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A GENERAL CONSTRUCTION OF FAMILY ALGEBRAIC STRUCTURES

LOÏC FOISSY, DOMINIQUE MANCHON, AND YUANYUAN ZHANG

ABSTRACT. We give a general account of family algebras over a finitely presented linear operad. In a family algebra, each operation of arity n is replaced by a family of operations indexed by Ω^n, where Ω is a set of parameters. We show that the operad, together with its presentation, naturally defines an algebraic structure on the set of parameters, which in turn is used in the description of the family version of the relations between operations. The examples of dendriform and duplicial family algebras (hence with two parameters) and operads are treated in detail, as well as the pre-Lie family case. Finally, free one-parameter duplicial family algebras are described, together with the extended duplicial semigroup structure on the set of parameters.

CONTENTS

1. INTRODUCTION

The first family algebra structures appeared in the literature in 2007: a natural example of Rota-Baxter family algebras of weight -1 was given by J. Gracia-Bondía, K. Ebrahimi-Fard and F. Patras in a paper on Lie-theoretic aspects of renormalization [11, Proposition 9.1] (see

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also [27]). The notion of Rota-Baxter family itself was suggested to the authors by Li Guo (see Footnote after Proposition 9.2 therein), who studied further these Rota-Baxter family algebras in [16], including the more general case of weight λ. They are associative algebras R over some field $\mathbf{k}$ together with a collection $(P_\omega)_{\omega\in\Omega}$ of linear endomorphisms indexed by a semigroup Ω such that the Rota-Baxter family relation

$$P_\alpha(a)P_\beta(b) = P_{\alpha\beta}\big(P_\alpha(a)b + aP_\beta(b) + \lambda ab\big)$$

holds for any $a, b \in R$ and $\alpha, \beta \in \Omega$. The example in [11] is given by the momentum renormalization scheme: here Ω is the additive semigroup of non-negative integers, and the operator P_ω associates to a Feynman diagram integral its Taylor expansion of order ω at vanishing exterior momenta. The simplest example we can provide, derived from the minimal subtraction scheme, is the algebra of Laurent series $R = \mathbf{k}[z^{-1}, z]]$, where, for any $\omega \in \Omega = \mathbb{Z}$, the operator P_ω is the projection onto the subspace $R_{<\omega}$ generated by $\{z^k,\ k < \omega\}$ parallel to the supplementary subspace $R_{\geq\omega}$ generated by $\{z^k,\ k \geq \omega\}$.

Other families of algebraic structures appeared more recently, such as dendriform and tridendriform family algebras [33, 34, 12], or pre-Lie family algebras [23]. The principle consists in replacing each product of the structure by a family of products, so that the operadic relations (Rota-Baxter, dendriform, pre-Lie, etc.) still hold in a "family" version taking the semigroup structure of the parameter set into account.

These various examples obviously called for a more conceptual approach. A major step in understanding family structures in general has been recently done by M. Aguiar, who defined family $\mathcal{P}$-algebras for any linear operad $\mathcal{P}$ [2]. In this setting, the semigroup Ω of parameters must be commutative unless the operad is nonsymmetric. An important point is that any n-ary operation gives rise to a family of operations parametrized by Ω^n rather than by Ω. In particular, in contrast with [33, 34, 23, 12], the natural family version of any binary operation requires two parameters. We start by revisiting M. Aguiar's two-parameter dendriform family algebras, finding that Ω is naturally given a structure of diassociative semigroup, also known as dimonoid [20].

This suggests that the natural algebraic structure of Ω should be determined in some way by the operad one starts with. This appears to be the case: we define family $\mathcal{P}$-algebras for any finitely presented linear operad $\mathcal{P}$, in a way which depends on the choice of the presentation. The definition makes sense when the parameter set Ω is endowed with a $\textcircled{\mathcal{P}}$-algebra structure, where $\textcircled{\mathcal{P}}$ is a set operad determined by $\mathcal{P}$ and its presentation. Following the lines of M. Aguiar, we define family $\mathcal{P}$-algebras indexed by Ω as *uniform Ω-graded $\mathcal{P}$-algebras*. The notion of Ω-graded $\mathcal{P}$-algebra, when Ω is a $\textcircled{\mathcal{P}}$-algebra, is defined via *color-mixing operads*, which are generalizations of the current-preserving operads of [29]. Let us stress at this stage the main difference between M. Aguiar's approach and ours: the parameter set Ω is not necessarily a semigroup. In fact, starting with the magmatic operad (i.e. the free operad generated by one single binary operation), the parameter set Ω is only required to be a magma. This is a simple consequence of Remark 5.2.

The paper is organized as follows: we investigate two-parameter dendriform family algebras over a fixed base field $\mathbf{k}$ in some detail in Section 2, as well as their duplicial counterparts, and we also give an example of two-parameter duplicial family algebra in terms of planar binary trees with $\Omega\times\Omega$-typed edges. To conclude this section, we give the generating series of the dimensions of the free two-parameter duplicial (or dendriform) family algebra with one generator, when the parameter set Ω is finite. We give a reminder of colored operads in A. Joyal's species formalism [18] in Section 3, and give a brief account of graded objects. Following a crucial idea in [2], we describe the uniformization functor $\mathcal{U}$ from ordinary (monochromatic) operads to colored operads (resp., with the same notations, from a suitable monoidal category to its graded version), and its right-adjoint, the completed forgetful functor $\overline{\mathcal{F}}$.

In Section 4, we study the pre-Lie case in some detail. The pre-Lie operad $\mathcal{P}$ gives rise to four different set operads, namely the associative operad, the twist-associative operad governing Thedy's rings with $x(yz) = (yx)z$ [31], an operad built from corollas governing rings with both $x(yz) = y(xz)$ and $(xy)z = (yx)z$, and finally the Perm operad governing rings with $x(yz) = y(xz) = (xy)z = (yx)z$, i.e. set-theoretical Perm algebras. This last operad is a quotient of the three others and gives rise to family pre-Lie algebras.

Color-mixing operads and the general definition of Ω-family algebras are given in Section 5, which is the main section of the paper. The construction depends on the operad presentation in an essential way, via the definition of the set-operadic equivalence relation generated by an ideal in a linear operad (Definition 5.1). It seems unlikely to us that a presentation-free approach can be undertaken. Finally, Section 6 is devoted to one-parameter family duplicial algebras, in the spirit of the recent article of the first author [12], in which the rich structure of extended diassociative semigroup (EDS) naturally appears. This approach encompasses the notion of matching families which was developed in [13] in the Rota–Baxter, dendriform and pre-Lie cases. We prove, similarly, that a similar (but weaker) structure of extended duplicial semigroup (EDuS) appears in the one-parameter version of family duplicial algebra. We further prove that planar binary trees with Ω-typed edges provide free one-parameter Ω-duplicial algebras for any EDuS Ω. This example, together with [12], raises the interesting question, to be addressed in a future work, of the algebraic structure naturally associated with the parameter set when considering one-parameter family algebraic structures in general.

Notation: In this paper, we fix a field $\mathbf{k}$ and assume that an algebra is a $\mathbf{k}$-algebra. The letter Ω will denote a set of indices, which will be endowed with various structures throughout the article.

2. Two-parameter dendriform and duplicial family algebras

2.1. Two-parameter Ω-dendriform algebras.

First, we borrow some concepts from J.-L. Loday [20], also used in the first author's recent article [12].

Definition 2.1. [20, Def. 1.1] **A diassociative semigroup**[1] is a triple $(\Omega, \leftarrow, \rightarrow)$, where Ω is a set and $\leftarrow, \rightarrow \colon \Omega \times \Omega \to \Omega$ are maps such that, for any $\alpha, \beta, \gamma \in \Omega$:

$$(\alpha \leftarrow \beta) \leftarrow \gamma = \alpha \leftarrow (\beta \leftarrow \gamma) = \alpha \leftarrow (\beta \rightarrow \gamma), \tag{1}$$

$$(\alpha \rightarrow \beta) \leftarrow \gamma = \alpha \rightarrow (\beta \leftarrow \gamma), \tag{2}$$

$$(\alpha \rightarrow \beta) \rightarrow \gamma = (\alpha \leftarrow \beta) \rightarrow \gamma = \alpha \rightarrow (\beta \rightarrow \gamma). \tag{3}$$

[1] The original term in [20] is *dimonoid*. We have preferred the terminology *diassociative semigroup* of [12], as no unit element is involved here.

Definition 2.2. Let Ω be a diassociative semigroup with two products $\leftarrow$ and $\rightarrow$. **A two-parameter Ω-dendriform algebra** is a family $(A, (\prec_{\alpha,\beta}, \succ_{\alpha,\beta})_{\alpha,\beta\in\Omega})$ where A is a vector space and

$$\prec_{\alpha,\beta}, \succ_{\alpha,\beta}\colon A \otimes A \to A$$

are bilinear binary products such that for any $x, y, z \in A$, for any $\alpha, \beta \in \Omega$,

$$(x \prec_{\alpha,\beta} y) \prec_{\alpha\leftarrow\beta,\gamma} z = x \prec_{\alpha,\beta\leftarrow\gamma} (y \prec_{\beta,\gamma} z) + x \prec_{\alpha,\beta\rightarrow\gamma} (y \succ_{\beta,\gamma} z), \tag{4}$$

$$(x \succ_{\alpha,\beta} y) \prec_{\alpha\rightarrow\beta,\gamma} z = x \succ_{\alpha,\beta\leftarrow\gamma} (y \prec_{\beta,\gamma} z), \tag{5}$$

$$x \succ_{\alpha,\beta\rightarrow\gamma} (y \succ_{\beta,\gamma} z) = (x \succ_{\alpha,\beta} y) \succ_{\alpha\rightarrow\beta,\gamma} z + (x \prec_{\alpha,\beta} y) \succ_{\alpha\leftarrow\beta,\gamma} z. \tag{6}$$

Remark 2.3. (a) If Ω is a set, we recover the definition of a two-parameter version of matching dendriform algebras [13] when we consider $(\Omega, \leftarrow, \rightarrow)$ as a diassociative semigroup, that is, for any $\alpha, \beta \in \Omega$,

$$\alpha \leftarrow \beta = \alpha, \quad \alpha \rightarrow \beta = \beta.$$

(b) If $(\Omega, \circledast)$ is a semigroup, we recover the definition of two-parameter dendriform family algebras given in [2] when we consider

$$\alpha \leftarrow \beta = \alpha \rightarrow \beta = \alpha \circledast \beta.$$

Two-parameter Ω-dendriform algebras are related to dendriform algebras and diassociative semigroups by the following proposition:

Proposition 2.4. *Let Ω be a set with two binary operations $\leftarrow$ and $\rightarrow$.*

(a) *Let A be a* **k***-vector space and let*

$$\prec_{\alpha,\beta}, \succ_{\alpha,\beta}\colon A \otimes A \to A$$

be two families of bilinear binary products indexed by $\Omega \times \Omega$. We define products $\prec$ and $\succ$ on the space $A \otimes \mathbf{k}\Omega$ by:

$$(x \otimes \alpha) \prec (y \otimes \beta) = (x \prec_{\alpha,\beta} y) \otimes (\alpha \leftarrow \beta), \tag{7}$$

$$(x \otimes \alpha) \succ (y \otimes \beta) = (x \succ_{\alpha,\beta} y) \otimes (\alpha \rightarrow \beta). \tag{8}$$

If $(A \otimes \mathbf{k}\Omega, \prec, \succ)$ is a dendriform algebra, then (4), (5) and (6) hold.

(b) *The following conditions are equivalent:*
   (i) *For any $(A, (\prec_{\alpha,\beta}, \succ_{\alpha,\beta})_{\alpha,\beta\in\Omega})$ where A is a* **k***-vector space and where (4), (5) and (6) hold, the vector space $A \otimes \mathbf{k}\Omega$ endowed with the binary operations $\prec$ and $\succ$ defined by (7) and (8) is a dendriform algebra,*
   (ii) *$(\Omega, \leftarrow, \rightarrow)$ is a diassociative semigroup.*

Proof. (a). Let us consider the three dendriform axioms:

$$\big((x \otimes \alpha) \prec (y \otimes \beta)\big) \prec (z \otimes \gamma) = (x \otimes \alpha) \prec \big((y \otimes \beta) \prec (z \otimes \gamma) + (y \otimes \beta) \succ (z \otimes \gamma)\big)$$

$$\big((x \otimes \alpha) \succ (y \otimes \beta)\big) \prec (z \otimes \gamma) = (x \otimes \alpha) \succ \big((y \otimes \beta) \prec (z \otimes \gamma)\big)$$

$$(x \otimes \alpha) \succ \big((y \otimes \beta) \succ (z \otimes \gamma)\big) = \big((x \otimes \alpha) \succ (y \otimes \beta) + (x \otimes \alpha) \prec (y \otimes \beta)\big) \succ (z \otimes \gamma).$$

The first one gives:

$$\begin{aligned}(x \prec_{\alpha,\beta} y) \prec_{\alpha\leftarrow\beta,\gamma} z \otimes (\alpha \leftarrow \beta) \leftarrow \gamma = x \prec_{\alpha,\beta\leftarrow\gamma} (y \prec_{\beta,\gamma} z) \otimes \alpha \leftarrow (\beta \leftarrow \gamma)\\ + x \prec_{\alpha,\beta\rightarrow\gamma} (y \succ_{\beta,\gamma} z) \otimes \alpha \rightarrow (\beta \leftarrow \gamma).\end{aligned} \tag{9}$$

Let $f : \mathbf{k}\Omega \longrightarrow \mathbf{k}$ be the linear map sending any $\delta \in \Omega$ to 1. Applying $\mathrm{Id}_A \otimes f$ to both sides of (9), we obtain (4). Similarly, the second dendriform axiom gives (5) and the last one gives (6).

(b). $(i) \Longrightarrow (ii)$. Let us consider the free 2-parameter Ω-dendriform algebra A on three generators x, y and z (from the operad theory, such an object exists). Let us fix α, β and γ in Ω. According to the relations defining 2-parameter Ω-dendriform algebras, $x \prec_{\alpha,\beta\leftarrow\gamma} (y \prec_{\beta,\gamma} z)$ and $x \prec_{\alpha,\beta\rightarrow\gamma} (y \succ_{\beta,\gamma} z)$ are linearly independent in A. Let $g : A \longrightarrow \mathbf{k}$ be a linear map such that

$$g(x \prec_{\alpha,\beta\leftarrow\gamma} (y \prec_{\beta,\gamma} z)) = 1,$$
$$g(x \prec_{\alpha,\beta\rightarrow\gamma} (y \succ_{\beta,\gamma} z)) = 0.$$

Applying $g \otimes \mathrm{Id}_{\mathbf{k}\Omega}$ on both sides of (9), we obtain that there exists a scalar λ such that $\lambda(\alpha \leftarrow \beta) \leftarrow \gamma = \alpha \leftarrow (\beta \leftarrow \gamma)$. As $(\alpha \leftarrow \beta) \leftarrow \gamma$ and $\alpha \leftarrow (\beta \leftarrow \gamma)$ are both elements of Ω, necessarily $\lambda = 1$. Using a linear map $h : A \longrightarrow \mathbf{k}$ such that

$$h(x \prec_{\alpha,\beta\leftarrow\gamma} (y \prec_{\beta,\gamma} z)) = 0,$$
$$h(x \prec_{\alpha,\beta\rightarrow\gamma} (y \succ_{\beta,\gamma} z)) = 1,$$

we obtain that $(\alpha \leftarrow \beta) \leftarrow \gamma = \alpha \rightarrow (\beta \leftarrow \gamma)$. The other axioms of diassociative semigroups are obtained in the same way from the second and third dendriform axioms.

$(ii) \Longrightarrow (i)$. If (ii) holds, (9) immediately implies that the first dendriform axiom is satisfied for any A. The second and third dendriform axioms are proved in the same way. □

Remark 2.5. We recover the ordinary (i.e. one-parameter) definitions of matching dendriform algebras in [13] (resp. dendriform family algebras in [34]) from Definition 2.2 if Ω is a set with diassociative semigroup structure given by $\alpha \leftarrow \beta = \alpha$ and $\alpha \rightarrow \beta = \beta$ for any $\alpha, \beta \in \Omega$ (resp. if $(\Omega, \circledast)$ is a semigroup with diassociative semigroup structure given by $\alpha \leftarrow \beta = \alpha \rightarrow \beta = \alpha \circledast \beta$ for any $\alpha, \beta \in \Omega$), if we suppose that $\prec_{\alpha,\beta}$ depends only on β and $\succ_{\alpha,\beta}$ depends only on α:

$$\prec_{\alpha,\beta} = \prec_\beta, \quad \succ_{\alpha,\beta} = \succ_\alpha \quad \text{for } \alpha, \beta \in \Omega.$$

A general definition of one-parameter dendriform family algebras encompassing both [13] and [34] has been recently proposed by the first author. We give the duplicial analogue in Section 6.

2.2. **Two-parameter Ω-duplicial algebras.** We can mimick step by step the construction of Paragraph 2.1:

Definition 2.6. [28, Paragraph 6.1] A **duplicial semigroup**[2] is a triple $(\Omega, \leftarrow, \rightarrow)$, where Ω is a set and $\leftarrow, \rightarrow : \Omega \times \Omega \to \Omega$ are maps such that, for any $\alpha, \beta, \gamma \in \Omega$:

$$(\alpha \leftarrow \beta) \leftarrow \gamma = \alpha \leftarrow (\beta \leftarrow \gamma),$$
$$(\alpha \rightarrow \beta) \leftarrow \gamma = \alpha \rightarrow (\beta \leftarrow \gamma),$$
$$(\alpha \rightarrow \beta) \rightarrow \gamma = \alpha \rightarrow (\beta \rightarrow \gamma). \tag{10}$$

Definition 2.7. Let Ω be a duplicial semigroup with two products $\leftarrow$ and $\rightarrow$. **A two-parameter Ω-duplicial algebra** is a family $(A, (\prec_{\alpha,\beta}, \succ_{\alpha,\beta})_{\alpha,\beta\in\Omega})$ where A is a vector space and

$$\prec_{\alpha,\beta}, \succ_{\alpha,\beta} : A \otimes A \to A$$

are bilinear binary products such that for any $x, y, z \in A$, for any $\alpha, \beta \in \Omega$,

$$(x \prec_{\alpha,\beta} y) \prec_{\alpha\leftarrow\beta,\gamma} z = x \prec_{\alpha,\beta\leftarrow\gamma} (y \prec_{\beta,\gamma} z), \tag{11}$$

[2] The original terminology in [28] is *duplex with the additional identity* $(\alpha \rightarrow \beta) \leftarrow \gamma = \alpha \rightarrow (\beta \leftarrow \gamma)$. A duplex is a set with two associative operations.

$$(x \succ_{\alpha,\beta} y) \prec_{\alpha\to\beta,\gamma} z = x \succ_{\alpha,\beta\leftarrow\gamma} (y \prec_{\beta,\gamma} z), \tag{12}$$

$$x \succ_{\alpha,\beta\to\gamma} (y \succ_{\beta,\gamma} z) = (x \succ_{\alpha,\beta} y) \succ_{\alpha\to\beta,\gamma} z. \tag{13}$$

Two-parameter Ω-duplicial algebras are related to duplicial algebras and duplicial semigroups the same way two-parameter Ω-dendriform algebras are related to dendriform algebras and dendriform semigroups by Proposition 2.4. The proof of this fact is entirely similar.

Now we give a concrete example of two-parameter Ω-duplicial algebra, which uses typed decorated planar binary trees [5, 34].

Definition 2.8. Let X and Ω be two sets. An **X-decorated $\Omega\times\Omega$-typed (abbreviated two-parameter typed decorated) planar binary tree** is a triple $T = (T, \mathrm{dec}, \mathrm{type})$, where

(a) T is a planar binary tree.
(b) $\mathrm{dec} : V(T) \to X$ is a map, where $V(T)$ stands for the set of internal vertices of T,
(c) $\mathrm{type} : IE(T) \to \Omega\times\Omega$ is a map, where $IE(T)$ stands for the set of internal edges of T.

Denote by $\mathrm{D}(X, \Omega)$ the set of two-parameter typed decorated planar binary trees. For any $s \in \mathrm{D}(X, \Omega)$ we denote by $\bar{s}$ the subjacent decorated tree, forgetting the types.

Definition 2.9. Let Ω be a set. For $\alpha, \beta \in \Omega$, first define

$$s \prec_{\alpha,\beta} t := \bar{s} \prec \bar{t} + \text{ following types} \tag{14}$$

which means grafting t on s at the rightmost leaf, and the types follow the rules below:

- the new edge is typed by the pair (α,β);
- any internal edge of t has its type moved as follows:
$$(\omega, \tau) \mapsto (\omega, \tau \leftarrow \beta);$$
- any internal edge of s has its type moved as follows:
$$(\omega, \tau) \mapsto (\alpha \leftarrow \omega, \tau);$$
- other edges keep their types unchanged.

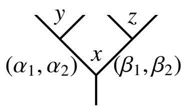
Similarly, we second define

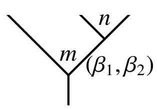
$$s \succ_{\alpha,\beta} t := \bar{s} \succ \bar{t} + \text{ following types} \tag{15}$$

which means grafting s on t at the leftmost leaf, and the types follow the following rules:

- the new edge is typed by the pair (α,β);
- any internal edge of t has its type moved as follows:
$$(\omega, \tau) \mapsto (\alpha \to \omega, \tau);$$
- any internal edge of s has its type moved as follows:
$$(\omega, \tau) \mapsto (\omega, \tau \to \beta);$$
- other edges keep their types unchanged.

Example 2.10. Let X and Ω be two sets. Let

$s =$ 

, and $t =$ 

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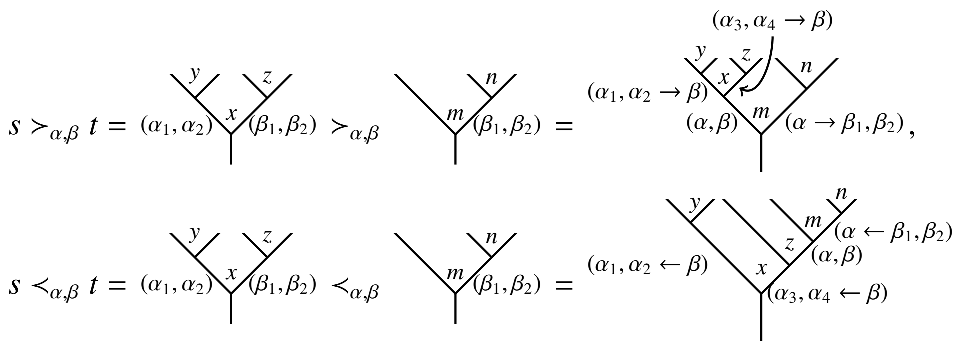
Then



Proposition 2.11. *Let X and Ω be two sets. The pair* $(\mathrm{D}(X,\,\Omega), (\prec_{\alpha,\beta}, \succ_{\alpha,\beta})_{\alpha,\beta\in\Omega})$ *is a two-parameter Ω-duplicial algebra.*

Proof. For $s,t,u \in \mathrm{D}(X,\,\Omega)$ and $\alpha,\beta,\gamma \in \Omega$, we first prove Eq. (11). Let us look at the right hand side of Eq. (11), that is, $s \prec_{\alpha,\beta\leftarrow\gamma} (t \prec_{\beta,\gamma} u)$. We divide the procedure into two steps.

- *First step*: we deal with $t \prec_{\beta,\gamma} u$, we have the new edge typed by (β,γ); the edges of u have their types (ω,τ) changed into $(\beta \leftarrow \omega,\tau)$; the edges of t have their types (ω,τ) changed into $(\omega,\tau \leftarrow \omega)$.
- *Second step*: we deal with $s \prec_{\alpha,\beta\leftarrow\gamma} (t \prec_{\beta,\gamma} u)$, which means grafting $t \prec_{\beta,\gamma} u$ on the rightmost leaf of s. The new edge has its type $(\alpha,\beta \leftarrow \gamma)$; the new edge of $t \prec_{\beta,\gamma} u$ produced in the first step has its type (β,γ) changed into $(\alpha \leftarrow \beta,\gamma)$; the edges of u have their types $(\beta \leftarrow \omega,\tau)$ changed into $(\alpha \leftarrow (\beta \leftarrow \omega),\tau)$; the edges of t have their types $(\omega,\tau \leftarrow \gamma)$ changed into $(\alpha \leftarrow \omega,\tau \leftarrow \gamma)$; the edges of s have their types (ω,τ) changed into $(\omega,\tau \leftarrow (\beta \leftarrow \gamma))$.

Let us now look at the left hand side of Eq. (11), that is, $(s \prec_{\alpha,\beta} t) \prec_{\alpha\leftarrow\beta,\gamma} u$. We also divide into the procedure two steps.

- *First step*: we deal with $s \prec_{\alpha,\beta} t$: we graft t on s, and the new edge typed by (α,β); the edges of t have their types (ω,τ) changed into $(\alpha \leftarrow \omega,\tau)$; the edges of s have their types (ω,τ) changed into $(\omega,\tau \leftarrow \beta)$.
- *Second step*: we deal with $(s \prec_{\alpha,\beta} t) \prec_{\alpha\leftarrow\beta,\gamma} u$. The new edge typed by $(\alpha \leftarrow \beta,\gamma)$; the new edge of $s \prec_{\alpha,\beta} t$ has its type (α,β) changed into $(\alpha,\beta \leftarrow \gamma)$; the edges of s have their types $(\omega,\tau \leftarrow \beta)$ changed into $(\omega,(\tau \leftarrow \beta) \leftarrow \gamma)$; the edges of t have their type $(\alpha \leftarrow \omega,\tau)$ changed into $(\alpha \leftarrow \omega,\tau \leftarrow \gamma)$; the edges of u have their types (ω,τ) changed into $((\alpha \leftarrow \beta) \leftarrow \omega,\tau)$.

Comparing both sides and using the duplicial semigroup axioms proves Equation (11).

Second, we prove Equation (12). We use a table for comparison.

$s \succ_{\alpha,\beta\leftarrow\gamma} (t \prec_{\beta,\gamma} u)$	$(s \succ_{\alpha,\beta} t) \prec_{\alpha\to\beta,\gamma} u$
the first step: $t \prec_{\beta,\gamma} u$	the first step: $s \succ_{\alpha,\beta} t$
new edge typed by (β,γ)	new edge typed by (α,β)
the edges of u $(\omega,\tau) \mapsto (\beta \leftarrow \omega, \tau)$	the edges of s $(\omega,\tau) \mapsto (\omega, \tau \to \beta)$
the edges of t $(\omega,\tau) \mapsto (\omega, \tau \leftarrow \gamma)$	the edges of t $(\omega,\tau) \mapsto (\alpha \to \omega, \tau)$
the second step: $s \succ_{\alpha,\beta\leftarrow\gamma} (t \prec_{\beta,\gamma} u)$	the second step: $(s \succ_{\alpha,\beta} t) \prec_{\alpha\to\beta,\gamma} u$
the new edge typed by $(\alpha, \beta \leftarrow \gamma)$	the new edge typed by $(\alpha \to \beta, \gamma)$
the new edge of $t \prec_{\beta,\gamma} u$ $(\omega,\gamma) \mapsto (\alpha \to \beta, \gamma)$	the new edge of $s \succ_{\alpha,\beta} t$ $(\alpha,\beta) \mapsto (\alpha, \gamma \leftarrow \gamma)$
the edges of t $(\omega, \tau \leftarrow \gamma) \mapsto (\alpha \to \omega, \tau \leftarrow \gamma)$	the edges of t $(\alpha \to \omega, \tau) \mapsto (\alpha \to \omega, \tau \leftarrow \gamma)$
the edges of u $(\beta \leftarrow \omega, \tau) \mapsto (\alpha \to (\beta \leftarrow \omega), \tau)$	the edges of u $(\omega,\tau) \mapsto ((\alpha \leftarrow \beta) \leftarrow \omega, \tau)$
the edges of s $(\omega,\tau) \mapsto (\omega, \tau \to (\beta \leftarrow \gamma))$	the edges of s $(\omega, \tau \to \beta) \mapsto (\omega, (\tau \to \beta) \leftarrow \gamma)$

So both the left hand side and right hand side coincide.

Last, Eq. (13) can be proved similarly to Eq. (11). Details are left to the reader. □

2.3. **On the operads of two-parameter duplicial or dendriform algebras.** Let us assume that the parameter set Ω is finite, and let us denote its cardinality by w. We denote by $\mathbf{Dend}^2_\Omega$, respectively $\mathbf{Dup}^2_\Omega$, the non-sigma operad of two-parameter dendriform, respectively duplicial, algebras.

Proposition 2.12. *For all $n \geqslant 1$, considering the dimensions*

$$r_n = \dim_{\mathbf{k}}(\mathbf{Dend}^2_\Omega(n)) = \dim_{\mathbf{k}}(\mathbf{Dup}^2_\Omega(n)),$$

and their generating series

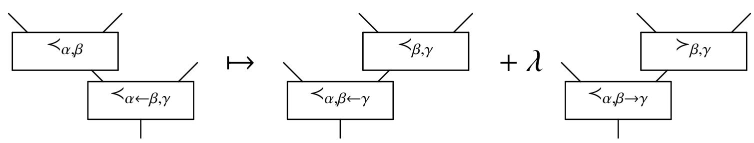
$$R(X) = \sum_{n=1}^{\infty} r_n X^n \in \mathbb{Q}[[X]],$$

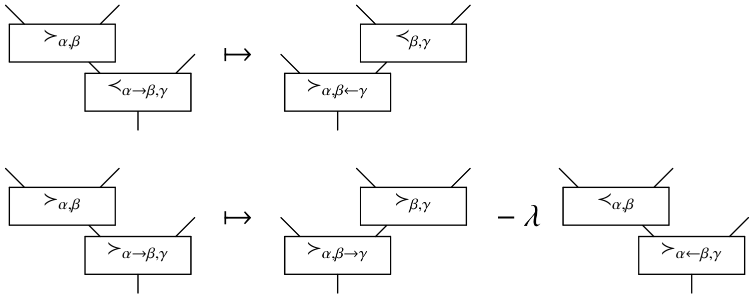
the following equation holds:

$$w^2(w-1)R^3 + w(wX + 2w - 2)R^2 + (2wX - 1)R + X = 0. \tag{16}$$

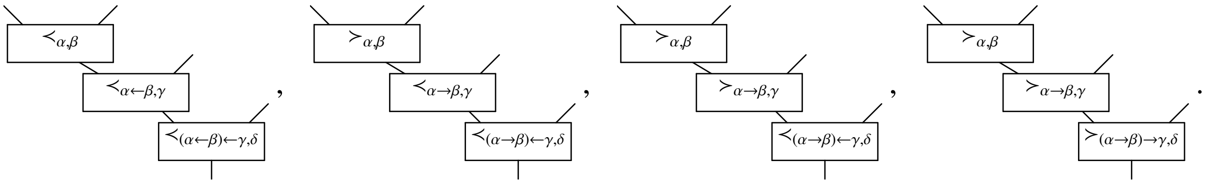
Moreover, the operads $\mathbf{Dend}^2_\Omega$ and $\mathbf{Dup}^2_\Omega$ are Koszul.

Proof. We shall use the rewriting method [8, 4] to obtain a PBW basis of the operad $\mathbf{Dend}^2_\Omega$ or $\mathbf{Dup}^2_\Omega$. In order to unify the proof of the two cases, let us take $\lambda \in \{0,1\}$. The presentation of the operad $\mathbf{Dend}^2_\Omega$ (if $\lambda = 1$) or $\mathbf{Dup}^2_\Omega$ (if $\lambda = 0$) gives the following three families of rewriting rules:

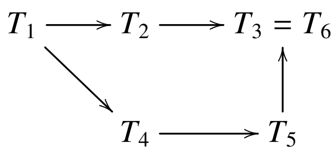




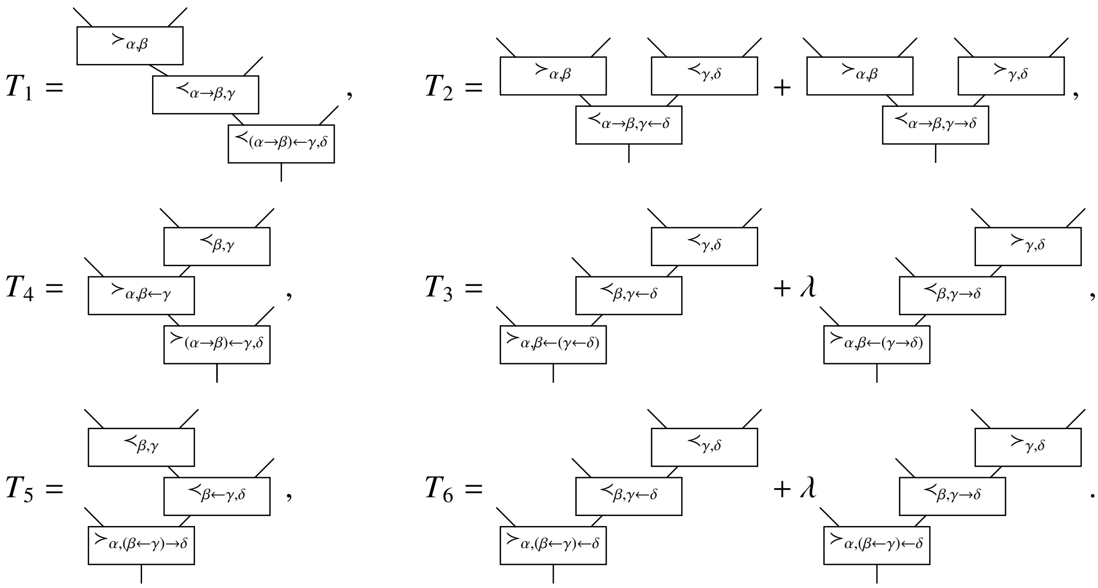
We obtain then four families of critical monomials:



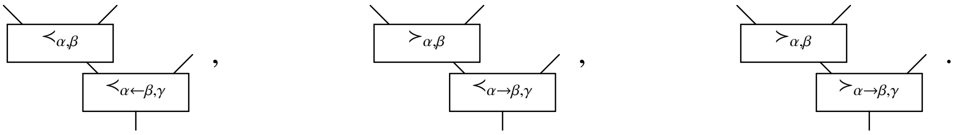
For each of them, one has to check that the diagram of rewritings is confluent. Here is one of them:



with:



The equality between T_3 and T_6 comes from the first axiom of duplicial semigroups if $\lambda = 0$ and from the first axiom of diassociative semigroup if $\lambda = 1$. We leave to the reader the three other confluent diagrams. As a consequence, these finitely generated quadratic operads are Koszul and we obtain a basis of the operads $\mathbf{Dend}^2_\Omega$ and $\mathbf{Dup}^2_\Omega$, given by planar binary trees the vertices of which are decorated by elements $\prec_{\alpha,\beta}$ or $\succ_{\alpha,\beta}$, with $\alpha,\beta \in \Omega$, avoiding the trees of the form



We denote by $R_\prec$ the formal series of such trees with root decorated by an element $\prec_{\alpha,\beta}$ and by $R_\succ$ the formal series of such trees with root decorated by an element $\succ_{\alpha,\beta}$, counted according to their number of leaves. Then:

$$\begin{aligned} R_\succ &= w^2(R_\prec + X)R + w(w-1)R_\succ R = w^2R^2 - wR_\succ R,\\ R_\prec &= w^2XR + w(w-1)(R_\succ + R_\prec)R = w^2R^2 - w(R-X)R,\\ R &= X + R_\prec + R_\succ. \end{aligned}$$

We obtain that:

$$R_\succ = \frac{w^2R^2}{1+wR}, \qquad R_\prec = w(w-1)R^2 + wXR.$$

Replacing in $R = R_\prec + R_\succ + X$, we obtain (16). □

For example:

$$\begin{aligned} r_1 &= 1, & r_2 &= 2w^2,\\ r_3 &= w^3(8w-3), & r_4 &= 2w^4(20w^2 - 15w + 2). \end{aligned}$$

Remark 2.13. If $w = 1$, one recovers duplicial and dendriform algebras, and $r_n(1)$ is the $n+1$ Catalan number Cat_{n+1}, sequence A000108 of the OEIS [30]. The sequences $r_n(w)$ for $w = 2$, 3 or 4 are not referenced (yet) in the OEIS.

Proposition 2.14. *Let $n \geqslant 1$.*

(a) *r_n is a polynomial in $\mathbb{Z}[w]$, of degree $2n-2$, and its leading coefficient is $2^{n-1}\mathrm{Cat}_n$.*
(b) *If $n \geqslant 2$, there exists a polynomial $t_n \in \mathbb{Z}[w]$, such that $r_n = w^n t_n$. Moreover, $t_n(0) = (-1)^n n$.*

Proof. By (16), if $n \geqslant 2$,

$$r_n = w^2(w-1)\sum_{i+j+k=n} r_ir_jr_k + w^2\sum_{i+j=n-1} r_ir_j + w(2w-2)\sum_{i+j=n} r_ir_j + 2wr_{n-1}. \tag{17}$$

Let us proceed by induction on n. The results are obvious if $n \leqslant 3$. Let us assume that $n \geqslant 4$ and the results at all ranks $< n$. By (17), obviously $r_n \in \mathbb{Z}[w]$. Moreover, by the induction hypothesis:

- The first term of (17) is of degree $\leqslant 3 + 2n - 6 = 2n - 3$.
- The second term of (17) is of degree $\leqslant 2 + 2n - 6 = 2n - 4$.
- The third term of (17) is of degree $\leqslant 2 + 2n - 4 = 2n - 2$; its coefficient of degree $2n-2$ is
$$2\sum_{i+j=n} 2^{i-1}\mathrm{Cat}_i 2^{j-1}\mathrm{Cat}_j = 2^{n-1}\sum_{i+j=n}\mathrm{Cat}_i = 2^{n-1}\mathrm{Cat}_n.$$
- The fourth term of (17) is of degree $\leqslant 1 + 2n - 4 = 2n - 3$.

Hence, r_n is of degree $2n-2$ and its leading coefficient is $2^{n-1}\mathrm{Cat}_n$. Still by the induction hypothesis:

- For the first term of (17):
  - If $i, j, k \geqslant 2$, then $w^2(w-1)r_ir_jr_k$ is a multiple of w^{n+2}.

  - If only one of i, j, k is equal to 1, then $w^2(w-1)r_ir_jr_k$ is a multiple of w^{n+1}.
  - If two of i, j, k are equal to 1, then the other one is equal to $n-2 \geqslant 2$ and $w^2(w-1)r_ir_jr_k$ is a multiple of w^n.

  Hence, this first term is a multiple of w^n and its contribution to the coefficent of w^n is
$$-3(-1)^{n-2}(n-2).$$
- For the second term of (17):
  - If $i, j \geqslant 2$, then $w^2r_ir_j$ is a multiple of w^{n+1}.
  - If one of i or j is equal to 1, then the second one is $n-2 \geqslant 2$ and $w^2r_ir_j$ is a multiple of w^n.

  Hence, this second term is a multiple of w^n and its contribution to the coefficent of w^n is
$$2(-1)^{n-2}(n-2).$$
- For the third term of (17):
  - If $i, j \geqslant 2$, then $w^2r_ir_j$ is a multiple of w^{n+1}.
  - If one of i or j is equal to 1, then the second one is $n-1 \geqslant 2$ and $w(2w-2)r_ir_j$ is a multiple of w^n.
  - If n is even, then any coefficent of r_n is even.

  Hence, this third term is a multiple of w^n and its contribution to the coefficent of w^n is
$$-2 \times 2(-1)^{n-1}(n-1).$$
- The last term of (17) is a multiple of w^n and its contribution to the coefficent of w^n is
$$2(-1)^{n-1}(n-1).$$

Finally, r_n is a multiple of w^n and the coefficient of w^n in r_n is
$$-3(-1)^n(n-2) + 2(-1)^n(n-2) + 4(-1)^n(n-1) - 2(-1)^n(n-1) = (-1)^n n.$$
Let us assume that n is even. Then, in $\mathbb{Z}/2\mathbb{Z}[w]$:
$$r_n \equiv w^2(w-1) \sum_{i+j+k=n} r_ir_jr_k + w^2 \sum_{i+j=n-1} r_ir_j + 0[2].$$
As n is even, in the first term, one or three of i, j, k are even, so $r_ir_jr_k \equiv 0[2]$; in the second term, one of i, j is even, so $r_ir_j \equiv 0[2]$. Finally, $r_n \equiv 0[2]$. □

3. Reminders on operads and colored operads in the species formalism

Colored operads are natural tools to be used in the description of algebraic structures on graded objects. We give a description of those in the colored species formalism, mainly following the presentation of [9]. We also give a reminder of the more familiar monochromatic case, i.e. ordinary operads, and we decribe a pair $(\mathcal{U}, \overline{\mathcal{F}})$ of adjoint functors from colored operads to monochromatic operads and vice-versa, along the lines of [2].

3.1. **Colored species.** Let $\mathcal{C}$ be a bicomplete symmetric monoidal category, i.e. with small limits and colimits, which in particular implies the existence of products and coproducts indexed by an arbitrary set. We further suppose that the monoidal product is distributive with respect to coproducts is each variables, i.e.
$$\left(\coprod_{i\in I} X_i\right) \otimes \left(\coprod_{j\in J} Y_j\right) = \coprod_{(i,j)\in I\times J} X_i \otimes Y_j$$

for any collections (X_i) and (Y_j) of objects indexed by arbitrary sets I and J. The unit for the monoidal product will be denoted by $\mathbf{1}$, or $\mathbf{1}_{\mathcal{C}}$ if the mention of the category must be precised.

Let $\mathcal{F}_\Omega$ be the category of Ω-*colored finite sets* defined as follows:

- objects are triples $(A, \underline{\alpha}, \omega)$ where A is a finite set, $\omega \in \Omega$ (the *output color*) and $\underline{\alpha} : A \to \Omega$ is a list of elements of Ω indexed by A (the *input colors*).
- morphisms are given by bijective maps from A onto B together with re-indexing of colors: a morphism
$$\varphi : (A, \underline{\alpha}, \omega) \longrightarrow (B, \underline{\beta}, \zeta)$$
is given by an underlying bijective map $\overline{\varphi} : A \to B$ under the two conditions that $\omega = \zeta$ and $\underline{\alpha} = \underline{\beta} \circ \varphi$, otherwise there is no morphism from $(A, \underline{\alpha}, \omega)$ to $(B, \underline{\beta}, \zeta)$.

Definition 3.1. An Ω-colored species $\mathcal{P}$ in the bicomplete monoidal category $\mathcal{C}$ is a contravariant functor $(A, \underline{\alpha}, \omega) \mapsto \mathcal{P}_{A,\underline{\alpha},\omega}$ from $\mathcal{F}_\Omega$ to $\mathcal{C}$. The Ω-colored species is *positive* if moreover $\mathcal{P}_{\emptyset,-,\omega} = 0_{\mathcal{C}}$ for any $\omega \in \Omega$, where $0_{\mathcal{C}}$ is the initial object.

The notion of colored species is originally due to A. Joyal [17]. This precise definition is borrowed from [9, Definition 2.2] which even provides a slightly more general framework: Ω-colored species correspond to (Ω, Ω)-collections therein.

Let us provide a brief summary of the monochromatic case: the category $\mathcal{F}_\Omega$ boils down to the category $\mathcal{F}$ of finite sets with bijections when the set Ω of colors is reduced to one element. We recover then the usual notion of (contravariant) species [18, 3, 25]. A $\mathcal{C}$-species is a contravariant functor from $\mathcal{F}$ into $\mathcal{C}$, where $\mathcal{F}$ is the category of finite sets with bijections as morphisms. We stick to *positive species*, i.e. species $\mathcal{P}$ such that $\mathcal{P}_\emptyset = 0_{\mathcal{C}}$, where $0_{\mathcal{C}}$ is the initial object of the monoidal category $\mathcal{C}$ [25]. We adopt M. Mendez' definition of an operad in the species formalism:

Definition 3.2. [17, Example 41], [25, Definition 3.1] An operad is a monoid in the category of positive species.

3.2. **Colored operads.**

Definition 3.3. The substitution product of two positive Ω-colored species is defined by

$$(\mathcal{P} \boxtimes \mathcal{Q})_{A,\underline{\alpha},\omega} := \coprod_{\pi \text{ set partition of } A} \ \coprod_{\underline{\gamma}:\pi\to\Omega} \mathcal{P}_{\pi,\underline{\gamma},\omega} \otimes \bigotimes_{B\in\pi} \mathcal{Q}_{B,\underline{\alpha}_{|B},\underline{\gamma}(B)}. \tag{18}$$

The substitution product $\boxtimes$ is also defined on morphisms and is associative, making the category of positive Ω-colored species a (non-symmetric) monoidal category. The unit is the colored species $\mathbf{1}$ defined by $\mathbf{1}_{A,\underline{\alpha},\omega} = \mathbf{1}_{\mathcal{C}}$ if $|A| = 1$ and $\underline{\alpha} = \omega$, and $\mathbf{1}_{A,\underline{\alpha},\omega} = 0$ otherwise. It can be written as

$$\mathbf{1} = \prod_{\omega\in\Omega} \mathbf{1}^\omega \tag{19}$$

where $\mathbf{1}^\omega$ is the colored species defined by $\mathbf{1}^\omega_{A,\underline{\alpha},\zeta} = \mathbf{1}_{\mathcal{C}}$ if $|A| = 1$ and $\underline{\alpha} = \zeta = \omega$, and $\mathbf{1}^\omega_{A,\underline{\alpha},\zeta} = 0$ otherwise. The colored species $\mathbf{1}^\omega$ is sometimes slightly abusively called *unit of color* ω.

Definition 3.4. A **colored operad** is a monoid in the monoidal category of positive Ω-colored species endowed with the substitution product.

The definition of operads in terms of partial compositions is originally due to M. Markl [24, Paragraph 1.3] and can be traced back to M. Gerstenhaber [14]. In the colored case, the global multiplication $\gamma : \mathcal{P} \boxtimes \mathcal{P} \to \mathcal{P}$ is declined into functorial partial compositions

$$\circ_a : \mathcal{P}_{A,\underline{\alpha},\omega} \otimes \mathcal{P}_{B,\underline{\beta},\zeta} \longrightarrow \begin{cases} \mathcal{P}_{A\sqcup_a B,\, \underline{\alpha}\sqcup_a\underline{\beta},\, \omega} & \text{if } \zeta = \underline{\alpha}(a), \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

subject to parallel and sequential associativity axioms, and there is a unit $e : \mathbf{1} \to \mathcal{P}$. Informally, the partial composition $\circ_a$ is nontrivial if and only if the output color of the second term matches the input color of the first term corresponding to $a \in A$, otherwise $\circ_a$ takes values in the terminal object $0_{\mathcal{C}}$.

For any set map $\kappa : \Omega \to \Omega'$, the color change functor from Ω'-colored species to Ω-colored species is defined by

$$(\kappa^*\mathcal{P})_{A,\underline{\alpha},\omega} := \mathcal{P}_{A,\, \kappa\circ\underline{\alpha},\, \kappa(\omega)} \tag{21}$$

for any $(A,\underline{\alpha},\omega) \in \mathcal{F}_\Omega$. It respects both monoidal products $\boxtimes$, hence restricts from Ω'-colored operads to Ω-colored operads. In particular, the case when $\Omega' = \{*\}$ contains a unique element shows that any ordinary (monochromatic) operad $\mathcal{Q}$ can be promoted to an Ω-colored operad $\mathcal{Q}^\Omega := \kappa^*\mathcal{Q}$, with $\kappa : \Omega \to \{*\}$. The colored operad $\mathcal{Q}^\Omega$ is said to be *uniform*. This functor $\mathcal{U} : \mathcal{Q} \to \mathcal{Q}^\Omega$ is left-adjoint to the *completed forgetful functor* $\overline{\mathcal{F}}$ from Ω-colored operads to ordinary operads, defined by

$$(\overline{\mathcal{F}}\mathcal{P})_A := \prod_{(\underline{\alpha},\omega)\in\Omega^A\times\Omega} \mathcal{P}_{A,\underline{\alpha},\omega}, \tag{22}$$

i.e. for any operad $\mathcal{Q}$ and for any colored operad $\mathcal{P}$ we have

$$\mathrm{Hom}(\mathcal{U}\mathcal{Q},\mathcal{P}) \simeq \mathrm{Hom}(\mathcal{Q},\overline{\mathcal{F}}\mathcal{P}).$$

3.3. **Categories of graded objects.** We keep the notations of the previous paragraph. The category $\mathcal{C}_\Omega$ of Ω-graded objects [2, Paragraph 2.2] is the category of collections $(V_\omega)_{\omega\in\Omega}$ of objects of $\mathcal{C}$. A $\mathcal{C}_\Omega$-morphism

$$\varphi : (V_\omega) \longrightarrow (W_\omega)$$

is a collection $(\varphi_\omega)_{\omega\in\Omega}$ of $\mathcal{C}$-morphisms $\varphi_\omega : V_\omega \to W_\omega$. This is not a monoidal category: indeed, the tensor product of two Ω-graded objects is a collection indexed by $\Omega\times\Omega$.

Remark 3.5. In the case when Ω is a semigroup, categories of Ω-graded objects can be given a monoidal structure by means of the Cauchy product [2, Paragraph 2.2]. We do not have this tool at our disposal here.

A well-known example of Ω-colored operad, in a closed bicomplete monoidal category $\mathcal{C}$, for example the category of $\mathbf{k}$-vector spaces, is given by $\mathrm{End}(\mathcal{V})$ where $\mathcal{V} = (V_\omega)_{\omega\in\Omega}$ is an Ω-graded object:

$$\mathrm{End}(\mathcal{V})_{A,\underline{\alpha},\omega} := \mathrm{Hom}_{\mathcal{C}}\left(\bigotimes_{a\in A} V_{\underline{\alpha}(a)},\, V_\omega\right), \tag{23}$$

where $\mathrm{Hom}_{\mathcal{C}}$ is the internal Hom [22, 26]. Details are standard and left to the reader. An *algebra over an Ω-colored operad* $\mathcal{P}$ is an Ω-graded object $\mathcal{V}$ together with a morphism of colored operads $\Phi : \mathcal{P} \to \mathrm{End}(\mathcal{V})$.

Definition 3.6. [2, Paragraph 2.2] An Ω-graded object $\mathcal{V} = (V_\omega)_\omega$ is **uniform** if all homogeneous components are identical, i.e. if there is an object V of $\mathcal{C}$ such that $V_\omega = V$ for any $\omega \in \Omega$. We write $\mathcal{V} = \mathcal{U}(V)$ in this case. This defines a functor $\mathcal{U} : \mathcal{C} \to \mathcal{C}_\Omega$, which has a left adjoint, the forgetful functor $\mathcal{F} : \mathcal{C}_\Omega \to \mathcal{C}$ defined by

$$\mathcal{F}(\mathcal{V}) := \coprod_{\omega\in\Omega} V_\omega,$$

which consists in forgetting the Ω-grading [2, Paragraph 2.4]. It has also a right adjoint, the completed forgetful functor $\overline{\mathcal{F}} : \mathcal{C}_\Omega \to \mathcal{C}$ defined by

$$\overline{\mathcal{F}}(\mathcal{V}) := \prod_{\omega\in\Omega} V_\omega.$$

4. Two-parameter Ω-pre-Lie algebras

The pre-Lie operad is no longer a set operad, hence new phenomena arise when seeking a compatible structure on the parameter set Ω. Indeed, four different associated set operads are involved. The first one is the well-known associative operad. The second one is the operad governing rings with the twist-associativity condition $x(yz) = (yx)z$, also known as Thedy rings [31]. The third one is the operad governing rings with both NAP relation $x(yz) = y(xz)$ and NAP' relation $(xy)z = (yx)z$. The fourth operad governs rings with all previous relations at once, this is the well-known Perm operad [6, 7].

4.1. **Four possibilities.** Let A be a vector space and let Ω be a set with a binary operation $\blacktriangleright$. Suppose that $A \otimes \mathbf{k}\Omega$ is endowed with an Ω-graded pre-Lie product:

$$(x\otimes\alpha) \rhd (y\otimes\beta) := x \rhd_{\alpha,\beta} y \otimes (\alpha \blacktriangleright \beta). \tag{24}$$

The pre-Lie axiom

$$(x\otimes\alpha)\rhd\Big((y\otimes\beta)\rhd(z\otimes\gamma)\Big)-\Big((x\otimes\alpha)\rhd(y\otimes\beta)\Big)\otimes(z\otimes\gamma) = (y\otimes\beta)\rhd\Big((x\otimes\alpha)\rhd(z\otimes\gamma)\Big)-\Big((y\otimes\beta)\rhd(x\otimes\alpha)\Big)\rhd(z\otimes\gamma)$$

together with the Ω-grading are equivalent to

$$\begin{aligned} & x \rhd_{\alpha,\beta\blacktriangleright\gamma} (y \rhd_{\beta,\gamma} z) \otimes \big(\alpha \blacktriangleright (\beta \blacktriangleright \gamma)\big) - (x \rhd_{\alpha,\beta} y) \rhd_{\alpha\blacktriangleright\beta,\gamma} z \otimes \big((\alpha \blacktriangleright \beta) \blacktriangleright \gamma\big) \\ & = y \rhd_{\beta,\alpha\blacktriangleright\gamma} (x \rhd_{\alpha,\gamma} z) \otimes \big(\beta \blacktriangleright (\alpha \blacktriangleright \gamma)\big) - (y \rhd_{\beta,\alpha} x) \rhd_{\beta\blacktriangleright\alpha,\gamma} z \otimes \big((\beta \blacktriangleright \alpha) \blacktriangleright \gamma\big). \end{aligned} \tag{25}$$

Eq. (25) induces four possible nontrivial different cases. The cardinality of the set

$$\{\alpha \blacktriangleright (\beta \blacktriangleright \gamma),\ (\alpha \blacktriangleright \beta) \blacktriangleright \gamma,\ \beta \blacktriangleright (\alpha \blacktriangleright \gamma),\ (\beta \blacktriangleright \alpha) \blacktriangleright \gamma\}$$

is equal to two for generic $\alpha,\beta,\gamma \in \Omega$ in the three first cases, and equal to one in the last case. We discard the degenerate case of cardinality four.

Case 1: let

$$\alpha \blacktriangleright (\beta \blacktriangleright \gamma) = (\alpha \blacktriangleright \beta) \blacktriangleright \gamma$$

for any $\alpha,\beta,\gamma \in \Omega$. Thus Ω is a semigroup. Then

$$x \rhd_{\alpha,\beta\blacktriangleright\gamma} (y \rhd_{\beta,\gamma} z) = (x \rhd_{\alpha,\beta} y) \rhd_{\alpha\blacktriangleright\beta,\gamma} z, \quad \text{for any } \alpha,\beta,\gamma \in \Omega$$

and we recover the notion of family associative algebra.

Case 2: let

$$\alpha \blacktriangleright (\beta \blacktriangleright \gamma) = (\beta \blacktriangleright \alpha) \blacktriangleright \gamma \tag{26}$$

for any $\alpha,\beta,\gamma \in \Omega$. Then Ω is a kind of “twisted associative semigroup”, a notion which has received little attention in the literature (see however [31] and [32]). We have then

$$x \rhd_{\alpha,\beta\blacktriangleright\gamma} (y \rhd_{\beta,\gamma} z) = -(y \rhd_{\beta,\alpha} x) \rhd_{\beta\blacktriangleright\alpha,\gamma} z \tag{27}$$

and we recover a notion of “family twisted associative algebra” modulo a minus sign.

Case 3: let

$$\begin{cases} \alpha \blacktriangleright (\beta \blacktriangleright \gamma) = \beta \blacktriangleright (\alpha \blacktriangleright \gamma), \\ (\alpha \blacktriangleright \beta) \blacktriangleright \gamma = (\beta \blacktriangleright \alpha) \blacktriangleright \gamma \end{cases} \tag{28}$$

for any $\alpha,\beta,\gamma \in \Omega$. The first relation is the NAP condition. We call NAP’ the second condition, and we call Ω a NAPNAP’ set. Then

$$\begin{cases} x \rhd_{\alpha,\beta\blacktriangleright\gamma} (y \rhd_{\beta,\gamma} z) = y \rhd_{\beta,\alpha\blacktriangleright\gamma} (x \rhd_{\alpha,\gamma} z), \\ (x \rhd_{\alpha,\beta} y) \rhd_{\alpha\blacktriangleright\beta,\gamma} z = (y \rhd_{\beta,\alpha} x) \rhd_{\beta\blacktriangleright\alpha,\gamma} z. \end{cases}$$

We obtain what we shall call *family NAPNAP’* algebras. The concept of NAP algebra (NAP standing for Non-Associative Permutative) was introduced by M. Livernet [19], and also a couple of years before by A. Dzhumadil’daev and C. Löfwall under the name ”left commutative algebras” [10, Section 7]. We use here the terminology NAP’ for the second relation, in absence of a better name in the literature (up to our knowledge).

Case 4: for any $\alpha,\beta,\gamma \in \Omega$,

$$\alpha \blacktriangleright (\beta \blacktriangleright \gamma) = (\alpha \blacktriangleright \beta) \blacktriangleright \gamma = \beta \blacktriangleright (\alpha \blacktriangleright \gamma) = (\beta \blacktriangleright \alpha) \blacktriangleright \gamma,$$

i.e. Ω is a set-theoretical Perm algebra. Then for any $x, y, z \in A$,

$$x \rhd_{\alpha,\beta\blacktriangleright\gamma} (y \rhd_{\beta,\gamma} z) - (x \rhd_{\alpha,\beta} y) \rhd_{\alpha\blacktriangleright\beta,\gamma} z = y \rhd_{\beta,\alpha\blacktriangleright\gamma} (x \rhd_{\alpha,\gamma} z) - (y \rhd_{\beta,\alpha} x) \rhd_{\beta\blacktriangleright\alpha,\gamma} z.$$

This relation is very similar to the pre-Lie one, and deserves the name “pre-Lie family”. We address this last case in Paragraph 4.4.

4.2. **The twist-associative operad TAs.**

Definition 4.1. Let $\mathcal{P}$ be the set species of non-diagonal ordered pairs, defined by

$$\begin{aligned} \mathcal{P}_{\{*\}} &= \{\mathbf{1}\}, \\ \mathcal{P}_A &= \{(a', a'') \in A \times A,\ a' \neq a''\}. \end{aligned}$$

for any finite set A of cardinal ≥ 2. For any bijection $\phi : A \to B$ where B is a finite set of the same cardinality than A, the relabeling isomorphism $\mathcal{P}_\phi : \mathcal{P}_B \to \mathcal{P}_A$ is defined by

$$\mathcal{P}_\phi(b', b'') = \big(\phi^{-1}(b'), \phi^{-1}(b'')\big).$$

Definition 4.2. Let A and B be two finite sets. Define partial compositions

$$\circ_a : \mathcal{P}_A \otimes \mathcal{P}_B \quad \to \mathcal{P}_{A\sqcup B\setminus\{a\}}, \ \text{ for } a \in A$$

as follows: for any ordered pair $(a', a'') \in \mathcal{P}_A$ and $(b', b'') \in \mathcal{P}_B$, we set

$$(a', a'') \circ_a (b', b'') = \begin{cases} (b'', a''), & \text{if } a = a'; \\ (a', b''), & \text{if } a = a''; \\ (a', a''), & \text{if } a \notin \{a', a''\}. \end{cases} \tag{29}$$

Partial compositions are extended to singletons by setting $\mathbf{1}$ as the unit.

Proposition 4.3. *The species $\mathcal{P}$ together with the partial compositions $\circ_a$ defined by Eq. (29) is an operad.*

Proof. Let A, B, C be three sets of cardinal ≥ 2, and let $x = (a', a'') \in A \times A, y = (b', b'') \in B \times B, z = (c', c'') \in C \times C$. When we prove sequential associativity, there are nine cases to consider.

1	$a = a', b = b'$	2	$a = a', b = b''$
3	$a = a', b \notin \{b', b''\}$	4	$a = a'', b = b'$
5	$a = a'', b = b''$	6	$a = a'', b \notin \{b', b''\}$
7	$a \notin \{a', a''\}, b = b'$	8	$a \notin \{a', a''\}, b = b''$
9	$a \notin \{a', a''\}, b \notin \{b', b''\}$		

Table: the nine cases for sequential associativity

The case-by-case proof is displayed in the following table:

	$(x \circ_a y) \circ_b z = \big((a', a'') \circ_a (b', b'')\big) \circ_b (c', c'')$	$x \circ_a (y \circ_b z) = (a', a'') \circ_a \big((b', b'') \circ_b (c', c'')\big)$
1	$(b'', a'') \circ_b (c', c'') = (b'', a'')$	$(a', a'') \circ_a (c'', b'') = (b'', a'')$
2	$(b'', a'') \circ_b (c', c'') = (c'', a'')$	$(a', a'') \circ_a (b', c'') = (c'', a'')$
3	$(b'', a'') \circ_b (c', c'') = (b'', a'')$	$(a', a'') \circ_a (b', b'') = (b'', a'')$
4	$(a', b'') \circ_b (c', c'') = (a', b'')$	$(a', a'') \circ_a (c'', b'') = (a', b'')$
5	$(a', b'') \circ_b (c', c'') = (a', c'')$	$(a', a'') \circ_a (b', c'') = (a', c'')$
6	$(a', b'') \circ_b (c', c'') = (a', b'')$	$(a', a'') \circ_a (b', b'') = (a', b'')$
7	$(a', a'') \circ_b (c', c'') = (a', a'')$	$(a', a'') \circ_a (c'', b'') = (a', a'')$
8	$(a', a'') \circ_b (c', c'') = (a', a'')$	$(a', a'') \circ_a (b', c'') = (a', a'')$
9	$(a', a'') \circ_b (c', c'') = (a', a'')$	$(a', a'') \circ_a (b', b'') = (a', a'')$

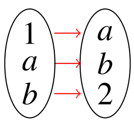
Parallel associativity is proved analogously, with only seven cases to consider. □

Proposition 4.4. *Let $A = \{1, 2\}$, let $\mu = (1, 2) \in \mathcal{P}_A$, and let $\overline{\mu} = (2, 1)$ be the other element of $\mathcal{P}_A$ obtained by permutation. The twist-associativity relation*

$$r = \mu \circ_2 \mu - \mu \circ_1 \bar{\mu} = 0 \tag{30}$$

holds in the operad $\mathcal{P}$.

Proof. Denoting by $\{a, b\}$ another copy of A (identifiying a with 1 and b with 2), both three-element sets $A \sqcup_2 A$ and $A \sqcup_1 A$ are identified by means of the bijection



in order to make Equation (30) consistent. We get then

$$\mu \circ_2 \mu = (1, 2) \circ_2 (a, b) = (1, b) \in \mathcal{P}_{\{1,a,b\}},$$

$$\mu \circ_1 \bar{\mu} = (1,2) \circ_1 (b,a) = (a,2) \in \mathcal{P}_{\{a,b,2\}},$$

hence $\mu \circ_2 \mu = \mu \circ_1 \bar{\mu}$ modulo the identification above. □

For later use, for any A, B finite sets we define the product $\blacktriangleright\colon \mathcal{P}_A \otimes \mathcal{P}_B \to \mathcal{P}_{A \sqcup B}$ by

$$\alpha \blacktriangleright \beta := (\mu \circ_1 \alpha) \circ_2 \beta.$$

An easy computation yields:

$$\mathbf{1} \blacktriangleright \mathbf{1} = \mu, \tag{31}$$

$$\mathbf{1} \blacktriangleright (x,y) = (*,y), \tag{32}$$

$$(x,y) \blacktriangleright \mathbf{1} = (y,*), \tag{33}$$

$$(x,y) \blacktriangleright (z,t) = (y,t) \tag{34}$$

for any $x, y \in A$ and $z, t \in B$ with $x \neq y$ and $z \neq t$. It is easily checked that the product verifies the twist-associative identity

$$\alpha \blacktriangleright (\beta \blacktriangleright \gamma) = (\beta \blacktriangleright \alpha) \blacktriangleright \gamma \tag{35}$$

for any finite sets A, B, C and for any $\alpha \in \mathcal{P}_A, \beta \in \mathcal{P}_B$ and $\gamma \in \mathcal{P}_C$.

Theorem 4.5. *The operad $\mathcal{P}$ of non-diagonal ordered pairs is isomorphic to the twist-associative operad $\mathcal{T} := \mathcal{M}/\langle r\rangle$, where $\langle r\rangle$ is the ideal generated by the twist-associative relation $r = \nu \circ_2 \nu - \nu \circ_1 \bar{\nu}$.*

Proof. We still adopt the notations in the proof of Proposition 4.4. The twist-associative operad is defined as the quotient of the magmatic operad $\mathcal{M}$ (the free operad generated by a single binary operation ν) by the ideal $\langle r\rangle$. Let A, B, C be three finite sets. Defining $\widetilde{\nu}$ as the image of ν in the quotient, we have

$$\alpha \rhd (\beta \rhd \gamma) = (\beta \rhd \alpha) \rhd \gamma \tag{36}$$

for any $\alpha \in \mathcal{T}_A, \beta \in \mathcal{T}_B$ and $\gamma \in \mathcal{T}_C$, where $\rhd$ is defined by

$$\alpha \rhd \beta := (\widetilde{\nu} \circ_1 \alpha) \circ_2 \beta.$$

As the ordered pair $\mu = (1,2)$ verifies the twist-associative relation (30), there is a unique surjective operad morphism $\Phi : \mathcal{T} \to \mathcal{P}$ such that $\Phi(\widetilde{\nu}) = \mu$. It obviously verifies

$$\Phi(\alpha \rhd \beta) = \Phi(\alpha) \blacktriangleright \Phi(\beta) \tag{37}$$

for any $\alpha \in \mathcal{T}_A$ and $\beta \in \mathcal{T}_B$. Let us prove that Φ is bijective. Define $\Psi_A : \mathcal{P}_A \to \mathcal{T}_A$ by induction on the arity $n = |A| \geq 2$. For $n = 1$ we set $\Psi(\mathbf{1}) = \mathbf{1}$, and for $n = 2$ it amounts to $\Psi(\mu) = \widetilde{\nu}$. Suppose that the inverse Ψ of Φ is well-defined (and hence bijective) up to arity n, and let A be of cardinality $n+1$. From (32), any $(x,y) \in \mathcal{P}_A$ can be written $(x,y) = \mathbf{1} \blacktriangleright (x',y)$, where $\mathbf{1} \in \mathcal{P}_{\{x\}}$ and $(x',y) \in \mathcal{P}_{A\setminus\{x\}}$. Hence we necessarily have

$$\Psi(x,y) = \mathbf{1} \rhd \Psi(x',y).$$

It is well defined because it does not depend on the choice of x'. Indeed, if another choice x'' is possible, then

$$(x,y) = \mathbf{1} \blacktriangleright (x'',y) = \mathbf{1} \blacktriangleright (\mathbf{1} \blacktriangleright (x',y)),$$

hence

$$\begin{aligned} \mathbf{1} \rhd \Psi(x'',y) &= \mathbf{1} \rhd (\mathbf{1} \rhd \Psi(x',y)), \qquad (x',y) \in A \setminus \{x,x''\} \\ &= \mathbf{1} \rhd (\mathbf{1} \rhd \Psi(x'',y)), \qquad (x'',y) \in A \setminus \{x,x'\} \text{ (by induction hypothesis),} \end{aligned}$$

$$= \mathbf{1} \triangleright \Psi(x', y) \quad \text{(again by induction hypothesis).}$$

We have

$$\Phi\Psi(x, y) = \mathbf{1} \triangleright \Phi\Psi(x', y) = \mathbf{1} \triangleright (x', y) = (x, y)$$

by induction hypothesis, hence $\Phi_A \Psi_A = \mathrm{Id}_{\mathcal{P}_A}$. Furthermore, for any partition $A = B \sqcup C$ and for any $\beta \in \mathcal{P}_B, \gamma \in \mathcal{P}_C$ we have

$$\Psi(\beta \blacktriangleright \gamma) = \Psi(\beta) \triangleright \psi(\gamma).$$

This is easily proven by induction on the cardinality of B, the case $|B| = 1$ being equivalent to the definition of Ψ: if $|B| \geq 2$ we write $\beta = \mathbf{1} \blacktriangleright \beta'$ and then

$$\begin{aligned}
\Psi(\beta \blacktriangleright \gamma) &= \Psi((\mathbf{1} \blacktriangleright \beta') \blacktriangleright \gamma) \\
&= \Psi(\beta' \blacktriangleright (\mathbf{1} \blacktriangleright \gamma)) \text{ (by (35))} \\
&= \Psi(\beta') \triangleright \Psi(\mathbf{1} \blacktriangleright \gamma) \text{ (by induction on } |B|) \\
&= \Psi(\beta') \triangleright (\mathbf{1} \triangleright \Psi(\gamma)) \\
&= (\mathbf{1} \triangleright \Psi(\beta')) \triangleright \Psi(\gamma) \text{ (by (36))} \\
&= \Psi(\beta) \triangleright \Psi(\gamma).
\end{aligned}$$

Now any $\alpha \in \mathcal{T}_A$ can be written $\beta \triangleright \gamma$ with $\beta \in \mathcal{T}_B$, $\gamma \in \mathcal{T}_C$, where B and C are two finite sets of cardinality $\leq n$ such that $A = B \sqcup C$. We have then

$$\Psi\Phi(\alpha) = \Psi\Phi(\beta \triangleright \gamma) = \Psi\Phi(\beta) \triangleright \Psi\Phi(\gamma) = \beta \triangleright \gamma = \alpha$$

again by induction hypothesis, hence $\Psi_A \Phi_A = \mathrm{Id}_{\mathcal{T}_A}$. This ends up the proof of Theorem 4.5. □

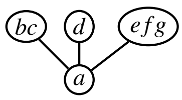
4.3. The operad NAPNAP′ of corollas.

Definition 4.6. A **corolla structure** β on a finite set B is a quasi-order admitting one unique minimum r, such that any element different from r is a maximum.

The unique minimum r is the *root* of the corolla. Any $b \neq r$ verifies $r \leq b$ but never $b \leq r$. The non-root elements are partitioned into *branches* $B_1, \ldots, B_p$, which are the equivalence classes (excluding the one of the root) under the relation $\sim$ defined by $b \sim b'$ if and only if $b \leq b'$ and $b' \leq b$. We shall write

$$\beta = [B_1, \ldots, B_p]_r.$$

For example, on the finite set $B : \{a, b, c, d, e, f, g\}$, the notation $\beta = [\{b, c\}, \{d\}, \{e, f, g\}]_a$ stands for the corolla

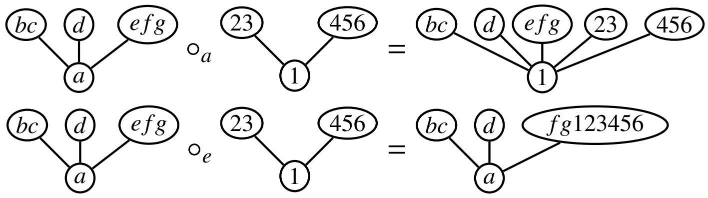
$\beta =$ 

Let $\mathbb{K}_B$ be the set of corolla structures on B. This forms a set species: any bijection $\varphi : B \to C$ induces a bijection $\mathbb{K}_\varphi : \mathbb{K}_C \to \mathbb{K}_B$ by relabeling.

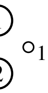
Now let us define the operad structure. Let B, C be two finite sets, let $b \in B$, let $\beta \in \mathbb{K}_B$ and $\gamma \in \mathbb{K}_C$. Let r be the root of the corolla γ. The partial composition $\beta \circ_b \gamma : \mathbb{K}_B \times \mathbb{K}_C \to \mathbb{K}_{B \sqcup_b C}$ is defined as follows:

- if b is the root of β, then $\beta \circ_b \gamma$ is the corolla on $B \sqcup_b C$ obtained by choosing r as the root, and by keeping all branches in $B \sqcup_b C \setminus \{r\}$. In particular, elements in $B \setminus \{b\}$ and elements in $C \setminus \{r\}$ belong to different branches, and thus are uncomparable.
- if b is not the root of β, then $\beta \circ_b \gamma$ is the corolla on $B \sqcup_b C$ obtained by replacing b by the whole C in the branch of b.

Let us give an example for better understanding.

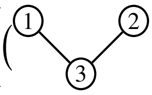
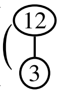


We leave it to the reader to show that $\mathbb{K}$ endowed with the partial compositions defined above is an operad, i.e. prove both sequential and parallel associativity axioms. Now define the product $\blacktriangleright$ on $\mathbb{K}$ by

$$\beta \blacktriangleright \gamma := (\ $$  $$\ \circ_1 \beta) \circ_2 \gamma.$$

Proposition 4.7. *The product $\blacktriangleright$ verifies for any $\alpha, \beta, \gamma \in \mathbb{K}$:*

(a) $\alpha \blacktriangleright (\beta \blacktriangleright \gamma) = \beta \blacktriangleright (\alpha \blacktriangleright \gamma)$,
(b) $(\alpha \blacktriangleright \beta) \blacktriangleright \gamma = (\beta \blacktriangleright \alpha) \blacktriangleright \gamma$.

Proof. Both sides of Equation (a) are equal to $\Big(\big($  $\circ_1 \alpha\big) \circ_2 \beta\Big) \circ_3 \gamma$, and both sides of Equation (b) are equal to $\Big(\big($  $\circ_1 \alpha\big) \circ_2 \beta\Big) \circ_3 \gamma$. Details are left to the reader. □

Theorem 4.8. *The operad $\mathbb{K}$ of corollas is the NAPNAP' operad.*

Proof. The NAPNAP' operad is defined as the quotient of the magmatic operad $\mathcal{M}$ by the NAP and NAP' relations, namely

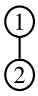
$$\text{NAPNAP'} := \mathcal{M}\Big/\Big\langle \mu \circ_2 \mu - \tau_{12}(\mu \circ_2 \mu),\ \mu \circ_1 \mu - \tau_{12}(\mu \circ_1 \mu)\Big\rangle.$$

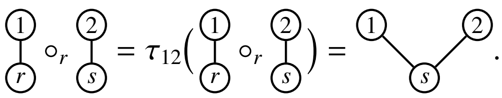
Defining $\overline{\mu}$ as the image of μ in the quotient, we further introduce the product $\triangleright$ on the NAPNAP' operad itself, defined by

$$\alpha \triangleright \beta := (\overline{\mu} \circ_1 \alpha) \circ_2 \beta.$$

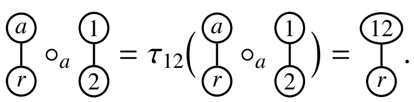
The NAP and NAP' relations for $\overline{\mu}$ yield analogous relations for $\triangleright$, namely

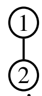
(a) $\alpha \triangleright (\beta \triangleright \gamma) = \beta \triangleright (\alpha \triangleright \gamma)$,
(b) $(\alpha \triangleright \beta) \triangleright \gamma = (\beta \triangleright \alpha) \triangleright \gamma$.

The corolla  respects both NAP and NAP' relations, namely

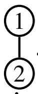


and



Hence the operad morphism $\widetilde{\Phi}$ from $\mathcal{M}$ onto $\mathbb{K}$ uniquely defined by $\widetilde{\Phi}(\mu) =$  vanishes on the ideal generated by the NAP and NAP' relations, giving rise to the unique surjective operad morphism

$$\Phi : \text{NAPNAP'} \longrightarrow \mathbb{K}$$

such that $\Phi(\mu) =$ . It is obvious that Φ changes the product $\triangleright$ into the product $\blacktriangleright$. It remains to prove that Φ is an isomorphism. We will prove the existence of an inverse $\Psi : \mathbb{K}_B \to \mathrm{NAPNAP'}_B$ of $\Phi : \mathrm{NAPNAP'}_B \to \mathbb{K}_B$ for any finite set B by induction on the cardinal of B. The cases where B has one or two elements are trivial. Suppose the result to be true up to n elements, and let B be of cardinal $n+1$. For any corolla structure β on B, there is $r \in B$ and a partition $B \setminus \{r\} = B_1 \sqcup \cdots \sqcup B_p$ such that

$$\beta = [B_1, \ldots, B_p]_r.$$

We now proceed by a secondary induction on p. If $p = 1$, we have $\beta = [B_1]_r = \beta' \blacktriangleright \{r\}$, where β' is any corolla structure on $B \setminus \{r\}$, and where we identify the one-element set $\{r\}$ with the only corolla structure which exists on it. We set:

$$\Psi(\beta') \triangleright \mathbf{1},$$

$\Phi\Psi(\beta) = \Phi\Psi(\beta') \blacktriangleright \Phi\Psi(\{r\}) = \beta$ by induction hypothesis. For $p \geq 2$ we have

$$\beta = [B_1, \ldots, B_p]_r = \beta_1 \blacktriangleright [B_2, \ldots, B_p]_r,$$

where β_1 is any corolla structure on B_1. We can define by induction hypothesis:

$$\Psi(\beta) := \Psi(\beta_1) \triangleright \Psi([B_2, \ldots, B_p]_r).$$

We have again $\Phi\Psi(\beta) = \beta$ for the same reasons. To make sure that $\Psi(\beta)$ is well defined, one has to prove that the result is invariant under permutation of the p branches. Invariance under permutation of the $p-1$ last ones is obvious by secondary induction hypothesis. To get invariance under permutation of B_1 and B_2, define

$$\Psi'(\beta) := \Psi(\beta_2) \triangleright \Psi([B_1, B_3 \ldots, B_p]_r)$$

where β_2 is any corolla structure on B_2. We then have

$$\begin{aligned}
\Psi(\beta) &= \Psi(\beta_1) \triangleright \Psi([B_2, \ldots, B_p]_r) \\
&= \Psi(\beta_1) \triangleright \big(\Psi(\beta_2) \triangleright \Psi([B_3, \ldots, B_p]_r)\big) \\
&= \Psi(\beta_2) \triangleright \big(\Psi(\beta_1) \triangleright \Psi([B_3, \ldots, B_p]_r)\big) \\
&= \Psi'(\beta).
\end{aligned}$$

Finally we also have $\Psi\Phi = \mathrm{Id}_{\mathrm{NAPNAP'}}$. It is easily proven by induction on arity, using $\Psi(\alpha \blacktriangleright \beta) = \Psi(\alpha) \triangleright \Psi(\beta)$. This completes the proof of Theorem 4.8. □

4.4. **Two-parameter Ω-pre-Lie algebras and the Perm operad.** Now we give the definition of two-parameter Ω-pre-Lie algebras. This requires that the product $\blacktriangleright$ on Ω fulfils the requirements of the fourth case of Paragraph 4.1:

Definition 4.9. Let Ω be a *set-theoretical perm algebra* [7, 6], i.e. a set with a product $\blacktriangleright$ such that

$$\alpha \blacktriangleright (\beta \blacktriangleright \gamma) = (\alpha \blacktriangleright \beta) \blacktriangleright \gamma = \beta \blacktriangleright (\alpha \blacktriangleright \gamma) = (\beta \blacktriangleright \alpha) \blacktriangleright \gamma \tag{38}$$

for any $\alpha, \beta, \gamma \in \Omega$, i.e. we ask that the product $\blacktriangleright$ is both associative and NAP. A **two-parameter Ω-pre-Lie algebra** is a family $(A, (\triangleright_{\alpha,\beta})_{\alpha,\beta\in\Omega})$ where A is a vector space and $\triangleright_{\alpha,\beta} : A \otimes A \to A$, such that for any $x, y, z \in A$ and $\alpha, \beta \in \Omega$, satisfying

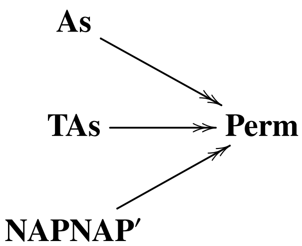
$$x \triangleright_{\alpha,\beta\blacktriangleright\gamma} (y \triangleright_{\beta,\gamma} z) - (x \triangleright_{\alpha,\beta} y) \triangleright_{\alpha\blacktriangleright\beta,\gamma} z = y \triangleright_{\beta,\alpha\blacktriangleright\gamma} (x \triangleright_{\alpha,\gamma} z) - (y \triangleright_{\beta,\alpha} x) \triangleright_{\beta\blacktriangleright\alpha,\gamma} z. \tag{39}$$

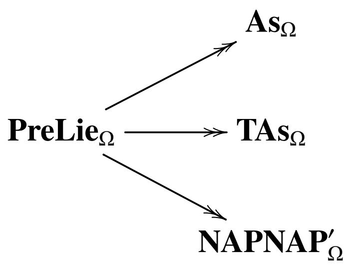
The Perm operad governing relations (38) has been described in [6]. In the species formalism, $\mathrm{Perm}_A := A$ for any finite set A, and the partial compositions are defined as follows: for any finite sets A, B, for any $a, a' \in A$ and $b' \in B$,

$$a' \circ_a b' = \begin{cases} b' & \text{if } a = a'; \\ a' & \text{if } a \neq a'. \end{cases}$$

Note that if $(\Omega, \blacktriangleright)$ is a set-theoretical Perm algebra, then it is also a semigroup, a twisted associative semigroup, and a set-theoretical NAPNAP' algebra. On the other hand, commutative semigroups are set-theoretical Perm algebras, and Relation (39) is the one verified by Ω-relative pre-Lie algebras [2, Paragraph 3.7] in this case. Hence, we can consider the operad $\mathbf{As}_\Omega$ of family associative algebras on Ω as in Case 1, $\mathbf{TAs}_\Omega$ of family twisted associative algebras on Ω defined by (27) as in Case 2, $\mathbf{NAPNAP}'_\Omega$ of family NAPNAP' algebras on Ω as in Case 3, and $\mathbf{PreLie}_\Omega$ of family pre-Lie algebras on Ω as in Case 4. Then, in an immediate way:

Proposition 4.10. *For any set-theoretical Perm algebra Ω, we have the two following diagrams.*





The four operads in the first diagram are set operads, and all arrows are surjective.

5. Color-mixing operads and family algebraic structures

We follow the lines of [2, Section 2], except that we consider gradings taking values in an arbitrary set Ω rather than in a semigroup.

5.1. Color-mixing operads.

Let Ω be a set of colors and $\mathcal{C}$ be a bicomplete monoidal category. Keeping the notations of Paragraphs 3.2 and 3.3, for any operad $\mathcal{P}$, all components $\mathcal{P}^{\Omega}_{A,\underline{\alpha},\omega}$ of the colored operad $\mathcal{P}^\Omega$ are isomorphic once the finite set A of inputs is fixed. This clearly contradicts the principle outlined in the introducting paragraph of this section, according to which the graded object $\mathcal{V} = (V_\omega)_{\omega\in\Omega}$ should be not only a $\mathcal{P}^\Omega$-algebra, but also a "graded $\mathcal{P}$-algebra" in some sense. This means that the output color ω should be a combination of the input colors $\underline{\alpha}$ in a way prescribed by the operad $\mathcal{P}$.

From now on, we consider linear colored operads, i.e. we stick to the case when $\mathcal{C}$ is the category of vector spaces over some field $\mathbf{k}$. The coproduct is now given by the usual direct sum $\oplus$. Guided by the dendriform and the pre-Lie examples detailed in the previous sections, we see that the color set Ω will be endowed with a $\textcircled{$\mathcal{P}$}$ -algebra structure, where $\textcircled{$\mathcal{P}$}$ is a set operad derived from the linear operad $\mathcal{P}$. For example, if $\mathcal{P}$ is the dendriform operad, $\textcircled{$\mathcal{P}$}$ is the the diassociative operad, and if $\mathcal{P}$ is the pre-Lie operad, $\textcircled{$\mathcal{P}$}$ is the Perm operad. When $\mathcal{P}$ is the linearization of a set operad $\mathbf{P}$, we should get $\textcircled{$\mathcal{P}$} = \mathbf{P}$, as the duplicial example suggests.

We suppose that $\mathcal{P}$ is of finite presentation, i.e. it can be written as

$$\mathcal{P} = \mathcal{M}_E \big/ \mathcal{R}, \tag{40}$$

where E is a set species of generators, $\mathbf{M}_E$ is the free set operad generated by E, and $\mathcal{R}$ is the operadic ideal of the linear operad $\mathcal{M}_E = \mathbf{k}.\mathbf{M}_E$ generated by a finite linearly independent collection $\mu^1, \ldots, \mu^N$ of elements. Each of these elements can be written as

$$\mu^i = \sum_{j=1}^{k_i} \lambda^i_j \mu^i_j,$$

where $(\mu^i_j)_j$ is a linearly independent collection of monomial expressions involving elements of E and partial compositions, and $\lambda^i_j \in \mathbf{k} - \{0\}$.

Definition 5.1. The **set operadic equivalence relation generated by** $\mathcal{R}$ is the finest equivalence relation Ⓡ on $\mathbf{M}_E$, compatible with the set operad structure, such that

$$\mu^i_p \text{Ⓡ} \mu^i_q \text{ for any } i \in \{1, \ldots, N\} \text{ and } p, q \in \{1, \ldots, k_i\}.$$

The **set operad associated to** $\mathcal{P}$ is the set operad

$$\text{Ⓟ} := \mathbf{M}_E \big/ \text{Ⓡ} .$$

Remark 5.2. When $\mathcal{P}$ is given by the linearization of a set operad $\mathbf{P}$, we have Ⓟ $= \mathbf{P}$.

Remark 5.3. The set operad Ⓟ depends on the presentation chosen for the linear operad $\mathcal{P}$. To see this, consider the free operad generated by three binary products μ, ν, ρ, and its quotient by the ideal generated by $\mu + \nu - \rho$, which is isomorphic to the free operad generated by μ and ν.

Remark 5.4. Let $\mathbf{Q}$ be a quadratic set operad, and let $\mathcal{P}$ be the Koszul dual [15] of its linearization. If $E = \mathbf{Q}_2$, the free set-operad $\mathbf{M}_E$ generated by E is combinatorially represented by binary trees with labeled leaves, and vertices decorated by elements of E. Let $\sim$ be the equivalence on $\mathbf{M}_E(3)$ such that $\mathbf{M}_E(3)/\sim\; = \mathbf{Q}(3)$. By definition of the Koszul dual, $\mathcal{P}$ is generated by E, and the relations

$$\sum_{T \in C} \pm T = 0$$

hold, where C is a class of $\sim$ and the signs $\pm$ depend only of the form of the tree. Applying Definition 5.1, we obtain that Ⓡ $=\sim$, so in this case, Ⓟ $= \mathbf{Q}$. This holds for example if $\mathbf{Q}$ is the associative, or permutative, or diassociative operad: then $\mathcal{P}$ is the operad of, respectively, associative, or pre-Lie, or dendriform algebras, with their usual presentations.

Proposition 5.5. *Let Ω be a set endowed with a Ⓟ-algebra structure. Considering the $\mathbf{M}_E$-algebra structure on Ω given by the operad morphism* ⓟ $: \mathbf{M}_E \twoheadrightarrow$ Ⓟ,

(a) *The colored subspecies $\widetilde{\mathbf{M}}^\Omega_E$ of $\mathbf{M}^\Omega_E$ defined by*

$$(\widetilde{\mathbf{M}}^\Omega_E)_{A,\underline{\alpha},\omega} := \{\mu \in (M_E)_A,\ \mu(\underline{\alpha}) = \omega\}$$

is a set colored suboperad of $\mathbf{M}^\Omega_E$.

(b) *The colored subspecies $\widetilde{\mathcal{M}}^\Omega_E$ of $\mathcal{M}^\Omega_E$ defined by*

$$(\widetilde{\mathcal{M}}^\Omega_E)_{A,\underline{\alpha},\omega} := \mathbf{k}.(\widetilde{\mathbf{M}}^\Omega_E)_{A,\underline{\alpha},\omega}$$

is a linear colored suboperad of $\mathcal{M}^\Omega_E$.

Proof. Let $(A, \underline{\alpha}, \omega)$ and $(B, \underline{\beta}, \zeta)$ be two Ω-colored finite sets. Let $\mu \in (\widetilde{\mathbf{M}}^{\Omega}_E)_{A,\underline{\alpha},\omega}$ and $\nu \in (\widetilde{\mathbf{M}}^{\Omega}_E)_{B,\underline{\beta},\zeta}$. We have then by definition of $\widetilde{\mathbf{M}}^{\Omega}_E$,

$$\mu(\underline{\alpha}) = \omega \quad \text{and} \quad \nu(\underline{\beta}) = \zeta.$$

Now let $a \in A$. The partial composition $\mu \circ_a \nu$ is defined in the colored operad $\mathbf{M}^{\Omega}_E$ if and only if $\zeta = \underline{\alpha}(a)$. In that case we obviously have

$$\omega = \mu(\underline{\alpha}) = (\mu \circ_a \nu)(\underline{\alpha} \sqcup_a \underline{\beta}),$$

hence $\mu \circ_a \nu \in (\widetilde{\mathbf{M}}^{\Omega}_E)_{A \sqcup_a B,\, \underline{\alpha} \sqcup_a \underline{\beta},\, \omega}$. The second assertion is an immediate consequence of the first. □

The following corollary is immediate:

Corollary 5.6. *Let π be the projection from the free linear operad $\mathcal{M}_E$ onto $\mathcal{P}$, and let $\widetilde{\pi}$ be the projection from $\mathcal{M}^{\Omega}_E$ onto $\mathcal{P}^{\Omega}$. Suppose that Ω is a* Ⓟ *-algebra. Then the colored subspecies $\widetilde{\mathcal{P}}^{\Omega} := \widetilde{\pi}(\widetilde{\mathcal{M}}^{\Omega}_E)$ of $\mathcal{P}^{\Omega}$ is a linear colored suboperad of $\mathcal{P}^{\Omega}$.*

Definition 5.7. The colored operad $\widetilde{\mathcal{P}}^{\Omega}$ is the **color-mixing operad** associated to the operad $\mathcal{P}$. It does depend on the presentation $\mathcal{P} = \mathcal{M}_E/\mathcal{R}$, and supposes a Ⓟ -algebra structure on Ω.

Remark 5.8. The notion of color-mixing operad was already approached in the case when Ω is a commutative semigroup : an Ω-colored operad in which the output color is the sum of the input colors was given the name *current-preserving operad* in [29].

5.2. **Graded algebras over a color-mixing operad and family structures.** Let Ω be a set, let $\mathbf{k}$ be a field, and let $\mathcal{P}$ be an operad in the category of $\mathbf{k}$-vector spaces. We keep the notations of the previous paragraphs, and in particular we fix a finite presentation $\mathcal{P} = \mathcal{M}_E/\mathcal{R}$.

Definition 5.9. An **Ω-graded $\mathcal{P}$-algebra** is an algebra over the Ω-colored operad $\widetilde{\mathcal{P}}^{\Omega}$, i.e. an Ω-graded $\mathbf{k}$-vector space $\mathcal{V}$ together with a morphism of colored operads $\widetilde{\mathcal{P}}^{\Omega} \to \mathrm{End}(\mathcal{V})$.

Let us remark that the notion of Ω-graded $\mathcal{P}$-algebra depends on the presentation of the operad $\mathcal{P}$. We are now ready to define Ω-family $\mathcal{P}$-algebras, which could also be also called Ω-relative $\mathcal{P}$-algebras in M. Aguiar's terminology [2, Definition 14]:

Definition 5.10. An **Ω-family $\mathcal{P}$-algebra** is an Ω-graded $\mathcal{P}$-algebra for which the underlying Ω-graded object is uniform.

Again, this notion depends on the presentation of $\mathcal{P}$.

Proposition 5.11. *Any Ω-family $\mathcal{P}$-algebra is an Ω-graded vector space $\mathcal{U}(V)$, where V is an algebra over the operad $\mathcal{F}(\widetilde{\mathcal{P}}^{\Omega})$.*

Proof. By definition, an Ω-family $\mathcal{P}$-algebra is given by a vector space V and a colored operad morphism

$$\Phi : \widetilde{\mathcal{P}}^{\Omega} \longrightarrow \mathrm{End}(\mathcal{U}(V)).$$

We have $\mathrm{End}(\mathcal{U}(V)) = \mathcal{U}(\mathrm{End}V)$ by Equation (23). The functor $\mathcal{U}$ of the left- (resp. right-) hand side is defined in Paragraph 3.3 (resp. 3.2). The functor $\mathcal{U}$ is right-adjoint to the forgetful functor $\mathcal{F}$ defined in Paragraph 3.2, hence there is a morphism of ordinary operads from $\mathcal{F}(\widetilde{\mathcal{P}}^{\Omega})$ to $\mathrm{End}V$. □

Finally, we recover the close link between algebras and family algebras which was already observed on the known examples, and established by M. Aguiar in the case when Ω is a semigroup [2, Paragraph 2.4]:

Proposition 5.12. *A uniform Ω-graded vector space $\mathcal{V} = \mathcal{U}(V)$ is an Ω-family $\mathcal{P}$-algebra if and only if the vector space $\mathcal{F}\mathcal{U}(V) = V \otimes \mathbf{k}\Omega$ is an Ω-graded $\mathcal{P}$-algebra.*

Proof. From Proposition 5.11, the vector space V is a $\mathcal{F}(\widetilde{\mathcal{P}^\Omega})$-algebra. It means that for any finite set A, any element ν of $\mathcal{P}_A$ gives rise to a collection $(\nu)_{\underline{\alpha},\omega}$ of A-ary products on V, indexed by the pairs $(\underline{\alpha}, \omega) \in \Omega^A \times \Omega$ such that there exists $\mu \in Ⓟ_A$ verifying $\mu(\underline{\alpha}) = \omega$. Due to the fact that there is only a finite number of such pairs for any given $\underline{\alpha}$, this collection gives rise to a unique A-ary product on $V \otimes \mathbf{k}\Omega$, which is Ω-graded with respect to the Ⓟ -algebra structure of Ω. The converse is obvious. □

6. Free one-parameter Ω-duplicial algebras

We give a general definition of one-parameter Ω-duplicial algebras in the spirit of [12], and we give the detailed construction of the free object in terms of Ω-typed trees.

Definition 6.1. An extended duplicial semigroup (briefly, EDuS) is a family $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$, where Ω is a set and $\leftarrow, \rightarrow, \triangleleft, \triangleright : \Omega \times \Omega \to \Omega$ are maps such that:

(a) $(\Omega, \leftarrow, \rightarrow)$ is a duplicial semigroup.
(b) For any $\alpha, \beta, \gamma \in \Omega$,

$$\alpha \triangleright (\beta \leftarrow \gamma) = \alpha \triangleright \beta, \tag{41}$$
$$(\alpha \rightarrow \beta) \triangleleft \gamma = \beta \triangleleft \gamma, \tag{42}$$
$$(\alpha \triangleleft \beta) \leftarrow \big((\alpha \leftarrow \beta) \triangleleft \gamma\big) = \alpha \triangleleft (\beta \leftarrow \gamma), \tag{43}$$
$$(\alpha \triangleleft \beta) \triangleleft \big((\alpha \leftarrow \beta) \triangleleft \gamma\big) = \beta \triangleleft \gamma, \tag{44}$$
$$\big(\alpha \triangleright (\beta \rightarrow \gamma)\big) \triangleright (\beta \triangleright \gamma) = \alpha \triangleright \beta, \tag{45}$$
$$\big(\alpha \triangleright (\beta \rightarrow \gamma)\big) \rightarrow (\beta \triangleright \gamma) = (\alpha \rightarrow \beta) \triangleright \gamma. \tag{46}$$

Remark 6.2. Any extended diassociative semigroup is an extended duplicial semigroup, but among the 10 axioms describing the compatibility between the arrows and the triangles in an extended diassociative semigroup (numbers 4 to 13 in [12]), only six of them survive in an EDuS (numbers 4, 5, 6, 7, 12 and 13).

Definition 6.3. Let $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$ be an EDuS. A one-parameter Ω-duplicial algebra is a family $(A, (\prec_\alpha, \succ_\alpha)_{\alpha\in\Omega})$, where A is a vector space and $\prec_\alpha, \succ_\alpha$: $A \otimes A \to A$ such that for any $x, y, z \in A$ and $\alpha, \beta \in \Omega$,

$$(x \prec_\alpha y) \prec_\beta z = x \prec_{\alpha\leftarrow\beta} (y \prec_{\alpha\triangleleft\beta} z), \tag{47}$$
$$x \succ_\alpha (y \prec_\beta z) = (x \succ_\alpha y) \prec_\beta z, \tag{48}$$
$$x \succ_\alpha (y \succ_\beta z) = (x \succ_{\alpha\triangleright\beta} y) \succ_{\alpha\rightarrow\beta} z. \tag{49}$$

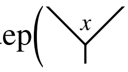
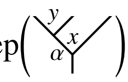
For any set Ω, in the spirit of [12], an Ω-matching duplicial algebra is a one-parameter Ω-duplicial algebra in the sense above, where the EDuS structure on Ω is given by

$$\alpha \leftarrow \beta = \alpha, \qquad \alpha \leftarrow \beta = \beta, \qquad \alpha \triangleleft \beta = \beta, \qquad \alpha \triangleright \beta = \alpha.$$

Now we describe free one-parameter Ω-duplicial algebras in terms of planar binary trees typed by Ω, that is, for which each internal edge is typed by single element of Ω. The set of Ω-typed X-decorated planar binary trees is denoted by $\mathbf{T}(X,\Omega)$. We denote by $\mathbf{T}^+(X,\Omega)$ the set of such trees different from the trivial tree $|$. For any $n \geq 0$, the set of trees with n internal vertices (and $n+1$ leaves) is denoted by $\mathbf{T}_n(X,\Omega)$. So we have

$$\mathbf{T}(X,\Omega) = \bigsqcup_{n\geqslant 0} \mathbf{T}_n(X,\Omega), \qquad \mathbf{T}^+(X,\Omega) = \bigsqcup_{n\geqslant 1} \mathbf{T}_n(X,\Omega).$$

The **depth** $\mathrm{dep}(T)$ of a rooted tree T is the maximal length of linear chains of vertices from the root to the leaves of the tree. For example,

dep() = 1 and dep() = 2.

Definition 6.4. Let $T_1, T_2 \in \mathbf{T}(X,\Omega)$, $x \in X$ and $\alpha, \beta \in \Omega$. We denote by $T_1 \vee_{x,(\alpha,\beta)} T_2$ the tree $T \in \mathbf{T}(X,\Omega)$ obtained by grafting T_1 and T_2 on a common root decorated by x. If $T_1 \neq |$, the type of the internal edge between the root of T and the root of T_1 is α. If $T_2 \neq |$, the type of internal edge between the root of T and the root of T_2 is β.

Remark 6.5. Note that any element $T \in \mathbf{T}_n(X,\Omega)$, with $n \geq 1$, can be written under the form

$$T = T_1 \vee_{x,(\alpha,\beta)} T_2,$$

with $T_1, T_2 \in \mathbf{T}(X,\Omega)$, $x \in X$ and $\alpha, \beta \in \Omega$. This writing is unique except if $T_1 =|$ or $T_2 =|$: in this case, one can change arbitrarily α or β. In order to solve this notational problem, we add an element denoted by 1 to Ω and we shall always assume that if $T_1 =|$, then $\alpha = 1$; if $T_2 =|$, then $\beta = 1$.

Definition 6.6. Let Ω be a set with four products $\leftarrow, \rightarrow, \lhd, \rhd$. We define binary operations $(\prec_\alpha, \succ_\alpha)_{\alpha\in\Omega}$ on $\mathbf{kT}^+(X,\Omega)$ recursively on $\mathrm{dep}(T) + \mathrm{dep}(U)$ by

(a) $| \prec_\omega T := T \succ_\omega | := T$ for $\omega \in \Omega$ and $T \in \mathbf{T}^+(X,\Omega)$.

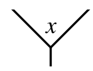
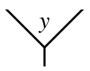
(b) For $T = T_1 \vee_{x,(\alpha_1,\alpha_2)} T_2$ and $U = U_1 \vee_{y,(\beta_1,\beta_2)} U_2$, define

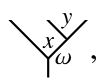
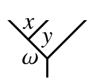
$$T \prec_\omega U := T_1 \vee_{x,(\alpha_1,\alpha_2\leftarrow\omega)} (T_2 \prec_{\alpha_2\lhd\omega} U), \tag{50}$$

$$T \succ_\omega U := (T \succ_{\omega\rhd\beta_1} U_1) \vee_{y,(\omega\rightarrow\beta_1,\beta_2)} U_2, \text{ where } \omega \in \Omega. \tag{51}$$

In the following, we employ the convention that

$$\omega \rhd 1 = 1 \lhd \omega = \omega \text{ and } \omega \rightarrow 1 = 1 \leftarrow \omega = \omega \text{ for } \omega \in \Omega. \tag{52}$$

Example 6.7. Let $T =$  and $U =$  with $x, y \in X$. For $\omega \in \Omega$, we have

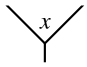
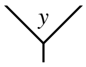
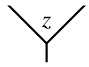
$T \prec_\omega U =$  , $\qquad T \succ_\omega U =$  .

Proposition 6.8. *Let X be a set and $(\Omega, \leftarrow, \rightarrow, \lhd, \rhd)$ be an EDuS. Then $(\mathbf{kT}^+(X,\Omega), (\prec_\omega, \succ_\omega)_{\omega\in\Omega})$ is an Ω-duplicial algebra.*

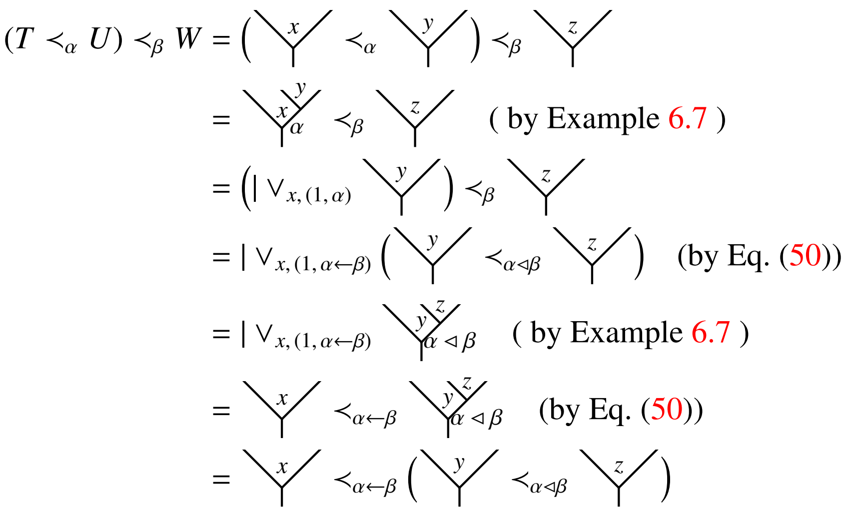
Proof. Let

$$T = T_1 \vee_{x,(\alpha_1,\alpha_2)} T_2, U = U_1 \vee_{y,(\beta_1,\beta_2)} U_2, W = W_1 \vee_{z,(\gamma_1,\gamma_2)} W_2 \in \mathbf{kT}^+(X,\Omega).$$

Then we apply induction on $\mathrm{dep}(T)+\mathrm{dep}(U)+\mathrm{dep}(W) \geq 3$. For the initial step $\mathrm{dep}(T)+\mathrm{dep}(U)+\mathrm{dep}(W) = 3$, we have

$T =$  , $U =$  and $W =$ 

and so



$$= T \prec_{\alpha\leftarrow\beta} (U \prec_{\alpha\triangleleft\beta} W),$$

verifying Eq. (47). Checking (48) and (49) is similar and left to the reader.

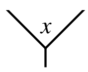
For the induction step, suppose $\mathrm{dep}(T) + \mathrm{dep}(U) + \mathrm{dep}(W) = k + 1 \geq 4$. First, we have

$$\begin{aligned}
(T \prec_\alpha U) \prec_\beta W &= \Big((T_1 \vee_{x,(\alpha_1,\alpha_2)} T_2) \prec_\alpha U\Big) \prec_\beta W\\
&= \Big(T_1 \vee_{x,(\alpha_1,\alpha_2\leftarrow\alpha))} (T_2 \prec_{\alpha_2\triangleleft\alpha} U)\Big) \prec_\beta W \quad \text{(by Eq. (50))}\\
&= T_1 \vee_{x,(\alpha_1,(\alpha_2\leftarrow\alpha)\leftarrow\beta)} \Big((T_2 \prec_{\alpha_2\triangleleft\alpha} U) \prec_{(\alpha_2\leftarrow\alpha)\triangleleft\beta} W\Big) \quad \text{(by Eq. (50))}\\
&= T_1 \vee_{x,(\alpha_1,(\alpha_2\leftarrow\alpha)\leftarrow\beta)} \Big(T_2 \prec_{(\alpha_2\triangleleft\alpha)\leftarrow\big((\alpha_2\leftarrow\alpha)\triangleleft\beta\big)} (U \prec_{(\alpha_2\triangleleft\alpha)\triangleleft\big((\alpha_2\leftarrow\alpha)\triangleleft\beta\big)} W)\\
&\qquad \text{(by the induction hypothesis)}\\
&= T_1 \vee_{x,(\alpha_1,\alpha_2\leftarrow(\alpha\leftarrow\beta))} \Big(T_2 \prec_{\alpha_2\triangleleft(\alpha\leftarrow\beta)} (U \prec_{\alpha\triangleleft\beta} W)\Big)\\
&\qquad \text{(by Eqs. (43-44) and definition (2.6))}\\
&= (T_1 \vee_{x,(\alpha_1,\alpha_2)} T_2) \prec_{\alpha\leftarrow\beta} (U \prec_{\alpha\triangleleft\beta} W) \quad \text{(by Eq. (50))}\\
&= T \prec_{\alpha\leftarrow\beta} (U \prec_{\alpha\triangleleft\beta} W)
\end{aligned}$$

thus checking (47). Second, we have

$$\begin{aligned}
T \succ_\alpha (U \prec_\beta W) &= T \succ_\alpha \Big((U_1 \vee_{y,(\beta_1,\beta_2)} U_2) \prec_\beta W\Big)\\
&= T \succ_\alpha \Big(U_1 \vee_{y,(\beta_1,\beta_2\leftarrow\beta)} (U_2 \prec_{\beta_2\triangleleft\beta} W)\Big) \quad \text{(by Eq. (51))}\\
&= (T \succ_{\alpha\triangleright\beta_1} U_1) \vee_{y,(\alpha\rightarrow\beta_1,\beta_2\leftarrow\beta)} (U_2 \prec_{\beta_2\triangleleft\beta} W)\\
&= \Big((T \succ_{\alpha\triangleright\beta_1} U_1) \vee_{y,(\alpha\rightarrow\beta_1,\beta_2)} U_2\Big) \prec_\beta W \quad \text{(by Eq. (50))}\\
&= \Big(T \succ_\alpha (U_1 \vee_{y,(\beta_1,\beta_2)} U_2)\Big) \prec_\beta W\\
&= (T \succ_\alpha U) \prec_\beta W,
\end{aligned}$$

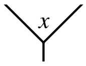
thus checking (48). Last, checking (49) is similar to checking (47) and left to the reader. □

Let $j : X \to \mathbf{k}\mathbf{T}^+(X,\Omega)$ be the map defined by $j(x) =$  for $x \in X$.

Theorem 6.9. *Let X be a set and let $(\Omega, \leftarrow, \rightarrow, \lhd, \rhd)$ be an EDuS. Then $(\mathbf{k}\mathbf{T}^{+}(X,\Omega), (\prec_\omega, \succ_\omega)_{\omega\in\Omega})$, together with the map j, is the free Ω-duplicial algebra on X.*

Proof. By Proposition 2.11, we are left to show that $(\mathbf{k}\mathbf{T}^{+}(X,\Omega), (\prec_\omega, \succ_\omega)_{\omega\in\Omega})$ satisfies the universal freeness property. For this, let $(D, (\prec'_\omega, \succ'_\omega)_{\omega\in\Omega})$ be an Ω-duplicial algebra and let us define a linear map

$$\bar{f} : \begin{cases} \mathbf{k}\mathbf{T}^{+}(X,\Omega) & \rightarrow & D \\ T & \mapsto & \bar{f}(T) \end{cases}$$

by induction on $\operatorname{dep}(T) \geq 1$. Let us write $T = T_1 \vee_{x,(\alpha_1,\alpha_2)} T_2$ with $x \in X$ and $\alpha_1, \alpha_2 \in \Omega$. For the initial step $\operatorname{dep}(T) = 1$, we have $T =$  for some $x \in X$ and define

$$\bar{f}(T) := f(x). \tag{53}$$

We define $\bar{f}(T)$ by the induction on $\operatorname{dep}(T) = k+1 \geq 2$. Note that T_1 and T_2 can not be $|$ simultaneously and define

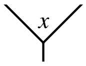
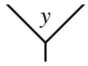
$$\bar{f}(T) := \bar{f}(T_1 \vee_{x,(\alpha_1,\alpha_2)} T_2) := \begin{cases} f(x) \prec'_{\alpha_2} \bar{f}(T_2), & \text{if } T_1 = | \neq T_2; \\ \bar{f}(T_1) \succ'_{\alpha_1} f(x), & \text{if } T_1 \neq | = T_2; \\ (\bar{f}(T_1) \succ'_{\alpha_1} f(x)) \prec'_{\alpha_2} \bar{f}(T_2), & \text{if } T_1 \neq | \neq T_2. \end{cases} \tag{54}$$

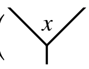
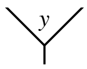
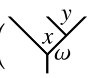
We are left to prove that $\bar{f}$ is a morphism of Ω-duplicial algebras:

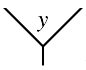
$$\bar{f}(T \prec_\omega U) = \bar{f}(T) \prec'_\omega \bar{f}(U) \ \text{ and } \ \bar{f}(T \succ_\omega U) = \bar{f}(T) \succ'_\omega \bar{f}(U),$$

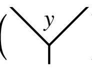
in which we only prove the first equation by induction on $\operatorname{dep}(T) + \operatorname{dep}(U) \geq 2$, as the proof of the second one is similar. Write

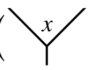
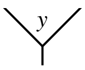
$$T = T_1 \vee_{x,(\alpha_1,\alpha_2)} T_2 \ \text{ and } \ U = U_1 \vee_{y,(\beta_1,\beta_2)} U_2.$$

For the initial step $\operatorname{dep}(T) + \operatorname{dep}(U) = 2$, we have $T =$  and $U =$  for some $x, y \in X$. So we have

$\bar{f}(T \prec_\omega U) = \bar{f}\big($  $\prec_\omega$  $\big) = \bar{f}\big($  $\big)$ (by Example 6.7)

$= \bar{f}\big(| \vee_{x,(1,\omega)}$  $\big)$

$= f(x) \prec'_\omega \bar{f}\big($  $\big)$ (by Eq. (54))

$= \bar{f}\big($  $\big) \prec'_\omega \bar{f}\big($  $\big)$ (by Eq. (53))

$= \bar{f}(T) \prec'_\omega \bar{f}(U).$

For the induction step of $\operatorname{dep}(T) + \operatorname{dep}(U) \geq 3$, we have four cases to consider.

Case 1: $T_1 \neq |$ and $T_2 \neq |$. Then

$$\begin{aligned} \bar{f}(T \prec_\omega U) &= \bar{f}\big((T_1 \vee_{x,(\alpha_1,\alpha_2)} T_2) \prec_\omega U\big) \\ &= \bar{f}\big(T_1 \vee_{x,(\alpha_1,\alpha_2\leftarrow\omega)} (T_2 \prec_{\alpha_2\lhd\omega} U)\big) \quad \text{(by Eq. (50))} \\ &= \big(\bar{f}(T_1) \succ'_{\alpha_1} f(x)\big) \prec'_{\alpha_2\leftarrow\omega} \bar{f}(T_2 \prec_{\alpha_2\lhd\omega} U) \quad \text{(by Eq. (54))} \\ &= \big(\bar{f}(T_1) \succ'_{\alpha_1} f(x)\big) \prec'_{\alpha_2\leftarrow\omega} \big(\bar{f}(T_1) \prec'_{\alpha_2\lhd\omega} \bar{f}(U)\big) \quad \text{(by the induction hypothesis)} \end{aligned}$$

$$
\begin{aligned}
&= \bar f(T_1) \succ'_{\alpha_1} \Big(f(x) \prec'_{\alpha_2 \leftarrow \omega} (\bar f(T_2) \prec'_{\alpha_2 \triangleleft \omega} \bar f(U))\Big) \quad \text{(by Eq. (48))}\\
&= \bar f(T_1) \succ'_{\alpha_1} \Big((f(x) \prec'_{\alpha_2} \bar f(T_2)) \prec'_{\omega} \bar f(U)\Big) \quad \text{(by Eq. (47))}\\
&= \Big(\bar f(T_1) \succ'_{\alpha_1} (f(x) \prec'_{\alpha_2} \bar f(T_2)\Big) \prec'_{\omega} \bar f(U) \quad \text{(by Eq. (48))}\\
&= \Big(\big(\bar f(T_1) \succ'_{\alpha_1} f(x)\big) \prec'_{\alpha_2} \bar f(T_2)\Big) \prec'_{\omega} \bar f(U) \quad \text{(by Eq. (48))}\\
&= \bar f(T) \prec'_{\omega} \bar f(U) \quad \text{(by Eq. (54))}.
\end{aligned}
$$

The three other cases, when at least one of the components T_1 or T_2 is equal to $|$, are simpler and left to the reader.

Let us prove the uniqueness of $\bar f$. Let $\bar g$ be another morphism from $\mathbf{kT}^+(X,\Omega)$ to D such that $\bar g\Big(\stackrel{x}{\curlyvee}\Big) = f(x)$. First, for any $a \in D$ and $\omega \in \Omega$, we define

$$a \succ_\omega 1 = 1 \prec_\omega a = 0, \qquad\qquad 1 \succ_\omega a = a \prec_\omega 1 = a.$$

For any $T \neq |$, let $T = T_1 \vee_{x,(\alpha_1,\alpha_2)} T_2$. In fact, the form $T = T_1 \succ_{\alpha_1} \stackrel{x}{\curlyvee} \prec_{\alpha_2} T_2$ include all the above four cases. We define

$$\bar g(|) = 1$$

and

$$\bar g(T) = \bar g\Big(T_1 \succ_{\alpha_1} \stackrel{x}{\curlyvee} \prec_{\alpha_2} T_2\Big) = \bar g(T_1) \succ'_{\alpha_1} f(x) \prec'_{\alpha_2} \bar g(T_2).$$

So $\bar f = \bar g$. This completes the proof. □

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