Generalized associative algebras

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Abstract

We study diverse parametrized versions of the operad of associative algebra, where the parameter are taken in an associative semigroup Ω (generalization of matching or family associative algebras) or in its cartesian square (two-parameters associative algebras). We give a description of the free algebras on these operads, study their formal series and prove that they are Koszul when the set of parameters is finite. We also study operadic morphisms between the operad of classical associative algebras and these objects, and links with other types of algebras (diassociative, dendriform, post-Lie...).

Keywords. Family associative algebras, matching associative algebras, two-parameters associative algebras, associative semigroups.

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Contents

Introduction

Recently, numerous parametrization of well-known operads were introduced. Choosing a set of parameters Ω, any product defining the considered operad is replaced by a bunch of products indexed by Ω and various relations are defined on them, mimicking the relations defining the initial operads. One can for example require that any linear span of the parametrized products also satisfy the relations of the initial operads: this is the *matching* parametrization. For example, matching Rota-Baxter algebras, associative, dendriform, prelie algebras are introduced in [33, 13]. Another way is to use one or more semigroup structures on Ω: this it the *family* parametrization. In this spirit, family Rota-Baxter algebras, dendriform, prelie algebras are introduced and studied in [1, 34, 35, 24]. A way to obtain both these parametrizations for dendriform algebras is introduced in [16], with the help of a generalization of diassociative semigroups, called extended diassociative semigroups (briefly, EDS). Finally, a two-parameters version is given for dendriform algebras and prelie algebras is described in [17].

Our aim in this paper is the study of these parametrizations for the operad of associative algebras, which surprisingly did not receive a lot of attention for now. We start with two-parameters associative algebras [17]. If $(\Omega, \rightarrow)$ is a semigroup, an Ω-two-parameters associative algebra is given products $*_{\alpha,\beta}$, with $\alpha, \beta \in \Omega$, satisfying the following axiom:

$$(x *_{\alpha,\beta} y) *_{\alpha\rightarrow\beta,\gamma} z = x *_{\alpha,\beta\rightarrow\gamma} (y *_{\beta,\gamma} z).$$

When $(\Omega, *) = (\mathbb{Z}/2\mathbb{Z}, \times)$, the two-parameters Ω-associative algebras were described in [4], as an operad on bicolored trees. When Ω is finite, the associated operad $\mathbf{As}^2_\Omega$ is finitely generated and quadratic. We prove that it is Koszul (Proposition 1.3), and describe its Poincaré-Hilbert formal series $P(X)$: if $|\Omega| = \omega > 1$, then

$$\begin{aligned}P(X) &= \frac{1-\omega X - \sqrt{1+2\omega(1-2\omega)X+\omega^2X^2}}{2\omega(\omega-1)}\\ &= X + \omega^2X^2 + (2\omega-1)\omega^3X^3 + (5\omega^2-5\omega+1)\omega^4X^4\\ &+ (2\omega-1)(7\omega^2-7\omega+1)\omega^5X^5 + (42\omega^4-84\omega^3+56\omega^2-14\omega+1)\omega^6X^6 + \dots\end{aligned}$$

We deduce a formula for the dimension $p_n(\omega)$ of $\mathbf{As}^2_\Omega(n)$ with the help of Narayana numbers, (Corollary 1.5), as well as properties of $p_n(\omega)$, seen as a polynomial in ω (Corollary 1.4). We also give a combinatorial description of Koszul dual of $\mathbf{As}^2_\Omega$ in terms of words (Proposition 1.7) and a description of free ${\mathbf{As}^2_\Omega}^!$-algebras.

In the second and third parts of this paper, we introduce and study Ω-associative algebras. Here, the set of parameters Ω is given two operations $\rightarrow$ and $\triangleright$. An Ω-associative algebra is given bilinear products $*_\alpha$, with $\alpha \in \Omega$, with the following axioms:

$$x *_\alpha (y *_\beta z) = (x *_{\alpha\triangleright\beta} y) *_{\alpha\rightarrow\beta} z.$$

In order to have a suitable parametrised operad, we impose that free Ω-associative algebras are of the form

$$T_A(V) = \bigoplus_{n=1}^{\infty} \mathbb{K}\Omega^{\otimes(n-1)} \otimes V^{\otimes n}.$$

Tensors of $\mathbb{K}\Omega^{\otimes(n-1)} \otimes V^{\otimes n}$ will be called A-typed words of length n and will be denoted $\alpha_1 \dots \alpha_{n-1} v_1 \dots v_n$. We impose that the products $*_\alpha$ satisfy, among other conditions, that for any $v_1, v_2 \in V$,

$$v_1 *_\alpha v_2 = \alpha v_1 v_2.$$

We prove in Theorem 3.5 that this holds if, and only if, the triple $(\Omega, \rightarrow, \triangleright)$ satisfies the following axioms:

$$\begin{aligned} \alpha \rightarrow (\beta \rightarrow \gamma) &= (\alpha \rightarrow \beta) \rightarrow \gamma, \\ (\alpha \triangleright (\beta \rightarrow \gamma)) \rightarrow (\beta \triangleright \gamma) &= (\alpha \rightarrow \beta) \triangleright \gamma, \\ (\alpha \triangleright (\beta \rightarrow \gamma)) \triangleright (\beta \triangleright \gamma) &= \alpha \triangleright \beta. \end{aligned}$$

Such a triple $(\Omega, \rightarrow, \triangleright)$ will be called an extended associative semigroup (briefly, EAS). For example:

- If Ω is a set, its trivial EAS structure is given, for any $\alpha, \beta \in \Omega$,

$$\alpha \rightarrow \beta = \beta \triangleright \alpha = \beta.$$

  In this case, the Ω-associative algebras are the matching associative algebras [33]; the particular case when Ω contains two elements appears also in [29]. The underlying operads are also used in [6].

- If $(\Omega, \rightarrow)$ is a semigroup, one can make it an EAS with, for any $\alpha, \beta \in \Omega$,

$$\alpha \triangleright \beta = \alpha.$$

  In this case, the Ω-associative algebras are the family associative algebras of [34].

- If $(\Omega, \star)$ is a group, one can make it an EAS with, for any $\alpha, \beta \in \Omega$,

$$\alpha \rightarrow \beta = \beta, \qquad \alpha \triangleright \beta = \alpha \star \beta^{\star -1}.$$

We give more examples of EAS, including a classification of EAS of cardinality two, in the second section. We in fact generalize these results in a linear setting: we first observe that if $(\Omega, \rightarrow, \triangleright)$ is a set with two operations, we consider the map

$$\phi : \left\{ \begin{array}{rcl} \Omega^2 & \longrightarrow & \Omega^2 \\ (\alpha, \beta) & \longrightarrow & (\alpha \rightarrow \beta, \alpha \triangleright \beta), \end{array} \right.$$

Then $(\Omega, \rightarrow, \triangleright)$ is an EAS if, and only if

$$(\mathrm{Id} \times \phi) \circ (\phi \times \mathrm{Id}) \circ (\mathrm{Id} \times \phi) = (\phi \times \mathrm{Id}) \circ (\mathrm{Id} \times \tau) \circ (\phi \times \mathrm{Id}),$$

where $\tau : \Omega^2 \longrightarrow \Omega^2$ is the usual flip:

$$\tau : \left\{ \begin{array}{rcl} \Omega^2 & \longrightarrow & \Omega^2 \\ (\alpha, \beta) & \longrightarrow & (\beta, \alpha). \end{array} \right.$$

This can easily be generalized in the category of vector spaces: a linear extended associative semigroup (briefly, ℓEAS) is a pair (A, Φ), where $\Phi : A \otimes A \longrightarrow A \otimes A$ is a linear map such as

$$(\mathrm{Id} \otimes \Phi) \circ (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Phi) = (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \tau) \circ (\Phi \otimes \mathrm{Id}),$$

where $\tau : A \otimes A \longrightarrow A \otimes A$ is the usual flip. In particular, if $(\Omega, \rightarrow, \triangleright)$ is an EAS, then its algebra $\mathbb{K}\Omega$ is an ℓEAS. We then introduce the notion of Φ-associative algebra (Definition 3.4) and we describe free Φ-associative algebras $T_\Phi(V)$ in term of tensor algebras in Theorem 3.5. In particular, as a vector space,

$$T_\Phi(V) = \bigoplus_{n=1}^{\infty} A^{\otimes(n-1)} \otimes V^{\otimes n}.$$

We prove in Proposition 3.6 that if V is a Φ-associative algebra, then $V \otimes A$ is naturally an associative algebra; if Φ is invertible, we prove conversely that any convenient associative product on $V \otimes A$ gives rise to a Φ-associative algebra structure on V. Following these results, we study the algebra structure of $T_\Phi(V) \otimes A$ and, if Φ is invertible, we prove that it is freely generated by $V \otimes A$ (Proposition 3.7).

The description of free Φ-algebras induce a combinatorial description of the operad $\mathbf{As}_\Phi$ of Φ-associative algebras (Proposition 3.8). We prove that, when A is finite-dimensional, that the operad $\mathbf{As}_\Phi$ is Koszul, and that its Koszul dual is the operad of $\mathbf{As}_{\Phi^*}$-algebras, generalizing a well-known result for the operad $\mathbf{As}$ of "classical" associative algebras (Proposition 3.9 and Theorem 3.10). We study operad morphisms between the operad of associative algebras and $\mathbf{As}_\Phi$, which is related to eigenvectors of Φ (Proposition 3.12). We then give results on operadic maps between the operads $\mathbf{As}$ and $\mathbf{As}_\Phi$, and between the operads $\mathbf{As}_\Omega^2$ and $\mathbf{As}_\Phi$ (Propositions 3.14 and 3.15). The paper ends with various links with other types of algebras, such that diassociative, post-Lie, dendriform, tridendriform or duplicial algebras, and their Koszul duals.

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Notations 0.1. Let $\mathbb{K}$ be a commutative field. Any vector space in this text will be taken over $\mathbb{K}$.

1 Two-parameters Ω-associative algebras

Notations 1.1. In all this section, $(\Omega, \rightarrow)$ is an associative semigroup.

1.1 Definition

In the spirit of the notion of two-parameters dendriform or duplicial algebras of [17], we now introduce the notion of two-parameters associative algebras, which can be found in [1]:

Definition 1.1. *A two-parameters Ω-associative algebra is a family $(V, (*_{\alpha,\beta})_{\alpha,\beta\in\Omega})$, where V is a vector space and, for any $(\alpha, \beta) \in \Omega^2$, $*_{\alpha,\beta} : V \otimes V \longrightarrow V$ is a linear map such that*

$$\forall \alpha, \beta, \gamma \in \Omega,\ \forall x, y, z \in V, \qquad (x *_{\alpha,\beta} y) *_{\alpha\rightarrow\beta,\gamma} z = x *_{\alpha,\beta\rightarrow\gamma} (y *_{\beta,\gamma} z).$$

Remark 1.1. If $|\Omega| = 1$, Ω-associative algebras are associative algebras.

Such a structure is related to $(\Omega, \rightarrow)$-graded associative products on $V \otimes \mathbb{K}\Omega$. For the sake of simplicity, we shall denote the tensor product $x \otimes \alpha$, with $x \in V$ and $\alpha \in \Omega$, by $x\alpha$.

Proposition 1.2. *Let V be a vector space, endowed with bilinear products $*_{\alpha,\beta}$ for any $(\alpha, \beta) \in \Omega^2$. We define a product $*$ on $V \otimes \mathbb{K}\Omega$ by*

$$\forall x, y \in V,\ \forall \alpha, \beta \in \Omega, \qquad x\alpha * y\beta = x *_{\alpha,\beta} y(\alpha \rightarrow \beta).$$

Then $*$ is associative if, and only if, $(V, (*_{\alpha,\beta})_{\alpha,\beta\in\Omega})$ is a two-parameters Ω-associative algebra.

Proof. For any $x, y, z \in V$, for any $\alpha, \beta, \gamma \in \Omega$,

$$\begin{aligned}(x\alpha * y\beta) * z\gamma &= (x *_{\alpha,\beta} y) *_{\alpha\rightarrow\beta,\gamma} z(\alpha \rightarrow \beta \rightarrow \gamma),\\ x\alpha * (y\beta * z\gamma) &= x *_{\alpha,\beta\rightarrow\gamma} (y *_{\beta,\gamma} z)(\alpha \rightarrow \beta \rightarrow \gamma).\end{aligned}$$

The result is then immediate. □

1.2 The operad of two-parameters Ω-associative algebras

We refer to [23, 25, 26, 27, 32] for notations and usual results on operads.

Notations 1.2. We denote by $\mathbf{As}^2_\Omega$ the nonsymmetric operad of two-parameters Ω-associative algebras. It is generated by $*_{\alpha,\beta} \in \mathbf{As}^2_\Omega(2)$, with $\alpha, \beta \in \Omega$, and the relations

$$\forall \alpha, \beta, \gamma \in \Omega, \qquad *_{\alpha\to\beta,\gamma} \circ_1 *_{\alpha,\beta} = *_{\alpha,\beta\to\gamma} \circ_2 *_{\beta,\gamma}.$$

We assume in this section that Ω is finite, of cardinality denoted by ω. Then the components of $\mathbf{As}^2_\Omega$ are finite-dimensional, and the following Proposition allows to inductively compute their dimension:

Proposition 1.3. *The operad $\mathbf{As}^2_\Omega$ is Koszul. For any $n \geqslant 1$, let us put $p_n = \dim_{\mathbb{K}}(\mathbf{As}^2_\Omega(n))$ and*

$$P(X) = \sum_{n=1}^{\infty} p_n X^n \in \mathbb{Q}[[X]].$$

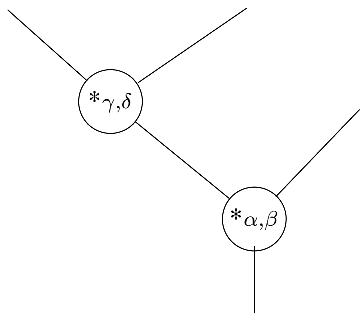
Then

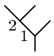
$$p_n = \omega(\omega-1)\sum_{k=1}^{n-1} p_k p_{n-k} + \omega p_{n-1}, \tag{1}$$

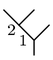
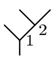
or equivalently, if $|\omega| \geqslant 2$,

$$P(X) = \frac{1-\omega X - \sqrt{1+2\omega(1-2\omega)X+\omega^2X^2}}{2\omega(\omega-1)}. \tag{2}$$

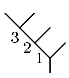
Proof. we shall use the rewriting method of [2, 23]. We shall write elements of the free nonsymmetric operad generated by $\mathbf{As}^2_\Omega(2)$ as planar trees which vertices are decorated by elements of Ω^2. We will write indices on the vertices on the trees and put the corresponding decorations between parentheses, and we delete the symbols $*$ in order to enlighten the notations. For example, the operadic tree



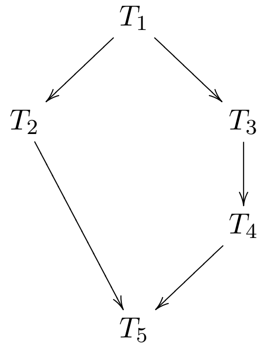
will be shortly written  $((\alpha,\beta),(\gamma,\delta))$. The rewriting rules are

 $((\alpha\to\beta,\gamma),(\alpha,\beta)) \longrightarrow$  $((\alpha,\beta\to\gamma),(\beta,\gamma))$

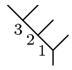
for any $\alpha, \beta, \gamma \in \Omega$. There is only one family of critical monomials, namely the monomials

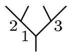
 $(((\alpha\to\beta)\to\gamma,\delta),(\alpha\to\beta,\gamma),(\alpha,\beta)),$

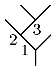
where $\alpha, \beta, \gamma \in \Omega$. Koszularity of $\mathbf{As}^2_\Omega$ comes from the confluence of the following diagram:

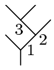


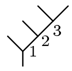
with

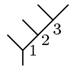
$T_1 =$  $(((\alpha \to \beta) \to \gamma, \delta), (\alpha \to \beta, \gamma), (\alpha, \beta)),$

$T_2 =$  $((\alpha \to \beta, \gamma \to \delta), (\alpha, \beta), (\gamma, \delta)),$

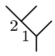
$T_3 =$  $((\alpha \to (\beta \to \gamma), \delta), (\alpha, \beta \to \gamma), (\beta, \gamma)),$

$T_4 =$  $((\alpha, (\beta \to \gamma) \to \delta), (\beta \to \gamma, \delta), (\beta, \gamma)),$

$T_5 =$  $((\alpha, (\beta \to \gamma) \to \delta), (\beta, \gamma \to \delta), (\gamma, \delta)),$

$=$  $((\alpha, \beta \to (\gamma \to \delta)), (\beta, \gamma \to \delta), (\gamma, \delta)).$

Hence, the operad $\mathbf{As}^2_\Omega$ is Koszul. Moreover, $\mathbf{As}^2_\Omega(n)$ has for basis the set of rooted planar binary trees with $n-1$ internal vertices decorated by Ω^2, avoiding subtrees

 $((\alpha \to \beta, \gamma), (\alpha, \beta))$

for any $\alpha, \beta, \gamma \in \Omega$. For any planar binary tree T, let us denote by p_T the number of decorations of the vertices of trees by elements of Ω^2, avoiding these subtrees. If T is a planar binary tree different from the tree $|$ (which is the unit of the operad $\mathbf{As}^2_\Omega$), we denote by T_l the left subtree born from the root of T, by T_r the right subtree born from the root of T, and we write $T = T_l \vee T_r$. Then, looking at the possible decorations of the root, we obtain

$$p_T = p_{T_l} p_{T_r} \omega \times \begin{cases} \omega \text{ if } T_2 = |, \\ \omega - 1 \text{ otherwise.} \end{cases}$$

Hence, if $n \geqslant 2$, denoting by $\mathcal{T}_n$ the set of planar binary rooted trees with $n-1$ internal vertices,

$$\begin{aligned} p_n &= \sum_{T \in \mathcal{T}_n} p_T \\ &= \omega^2 \sum_{T \in \mathcal{T}_{n-1}} p_T + \omega(\omega - 1) \sum_{k=2}^{n-1} \sum_{T_l \in \mathcal{T}_k} \sum_{T_r \in \mathcal{T}_{n-k}} p_{T_l} p_{T_r} \\ &= \omega^2 p_{n-1} + \omega(\omega - 1) \sum_{k=2}^{n-1} p_k p_{n-k} \\ &= \omega(\omega - 1) \sum_{k=1}^{n-1} p_k p_{n-k} + \omega p_{n-1}, \end{aligned}$$

which gives (1). Summing over n, with $p_1 = 1$, we obtain

$$P(X) = \omega(\omega-1)P(X)^2 + \omega X P(X) + X. \tag{3}$$

if $\omega = 1$, we obtain that

$$P(X) = XP(X) + X,$$

so

$$P(X) = \frac{X}{1-X} = \sum_{n=1}^{\infty} X^n,$$

recovering the formal series of the nonsymmetric operad of associative algebras. If $\omega \geqslant 2$, solving (3), with the initial condition $P(0) = 0$, we obtain (2). $\square$

Example 1.1.

$$\begin{aligned}
p_2(\omega) &= \omega^2,\\
p_3(\omega) &= (2\omega-1)\omega^3,\\
p_4(\omega) &= (5\omega^2-5\omega+1)\omega^4,\\
p_5(\omega) &= (2\omega-1)(7\omega^2-7\omega+1)\omega^5,\\
p_6(\omega) &= (42\omega^4-84\omega^3+56\omega^2-14\omega+1)\omega^6,\\
p_7(\omega) &= (2\omega-1)(66\omega^4-132\omega^3+84\omega^2-18\omega+1)\omega^7,\\
p_8(\omega) &= (429\omega^6-1287\omega^5+1485\omega^4-825\omega^3+225\omega^2-27\omega+1)\omega^8,\\
p_9(\omega) &= (2\omega-1)(715\omega^6-2145\omega^5+2431\omega^4-1287\omega^3+319\omega^2-33\omega+1)\omega^9.
\end{aligned}$$

This gives

$\omega\backslash n$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	4	24	176	1440	12608	115584
3	1	9	135	2511	52245	1164213	27173475
4	1	16	448	15616	609280	25464832	1114882048
5	1	25	1125	63125	3965625	266890625	18816328125
6	1	36	2376	195696	18048096	1783238976	184576081536
7	1	49	4459	506611	64454845	8785674373	1254546699679
8	1	64	7680	1150976	193167360	34733293568	6542642380800
9	1	81	12393	2368521	506935665	116245810017	27925350157593

Remark 1.2. 1. If $\omega = 2$, the sequence $(p_n)_{n\geqslant 2}$ is referenced as A156017 in the OEIS. This is the sequence of dimensions of an operad given in [4], generated by four products $\prec$, $\succ$, $\circ$ and $\odot$, with eight relations, see (21) in [4]. This is a special example of a type of two-parameters Ω-associative algebras, with $\Omega = (\mathbb{Z}/2\mathbb{Z}, \times)$ and

$$*_{(\overline{0},\overline{0})} = \circ, \qquad *_{(\overline{0},\overline{1})} = \prec, \qquad *_{(\overline{1},\overline{0})} = \succ, \qquad *_{(\overline{1},\overline{1})} = \odot.$$

Moreover,

$$P(X)_{|\omega=2} = \frac{1-2X-\sqrt{1-6(2X)+(2X)^2}}{4},$$

so for any $n \geqslant 2$, $p_n = 2^{n-1}\mathrm{schr}_n$, where schr_n is the n-th large Schröder number (sequence A006318 in the OEIS):

n	1	2	3	4	5	6	7	8	9	10
schr_n	1	2	6	22	90	394	1806	8558	41586	206098

Corollary 1.4. *Let $n \geqslant 1$.*

1. *p_n is a polynomial in $\mathbb{Z}[\omega]$, of degree $2n-2$. Its leading coefficient is the n-th Catalan number* cat_n *(Sequence A000108 in the OEIS):*

n	1	2	3	4	5	6	7	8	9	10
cat_n	1	1	2	5	14	42	132	429	1430	4862

2. *If $n \geqslant 2$, there exists a polynomial $q_n \in \mathbb{Z}[\omega]$, such that $p_n = \omega^n q_n$. Moreover, $q_n(0) = (-1)^n$.*

3. *If n is odd and $\geqslant 3$, then $p_n\left(\frac{1}{2}\right) = 0$.*

Proof. 1. and 2. We proceed by induction on n. As $p_1 = 1$, this is obvious. Let us assume the result at all ranks $< n$, with $n \geqslant 2$. The induction hypothesis gives that the following is a polynomial in $\mathbb{Z}[\omega]$, of degree $2n-2$,

$$\omega(\omega-1)\sum_{k=1}^{n-1} p_k p_{n-k}.$$

Its leading term is

$$\sum_{k=1}^{n-1} \mathrm{cat}_k \mathrm{cat}_{n-k} = \mathrm{cat}_n.$$

We also obtain that ωp_{n-1} is a polynomial in $\mathbb{Z}[\omega]$, of degree $2n-3$. Summing in (1), we obtain the first point for p_n. Still by (1),

$$\begin{aligned}
p_n &= \omega(\omega-1)\sum_{k=2}^{n-2} p_k p_{n-k} + 2\omega(\omega-1)p_{n-1} + \omega p_{n-1} \\
&= \omega(\omega-1)\sum_{k=2}^{n-2} p_k p_{n-k} + \omega(2\omega-1)p_{n-1} \\
&= \omega^{n+1}(\omega-1)\sum_{k=2}^{n-2} q_k q_{n-k} + \omega^n(2\omega-1)q_{n-1} \\
&= \omega^n \left(\underbrace{\omega(\omega-1)\sum_{k=2}^{n-2} q_k q_{n-k} + (2\omega-1)q_{n-1}}_{q_n} \right).
\end{aligned}$$

Moreover, $q_n(0) = 0 - q_{n-1}(0) = (-1)^n$, which proves the second point.

3. For $\omega = \frac{1}{2}$, we obtain

$$P(X)_{|\omega=\frac{1}{2}} = X - 2 + \sqrt{4+X^2} = X + 2\sum_{k=2}^{\infty} \frac{(-1)^k(2k-2)!}{2^{4k-1}k!(k-1)!} X^{2k}.$$ □

Corollary 1.5. *For any $n \geqslant 2$,*

$$p_n = \frac{\omega^n}{n-1}\left(\sum_{k=1}^{n-1} \binom{n-1}{k}\binom{n-1}{k-1}\omega^{n-1-k}(\omega-1)^{k-1}\right).$$

Proof. Let us consider the Narayana numbers [31]:

$$\forall k, n \geqslant 1, \qquad N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1},$$

and their formal series

$$\mathbf{N}(z,t) = \sum_{k,n\geqslant 1} N(n,k) z^n t^{k-1} = \frac{1 - z(t+1) - \sqrt{1 - 2z(t+1) + z^2(t-1)^2}}{2tz}.$$

Then

$$\begin{aligned} P(X) &= X + X\mathbf{N}\left(\omega^2 X, \frac{\omega - 1}{\omega}\right) \\ &= X + \sum_{k,n\geqslant 1} N(n,k)\omega^{2n} X^{n+1} \left(\frac{\omega-1}{\omega}\right)^{k-1} \\ &= X + \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} N(n-1,k)\omega^{2n-2}\left(\frac{\omega-1}{\omega}\right)^{k-1}\right) X^n \\ &= X + \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} N(n-1,k)\omega^{2n-1-k}(\omega-1)^{k-1}\right) X^n. \end{aligned}$$

□

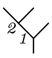
Remark 1.3. These numbers appear in [5], where they are interpreted in terms of Catalan paths. More precisely, with the notations of [5, Definition (1.13)],

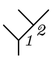
$$p_n = \frac{1}{n-1}\sum_{k=1}^{n-1} \binom{n-1}{k}\binom{n-1}{k-1} (\omega^2)^{n-k}(\omega(\omega-1))^{k-1} = C_{n-1}^{(\omega^2,\, \omega(\omega-1))}.$$

1.3 Koszul dual of $\mathbf{As}^2_\Omega$

In all this paragraph, $(\Omega, \rightarrow)$ is a finite semigroup.

Proposition 1.6. *Koszul dual $\mathbf{As}^{2\,!}_\Omega$ of the operad $\mathbf{As}^2_\Omega$ is the quotient of $\mathbf{As}^2_\Omega$ by the trees*

 $((\alpha,\beta),(\gamma,\delta)) = (\alpha,\beta)\circ_1(\gamma,\delta)$ *with* $\alpha \rightarrow \beta \neq \gamma$,

 $((\alpha,\beta),(\gamma,\delta)) = (\alpha,\beta)\circ_2(\gamma,\delta)$ *with* $\beta \neq \gamma \rightarrow \delta$.

In other terms, a $\mathbf{As}^{2\,!}_\Omega$-algebra is a family $(V, (*_{\alpha,\beta})_{\alpha,\beta\in\Omega})$, where V is a vector space and, for any $(\alpha,\beta) \in \Omega^2$, $*_{\alpha,\beta} : V \otimes V \longrightarrow V$ is a linear map such that

$$\begin{aligned} &\forall \alpha,\beta,\gamma \in \Omega,\ \forall x,y,z \in V, && (x *_{\alpha,\beta} y) *_{\alpha\rightarrow\beta,\gamma} z = x *_{\alpha,\beta\rightarrow\gamma} (y *_{\beta,\gamma} z), \\ &\forall \alpha,\beta,\gamma,\delta \in \Omega,\ \forall x,y,z \in V, && (x *_{\alpha,\beta} y) *_{\gamma,\delta} z = 0 \text{ if } \alpha \rightarrow \beta \neq \gamma, \\ & && x *_{\alpha,\beta} (y *_{\gamma,\delta} z) = 0 \text{ if } \beta \neq \gamma \rightarrow \delta. \end{aligned}$$

For any $n \geqslant 2$, $\dim_{\mathbb{K}}(\mathbf{As}^{2\,!}_\Omega(n)) = \omega^n$.

Proof. The presentation of $\mathbf{As}^{2\,!}_\Omega$ comes from a direct computation. Let us denote by $Q(X)$ the Poincaré-Hilbert formal series of $\mathbf{As}^{2\,!}_\Omega$. As $\mathbf{As}^2_\Omega$ is Koszul, $Q(X)$ is the inverse for the composition of $-P(-X)$. From (3):

$$X = \frac{P(X) - \omega(\omega-1)P(X)^2}{\omega P(X) + 1} = \frac{-P(-X) + \omega(\omega-1)(-P(-X))^2}{1 - \omega(-P(-X))},$$

so $Q(X)$ is given by

$$Q(X) = \frac{\omega(\omega-1)X^2 + X}{1-\omega X} = X + \sum_{n=2}^{\infty} \omega^n X^n. \qquad \square$$

Let us give a combinatorial presentation of $\mathbf{As}_\Omega^{2\,!}$:

Proposition 1.7. *For any $n \geqslant 1$, let us put $\mathbf{P}(n) = (\mathbb{K}\Omega)^{\otimes n}$. Elements of $\mathbf{P}(n)$ are linear spans of words $\alpha_1 \dots \alpha_n$ in Ω. Then $\mathbf{P} = (\mathbf{P}(n))_{n\geqslant 1}$ is given a structure of nonsymmetric operad with the following composition: for any $\alpha_1, \dots, \alpha_n \in \Omega$, for any $\beta_{i,j} \in \Omega$,*

$$\alpha_1 \dots \alpha_n \circ (\beta_{1,1} \dots \beta_{1,k_1}, \dots, \beta_{n,1} \dots \beta_{n,k_n}) = \left(\prod_{i=1}^{n} \delta_{\alpha_i, \beta_{i,1} \to \dots \to \beta_{i,k_i}}\right) \beta_{1,1} \dots \beta_{1,k_1} \dots \beta_{n,1} \dots \beta_{n,k_n}.$$

The unit is

$$I = \sum_{\alpha \in \Omega} \alpha.$$

We define a suboperad $\mathbf{P}_0$ isomorphic to $\mathbf{As}_\Omega^{2\,!}$ by

$$\mathbf{P}_0(n) = \begin{cases} \mathbb{K}I \text{ if } n = 1, \\ \mathbf{P}(n) \text{ if } n \geqslant 2. \end{cases}$$

Proof. For any word $w = \alpha_1 \dots \alpha_n$ in α, we put $|w| = \alpha_1 \to \dots \to \alpha_n$. The composition $\circ$ can be rewritten in the following way: for any words $w, w_1, \dots, w_n$ with letters in Ω, w being of length n,

$$w \circ (w_1, \dots, w_n) = \delta_{w, |w_1| \dots |w_n|} w_1 \dots w_n.$$

Let us prove the associativity: for any word w of length n, w_i, $1 \leqslant i \leqslant n$ of respective lengths k_i, $w_{i,j}$ with $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k_i$, all with letters in Ω,

$$\begin{aligned} & w \circ (w_1 \circ (w_{1,1}, \dots, w_{1,k_1}), \dots, w_n \circ (w_{n,1}, \dots, w_{n,k_n})) \\ &= \left(\prod_{i=1}^{n} \delta_{w_i, |w_{i,1}| \dots |w_{i,k_i}|} \delta_{w, \left||w_{1,1}| \dots |w_{1,k_1}|\right| \dots \left||w_{n,1}| \dots |w_{n,k_n}|\right|}\right) w_{1,1} \dots w_{n,k_n} \\ &= \left(\delta_{w_1 \dots w_n, |w_{1,1}| \dots |w_{n,k_n}|} \delta_{w, |w_1| \dots |w_n|}\right) w_{1,1} \dots w_{n,k_n} \\ &= (w \circ (w_1, \dots, w_n)) \circ (w_{1,1}, \dots, w_{n,k_n}). \end{aligned}$$

Let us prove that I is a unit. Let $w = \alpha_1 \dots \alpha_n$ be a word with letters in Ω.

$$\begin{aligned} I \circ \alpha_1 \dots \alpha_n &= \sum_{\alpha \in \Omega} \delta_{\alpha, |\alpha_1 \dots \alpha_n} \alpha_1 \dots \alpha_n = \alpha_1 \dots \alpha_n, \\ \alpha_1 \dots \alpha_n \circ (I, \dots, I) &= \sum_{\beta_1, \dots, \beta_n \in \Omega} \prod_{i=1}^{n} \delta_{\alpha_i, \beta_i} \beta_1 \dots \beta_n \\ &= \alpha_1 \dots \alpha_n. \end{aligned}$$

So $\mathbf{P}$ is an operad. Obviously, $\mathbf{P}_0$ is a suboperad. Moreover, for any word $\alpha_1 \dots \alpha_n$ of length $\geqslant 3$,

$$\alpha_1 \dots \alpha_n = (\alpha_1 \to \alpha_2)\alpha_3 \dots \alpha_n \circ (\alpha_1\alpha_2, I, \dots, I).$$

A direct induction then proves that $\mathbf{P}_0$ is generated by $\mathbf{P}(2)$. Moreover, for any $\alpha, \beta, \gamma, \delta \in \Omega$,

$$\begin{aligned} (\alpha \to \beta)\gamma \circ (\alpha\beta, I) = \alpha(\beta \to \gamma) \circ (I, \beta\gamma) &= \alpha\beta\gamma, \\ \gamma\delta \circ (\alpha\beta, I) &= 0 \text{ if } \alpha \to \beta \neq \gamma, \\ \alpha\beta \circ (I, \gamma\delta) &= 0 \text{ if } \gamma \to \delta \neq \beta. \end{aligned}$$

Therefore, the relations defining $\mathbf{As}_\Omega^{2\,!}$ are satisfied in $\mathbf{P}_0$. Hence, there exists a surjective operad morphism

$$\begin{cases} \mathbf{As}_\Omega^{2\,!} & \longrightarrow & \mathbf{P}_0 \\ *_{\alpha,\beta} & \longrightarrow & \alpha\beta. \end{cases}$$

Comparing the formal series of $\mathbf{As}_\Omega^{2\,!}$ and $\mathbf{P}_0$, we deduce that this is an isomorphism. □

Corollary 1.8. *Let V be a vector space. The free $\mathbf{As}_\Omega^{2\,!}$-algebra generated by V is*

$$T_\Omega^2(V) = V \oplus \bigoplus_{n=2}^{\infty} (\mathbb{K}\Omega \otimes V)^{\otimes n}.$$

The product $*_{\alpha,\beta}$ is given in the following way: for any $u, v \in V$, for any $\alpha_1 u_1, \ldots, \alpha_k u_k$, $\beta_1 v_1, \ldots, \beta_l v_l \in \mathbb{K}\Omega \otimes V$, with $k, l \geqslant 1$,

$$\begin{aligned} u *_{\alpha,\beta} v &= \alpha u \beta v, \\ \alpha_1 u_1 \ldots \alpha_k u_k *_{\alpha,\beta} v &= (\delta_{\alpha,\alpha_1\to\ldots\to\alpha_k})\, \alpha_1 u_1 \ldots \alpha_k u_k \beta v, \\ u *_{\alpha,\beta} \beta_1 v_1 \ldots \beta_l v_l &= (\delta_{\beta,\beta_1\to\ldots\to\beta_l})\, \alpha u \beta_1 v_1 \ldots \beta_l v_l, \\ \alpha_1 u_1 \ldots \alpha_k u_k *_{\alpha,\beta} \beta_1 v_1 \ldots \beta_l v_l &= (\delta_{\alpha,\alpha_1\to\ldots\to\alpha_k} \delta_{\beta,\beta_1\to\ldots\to\beta_l})\, \alpha_1 u_1 \ldots \alpha_k u_k \beta_1 v_1 \ldots \beta_l v_l. \end{aligned}$$

2 Extended associative semigroups

We here give the definition and few examples of extended associative semigroups. More results can be found in [15].

2.1 Definition and examples

Definition 2.1. *An associative extended semisgroup (briefly, EAS) is a triple $(\Omega, \to, \triangleright)$, where Ω is a nonempty set and $\to, \triangleright : \Omega^2 \longrightarrow \Omega$ are maps such that, for any $\alpha, \beta, \gamma \in \Omega$,*

$$\alpha \to (\beta \to \gamma) = (\alpha \to \beta) \to \gamma, \tag{4}$$

$$(\alpha \triangleright (\beta \to \gamma)) \to (\beta \triangleright \gamma) = (\alpha \to \beta) \triangleright \gamma, \tag{5}$$

$$(\alpha \triangleright (\beta \to \gamma)) \triangleright (\beta \triangleright \gamma) = \alpha \triangleright \beta. \tag{6}$$

Example 2.1.

1. Let Ω be a set. We put

   $$\forall \alpha, \beta \in \Omega, \qquad \begin{cases} \alpha \to \beta = \beta, \\ \alpha \triangleright \beta = \alpha. \end{cases}$$

   Then $(\Omega, \to, \triangleright)$ is an EAS, denoted by $\mathbf{EAS}(\Omega)$.

2. Let $(\Omega, \star)$ be an associative semigroup. We put

   $$\forall \alpha, \beta \in \Omega, \qquad \alpha \triangleright \beta = \alpha.$$

   Then $(\Omega, \star, \triangleright)$ is an EAS, which we denote by $\mathbf{EAS}(\Omega, \star)$.

3. Let $(\Omega, \star)$ be a group. We put, for any $\alpha, \beta \in \Omega$,

   $$\alpha \to \beta = \beta, \qquad \alpha \triangleright \beta = \alpha \star \beta^{\star -1}.$$

   Then $(\Omega, \to, \triangleright)$ is an EAS, denoted by $\mathbf{EAS}'(\Omega, \star)$.

Definition 2.2. *Let $(\Omega, \rightarrow, \triangleright)$ be an EAS. We shall say that it is nondegenerate if the following map is bijective:*

$$\phi : \left\{ \begin{array}{rcl} \Omega^2 & \longrightarrow & \Omega^2 \\ (\alpha, \beta) & \longrightarrow & (\alpha \rightarrow \beta, \alpha \triangleright \beta). \end{array} \right.$$

Example 2.2.

1. Let Ω be a set. In $\mathbf{EAS}(\Omega)$, for any $\alpha, \beta \in \Omega$, $\phi(\alpha, \beta) = (\beta, \alpha)$, so $\mathbf{EAS}(\Omega)$ is nondegenerate, and $\phi^{-1} = \phi$.

2. Let $(\Omega, \star)$ be a group. Then $\mathbf{EAS}(\Omega, \star)$ is nondegenerate. indeed, for any $\alpha, \beta \in \Omega$, $\phi(\alpha, \beta) = (\alpha \star \beta, \alpha)$, so ϕ is a bijection, of inverse given by $\phi^{-1}(\alpha, \beta) = (\beta, \beta^{\star -1} \star \alpha)$.

3. Let $(\Omega, \star)$ be an associative semigroup with the right inverse condition. Then $\mathbf{EAS}'(\Omega, \star)$ is nondegenerate. Indeed, for any $\alpha, \beta \in \Omega$, $\phi(\alpha, \beta) = (\beta, \alpha \triangleright \beta)$, so ϕ is a bijection, of inverse given by $\phi^{-1}(\alpha, \beta) = (\beta \star \alpha, \alpha)$.

2.2 EAS of cardinality two

Here is a classification of EAS of cardinality two. The underlying set is $\Omega = \{a, b\}$ and the products will be given by two matrices

$$\begin{pmatrix} a \rightarrow a & a \rightarrow b \\ b \rightarrow a & b \rightarrow b \end{pmatrix}, \qquad \begin{pmatrix} a \triangleright a & a \triangleright b \\ b \triangleright a & b \triangleright b \end{pmatrix}.$$

We shall use the two maps

$$\pi_a : \left\{ \begin{array}{rcl} \Omega & \longrightarrow & \Omega \\ \alpha & \longrightarrow & a, \end{array} \right. \qquad \pi_b : \left\{ \begin{array}{rcl} \Omega & \longrightarrow & \Omega \\ \alpha & \longrightarrow & b. \end{array} \right.$$

We respect the indexation of EDS of [16].

Case	$\to$	$\rhd$	Description
A1	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\mathbf{EAS}(\Omega, \to, \pi_a)$
A2	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\mathbf{EAS}(\Omega, \to)$
C1	$\begin{pmatrix} a & a \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\mathbf{EAS}(\Omega, \to, \pi_{\overline{0}})$
C3	$\begin{pmatrix} a & a \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\mathbf{EAS}(\mathbb{Z}/2\mathbb{Z}, \times)$
C5	$\begin{pmatrix} a & a \\ a & b \end{pmatrix}$	$\begin{pmatrix} b & b \\ b & b \end{pmatrix}$	$\mathbf{EAS}(\mathbb{Z}/2\mathbb{Z}, \pi_{\overline{1}})$
C6	$\begin{pmatrix} a & a \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & a \end{pmatrix}$	
E1′ − E2′	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\mathbf{EAS}(\Omega, \to, \pi_a)$
E3′	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\mathbf{EAS}(\Omega, \to)$
F1	$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\mathbf{EAS}(\Omega, \to, \pi_a)$
F3	$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\mathbf{EAS}(\Omega)$
F4	$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$	$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$	$\mathbf{EAS}'(\mathbb{Z}/2\mathbb{Z}, +)$
H1	$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\mathbf{EAS}(\mathbb{Z}/2\mathbb{Z}, +, \pi_{\overline{0}})$
H2	$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$	$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$	$\mathbf{EAS}(\mathbb{Z}/2\mathbb{Z}, +)$

The nondegenerate EAS are F_3, F_4 and H_2.

3 Generalized associative algebras

3.1 Discrete and linear versions

Definition 3.1. *Let $(\Omega, \to, \rhd)$ be a set with two binary operations. Let $(V, (*_\alpha)_{\alpha\in\Omega})$ be a family such that V is a vector space and, for any $\alpha \in \Omega$, $*_\alpha : V \otimes V \longrightarrow V$ is a linear map. We shall say that it is an Ω-associative algebra if*

$$\forall x, y, z \in V,\ \forall \alpha, \beta \in \Omega, \qquad x *_\alpha (y *_\beta z) = (x *_{\alpha\rhd\beta} y) *_{\alpha\to\beta} z. \tag{7}$$

Example 3.1. 1. If Ω is a set, for $\mathbf{EAS}(\Omega)$, (7) becomes

$$\forall x, y, z \in V,\ \forall \alpha, \beta \in \Omega, \qquad x *_\alpha (y *_\beta z) = (x *_\alpha y) *_\beta z.$$

As a consequence, any linear span of $*_\alpha$ is associative. We recover the notion of matching associative algebra [33].

2. If $(\Omega, \star)$ is a semigroup, for $\mathbf{EAS}(\Omega, \star)$, (7) becomes

$$\forall x, y, z \in V,\ \forall \alpha, \beta \in \Omega, \qquad x *_\alpha (y *_\beta z) = (x *_\alpha y) *_{\alpha\star\beta} z.$$

These are $(\Omega, \star)$-family associative algebras.

Remark 3.1. This does not include the multiassociative and the dual multiassociative algebras introduced by Giraudo in [18]. We shall see that the dimension of the n-th components of the operad of Ω-associative algebras is $|\Omega|^{n-1}$ for any $n \geqslant 2$, whereas it is constant for dual multiassociative algebras and described by Narayana numbers for dual multiassociative algebras.

In order to linearize these axioms, let us first consider the following lemma, proved in [14]:

Lemma 3.2. *Let $(\Omega, \rightarrow, \triangleright)$ be a set with two binary operations. We consider the maps*

$$\phi : \begin{cases} \Omega^2 & \longrightarrow & \Omega^2 \\ (\alpha, \beta) & \longrightarrow & (\alpha \rightarrow \beta, \alpha \triangleright \beta), \end{cases} \qquad \tau : \begin{cases} \Omega^2 & \longrightarrow & \Omega^2 \\ (\alpha, \beta) & \longrightarrow & (\beta, \alpha). \end{cases}$$

Then $(\Omega, \rightarrow, \triangleright)$ is an EAS if, and only if

$$(\mathrm{Id} \times \phi) \circ (\phi \times \mathrm{Id}) \circ (\mathrm{Id} \times \phi) = (\phi \times \mathrm{Id}) \circ (\mathrm{Id} \times \tau) \circ (\phi \times \mathrm{Id}). \tag{8}$$

This naturally leads to the following definition:

Definition 3.3. *Let A be a vector space and let $\Phi : A \otimes A \longrightarrow A \otimes A$ be a linear map. We shall say that (A, Φ) is a linear extended associative semigroup (briefly, ℓEAS) if*

$$(\mathrm{Id} \otimes \Phi) \circ (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Phi) = (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \tau) \circ (\Phi \otimes \mathrm{Id}), \tag{9}$$

where $\tau : A \otimes A \longrightarrow A \otimes A$ is the usual flip:

$$\tau : \begin{cases} A \otimes A & \longrightarrow & A \otimes A \\ a \otimes b & \longrightarrow & b \otimes a. \end{cases}$$

We shall say that (A, Φ) is nondegenerate if Φ is invertible.

Example 3.2.

1. Let $(\Omega, \rightarrow, \triangleright)$ be an EAS and let $A = \mathbb{K}\Omega$ be its algebra, that is to say the vector space generated by Ω. We define

$$\Phi : \begin{cases} \mathbb{K}\Omega \otimes \mathbb{K}\Omega & \longrightarrow & \mathbb{K}\Omega \otimes \mathbb{K}\Omega \\ \alpha \otimes \beta & \longrightarrow & (\alpha \rightarrow \beta) \otimes (\alpha \triangleright \beta), \end{cases}$$

   where $\alpha, \beta \in \Omega$. Lemma 3.2 implies that $(\mathbb{K}\Omega, \Phi)$ is an ℓEAS, which we call the linearization of $(\Omega, \rightarrow, \triangleright)$.

2. Here are examples of ℓEAS of dimension 2, which are not linearization of an EAS. In these examples, A is a two-dimensional space with basis (x, y), and the maps Φ are given by

their matrices in the basis $(x \otimes x, x \otimes y, y \otimes x, y \otimes y)$.

$$\begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&0\\0&0&\lambda&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&0&1&0\\0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&1&0 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&1&1&0\\0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&0&1&0\\0&0&1&0 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&1&0&0\\0&1&-1&0 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&1&0\\0&0&-1&0\\0&1&-1&0\\0&0&2&1 \end{pmatrix},$$

where λ is a scalar. More details on these examples car be found in [15].

Notations 3.1. Let (A, Φ) be a pair, such that A is a vector space and $\Phi : A \otimes A \longrightarrow A \otimes A$ is a linear map. We use the Sweedler notation:

$$\Phi(a \otimes b) = \sum a' \to b' \otimes a'' \rhd b''.$$

Note that the operations $\to$ and $\rhd$ may not exist, nor the coproducts $a' \otimes a''$ or $b' \otimes b''$. With this notation, (9) can be rewritten as

$$\sum\sum\sum a' \to (b' \to c')' \otimes (a'' \rhd (b' \to c')'')' \to (b'' \rhd c'')' \otimes (a'' \rhd (b' \to c'))'' \rhd (b'' \rhd c'')'' \quad \text{(9')}$$
$$= \sum\sum (a' \to b')' \to c' \otimes (a' \to b')'' \rhd c'' \otimes a'' \rhd b''.$$

Definition 3.4. *Let V be a vector space and $*$ a linear map:*

$$* : \left\{ \begin{array}{rcl} A & \longrightarrow & \hom(V \otimes V, V) \\ a & \longrightarrow & *_a : \left\{ \begin{array}{rcl} V \otimes V & \longrightarrow & v \\ x \otimes y & \longrightarrow & x *_a y. \end{array} \right. \end{array} \right.$$

We shall say that $(V, *)$ is a Φ-associative algebra if

$$\forall x, y, z \in V,\ \forall a, b \in A, \qquad x *_a (y *_b z) = \sum (x *_{a'' \rhd b''} y) *_{a' \to b'} z. \tag{10}$$

We shall say that $(V, *)$ is an opposite Φ-associative algebra if

$$\forall x, y, z \in V,\ \forall a, b \in A, \qquad \sum x *_{a' \to b'} (y *_{a'' \rhd b''} z) = (x *_b y) *_a z. \tag{11}$$

Remark 3.2.

1. We define $*^{op}$ by $x *_a^{op} y = y *_a x$. Then $(V, *)$ is a Φ-associative algebra if, and only if, $(V, *^{op})$ is an opposite Φ-associative algebra.
2. If Φ is invertible, opposite Φ-associative algebras and Φ^{-1}-associative algebras are the same.

Remark 3.3. If $(\Omega, \to, \triangleright)$ is an EAS and if $(\mathbb{K}\Omega, \Phi)$ is its linearization, then the categories of Ω-associative algebras and of Φ-associative algebras are isomorphic: if $(V, (*_\alpha)_{\alpha\in\Omega})$ is an Ω-algebra, then we obtain a Φ-associative algebra with the map

$$* : \begin{cases} \mathbb{K}\Omega & \longrightarrow & \mathrm{hom}(V \otimes V, V) \\ \alpha \in \Omega & \longrightarrow & *_\alpha. \end{cases}$$

In this way, Ω-associative algebras can be seen as particular examples of Φ-associative algebras.

3.2 Free objects

Notations 3.2. Let V be a vector space. We put

$$T_A(V) = \bigoplus_{n=1}^{\infty} A^{\otimes(n-1)} \otimes V^{\otimes n}.$$

If $a_1, \ldots, a_{n-1} \in A$, $x_1, \ldots, x_n \in V$, we shall denote their tensor product in $A^{\otimes(n-1)} \otimes V^{\otimes n}$ by $a_1 \ldots a_{n-1} x_1 \ldots x_n$. Such a tensor will be called an *A-typed word of length n*. We shall use the map

$$\cdot : \begin{cases} T_A(V) \otimes A \otimes V & \longrightarrow & T_A(V) \\ a_1 \ldots a_{n-1} x_1 \ldots x_n \otimes a \otimes x & \longrightarrow & a_1 \ldots a_{n-1} x_1 \ldots x_n \cdot ax = a a_1 \ldots a_{n-1} a x_1 \ldots x_n x. \end{cases}$$

Theorem 3.5. *For any vector space V, we define bilinear products $*_a$ on $T_A(V)$ in the following way, by induction on the length of A-typed words:*

$$w *_a z = w \cdot az, \qquad u *_a (v \cdot bz) = \sum (u *_{a'' \triangleright b''} v) \cdot (a' \to b')z,$$

where $u, v, w \in T_A(V)$, $z \in V$ and $a, b \in A$. The following conditions are equivalent:

1. *(A, Φ) is an ℓEAS.*
2. *For any vector space V, $(T_A(V), *)$ is a Φ-associative algebra.*
3. *There exists a nonzero vector space V such that $(T_A(V), *)$ is a Φ-associative algebra.*

Moreover, if these conditions hold, then $(T_A(V), *)$ is the free Φ–associative algebra generated by V.

Proof. Obviously, 2. $\Longrightarrow$ 3.

3. $\Longrightarrow$ 1. Let x, y, z, t be four nonzero elements of any nonzero vector space V. For any $a, b, c \in A$,

$$\begin{aligned} x *_a (y *_b czt) &= \sum\sum\sum (a'' \triangleright (b' \to c'))'' \triangleright (b'' \triangleright c'')'' \\ &\qquad (a'' \triangleright (b' \to c')'')' \to (b'' \triangleright c'')' \\ &\qquad a' \to (b' \to c')' xyzt, \\ \sum (x *_{a' \triangleright b'} y) *_{a'' \to b''} czt &= \sum\sum (a'' \triangleright b'')((a' \to b')'' \triangleright c'')((a' \to b')' \to c')xyzt. \end{aligned}$$

This immediately gives (9).

1. $\Longrightarrow$ 2. Let us prove that for any A-typed words u, v, w, for any $a, b \in A$,

$$u *_a (v *_b w) = \sum (u *_{a' \triangleright b'} v) *_{a'' \to b''} w.$$

We proceed by induction on the length n of w. If $n = 1$, we put $w = z \in V$.

$$u *_a (v *_b w) = u *_a (v \cdot bz) = \sum (u *_{a' \rhd b'} v) \cdot (a'' \to b'')z = \sum (u *_{a' \rhd b'} v) *_{a'' \to b''} z.$$

Let us assume the result at rank $n - 1$. We put $w = w' \cdot cz$.

$$\begin{aligned}
u *_a (v *_b w) &= \sum u *_a ((v *_{b' \rhd c'} w') \cdot (b'' \to c'')z) \\
&= \sum \sum u *_{a'' \rhd (b' \to c')''} (v *_{b'' \rhd c''} w') \cdot a' \to (b' \to c')''z \\
&= \sum \sum \sum (u *_{(a'' \rhd (b' \to c')'')'' \rhd (b'' \rhd c'')''} v) *_{(a'' \rhd (b' \to c')'')' \to (b'' \rhd c'')'} w' \cdot a' \to (b' \to c')'z \\
&= \sum \sum (u *_{a'' \rhd b''} v) *_{(a' \to b')'' \rhd c''} w' \cdot (a' \to b')' \to c'z \\
&= \sum (u *_{a'' \rhd b''} v) *_{a' \to b'} (w' \cdot cz) \\
&= \sum (u *_{a'' \rhd b''} v) *_{a' \to b'} w.
\end{aligned}$$

We use the induction hypothesis for the third equality and (9) for the fourth one.

Freeness. Let us assume that conditions 1 and 2 hold. Let W be an A-associative algebra, and $\theta : V \longrightarrow W$ be any linear map. Let us prove that it can be uniquely extended as a Φ-associative algebra morphism Θ from $T_A(V)$ to W.

Existence of Θ. We inductively define $\Theta(w)$ for any A-typed word w by induction on its length n. If $n = 1$, then $w \in V$ and we put $\Theta(w) = \theta(w)$. Otherwise, let us write $w = w' \cdot az$. We put

$$\Theta(w) = \Theta(w') *_a \theta(z).$$

Let u, v be two A-typed words, and let us prove that for any $a \in A$, $\Theta(u *_a v) = \Theta(u) *_a \Theta(v)$. We proceed by induction on the length n of v. If $n = 1$, then by definition of Θ, this is true. Otherwise, let us put $v = v' \cdot bz$. Then

$$\begin{aligned}
\Theta(u *_a v) &= \sum \Theta((u *_{a'' \rhd b''} v') \cdot (a' \to b')z) \\
&= \sum \Theta(u *_{a'' \rhd b''} v') *_{a' \to b'} \theta(z) \\
&= \sum (\Theta(u) *_{a'' \rhd b''} \Theta(v')) *_{a' \to b'} \theta(z) \\
&= \Theta(u) *_a (\Theta(v') *_b \theta(z)) \\
&= \Theta(u) *_a \Theta(v).
\end{aligned}$$

So Θ is a Φ-associative algebra morphism.

Uniqueness of Θ. If Θ' is another morphism extending θ, for any A-typed word u, for any $a \in A$, for any $z \in V$,

$$\Theta'(u \cdot az) = \Theta'(u *_a z) = \Theta'(u) *_a \theta(z).$$

An easy induction on the length proves that for any A- typed word u, $\Theta'(u) = \Theta(u)$. □

3.3 Links with associative algebras

Proposition 3.6. *Let (A, Φ) such that A is a vector space and $\Phi : A \otimes A \longrightarrow A \otimes A$ is a linear map. We assume that V is a vector space and $* : A \longrightarrow \hom(V \otimes V, V)$ is a linear map. We define a product on $V \otimes A$ by*

$$\forall x, y \in V,\ \forall a, b \in A, \qquad xa \star yb = \sum x *_{a'' \rhd b''} ya' \to b'.$$

Then

1. *If (A,Φ) is an ℓEAS and $(V,*)$ is a Φ-associative algebra, then $(V\otimes A,\star)$ is an associative algebra.*
2. *If (A,Φ) is a nondegenerate ℓEAS and $(V\otimes A,\star)$ is an associative algebra, then $(V,*)$ is a Φ-associative algebra.*
3. *Let V be a nonzero vector space. If $(V\otimes T_A(V),\star)$ is an associative algebra, then (A,Φ) is an ℓEAS.*

Proof. Let $a,b,c\in A$ and $x,y,z\in V$. In $V\otimes A$,

$$\begin{aligned} xa\star(yb\star zc)&=\sum\sum x*_{a''\rhd(b'\to c')''}(y*_{b''\rhd c''}z)a'\to(b'\to c')',\\ (xa\star yb)\star zc&=\sum\sum(x*_{a''\rhd b''}y)*_{(a'\to b')''\rhd c''}z(a'\to b')'\to c'. \end{aligned}\tag{12}$$

1. We put

$$X=\sum\sum a'\to(b'\to c')'\otimes a''\rhd(b'\to c')''\otimes b''\rhd c''=(\Phi\otimes \mathrm{Id})\circ(\mathrm{Id}\otimes\Phi)(a\otimes b\otimes c).$$

By (9),

$$\begin{aligned}(\mathrm{Id}\otimes\Phi)(X)&=(\Phi\otimes\mathrm{Id}_A)\circ(\mathrm{Id}_A\otimes\tau)\circ(\Phi\otimes\mathrm{Id}_A)(a\otimes b\otimes c)\\ &=\sum\sum(a'\to b')'\to c'\otimes(a'\to b')''\rhd c''\otimes a''\rhd b''.\end{aligned}$$

As V is Φ-associative, we obtain that $\star$ is associative.

2. By composition, the following map is bijective:

$$\Psi=(\Phi\otimes\mathrm{Id}_A)\circ(\mathrm{Id}_A\times\Phi):\left\{\begin{array}{rcl} V^{\otimes 3}&\longrightarrow&V^{\otimes 3}\\ a\otimes b\otimes c&\longrightarrow&a'\to(b'\to c')'\otimes a''\rhd(b'\to c')''\otimes b''\rhd c''.\end{array}\right.$$

Let $a\otimes b\otimes c\in A^{\otimes 3}$ and $a_1\otimes b_1\otimes c_1=\Psi^{-1}(a\otimes b\otimes c)$. For any $x,y,z\in A$,

$$\begin{aligned} xa_1\star(yb_1\star zc_1)&=(x*_b(y*_c z))\,a,\\ (xa_1\star yb_1)\star zc_1&=\sum((x*_{b''\rhd c''}y)*_{b'\to c'}z)\,a.\end{aligned}$$

The associativity of $\star$ induces the axiom of Φ-associative algebra for V.

3. Let $x,y,z\in V$, nonzero (not necessarily distinct). From the associativity of $\star$, we immediately deduce from (12) that

$$\begin{aligned}&\sum\sum\sum a'\to(b'\to c')'\otimes(a''\rhd(b'\to c')'')'\to(b''\rhd c'')'\otimes(a''\rhd(b'\to c')'')''\rhd(b''\rhd c)''\\ &=\sum\sum(a'\to b')'\to c'\otimes(a'\to b')''\rhd c''\otimes a''\rhd b''.\end{aligned}$$ □

So (A,Φ) is an ℓEAS.

Remark 3.4. As a corollary, if $(\Omega,\to,\rhd)$ is an EAS, then Ω-associative algebras are 2-parameters associative algebras with $*_{\alpha,\beta}=*_{\alpha\rhd\beta}$. This will be formalized in Proposition 3.14 by an operad morphism.

Proposition 3.7. *Let (A,Φ) be an ℓEAS and let V be a nonzero vector space.*

1. *The following conditions are equivalent:*
   (a) *The associative algebra $T_A(V)\otimes A$ is generated by $V\otimes A$.*
   (b) *Φ is surjective.*

2. *The following conditions are equivalent:*

   (a) *The subalgebra $T_A(V)\otimes A$ generated by $V\otimes A$ is free.*

   (b) *Φ is injective.*

Proof. We denote by W the subalgebra of $T_A(V)\otimes A$ generated by $V\otimes A$. Note that it is graded by the length of words.

1. $(a)\Longrightarrow(b)$. Let $a\otimes b\in A^{\otimes 2}$. Let us choose a nonzero element x of V. Then $xxab\in A$. Because of the graduation, we can write this element under the form

$$xxab=\sum_{i=1}^{n}x_ia_i\star y_ib_i=\sum_{i=1}^{n}\sum x_iy_i(a_i''\rhd b_i'')(a_i'\to b_i').$$

Applying an element f of V^* such that $f(x)=1$, we obtain

$$\Phi\left(\sum_{i=1}^{n}f(x_i)f(y_i)a_i\otimes b_i\right)=a\otimes b,$$

so Φ is surjective.

1. $(b)\Longrightarrow(a)$. Let $x_1\ldots x_na_1\ldots a_n$ be a word of length n, and let us prove that it belongs to W by induction on n. This is obvious if $n=1$. Otherwise, there exists $x=\sum b_{n-1}\otimes b_n\in A^{\otimes 2}$, such that

$$\Phi\left(\sum b_{n-1}\otimes b_n\right)=a_n\otimes a_{n-1}.$$

By the induction hypothesis, $x_1\ldots x_{n-1}a_1\ldots a_{n-2}b_{n-1}\in W$, so

$$\begin{aligned}\sum x_1\ldots x_{n-1}a_1\ldots a_{n-2}b_{n-1}\star x_nb_n&=\sum\sum x_1\ldots x_na_1\ldots a_{n-2}(b_{n-1}''\rhd b_n'')(b_{n-1}'\to b_n')\\&=x_1\ldots x_na_1\ldots a_n\in W.\end{aligned}$$

2. $(a)\Longrightarrow(b)$. Because of the graduation, W is freely generated by $V\otimes A$. Let x be a nonzero element of V. If $\sum a_n\otimes b_n\neq 0$, by freeness, $\sum xa_n\star xb_n\neq 0$ and

$$\sum xa_n\star xb_n=\sum\sum xx(a_n''\rhd b_n'')(a_n'\to b_n')\neq 0,$$

So $\Phi\left(\sum a_n\otimes b_n\right)\neq 0$.

2. $(b)\Longrightarrow(a)$. We shall use the map

$$\Phi'=\tau\circ\Phi:\left\{\begin{array}{rcl}A\otimes A&\longrightarrow&A\otimes A\\a\otimes b&\longrightarrow&\sum a''\rhd b''\otimes a'\to b'.\end{array}\right.$$

As Φ is injective, Φ' is injective. Let $x_1,\ldots,x_n\in V$ and let $a_1,\ldots,a_n\in A$. An easy induction on n proves that

$$\begin{aligned}&x_1a_1\star\ldots\star x_na_n\\&=x_1\ldots x_n\left(\mathrm{Id}_A^{\otimes(n-2)}\otimes\Phi'\right)\circ\left(\mathrm{Id}_A^{\otimes(n-3)}\otimes\Phi'\otimes\mathrm{Id}_A\right)\circ\ldots\circ\left(\Phi'\otimes\mathrm{Id}_A^{\otimes(n-2)}\right)(a_1\otimes\ldots\otimes a_n).\end{aligned}$$

As a consequence, the following algebra map is injective:

$$\left\{\begin{array}{rcl}T(V\otimes A)&\longrightarrow&T_A(V)\otimes A\\x_1a_1\ldots x_na_n&\longrightarrow&x_1a_1\star\ldots\star x_na_n.\end{array}\right.$$

So the image of this morphism, which is W, is freely generated by $V\otimes A$. □

Remark 3.5. Consequently, for any vector space V, $(T_A(V),\star)$ is freely generated by $V\otimes A$ if, and only if, (A,Φ) is nondegenerate.

3.4 Operadic aspects and Koszul duality

In this section, (A, Φ) is an ℓEAS.

Notations 3.3. We denote the nonsymmetric operad of Φ-associative algebras by $\mathbf{As}_\Phi$, and the nonsymmetric operad of opposite Φ-associative algebras by $\mathbf{As}'_\Phi$. In other words, $\mathbf{As}_\phi$ is the nonsymetric operad generated by $A = \mathbf{As}_\phi(2)$, with the relations

$$a \circ_2 b = \sum a' \to b' \circ_1 a'' \rhd b'',$$

whereas $\mathbf{As}'_\phi$ is the nonsymetric operad generated by $A = \mathbf{As}'_\phi(2)$, with the relations

$$a \circ_1 b = \sum a' \to b' \circ_2 a'' \rhd b''.$$

We denote by $\mathrm{Sym}\mathbf{As}_\Phi$, respectively by $\mathrm{Sym}\mathbf{As}'_\Phi$, the operad of Φ-associative algebras, respectively of opposite Φ-associative algebras.

Remark 3.6.

1. $\mathrm{Sym}\mathbf{As}_\phi$, respectively $\mathrm{Sym}\mathbf{As}'_\Phi$, is the symmetrisation of the nonsymmetric operad $\mathbf{As}_\Phi$, respectively $\mathbf{As}'_\phi$.
2. If Φ is nondegenerate, then $\mathbf{As}'_\Phi = \mathbf{As}_{\Phi^{-1}}$.
3. $\mathrm{Sym}\mathbf{As}_\Phi$ and $\mathrm{Sym}\mathbf{As}'_\Phi$ are isomorphic operads, through the morphism

$$\begin{cases} \mathrm{Sym}\mathbf{As}_\Phi & \longrightarrow & \mathrm{Sym}\mathbf{As}'_\Phi \\ a \in A & \longrightarrow & a^{op} = a^{(12)}. \end{cases}$$

From the description of free Φ-associative algebras, we obtain a combinatorial description of $\mathbf{As}_\Phi$:

Proposition 3.8. *For any $n \geqslant 1$, $\mathbf{As}_\Phi(n)$ is the vector space $A^{\otimes(n-1)}$. For any $a_k \ldots a_1 \in A^{\otimes k} = \mathbf{As}_\Phi(k+1)$, for any $b_l \ldots b_1 \in A^{\otimes l} = \mathbf{As}_\Phi(l+1)$, for any $i \in [k+1]$,*

$$a_k \ldots a_1 \circ_i b_l \ldots b_1$$
$$= \begin{cases} b_l \ldots b_1 a_k \ldots a_1 \text{ if } i = 1, \\ a_k \ldots a_i(\Phi \otimes \mathrm{Id}^{\otimes(l-2)}) \circ \ldots \circ (\mathrm{Id} \otimes \Phi \otimes \mathrm{Id}^{\otimes(l-3)}) \circ (\mathrm{Id}^{\otimes(l-2)} \otimes \Phi)(a_{i-1} b_l \ldots b_1) a_{i-2} \ldots a_1 \\ \qquad \text{if } i \geqslant 2. \end{cases}$$

Example 3.3. Let us consider linearizations of EAS.

1. For $\mathbf{EAS}(A, \star)$, this simplifies as

$$\alpha_1 \ldots \alpha_k \circ_i \beta_1 \ldots \beta_l = \alpha_1 \ldots \alpha_{i-1}(\alpha_{i-1} \star \beta_1) \ldots (\alpha_{i-1} \star \beta_l)\alpha_i \ldots \alpha_k.$$

2. For $\mathbf{EAS}(\Omega)$, this simplifies as

$$\alpha_1 \ldots \alpha_k \circ_i \beta_1 \ldots \beta_l = \alpha_1 \ldots \alpha_{i-1}\beta_1 \ldots \beta_l \alpha_i \ldots \alpha_k.$$

   This operad is used in [6]. When Ω has two elements, this gives the operad of duplexes of vertices of cubes defined in [29, Section 6.3].

3. If $(A, \star)$ is a group, we obtain, for $\mathbf{EAS}'(A, \star)$, that

$$\alpha_1 \ldots \alpha_k \circ_i \beta_1 \ldots \beta_l = \alpha_1 \ldots \alpha_{i-2}\left(\alpha_{i-1} \star \beta_l^{-1} \star \ldots \star \beta_1^{-1}\right) \beta_1 \ldots \beta_l \alpha_i \ldots \alpha_k.$$

Proposition 3.9. *Let us assume that A is finite-dimensional.*

1. *Koszul dual of the nonsymmetric operad* $\mathbf{As}_\Phi$ *is isomorphic to* $\mathbf{As}'_{\Phi^*}$.
2. *Koszul dual of the nonsymmetric operad* $\mathbf{As}'_\Phi$ *is isomorphic to* $\mathbf{As}_{\Phi^*}$.
3. *Koszul dual of the operad* $\mathrm{Sym}\mathbf{As}_\Phi$ *is isomorphic to* $\mathrm{Sym}\mathbf{As}_{\Phi^*}$.

Proof. 1. We identify $\mathbf{As}_\Phi(2)^* = A$ and A^*. This identification induces a pairing between the free nonsymmetric operad $\mathbf{F}_A$ generated by A and the free nonsymmetric operad $\mathbf{F}_{A^*}$ generated by A^*. In particular, if $a, b \in A$, $f, g \in A^*$,

$$\langle f \circ_1 g, a \circ_1 b\rangle = f(a)g(b), \qquad \langle f \circ_2 g, a \circ_2 b\rangle = -f(a)g(b),$$
$$\langle f \circ_1 g, a \circ_2 b\rangle = 0, \qquad \langle f \circ_2 g, a \circ_1 b\rangle = 0.$$

We denote by I the space of relations of $\mathbf{As}_\Phi(3)$: this is the subspace of $\mathbf{F}_A$ generated by the elements

$$\sum a' \to b' \circ_1 a'' \rhd b'' - a \circ_2 b,$$

with $a, b \in A$. Note that $\mathbf{As}^!_\Phi$ is the quotient of $\mathbf{F}_{A^*}$ by the operadic ideal generated by $I^\perp$. We also denote by I' the space of relations of $\mathbf{As}_{\Phi^*}(3)$: this is the subspace of $\mathbf{F}_{A^*}$ generated by the elements

$$\sum f' \to g' \circ_2 f'' \rhd g'' - f \circ_1 g,$$

with $f, g \in A^*$. Let $a, b \in A$ and $f, g \in A^*$.

$$\begin{aligned}&\langle \sum f' \to g' \circ_2 f'' \rhd g'' - f \circ_1 g, \sum a' \to b' \circ_1 a'' \rhd b'' - a \circ_2 b\rangle\\ &= -\Phi^*(f \otimes g)(a \otimes b) - (f \otimes g)(\Phi(a \otimes b))\\ &= 0,\end{aligned}$$

so $I' \subseteq I^\perp$. Moreover,

$$\dim(\mathbf{F}_A(3)) = 2\dim(A)^2, \qquad \dim(I) = \dim(I') = \dim(A)^2,$$

so $\dim(I^\perp) = 2\dim(A)^2 - \dim(A)^2 = \dim(A)^2 = \dim(I')$ and finally $I^\perp = I'$.

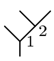
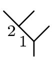
2. By duality.
3. By symmetrisation, $(\mathrm{Sym}\mathbf{As}_\Phi)^! = \mathrm{Sym}\mathbf{As}'_{\Phi^*}$, which is isomorphic to $\mathrm{Sym}\mathbf{As}_{\Phi^*}$, see Remark 3.6. □

Example 3.4. Let Ω be a finite EAS. Koszul dual of the operad $\mathbf{As}_A$ of Ω-associative algebra is generated by the products $\star_\alpha$, with $\alpha \in \Omega$, and the relations

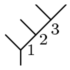
$$\forall \alpha, \beta \in \Omega, \qquad \left(\sum_{\substack{(\alpha',\beta') \in \Omega^2,\\ \phi(\alpha',\beta')=(\alpha,\beta)}} \star_{\alpha'} \circ (I, \star_{\beta'}) \right) = \star_\alpha(\star_\beta, I).$$

Theorem 3.10. *If A is finite-dimensional, the nonsymmetric operads* $\mathbf{As}_\Phi$ *and* $\mathbf{As}'_\Phi$ *as well as the operad* $\mathrm{Sym}\mathbf{As}_\Phi$ *are Koszul.*

Proof. We shall use the rewriting method of [2, 23]. We shall write elements of the free nonsymmetric operad generated by $\mathbf{As}_A(2)$ as planar trees which vertices are decorated by elements of A. The rewriting rules are

 $(a, b) \longrightarrow \sum$  $(a'' \to b'', a'' \rhd b'')$

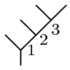
for any $a, b \in A$. There is only one family of critical monomials, which are the trees

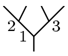
 (a, b, c)

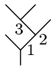
with $a, b, c \in A$. Koszularity of $\mathbf{As}_A$ comes from the confluence of the following diagram:

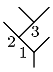
$$\begin{array}{ccccc} & & T_1 & & \\ & \swarrow & & \searrow & \\ T_2 & & & & T_3 \\ & \searrow & & & \downarrow \\ & & & & T_4 \\ & & & \swarrow & \\ & & T_5 & & \end{array} \tag{13}$$

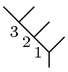
with

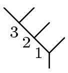
$T_1 =$  $(a, b, c),$

$T_2 = \sum$  $(a' \to b', a'' \rhd b'', c),$

$T_3 = \sum$  $(a, b' \to c', b'' \rhd c''),$

$T_4 = \sum\sum$  $(a' \to (b' \to c')', a'' \rhd (b' \to c')'', b'' \rhd c'')$

$T_5 = \sum\sum$  $((a' \to b')' \to c', (a' \to b')'' \rhd c'', a'' \rhd b'')$

$= \sum\sum\sum$  $(a' \to (b' \to c')', (a'' \rhd (b' \to c')'')' \to (b'' \rhd c'')', (a'' \rhd (b' \to c')'')'' \rhd (b'' \rhd c'')'').$

The equality between the two expressions of T_5 is equivalent to (9). $\square$

Here is another application of Diagram (13):

Proposition 3.11. *Let* $\mathbf{P}$ *be a nonsymmetric set operad such that for any* $n \geqslant 1$, *the following map is a linear isomorphism:*

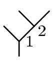
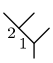
$$\iota_n : \left\{ \begin{array}{rcl} \mathbf{P}(2)^{\otimes(n-1)} & \longrightarrow & \mathbf{P}(n) \\ p_1 \otimes \ldots \otimes p_{n-1} & \longrightarrow & p_1 \circ_1 (p_2 \circ_1 (\ldots \circ_1 (p_{n-2} \circ_1 p_{n-1}) \ldots)). \end{array} \right.$$

Then there exists an ℓEAS (A, Φ) *such that* $\mathbf{P}$ *is isomorphic to* $\mathbf{As}_\Phi$.

Proof. We put $A = \mathbf{P}(2)$ as a vector space. As ι_3 is bijective, for any $a \otimes b \in A \otimes A$, there exists a unique $\Phi(a \otimes b) = \sum a' \to b' \otimes a'' \rhd b'' \in A \otimes A$ such that

$$a \circ_2 b = \sum (a' \to b') \circ_1 (a'' \rhd b''),$$

or, equivalently,

 $(a, b) = \sum$  $(a' \to b', a'' \rhd b'').$

For any $a, b, c \in A$, let us compute $a \circ_2 (b \circ_2 c)$ into two different ways. This element is the tree T_1 of (13), and, following the two paths of this diagram, we obtain that, in $\mathbf{P}(3)$,

$$\sum\sum (a' \to b')' \to c' \circ_1 ((a' \to b')'' \rhd c'' \circ_1 (a'' \rhd b''))$$
$$= \sum\sum\sum a' \to (b' \to c')' \circ_1 (a'' \rhd (b' \to c')'')' \to (b'' \rhd c'')' \circ ((a'' \rhd (b' \to c')'')'' \rhd (b'' \rhd c'')'')).$$

As ι_4 is an isomorphism, we obtain the axioms of ℓEAS for (A, Φ). Hence, we obtain an operad isomorphism from $\mathbf{As}_\Phi$ to $\mathbf{P}$, sending $*_a$ to a for any $a \in A$. □

3.5 Associative products

We now look for operad morphisms from the operad of associative algebras to the operad $\mathrm{Sym}\mathbf{As}_\Phi$, where (A, Φ) is an ℓEAS, or equivalently to products $m \in \mathrm{Sym}\mathbf{As}_\Phi(2)$ which are associative, that is to say such that $m \circ_1 m = m \circ_2 m$.

Proposition 3.12. *Let (A, Φ) be an ℓEAS. The associative products in $\mathrm{Sym}\mathbf{As}_\Phi(2)$ are the elements of the form*

$$m = *_a \qquad \text{or} \qquad m = *_a^{op},$$

where $a \in A$ is such that $\Phi(a \otimes a) = a \otimes a$. The products $m \in \mathrm{Sym}\mathbf{As}_\Phi(2)$ such that $m \circ_2 m = 0$ are the elements of the form

$$m = *_a,$$

where $a \in A$ is such that $\Phi(a \otimes a) = 0$.

Proof. Let $m = *_a + *_b^{op} \in \mathrm{Sym}\mathbf{As}_\Phi(2)$. Let $V = T_A(\mathrm{Vect}(x, y, z))$ be the free Φ-associative algebra generated by three elements x, y, z. In V,

$$\begin{aligned} m \circ (\mathrm{Id} \otimes m)(x \otimes y \otimes z) &= m(x \otimes (ayz + bzy)) \\ &= \tau \circ \Phi(a \otimes a)xyz + abyzx + \tau \circ \Phi(a \otimes b)xzy + bbzyx, \\ m \circ (m \otimes \mathrm{Id})(x \otimes y \otimes z) &= m((axy + byx) \otimes z) \\ &= aaxyz + bayxz + \tau \circ \Phi(b \otimes a)zxy + \tau \circ \Phi(b \otimes b)zxy. \end{aligned}$$

1. If m is associative, identifying the terms in yzx, we find $a \otimes b = 0$, so $a = 0$ or $b = 0$. Identifying the terms in xyz, we find that $\tau \circ \Phi(a \times a) = a \otimes a$. Similarly, the identification of the terms in zyx gives that $\Phi(b \otimes b) = b \otimes b$. Conversely, if $\Phi(a) = a \otimes a$, then

$$a \circ_1 a = aa, \qquad a \circ_2 a = \Phi(a \otimes a) = aa,$$

   so $*_a$ is associative, and its opposite $*_a^{op}$ is associative too.

2. If $m \circ_2 m = 0$, identifying the term in zyx, we find that $b = 0$. Identifying the term in xyz, we find that $\tau \circ \Phi(a \times a) = 0$. Conversely, if $\Phi(a \otimes a) = 0$, then

$$a \circ_2 a = \Phi(a \otimes a) = 0.$$

□

Remark 3.7. If (A, Φ) is the linearization of an EAS $(\Omega, \to, \rhd)$, we obtain that:

- The associative elements $m \in \mathrm{Sym}\mathbf{As}_\Phi(2)$ are the elements of the form

$$m = \sum_{\alpha \in \Omega} \lambda_\alpha *_\alpha \qquad \text{or} \qquad m = \sum_{\alpha \in \Omega} \lambda_\alpha *_\alpha^{op},$$

  such that

$$\forall (\alpha, \beta) \in \Omega^2, \qquad \lambda_\alpha \lambda_\beta = \sum_{\substack{(\gamma,\delta) \in \Omega^2, \\ \phi(\gamma,\delta) = (\alpha,\beta)}} \lambda_\gamma \lambda_\delta. \tag{14}$$

- The elements $m \in \mathrm{Sym}\mathbf{As}_\Phi(2)$ such that $m \circ_2 m = 0$ are the elements of the form

$$m = \sum_{\alpha \in \Omega} \lambda_\alpha *_\alpha,$$

  such that

$$\forall (\alpha, \beta) \in \Omega^2, \qquad \sum_{\substack{(\gamma,\delta) \in \Omega^2, \\ \phi(\gamma,\delta)=(\alpha,\beta)}} \lambda_\gamma \lambda_\delta = 0. \tag{15}$$

Example 3.5. Working with $\mathbf{EAS}(\Omega)$, then $\phi(\alpha, \beta) = (\beta, \alpha)$ and Condition (14) is empty: any linear combination of $*_\alpha$ is associative, as well as their opposite.

Example 3.6. Let us give the associative products for EAS of cardinality two. We only mention the spans of $*_\alpha$, their opposite should be added. Here, λ, μ are scalars.

Cases	Associative products	$m \circ_2 m = 0$
A1	$\lambda *_a$	$\lambda(*_a - *_b)$
A2	$\lambda *_a$	$\lambda(*_a - *_b)$
C1	$\lambda *_a$	0
C3	$\lambda *_a,\ \lambda *_b$	0
C5	$\lambda *_b$	0
C6	$\lambda *_a$	0
$\mathbf{E1}' - \mathbf{E2}'$	$\lambda *_a$	$\lambda(*_a - *_b)$
$\mathbf{E3}'$	$\lambda *_a\ \lambda *_b$	$\lambda(*_a - *_b)$
F1	$\lambda *_a$	$\lambda(*_a - *_b)$
F3	$\lambda *_a + \mu *_b$	0
F4	$\lambda(*_a + *_b),\ \lambda *_a$	0
H1	$\lambda *_a$	0
H2	$\lambda(*_a + *_b),\ \lambda *_a$	0

Corollary 3.13. *Let $(\Omega, \star)$ be a group. The nonzero associative products in $\mathrm{Sym}\mathbf{As}_{\mathbf{EAS}(\Omega,\star)}$ or in $\mathrm{Sym}\mathbf{As}_{\mathbf{EAS}'(\Omega,\star)}$ are the elements of one of the form*

$$\lambda \sum_{\alpha \in H} *_\alpha \qquad \text{or} \qquad \lambda \sum_{\alpha \in H} *_\alpha^{op},$$

where λ is a nonzero scalar and H is a subgroup of Ω.

Proof. Case of $\mathbf{EAS}(\Omega, \star)$. Then (14) becomes

$$\forall (\alpha, \beta) \in \Omega^2, \qquad \lambda_{\alpha \star \beta} \lambda_\alpha = \lambda_\alpha \lambda_\beta.$$

Let $H = \{\alpha \in \Omega, \lambda_\alpha \neq 0\}$. We assume that H is nonempty. If $\alpha \in H$, for any $\beta \in H$, $\lambda_{\alpha \star \beta} = \lambda_\beta$. In particular:

- If $\beta \in H$, then $\alpha \star \beta \in H$.
- If $\beta = e_\Omega$, then $\lambda_\alpha = \lambda_{e_\Omega} \neq 0$: $e_\Omega \in H$.
- If $\beta = \alpha^{\star -1}$, $\lambda_{e_\Omega} = \lambda_{\alpha^{\star -1}} \neq 0$: $\alpha^{\star -1} \in H$.

Therefore, H is a subgroup of Ω. Let $\alpha, \beta \in I$, then $\alpha' = \alpha \star \beta^{-1} \in H$. From (14), we deduce that $\lambda_{\alpha' \star \beta} = \lambda_\alpha = \lambda_\beta$. Let λ be the common value of λ_α for any $\alpha \in H$; the result is the immediate.

Case of $\mathbf{EAS}'(\Omega, \star)$. Then (14) becomes

$$\forall (\alpha, \beta) \in \Omega^2, \qquad \lambda_{\alpha \star \beta^{\star -1}} \lambda_\alpha = \lambda_\alpha \lambda_\beta.$$

The proof is similar to the case of $\mathbf{EAS}(\Omega, \star)$. □

3.6 Operadic morphisms between $\mathbf{As}^2_\Omega$ and $\mathbf{As}_\Omega$

Proposition 3.14. *Let $(\Omega, \rightarrow, \triangleright)$ and let (A, Φ) be its linearization, that is to say $A = \mathbb{K}\Omega$ and*

$$\Phi : \begin{cases} A \otimes A & \longrightarrow & A \otimes A \\ \alpha \otimes \beta & \longrightarrow & \alpha \rightarrow \beta \otimes \alpha \triangleright \beta, \end{cases}$$

where $\alpha, \beta \in \Omega$. The following defines an operad morphism:

$$\Theta_\Omega : \begin{cases} \mathbf{As}^2_\Omega & \longrightarrow & \mathbf{As}_\Phi \\ *_{\alpha,\beta} & \longrightarrow & *_{\alpha \triangleright \beta}. \end{cases}$$

Proof. Let us consider an Ω-associative algebra $(A, (*_\alpha)_{\alpha \in \Omega})$. For any $(\alpha, \beta) \in \Omega^2$, we put $*_{\alpha,\beta} = *_{\alpha \triangleright \beta}$. Then, for any $x, y, z \in A$,

$$\begin{aligned} x *_{\alpha,\beta \rightarrow \gamma} (y *_{\beta,\gamma} z) &= x *_{\alpha \triangleright (\beta \rightarrow \gamma)} (y *_{\beta \rightarrow \gamma} z) \\ &= (x *_{(\alpha \triangleright (\beta \rightarrow \gamma)) \triangleright (\beta \rightarrow \gamma)} y) *_{(\alpha \triangleright (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow \gamma)} z \\ &= (x *_{(\alpha \rightarrow \beta) \triangleright \gamma} y) *_{\alpha \triangleright \beta} z \\ &= (x *_{\alpha \rightarrow \beta,\gamma} y) *_{\alpha,\beta} z. \end{aligned}$$

Hence, $(A, (*_{\alpha,\beta})_{\alpha,\beta, \in \Omega})$ is a 2-parameter Ω-associative algebra, which implies the existence of the operadic morphism Θ_Ω. □

Proposition 3.15. *Let $(\Omega, \rightarrow)$ be an associative semigroup with the right inverse condition. We consider the EAS $\Omega' = \mathbf{EAS}(\Omega, \rightarrow) \times \mathbf{EAS}'(\Omega, \rightarrow)$ and denote by (A', Φ') its linearization. The following defines a surjective operad morphism:*

$$\Theta'_\Omega : \begin{cases} \mathbf{As}^2_\Omega & \longrightarrow & \mathbf{As}_{\Phi'} \\ *_{\alpha,\beta} & \longrightarrow & *_{(\alpha,\beta)}. \end{cases}$$

Proof. The EAS structure of Ω' is given by

$$\forall (\alpha, \beta, \gamma, \delta) \in \Omega^4, \qquad \begin{aligned} (\alpha, \beta) \rightarrow (\gamma, \delta) &= (\alpha \rightarrow \gamma, \delta), \\ (\alpha, \beta) \triangleright (\gamma, \delta) &= (\alpha, \beta \triangleright \delta). \end{aligned}$$

Let $(A, (*_{(\alpha,\beta)})_{(\alpha,\beta) \in \Omega^2})$ be an Ω'-associative algebra. For any $(\alpha, \beta) \in \Omega'$, we put $*_{\alpha,\beta} = *_{\alpha,\beta}$. Then, for any $x, y, z \in A$, using the right inverse property for the second equality,

$$\begin{aligned} (x *_{\alpha,\beta} y) *_{\alpha \rightarrow \beta,\gamma} z &= (x *_{(\alpha,\beta)} y) *_{(\alpha \rightarrow \beta,\gamma)} z \\ &= (x *_{(\alpha,(\beta \rightarrow \gamma) \triangleright \gamma)} y) *_{(\alpha \rightarrow \beta,\gamma)} z \\ &= (x *_{(\alpha,\beta \rightarrow \gamma) \triangleright (\beta,\gamma)} y) *_{(\alpha,\beta \rightarrow \gamma) \rightarrow (\beta,\gamma)} z \\ &= x *_{(\alpha,\beta \rightarrow \gamma)} (y *_{(\beta,\gamma)} z) \\ &= x *_{\alpha,\beta \rightarrow \gamma} (y *_{\beta,\gamma} z). \end{aligned}$$

Hence, $(A, (*_{\alpha,\beta})_{\alpha,\beta \in \Omega})$ is a 2-parameter Ω-associative algebra. This implies the existence of the morphism Θ'_Ω. □

Remark 3.8. Except if $\omega = |\Omega| = 1$, this morphism is not bijective: the dimension of $\mathbf{As}^2_\Omega(3)$ is $(2\omega - 1)\omega^3$, whereas the dimension of $\mathbf{As}_{\Phi'}(3)$ is ω^4.

4 Links with other operads

4.1 Post-Lie and ComTriAs algebras

Let us consider post-Lie algebras [30], see also [7, 8, 9, 10, 11, 12, 19] for applications and developments. Recall that a post-Lie algebra is a family $(A, *, \{,\})$ where A is a vector space and $*$ and $\star$ are bilinear products on A such that $(A, \{,\})$ is a Lie algebra and, for any $x, y, z \in A$,

$$\begin{aligned} x * \{y, z\} &= (x * y) * z - x * (y * z) - (x * z) * y + x * (z * y), \\ \{x, y\} * z &= \{x * z, y\} + \{x, y * z\}. \end{aligned}$$

Let us start with the Koszul dual of the operad of post-Lie algebras, namely the operad of ComTriAs algebras [36]:

Definition 4.1. *A ComTriAs algebra is a family $(A, \cdot, \star)$, where A is a vector space and $\cdot$ and $\star$ are bilinear products on A such that for any $x, y, z \in A$,*

$$\begin{aligned} x \cdot y &= y \cdot x, \\ (x \cdot y) \cdot z &= x \cdot (y \cdot z), \\ (x \star y) \star z &= x \star (y \star z), \\ (x \star y) \star z &= x \star (y \cdot z), \\ (x \cdot y) \star z &= x(y \star z). \end{aligned}$$

Note that the products $\cdot$ and $\star$ are respectively denoted by $\perp$ and $\dashv$ in [36].

Apart from the first one, these axioms are the ones of a particular example of generalized associative algebra:

Proposition 4.2. *Let Ω be the EAS* **C3** *(that is to say the EAS associated to the semigroup $(\mathbb{Z}/2\mathbb{Z}, \times)$. Then any ComTriAs algebra $(A, \cdot, *)$ is an Ω-associative algebra, with $*_{\overline{0}} = \star$ and $*_{\overline{1}} = \cdot$.*

Consequently, we obtain an operad morphism from the operad of Ω-associative algebra to the operad of ComTriAs algebras. Using Koszul duality:

Corollary 4.3. *Let (V, Φ) be the ℓEAS dual to* **C3***: V is two-dimensional, with basis (e_1, e_2), and the basis of Φ is the basis $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$ is*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then any opposite Φ-associative algebra is a post-Lie algebra, with, for any $x, y \in A$,

$$\{x, y\} = x *_2 y - y *_2 x, \qquad x * y = x *_1 y.$$

We conjecture that the associated operad morphism from the operad of post-Lie algebras into the operad of opposite Φ-associative algebra is injective.

4.2 Diassociative and dendriform algebras

Definition 4.4. *[20] A diassociative algebra is a family $(A, \dashv, \vdash)$ where A is a vector space and $\dashv$ and $\vdash$ are bilinear products on A such that for any $x, y, z \in A$,*

$$(x \dashv y) \dashv z = x \dashv (y \dashv z), \tag{16}$$
$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \tag{17}$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{18}$$
$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \tag{19}$$
$$(x \vdash y) \vdash z = x \vdash (y \vdash z). \tag{20}$$

Proposition 4.5. *Let $(A, \dashv, \vdash)$ be a diassociative algebra.*

1. *$(A, \dashv, \vdash)$ is an opposite Ω-associative algebra, with the EAS laws*

$\rightarrow$	$\dashv$	$\vdash$
$\dashv$	$\dashv$	$\vdash$
$\vdash$	$\vdash$	$\vdash$

$\triangleright$	$\dashv$	$\vdash$
$\dashv$	$\dashv$	$\dashv$
$\vdash$	$\vdash$	$\vdash$

or

$\triangleright$	$\dashv$	$\vdash$
$\dashv$	$\vdash$	$\dashv$
$\vdash$	$\vdash$	$\vdash$

These EAS are isomorphic to **C3** *and* **C6**.

2. *$(A, \dashv, \vdash)$ is an Ω-associative algebra, with the EAS laws*

$\rightarrow$	$\dashv$	$\vdash$
$\dashv$	$\dashv$	$\dashv$
$\vdash$	$\dashv$	$\vdash$

$\triangleright$	$\dashv$	$\vdash$
$\dashv$	$\dashv$	$\dashv$
$\vdash$	$\vdash$	$\dashv$

or

$\triangleright$	$\dashv$	$\vdash$
$\dashv$	$\dashv$	$\dashv$
$\vdash$	$\vdash$	$\vdash$

These EAS are isomorphic to **C6** *and* **C3**.

Proof.
1. This is a reformulation of axioms (16), (18), (19) and (20), and of axioms (17), (18), (19) and (20).
2. This is a reformulation of axioms (16), (17), (18) and (19), and of axioms (16), (17), (18) and (20). □

Using Koszul duality, we obtain dendriform algebras: recall that a dendriform algebra is a family $(A, \prec, \succ)$ where A is a vector space and $\prec$ and $\succ$ are bilinear products on A such that for any $x, y, z \in A$,

$$(x \prec y) \prec z = x \prec (y \prec z + y \succ z),$$
$$(x \succ y) \prec z = x \succ (y \prec z),$$
$$x \succ (y \succ z) = (x \prec y + x \succ y) \succ z.$$

Corollary 4.6.
1. *Let (V, Φ) be one of the two following 2-dimensional ℓ-EAS, where the matrix of Φ is expressed in the basis $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then any Φ-associative algebra is a dendriform algebra, with $\prec = *_1$ and $\succ = *_2$.

2. *Let (V, Φ) is one of the two following 2-dimensional ℓ-EAS, where the matrix of Φ is expressed in the basis $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then any opposite Φ-associative algebra is a dendriform algebra, with $\prec = *_1$ and $\succ = *_2$.

4.3 Triassociative and tridendriform algebras

Definition 4.7. *[22] A triassociative algebra is a family $(A, \dashv, \vdash, \perp)$ where A is a vector space and $\dashv$, $\vdash$ and $\perp$ are bilinear products on A such that for any $x, y, z \in A$,*

$$(x \dashv y) \dashv z = x \dashv (y \dashv z), \tag{21}$$
$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \tag{22}$$
$$(x \dashv y) \dashv z = x \dashv (y \perp z), \tag{23}$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{24}$$
$$(x \perp y) \dashv z = x \perp (y \dashv z), \tag{25}$$
$$(x \dashv y) \perp z = x \perp (y \vdash z), \tag{26}$$
$$(x \vdash y) \perp z = x \vdash (y \perp z), \tag{27}$$
$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \tag{28}$$
$$(x \perp y) \vdash z = x \vdash (y \vdash z), \tag{29}$$
$$(x \vdash y) \vdash z = x \vdash (y \vdash z). \tag{30}$$

Proposition 4.8. *Let $(A, \dashv, \vdash, \perp)$ be a triassociative algebra.*

1. *$(A, \dashv, \vdash, \perp)$ is an opposite Ω-associative algebra, with the EAS laws*

$\rightarrow$	$\dashv$	$\vdash$	$\perp$
$\dashv$	$\dashv$	$\vdash$	$\perp$
$\vdash$	$\vdash$	$\vdash$	$\vdash$
$\perp$	$\perp$	$\vdash$	$\perp$

$\rhd$	$\dashv$	$\vdash$	$\perp$
$\dashv$	$\dashv$	$\dashv$	$\dashv$
$\vdash$	$\vdash$	$\vdash$	$\vdash$
$\perp$	$\vdash$	$\perp$	$\perp$

or

$\rhd$	$\dashv$	$\vdash$	$\perp$
$\dashv$	$\vdash$	$\dashv$	$\dashv$
$\vdash$	$\vdash$	$\vdash$	$\vdash$
$\perp$	$\vdash$	$\perp$	$\perp$

or

$\rhd$	$\dashv$	$\vdash$	$\perp$
$\dashv$	$\perp$	$\dashv$	$\dashv$
$\vdash$	$\vdash$	$\vdash$	$\vdash$
$\perp$	$\vdash$	$\perp$	$\perp$

2. *$(A, \dashv, \vdash, \perp)$ is an Ω-associative algebra, with the EAS laws*

$\rightarrow$	$\vdash$	$\dashv$	$\perp$
$\vdash$	$\vdash$	$\dashv$	$\perp$
$\dashv$	$\dashv$	$\dashv$	$\dashv$
$\perp$	$\perp$	$\dashv$	$\perp$

$\rhd$	$\vdash$	$\dashv$	$\perp$
$\vdash$	$\vdash$	$\vdash$	$\vdash$
$\dashv$	$\dashv$	$\dashv$	$\dashv$
$\perp$	$\dashv$	$\perp$	$\perp$

or

$\rhd$	$\vdash$	$\dashv$	$\perp$
$\vdash$	$\dashv$	$\vdash$	$\vdash$
$\dashv$	$\dashv$	$\dashv$	$\dashv$
$\perp$	$\dashv$	$\perp$	$\perp$

or

$\rhd$	$\vdash$	$\dashv$	$\perp$
$\vdash$	$\perp$	$\vdash$	$\vdash$
$\dashv$	$\dashv$	$\dashv$	$\dashv$
$\perp$	$\dashv$	$\perp$	$\perp$

Proof.

1. This is a reformulation of axioms((21) or (22) or (23)) and (24) – (30).
2. This is a reformulation of axioms (21) – (27), and ((28) or (29) or (30)). □

Using Koszul duality, we obtain tridendriform algebras [20, 3, 28], that is to say families $(A, \prec, \succ, \cdot)$ where A is a vector space and $\prec$, $\succ$ and $\cdot$ are bilinear products on A such that for any $x, y, z \in A$,

$$\begin{aligned}
(x \prec y) \prec z &= x \prec (y \prec z + y \succ z + y \cdot z),\\
(x \succ y) \prec z &= x \succ (y \prec z),\\
x \succ (y \succ z) &= (x \prec y + x \succ y + x \cdot y) \succ z,\\
(x \succ y) \cdot z &= x \succ (y \cdot z),\\
(x \prec y) \cdot z &= x \cdot (y \succ z),\\
(x \cdot y) \prec z &= x \cdot (y \prec z),\\
(x \cdot y) \cdot z &= x \cdot (y \cdot z).
\end{aligned}$$

Corollary 4.9. *Let (V, Φ) is one of the three following 3-dimensional ℓ-EAS, where the matrix of Φ is expressed in the basis $(e_1 \otimes e_1, e_1 \otimes e_2, e_1 \otimes e_3, e_2 \otimes e_1, e_2 \otimes e_2, e_2 \otimes e_3, e_3 \otimes e_1, e_3 \otimes e_2, e_3 \otimes e_3)$:*

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

Then any Φ-associative algebra is a tridendriform algebra, with $\prec = *_1$, $\succ = *_2$ and $\perp = *_3$. Any opposite Φ-associative algebra is a tridendriform algebra, with $\prec = *_2$, $\succ = *_1$ and $\perp = *_3$.

4.4 Dual duplicial and duplicial algebras

Definition 4.10. *[36] A dual duplicial algebra is a family $(A, \prec, \succ)$ where A is a vector space and $\prec$ and $\succ$ are bilinear products on A such that for any $x, y, z \in A$,*

$$(x \prec y) \prec z = x \prec (y \prec z), \tag{31}$$

$$(x \prec x) \succ z = 0, \tag{32}$$

$$(x \succ y) \prec z = x \succ (y \prec z), \tag{33}$$

$$0 = x \prec (y \succ z), \tag{34}$$

$$(x \succ y) \succ z = x \succ (y \succ z). \tag{35}$$

Proposition 4.11. *Let (V, Φ)be the following 2-dimensional ℓ-EAS, where the matrix of Φ is expressed in the basis $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$:*

$$\begin{pmatrix} 1&0&0&0\\ 0&0&0&0\\ 0&1&0&0\\ 0&0&0&1 \end{pmatrix}.$$

Then any dual duplicial algebra $(A, \prec, \succ)$ is an opposite Φ-associative algebra, with $*_1 = \prec$ and $*_2 = \succ$, and aΦ-associative algebra, with $*_1 = \succ$ and $*_2 = \prec$.

Proof. This is a reformulation of axioms (31) – (33) and (35), and of axioms (31) and (33) – (35). □

Using Koszul duality, we recover duplicial algebra [21], that is to say families $(A, \prec, \succ)$ where A is a vector space and $\prec$ and $\succ$ are bilinear products on A such that for any $x, y, z \in A$,

$$(x \prec y) \prec z = x \prec (y \prec z),$$
$$(x \succ y) \prec z = x \succ (y \prec z),$$
$$x \succ (y \succ z) = (x \succ y) \succ z.$$

Corollary 4.12. *Let (V, Φ)be the following 2-dimensional ℓ-EAS, where the matrix of Φ is expressed in the basis $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$:*

$$\begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&1 \end{pmatrix}.$$

Then any Φ-associative algebra is a duplicial algebra, with $*_1 = \prec$ and $*_2 = \succ$, and any opposite Φ-associative algebra is a duplicial algebra, with $*_1 = \succ$ and $*_2 = \prec$.

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