j) \text{ and } \beta_G(e_i) < j, \\ \beta_G(e_i) - 1 & \text{if } e = (e_i, f_j) \text{ and } \beta_G(e_i) \geq j, \\ \beta_G(e) & \text{if } e \neq (e_i, f_j) \text{ and } \beta_G(e) < j, \\ \beta_G(e) - 1 & \text{if } e \neq (e_i, f_j) \text{ and } \beta_G(e) \geq j. \end{cases}
\end{aligned}$$

5. If $e_i \in IO(G)$, $f_j \in IO(G)$ and $e_i = f_j$, then:

$$\begin{aligned}
V(G') &= V(G), & E(G') &= E(G), \\
I(G') &= I(G), & O(G') &= O(G), \\
IO(G') &= IO(G) \setminus \{(e_i, f_j)\}, & L(G') &= L(G) \sqcup \{(e_i, f_j)\}, \\
s_{G'}(e) &= s_G(e), & t_{G'}(e) &= t_G(e), \\
\alpha_{G'}(e) &= \begin{cases} \alpha_G(e) & \text{if } \alpha_G(e) < i, \\ \alpha_G(e) - 1 & \text{if } \alpha_G(e) \geq i, \end{cases} & \beta_{G'}(e) &= \begin{cases} \beta_G(e) & \text{if } \beta_G(e) < j, \\ \beta_G(e) - 1 & \text{if } \beta_G(e) \geq j. \end{cases}
\end{aligned}$$

C Appendix: full proofs

C.1 Proof of Theorem 3.2.1

Proof. Let us define $\Phi(G)$ for any graph G by induction on $n = |V(G)|$, such that for any permutation $\sigma \in \mathfrak{S}_{i(G)}$, $\tau \in \mathfrak{S}_{o(G)}$,

$$\Phi(\sigma \cdot G \cdot \tau) = \sigma \cdot \Phi(G) \cdot \tau.$$

If $n = 0$, there exists a unique permutation $\gamma \in \mathfrak{S}_k$ such that $G = \gamma \cdot I_k$. We put

$$\Phi(G) = \gamma \cdot I_k,$$

where we used the same notation I_k for the units of \mathbf{Gr} and P .

If $\sigma, \tau \in \mathfrak{S}_k$:

$$\begin{aligned}
\Phi(\sigma \cdot G \cdot \tau) &= \Phi((\sigma\gamma) \cdot I_k \cdot \tau) \\
&= \Phi((\sigma\gamma\tau) \cdot I_k) \\
&= (\sigma\gamma\tau) \cdot I_k \\
&= \sigma \cdot (\gamma \cdot I_k) \cdot \tau \\
&= \sigma \cdot \Phi(G) \cdot \tau.
\end{aligned}$$

Let us assume that $\Phi(G')$ is defined for any graph G' such that $|V(G')| < n$. Let

$$G = \gamma \cdot (G_1 * \dots * G_k * I_p) \circ G_0 * \mathcal{O}^{*\ell}$$

be a minimal decomposition of G . If G is indecomposable, we set $\Phi(G) = \phi(G)$. Otherwise, as $V(G_1) \neq \emptyset$, $|V(G_0)| < n$. We put:

$$\Phi(G) = \gamma \cdot (\phi(G_1) * \dots * \phi(G_k) * I_p) \circ \Phi(G_0) * \phi(\mathcal{O})^{*\ell}.$$

Let us first prove that this does not depend on the choice of the minimal decomposition of G . Starting from a minimal decomposition of G , one obtains all possible minimal decompositions of G by a finite sequence of operations of type A and B:

- Type A: changing the indexations of the input and output edges of the graphs G_i . We obtain a minimal decomposition $G = \gamma' \cdot (G'_1 * \dots * G'_k * I_p) \circ G'_0 * \mathcal{O}^{*\ell}$, such that there exists permutations α_i, β_i , with:

$$\begin{aligned} G'_i &= \alpha_i \cdot G_i \cdot \beta_i, \\ G'_0 &= (\beta_1^{-1} \otimes \dots \otimes \beta_k^{-1} \otimes \text{Id}_p) \cdot G_0, \\ \alpha' &= \alpha(\alpha_1^{-1} \otimes \dots \otimes \alpha_k^{-1} \otimes \text{Id}_p). \end{aligned}$$

- Type B: permuting G_l and G_{l+1} for $l \in [k-1]$. We obtain another minimal decomposition $G = \gamma' \cdot (G'_1 * \dots * G'_k * I_p) \circ G'_0 * \mathcal{O}^{*\ell}$, with:

$$\begin{aligned} G'_i &= \begin{cases} G_{l+1} & \text{if } i = l, \\ G_l & \text{if } i = l + 1, \\ G_i & \text{otherwise;} \end{cases} \\ G'_0 &= (\text{Id}_{i(G_1)+\dots+i(G_{l-1})} \otimes c_{i(G_{l+1}),i(G_l)} \otimes \text{Id}_{i(G_{l+2})+\dots+i(G_k)+p}) \circ G_0, \\ \gamma' &= \gamma(\text{Id}_{o(G_1)+\dots+o(G_{l-1})} \otimes c_{o(G_l),o(G_{l+1})} \otimes \text{Id}_{o(G_{l+2})+\dots+o(G_k)+p}). \end{aligned}$$

Let $G = \gamma \cdot (G'_1 * \dots * G'_k * I_p) \circ G'_0 * \mathcal{O}^{*\ell'}$ be another minimal decomposition of G . Then $\ell = \ell'$ is the number of loops of G . It is enough to prove that

$$\gamma \cdot (\phi(G_1) * \dots * \phi(G_k) * I_p) \circ \Phi(G_0) = \gamma' \cdot (\phi(G'_1) * \dots * \phi(G'_k) * I_p) \circ \Phi(G'_0).$$

We can assume that G' is obtained from G by a single operation of type A or of type B. If it is of type A:

$$\begin{aligned} &\gamma' \cdot (\phi(G'_1) * \dots * \phi(G'_k) * I_p) \circ \Phi(G'_0) \\ &= \gamma \cdot (\alpha_1^{-1} \otimes \dots \otimes \alpha_k^{-1} \otimes \text{Id}_p) \cdot (\phi(\alpha_1 \cdot G_1 \cdot \beta_1) \otimes \dots \otimes \phi(\alpha_k \cdot G_k \cdot \beta_k) * I_p) \\ &\circ \Phi((\beta_1^{-1} \otimes \dots \otimes \beta_k^{-1} \otimes \text{Id}_p) \cdot G_0) \\ &= \gamma \cdot (\alpha_1^{-1} \otimes \dots \otimes \alpha_k^{-1} \otimes \text{Id}_p) \cdot (\alpha_1 \cdot \phi(G_1) \cdot \beta_1 \otimes \dots \otimes \alpha_k \cdot \phi(G_k) \cdot \beta_k * I_p) \\ &\circ ((\beta_1^{-1} \otimes \dots \otimes \beta_k^{-1} \otimes \text{Id}_p) \cdot \Phi(G_0)) \\ &= \gamma(\alpha_1^{-1} \otimes \dots \otimes \alpha_k^{-1} \otimes \text{Id}_p)(\alpha_1 \otimes \dots \otimes \alpha_k \otimes \text{Id}_p) \cdot (\phi(G_1) \otimes \dots \otimes \phi(G_k) * I_p) \\ &\circ (\beta_1 \otimes \dots \otimes \beta_k \otimes \text{Id}_p)(\beta_1^{-1} \otimes \dots \otimes \beta_k^{-1} \otimes \text{Id}_p) \cdot \Phi(G_0) \\ &= \gamma \cdot (\phi(G_1) \otimes \dots \otimes \phi(G_k) * I_p) \circ \Phi(G_0). \end{aligned}$$

If it is of type B:

$$\begin{aligned}
& \gamma' \cdot (\phi(G'_1) * \dots * \phi(G'_k) * I_p) \circ \Phi(G'_0) \\
&= \gamma(\text{Id}_{o(G_1)+\dots+o(G_{l-1})} \otimes c_{o(G_l),o(G_{l+1})} \otimes \text{Id}_{o(G_{l+2})+\dots+o(G_k)+p}) \\
&\cdot (\phi(G_1) * \dots * \phi(G_{l+1}) * \phi(G_l) * \dots * \phi(G_k) * I_p) \\
&\circ \Phi((\text{Id}_{i(G_1)+\dots+i(G_{l-1})} \otimes c_{i(G_{l+1}),i(G_l)} \otimes \text{Id}_{i(G_{l+2})+\dots+i(G_k)+p}) \cdot G_0) \\
&\gamma' \cdot (\phi(G'_1) * \dots * \phi(G'_k) * I_p) \circ \Phi(G'_0) \\
&= \gamma(\text{Id}_{o(G_1)+\dots+o(G_{l-1})} \otimes c_{o(G_l),o(G_{l+1})} \otimes \text{Id}_{o(G_{l+2})+\dots+o(G_k)+p}) \\
&\cdot (\phi(G_1) * \dots * \phi(G_{l+1}) * \phi(G_l) * \dots * \phi(G_k) * I_p) \\
&\circ (\text{Id}_{i(G_1)+\dots+i(G_{l-1})} \otimes c_{i(G_{l+1}),i(G_l)} \otimes \text{Id}_{i(G_{l+2})+\dots+i(G_k)+p}) \cdot \Phi(G_0) \\
&= \gamma \cdot (\phi(G_1) * \dots * c_{o(G_l),o(G_{l+1})} \cdot (\phi(G_{l+1}) * \phi(G_l)) \cdot c_{i(G_{l+1}),i(G_l)} * \dots * \phi(G_k) * I_p) \circ \Phi(G_0) \\
&= \gamma \cdot (\phi(G_1) * \dots * \phi(G_l) * \phi(G_{l+1})) * \dots * \phi(G_k) * I_p \circ \Phi(G_0) \\
&= \gamma \cdot (\phi(G_1) \otimes \dots \otimes \phi(G_k) * I_p) \circ \Phi(G_0).
\end{aligned}$$

So $\Phi(G)$ is well-defined. Let $\sigma \in \mathfrak{S}_{o(G)}$ and $\tau \in \mathfrak{S}_{i(G)}$. We put $H = \sigma \cdot G \cdot \tau$. A minimal decomposition of H is given by:

$$H_0 = G_0 \cdot \tau, \quad H_i = G_i \text{ if } i \in [k], \quad \gamma' = \sigma\gamma.$$

Hence:

$$\begin{aligned}
\Phi(H) &= \sigma\gamma \cdot (\phi(G_1) * \dots * \phi(G_k) * I_p) \circ \Phi(G_0 \cdot \tau) * \phi(\mathcal{O})^{*\ell} \\
&= \sigma\gamma \cdot (\phi(G_1) * \dots * \phi(G_k) * I_p) \circ \Phi(G_0) \cdot \tau * \phi(\mathcal{O})^{*\ell} \\
&= \sigma \cdot (\gamma \cdot (\phi(G_1) * \dots * \phi(G_k) * I_p) \circ \Phi(G_0)) \cdot \tau * \phi(\mathcal{O})^{*\ell} \\
&= \sigma \cdot \Phi(G) \cdot \tau.
\end{aligned}$$

Consequently, we have defined a map $\Phi : \mathbf{Gr}^\cup \longrightarrow P$, extending the morphism ϕ of $\mathfrak{S} \times \mathfrak{S}^{op}$ -modules. Let us prove that it is compatible with both concatenations.

Let G and G' be two graphs, both with no loop. Let us prove that $\Phi(G * G') = \Phi(G) * \Phi(G')$ by induction on $n' = |V(G')|$. If $n' = 0$, there exists $\tau' \in \mathfrak{S}_q$ and $\ell' \in \mathbb{N}_0$, such that $G' = \sigma' \cdot I_q$. We proceed by induction on $n = |V(G)|$. If $n = 0$, there exists $\tau \in \mathfrak{S}_p$, such that $G = \sigma \cdot I_p$. Then $G * G' = (\sigma \otimes \sigma') \cdot I_{p+q}$, and:

$$\begin{aligned}
\Phi(G * G') &= (\sigma \otimes \sigma') \cdot I_{p+q} \\
&= (\sigma \otimes \sigma') \cdot (I_p * I_q) \\
&= (\sigma \cdot I_p) * (\sigma' \cdot I_q) \\
&= \Phi(G) * \Phi(G').
\end{aligned}$$

Otherwise, let $G = \gamma \cdot (G_1 * \dots * G_k * I_p) \circ G_0$ be a minimal decomposition of G . A minimal decomposition of $G * G'$ is:

$$G * G' = (\alpha \otimes \sigma') \cdot (G_1 * \dots * G_k * I_{p+q}) \circ (G_0 * I_q),$$

so, using the induction hypothesis on G_0 :

$$\begin{aligned}
\Phi(G * G') &= (\gamma \otimes \sigma') \cdot (\phi(G_1) * \dots * \phi(G_k) * I_{p+q}) \circ \Phi(G_0 * I_q) \\
&= (\gamma \otimes \sigma') \cdot \phi(G_1) * \dots * \phi(G_k) * I_p * I_q \circ (\Phi(G_0) * I_q) \\
&= (\gamma \cdot (\phi(G_1) * \dots * \phi(G_k) * I_p) \circ \Phi(G_0)) * (\sigma' \cdot I_q) \\
&= \Phi(G) * \Phi(G').
\end{aligned}$$

So the result holds at rank $n' = 0$.

Let us assume the results hold at any rank $< n'$. Let us consider minimal decompositions of G and G' :

$$G = \gamma \cdot (G_1 * \dots * G_k * I_p) \circ G_0, \quad G' = \gamma' \cdot (G'_1 * \dots * G'_l * I_q) \circ G'_0,$$

with the convention $k = 0$ if $V(G) = \emptyset$. We obtain a minimal decomposition of $G * G'$:

$$\begin{aligned} G * G' &= (\gamma \otimes \gamma')(\text{Id}_{o(G_1)+\dots+o(G_k)} \otimes c_{p,o(G'_1)+\dots+o(G'_l)} \otimes \text{Id}_q) \\ &\quad \cdot (G_1 * \dots * G_k * G'_1 * \dots * G'_l * I_{p+q}) \\ &\quad \circ ((\text{Id}_{i(G_1)+\dots+i(G_k)} \otimes c_{i(G'_1)+\dots+i(G'_l),p} \otimes \text{Id}_q)) \cdot (G_0 * G'_0)). \end{aligned}$$

We apply the induction assumption $\Phi(G * G') = \Phi(G) * \Phi(G')$ for $|V(G')| < n'$ to G'_0 whose number of vertices is smaller than that of G' and hence smaller than n' .

$$\begin{aligned} \Phi(G * G') &= (\gamma \otimes \gamma')(\text{Id}_{o(G_1)+\dots+o(G_k)} \otimes c_{p,o(G'_1)+\dots+o(G'_l)} \otimes \text{Id}_q) \\ &\quad \cdot (\phi(G_1) * \dots * \phi(G_k) * \phi(G'_1) * \dots * \phi(G'_l) * I_{p+q}) \\ &\quad \circ \Phi((\text{Id}_{i(G_1)+\dots+i(G_k)} \otimes c_{i(G'_1)+\dots+i(G'_l),p} \otimes \text{Id}_q)) \cdot (G_0 * G'_0)) \\ &= (\gamma \otimes \gamma')(\text{Id}_{o(G_1)+\dots+o(G_k)} \otimes c_{p,o(G'_1)+\dots+o(G'_l)} \otimes \text{Id}_q) \\ &\quad \cdot (\phi(G_1) * \dots * \phi(G_k) * \phi(G'_1) * \dots * \phi(G'_l) * I_{p+q}) \\ &\quad \circ (\text{Id}_{i(G_1)+\dots+i(G_k)} \otimes c_{i(G'_1)+\dots+i(G'_l),p} \otimes \text{Id}_q)) \cdot \Phi(G_0 * G'_0) \\ &= (\gamma \otimes \gamma')(\text{Id}_{o(G_1)+\dots+o(G_k)} \otimes c_{p,o(G'_1)+\dots+o(G'_l)} \otimes \text{Id}_q) \\ &\quad \cdot (\phi(G_1) * \dots * \phi(G_k) * \phi(G'_1) * \dots * \phi(G'_l) * I_p * I_q) \\ &\quad \circ (\text{Id}_{i(G_1)+\dots+i(G_k)} \otimes c_{i(G'_1)+\dots+i(G'_l),p} \otimes \text{Id}_q)) \cdot (\Phi(G_0) * \Phi(G'_0)) \\ &= (\gamma \otimes \gamma') \cdot (\phi(G_1) * \dots * \phi(G_k) * I_p * \phi(G'_1) * \dots * \phi(G'_l) * I_q) \circ (\Phi(G_0) * \Phi(G'_0)) \\ &= (\gamma \cdot (\phi(G_1) * \dots * \phi(G_k) * I_p) \circ \Phi(G_0)) * (\gamma' \cdot (\phi(G'_1) * \dots * \phi(G'_l) * I_q) \circ (\Phi(G'_0))) \\ &= \Phi(G) * \Phi(G'). \end{aligned}$$

So if G and G' are both graphs with no loop, $\Phi(G * G') = \Phi(G) * \Phi(G')$.

Let G, G' be two graphs, both with no loop. Let us prove that $\Phi(G' \circ G) = \Phi(G') \circ \Phi(G)$. We proceed by induction on $n = |V(G)| + |V(G')|$. If $V(G') = \emptyset$, there exists a permutation $\sigma \in \mathfrak{S}_p$ such that $G' = \sigma \cdot I_k$. Then:

$$\Phi(G' \circ G) = \Phi(\sigma \cdot G) = \sigma \cdot \Phi(G) = \sigma \cdot (I_p \circ \Phi(G)) = (\sigma \cdot I_p) \circ \Phi(G) = \Phi(G') \circ \Phi(G).$$

Similarly, if $V(G) = \emptyset$, $\Phi(G' \circ G) = \Phi(G') \circ \Phi(G)$. Thus we have proved the cases $n = 0$ and 1 .

Let us assume it holds up to rank N and take G and G' such that $n = N + 1$. By the previous argument, $\Phi(G \circ G') = \Phi(G) \circ \Phi(G')$ if $V(G) = \emptyset$ or $V(G') = \emptyset$. We now assume that $V(G)$ and $V(G')$ are nonempty. Let us consider minimal decompositions of G and G' :

$$G = \gamma \cdot (G_1 * \dots * G_k * I_p) \circ G_0 \quad G' = \gamma' \cdot (G'_1 * \dots * G'_l * I_q) \circ G'_0.$$

In $G' \circ G$, the output edges of G are glued with an input or an input-output edge of G' . In particular, for any i , output edges of G_i are glued with input edges or input-output edges of G' . Up to a change of indexation we assume that there is some r such that:

- For all $i \leq r$, at least one output edge of G_i is glued with an input edge of G' .
- If $i > r$, all output edges of G_i are glued with input-output edges of G' .

A *particular sub-case*. We assume that the input-output edges of G' glued with an output of one of the G_i are the input edges of G' with the greatest indices. Then $G'_0 = G''_0 * I_{s+o(G_{r+1})+\dots+o(G_k)}$ for a certain s . Moreover, γ can be written as $\gamma = \gamma_1 \otimes \gamma_2$, such that a minimal decomposition of $H = G' \circ G$ is given by:

$$\begin{aligned} H_0 &= (\text{Id}_{i(G'_1)+\dots+i(G'_l)} \otimes c_{i(G_{r+1})+\dots+i(G_k)+p,s}) \cdot G'_0 \\ &\quad \circ (\gamma_1 \cdot (G_1 * \dots * G_r * I_{i(G_{r+1})+\dots+i(G_k)+p}) \cdot G_0), \\ (H_1, \dots, H_m) &= (G'_1, \dots, G'_l, G_{r+1}, \dots, G_k), \\ \gamma'' &= \gamma' (\text{Id}_{o(G'_1)+\dots+o(G'_l)} \otimes c_{s,o(G_{r+1})+\dots+o(G_k)+p}) (\text{Id}_{o(G'_1)+\dots+o(G'_l)+s} \otimes \gamma_2). \end{aligned}$$

Applying the induction hypothesis on G_0 and G'_0 :

$$\begin{aligned} \Phi(H) &= \gamma' (\text{Id}_{o(G'_1)+\dots+o(G'_l)} \otimes c_{s,o(G_{r+1})+\dots+o(G_k)+p}) (\text{Id}_{o(G'_1)+\dots+o(G'_l)+s} \otimes \gamma_2) \\ &\quad \cdot ((\phi(G'_1) * \dots * \phi(G'_l) * \phi(G_{r+1}) * \dots * \phi(G_k))) \\ &\quad \circ \Phi((\text{Id}_{i(G'_1)+\dots+i(G'_l)} \otimes c_{i(G_{r+1})+\dots+i(G_k)+p,s}) \cdot G'_0) \\ &\quad \circ (\gamma_1 \cdot (\phi(G_1) * \dots * \phi(G_r) * I_{i(G_{r+1})+\dots+i(G_k)+p}) \cdot G_0) \\ &= \gamma' (\text{Id}_{o(G'_1)+\dots+o(G'_l)} \otimes c_{s,o(G_{r+1})+\dots+o(G_k)+p}) (\text{Id}_{o(G'_1)+\dots+o(G'_l)+s} \otimes \gamma_2) \\ &\quad \cdot ((\phi(G'_1) * \dots * \phi(G'_l) * \phi(G_{r+1}) * \dots * \phi(G_k))) \\ &\quad \circ (\text{Id}_{i(G'_1)+\dots+i(G'_l)} \otimes c_{i(G_{r+1})+\dots+i(G_k)+p,s}) \cdot \Phi(G'_0) \\ &\quad \circ (\gamma_1 \cdot (\phi(G_1) * \dots * \phi(G_r) * I_{i(G_{r+1})+\dots+i(G_k)+p}) \cdot \Phi(G_0)) \\ &= (\gamma \cdot (\phi(G_1) * \dots * \phi(G_k) * I_p) \circ \Phi(G_0)) \circ (\gamma' \cdot (\phi(G'_1) * \dots * \phi(G'_l) * I_q) \circ \Phi(G'_0)) \\ &= \Phi(G) \circ \Phi(G'). \end{aligned}$$

General case. There exists a permutation σ , such that if $H' = G' \cdot \sigma^{-1}$ and $H = \sigma \cdot G$, then the condition of the particular sub-case holds for (H, H') . Then:

$$\begin{aligned} \Phi(G' \circ G) &= \Phi((G' \cdot \sigma^{-1} \sigma) \circ G) \\ &= \Phi((G' \cdot \sigma^{-1}) \circ (\sigma \cdot G)) \\ &= \Phi(G' \cdot \sigma^{-1}) \circ \Phi(\sigma \cdot G) \quad \text{since the subcase holds} \\ &= (\Phi(G') \cdot \sigma^{-1}) \circ (\sigma \cdot \Phi(G)) \\ &= (\Phi(G') \cdot \sigma^{-1} \sigma) \cdot \Phi(G) \\ &= \Phi(G') \cdot \Phi(G). \end{aligned}$$

Finally, if G and G' are both graphs with no loop, $\Phi(G \circ G') = \Phi(G) \circ \Phi(G')$.

Let us finish this proof by considering loops. First, if H is a graph, there exist a (unique) graph with no loop and a (unique) integer ℓ , such that $H = G * \mathcal{O}^{*\ell}$. Let

$$G = \gamma \cdot (G_1 * \dots * G_k * I_p) \circ G_0$$

be a minimal decomposition of G . Then a minimal decomposition of H is:

$$H = \gamma \cdot (G_1 * \dots * G_k * I_p) \circ G_0 * \mathcal{O}^{*\ell}$$

, so

$$\Phi(H) = \gamma \cdot (\phi(G_1) * \dots * \phi(G_k) * I_p) \circ \Phi(G_0) * \phi(\mathcal{O})^\ell = \Phi(G) * \phi(\mathcal{O})^\ell.$$

Hence, if H and H' are two graphs, let us consider graphs G and G' with no loop and integers ℓ and ℓ' , such that

$$H = G * \mathcal{O}^{*\ell}, \quad H' = G' * \mathcal{O}^{*\ell'}.$$

Then $H * H' = G * G' * \mathcal{O}^{*(\ell+\ell')}$ and $G * G'$ is a graph with no loop. Hence, by commutativity of the horizontal concatenation of the product of P :

$$\begin{aligned}\Phi(H * H') &= \Phi(G * G') * \phi(\mathcal{O})^{*(\ell+\ell')} \\ &= \Phi(G) * \Phi(G') * \phi(\mathcal{O})^{*\ell} * \phi(\mathcal{O})^{*\ell'} \\ &= \Phi(G) * \phi(\mathcal{O})^{*\ell} * \Phi(G') * \phi(\mathcal{O})^{*\ell'} \\ &= \Phi(H) * \Phi(H').\end{aligned}$$

So Φ is compatible with the horizontal concatenation.

If moreover, $H \in \mathbf{Gr}^\cup(l, m)$ and $H' \in \mathbf{Gr}^\cup(k, l)$, then $H \circ H' = (G \circ G') * \mathcal{O}^{*(\ell+\ell')}$, and $G \circ G'$ is a graph with no loop. By the compatibility of the two concatenations of P :

$$\begin{aligned}\Phi(H \circ H') &= \Phi(G \circ G') * \phi(\mathcal{O})^{*(\ell+\ell')} \\ &= (\Phi(G) \circ \Phi(G')) * \phi(\mathcal{O})^{*\ell} * \phi(\mathcal{O})^{*\ell'} \\ &= (\Phi(G) * \phi(\mathcal{O})^{*\ell}) \circ (\Phi(G') * \phi(\mathcal{O})^{*\ell'}) \\ &= \Phi(H) \circ \Phi(H').\end{aligned}$$

So Φ is compatible with the vertical concatenation. □

C.2 Proof of Theorem 5.4.1

Proof. We first define $\Phi(G)$ for any graph such that, if $G \in \mathbf{Gr}^\cup(k, l)$, for any $(\sigma, \tau) \in \mathfrak{S}_l \times \mathfrak{S}_k$, $\Phi(\sigma \cdot G \cdot \tau) = \sigma \cdot \Phi(G) \cdot \tau$. We proceed by induction on the number N of internal edges of G . If $N = 0$, then G can be written (non uniquely) as

$$G = \mathcal{O}^{*p} * \sigma \cdot (I^{*q} * G_{k_1, l_1} * \dots * G_{k_r, l_r}) \cdot \tau,$$

where $p, q, r \in \mathbb{N}_0$ are unique, $(k_i, l_i) \in \mathbb{N}_0^2$ for any i , unique up to a permutation, and $\sigma \in \mathfrak{S}_{q+k_1+\dots+k_r}$, $\tau \in \mathfrak{S}_{q+l_1+\dots+l_r}$. We then put:

$$\Phi(G) = t_{1,1}(I)^{*p} * \sigma \cdot (I^{*q} * x_{k_1, l_1} * \dots * x_{k_r, l_r}) \cdot \tau.$$

Let us prove that this does not depend of the choice of the writing of G . As this is up to a permutation of the vertices and of the choice of σ and τ , we can go from one decomposition of G to any other one in a finite steps among the following two cases:

1. We consider two writing of G of the form

$$\begin{aligned}G &= \mathcal{O}^{*p} * \sigma \cdot (I^{*q} * G_{k_1, l_1} * \dots * G_{k_i, l_i} * G_{k_{i+1}, l_{i+1}} * \dots * G_{k_r, l_r}) \cdot \tau, \\ G &= \mathcal{O}^{*p} * \sigma' \cdot (I^{*q} * G_{k_1, l_1} * \dots * G_{k_{i+1}, l_{i+1}} * G_{k_i, l_i} * \dots * G_{k_r, l_r}) \cdot \tau',\end{aligned}$$

with

$$\begin{aligned}\sigma' &= \sigma(\text{Id}_{q+l_1+\dots+l_{i-1}} \otimes c_{l_i, l_{i+1}} \otimes \text{Id}_{l_{i+2}+\dots+l_r}), \\ \tau' &= (\text{Id}_{q+k_1+\dots+k_{i-1}} \otimes c_{k_{i+1}, k_i} \otimes \text{Id}_{k_{i+2}+\dots+k_r})\tau.\end{aligned}$$

Then, by commutativity of $*$:

$$\begin{aligned}\sigma' \cdot (I^{*q} * x_{k_1, l_1} * \dots * x_{k_r, l_r}) \cdot \tau' \\ &= \sigma \cdot (I^{*q} * x_{k_1, l_1} * \dots * c_{l_i, l_{i+1}} \cdot (x_{k_{i+1}, l_{i+1}} * x_{k_i, l_i}) \cdot c_{k_{i+1}, k_i} * \dots * x_{k_r, l_r}) \cdot \tau \\ &= \sigma \cdot (I^{*q} * x_{k_1, l_1} * \dots * x_{k_i, l_i} * x_{k_{i+1}, l_{i+1}} * \dots * x_{k_r, l_r}) \cdot \tau.\end{aligned}$$

2. We consider two writings of G of the form

$$\begin{aligned} G &= \mathcal{O}^{*p} * \sigma \cdot (I^{*q} * G_{k_1, l_1} * \dots * G_{k_r, l_r}) \cdot \tau, \\ G &= \mathcal{O}^{*p} * \sigma' \cdot (I^{*q} * G_{k_1, l_1} * \dots * G_{k_r, l_r}) \cdot \tau', \end{aligned}$$

with

$$\sigma' = \sigma(\sigma_0 \otimes \sigma_1 \otimes \dots \otimes \sigma_r), \quad \tau' = (\sigma_0^{-1} \otimes \tau_1 \otimes \dots \otimes \tau_r) \tau',$$

with $\sigma_0 \in \mathfrak{S}_q$, $\sigma_i \in \mathfrak{S}_{k_i}$ and $\tau_i \in \mathfrak{S}_{l_i}$ if $i \geq 1$. Using the commutativity of $*$ and the invariance of the $x_{k,l}$:

$$\begin{aligned} &\sigma' \cdot (I^{*q} * x_{k_1, l_1} * \dots * x_{k_r, l_r}) \cdot \tau' \\ &= \sigma \cdot (\sigma_0 \cdot I^{*q} \cdot \sigma_0^{-1} * \sigma_1 \cdot x_{k_1, l_1} \cdot \tau_1 * \dots * \sigma_r \cdot x_{k_r, l_r} \cdot \tau_r) \cdot \tau \\ &= \sigma \cdot (I^{*q} * x_{k_1, l_1} * \dots * x_{k_r, l_r}) \cdot \tau. \end{aligned}$$

Hence, $\Phi(G)$ is well-defined. Moreover, of $\tau' \in \mathfrak{S}_k$, $\sigma' \in \mathfrak{S}_l$, choosing a writing of G of the form

$$G = \mathcal{O}^{*p} * \sigma \cdot (I^{*q} * G_{k_1, l_1} * \dots * G_{k_i, l_i} * G_{k_{i+1}, l_{i+1}} * \dots * G_{k_r, l_r}) \cdot \tau,$$

a writing of $G' = \sigma' \cdot G \cdot \tau'$ is

$$\mathcal{O}^{*p} * \sigma' \sigma \cdot (I^{*q} * G_{k_1, l_1} * \dots * G_{k_r, l_r}) \cdot \tau \tau',$$

and, by definition of $\Phi(G')$:

$$\begin{aligned} \Phi(G') &= t_{1,1}(I)^{*p} * \sigma' \sigma \cdot (I^{*q} * x_{k_1, l_1} * \dots * x_{k_r, l_r}) \cdot \tau \tau' \\ &= \sigma' \cdot (t_{1,1}(I)^{*p} * \sigma \cdot (I^{*q} * x_{k_1, l_1} * \dots * x_{k_r, l_r}) \cdot \tau) * \tau' \\ &= \sigma' \cdot \Phi(G) \cdot \tau'. \end{aligned}$$

Let us assume now that $\Phi(G')$ is defined for any graph with $N - 1$ internal edges, for a given $N \geq 1$. Let G be a graph with N internal edges and let e be one of these edges. Let G_e be a graph obtained by cutting this edge in two:

1. $V(G_e) = V(G)$.
2. $E(G_e) = E(G) \setminus \{e\}$, $I(G_e) = I(G) \sqcup \{e\}$, $O(G_e) = O(G) \sqcup \{e\}$, $IO(G_e) = IO(G)$, $L(G_e) = L(G)$.
3. $s_{G_e} = s_G$ and $t_{G_e} = t_G$.
4. For any $e' \in I(G_e) \sqcup IO(G_e)$, for any $f' \in O(G_e) \sqcup IO(G_e)$:

$$\alpha_{G_e}(e') = \begin{cases} 1 & \text{if } e' = e, \\ \alpha_G(e') + 1 & \text{if } e' \neq e, \end{cases} \quad \beta_{G_e}(f') = \begin{cases} 1 & \text{if } f' = e, \\ \beta_G(f') + 1 & \text{if } f' \neq e. \end{cases}$$

Then $G = t_{1,1}(G_e)$ and G_e has $N - 1$ internal edges. We then put:

$$\Phi(G) = t_{1,1} \circ \Phi(G_e).$$

Let us prove that this does not depend of the choice of e . If e' is another internal edge of G , then:

$$(G_e)_{e'} = (12) \cdot (G_{e'})_e \cdot (12),$$

which implies, by definition of $\Phi(G_e)$ and $\Phi(G_{e'})$:

$$\begin{aligned}
t_{1,1} \circ \Phi(G_e) &= t_{1,1} \circ t_{1,1} \circ \Phi((G_e)_{e'}) \\
&= t_{1,1} \circ t_{1,1} \circ ((12) \cdot \Phi((G_{e'})_e) \cdot (12)) \\
&= t_{1,1} \circ t_{2,2} \circ \Phi((G_{e'})_e) \\
&= t_{1,1} \circ t_{1,1} \circ \Phi((G_{e'})_e) \\
&= t_{1,1} \circ \Phi(G_{e'}).
\end{aligned}$$

So $\Phi(G)$ is well-defined. Let $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_l$. Then:

$$(\sigma \cdot G \cdot \tau)_e = ((1) \otimes \sigma) \cdot (G_e) \cdot ((1) \otimes \tau),$$

so:

$$\begin{aligned}
\Phi(\sigma \cdot G \cdot \tau) &= t_{1,1} \circ \Phi((\sigma \cdot G \cdot \tau)_e) \\
&= t_{1,1}((1) \otimes \sigma) \cdot \Phi(G_e) \cdot ((1) \otimes \tau) \\
&= ((1) \otimes \sigma)_1 \cdot t_{1,1} \circ \Phi(G_e) \cdot ((1) \otimes \tau)_1 \\
&= \sigma \cdot \Phi(G) \cdot \tau.
\end{aligned}$$

where, for $\sigma \in \mathfrak{S}_k$ we use σ_i for the permutation in \mathfrak{S}_{k-1} defined by

$$\sigma_i(j) = \begin{cases} \sigma(j) & \text{if } j \leq i-1, \\ \sigma(j-1) & \text{if } j \geq i. \end{cases}$$

where $((1) \otimes \tau)_1$ is defined by (14).

We have therefore defined a map $\Phi : \mathbf{GGr} \rightarrow P$, compatible with the action of the symmetric groups. Let us prove that for any graphs G, G' ,

$$\Phi(G * G') = \Phi(G) * \Phi(G').$$

We proceed by induction on the number N of internal edges of $G * G'$. If $N = 0$, we put:

$$\begin{aligned}
G &= \mathcal{O}^{*p} * \sigma \cdot (I^{*q} * G_{k_1, l_1} * \dots * G_{k_r, l_r}) \cdot \tau, \\
G' &= \mathcal{O}^{*p'} * \sigma' \cdot (I^{*q'} * G_{k'_1, l'_1} * \dots * G_{k'_{r'}, l'_{r'}}) \cdot \tau'.
\end{aligned}$$

We obtain:

$$\begin{aligned}
G * G' &= \mathcal{O}^{*(p+p')} * (\sigma \otimes \sigma') * (\text{Id}_q \otimes c_{k_1+\dots+k_r, q'} \otimes \text{Id}_{k'_1+\dots+k'_{r'}}) \\
&\quad \cdot (I^{q+q'} * G_{k_1, l_1} * \dots * G_{k'_{r'}, l'_{r'}}) \cdot (\text{Id}_q \otimes c_{q', l_1+\dots+l_r} \otimes \text{Id}_{l'_1+\dots+l'_{r'}}),
\end{aligned}$$

which gives, by commutativity of $*$:

$$\begin{aligned}
\Phi(G * G') &= t_{1,1}(I)^{*(p+p')} * (\sigma \otimes \sigma') * (\text{Id}_q \otimes c_{l_1+\dots+l_r, q'} \otimes \text{Id}_{l'_1+\dots+l'_{r'}}) \\
&\quad \cdot (I^{q+q'} * x_{k_1, l_1} * \dots * x_{k'_{r'}, l'_{r'}}) \cdot (\text{Id}_q \otimes c_{q', k_1+\dots+k_r} \otimes \text{Id}_{k'_1+\dots+k'_{r'}}) \\
&= t_{1,1}(I)^{*p} * \sigma \cdot (I^{*q} * x_{k_1, l_1} * \dots * x_{k_r, l_r}) \cdot \tau \\
&\quad * t_{1,1}(I)^{*p'} * \sigma' \cdot (I^{*q'} * x_{k'_1, l'_1} * \dots * x_{k'_{r'}, l'_{r'}}) \cdot \tau' \\
&= \Phi(G) * \Phi(G').
\end{aligned}$$

If $N \geq 1$, let us take an internal edge e of $G * G'$. If e is an internal edge of G , then $(G * G')_e = G_e * G'$, and:

$$\Phi(G * G') = t_{1,1} \circ \Phi((G * G')_e) = t_{1,1} \circ \Phi(G_e * G') = t_{1,1}(\Phi(G_e) * G') = t_{1,1} \circ \Phi(G_e) * \Phi(G') = \Phi(G) * \Phi(G').$$

If e is an internal edge of G' , we obtain similarly that $\Phi(G' * G) = \Phi(G') * \Phi(G)$. The result then follows from the commutativity of $*$ (axiom 2.(d) of Definition 5.1.1). So Φ is compatible with $*$.

It remains to prove the compatibility of Φ with the partial trace maps. By Lemma 5.1.4, it is enough to prove that Φ is compatible with $t_{1,1}$. Let $G \in \mathbf{Gr}^\cup(k, l)$ be a graph, $e_1 = \alpha^{-1}(1)$, $f_1 = \beta^{-1}(1)$. We put $G' = t_{1,1}(G)$ and $e = \{e_1, f_1\}$ be the edge of G' created in the process. There are five different cases:

1. If $e_1 \in I(G)$ and $f_1 \in O(G)$, then $e \in E(G')$ and $G'_e = G$. By construction of $\Phi(G')$:

$$\Phi \circ t_{1,1}(G) = \Phi(G') = t_{1,1} \circ \Phi(G'_e) = t_{1,1} \circ \Phi(G).$$

2. If $e_1 \in IO(G)$ and $f_1 \in O(G)$, let us put $j = \beta(e_1)$. Then there exists a graph H such that $(1, j) \cdot G = I * H$. Then:

$$t_{1,1}(G) = t_{1,1}((1, j) \cdot (I * H)) = (1, \dots, j) \cdot (t_{1,i}(I * H)) = (1, \dots, j) \cdot H,$$

so:

$$\begin{aligned} t_{1,1} \circ \Phi(G) &= t_{1,1}((1, j) \cdot (I * \Phi(H))) \\ &= (1, j)(1, \dots, j-1) \cdot \Phi(H) \\ &= (1, \dots, j) \cdot \Phi(H) \\ &= \Phi((1, \dots, j) \cdot H) \\ &= \Phi \circ t_{1,1}(G). \end{aligned}$$

3. If $e_1 \in I(G)$ and $f_1 \in IO(G)$: similar computation.
4. If $e_1, f_1 \in IO(G)$, with $e_1 \neq f_1$: similar computation.
5. If $e_1 = f_1$ in $IO(G)$, then $G = I * H$ for a certain graph G and $t_{1,1}(G) = \mathcal{O} * H$. Then:

$$\Phi \circ t_{1,1}(G) = \Phi(\mathcal{O}) * \Phi(H) = t_{1,1} \circ \Phi(I) * \Phi(H) = t_{1,1}(\Phi(I) * \Phi(H)) = t_{1,1} \circ \Phi(G).$$

So Φ is compatible with the partial trace maps. □

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