On extended associative semigroups

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Abstract

We study extended associative semigroups (briefly, EAS), an algebraic structure used to define generalizations of the operad of associative algebras, and the subclass of commutative extended diassociative semigroups (briefly, CEDS), which are used to define generalizations of the operad of pre-Lie algebras. We give families of examples based on semigroups or on groups, as well as a classification of EAS of cardinality two. We then define linear extended associative semigroups as linear maps satisfying a variation of the braid equation. We explore links between linear EAS and bialgebras and Hopf algebras. We also study the structure of nondegenerate finite CEDS and show that they are obtained by semidirect and direct products involving two groups.

Keywords. Semigroups, diassociative semigroups, braid equation

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Contents

1 Introduction

Recently, numerous parametrizations of well-known operads were introduced. Choosing a set Ω of parameters, any product defining the considered operad is replaced by a bunch of products indexed by Ω, and various relations are defined on them, mimicking the relations defining the initial operads. One can first require that any linear spans of the parametrized products also satisfy the relations of the initial operads this is the *matching* parametrization. For example, matching Rota-Baxter algebras, associative, dendriform, pre-Lie algebras are introduced in [10], see also [2] for pre-Lie algebras. Another way is the use of one or more semigroup structures on Ω: this it the *family* parametrization. For example, family Rota-Baxter algebras, dendriform, pre-Lie algebras are introduced and studied in [11, 12, 7]. A way to obtain both these parametrizations for dendriform algebras is introduced in [3], with the help of a generalization of diassociative semigroups, namely extended diassociative semigroups (EDS). A similar result is obtained for pre-Lie algebras in [5], with the notion of commutative extended semigroup (CEDS). A two-parameter version for dendriform algebras and pre-Lie algebras is described in [6]. Finally, in [4], the parametrization of the associative operad is introduced, with the notion of extended associated semigroup (EAS).

We study in this paper CEDS and EAS used for the parametrization of the pre-Lie and the associative operads. An EAS is a set Ω with two operations $\rightarrow$ and $\triangleright$, satisfying the following axioms:

$$\begin{aligned}\alpha \rightarrow (\beta \rightarrow \gamma) &= (\alpha \rightarrow \beta) \rightarrow \gamma,\\ (\alpha \triangleright (\beta \rightarrow \gamma)) \rightarrow (\beta \triangleright \gamma) &= (\alpha \rightarrow \beta) \triangleright \gamma,\\ (\alpha \triangleright (\beta \rightarrow \gamma)) \triangleright (\beta \triangleright \gamma) &= \alpha \triangleright \beta.\end{aligned}$$

In particular, $(\Omega, \rightarrow)$ is an associative semigroup. CEDS are EAS satisfying the complementary axioms

$$\begin{aligned}&\alpha \rightarrow (\beta \rightarrow \gamma) = (\alpha \rightarrow \beta) \rightarrow \gamma = (\beta \rightarrow \alpha) \rightarrow \gamma,\\ &\alpha \triangleright (\beta \rightarrow \gamma) = \alpha \triangleright \gamma,\end{aligned}$$

and dual CEDS are EAS satisfying the complementary axioms

$$\begin{aligned}(\alpha \triangleright \beta) \rightarrow \gamma &= \alpha \rightarrow \gamma,\\ (\alpha \triangleright \beta) \triangleright \gamma &= (\alpha \triangleright \gamma) \triangleright \beta.\end{aligned}$$

Here are particular examples of EAS:

- If Ω is a set, putting

  $$\forall \alpha, \beta \in \Omega, \qquad \alpha \rightarrow \beta = \beta, \qquad \alpha \triangleright \beta = \alpha,$$

  we obtain an EAS, which is both a CEDS and a dual CEDS, denoted by $\mathbf{EAS}(\Omega)$. This EAS leads to the notion of matching (pre-Lie, associative, dendriform...) algebras.

- If $(\Omega, \rightarrow)$ is an associative semigroup, it is an EAS with

  $$\forall \alpha, \beta \in \Omega, \qquad \alpha \triangleright \beta = \alpha.$$

  This EAS is denoted by $\mathbf{EAS}(\Omega, \rightarrow)$. This leads to the notion of $(\Omega, \rightarrow)$-family (associative, pre-Lie if Ω is commutative, dendriform...) algebras.

- If $(\Omega, \star)$ is a group, it is a CEDS, with

  $$\forall \alpha, \beta \in \Omega, \qquad \alpha \rightarrow \beta = \beta, \qquad \alpha \triangleright \beta = \alpha \star \beta^{\star -1},$$

  It is denoted by $\mathbf{EAS}'(\Omega, \star)$. It is a dual CEDS if, and only if, $(\Omega, \star)$ is an abelian group.

There are more EAS, and in particular we give a classification of the thirteen EAS of cardinality 2. But these three examples are specially interesting: we prove in the third section of this paper that any finite nondegenerate CEDS is the direct product of a semidirect product $\mathbf{EAS}(\Omega_1, *) \rtimes \mathbf{EAS}'(\Omega_2, \star)$ with a CEDS $\mathbf{EAS}(\Omega_3)$, see Theorem 3.14.

In the fourth section of this paper, we are interested in a linear version of EAS, CEDS and dual CEDS, based on a linear version of Lemma 2.6. An ℓEAS is pair (A, Φ), where A is a vector space and $\Phi : A \otimes A \longrightarrow A \otimes A$ is a linear map satisfying the *ℓEAS braid equation*:

$$(\mathrm{Id} \otimes \Phi) \circ (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Phi) = (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \tau) \circ (\Phi \otimes \mathrm{Id}).$$

An ℓEAS is an ℓCEDS if it satisfies *the commutation relation*:

$$(\mathrm{Id} \otimes \Phi) \circ (\mathrm{Id} \otimes \tau) \circ (\tau \otimes \mathrm{Id}) \circ (\Phi \otimes \mathrm{Id}) = (\tau \otimes \mathrm{Id}) \circ (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Phi) \circ (\mathrm{Id} \otimes \tau),$$

and is a dual ℓCEDS if it satisfies the *dual commutation relation*:

$$(\Phi \otimes \mathrm{Id}) \circ (\tau \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \tau) \circ (\mathrm{Id} \otimes \Phi) = (\mathrm{Id} \otimes \tau) \circ (\mathrm{Id} \otimes \Phi) \circ (\Phi \otimes \mathrm{Id}) \circ (\tau \otimes \mathrm{Id}).$$

In particular, let $(\Omega, \rightarrow, \triangleright)$ be a set with two operations. We denote by $\mathbb{K}\Omega$ the vector space generated by Ω and we define $\Phi : \mathbb{K}\Omega \otimes \mathbb{K}\Omega \longleftrightarrow \mathbb{K}\Omega \otimes \mathbb{K}\Omega$ by

$$\forall \alpha, \beta \in \Omega, \qquad \Phi(\alpha \otimes \beta) = (\alpha \rightarrow \beta) \otimes (\alpha \triangleright \beta).$$

Then $(\mathbb{K}\Omega, \Phi)$ is an ℓEAS (respectively an ℓCEDS, a dual ℓCEDS) if, and only if, $(\Omega, \rightarrow, \triangleright)$ is an EAS (respectively a CEDS, a dual CEDS). Other examples of ℓEAS of dimension 2 are given is Example 4.1.

In the last section, we introduce two functors taking their values in the category of ℓEAS. The first one (Proposition 5.1) is defined on the category of bialgebras (not necessarily unitary nor counitary) and generalizes the construction of $\mathbf{EAS}(\Omega, \rightarrow)$. The second one (Proposition 5.7) is defined is on the category of Hopf algebras and generalizes the construction $\mathbf{EAS}'(\Omega, *)$. These objects are studied with the help of left units and counits (Definition 4.3): if (A, Φ) is an ℓEAS, an element $a \in A$ is a left unit if for any $b \in A$, $\Phi(a \otimes b) = b \otimes a$. An element $f \in A^*$ is a left counit if for any $a, b \in A$, $(f \otimes \mathrm{Id}) \circ \Phi(a \otimes b) = f(b)a$. In the case of an ℓEAS coming from a Hopf algebra, this is closely related to the notion of right integral (Proposition 5.11). We prove in Theorem 5.12 that we can associate to any convenient pair (a, f) of a unit and a counit a bialgebra structure on A, recovering in this way ℓEAS coming from a bialgebra. This is finally applied to ℓEAS defined from Hopf algebras of groups.

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Notations 1.1. $\mathbb{K}$ is a commutative field. All the vector spaces in this text are taken over $\mathbb{K}$.

2 Extended (di)associative semigroups

2.1 Commutative extended diassociative semigroup

Let us first recall this definition of [3], where it is related to a parametrization of the operad of dendriform algebras:

Definition 2.1. *1. A* diassociative semigroup *is a family* $(\Omega, \leftarrow, \rightarrow)$*, where* Ω *is a nonempty set and* $\leftarrow, \rightarrow: \Omega \times \Omega \longrightarrow \Omega$ *are maps such that, for any* $\alpha, \beta, \gamma \in \Omega$*:*

$$(\alpha \leftarrow \beta) \leftarrow \gamma = \alpha \leftarrow (\beta \leftarrow \gamma) = \alpha \leftarrow (\beta \rightarrow \gamma), \tag{1}$$
$$(\alpha \rightarrow \beta) \leftarrow \gamma = \alpha \rightarrow (\beta \leftarrow \gamma), \tag{2}$$
$$(\alpha \rightarrow \beta) \rightarrow \gamma = (\alpha \leftarrow \beta) \rightarrow \gamma = \alpha \rightarrow (\beta \rightarrow \gamma). \tag{3}$$

An extended diassociative semigroup *(briefly, EDS) is a family* $(\Omega, \leftarrow, \rightarrow, \lhd, \rhd)$*, where* Ω *is a nonempty set and* $\leftarrow, \rightarrow, \lhd, \rhd : \Omega \times \Omega \longrightarrow \Omega$ *are maps such that:*

(a) $(\Omega, \leftarrow, \rightarrow)$ *is a diassociative semigroup.*

(b) For any $\alpha, \beta, \gamma \in \Omega$*:*

$$\alpha \rhd (\beta \leftarrow \gamma) = \alpha \rhd \beta, \tag{4}$$
$$(\alpha \rightarrow \beta) \lhd \gamma = \beta \lhd \gamma, \tag{5}$$

$$(\alpha \lhd \beta) \leftarrow ((\alpha \leftarrow \beta) \lhd \gamma) = \alpha \lhd (\beta \leftarrow \gamma), \tag{6}$$
$$(\alpha \lhd \beta) \lhd ((\alpha \leftarrow \beta) \lhd \gamma) = \beta \lhd \gamma, \tag{7}$$
$$(\alpha \lhd \beta) \rightarrow ((\alpha \leftarrow \beta) \lhd \gamma) = \alpha \lhd (\beta \rightarrow \gamma), \tag{8}$$
$$(\alpha \lhd \beta) \rhd ((\alpha \leftarrow \beta) \lhd \gamma) = \beta \rhd \gamma, \tag{9}$$

$$(\alpha \rhd (\beta \rightarrow \gamma)) \leftarrow (\beta \rhd \gamma) = (\alpha \leftarrow \beta) \rhd \gamma, \tag{10}$$
$$(\alpha \rhd (\beta \rightarrow \gamma)) \lhd (\beta \rhd \gamma) = \alpha \lhd \beta, \tag{11}$$
$$(\alpha \rhd (\beta \rightarrow \gamma)) \rightarrow (\beta \rhd \gamma) = (\alpha \rightarrow \beta) \rhd \gamma, \tag{12}$$
$$(\alpha \rhd (\beta \rightarrow \gamma)) \rhd (\beta \rhd \gamma) = \alpha \rhd \beta. \tag{13}$$

An EDS $(\Omega, \leftarrow, \rightarrow, \lhd, \rhd)$ is commutative if for any $\alpha, \beta \in \Omega$:

$$\alpha \leftarrow \beta = \beta \rightarrow \alpha, \qquad \alpha \lhd \beta = \beta \rhd \alpha. \tag{14}$$

Let us reformulate the definition of commutative EDS:

Proposition 2.2. *A commutative EDS (briefly, CEDS) is a triple* $(\Omega, \rightarrow, \rhd)$*, where* Ω *is a nonempty set and* $\rightarrow, \rhd : \Omega^2 \longrightarrow \Omega$ *are maps such that, for any* $\alpha, \beta, \gamma \in \Omega$*:*

$$\alpha \rightarrow (\beta \rightarrow \gamma) = (\alpha \rightarrow \beta) \rightarrow \gamma = (\beta \rightarrow \alpha) \rightarrow \gamma, \tag{15}$$
$$\alpha \rhd (\beta \rightarrow \gamma) = \alpha \rhd \gamma, \tag{16}$$
$$(\alpha \rhd \gamma) \rightarrow (\beta \rhd \gamma) = (\alpha \rightarrow \beta) \rhd \gamma, \tag{17}$$
$$(\alpha \rhd \gamma) \rhd (\beta \rhd \gamma) = \alpha \rhd \beta. \tag{18}$$

Proof. Replacing $\leftarrow$ and $\lhd$ in (1)-(13) with the help of (14), we find (15)-(18). □

Definition 2.3. *[4] An* associative extended semigroup *(briefly, EAS) is a triple* $(\Omega, \rightarrow, \rhd)$*, where* Ω *is a nonempty set and* $\rightarrow, \rhd : \Omega^2 \longrightarrow \Omega$ *are maps such that, for any* $\alpha, \beta, \gamma \in \Omega$*:*

$$\alpha \rightarrow (\beta \rightarrow \gamma) = (\alpha \rightarrow \beta) \rightarrow \gamma, \tag{19}$$
$$(\alpha \rhd (\beta \rightarrow \gamma)) \rightarrow (\beta \rhd \gamma) = (\alpha \rightarrow \beta) \rhd \gamma, \tag{12}$$
$$(\alpha \rhd (\beta \rightarrow \gamma)) \rhd (\beta \rhd \gamma) = \alpha \rhd \beta. \tag{13}$$

Remark 2.1. Let $(\Omega, \to, \leftarrow, \rhd, \lhd)$ be an EDS. Then $(\Omega, \to, \rhd)$ is an EAS, called the *right part* of the EDS $(\Omega, \to, \leftarrow, \rhd, \lhd)$. We obtain a commutative triangle of functors

$$\begin{array}{ccc} \mathbf{CEDS} & \hookrightarrow & \mathbf{EAS} \\ \big\downarrow & \nearrow_{\text{right part}} & \\ \mathbf{EDS} & & \end{array}$$

We shall see that not all the EAS are right parts of an EDS (see case **C6** in the classification of EAS of cardinality 2 in the next paragraph).

Example 2.1. 1. Let Ω be a set. We put:

$$\forall(\alpha, \beta) \in \Omega^2, \qquad \begin{cases} \alpha \to \beta = \beta, \\ \alpha \rhd \beta = \alpha. \end{cases}$$

Then $(\Omega, \to, \rhd)$ is an EAS, denoted by $\mathbf{EAS}(\Omega)$. It is a CEDS.

2. Let $(\Omega, \star)$ be an associative semigroup and let $\pi : \Omega \longrightarrow \Omega$ be an endomorphism of associative semigroup such that $\pi^2 = \pi$. We put:

$$\forall \alpha, \beta \in \Omega, \qquad \alpha \rhd \beta = \pi(\alpha).$$

It is an EAS, which we denote by $\mathbf{EAS}(\Omega, \star, \pi)$. It is a CEDS if, and only if, for any $\alpha, \beta, \gamma \in \Omega$:

$$(\alpha \star \beta) \star \gamma = (\beta \star \alpha) \star \gamma.$$

We shall simply denote $\mathbf{EAS}(\Omega, \star)$ instead of $\mathbf{EAS}(\Omega, \star, \mathrm{Id}_\Omega)$. In particular, if $(\Omega, \star)$ is a group, then $\mathbf{EAS}(\Omega, \star)$ is a CEDS if, and only if, $(\Omega, \star)$ is abelian, which proves that not all EAS are CEDS.

3. Let Ω be a set with an operation $\rhd$ such that, for any $\alpha, \beta, \gamma \in \Omega$:

$$(\alpha \rhd \gamma) \rhd (\beta \rhd \gamma) = \alpha \rhd \beta.$$

We then put:

$$\forall(\alpha, \beta) \in \Omega^2, \qquad \alpha \to \beta = \beta.$$

Then $(\Omega, \to, \rhd)$ is a CEDS (so is an EAS). This holds for example if $(\Omega, \star)$ is an associative semigroup with the *right inverse condition*:

$$\forall(\beta, \gamma) \in \Omega^2,\ \exists! \alpha \in \Omega,\ \alpha \star \beta = \gamma.$$

This unique α is denoted by $\gamma \rhd \beta$. Then, for any $\alpha, \beta, \gamma \in \Omega$:

$$\begin{aligned} ((\alpha \rhd \gamma) \rhd (\beta \rhd \gamma)) \star \beta &= ((\alpha \rhd \gamma) \rhd (\beta \rhd \gamma)) \star ((\beta \rhd \gamma) \star \gamma) \\ &= (((\alpha \rhd \gamma) \rhd (\beta \rhd \gamma)) \star (\beta \rhd \gamma)) \star \gamma \\ &= (\alpha \rhd \gamma) \star \gamma \\ &= \alpha, \end{aligned}$$

so $(\alpha \rhd \gamma) \rhd (\beta \rhd \gamma) = \alpha \rhd \beta$. This EAS is denoted by $\mathbf{EAS}'(\Omega, \star)$. The right inverse condition holds for example if $(\Omega, \star)$ is a group, and then:

$$\alpha \rhd \beta = \alpha \star \beta^{\star -1}.$$

It also holds for semigroups which are not groups. For example, if Ω is a nonempty set, we give it an associative product defined by:

$$\forall \alpha, \beta \in \Omega, \qquad \alpha \star \beta = \alpha.$$

It satisfies the right inverse condition and, for any $\alpha, \beta \in \Omega$, $\alpha \rhd \beta = \alpha$. Note that for this example, $\mathbf{EAS}'(\Omega, \star) = \mathbf{EAS}(\Omega)$.

Definition 2.4. *Let* $(\Omega, \to, \rhd)$ *be an EAS. We shall say that it is* nondegenerate *if the following map is bijective:*

$$\phi : \begin{cases} \Omega^2 & \longrightarrow & \Omega^2 \\ (\alpha, \beta) & \longrightarrow & (\alpha \to \beta, \alpha \rhd \beta). \end{cases}$$

If Ω is a nondegenerate EAS, the structure implied on Ω by ϕ^{-1} will be studied in the next paragraph.

Example 2.2.

1. Let Ω be a set. In $\mathbf{EAS}(\Omega)$, for any $\alpha, \beta \in \Omega$, $\phi(\alpha, \beta) = (\beta, \alpha)$, so $\mathbf{EAS}(\Omega)$ is nondegenerate.
2. Let $(\Omega, \star)$ be a group. Then $\mathbf{EAS}(\Omega, \star)$ is nondegenerate. Indeed, in this case, $\phi(\alpha, \beta) = (\alpha \star \beta, \alpha)$, so ϕ is a bijection, of inverse given by $\phi^{-1}(\alpha, \beta) = (\beta, \beta^{\star -1} \star \alpha)$.
3. Let $(\Omega, \star)$ be an associative semigroup with the right inverse condition. Then $\mathbf{EAS}'(\Omega, \star)$ is nondegenerate. Indeed, in this case, $\phi(\alpha, \beta) = (\beta, \alpha \rhd \beta)$, so ϕ is a bijection, of inverse given by $\phi^{-1}(\alpha, \beta) = (\beta \star \alpha, \alpha)$.

2.2 Dual commutative extended semigroups

Definition 2.5. *Let* $(\Omega, \to, \rhd)$ *be a map with two binary operations. We shall say that it is a* dual CEDS *if, for any* $\alpha, \beta, \gamma \in \Omega$*:*

$$(\alpha \to \beta) \to \gamma = \alpha \to (\beta \to \gamma), \tag{20}$$

$$(\alpha \rhd (\beta \to \gamma)) \to (\beta \rhd \gamma) = (\alpha \to \beta) \rhd \gamma, \tag{12}$$

$$(\alpha \rhd (\beta \to \gamma)) \rhd (\beta \rhd \gamma) = \alpha \rhd \beta, \tag{13}$$

$$(\alpha \rhd \beta) \to \gamma = \alpha \to \gamma, \tag{21}$$

$$(\alpha \rhd \beta) \rhd \gamma = (\alpha \rhd \gamma) \rhd \beta. \tag{22}$$

Example 2.3.

1. If Ω is a set, then $\mathbf{EAS}(\Omega)$ is a dual CEDS.
2. If $(\Omega, \star)$ is a semigroup and $\pi : \Omega \longrightarrow \Omega$ is a semigroup morphism such that $\pi^2 = \pi$, then $\mathbf{EAS}(\Omega, \star, \pi)$ is a dual CEDS if, and only if,

$$\forall \alpha, \beta \in \Omega, \qquad \pi(\alpha) \star \beta = \alpha \star \beta.$$

   In particular, $\mathbf{EAS}(\Omega, \star)$ is a dual CEDS.
3. If $(\Omega, \star)$ is a semigroup with the right inverse condition, then $\mathbf{EAS}'(\Omega, \star)$ is a dual CEDS if, and only if:

$$\forall \alpha, \beta, \gamma \in \Omega, \qquad (\alpha \rhd \beta) \rhd \gamma = (\alpha \rhd \gamma) \rhd \beta.$$

   This is is equivalent to:

$$\forall \alpha, \beta, \gamma \in \Omega, \qquad \alpha \star \beta \star \gamma = \alpha \star \gamma \star \beta.$$

   In the case where $(\Omega, \star)$ is a group, $\mathbf{EAS}'(\Omega, \star)$ is a dual CEDS if, and only if, $(\Omega, \star)$ is abelian.

The following lemma, proved in [5], is a reformulation of the axioms of EAS, CEDS and dual CEDS with the help of the map ϕ:

Lemma 2.6. *Let* $(\Omega, \to, \rhd)$ *be a set with two binary operations. We consider the maps*

$$\phi : \begin{cases} \Omega^2 & \longrightarrow & \Omega^2 \\ (\alpha, \beta) & \longrightarrow & (\alpha \to \beta, \alpha \rhd \beta), \end{cases} \qquad \tau : \begin{cases} \Omega^2 & \longrightarrow & \Omega^2 \\ (\alpha, \beta) & \longrightarrow & (\beta, \alpha). \end{cases}$$

Then:

1. $(\Omega, \to, \rhd)$ *is an EAS if, and only if:*

$$(\mathrm{Id} \times \phi) \circ (\phi \times \mathrm{Id}) \circ (\mathrm{Id} \times \phi) = (\phi \times \mathrm{Id}) \circ (\mathrm{Id} \times \tau) \circ (\phi \times \mathrm{Id}). \tag{23}$$

2. $(\Omega, \to, \rhd)$ *is a CEDS if, and only if:*

$$(\mathrm{Id} \times \phi) \circ (\phi \times \mathrm{Id}) \circ (\mathrm{Id} \times \phi) = (\phi \times \mathrm{Id}) \circ (\mathrm{Id} \times \tau) \circ (\phi \times \mathrm{Id}), \tag{23}$$
$$(\mathrm{Id} \times \phi) \circ (\mathrm{Id} \times \tau) \circ (\tau \times \mathrm{Id}) \circ (\phi \times \mathrm{Id}) = (\tau \times \mathrm{Id}) \circ (\phi \times \mathrm{Id}) \circ (\mathrm{Id} \times \phi) \circ (\mathrm{Id} \times \tau). \tag{24}$$

$(\Omega, \to, \rhd)$ *is a dual CEDS if, and only if:*

$$(\mathrm{Id} \times \phi) \circ (\phi \times \mathrm{Id}) \circ (\mathrm{Id} \times \phi) = (\phi \times \mathrm{Id}) \circ (\mathrm{Id} \times \tau) \circ (\phi \times \mathrm{Id}), \tag{23}$$
$$(\phi \times \mathrm{Id}) \circ (\tau \times \mathrm{Id}) \circ (\mathrm{Id} \times \tau) \circ (\mathrm{Id} \times \phi) = (\mathrm{Id} \times \tau) \circ (\mathrm{Id} \times \phi) \circ (\phi \times \mathrm{Id}) \circ (\tau \times \mathrm{Id}). \tag{25}$$

With this reformulation, the following result becomes immediate, as inversing (23) gives (23) again and inversing (24) gives (25):

Proposition 2.7. *Let* $(\Omega, \to, \rhd)$ *be a set with two binary operations. We shall say that* $(\Omega, \to, \rhd)$ *is nondegenerate if the map* ϕ *of Definition 2.4 is a bijection. If so, we put:*

$$\phi^{-1} : \left\{ \begin{array}{rcl} \Omega^2 & \longrightarrow & \Omega^2 \\ (\alpha, \beta) & \longrightarrow & (\alpha \rightharpoondown \beta, \alpha \blacktriangleright \beta). \end{array} \right.$$

Then $(\Omega, \to, \rhd)$ *is an EAS (respectively a CEDS, a dual CEDS) if, and only if,* $(\Omega, \rightharpoondown, \blacktriangleright)$ *is an EAS (respectively a dual CEDS, a CEDS).*

2.3 EAS of cardinality two

Here is a classification of EAS of cardinality two, which we obtained by an exhaustive study of the 2^8 possibilities of pairs of operations. The underlying set is $\Omega = \{X, Y\}$ and the products will be given by the pair of matrices

$$\begin{pmatrix} X \to X & X \to Y \\ Y \to X & Y \to Y \end{pmatrix}, \qquad \begin{pmatrix} X \rhd X & X \rhd Y \\ Y \rhd X & Y \rhd Y \end{pmatrix}.$$

We shall use the two maps:

$$\pi_X : \left\{ \begin{array}{rcl} \Omega & \longrightarrow & \Omega \\ \alpha & \longrightarrow & X, \end{array} \right. \qquad \pi_Y : \left\{ \begin{array}{rcl} \Omega & \longrightarrow & \Omega \\ \alpha & \longrightarrow & Y. \end{array} \right.$$

We respect the indexation of EDS of [3].

Case	$\to$	$\rhd$	Description	Comments
A1	$\begin{pmatrix} X & X \\ X & X \end{pmatrix}$	$\begin{pmatrix} X & X \\ X & X \end{pmatrix}$	$\mathbf{EAS}(\Omega, \to, \pi_X)$	CEDS, dual CEDS, right part of **D1**
A2	$\begin{pmatrix} X & X \\ X & X \end{pmatrix}$	$\begin{pmatrix} X & X \\ Y & Y \end{pmatrix}$	$\mathbf{EAS}(\Omega, \to)$	CEDS, dual CEDS, right part of **D2**
C1	$\begin{pmatrix} X & X \\ X & Y \end{pmatrix}$	$\begin{pmatrix} X & X \\ X & X \end{pmatrix}$	$\mathbf{EAS}(\Omega, \to, \pi_X)$	CEDS, right part of **C4**
C3	$\begin{pmatrix} X & X \\ X & Y \end{pmatrix}$	$\begin{pmatrix} X & X \\ Y & Y \end{pmatrix}$	$\mathbf{EAS}(\mathbb{Z}/2\mathbb{Z}, \times)$	CEDS, dual CEDS
C5	$\begin{pmatrix} X & X \\ X & Y \end{pmatrix}$	$\begin{pmatrix} Y & Y \\ Y & Y \end{pmatrix}$	$\mathbf{EAS}((\mathbb{Z}/2\mathbb{Z}, \times), \pi_Y)$	CEDS, right part of **C2**
C6	$\begin{pmatrix} X & X \\ X & Y \end{pmatrix}$	$\begin{pmatrix} X & X \\ Y & X \end{pmatrix}$		

Case	$\to$	$\rhd$	Description	Comments
E1′ – E2′	$\begin{pmatrix} X & X \\ Y & Y \end{pmatrix}$	$\begin{pmatrix} X & X \\ X & X \end{pmatrix}$	$\mathbf{EAS}(\Omega, \to, \pi_X)$	right part of **E1** and **E2**
E3′	$\begin{pmatrix} X & X \\ Y & Y \end{pmatrix}$	$\begin{pmatrix} X & X \\ Y & Y \end{pmatrix}$	$\mathbf{EAS}(\Omega, \to)$	dual CEDS, right part of **E3**
F1	$\begin{pmatrix} X & Y \\ X & Y \end{pmatrix}$	$\begin{pmatrix} X & X \\ X & X \end{pmatrix}$	$\mathbf{EAS}(\Omega, \to, \pi_X)$	CEDS, dual CEDS, right part of **B1**, **F2**, **G1** and **G2**
F3	$\begin{pmatrix} X & Y \\ X & Y \end{pmatrix}$	$\begin{pmatrix} X & X \\ Y & Y \end{pmatrix}$	$\mathbf{EAS}(\Omega)$	CEDS, dual CEDS, nondegenerate, right part of **B2** and **G3**
F4	$\begin{pmatrix} X & Y \\ X & Y \end{pmatrix}$	$\begin{pmatrix} X & Y \\ Y & X \end{pmatrix}$	$\mathbf{EAS}'(\mathbb{Z}/2\mathbb{Z}, +)$	CEDS, dual CEDS, nondegenerate, right part of **F5**
H1	$\begin{pmatrix} X & Y \\ Y & X \end{pmatrix}$	$\begin{pmatrix} X & X \\ X & X \end{pmatrix}$	$\mathbf{EAS}(\mathbb{Z}/2\mathbb{Z}, +, \pi_X)$	CEDS
H2	$\begin{pmatrix} X & Y \\ Y & X \end{pmatrix}$	$\begin{pmatrix} X & X \\ Y & Y \end{pmatrix}$	$\mathbf{EAS}(\mathbb{Z}/2\mathbb{Z}, +)$	CEDS, dual CEDS, nondegenerate

For the cases **C3**, **C5**, **F4**, **H1** and **H2**, Ω is identified with $\mathbb{Z}/2\mathbb{Z}$, X being $\overline{0}$ and Y being $\overline{1}$.

Remark 2.2. With similar methods, it is possible to prove that there are three nondegenerate EAS of cardinality 3 up to isomorphism: $\mathbf{EAS}(\{1,2,3\})$, $\mathbf{EAS}(\mathbb{Z}/3\mathbb{Z}, +)$ and $\mathbf{EAS}'(\mathbb{Z}/3\mathbb{Z}, +)$. All of them are both CEDS and dual CEDS.

3 Structure of nondegenerate finite CEDS

3.1 Preliminary results

Lemma 3.1. *let Ω be a finite nondegenerate EAS.*

1. *Let Ω' be a sub-EAS of Ω. Then Ω' is nondegenerate.*
2. *Let $\sim$ be an equivalence on Ω, compatible with the EAS structure. Then the quotient EAS $\Omega/\sim$ is nondegenerate.*

Proof. 1. By restriction, $\phi_{\Omega'} = (\phi_\Omega)_{|\Omega'^2}$ is injective. As Ω' is finite, it is a bijection. So Ω' is non degenerate.

2. Let $\pi : \Omega \longrightarrow \Omega/\sim$ be the canonical surjection. Then $\phi_{\Omega/\sim} \circ \pi = (\pi \otimes \pi) \circ \phi_\Omega$. As ϕ is surjective, $\phi_{\Omega/\sim}$ is surjective. As $\Omega/\sim$ is finite, it is a bijection. So $\Omega/\sim$ is nondegenerate. ☐

Definition 3.2. *Let $(\Omega, \to, \rhd)$ be an EAS. For any $\alpha \in \Omega$, we put:*

$$\phi_\alpha : \left\{ \begin{array}{rcl} \Omega & \longrightarrow & \Omega \\ \beta & \longrightarrow & \alpha \to \beta, \end{array} \right. \qquad \psi_\alpha : \left\{ \begin{array}{rcl} \Omega & \longrightarrow & \Omega \\ \beta & \longrightarrow & \beta \rhd \alpha. \end{array} \right.$$

We shall say that $(\Omega, \to, \rhd)$ is strongly nondegenerate if for any $\alpha \in \Omega$, ϕ_α is bijective.

Remark 3.1. As the product $\to$ is associative, for any $\alpha, \beta \in \Omega$,

$$\phi_\alpha \circ \phi_\beta = \phi_{\alpha \to \beta}.$$

Lemma 3.3. *Let $(\Omega, *)$ be an associative semigroup. The following conditions are equivalent:*

1. $\mathbf{EAS}(\Omega, *)$ *is nondegenerate.*
2. $\mathbf{EAS}(\Omega, *^{op})$ *is strongly nondegenerate.*
3. $(\Omega, *^{op})$ *has the right inverse condition.*

Proof. Let α, β, γ, $\delta \in \Omega$. Then:

$$\phi(\alpha, \beta) = (\gamma, \delta) \Longleftrightarrow \begin{cases} \alpha * \beta = \gamma, \\ \alpha = \delta. \end{cases}$$

So:

$$\begin{aligned} \phi \text{ is bijective} &\Longleftrightarrow \forall(\gamma, \delta) \in \Omega^2,\ \exists! \beta \in \Omega,\ \delta * \beta = \gamma \\ &\Longleftrightarrow \text{in } \mathbf{EAS}(\Omega, *^{op}),\ \forall \delta \in \Omega,\ \phi_\delta \text{ is bijective} \\ &\Longleftrightarrow (\Omega, *^{op}) \text{ has the right inverse condition.} \end{aligned}$$

□

Lemma 3.4. *Let $(\Omega, \rightarrow, \rhd)$ be a finite nondegenerate CEDS. Then it is strongly nondegenerate.*

Proof. Let $\alpha, \gamma, \gamma' \in \Omega$ such that $\phi_\alpha(\gamma) = \phi_\alpha(\gamma')$. In other words, $\alpha \rightarrow \gamma = \alpha \rightarrow \gamma'$. By (16):

$$\alpha \rhd \gamma = \alpha \rhd (\alpha \rightarrow \gamma) = \alpha \rhd (\alpha \rightarrow \gamma') = \alpha \rhd \gamma'.$$

Therefore, $\phi(\alpha, \gamma) = \phi(\alpha, \gamma')$. As ϕ is injective, $\gamma = \gamma'$, so ϕ_α is injective. As Ω is finite, ϕ_α is bijective. □

Lemma 3.5. *Let $\Omega = (\Omega, \rightarrow, \rhd)$ be a nondegenerate EAS, such that:*

$$\forall \alpha, \beta \in \Omega, \qquad \alpha \rightarrow \beta = \beta.$$

There exists a product $*$ on Ω, making it a semigroup with the right inverse condition, such that $\Omega = \mathbf{EAS}'(\Omega, *)$. For any $\beta \in \Omega$, ψ_β is bijective and its inverse is:

$$\phi'_\beta : \begin{cases} \Omega & \longrightarrow & \Omega \\ \alpha & \longrightarrow & \alpha * \beta. \end{cases}$$

Moreover, for any $\beta, \gamma \in \Omega$:

$$\psi_\beta \circ \psi_\gamma = \psi_{\beta * \gamma}, \qquad \psi_{\beta \rhd \gamma} = \psi_\beta \circ \psi_\gamma^{-1}. \tag{26}$$

Proof. Note that for any $\alpha \in \Omega$, $\phi_\alpha = \mathrm{Id}_\Omega$. Let $\alpha, \beta, \gamma, \delta \in \Omega$. Then:

$$\phi(\alpha, \beta) = (\gamma, \delta) \Longleftrightarrow \begin{cases} \beta = \gamma, \\ \alpha \rhd \beta = \delta. \end{cases}$$

Hence:

$$\begin{aligned} \phi \text{ is bijective} &\Longleftrightarrow \forall(\gamma, \delta) \in \Omega^2,\ \exists! \alpha \in \Omega,\ \alpha \rhd \gamma = \delta \\ &\Longleftrightarrow \forall \gamma \in \Omega,\ \psi_\gamma \text{ is bijective.} \end{aligned}$$

Putting $\phi^{-1}(\alpha, \beta) = (\alpha \rightharpoondown \beta, \alpha \blacktriangleright \beta)$, by Proposition 2.7 $(\Omega, \rightharpoondown, \blacktriangleright)$ is an EAS, so $\rightharpoondown$ is associative. Moreover, $\phi^{-1}(\alpha, \beta) = (\alpha \rightharpoondown \beta, \alpha)$, so $(\Omega, \rightharpoondown, \blacktriangleleft) = \mathbf{EAS}(\Omega, \rightharpoondown)$. By Lemma 3.3, if $* = \rightharpoondown^{op}$, then $*$ has the right inverse condition. Moreover, for any $\alpha, \beta \in \Omega$:

$$\phi^{-1} \circ \phi(\alpha, \beta) = \phi^{-1}(\beta, \alpha \rhd \beta) = ((\alpha \rhd \beta) * \beta, \beta) = (\alpha, \beta).$$

Hence, the unique element $\gamma \in \Omega$ such that $\gamma * \beta = \alpha$ is $\alpha \rhd \beta$: consequently, $\Omega = \mathbf{EAS}'(\Omega, *)$. Moreover, for any $\alpha, \beta \in \Omega$,

$$\phi'_\beta \circ \psi_\beta(\alpha) = (\alpha \rhd \beta) * \beta = \alpha.$$

So $\phi'_\beta \circ \psi_\beta = \mathrm{Id}_\Omega$. As ψ_β is bijective, $\psi_\beta^{-1} = \phi'_\beta$.

Let $\beta, \gamma \in \Omega$. Then, for any $\alpha \in \Omega$,

$$\phi'_\gamma \circ \phi'_\beta(\alpha) = \alpha * \beta * \gamma = \phi'_{\beta*\gamma}.$$

So $\phi'_\gamma \circ \phi'_\beta = \phi'_{\beta*\gamma}$. Inversing, $\psi_\beta \circ \psi_\gamma = \psi_{\beta*\gamma}$. As a consequence,

$$\psi_{\beta\rhd\gamma} \circ \psi_\gamma = \psi_{(\beta\rhd\gamma)*\gamma} = \psi_\beta,$$

which induces the last formula. □

Lemma 3.6. *Let $\Omega = (\Omega, \rightarrow, \rhd)$ be a nondegenerate EAS such that for any $\alpha, \beta \in \Omega$,*

$$\alpha \rightarrow \beta = \beta.$$

Then $\Omega_\psi = \{\psi_\alpha,\ \alpha \in \Omega\}$ is a subgroup of the group of permutations of Ω.

Proof. Direct consequence of (26). □

Proposition 3.7. *Let $\Omega = \mathbf{EAS}'(\Omega, *)$, where $(\Omega, *)$ is a finite semigroup with the right inverse condition. We define an equivalence $\sim$ on Ω by $\alpha \sim \beta$ if $\psi_\alpha = \psi_\beta$. Then:*

1. *$\sim$ is compatible with the EAS structure of Ω. Therefore, $\Omega/\sim$ is an EAS.*
2. *There exists a product $\star$ on $\Omega/\sim$, making it a group, such that $\Omega/\sim = \mathbf{EAS}'(\Omega/\sim, \star)$.*
3. *There exists a sub-EAS Ω_0 of Ω, such that the restriction to Ω_0 of the canonical surjection $\pi : \Omega \longrightarrow \Omega/\sim$ is an isomorphism.*

Proof. 1. Let $\alpha, \beta \in \Omega$, such that $\alpha \sim \beta$. Then $\psi_\alpha = \psi_\beta$. Let $\gamma \in \Omega$. Then $\alpha \rightarrow \gamma = \beta \rightarrow \gamma = \gamma$, and $\gamma \rightarrow \alpha = \alpha \sim \beta = \gamma \rightarrow \beta$. As $\psi_\alpha = \psi_\beta$, $\gamma \rhd \alpha = \gamma \rhd \beta$. Moreover, by Lemma 3.5:

$$\psi_{\alpha\rhd\gamma} = \psi_\alpha \circ \psi_\gamma^{-1} = \psi_\beta \circ \psi_\gamma^{-1} = \psi_{\beta\rhd\gamma},$$

so $\alpha \rhd \gamma \sim \beta \rhd \gamma$: $\sim$ is compatible with the EAS structure.

2. By Lemma 3.1, $\Omega/\sim$ is nondegenerate. By Lemma 3.5, there exists a product $\star$ satisfying the right inverse condition, such that $\Omega/\sim = \mathbf{EAS}'(\Omega/\sim, \star)$. We consider the map

$$\psi : \begin{cases} (\Omega/\sim, \star) & \longrightarrow & (\mathfrak{S}_{\Omega/\sim}, \circ) \\ \overline{\alpha} & \longrightarrow & \psi_{\overline{\alpha}}. \end{cases}$$

By Lemma 3.5, this is a semigroup morphism. Let us prove that it is injective. We assume that $\psi_{\overline{\alpha}} = \psi_{\overline{\beta}}$. In other words, for any $\gamma \in \Omega$, $\gamma \rhd \alpha \sim \gamma \rhd \beta$, or equivalently, $\psi_{\gamma\rhd\alpha} = \psi_{\gamma\rhd\beta}$. Moreover:

$$\psi_{\gamma\rhd\alpha} = \psi_\gamma \circ \psi_\alpha^{-1} = \psi_{\gamma\rhd\beta} = \psi_\gamma \circ \psi_\beta^{-1}.$$

As ψ_γ is bijective, $\psi_\alpha = \psi_\beta$, so $\overline{\alpha} = \overline{\beta}$.

By Lemma 3.6, there exists $e \in \Omega/\sim$, such that $\psi_e = \mathrm{Id}_{\Omega/\sim}$. For any $\overline{\alpha} \in \Omega/\sim$,

$$\psi_{e\rhd\overline{\alpha}} = \psi_e \circ \psi_{\overline{\alpha}}^{-1} = \psi_{\overline{\alpha}}^{-1},$$

so $\psi(\Omega/\sim)$ is a subgroup of $\mathfrak{S}_{\Omega/\sim}$. Consequently, $(\Omega/\sim, \star)$ is a group.

3. By Lemma 3.6, there exists $\beta_0 \in \Omega$ such that $\psi_{\beta_0} = \mathrm{Id}_\Omega$. We put:

$$\Omega_0 = \{\beta_0 \rhd \alpha,\ \alpha \in \Omega\} = \{\psi_\alpha(\beta_0),\ \alpha \in \Omega\}.$$

As the product $\rightarrow$ of Ω is trivial, this is a sub-semigroup of $(\Omega, \rightarrow)$. Let $\beta_0 \rhd \alpha, \beta_0 \rhd \beta \in \Omega_0$.

$$(\beta_0 \rhd \alpha) \rhd (\beta_0 \rhd \beta) = \psi_{\beta_0 \rhd \gamma} \circ \psi_\alpha(\beta_0) = \psi_{(\beta_0 \rhd \gamma) * \alpha}(\beta_0) \in \Omega_0,$$

so Ω_0 is a sub-EAS of Ω.

Let us assume that $\beta_0 \rhd \alpha \sim \beta_0 \rhd \beta$. Then:

$$\begin{aligned} \psi_{\beta_0 \rhd \alpha} &= \psi_{\beta_0} \circ \psi_\alpha^{-1} = \psi_\alpha^{-1} \\ = \psi_{\beta_0 \rhd \beta} &= \psi_{\beta_0} \circ \psi_\beta^{-1} = \psi_\beta^{-1}, \end{aligned}$$

so $\psi_\alpha = \psi_\beta$. Hence, $\beta_0 \rhd \alpha = \beta_0 \rhd \beta$, which proves that $\pi_{|\Omega_0}$ is injective. By Lemma 3.6, there exists $\beta \in \Omega$ such that $\psi_\beta = \psi_\alpha^{-1}$. We consider $\beta_0 \rhd \beta \in \Omega_0$. Then:

$$\psi_{\beta_0 \rhd \beta} = \psi_{\beta_0} \circ \psi_\beta^{-1} = \psi_\alpha,$$

so $\beta_0 \rhd \beta \sim \alpha$. Hence, $\pi_{|\Omega_0}$ is surjective. □

Theorem 3.8. *Let $\Omega = \mathbf{EAS}'(\Omega, *)$, where $(\Omega, *)$ is a finite semigroup with the right inverse condition. There exists a group $(\Omega_0, \star)$ and a set Ω_1 such that*

$$\Omega \approx \mathbf{EAS}(\Omega_1) \times \mathbf{EAS}'(\Omega_0, \star).$$

Proof. We keep the notations of the proof of Proposition 3.7. As the sub-EAS Ω_0 is isomorphic to $\Omega/\sim$, it is a group for the law $*$, and $\Omega_0 = \mathbf{EAS}'(\Omega_0, *)$. Let e be the unit of the group $(\Omega/\sim, \star)$. We consider:

$$\Omega_1 = \{\alpha \in \Omega,\ \overline{\alpha} = e\}.$$

Let us prove that $\Omega_1 = \{\alpha \in \Omega,\ \psi_\alpha = \mathrm{Id}_\Omega\}$.
$\supseteq$: if $\psi_\alpha = \mathrm{Id}_\Omega$, for any $\beta \in \Omega$, $\overline{\beta} \star \overline{\alpha}^{*-1} = \overline{\psi_\alpha(\beta)} = \overline{\beta}$, so $\overline{\alpha} = e$ and $\alpha \in \Omega_1$.
$\subseteq$: if $\overline{\alpha} = e$, then for any $\beta \in \Omega$, $\overline{\beta \rhd \alpha} = \overline{\beta}$, so $\beta \rhd \alpha \sim \beta$: in other words, $\psi_{\beta \rhd \alpha} = \psi_\beta$. Then:

$$\psi_{\beta \rhd \alpha} = \psi_\beta \circ \psi_\alpha^{-1} = \psi_\beta.$$

As ψ_β is bijective, $\psi_\alpha = \mathrm{Id}_\Omega$.

Therefore, for any $\alpha \in \Omega$, for any $\beta \in \Omega_1$,

$$\alpha \rhd \beta = \psi_\beta(\alpha) = \alpha.$$

As a consequence, $\Omega_1 = \mathbf{EAS}(\Omega_1)$. We consider the map:

$$\theta : \begin{cases} \Omega_1 \times \Omega_0 & \longrightarrow & \Omega \\ (\alpha, \beta) & \longrightarrow & \alpha * \beta. \end{cases}$$

Let us prove that θ is injective. If $\theta(\alpha, \beta) = \theta(\alpha', \beta')$, in $\Omega/\sim$:

$$\overline{\alpha} \star \overline{\beta} = \overline{\beta} = \overline{\alpha'} \star \overline{\beta'} = \overline{\beta'}.$$

As $\pi_{|\Omega_0}$ is injective, $\beta = \beta'$. Because of the right inverse condition for $*$, $\alpha = \alpha'$.

Let us prove that θ is surjective. Let $\gamma \in \Omega$. There exists a unique $\beta \in \Omega_0$ such that $\psi_{\overline{\gamma}} = \psi_{\overline{\beta}}$. We put $\alpha = \gamma \rhd \beta$, so $\gamma = \alpha * \beta$. Moreover:

$$\psi_{\overline{\alpha}} = \psi_{\overline{\gamma}} \circ \psi_{\overline{\beta}}^{-1} = \mathrm{Id}_{\Omega/\sim}.$$

So $e \rhd \overline{\alpha} = \overline{\alpha}^{\star -1} = \psi_{\overline{\alpha}}(e) = e$, and finally $\alpha \in \Omega_1$.

Let (α, β) and $(\alpha', \beta') \in \Omega_1 \times \Omega_0$. In Ω, as $\alpha' \in \Omega_1$,

$$\alpha * \beta * \alpha' * \beta' = \alpha * (\beta * \beta'),$$

which implies that

$$(\alpha * \beta) \rhd (\alpha' * \beta') = \alpha * (\beta * \beta'^{*-1}).$$

So θ is an isomorphism of EAS from $\mathbf{EAS}(\Omega_1) \times \mathbf{EAS}'(\Omega_0, *)$ to Ω. □

3.2 Nondegenerate finite CEDS

Lemma 3.9. *Let $(\Omega, \rightarrow, \rhd)$ be a strongly nondegenerate finite EAS. Then $\Omega_\phi = (\{\phi_\alpha, \alpha \in \Omega\}, \circ)$ is a group. The following map is a surjective morphism of semigroups:*

$$\phi : \left\{ \begin{array}{rcl} (\Omega, \rightarrow) & \longrightarrow & \Omega_\phi \\ \alpha & \longrightarrow & \phi_\alpha. \end{array} \right.$$

Proof. We already observed that for any $\alpha, \beta \in \Omega$, $\phi_\alpha \circ \phi_\beta = \phi_{\alpha \rightarrow \beta}$, so ϕ is a semigroup morphism. By hypothesis, for any $\alpha \in \Omega$, ϕ_α is a bijection, so belongs the symmetric group $\mathfrak{S}_\Omega$ of permutations of Ω. As Ω is finite, for any $\alpha \in \Omega$, there exists $n \geqslant 2$ such that $\phi_\alpha^n = \mathrm{Id}_\Omega$. Then $\phi_{\alpha^{\rightarrow n}} = \mathrm{Id}_\Omega$, so Ω_ϕ is a monoid. Putting $\beta = \alpha^{\rightarrow (n-1)}$, $\phi_\beta \circ \phi_\alpha = \phi_\alpha \circ \phi_\beta = \mathrm{Id}_\Omega$, so Ω_ϕ is a group. □

Proposition 3.10. *Let $\Omega = (\Omega, \rightarrow, \rhd)$ be a finite nondegenerate EAS, such that for any $\alpha \in \Omega$, ϕ_α is a bijection. We put:*

$$\Omega^{\rightarrow} = \{\alpha \in \Omega,\ \phi_\alpha = \mathrm{Id}_\Omega\}, \qquad \Omega^{\rhd} = \{\beta \in \Omega,\ \psi_\beta = \mathrm{Id}_\Omega\}.$$

Then:

1. *$\Omega^{\rightarrow}$ is a nondegenerate sub-EAS of Ω.*
2. *If $\Omega^{\rhd}$ is nonempty, it is a nondegenerate sub-EAS of Ω.*
3. *If $\Omega^{\rhd}$ is nonempty, then $\Omega^{\rhd} \cap \Omega^{\rightarrow}$ is nonempty.*
4. *If Ω is a CEDS, $\Omega^{\rhd}$ is nonempty.*

Proof. 1. Recall that for any $\alpha, \beta \in \Omega$, $\phi_\alpha \circ \phi_\beta = \phi_{\alpha \rightarrow \beta}$. This easily implies that $\Omega^{\rightarrow}$ is stable under $\rightarrow$. By Lemma 3.9, there exists $\alpha \in \Omega$, such that $\phi_\alpha = \mathrm{Id}_\Omega$, so $\Omega^{\rightarrow}$ is nonempty.

Let $\alpha, \beta \in \Omega^{\rightarrow}$. Let us consider $\gamma \in \Omega$. As ϕ is bijective, there exists $\beta', \gamma' \in \Omega$ such that:

$$(\beta' \rightarrow \gamma', \beta' \rhd \gamma') = (\beta, \gamma).$$

Then:

$$\phi_{\alpha \rhd \beta}(\gamma) = (\alpha \rhd \beta) \rightarrow \gamma = (\alpha \rhd (\beta' \rightarrow \gamma')) \rightarrow (\beta' \rhd \gamma') = \beta' \rhd \gamma' = \gamma.$$

So $\phi_{\alpha \rhd \beta} = \mathrm{Id}_\Omega$ and $\alpha \rhd \beta \in \Omega^{\rightarrow}$. By Lemma 3.1, $\Omega^{\rightarrow}$ is a nondegenerate sub-EAS.

2. Let $\beta, \gamma \in \Omega^{\rhd}$. As $\psi_\gamma = \mathrm{Id}_\Omega$, $\beta \rhd \gamma = \beta \in \Omega^{\rhd}$. For any $\alpha \in \Omega$, by (12) and (13):

$$(\alpha \rhd (\beta \to \gamma)) \to (\beta \rhd \gamma) = (\alpha \to \beta) \rhd \gamma = \alpha \to \beta,$$
$$(\alpha \rhd (\beta \to \gamma)) \rhd (\beta \rhd \gamma) = \alpha \rhd \beta.$$

So $\phi(\alpha \rhd (\beta \to \gamma), \beta \rhd \gamma) = \phi(\alpha, \beta)$. As ϕ is injective, $\alpha \rhd (\beta \to \gamma) = \alpha$, so $\psi_{\beta \to \gamma} = \mathrm{Id}_\Omega$ and $\beta \to \gamma \in \Omega^{\rhd}$. If $\Omega^{\rhd}$ is nonempty, by Lemma 3.1, it is nondegenerate.

3. Let us take $\alpha \in \Omega^{\rhd}$. The permutation ϕ_α is of finite order as Ω is finite, so there exists $n \geqslant 2$, such that $\phi_\alpha^n = \phi_{\alpha^{\to n}} = \mathrm{Id}_\Omega$. Putting $\beta = \alpha^{\to n}$, then $\beta \in \Omega^{\rhd}$ (as it is a sub-CEDS) and $\beta \in \Omega^{\to}$ as $\phi_\beta = \mathrm{Id}_\Omega$.

4. Let us consider the EAS associated to the inverse of ϕ (Proposition 2.7), which we denote $(\Omega, \rightharpoondown, \blacktriangleright)$. By the first point, there exists $\alpha \in \Omega$ such that for any $\beta \in \Omega$, $\alpha \rightharpoondown \beta = \beta$. In other words, for any $\beta \in \Omega$,

$$\phi^{-1}(\alpha, \beta) = (\beta, \alpha \blacktriangleright \beta).$$

This implies that $\phi_\beta(\alpha \blacktriangleright \beta) = \alpha$. The inverse of the bijection ϕ_β is the map:

$$\phi'_\beta : \begin{cases} \Omega & \longrightarrow & \Omega \\ \alpha & \longrightarrow & \alpha \blacktriangleright \beta. \end{cases}$$

As Ω is finite, there exists $\beta' \in \Omega$ such that $\phi_\beta^{-1} = \phi_{\beta'}$. Hence:

$$\beta = \beta \rhd (\alpha \blacktriangleright \beta) = \beta \rhd (\beta' \to \alpha) = \beta \rhd \alpha,$$

by (16). So $\alpha \in \Omega^{\rhd}$. □

Proposition 3.11. *Let $(\Omega, \to, \rhd)$ be a finite nondegenerate CEDS.*

1. *We define an equivalence $\equiv$ on Ω by*

$$\beta \equiv \beta' \text{ if } \exists \alpha \in \Omega,\ \beta' = \alpha \to \beta.$$

   This equivalence is compatible with the CEDS structure. Therefore, $\Omega/\equiv$ is a CEDS.

2. *The restriction to $\Omega^{\to}$ of the canonical surjection $\pi : \Omega \longrightarrow \Omega/\equiv$ is an isomorphism.*

3. *$\Omega = \Omega^{\rhd} \to \Omega^{\to}$.*

Proof. 1. The relation $\equiv$ can be reformulated as: there exists $\phi_\alpha \in \Omega_\phi$, such that $\phi_\alpha(\beta) = \beta'$. By Lemmas 3.4 and 3.9, Ω_ϕ is a group. This easily implies that $\equiv$ is an equivalence: its classes are the orbits of the action of the group Ω_ϕ over Ω.

Let us assume that $\beta \equiv \beta'$: we put $\beta' = \alpha \to \beta$. Let $\gamma \in \Omega$. Then $\gamma \to \beta \equiv \beta \equiv \beta' \equiv \gamma \to \beta'$ by definition of $\equiv$. Moreover, $\beta' \to \gamma = \alpha \to (\beta \to \gamma)$, so $\beta' \to \gamma \equiv \beta \to \gamma$. By (13):

$$\beta' \rhd \gamma = (\alpha \to \gamma) \rhd \gamma = (\alpha \rhd (\beta \to \gamma)) \to (\beta \rhd \gamma) \equiv \beta \rhd \gamma,$$
$$\gamma \rhd \beta' = \gamma \rhd (\alpha \to \beta) = \gamma \rhd \beta.$$

So $\equiv$ is compatible with the CEDS structure.

2. Let $\alpha \in \Omega$. As ϕ_α is bijective, there exists a unique $\beta \in \Omega$ such that $\alpha \to \beta = \alpha$. Then

$$\phi_\alpha = \phi_{\alpha \to \beta} = \phi_\alpha \circ \phi_\beta.$$

As ϕ_α is bijective, $\phi_\beta = \mathrm{Id}_\Omega$, so $\beta \in \Omega^{\to}$ and $\alpha \equiv \beta$. This proves that $\pi_{|\Omega^{\to}}$ is surjective.

Let $\beta, \beta' \in \Omega^{\rightarrow}$, such that $\beta \equiv \beta'$. There exists $\alpha \in \Omega$ such that $\alpha \rightarrow \beta = \beta'$. Then:

$$\mathrm{Id}_\Omega = \phi_{\beta'} = \phi_\alpha \circ \phi_\beta = \phi_\alpha,$$

so $\phi_\alpha = \mathrm{Id}_\Omega$. We deduce that $\beta' = \phi_\alpha(\beta) = \beta$. Hence, $\pi_{|\Omega^{\rightarrow}}$ is injective.

3. By Proposition 3.10, there exists $\beta_0 \in \Omega^{\rhd} \cap \Omega^{\rightarrow}$. Let $\beta \in \Omega$. As $\pi_{|\Omega^{\rightarrow}}$ is bijective, there exists $\beta_1 \in \Omega^{\rightarrow}$, such that $\beta \equiv \beta_1$. We put $\beta = \alpha \rightarrow \beta_1$. As $\beta_0 \in \Omega^{\rightarrow}$:

$$\beta = \alpha \rightarrow \beta_0 \rightarrow \beta_1.$$

Moreover, for any $\gamma \in \Omega$, as $\beta_0 \in \Omega^{\rhd}$, by (16):

$$\gamma \rhd (\alpha \rightarrow \beta_0) = \gamma \rhd \beta_0 = \gamma,$$

so $\alpha \rightarrow \beta_0 \in \Omega^{\rhd}$. □

Proposition 3.12. *Let $(\Omega, \rightarrow, \rhd)$ be a finite nondegenerate CEDS. We define an equivalence on $\Omega^{\rhd}$ by:*

$$\alpha' \sim \alpha'' \Longleftrightarrow \exists \alpha \in \Omega^{\rightarrow},\ \alpha'' = \alpha' \rightarrow \alpha.$$

1. *This equivalence is compatible with the CEDS structure, and $\Omega' = \Omega^{\rhd}/\sim$ is a nondegenerate CEDS. Moreover, $(\Omega', \rightarrow)$ is an abelian group and $\Omega' = \mathbf{EAS}(\Omega', \rightarrow)$.*
2. *The following map is a semigroup isomorphism:*

$$\theta : \left\{ \begin{array}{rcl} (\Omega' \times \Omega^{\rightarrow}, \rightarrow) & \longrightarrow & (\Omega, \rightarrow) \\ (\overline{\alpha}, \beta) & \longrightarrow & \alpha \rightarrow \beta. \end{array} \right.$$

Proof. We consider

$$\Theta : \left\{ \begin{array}{rcl} \Omega^{\rhd} \times \Omega^{\rightarrow} & \longrightarrow & \Omega \\ (\alpha, \beta) & \longrightarrow & \alpha \rightarrow \beta. \end{array} \right.$$

By Proposition 3.11, it is surjective. Let us prove that $\Theta(\alpha', \beta') = \Theta(\alpha'', \beta'')$ if, and only if, $\alpha' \sim \alpha''$ and $\beta' = \beta''$.

Let us assume that $\Theta(\alpha', \beta') = \Theta(\alpha'', \beta'')$. As $\phi_{\alpha'}$ is bijective, there exists $\alpha \in \Omega$, $\alpha'' = \alpha' \rightarrow \alpha$. As $\alpha' \rightarrow \beta' = \alpha'' \rightarrow \beta''$ and $\beta', \beta'' \in \Omega^{\rightarrow}$,

$$\phi_{\alpha'} = \phi_{\alpha'} \circ \phi_{\beta'} = \phi_{\alpha' \rightarrow \beta'} = \phi_{\alpha'' \rightarrow \beta''} = \phi_{\alpha''}.$$

Hence:

$$\phi_{\alpha''} = \phi_{\alpha'} \circ \phi_\alpha = \phi_{\alpha'}.$$

As $\phi_{\alpha'}$ is bjective, $\phi_\alpha = \mathrm{Id}_\Omega$, so $\alpha \in \Omega^{\rightarrow}$: we obtain that $\alpha' \equiv \alpha''$. As $\phi_{\alpha'} = \phi_{\alpha''}$, $\phi_{\alpha'}(\beta') = \phi_{\alpha''}(\beta'') = \phi'_\alpha(\beta'')$. As $\phi_{\alpha'}$ is injective, $\beta = \beta'$. Conversely, if $\alpha \in \Omega^{\rightarrow}$, $\alpha' \rightarrow \alpha \rightarrow \beta' = \alpha' \rightarrow \beta'$.

As a consequence, $\sim$ is indeed an equivalence, θ is well-defined and is a bijection. It remains to show that $\sim$ is compatible with the CEDS structure of $\Omega^{\rhd}$. Let $\alpha', \alpha'' \in \Omega^{\rhd}$, such that $\alpha' \sim \alpha''$. We put $\alpha'' = \alpha' \rightarrow \alpha$, with $\alpha \in \Omega^{\rightarrow}$. Let $\beta \in \Omega^{\rhd}$. Then:

$$\alpha' \rhd \beta = \alpha' \sim \alpha'' = \alpha'' \rhd \beta, \qquad \beta \rhd \alpha' = \beta = \beta \rhd \alpha''.$$

Moreover:

$$\alpha'' \rightarrow \beta = \alpha'' \rightarrow \alpha \rightarrow \beta = \alpha'' \rightarrow \beta, \qquad \beta \rightarrow \alpha'' = \beta \rightarrow \alpha' \rightarrow \alpha \sim \beta \rightarrow \alpha'.$$

Therefore, $\Omega^{\rhd}/\sim$ is a CEDS. By Lemma 3.1, it is nondegenerate.

Let $\alpha, \alpha' \in \Omega^{\rhd}$ and $\beta, \beta' \in \Omega^{\rightarrow}$. As $\beta \in \Omega^{\rightarrow}$:

$$\begin{aligned}\theta(\overline{\alpha}, \beta) \rightarrow \theta(\overline{\alpha'}, \beta') &= \alpha \rightarrow \beta \rightarrow \alpha' \rightarrow \beta' \\ &= \alpha \rightarrow \alpha' \rightarrow \beta' \\ &= \alpha \rightarrow \alpha' \rightarrow \beta \rightarrow \beta' \\ &= \theta(\overline{\alpha} \rightarrow \overline{\alpha'}, \beta \rightarrow \beta').\end{aligned}$$

So θ is an isomorphism for the products $\rightarrow$.

Let us now study the CEDS Ω'. By definition of $\Omega^{\rhd}$, for any $\overline{\alpha}$, $\overline{\beta} \in \Omega'$, $\overline{\alpha} \rhd \overline{\beta} = \overline{\alpha}$, so $\Omega' = \mathbf{EAS}'(\Omega', \rightarrow)$. By Proposition 3.10, $\Omega^{\rightarrow}$ is nonempty. Let us prove that

$$\Omega'^{\rightarrow} = \{\overline{\alpha},\ \alpha \in \Omega^{\rhd} \cap \Omega^{\rightarrow}\}.$$

$\supseteq$ is obvious. Let us take $\overline{\alpha} \in \Omega'^{\rightarrow}$. Then, for any $\beta \in \Omega^{\rhd}$, $\overline{\alpha} \rightarrow \overline{\beta} = \overline{\beta}$: there exists $\gamma \in \Omega^{\rightarrow}$, $\alpha \rightarrow \beta = \beta \rightarrow \gamma$. Therefore:

$$\phi_\alpha \circ \phi_\beta = \phi_\beta \circ \phi_\gamma = \phi_\beta,$$

as $\phi_\gamma = \mathrm{Id}_\Omega$. As ϕ_β is a bijection, $\phi_\alpha = \mathrm{Id}_\Omega$, so $\alpha \in \Omega^{\rightarrow}$. Let $\alpha, \beta \in \Omega^{\rhd} \cap \Omega^{\rightarrow}$. As ϕ_α is bijective, there exists $\beta' \in \Omega$, $\alpha \rightarrow \beta' = \beta$. Then:

$$\mathrm{Id}_\Omega = \phi_\beta = \phi_\alpha \circ \phi_{\beta'} = \phi_{\beta'},$$

so $\beta' \in \Omega^{\rightarrow}$: we proved that $\alpha \sim \beta$. As a conclusion, there exists a unique $\overline{e} \in \Omega'$, such that for any $\overline{\alpha} \in \Omega'$, $\overline{e} \rightarrow \overline{\alpha} = \overline{\alpha}$.

Let us choose $\overline{\alpha} \in \Omega'$. As ϕ_α is bijective, there exists $\overline{e'} \in \Omega'$ such that $\overline{\alpha} \rightarrow \overline{e'} = \overline{\alpha}$. Let $\overline{\beta} \in \Omega'$. Then $\overline{\alpha} \rightarrow \overline{e'} \rightarrow \overline{\beta} = \overline{\alpha} \rightarrow \overline{\beta}$: in other words, $\alpha \rightarrow e' \rightarrow \beta \sim \alpha \rightarrow \beta$, and there exists $\gamma \in \Omega^{\rightarrow}$, such that $\alpha \rightarrow e' \rightarrow \beta = \alpha \rightarrow \beta \rightarrow \gamma$. As ϕ_α is injective, $e' \rightarrow \beta = \beta \rightarrow \gamma \sim \beta$, so $\overline{e'} \rightarrow \overline{\beta} = \overline{\beta}$ for any $\overline{\beta} \in \Omega'$. By unicity of $\overline{e}$, $\overline{e'} = \overline{e}$: for any $\overline{\alpha} \in \Omega'$, $\overline{\alpha} \rightarrow \overline{e} = \overline{\alpha}$, so $\overline{e}$ is a unit of $(\Omega', \rightarrow)$. By (15), for $\gamma = \overline{e}$, we deduce that $(\Omega', \rightarrow)$ is an abelian monoid. Let $\overline{\alpha} \in \Omega'$. As ϕ_α is surjective, there exists $\overline{\alpha}' \in \Omega'$ such that $\overline{\alpha} \rightarrow \overline{\alpha}' = \overline{e}$. So $(\Omega', \rightarrow)$ is a group. □

Proposition 3.13. *Let $(\Omega, *)$ be an associative semigroup such that for any $\alpha, \beta, \gamma \in \Omega$,*

$$\alpha * \beta * \gamma = \beta * \alpha * \gamma.$$

Let $(\Omega', \rightarrow, \rhd)$ be a CEDS, and $\prec\colon \Omega \times \Omega' \longrightarrow \Omega$ be a map such that for any $\alpha, \beta \in \Omega$, for any $\beta', \gamma' \in \Omega'$,

$$\alpha \prec (\beta' \rightarrow \gamma') = \alpha \prec \gamma', \tag{27}$$
$$(\alpha * \beta) \prec \gamma' = (\alpha \prec \gamma') * (\beta \prec \gamma'), \tag{28}$$
$$(\alpha \prec \gamma') \prec (\beta' \rhd \gamma') = \alpha \prec \beta'. \tag{29}$$

We define a structure of CEDS on $\Omega \times \Omega'$ in the following way: for any (α, α'), $(\beta, \beta') \in \Omega \times \Omega'$,

$$(\alpha, \alpha') \rightarrow (\beta, \beta') = (\alpha * \beta, \alpha' \rightarrow \beta'), \qquad (\alpha, \alpha') \rhd (\beta, \beta') = (\alpha \prec \beta', \alpha' \rhd \beta').$$

This CEDS is denoted by $\Omega \rtimes_\prec \Omega'$.

Proof. Direct verifications. □

Remark 3.2. If for any $(\alpha, \alpha') \in \Omega \times \Omega'$, $\alpha \prec \alpha' = \alpha$, we recover the direct product $\Omega \times \Omega'$ of EAS.

Theorem 3.14. *Let Ω be a finite nondegenerate CEDS. There exists an abelian group $(\Omega_1, *)$, a group $(\Omega_2, \star)$, a left action $\succ \colon \Omega_2 \times \Omega_1 \longrightarrow \Omega_1$ of $(\Omega_2, \star)$ on $(\Omega_1, *)$ by group automorphisms, and a nonempty set Ω_3 such that Ω is of the form*

$$(\mathbf{EAS}(\Omega_1, *) \rtimes_{\succ} \mathbf{EAS}'(\Omega_2, \star)) \times \mathbf{EAS}(\Omega_3),$$

with the products given by

$$\begin{aligned}(\alpha_1, \alpha_2, \alpha_3) \to (\beta_1, \beta_2, \beta_3) &= (\alpha_1 * \beta_1, \beta_2, \beta_3),\\ (\alpha_1, \alpha_2, \alpha_3) \rhd (\beta_1, \beta_2, \beta_3) &= (\beta_2 \succ \alpha_1, \alpha_2 \star \beta_2^{\star -1}, \alpha_3).\end{aligned}$$

Proof. Let us consider the map θ of Proposition 3.12. For any $\alpha, \alpha' \in \Omega^{\rhd}$, for any $\beta, \beta' \in \Omega^{\to}$, by (16) and (17):

$$(\alpha \to \beta) \rhd (\alpha' \to \beta') = (\alpha \to \beta) \rhd \beta' = (\alpha \rhd \beta') \to (\beta \rhd \beta').$$

Let $\beta_0 \in \Omega^{\rhd} \cap \Omega^{\to}$. Then, as $\beta_0 \in \Omega^{\to}$:

$$(\alpha \to \beta) \rhd (\alpha' \to \beta') = \underbrace{(\alpha \rhd \beta') \to \beta_0}_{=\gamma_1} \to \underbrace{(\beta \rhd \beta')}_{=\gamma_2}.$$

Obviously, $\gamma_2 \in \Omega^{\to}$. For any $\gamma \in \Omega$, by (16):

$$\gamma \rhd \gamma_1 = \gamma \rhd \beta_0 = \gamma,$$

so $\gamma_1 \in \Omega^{\rhd}$. We then put, for any $\overline{\alpha} \in \Omega'$, for any $\beta \in \Omega^{\to}$:

$$\overline{\alpha} \prec \beta = \overline{\alpha \rhd \beta} \to \overline{\beta_0}.$$

Then, for any $\overline{\alpha}$, $\overline{\alpha'} \in \Omega'$, for any $\beta, \gamma \in \Omega^{\to}$:

$$\theta(\overline{\alpha}, \beta) \rhd \theta(\overline{\alpha'}, \beta') = \theta(\overline{\alpha} \prec \beta', \beta \rhd \beta').$$

Then:

$$\overline{\alpha} \prec (\beta \to \gamma) = \overline{\alpha \rhd (\beta \to \gamma)} \to \overline{\beta_0} = \overline{\alpha \rhd \gamma} \to \overline{\beta_0} = \overline{\alpha} \prec \gamma,$$

which proves (27). As $\beta_0 \in \Omega^{\to}$:

$$\begin{aligned}(\overline{\alpha} \to \overline{\beta}) \prec \gamma &= \overline{(\alpha \to \beta) \rhd \gamma} \to \overline{\beta_0}\\ &= \overline{\alpha \rhd \gamma} \to \overline{\beta \rhd \gamma} \to \overline{\beta_0}\\ &\overline{\alpha \rhd \gamma} \to \overline{\beta_0} \to \overline{\beta \rhd \gamma} \to \overline{\beta_0}\\ &= (\overline{\alpha} \prec \gamma) \to (\overline{\beta} \prec \gamma),\end{aligned}$$

which proves (28).

$$\begin{aligned}(\overline{\alpha} \prec \gamma) \prec (\beta \rhd \gamma) &= \overline{(\alpha \rhd \gamma) \rhd (\beta \rhd \gamma)} \to \overline{\beta_0 \rhd (\beta \rhd \gamma)} \to \overline{\beta_0}\\ &= \overline{\alpha \rhd \beta} \to \overline{\beta_0 \rhd (\beta \rhd \gamma)} \to \overline{\beta_0}\\ &= \overline{\alpha \rhd \beta} \to \overline{\beta_0}\\ &= \overline{\alpha} \prec \beta,\end{aligned}$$

which proves (29). For the last equality, we used that $\beta_0 \rhd (\beta \rhd \gamma) \in \Omega^{\to}$, as β, β_0 and γ belong to $\Omega^{\to}$.

We finally obtain that θ is an isomorphism between $\Omega' \rtimes_{\prec} \Omega^{\rightarrow}$ and Ω. We put $\Omega' = \mathbf{EAS}(\Omega_1, *)$. From Theorem 3.8, we obtain a decomposition of $\Omega^{\rightarrow}$ of the form $\mathbf{EAS}'(\Omega_2, \star) \times \mathbf{EAS}(\Omega_3)$. The map $\prec: \Omega_1 \times \Omega_2 \times \Omega_3 \longrightarrow \Omega_1$ satisfies (27)-(29). In this particular case, (27) becomes trivial, and (28), (29) can be reformulated in this way: for any $\alpha_1, \beta_1 \in \Omega_1$, $\beta_2, \gamma_2 \in \Omega_2$, $\beta_3, \gamma_3 \in \Omega_3$,

$$
\begin{aligned}
(\alpha_1 * \beta_1) \prec (\gamma_2, \gamma_3) &= (\alpha_1 \prec (\gamma_2, \gamma_3)) * (\beta_1 \prec (\gamma_2, \gamma_3)),\\
(\alpha_1 \prec (\gamma_2, \gamma_3)) \prec (\beta_2, \beta_3) &= \alpha_1 \prec (\beta_2 \star \gamma_2, \beta_3).
\end{aligned}
$$

The products of Ω are given in this way: for any $\alpha_i, \beta_i \in \Omega_i$, with $1 \leqslant i \leqslant 3$,

$$
\begin{aligned}
(\alpha_1, \alpha_2, \alpha_3) \rightarrow (\beta_1, \beta_2, \beta_3) &= (\alpha_1 * \beta_1, \beta_2, \beta_3),\\
(\alpha_1, \alpha_2, \alpha_3) \triangleright (\beta_1, \beta_2, \beta_3) &= (\alpha_1 \prec (\beta_2, \beta_3), \alpha_2 \star \beta_2^{\star -1}, \alpha_3).
\end{aligned}
$$

For any $(\beta_2, \beta_3) \in \Omega_2 \times \Omega_3$, we consider:

$$
\psi^{\prec}_{\beta_2,\beta_3} : \left\{ \begin{array}{rcl} \Omega_1 & \longrightarrow & \Omega_1 \\ \alpha_1 & \longrightarrow & \alpha_1 \prec (\beta_2, \beta_3). \end{array} \right.
$$

As Ω is nondegenerate, necessarily ψ_{β_2,β_3} is injective. As Ω is finite, ψ_{β_2,β_3} is a bijection. Moreover, by (30), for any (β_2, β_3), $(\gamma_2, \gamma_3) \in \Omega_2 \times \Omega_3$,

$$
\psi^{\prec}_{\beta_2,\beta_3} \circ \psi^{\prec}_{\gamma_2,\gamma_3} = \psi^{\prec}_{\beta_2 \star \gamma_2,\beta_3}.
$$

For $\beta_2 = \gamma_2$ being the unit e of Ω_2 and $\beta_3 = \gamma_3$, we obtain that $\left(\psi^{\prec}_{e,\beta_3}\right)^2 = \psi^{\prec}_{e,\beta_3}$. As it is a bijection, $\psi^{\prec}_{e,\beta_3} = \mathrm{Id}_{\Omega_1}$ for any $\beta_3 \in \Omega_3$. Hence:

$$
\psi^{\prec}_{e,\beta_3} \circ \psi^{\prec}_{\gamma_2,\gamma_3} = \psi^{\prec}_{\gamma_2,\gamma_3} = \psi^{\prec}_{\gamma_2,\beta_3},
$$

so ψ_{β_2,β_3} does not depend on β_3. We denote this map by ψ_{β_2}. Note that we proved that $\psi_{\beta_{e_{\Omega_2}}} = \mathrm{Id}_{\Omega_1}$. We put, for any $\alpha_1 \in \Omega_1$, $\beta_2 \in \Omega_2$, $\alpha_1 \prec \beta_2 = \psi^{\prec}_{\beta_2}(\alpha_1)$. We finally obtain that the products in Ω are given in this way:

$$
\begin{aligned}
(\alpha_1, \alpha_2, \alpha_3) \rightarrow (\beta_1, \beta_2, \beta_3) &= (\alpha_1 * \beta_1, \beta_2, \beta_3),\\
(\alpha_1, \alpha_2, \alpha_3) \triangleright (\beta_1, \beta_2, \beta_3) &= (\alpha_1 \prec \beta_2, \alpha_2 \star \beta_2^{\star -1}, \alpha_3).
\end{aligned}
$$

So $\Omega = (\mathbf{EAS}(\Omega_1, *) \rtimes_{\prec} \mathbf{EAS}'(\Omega_2, \star)) \times \mathbf{EAS}(\Omega_3)$.

In the particular case of $\mathbf{EAS}(\Omega_1, *) \rtimes_{\prec} \mathbf{EAS}'(\Omega_2, \star)$, (27) is trivial, and (28), (29) can be reformulated in this way: for any $\alpha_1, \beta_1 \in \Omega_1$, $\beta_2, \gamma_2 \in \Omega_2$,

$$
\begin{aligned}
(\alpha_1 * \beta_1) \prec \gamma_2 &= (\alpha_1 \prec \gamma_2) * (\beta_1 \prec \gamma_2),\\
(\alpha_1 \prec \gamma_2) \prec \beta_2 &= \alpha_1 \prec (\beta_2 \star \gamma_2).
\end{aligned}
$$

As $\psi_{e_{\Omega_2}} = \mathrm{Id}_{\Omega_1}$, the following map is a left action of $(\Omega_2, \star)$ on $(\Omega_1, *)$ by group automorphisms:

$$
\succ: \left\{ \begin{array}{rcl} \Omega_2 \times \Omega_1 & \longrightarrow & \Omega_1 \\ (\beta_2, \alpha_1) & \longrightarrow & \beta_2 \succ \alpha_1 = \alpha_1 \prec \beta_2. \end{array} \right.
$$

The formulas for the products in Ω are then immediate. □

Remark 3.3. Consequently, we have a semidirect product of groups $(\Omega_1, *) \rtimes_{\succ} (\Omega_2, \star)$.

Inverting the corresponding maps ϕ, we obtain:

Corollary 3.15. *Let Ω be a finite nondegenerate CEDS. There exists an abelian group $(\Omega_1, *)$, a group $(\Omega_2, \star)$, a right action $\prec: \Omega_1 \times \Omega_2 \longrightarrow \Omega_1$ of $(\Omega_2, \star)$ on $(\Omega_1, *)$ by group automorphisms, and a nonempty set Ω_3 such that Ω is of the form*

$$(\mathbf{EAS}(\Omega_2, \star) \ltimes_{\prec} (\mathbf{EAS}'(\Omega_1, *)) \times \mathbf{EAS}(\Omega_3),$$

with the products given by

$$\begin{aligned}(\alpha_2, \alpha_1, \alpha_3) \rightarrow (\beta_2, \beta_1, \beta_3) &= (\alpha_2 \star \beta_2, \beta_1 \prec \alpha_2, \beta_3),\\ (\alpha_2, \alpha_1, \alpha_3) \triangleright (\beta_2, \beta_1, \beta_3) &= (\alpha_2, \alpha_1 * (\beta_1^{-1} \prec \alpha_2^{-1}), \alpha_3).\end{aligned}$$

Remark 3.4. The inverse dual CEDS of the CEDS $(\mathbf{EAS}(\Omega_1, *) \rtimes_{\succ} \mathbf{EAS}'(\Omega_2, \star)) \times \mathbf{EAS}(\Omega_3)$ is $(\mathbf{EAS}(\Omega_2, \star^{op}) \ltimes_{\succ^{op}} (\mathbf{EAS}'(\Omega_1, *)) \times \mathbf{EAS}(\Omega_3)$.

4 Linear extended associative semigroups

4.1 Definitions and example

The notions of ℓEAS, ℓCEDS and dual ℓCEDS are introduced in [5, Definition 1.5], as a linear version of Lemma 2.6:

Definition 4.1. *Let A be a vector space and let $\Phi : A \otimes A \longrightarrow A \otimes A$ be a linear map.*

1. *We shall say that (A, Φ) is a linear extended associative semigroup (briefly, ℓEAS) if:*

$$(\mathrm{Id} \otimes \Phi) \circ (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Phi) = (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \tau) \circ (\Phi \otimes \mathrm{Id}). \tag{30}$$

2. *We shall say that (A, Φ) is a linear commutative extended diassociative semigroup (briefly, ℓCEDS) if:*

$$(\mathrm{Id} \otimes \Phi) \circ (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Phi) = (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \tau) \circ (\Phi \otimes \mathrm{Id}), \tag{30}$$
$$(\mathrm{Id} \otimes \Phi) \circ (\mathrm{Id} \otimes \tau) \circ (\tau \otimes \mathrm{Id}) \circ (\Phi \otimes \mathrm{Id}) = (\tau \otimes \mathrm{Id}) \circ (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Phi) \circ (\mathrm{Id} \otimes \tau). \tag{31}$$

3. *We shall say that (A, Φ) is a linear dual commutative extended diassociative semigroup (briefly, dual ℓCEDS) if:*

$$(\mathrm{Id} \otimes \Phi) \circ (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Phi) = (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \tau) \circ (\Phi \otimes \mathrm{Id}), \tag{30}$$
$$(\Phi \otimes \mathrm{Id}) \circ (\tau \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \tau) \circ (\mathrm{Id} \otimes \Phi) = (\mathrm{Id} \otimes \tau) \circ (\mathrm{Id} \otimes \Phi) \circ (\Phi \otimes \mathrm{Id}) \circ (\tau \otimes \mathrm{Id}). \tag{32}$$

If (A, Φ) is an ℓEAS (respectively an ℓCEDS or a dual ℓCEDS), we shall say that it is nondegenerate if Φ is bijective.

Note that, by definition, ℓCEDS and dual ℓCEDS are ℓEAS.

Example 4.1. 1. Let $(\Omega, \rightarrow, \triangleright)$ be an EAS (respectively, a CEDS, a dual CEDS). Let $A = \mathbb{K}\Omega$ be the vector space generated by Ω. We define:

$$\Phi : \left\{ \begin{array}{rcl} A \otimes A & \longrightarrow & A \otimes A \\ a \otimes b & \longrightarrow & (a \rightarrow b) \otimes (a \triangleright b). \end{array} \right.$$

Then (A, Φ) is an ℓEAS (respectively, an ℓCEDS, a dual ℓCEDS), which we call the linearization of $(\Omega, \rightarrow, \triangleright)$. It is a nondegenerate ℓEAS if, and only if, Ω is a nondegenerate EAS.

2. All the ℓEAS are not of the form $\mathbb{K}\Omega$. For example, if A is a two-dimensional space with basis (x, y), the maps given by the following matrices in the basis $(x\otimes x, x\otimes y, y\otimes x, y\otimes y)$ are ℓEAS:

$$M_1 = \begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0&0&0&0\\0&0&a&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \qquad M_5 = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{pmatrix}, \qquad M_6 = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix},$$

$$M_7 = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix}, \qquad M_8 = \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}, \qquad M_9 = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&1&0 \end{pmatrix},$$

$$M_{10} = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&1&1&0\\0&0&0&0 \end{pmatrix}, \qquad M_{11} = \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&0&1&0\\0&0&1&0 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&1&0&0\\0&1&-1&0 \end{pmatrix},$$

$$M_{13} = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}, \qquad M_{14} = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&1 \end{pmatrix}, \qquad M_{15} = \begin{pmatrix} 1&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&1 \end{pmatrix},$$

$$M_{16} = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&1&1 \end{pmatrix}, \qquad M_{17} = \begin{pmatrix} 1&0&1&0\\0&0&-1&0\\0&1&-1&0\\0&0&2&1 \end{pmatrix}, \qquad M_{18} = \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix},$$

where a is a scalar. Moreover:

- The ℓCEDS in this list are the M_i's with

$$i \in \{1, 2, 3, 4, 5, 9, 10, 13, 14, 16, 17, 18\}.$$

- The dual ℓCEDS in this list are are the M_i's with

$$i \in \{1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 14, 15, 16, 17, 18\}.$$

These ℓEAS are in fact the EAS of dimension 2 which have a basis of special vectors, see Definition 4.3.

Notations 4.1. Let (A, Φ) be an ℓEAS. We use the Sweedler notation:

$$\Phi(a \otimes b) = \sum a' \to b' \otimes a'' \rhd b''.$$

Note that the operations $\to$ and $\rhd$ do not necessarily exist, nor the coproducts $a' \otimes a''$ or $b' \otimes b''$. With this notation, (30) can be rewritten as:

$$\begin{aligned} &\sum\sum\sum a' \to (b' \to c')' \otimes (a'' \rhd (b' \to c')'')' \to (b'' \rhd c'')' \otimes (a'' \rhd (b' \to c'))'' \rhd (b'' \rhd c'')'' \quad (30') \\ &= \sum\sum (a' \to b')' \to c' \otimes (a' \to b')'' \rhd c'' \otimes a'' \rhd b''. \end{aligned}$$

Similarly, (31) and (32) are rewritten as:

$$\sum\sum a'' \rhd (c' \to b')'' \otimes a' \to (c'' \to b'')' \otimes c'' \rhd b'' \qquad (31')$$
$$= \sum\sum a'' \rhd b'' \otimes c' \to (a' \to b')' \otimes c'' \rhd (a' \to b')'',$$

$$\sum\sum (b'' \rhd c'')' \to a' \otimes (b'' \rhd c'')'' \rhd a'' \otimes b' \to c' \qquad (32')$$
$$= \sum\sum b' \to a' \otimes (b'' \rhd a'')'' \rhd c'' \otimes (b'' \rhd a'')' \to c'.$$

By transposition of (30), (31) and (32):

Proposition 4.2. *Let V be a finite-dimensional space and $\Phi : V \otimes V \longrightarrow V \otimes V$ be a linear map. We consider $\Phi^* : V^* \otimes V^* = (V \otimes V)^* \longrightarrow (V \otimes V)^* = V^* \otimes V^*$. Then (V, Φ) is an ℓEAS [respectively an ℓCEDS, a dual ℓCEDS] if, and only if, (V^*, Φ^*) is an ℓEAS [respectively a dual ℓCEDS, an ℓCEDS].*

Example 4.2.

1. As their matrices are symmetric, the ℓEAS M_3, M_6, M_{13} and M_{18} are self-dual, through the pairing which matrix in the basis (x, y) is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

With the same pairing, the dual of M_4 is M_5 and the dual of M_8 is M_{10}. The ℓEAS M_2 and M_{14} are also self-dual, through the pairing which matrix in the basis (x, y) is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The ℓEAS M_{16} and M_{17} are self-dual[1], through the pairings which matrix in the basis (x, y) are respectively

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

The duals of M_1, M_7, M_9, M_{11}, M_{12} and M_{15} are not isomorphic to any M_i's.

2. Let Ω be a finite EAS and $A = \mathbb{K}\Omega$ be the associated ℓEAS. The dual A^* is identified with the space $\mathbb{K}^\Omega$ of maps from Ω to $\mathbb{K}$, with the dual basis $(\delta_\alpha)_{\alpha \in \Omega}$ of the basis Ω of $\mathbb{K}\Omega$. Then, for any $\alpha, \beta \in \Omega$:

$$\Phi^*(\delta_\alpha \otimes \delta_\beta) = \sum_{(\gamma,\delta) \in \phi^{-1}(\alpha,\beta)} \delta_\gamma \otimes \delta_\delta.$$

This is usually not the linearization of an EAS, except if Ω is nondegenerate: in this case, we recover the linearization of $(\Omega, \rightharpoonup, \blacktriangleright)$ of Proposition 2.7.

4.2 Special vectors, left units and counits

Definition 4.3. *Let (A, Φ) be an ℓEAS.*

1. *Let $a \in A$. We shall say that a is a left unit of (A, Φ) if for any $b \in A$, $\Phi(a \otimes b) = b \otimes a$.*
2. *Let $f \in A^*$. We shall say that f is a left counit of (A, Φ) if $(f \otimes \mathrm{Id}) \circ \Phi = \mathrm{Id} \otimes f$.*
3. *Let $a \in A$ and $\lambda \in \mathbb{K}$. We shall say that a is a special vector of (A, Φ) of eigenvalue λ if $\Phi(a \otimes a) = \lambda a \otimes a$.*

[1]For M_{17}, this holds if the characteristic of the base field is not 2.

Remark 4.1. Let (A, Φ) be an ℓEAS.

1. Any left unit of (A, Φ) is a special vector of eigenvalue 1.
2. If A is finite dimensional, its left counits are the left units of (A^*, Φ^*).
3. The set of left units is a subspace of A and the set of left counits a subspace of A^*. The set of special vectors of a given eigenvalue is generally not a subspace of A.

Lemma 4.4. *Let (A, Φ) be an ℓEAS and $a \in A$ be a nonzero special vector of (A, Φ). Then its eigenvalue λ is 0 or 1.*

Proof. Let us apply (30) to $a \otimes a \otimes a$. This gives $\lambda^3 a \otimes a \otimes a = \lambda^2 a \otimes a \otimes a$. As $a \neq 0$, $\lambda = 0$ or 1. □

Example 4.3. 1. Let us give special vectors, left units and left counits for the thirteen ℓEAS associated to the thirteen EAS of cardinality 2. In each case, we give a basis of the spaces of left units and left counits; λ, μ and ν are scalars. The dual basis of the basis (X, Y) of $\mathbb{K}\Omega$ is denoted by (X^*, Y^*).

Case	Special vectors of eigenvalue 1	Special vectors of eigenvalue 0	Left units	Left conits
A1	λX	$\nu(X-Y)$	$\emptyset$	$\emptyset$
A2	λX	$\nu(X-Y)$	$\emptyset$	(X^*+Y^*)
C1	λX	0	$\emptyset$	$\emptyset$
C3	$\lambda X, \mu Y$	0	(Y)	(X^*+Y^*)
C5	μY	0	(Y)	$\emptyset$
C6	λX	0	$\emptyset$	$\emptyset$
$\mathbf{E'1-E'2}$	λX	$\nu(X-Y)$	$\emptyset$	$\emptyset$
$\mathbf{E'3}$	$\lambda X, \mu Y$	$\nu(X-Y)$	$\emptyset$	(X^*+Y^*)
F1	λX	$\nu(X-Y)$	(X)	$\emptyset$
F3	$\lambda X+\mu Y$	0	(X,Y)	(X^*,Y^*)
F4	$\lambda X, \nu(X+Y)$	0	$(X+Y)$	(X^*)
H1	λX	0	(X)	$\emptyset$
H2	$\lambda X, \nu(X+Y)$	0	(X)	(X^*+Y^*)

Some of them have a basis of special vectors: let us determine their matrices in such a basis. We recover in this way some matrices of Example 4.1:

- For **A1**, in the basis $(X, X-Y)$, we obtain M_3.
- For **A2**, in the basis $(X, X-Y)$, and for **F1**, in the basis $(X-Y, X)$, we obtain M_4.
- For **C3**, in the basis (Y, X), we obtain M_{16}.
- For $\mathbf{E'1-E'2}$, in the basis $(X, X-Y)$, we obtain M_6.
- For $\mathbf{E'3}$, in the basis $(X, Y-X)$, we obtain M_{11}.
- For **F3**, in the basis (X, Y), we obtain M_{18}.
- For **F4** and **H2**, in the basis $(X-Y, X)$, we obtain M_{17}.

Hence, the ℓEAS associated to **A2** and **F1** are isomorphic, whereas the EAS **A2** and **F1** are not. As similar situation holds for **F4** and **H2**.

2. It is possible to show that any 2-dimensional ℓEAS with a basis of special vectors is isomorphic to one of the eighteen cases of Example 4.1. For all of them, let us give special vectors, left units and left counits for the eighteen cases of Example 4.1. In each case, we

give a basis of the spaces of left units and left counits; λ, μ and ν are scalars. For M_2, we assume that the parameter a is nonzero (otherwise, $\Phi = 0$).

Case	Special vectors of eigenvalue 1	Special vectors of eigenvalue 0	Left units	Left counits
M_1	0	$\lambda x,\ \mu y$	$\emptyset$	$\emptyset$
M_2	0	$\lambda x,\ \mu y$	$\emptyset$	$\emptyset$
M_3	λx	μy	$\emptyset$	$\emptyset$
M_4	λx	μy	$\emptyset$	(x^*)
M_5	λx	μy	(x)	$\emptyset$
M_6	λx	μy	$\emptyset$	$\emptyset$
M_7	λx	μy	$\emptyset$	$\emptyset$
M_8	λx	μy	$\emptyset$	(x^*)
M_9	λx	μy	(x)	$\emptyset$
M_{10}	λx	μy	(x)	$\emptyset$
M_{11}	$\lambda x, \nu(x+y)$	μy	$\emptyset$	(x^*)
M_{12}	λx	μy	$\emptyset$	$\emptyset$
M_{13}	$\lambda x,\ \mu y$	0	$\emptyset$	$\emptyset$
M_{14}	$\lambda x,\ \mu y$	0	(x)	(y^*)
M_{15}	$\lambda x,\ \mu y$	$\nu(x-y)$	$\emptyset$	(x^*+y^*)
M_{16}	$\lambda x,\ \mu y$	0	(x)	(x^*+y^*)
M_{17}	$\lambda x,\ \mu y$	0	(x)	(x^*+y^*)
M_{18}	$\lambda x+\mu y$	0	(x, y)	(x^*, y^*)

Among them, M_{11} has three lines of special vectors. In the basis $(x+y, x)$ its matrix is M_{15}, so M_{11} and M_{15} are isomorphic.

4.3 Left units and counits of finite nondegenerate CEDS

Proposition 4.5. *Let $(\Omega_1, *)$ be an abelian finite group, $(\Omega_2, \star)$ be a finite group, and Ω_3 be a finite set. We denote by e_1 and e_2 the units of Ω_1 and Ω_2.*

1. *Let (A, Φ) be the linearization of the CEDS*

$$(\mathbf{EAS}(\Omega_1, *) \rtimes_{\succ} \mathbf{EAS}'(\Omega_2, \star)) \times \mathbf{EAS}(\Omega_3)$$

of Theorem 3.14.

   (a) *The special vectors of eigenvalue 1 of (A, Φ) are the vectors of the form*

$$a = \sum_{(\alpha_1,\alpha_2,\alpha_3)\in H_1\times H_2\times\Omega_3} g(\alpha_3)(\alpha_1, \alpha_2, \alpha_3),$$

where H_1 is a subgroup of Ω_1, H_2 is a subgroup of Ω_2, such that $H_2 \succ H_1 \subseteq H_1$, and $g : \Omega_3 \longrightarrow \mathbb{K}$ is a map.

   (b) *The left units of (A, Φ) are the vectors of the form*

$$a = \sum_{(\alpha_2,\alpha_3)\in\Omega_2\times\Omega_3} g(\alpha_3)(e_1, \alpha_2, \alpha_3),$$

where $g : \Omega_3 \longrightarrow \mathbb{K}$ is a map.

   (c) *The left counits of (A, Φ) are the linear forms f such that for any $(\alpha_1, \alpha_2, \alpha_3) \in \Omega$:*

$$f(\alpha_1, \alpha_2, \alpha_3) = \delta_{\alpha_2,e_2} g(\alpha_3),$$

where $g : \Omega_3 \longrightarrow \mathbb{K}$ is a map.

2. Let (A, Φ) be the linearization of the CEDS

$$(\mathbf{EAS}(\Omega_2, \star) \ltimes_{\prec} \mathbf{EAS}'(\Omega_1, *)) \times \mathbf{EAS}(\Omega_3)$$

of Corollary 3.15.

(a) The special vectors of eigenvalue 1 of (A, Φ) are the vectors of the form

$$a = \sum_{(\alpha_1,\alpha_2,\alpha_3)\in H_1\times H_2\times\Omega_3} g(\alpha_3)(\alpha_2, \alpha_1, \alpha_3),$$

where H_1 is a subgroup of Ω_1, H_2 is a subgroup of Ω_2, such that $H_1 \prec H_2 \subseteq H_1$, and $g : \Omega_3 \longrightarrow \mathbb{K}$ is a map.

(b) The left units of (A, Φ) are the vectors of the form

$$a = \sum_{(\alpha_1,\alpha_3)\in\Omega_1\times\Omega_3} g(\alpha_3)(e_2, \alpha_1, \alpha_3),$$

where $g : \Omega_3 \longrightarrow \mathbb{K}$ is a map.

(c) The left counits of (A, Φ) are the linear forms f such that for any $(\alpha_1, \alpha_2, \alpha_3) \in \Omega$:

$$f(\alpha_1, \alpha_2, \alpha_3) = \delta_{\alpha_1,e_1} g(\alpha_3),$$

where $g : \Omega_3 \longrightarrow \mathbb{K}$ is a map.

Proof. 1.(a) Let a be a nonzero vector of A, which we write as

$$a = \sum_{(\alpha_1,\alpha_2,\alpha_3)\in\Omega} a_{(\alpha_1,\alpha_2,\alpha_3)}(\alpha_1, \alpha_2, \alpha_3).$$

Then a is a special vector of eigenvalue 1 if, and only if, for any $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3) \in \Omega$,

$$a_{(\alpha_1,\alpha_2,\alpha_3)} a_{(\beta_1,\beta_2,\beta_3)} = a_{(\alpha_2^{-1}\succ\beta_1,\beta_2\star\alpha_2,\beta_3)} a_{((\alpha_2^{-1}\succ\beta_1^{-1})*\alpha_1,\alpha_2,\alpha_3)}. \tag{33}$$

We put:

$$\begin{aligned}
\forall(\alpha_2, \alpha_3) \in \Omega_2 \times \Omega_3, \qquad & H_1(\alpha_2, \alpha_3) = \{\alpha_1 \in \Omega_1,\ a_{(\alpha_1,\alpha_2,\alpha_3)} \neq 0\},\\
\forall\alpha_3 \in \Omega_3, \qquad & H_2(\alpha_3) = \{\alpha_2 \in \Omega_2,\ H_1(\alpha_2, \alpha_3) \neq \emptyset\}.,\\
& H_3 = \{\alpha_3 \in \Omega_3,\ H_2(\alpha_3) \neq \emptyset\}.
\end{aligned}$$

We shall also consider the map:

$$g : \left\{ \begin{array}{rcl} \Omega_3 & \longrightarrow & \mathbb{K} \\ \alpha_3 & \longrightarrow & g(\alpha_3) = a_{e_1,e_2,\alpha_3}. \end{array} \right.$$

Let us first prove that if $\alpha_3 \in H_3$, then $H_2(\alpha_3)$ is a subgroup of Ω_2. Let $\alpha_2, \beta_2 \in H_2(\alpha_3)$ (which is nonempty as $\alpha_3 \in H_3$). Let $\alpha_1, \beta_1 \in \Omega_1$, such that $a_{(\alpha_1,\alpha_2,\alpha_3)} \neq 0$ and $a_{(\beta_1,\beta_2,\alpha_3)} \neq 0$. By (33),

$$a_{(\alpha_2^{-1}\succ\beta_1,\beta_2\star\alpha_2,\alpha_3)} \neq 0,$$

so $\beta_2 \star \alpha_2 \in H_2(\alpha_3)$. As Ω_2 is finite, $H_2(\alpha_3)$ is a subgroup of Ω_2.

Let us prove that if $\alpha_3 \in H_3$, then $H_1(e_2, \alpha_3)$ is a subgroup of Ω_1 and, moreover, for any $\alpha_1 \in H_1(e_2, \alpha_3)$, $a_{(\alpha_1,e_2,\alpha_3)} = g(\alpha_3)$. As $H_2(\alpha_3)$ is a subgroup of Ω_2, it contains e_2, so $H_1(e_2, \alpha_3) \neq \emptyset$. Let $\alpha_1, \beta_1 \in H_1(e_2, \alpha_3)$. By (33):

$$a_{(\alpha_1,e_2,\alpha_3)} a_{(\beta_1,e_2,\alpha_3)} = a_{(\beta_1,e_2,\alpha_3)} a_{(\beta_1^{-1}*\alpha_1,e_2,\alpha_3)} \neq 0.$$

Hence, $\beta_1^{-1} * \alpha_1 \in H_1(e_2, \alpha_3)$. Taking $\alpha_1 = \beta_1$, we obtain

$$a_{(\alpha_1,e_2,\alpha_3)} a_{(\alpha_1,e_2,\alpha_3)} = a_{(\alpha_1,e_2,\alpha_3)} a_{(e_1,e_2,\alpha_3)} \neq 0,$$

so $a_{(\alpha_1,e_2,\alpha_3)} = a_{(e_1,e_2,\alpha_3)} = g(\alpha_3)$.

Let us prove that if $\alpha_3, \beta_3 \in H_3$ and $\beta_2 \in H_2(\beta_3)$, then $H_1(\beta_2, \beta_3) \subseteq H_1(e_2, \alpha_3)$. Let $\beta_1 \in H_1(\beta_2, \beta_3)$. Then $a_{(\beta_1,\beta_2,\beta_3)} \neq 0$. As $H_1(e_1, \beta_2)$ is a subgroup of Ω_1, it contains e_1, so $a_{(e_1,e_2,\alpha_3)} \neq 0$. By (33):

$$a_{(e_1,e_2,\alpha_3)} a_{(\beta_1,\beta_2,\beta_3)} = g(\alpha_3) a_{(\beta_1,\beta_2,\beta_3)} = a_{(\beta_1,\beta_2,\beta_3)} a_{(\beta_1^{-1},e_2,\alpha_3)} \neq 0,$$

so $\beta_1^{-1} \in H_1(e_2, \alpha_3)$. As this is a subgroup of Ω_1, $\beta_1 \in H_1(e_2, \alpha_3)$.

As a consequence, for $\beta_2 = e_2$, we obtain by symmetry that for any $\alpha_3, \beta_3 \in H_3$, $H_1(e_2, \alpha_3) = H_1(e_2, \beta_3)$. Therefore, there exists a subgroup H_1 of Ω_1 such that for any $\alpha_3 \in \Omega_3$, $H_1(e_2, \alpha_3) = H_1$.

Let us prove that for any $\alpha_3 \in H_3$, $H_2(\alpha_3) \succ H_1 \subseteq H_1$. Let $\beta_1 \in H_1 = H_1(e_2, \alpha_3)$, then $a_{(\beta_1,e_2,\alpha_3)} \neq 0$. Let $\alpha'_2 \in H_2(\alpha_3)$. We put $\alpha_2 = \alpha_2'^{-1} \in H_2(\alpha_3)$. there exists $\alpha_1 \in H_1(\alpha_2, \alpha_3)$, such that $a_{(\alpha_1,\alpha_2,\alpha_3)} \neq 0$. By (33):

$$a_{(\alpha_1,\alpha_2,\alpha_3)} a_{(\beta_1,e_2,\alpha_3)} = a_{(\beta_1,\alpha_2,\alpha_3)} a_{((\alpha_2^{-1} \succ \beta_1^{-1}) * \alpha_1, e_2, \alpha_3)} \neq 0.$$

So $(\alpha_2^{-1} \succ \beta_1^{-1}) * \alpha_1 \in H_1(e_2, \alpha_3) = H_1$. Moreover, as $H_1(\alpha_2, \alpha_2) \subseteq H_1$:

$$\alpha_2^{-1} \succ \beta_1^{-1} = \alpha'_2 \succ \beta_1^{-1} \in H_1.$$

Its inverse $\alpha'_2 \succ \beta_1$ is also an element of H_1, so $H_2(\alpha_3) \succ H_1 \subseteq H_1$.

Let us prove that for any $\alpha_3 \in H_3$, for any $\alpha_2 \in H_2(\alpha_3)$, $H_1(\alpha_2, \alpha_3) = H_1$. Let $\alpha_1 \in H_1 = H_1(e_2, \alpha_3)$ and $\beta_1 \in H_1(\alpha_2, \alpha_3)$. By (33):

$$a_{(\alpha_1,e_2,\alpha_3)} a_{(\beta_1,\alpha_2,\alpha_3)} = a_{(\beta_1,\alpha_2,\alpha_3)} a_{(\beta_1^{-1} * \alpha_1,\alpha_2,\alpha_3)} \neq 0,$$

so $\beta_1^{-1} * \alpha_1 \in H_1(\alpha_2, \alpha_3)$. We obtain an injective map

$$\begin{cases} H_1 & \longrightarrow & H_1(\alpha_2, \alpha_3) \\ \alpha_1 & \longrightarrow & \beta_1^{-1} * \alpha_1. \end{cases}$$

Hence, $|H_1| \leqslant |H_1(\alpha_2, \alpha_3)|$. We already proved that $H_1(\alpha_2, \alpha_3) \subseteq H_1$, so $H_1 = H_1(\alpha_2, \alpha_3)$.

We now prove that there exists a subgroup H_2 of Ω_2 such that for any $\alpha_3 \in H_3$, $H_2(\alpha_3) = H_2$, and that, moreover, for any $\alpha_2 \in H_2$, for any $\alpha_3 \in H_3$, $a_{(e_1,\alpha_2,\alpha_3)} = g(\alpha_3)$. Let $\alpha_3, \beta_3 \in H_3$. Let $\alpha_2 \in H_2(\alpha_3)$. As $e_1 \in H_1(\alpha_2, \alpha_3) = H_1$, $a_{(e_1,\alpha_2,\alpha_3)} \neq 0$. As $e_2 \in H_2(\alpha_3)$, $a_{(e_1,e_2,\beta_3)} \neq 0$. By (33):

$$a_{(e_1,\alpha_2,\alpha_3)} a_{(e_1,e_2,\beta_3)} = a_{(e_1,\alpha_2,\beta_3)} a_{(e_1,\alpha_2\alpha_3)} \neq 0,$$

so $\alpha_2 \in H_2(\beta_3)$. We proved that $H_2(\alpha_3) \subseteq H_2(\beta_3)$: by symmetry, $H_2(\alpha_3) = H_2(\beta_3)$, which prove the existence of H_2. Moreover, as $a_{(e_1,\alpha_2,\alpha_3)} \neq 0$ and $a_{(e_1,e_2,\beta_3)} = g(\beta_3)$, we obtain that $a_{(e_1,\alpha_2,\beta_3)} = g(\beta_3)$.

We proved that for any $(\alpha_1, \alpha_2, \alpha_3) \in \Omega$,

$$a_{(\alpha_1,\alpha_2,\alpha_3)} \neq 0 \Longleftrightarrow (\alpha_1, \alpha_2, \alpha_3) \in H_1 \times H_2 \times H_3.$$

Let $\alpha_3, \beta_3 \in H_3$, $\beta_1 \in H_1$, $\alpha_2 \in H_2$. We put $\alpha_1 = \alpha_2^{-1} \succ \beta_1$. Then, By (33):

$$g(\alpha_3)g(\beta_3) = a_{(\beta_1,\alpha_2,\alpha_3)}a_{(e_1,\alpha_2,\beta_3)} = a_{(\beta_1,\alpha_2,\alpha_3)}g(\beta_3),$$

so $a_{(\beta_1,\alpha_2,\alpha_3)} = g(\alpha_3)$. we proved that a has the announced form.

Conversely, if a is of the announced form,

$$\begin{aligned}\Phi(a \otimes a) &= \sum_{(\alpha_1,\alpha_2,\alpha_3),(\beta_1,\beta_2,\beta_3)\in H_1\times H_2\times\Omega_3} g(\alpha_3)g(\beta_3)(\alpha_1 * \beta_1, \beta_2, \beta_3) \otimes (\beta_2 \succ \alpha_2, \alpha_2 * \beta_2^{-1}, \alpha_3)\\ &= \sum_{(\alpha_1,\alpha_2,\alpha_3),(\beta_1,\beta_2,\beta_3)\in H_1\times H_2\times\Omega_3} (\beta_1, \beta_2, \beta_3) \otimes (\alpha_1, \alpha_2, \alpha_3)\\ &= a \otimes a.\end{aligned}$$

2.(a) If (A, Φ) is the linearization of $(\mathbf{EAS}(\Omega_2, \star) \ltimes_{\prec} \mathbf{EAS}'(\Omega_1, *)) \times \mathbf{EAS}(\Omega_3)$, then (A, Φ^{-1}) is the linearization of $(\mathbf{EAS}(\Omega_1, *) \rtimes_{\prec^{op}} \mathbf{EAS}'(\Omega_2, \star^{op})) \times \mathbf{EAS}(\Omega_3)$. The result then comes from the observation that the special vectors of (A, Φ) and (A, Φ^{-1}) are the same.

1.(b) Let a be a left unit of A. Then it is a special vector, which we write as

$$a = \sum_{(\alpha_1,\alpha_1,\alpha_3)\in H_1\times H_2\times\Omega_3} g(\alpha_3)(\alpha_1, \alpha_1, \alpha_3).$$

For any $b = (\beta_1, \beta_2, \beta_3) \in \Omega$:

$$\begin{aligned}\Phi(a \otimes b) &= \sum_{(\alpha_1,\alpha_1,\alpha_3)\in H_1\times H_2\times\Omega_3} g(\alpha_3)(\alpha_1 * \beta_1, \beta_2, \beta_3) \otimes (\beta_2 \succ \alpha_1, \alpha_2 * \beta_2^{-1}, \alpha_3)\\ = b \otimes a &= \sum_{(\alpha_1,\alpha_1,\alpha_3)\in H_1\times H_2\times\Omega_3} g(\alpha_3)(\beta_1, \beta_2, \beta_3) \otimes (\alpha_1, \alpha_1, \alpha_3).\end{aligned}$$

Taking $\beta_1 = e_1$, we obtain that for any $\alpha_1 \in H_1$, $\alpha_1 = e_1$, so $H_1 = \{e_1\}$. Moreover, for any $\beta_2 \in \Omega_2$:

$$\sum_{(\alpha_2,\alpha_3)\in H_2\times\Omega_3} g(\alpha_3)(\alpha_2 \star \beta_2^{-1}, \alpha_3) = \sum_{(\alpha_2,\alpha_3)\in H_2\times\Omega_3} g(\alpha_3)(\alpha_2, \alpha_3),$$

so for any $\alpha_2 \in H_2$, $\alpha_2 \star \beta_2^{-1} \in H_2$. In particular, for $\alpha_2 = e_2$, $\beta_2^{-1} \in H_2$ and finally $\beta_2 \in H_2$: $H_2 = \Omega_2$. The converse application is immediate.

2.(b) Similar proof.

1.(c) and 2.(c) The left counits of (A, Φ) are the left units of (A^*, Φ^*), which is isomorphic to (A, Φ^{-1}). The resut comes from 2.(b). and 1.(b). □

5 From bialgebras to ℓEAS

We refer to [1, 8] for classical results and notations on bialgebras and Hopf algebras.

5.1 A functor from bialgebras to ℓEAS

Proposition 5.1. *Let (A, m, Δ) be a bialgebra, not necessarily unitary nor counitary. We define $\Phi : A \otimes A \longrightarrow A \otimes A$ by:*

$$\forall a, b \in A, \qquad \Phi(a \otimes b) = (m \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \tau) \circ (\Delta \otimes \mathrm{Id}_A)(a \otimes b) = \sum a^{(1)}b \otimes a^{(2)},$$

with Sweedler's notation $\Delta(a) = \sum a^{(1)} \otimes a^{(2)}$. Then (A, Φ) is an ℓEAS, denoted by $\ell\mathbf{EAS}(A, m, \Delta)$.

Proof. For any $a,b,c \in A$:

$$\begin{aligned}(\mathrm{Id}\otimes\Phi)\circ(\Phi\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\Phi)(a\otimes b\otimes c) &= (\Phi\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\tau)\circ(\Phi\otimes\mathrm{Id})(a\otimes b\otimes c)\\ &= \sum\sum a^{(1)}b^{(1)}c\otimes a^{(2)}b^{(2)}\otimes a^{(3)}.\end{aligned}$$ □

Example 5.1. 1. Let $(\Omega,\star)$ be a semigroup. We take $A=\mathbb{K}\Omega$, with its usual bialgebra structure: the product m obtained by linearization of $\star$ and the coproduct Δ defined by

$$\forall \alpha\in\Omega, \qquad \Delta(\alpha)=\alpha\otimes\alpha.$$

Then (A,m,Δ) is a bialgebra, unitary if, and only if Ω is a monoid, and counitary. In $\ell\mathbf{EAS}(A,m,\Delta)$, for any $\alpha,\beta\in\Omega$:

$$\Phi(\alpha\otimes\beta)=\alpha\star\beta\otimes\alpha.$$

We recover the linearization of $\mathbf{EAS}(\Omega,\star)$.

2. Let A be a vector space, $1_A\in A$ and $\varepsilon\in A^*$ such that $\varepsilon(1_A)=1$. We define a product and a coproduct on A by:

$$\begin{aligned}&\forall a,b\in A, && a\cdot b=\varepsilon(a)b,\\ &\forall a\in A, && \Delta(a)=1_A\otimes a.\end{aligned}$$

Then (A,m,Δ) is a bialgebra, with a left unit 1_A and a left counit ε. It is unitary if, and only if A is one-dimensional; it is counitary if, and only if, A is one-dimensional. In $\ell\mathbf{EAS}(A,m,\Delta)$, for any $a,b\in A$, $\Phi(a\otimes b)=b\otimes a$.

Proposition 5.2. *Let (A,m,Δ) be a bialgebra, not necessarily unitary nor counitary.*

1. *Let us consider the following conditions:*
   - *(a) $\ell\mathbf{EAS}(A,m,\Delta)$ is an ℓCEDS.*
   - *(b) For any $a,b,c\in A$, $\sum\sum a^{(1)}b^{(1)}c\otimes a^{(2)}\otimes b^{(2)}=\sum\sum b^{(1)}a^{(1)}c\otimes a^{(2)}\otimes b^{(2)}$.*
   - *(c) For any $a,b,c\in A$, $abc=bac$.*
   - *(d) m is commutative.*

   Then (d) $\Longrightarrow$ (c) $\Longrightarrow$ (b) $\Longleftrightarrow$ (a). If (A,Δ) has a right counit, then (c) $\Longleftrightarrow$ (a). If (A,m,Δ) has a right counit and a right unit, then (d) $\Longleftrightarrow$ (a).

2. *Let us consider the following conditions:*
   - *(a) $\ell\mathbf{EAS}(A,m,\Delta)$ is a dual ℓCEDS.*
   - *(b) For any $a,b,c\in A$, $\sum a^{(1)}b\otimes a^{(2)}c\otimes a^{(3)}=\sum a^{(2)}b\otimes a^{(1)}c\otimes a^{(3)}$.*
   - *(c) $(\Delta\otimes\mathrm{Id})\circ\Delta=(\tau\otimes\mathrm{Id})\circ(\Delta\otimes\mathrm{Id})\circ\Delta$.*
   - *(d) Δ is cocommutative.*

   Then (d) $\Longrightarrow$ (c) $\Longrightarrow$ (b) $\Longleftrightarrow$ (a). If (A,m) has a right unit, then (c) $\Longleftrightarrow$ (a). If (A,m,Δ) has a right counit and a right unit, then (d) $\Longleftrightarrow$ (a).

Proof. 1. Obviously, $(d)\Longrightarrow(c)\Longrightarrow(b)$. Let $a,b,c\in A$.

$$\begin{aligned}(\mathrm{Id}\otimes\Phi)\circ(\mathrm{Id}\otimes\tau)\circ(\tau\otimes\mathrm{Id})\circ(\Phi\otimes\mathrm{Id})(b\otimes c\otimes a) &= \sum\sum b^{(2)}\otimes a^{(1)}b^{(1)}c\otimes a^{(2)},\\ (\tau\otimes\mathrm{Id})\circ(\Phi\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\Phi)\circ(\mathrm{Id}\otimes\tau)(b\otimes c\otimes a) &= \sum\sum b^{(2)}\otimes b^{(1)}a^{(1)}c\otimes a^{(2)},\end{aligned}$$

so $(a) \Longleftrightarrow (b)$. If (b) is satisfied and if (A,Δ) has a right counit ε, applying $(\mathrm{Id}\otimes\varepsilon\otimes\varepsilon)$ to (b), we obtain (c). If (c) is satisfied and (A,m) has a right unit 1_A, taking $c=1_A$ in (c), we obtain (d).

2. Obviously, $(d) \Longrightarrow (c) \Longrightarrow (b)$. Let $a,b,c\in A$.

$$\begin{aligned}(\Phi\otimes\mathrm{Id})\circ(\tau\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\tau)\circ(\mathrm{Id}\otimes\Phi)(b\otimes a\otimes c)&=\sum a^{(2)}b\otimes a^{(3)}\otimes a^{(1)}c,\\(\mathrm{Id}\otimes\tau)\circ(\mathrm{Id}\otimes\Phi)\circ(\Phi\otimes\mathrm{Id})\circ(\tau\otimes\mathrm{Id})(b\otimes a\otimes c)&=\sum a^{(1)}b\otimes a^{(3)}\otimes a^{(2)}c,\end{aligned}$$

so $(a) \Longleftrightarrow (b)$. If (b) is satisfied and if (A,m) has a right unit 1_A, taking $b=c=1_A$ in (b), we obtain (c). If (c) is satisfied and (A,Δ) has a right counit ε, applying $(\mathrm{Id}\otimes\mathrm{Id}\otimes\varepsilon)$ to (c), we obtain (d). □

Proposition 5.3. *Let (A,m,Δ) be a finite-dimensional bialgebra, not necessarily unitary nor counitary. Then $\ell\mathbf{EAS}(A,m,\Delta)^*=\ell\mathbf{EAS}(A^*,\Delta^*,m^*)$.*

Proof. Let $f,g\in A^*$. For any $a,b\in A$:

$$\begin{aligned}\Phi^*&=((m\otimes\mathrm{Id}_A)\circ(\mathrm{Id}_A\otimes\tau)\circ(\Delta\otimes\mathrm{Id}_A))^*\\&=(\Delta^*\otimes\mathrm{Id}_{A^*})\circ(\mathrm{Id}_{A^*}\otimes\tau)\circ(m^*\otimes\mathrm{Id}_{A^*}).\end{aligned}$$

Therefore, $\ell\mathbf{EAS}(A,m,\Delta)^*=\ell\mathbf{EAS}(A^*,\Delta^*,m^*)$. □

Proposition 5.4. *Let (A,m,Δ) be a bialgebra.*

1. *We assume that (A,m) has a right unit 1_A.*
   - *If 1_A is not a unit of (A,m), the unique left unit of $\ell\mathbf{EAS}(A,m,\Delta)$ is 0. If 1_A is a unit of (A,m), then the left units of $\ell\mathbf{EAS}(A,m,\Delta)$ are the elements $a\in A$ such that $\Delta(a)=1_A\otimes a$.*
2. *We assume that (A,Δ) has a right counit ε_A.*
   - *If ε_A is not a unit of (A,Δ), the unique left counit of $\ell\mathbf{EAS}(A,m,\Delta)$ is 0. If ε_A is a counit of (A,Δ), then the left counits of $\ell\mathbf{EAS}(A,m,\Delta)$ are the elements $\lambda\in A^*$ such that $\lambda\circ m=\varepsilon\otimes\lambda$.*

Proof. 1. Let us assume that $\ell\mathbf{EAS}(A,m,\Delta)$ has a nonzero left unit a. Let us choose $\lambda\in A^*$ such that $\lambda(a)=1$. For any $b\in A$:

$$(\mathrm{Id}\otimes\lambda)\circ\Phi(a\otimes b)=\underbrace{\left(\sum a^{(1)}\lambda\left(a^{(2)}\right)\right)}_{=a'}b=(\mathrm{Id}\otimes\lambda)(b\otimes a)=b\lambda(a)=b,$$

so a' is a left unit of (A,m). Then $a'1_A=a'=1_A$, so $a'=1_A$ is a unit. Moreover, for $b=1_A$,

$$\Phi(a\otimes 1_A)=\sum a^{(1)}1_A\otimes a^{(2)}=\Delta(a)=1_A\otimes a.$$

Conversely, if 1_A is a unit of (A,m) and $\Delta(a)=1_A\otimes a$, then a is clearly a left unit of $\ell\mathbf{EAS}(A,m,\Delta)$.

2. Let us assume that $\ell\mathbf{EAS}(A,m,\Delta)$ has a nonzero left counit λ. Let us choose $b\in A$ such that $\lambda(b)=1$. For any $a\in A$:

$$(\lambda\otimes\mathrm{Id})\circ\Phi(a\otimes b)=\sum\lambda\left(a^{(1)}b\right)a^{(2)}=a\lambda(b)=a.$$

If we define $\lambda' : A \longrightarrow \mathbb{K}$ by $\lambda'(a) = \lambda(ab)$, then λ' is a left counit of (A, Δ). As ε_A is a right counit of (A, Δ),

$$(\lambda' \otimes \varepsilon_A) \circ \Delta = \lambda' = \varepsilon_A,$$

so $\lambda' = \varepsilon_A$ is a counit of (A, Δ). Moreover, for any $a, b \in A$:

$$(\lambda \otimes \varepsilon_A) \circ \Phi(a \otimes b) = \varepsilon_A(a)\lambda(b) = \sum \lambda\left(a^{(1)}b\right) \varepsilon_A\left(a^{(2)}\right) = \lambda(ab),$$

so $\lambda \circ m = \varepsilon_A \otimes \lambda$. Conversely, if ε_A is a counit of (A, Δ) and $\lambda \circ m = \varepsilon_A \otimes \lambda$, then for any $a, b \in A$:

$$(\lambda \otimes \mathrm{Id}) \circ \Phi(a \otimes b) = \sum \lambda\left(a^{(2)}b\right) a^{(2)} = \sum \varepsilon_A\left(a^{(1)}\right) \lambda(b)a = \lambda(b)a,$$

so λ is a left counit of $\ell\mathbf{EAS}(A, m, \Delta)$. □

More generally, we can obtain others ℓEAS with the help of a bialgebra projection or with certain linear forms:

Proposition 5.5. *Let (A, m, Δ) be a bialgebra, not necessarily unitary nor counitary, and $\pi : A \longrightarrow A$ be a bialgebra morphism such that $\pi^2 = \pi$. We define $\Phi : A \otimes A \longrightarrow A \otimes A$ by:*

$$\forall a, b \in A, \qquad \Phi(a \otimes b) = (m \otimes \pi) \circ (\mathrm{Id}_A \otimes \tau) \circ (\Delta \otimes \mathrm{Id}_A)(a \otimes b) = \sum a^{(1)}b \otimes \pi\left(a^{(2)}\right).$$

Then (A, Φ) is an ℓEAS.

Proof. We define $\delta = (\mathrm{Id} \otimes \pi) \circ \Delta$. Then (A, m, δ) is a bialgebra. Note that it is not counitary, except if (A, Δ) is counitary and $\pi = \mathrm{Id}_A$. We can then apply Proposition 5.1 to (A, m, δ). □

Example 5.2. Let $(\Omega, \star)$ be a semigroup and $\pi : \Omega \longrightarrow \Omega$ be a semigroup morphism such that $\pi^2 = \pi$. We take $A = \mathbb{K}\Omega$, with its usual bialgebra structure. Then in $\ell\mathbf{EAS}(A, m, \Delta)$, for any $\alpha, \beta \in \Omega$:

$$\Phi(\alpha \otimes \beta) = \alpha \star \beta \otimes \pi(\alpha).$$

We recover the linearization of $\mathbf{EAS}(\Omega, \star, \pi)$.

Proposition 5.6. *Let (A, Δ) be a coalgebra, not necessarily counitary, and $f \in A^*$ such that $(f \otimes f) \circ \Delta = f$. We put, for any $a, b \in A$:*

$$\Phi(a \otimes b) = \sum f\left(a^{(1)}\right) b \otimes a^{(2)}.$$

Then (A, Φ) is an ℓCEDS.

Proof. We define a product on A by $a \star b = f(a)b$. It is associative. Moreover, for any $a, b \in A$, as $(f \otimes f) \circ \Delta = f$:

$$\begin{aligned}
\Delta(a \star b) &= f(a) \sum b^{(1)} \otimes b^{(2)} \\
&= \sum f\left(a^{(1)}\right) f\left(a^{(2)}\right) \sum b^{(1)} \otimes b^{(2)} \\
&= \sum \sum f\left(a^{(1)}\right) b^{(1)} \otimes f\left(a^{(2)}\right) b^{(2)} \\
&= \Delta(a) \star \Delta(b),
\end{aligned}$$

so $(A, \star, \Delta)$ is a bialgebra, and $(A, \Phi) = \ell\mathbf{EAS}(A, \star, \Delta)$. Moreover, for any $a, bc \in A$,

$$a \star b \star c = f(a)f(b)c = f(b)f(c)a = b \star a \star c.$$

By Proposition 5.2, (A, Φ) is an ℓCEDS. □

Example 5.3. Let Ω be a set, $A = \mathbb{K}\Omega$ be the associated coalgebra (where any $\alpha \in \Omega$ is a group-like element), and $\Omega' \subseteq \Omega$ be any set. We define the linear form $f : A \longrightarrow \mathbb{K}$ by

$$\forall \alpha \in \Omega, \qquad f(\alpha) = \begin{cases} 1 \text{ if } \alpha \in \Omega', \\ 0 \text{ otherwise.} \end{cases}$$

For any $\alpha \in \Omega$, $(f \otimes f) \circ \Delta(\alpha) = f(\alpha)^2 = f(\alpha)$, so we obtain an ℓCEDS such that for any $\alpha, \beta \in \Omega$:

$$\Phi(\alpha \otimes \beta) = \begin{cases} \beta \otimes \alpha \text{ if } \alpha \in \Omega', \\ 0 \text{ otherwise.} \end{cases}$$

5.2 A functor from Hopf algebras to ℓEAS

Proposition 5.7. *Let (A, m, Δ) be a Hopf algebra, of antipode S. We define $\Phi : A\otimes A \longrightarrow A\otimes A$ by:*

$$\forall a, b \in A, \quad \Phi(a \otimes b) = (\mathrm{Id}_A \otimes m) \circ (\mathrm{Id}_A \otimes S \otimes \mathrm{Id}_A) \circ (\Delta \otimes \mathrm{Id}) \circ \tau(a \otimes b) = \sum b^{(1)} \otimes S\left(b^{(2)}\right) a.$$

Then (A, Φ) is an ℓEAS, denoted by $\ell\mathbf{EAS}'(A, m, \Delta)$. It is nondegenerate, and $(A, \Phi^{-1}) = \ell\mathbf{EAS}(A, m, \Delta^{op})$.

Proof. Let $a, b, c \in A$.

$$\begin{aligned}
&(\mathrm{Id} \otimes \Phi) \circ (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Phi)(a \otimes b \otimes c) \\
&= \sum\sum c^{(1)} \otimes S\left(c^{(3)}\right)^{(1)} b^{(1)} \otimes S\left(S\left(c^{(3)}\right)^{(2)} b^{(2)}\right) S\left(c^{(2)}\right) a \\
&= \sum\sum c^{(1)} \otimes S\left(c^{(4)}\right) b^{(1)} \otimes S\left(S\left(c^{(3)}\right) b^{(2)}\right) S\left(c^{(2)}\right) a \\
&= \sum\sum c^{(1)} \otimes S\left(c^{(4)}\right) b^{(1)} \otimes S\left(c^{(2)} S\left(c^{(3)}\right) b^{(2)}\right) a \\
&= \sum\sum c^{(1)} \otimes S\left(c^{(2)}\right) b^{(1)} \otimes S\left(b^{(2)}\right) a \\
&= (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \tau) \circ (\Phi \otimes \mathrm{Id})(a \otimes b \otimes c).
\end{aligned}$$

So (A, Φ) is an ℓEAS.

Let $(A, \Psi) = \ell\mathbf{EAS}(A, m, \Delta^{op})$: for any $a, b \in A$, $\Psi(a \otimes b) = \sum a^{(2)} b \otimes a^{(1)}$. Then:

$$\begin{aligned}
\Phi \circ \Psi(a \otimes b) &= \sum a^{(1)} \otimes S\left(a^{(2)}\right) a^{(3)} b = a \otimes b, \\
\Psi \circ \Phi(a \otimes b) &= \sum b^{(1)} S\left(b^{(2)}\right) a \otimes b^{(3)} = a \otimes b.
\end{aligned}$$

So Φ is bijective, of inverse Ψ. □

Example 5.4. Let $(G, \star)$ be a group and let $A = \mathbb{K}G^{op}$ be the Hopf algebra of the opposite of this group. A basis of $\ell\mathbf{EAS}'(A, m, \Delta)$ is given by G itself and, for any $\alpha, \beta \in G$:

$$\Phi(\alpha \otimes \beta) = \beta \otimes \alpha \star \beta^{-1}.$$

We recover in this way the linearisation of $\mathbf{EAS}'(\Omega, \star)$.

Corollary 5.8. *Let (A, m, Δ) be a bialgebra, such that (A, m, Δ^{op}) is a bialgebra. Then $(A, \Phi) = \ell\mathbf{EAS}(A, m, \Delta)$ is nondegenerate and $(A, \Phi^{-1}) = \ell\mathbf{EAS}'(A, m, \Delta^{op})$.*

Proposition 5.9. *Let (A, m, Δ) be a Hopf algebra.*

1. *Then* $\ell\mathbf{EAS}'(A,m,\Delta)$ *is an* ℓ*CEDS if, and only if,* $\Delta\circ S=\Delta^{op}\circ S$.

2. *Then* $\ell\mathbf{EAS}'(A,m,\Delta)$ *is a dual* ℓ*CEDS if, and only if* $S\circ m=S\circ m^{op}$.

Proof. 1. Let $a,b,c\in A$.

$$(\mathrm{Id}\otimes\Phi)\circ(\mathrm{Id}\otimes\tau)\circ(\tau\otimes\mathrm{Id})\circ(\Phi\otimes\mathrm{Id})(b\otimes c\otimes a)=\sum S\left(b^{(3)}\right)a\otimes b^{(1)}\otimes S\left(b^{(2)}\right)c, \tag{34}$$

$$(\tau\otimes\mathrm{Id})\circ(\Phi\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\Phi)\circ(\mathrm{Id}\otimes\tau)(b\otimes c\otimes a)=\sum S\left(b^{(2)}\right)a\otimes b^{(1)}\otimes S\left(b^{(3)}\right)c.$$

If $\Delta\circ S=\Delta^{op}\circ S$, then:

$$\begin{aligned}\sum b^{(1)}\otimes S\left(b^{(2)}\right)\otimes S\left(b^{(3)}\right)&=\sum\sum b^{(1)}\otimes S\left(b^{(2)}\right)^{(2)}\otimes S\left(b^{(2)}\right)^{(1)}\\&=\sum\sum b^{(1)}\otimes S\left(b^{(2)}\right)^{(1)}\otimes S\left(b^{(2)}\right)^{(2)}\\&=\sum b^{(1)}\otimes S\left(b^{(3)}\right)\otimes S\left(b^{(2)}\right),\end{aligned}$$

which implies that (A,Φ) is an ℓCEDS. Conversely, taking $a=c=1_A$, we obtain in (34):

$$\sum S\left(b^{(3)}\right)\otimes b^{(1)}\otimes S\left(b^{(2)}\right)=\sum S\left(b^{(2)}\right)\otimes b^{(1)}\otimes S\left(b^{(3)}\right).$$

Applying $\mathrm{Id}\otimes\varepsilon\otimes\mathrm{Id}$, we obtain:

$$\begin{aligned}\Delta\circ S(b)&=\sum S(b)^{(1)}\otimes S(b)^{(2)}\\&=\sum S\left(b^{(2)}\right)\otimes S\left(b^{(1)}\right)\\&=\sum S\left(b^{(1)}\right)\otimes S\left(b^{(2)}\right)\\&=\sum S(b)^{(2)}\otimes S(b)^{(1)}\\&=\Delta^{op}\circ S(b).\end{aligned}$$

2. Let $a,b,c\in A$.

$$\begin{aligned}(\Phi\otimes\mathrm{Id})\circ(\tau\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\tau)\circ(\mathrm{Id}\otimes\Phi)(b\otimes a\otimes c)&=\sum\sum a^{(1)}\otimes S\left(a^{(2)}\right)S\left(c^{(2)}\right)b\otimes c^{(1)},\\(\mathrm{Id}\otimes\tau)\circ(\mathrm{Id}\otimes\Phi)\circ(\Phi\otimes\mathrm{Id})\circ(\tau\otimes\mathrm{Id})(b\otimes a\otimes c)&=\sum\sum a^{(1)}\otimes S\left(c^{(2)}\right)S\left(a^{(2)}\right)b\otimes c^{(1)}.\end{aligned}$$

If $S\circ m=S\circ m^{op}$, then $m^{op}\circ(S\otimes S)=m\circ(S\otimes S)$, which implies that (A,Φ) is a dual ℓCEDS. Conversely, taking $b=1$ and applying $\varepsilon\otimes\mathrm{Id}\otimes\varepsilon$, we obtain:

$$S(a)S(c)=S(c)S(a),$$

so $m^{op}\circ S=m\circ(S\otimes S)=m^{op}\circ(S\otimes S)=S\circ m$. ☐

Remark 5.1. In particular, if S is invertible, then $\ell\mathbf{EAS}'(A,m,\Delta)$ is an ℓCEDS if, and only if, (A,m) is commutative; it is a dual ℓCEDS if, and only if, (A,Δ) is cocommutative.

Proposition 5.10. *Let* (A,m,Δ) *be a finite-dimensional Hopf algebra. Then* $\ell\mathbf{EAS}'(A,m,\Delta)^*=\ell\mathbf{EAS}'(A^*,\Delta^{*,op},m^{*,op})$.

Proof. Let $f,g\in A^*$. For any $a,b\in A$:

$$\begin{aligned}\Phi^*(f\otimes g)(a\otimes b)&=(f\otimes g)(\Phi(a\otimes b))\\&=\sum(f\otimes g)\left(b^{(1)}\otimes S\left(b^{(2)}a\right)\right)\\&=\sum\sum\left(f\otimes g^{(1)}\otimes g^{(2)}\right)\left(b^{(1)}\otimes S\left(b^{(2)}\right)\otimes a\right)\\&=\sum\left(f\otimes S^*\left(g^{(1)}\right)\otimes g^{(2)}\right)\left(b^{(1)}\otimes b^{(2)}\otimes a\right)\\&=\sum\left(g^{(2)}\otimes fS^*\left(g^{(1)}\right)\right)(a\otimes b),\end{aligned}$$

so $\Phi^*(f\otimes g)=\sum g^{(2)}\otimes fS^*\left(g^{(1)}\right)$, which is the ℓEAS attached to the Hopf algebra $(A^*,\Delta^{*,op},m^{*,op})$, whose antipode is S^*. □

Recall from [9] that a right integral of a Hopf algebra (A,m,Δ) is a linear map $f\in A^*$ such that for any $\mu\in A^*$,

$$(\lambda\otimes\mu)\circ\Delta=\mu(1_A)\lambda.$$

Proposition 5.11. *Let (A,m,Δ) be a Hopf algebra.*

1. *Let $a\in A$. It is a left unit of $\ell\mathbf{EAS}'(A,m,\Delta)$ if, and only if for any $b\in A$, $S(b)a=\varepsilon(b)a$.*
2. *Let $\lambda\in A^*$. It is a left counit of $\ell\mathbf{EAS}'(A,m,\Delta)$ if, and only if, for any $a\in A$, $\sum\lambda\left(b^{(1)}\right)S\left(b^{(2)}\right)=\lambda(b)1_A$. In particular, right integrals on (A,m,Δ) are left counit of $\ell\mathbf{EAS}'(A,m,\Delta)$; if S is invertible, then the converse is true.*

Proof. 1. Let $a\in A$. Then its a left unit if, and only if, for any $b\in A$, $\sum b^{(1)}\otimes S\left(b^{(2)}\right)a=b\otimes a$. Applying $\varepsilon\otimes\mathrm{Id}$, if a is a left unit, for any $b\in B$, $S(b)a=\varepsilon(b)\otimes a$. Conversely, if this holds, then for any $b\in B$:

$$\Phi(a\otimes b)=\sum b^{(1)}\otimes S\left(b^{(1)}\right)a=\sum b^{(1)}\otimes\varepsilon\left(b^{(2)}\right)a=b\otimes a.$$

2. Let $\lambda\in A^*$. It is a left counit if, and only if, for any $a,b\in A$:

$$\sum\lambda\left(b^{(1)}\right)S\left(b^{(2)}\right)a=a\lambda(b).$$

If λ is a left counit, taking $a=1_A$, we obtain that for any $b\in A$, $\lambda\left(b^{(1)}\right)S\left(b^{(2)}\right)=\lambda(b)1_A$. Conversely, if this holds, then for any $a,b\in A$,

$$(\lambda\otimes\mathrm{Id})\circ\Phi(a\otimes b)=\sum\lambda\left(b^{(1)}\right)S\left(b^{(2)}\right)a=\lambda(b)a=(\mathrm{Id}\otimes\lambda)(a\otimes b),$$

so λ is a left counit.

Let us assume that λ is a right integral of (A,m,Δ). For any $b\in A$, for any $\mu\in A^*$:

$$\sum\lambda\left(b^{(1)}\right)\mu\left(S\left(b^{(2)}\right)\right)=(\lambda\otimes\mu\circ S)\circ\Delta(b)\qquad=\mu\circ S(1_A)\lambda(b)=\mu(1_A)\lambda(b).$$

As this holds for any $\mu\in A^*$, $\sum\lambda\left(b^{(1)}\right)S\left(b^{(2)}\right)=\lambda(b)1_A$, so λ is a right integral. Let us now assume that S is invertible and that λ is a left counit. Let $\nu\in A^*$. For any $b\in A$, if $\mu=\nu\circ S^{-1}$:

$$\begin{aligned}\sum\lambda\left(b^{(1)}\right)\nu\left(b^{(2)}\right)&=\sum\lambda\left(b^{(1)}\right)\mu\circ S\left(b^{(2)}\right)\\&=\lambda(b)\mu(1_A)\\&=\lambda(b)\nu\circ S^{-1}(1_A)\\&=\lambda(b)\nu(1_A).\end{aligned}$$

So λ is a right integral. □

5.3 From left units and counits to bialgebras

Theorem 5.12. *Let (A,Φ) be an ℓEAS.*

1. *If a is a special vector of eigenvalue 1 of (A,Φ) then $\Delta_a:A\longrightarrow A\otimes A$ defined by $\Delta_a(b)=\Phi(b\otimes a)$ is a coassociative coproduct.*
2. *If ε is a special vector of eigenvalue 1 of $(A,\Phi)^*$, that is to say if $(\varepsilon\otimes\varepsilon)\circ\Phi=\varepsilon\otimes\varepsilon$, then $m_\varepsilon:A\otimes A\longrightarrow A$ defined by $m_\varepsilon=(\mathrm{Id}\otimes\varepsilon)\circ\Phi$ is an associative coproduct.*

3. *If a is a left unit of (A, Φ) and ε is a left counit of (A, Φ) such that $\varepsilon(a) = 1$, then $(A, m_\varepsilon, \Delta_a)$ is a bialgebra, with a as a left unit and ε as a left counit. Moreover, $(A, \Phi) = \ell\mathbf{EAS}(A, m_\varepsilon, \Delta_a)$.*

Proof. 1. For any $b \in A$:

$$\begin{aligned}(\mathrm{Id} \otimes \Phi) \circ (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Phi)(b \otimes a \otimes a) &= (\mathrm{Id} \otimes \Phi) \circ (\Phi \otimes \mathrm{Id})(b \otimes a \otimes a)\\ &= (\mathrm{Id} \otimes \Phi)(\Delta_a(b) \otimes a)\\ &= (\mathrm{Id} \otimes \Delta_a) \circ \Delta_a(b),\end{aligned}$$

$$\begin{aligned}(\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \tau) \circ (\Phi \otimes \mathrm{Id})(b \otimes a \otimes a) &= (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \tau)(\Delta_a(b) \otimes a)\\ &= (\Delta_a \otimes \mathrm{Id}) \circ \Delta_a(b).\end{aligned}$$

Hence, Δ_a is coassociative.

2. We obtain, as ε is a special vector of eigenvalue 1 of $(A, \Phi)^*$:

$$\begin{aligned}(\mathrm{Id} \otimes \varepsilon \otimes \varepsilon) \circ (\mathrm{Id} \otimes \Phi) \circ (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Phi) &= (\mathrm{Id} \otimes \varepsilon \otimes \varepsilon) \circ (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Phi)\\ &= (\mathrm{Id} \otimes \varepsilon) \circ \Phi \circ ((\mathrm{Id} \otimes \varepsilon) \circ \Phi))\\ &= m_\varepsilon \circ (\mathrm{Id} \otimes m_\varepsilon),\end{aligned}$$

$$\begin{aligned}(\mathrm{Id} \otimes \varepsilon \otimes \varepsilon) \circ (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \tau) \circ (\Phi \otimes \mathrm{Id}) &= (\mathrm{Id} \otimes \varepsilon) \circ \Phi \circ ((\mathrm{Id} \otimes \varepsilon) \otimes \mathrm{Id})\\ &= m_\varepsilon \circ (m_\varepsilon \otimes \mathrm{Id}).\end{aligned}$$

As a consequence, m_ε is associative.

3. As a is a left unit, it is a special vector of eigenvalue 1 of (A, Φ), so Δ_a is coassociative. Moreover, for any $b \in A$:

$$(\varepsilon \otimes \mathrm{Id}) \circ \Delta_a(b) = (\varepsilon \otimes \mathrm{Id}) \circ \Phi(b \otimes a) = (\mathrm{Id} \otimes \varepsilon)(b \otimes a) = b\varepsilon(a) = b,$$

so ε is a left counit of Δ_a. As ε is a left counit, it is a special vector of eigenvalue 1 of $(A, \Phi)^*$, so m_ε is associative. Moreover, for any $b \in A$:

$$m_\varepsilon(a \otimes b) = (\mathrm{Id} \otimes \varepsilon) \circ \Phi(a \otimes b) = (\mathrm{Id} \otimes \varepsilon)(b \otimes a) = b\varepsilon(a) = b.$$

So a is a left unit of m_ε.

Let $b_1, b_2 \in A$.

$$\begin{aligned}
\Delta_a(b_1b_2) &= (\mathrm{Id}\otimes\varepsilon\otimes\mathrm{Id}\otimes\varepsilon)\circ(\Phi\otimes\Phi)\circ(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})\circ(\Phi\otimes\Phi)\circ(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})(b_1\otimes b_2\otimes a\otimes a)\\
&= (\mathrm{Id}\otimes\varepsilon\otimes\mathrm{Id}\otimes\varepsilon)\circ(\mathrm{Id}\otimes\mathrm{Id}\otimes\Phi)\circ(\Phi\otimes\mathrm{Id}\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})\circ(\Phi\otimes\mathrm{Id}\otimes\mathrm{Id})\\
&\circ(\mathrm{Id}\otimes\mathrm{Id}\otimes\Phi)\circ(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})(b_1\otimes b_2\otimes a\otimes a)\\
&= (\mathrm{Id}\otimes\varepsilon\otimes\mathrm{Id}\otimes\varepsilon)\circ(\mathrm{Id}\otimes\mathrm{Id}\otimes\Phi)\circ(\mathrm{Id}\otimes\Phi\otimes\mathrm{Id})\circ(\Phi\otimes\mathrm{Id}\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\Phi\otimes\mathrm{Id})\\
&\circ(\mathrm{Id}\otimes\mathrm{Id}\otimes\Phi)\circ(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})(b_1\otimes b_2\otimes a\otimes a)\\
&= (\mathrm{Id}\otimes\mathrm{Id}\otimes\varepsilon)\circ(\mathrm{Id}\otimes\Phi)\circ(\mathrm{Id}\otimes((\varepsilon\otimes\mathrm{Id})\circ\Phi)\otimes\mathrm{Id})\circ(\Phi\otimes\mathrm{Id}\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\Phi\otimes\mathrm{Id})\\
&\circ(\mathrm{Id}\otimes\mathrm{Id}\otimes\Phi)\circ(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})(b_1\otimes b_2\otimes a\otimes a)\\
&= (\mathrm{Id}\otimes\mathrm{Id}\otimes\varepsilon)\circ(\mathrm{Id}\otimes\Phi)\circ(\mathrm{Id}\otimes\mathrm{Id}\otimes\varepsilon\otimes\mathrm{Id})\circ(\Phi\otimes\mathrm{Id}\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\Phi\otimes\mathrm{Id})\\
&\circ(\mathrm{Id}\otimes\mathrm{Id}\otimes\Phi)\circ(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})(b_1\otimes b_2\otimes a\otimes a)\\
&= (\mathrm{Id}\otimes\mathrm{Id}\otimes\varepsilon)\circ(\mathrm{Id}\otimes\Phi)\circ(\Phi\otimes\mathrm{Id}\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes(\mathrm{Id}\otimes\varepsilon)\circ\Phi\otimes\mathrm{Id})\\
&\circ(\mathrm{Id}\otimes\mathrm{Id}\otimes\Phi)(b_1\otimes a\otimes b_2\otimes a)\\
&= (\mathrm{Id}\otimes\mathrm{Id}\otimes\varepsilon)\circ(\mathrm{Id}\otimes\Phi)\circ(\Phi\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\Phi)(b_1\otimes b_2\otimes a)\\
&= \Phi\circ((\mathrm{Id}\otimes\varepsilon)\circ\Phi\otimes\mathrm{Id})(b_1\otimes b_2\otimes a)\\
&= \Phi(m_\varepsilon(b_1\otimes b_2)\otimes a)\\
&= \Delta_a(m_\varepsilon(b_1\otimes b_2)).
\end{aligned}$$

So $(A, m_\varepsilon, \Delta_a)$ is a bialgebra. Let $(A, \Psi) = \ell\mathbf{EAS}(A, m_\varepsilon, \Delta_a)$. For any $b_1, b_2 \in A$:

$$\begin{aligned}
\Psi(b_1\otimes b_2) &= (\mathrm{Id}\otimes\varepsilon\otimes\mathrm{Id})\circ(\Phi\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\tau)\circ(\Phi\otimes\mathrm{Id})(b_1\otimes a\otimes b_2)\\
&= (\mathrm{Id}\otimes\varepsilon\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\Phi)\circ(\Phi\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\Phi)(b_1\otimes a\otimes b_2)\\
&= (\mathrm{Id}\otimes\mathrm{Id}\otimes\varepsilon)\circ(\Phi\otimes\mathrm{Id})(b_1\otimes b_2\otimes a)\\
&= \Phi(b_1\otimes b_2)\varepsilon(a)\\
&= \Phi(b_1\otimes b_2).
\end{aligned}$$

Therefore, $(A, \Phi) = \ell\mathbf{EAS}(A, m_\varepsilon, \Delta_a)$. □

Example 5.5. This can be applied for ℓEAS M_{16}, M_{17} and M_{18} of Example 4.1.

- For M_{16}, taking $a = x$ and $\varepsilon = x^* + y^*$, we obtain:

$$\Delta_a(x) = x\otimes x, \qquad \Delta_a(y) = y\otimes y,$$

$$\begin{aligned}
m_\varepsilon(x\otimes x) &= x, & m_\varepsilon(x\otimes y) &= y,\\
m_\varepsilon(y\otimes x) &= y, & m_\varepsilon(y\otimes y) &= y.
\end{aligned}$$

  This is the bialgebra of the semigroup $(\mathbb{Z}/2\mathbb{Z}, \times)$, with $x = \overline{1}$ and $y = \overline{0}$: we recover the linearization of **C3**.

- For M_{17}, taking $a = x$ and $\varepsilon = x^* + y^*$, we obtain:

$$\Delta_a(x) = x\otimes x, \qquad \Delta_a(y) = x\otimes x - x\otimes y - y\otimes x + 2y\otimes y,$$

$$\begin{aligned}
m_\varepsilon(x\otimes x) &= x, & m_\varepsilon(x\otimes y) &= y,\\
m_\varepsilon(y\otimes x) &= x, & m_\varepsilon(y\otimes y) &= y.
\end{aligned}$$

Putting $y' = -x + 2y^2$, we obtain:

$$\Delta_a(x) = x \otimes x, \qquad \Delta_a(y') = y' \otimes y',$$

$$\begin{aligned} m_\varepsilon(x \otimes x) &= x, & m_\varepsilon(x \otimes y') &= y', \\ m_\varepsilon(y' \otimes x) &= y', & m_\varepsilon(y' \otimes y') &= x. \end{aligned}$$

This is the bialgebra of the semigroup $(\mathbb{Z}/2\mathbb{Z}, +)$, with $x = \overline{0}$ and $y = \overline{1}$: we recover the linearization of **H2**.

- For M_{18}, we can take any $a \in A$ and any $\varepsilon \in A^*$ such that $\varepsilon(a) = 1$. For any $b, c \in A$,

$$\Delta_a(b) = a \otimes b, \qquad m_\varepsilon(b \otimes c) = \varepsilon(b)c.$$

5.4 Applications to nondegenerate finite CEDS

From Proposition 4.5:

Proposition 5.13. *Let $(\Omega, \rightarrow, \triangleright)$ be a nondegenerate finite CEDS, which we write following Theorem 3.14 under the form*

$$(\mathbf{EAS}(\Omega_1, *) \rtimes_{\succ} \mathbf{EAS}'(\Omega_2, \star)) \times \mathbf{EAS}(\Omega_3).$$

Let $g, h : \Omega_3 \longrightarrow \mathbb{K}$ be two maps such that:

$$\sum_{\alpha_3 \in \Omega_3} g(\alpha_3)h(\alpha_3) = 1.$$

We define a product and a coproduct on $\mathbb{K}\Omega$, putting, for any $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3) \in \Omega$:

$$\begin{aligned} (\alpha_1, \alpha_2, \alpha_3) \cdot (\beta_1, \beta_2, \beta_3) &= \delta_{\alpha_2, \beta_2} g(\alpha_3)(\alpha_1 * \beta_1, \beta_2, \beta_3), \\ \Delta(\alpha_1, \alpha_2, \alpha_3) &= \sum_{(\beta_2, \beta_3) \in \Omega_2 \times \Omega_3} h(\beta_3)(\alpha_1, \beta_2, \beta_3) \otimes (\beta_2 \succ \alpha_1, \alpha_2 \star \beta_2^{-1}, \alpha_3). \end{aligned}$$

Then $(\mathbb{K}\Omega, \cdot, \Delta)$ is a bialgebra and the linearization of Ω is $\ell\mathbf{EAS}(\mathbb{K}\Omega, \cdot, \Delta)$.

Proof. By Proposition 4.5, the following is a left unit of $\mathbb{K}\Omega$:

$$a = \sum_{(\alpha_2, \alpha_3) \in \Omega_2 \times \Omega_3} h(\alpha_3)(e_1, \alpha_2, \alpha_3),$$

and the following map is a left counit of $\mathbb{K}\Omega$:

$$\varepsilon : \left\{ \begin{array}{rcl} \mathbb{K}\Omega & \longrightarrow & \mathbb{K} \\ (\alpha_1, \alpha_2, \alpha_3) & \longrightarrow & \delta_{\alpha_2, e_2} g(\alpha_3). \end{array} \right.$$

By hypothesis, $\varepsilon(a) = 1$. The result comes from a direct application of Theorem 5.12. □

Similarly:

Proposition 5.14. *Let $(\Omega, \rightarrow, \triangleright)$ be a nondegenerate finite dual CEDS, which we write following Corollary 3.15 under the form*

$$(\mathbf{EAS}(\Omega_2, \star) \ltimes_{\prec} (\mathbf{EAS}'(\Omega_1, *)) \times \mathbf{EAS}(\Omega_3),$$

[2] if the characteristic of the base field $\mathbb{K}$ is not 2.

Let $g, h : \Omega_3 \longrightarrow \mathbb{K}$ be two maps such that:

$$\sum_{\alpha_3 \in \Omega_3} g(\alpha_3) h(\alpha_3) = 1.$$

We define a product and a coproduct on $\mathbb{K}\Omega$, putting, for any $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3) \in \Omega$:

$$(\alpha_1, \alpha_2, \alpha_3) \cdot (\beta_1, \beta_2, \beta_3) = \delta_{\alpha_1, \beta_1} g(\alpha_3)(\alpha_2 * \beta_2, \beta_1 \prec \alpha_2, \beta_3),$$
$$\Delta(\alpha_2, \alpha_1, \alpha_3) = \sum_{(\beta_1, \beta_3) \in \Omega_1 \times \Omega_3} h(\beta_3)(\alpha_2, \beta_1 \prec \alpha_2, \beta_3) \otimes (\alpha_2, \alpha_1 \star (\beta_1^{-1} \prec \alpha_2^{-1}), \alpha_3).$$

Then $(\mathbb{K}\Omega, \cdot, \Delta)$ is a bialgebra and the linearization of Ω is $\ell\mathbf{EAS}(\mathbb{K}\Omega, \cdot, \Delta)$.

5.5 Applications to Hopf algebras of groups

In all this paragraph, G is a group. We denote by $\mathbb{K}G$ the associated Hopf algebra. If G is finite, we denote by $\mathbb{K}^G$ the Hopf algebra of functions over G, with its basis $(\delta_g)_{g \in G}$, dual of the basis G of $\mathbb{K}G$.

Corollary 5.15. *If G is finite, then $\ell\mathbf{EAS}'(\mathbb{K}G)$ is isomorphic to $\ell\mathbf{EAS}(\mathbb{K}^G)$, and $\ell\mathbf{EAS}'(\mathbb{K}^G)$ is isomorphic to $\ell\mathbf{EAS}(\mathbb{K}G^{op})$.*

Proof. As G is finite, $a = \sum_{g \in G} g$ is a right integral of $\mathbb{K}^G$, so is a left unit of $\ell\mathbf{EAS}'(\mathbb{K}G)$. If e_G is the unit of the group G, then $\varepsilon = \delta_{e_G}$ is a right integral of $\mathbb{K}G$, so is a left counit of $\ell\mathbf{EAS}'(\mathbb{K}G)$. As $\varepsilon(a) = 1$, $\ell\mathbf{EAS}'(\mathbb{K}G) = \ell\mathbf{EAS}(\mathbb{K}G, m_\varepsilon, \Delta_a)$. For any $g, h \in G$:

$$m_\varepsilon(g \otimes h) = (\mathrm{Id} \otimes \delta_{e_G}) \circ \Phi(g \otimes h) = h\delta_{e_G}(h^{-1}g) = \delta_{g,h} h.$$

For any $g \in G$:

$$\Delta_a(g) = \sum_{h \in G} \Phi(g \otimes h) = \sum_{h \in G} h \otimes h^{-1} g = \sum_{\substack{g_1, g_2 \in G, \\ g_1 g_2 = g}} g_1 \otimes g_2.$$

So $(\mathbb{K}G, m_\varepsilon, \Delta_a)$ is isomorphic to $\mathbb{K}^G$, via the map sending g to δ_g, for any $g \in G$.

By duality, a is a left counit of $\ell\mathbf{EAS}'(\mathbb{K}^G)$ and ε is a left unit of $\ell\mathbf{EAS}'(\mathbb{K}^G)$. For any $g, h \in G$:

$$m_a(\delta_g \otimes \delta_h) = (\mathrm{Id} \otimes a) \circ \Phi(g \otimes h) = \sum_{\substack{h_1, h_2 \in G, \\ h_1 h_2 = h}} \delta_{h_1} \otimes \delta_{h_2^{-1}} \delta_g(a) = \delta_{hg}.$$

For any $g \in G$:

$$\Delta_\varepsilon(\delta_g) = \Phi(\delta_g \otimes \delta_{e_G}) = \sum_{h \in G} \delta_h \otimes \delta_h \delta_g = \sum_{h \in G} \delta_h \otimes \delta_{g,h} \delta_h = \delta_g \otimes \delta_g.$$

So $(\mathbb{K}^G, m_a, \Delta_\varepsilon)$ is isomorphic to $\mathbb{K}G^{op}$ via the map sending δ_g to g, for any $g \in G$. □

Proposition 5.16.
1. *The nonzero special vectors of eigenvalue 1 of $\ell\mathbf{EAS}(\mathbb{K}G)$ and of $\ell\mathbf{EAS}'(\mathbb{K}G)$ are the elements*
$$\lambda \sum_{\alpha \in H} \alpha,$$
where λ is a nonzero scalar and H is a subgroup of G.
2. *If G is finite, the nonzero special vectors of eigenvalue 1 of $\ell\mathbf{EAS}(\mathbb{K}^G)$ and of $\ell\mathbf{EAS}'(\mathbb{K}^G)$ are the elements*
$$\lambda \sum_{\alpha \in H} \delta_\alpha,$$
where λ is a nonzero scalar and H is a subgroup of G.

Proof. Any $a \in A$ can be written under the form $a = \sum_{\alpha \in G} \lambda_\alpha \alpha$. Then:

$$\begin{aligned}
&a \text{ is a special vector of eigenvalue 1 of } \ell\mathbf{EAS}(\mathbb{K}G)\\
&\Longleftrightarrow \sum_{\alpha,\beta \in G} a_\alpha a_\beta \alpha \otimes \beta = \sum_{\alpha,\beta \in G} a_\alpha a_\beta \alpha\beta \otimes \alpha\\
&\Longleftrightarrow \sum_{\alpha,\beta \in G} a_\alpha a_\beta \alpha \otimes \beta = \sum_{\alpha,\beta \in G} a_\beta a_{\beta^{-1}\alpha} \alpha\beta \otimes \alpha\\
&\Longleftrightarrow \forall \alpha, \beta \in G,\ a_\beta(a_\alpha - a_{\beta^{-1}\alpha}) = 0.
\end{aligned}$$

Let a be a nonzero special vector of eigenvalue 1 of $\ell\mathbf{EAS}(\mathbb{K}G)$. Let us put $a_{1_G} = \lambda$ and $H = \{\alpha \in G,\ a_\alpha \neq 0\}$. Let $\alpha = \beta \in H$. As $a_\beta \neq 0$, we obtain $a_\alpha = a_{1_G} = \lambda$, so $1_G \in H$ and $\lambda \neq 0$. For any $\beta \in H$, taking $\alpha = 1_G$, we obtain $a_{\beta^{-1}} = \lambda$, so $\beta^{-1} \in H$. If $\alpha, \beta \in H$, we obtain that $a_{\beta^{-1}\alpha} = a_\alpha \neq 0$, so $\beta^{-1}\alpha \in H$. Hence, H is a subgroup and $a = \lambda \sum_{\alpha \in H} \alpha$.

$$\begin{aligned}
&a \text{ is a special vector of eigenvalue 1 of } \ell\mathbf{EAS}'(\mathbb{K}G)\\
&\Longleftrightarrow \sum_{\alpha,\beta \in G} a_\alpha a_\beta \alpha \otimes \beta = \sum_{\alpha,\beta \in G} a_\alpha a_\beta \beta \otimes \beta^{-1}\alpha\\
&\Longleftrightarrow \sum_{\alpha,\beta \in G} a_\alpha a_\beta \alpha \otimes \beta = \sum_{\alpha,\beta \in G} a_\beta a_{\alpha\beta} \alpha\beta \otimes \alpha\\
&\Longleftrightarrow \forall \alpha, \beta \in G,\ a_\alpha(a_\beta - a_{\alpha\beta}) = 0.
\end{aligned}$$

Let a be a nonzero special vector of eigenvalue 1 of $\ell\mathbf{EAS}'(\mathbb{K}G)$. Let us put $a_{1_G} = \lambda$ and $H = \{\alpha \in G,\ a_\alpha \neq 0\}$. Let $\alpha = \beta \in H$. If $\alpha \in H$, for $\beta = 1_G$, we obtain $a_{1_G} = a_\alpha = \lambda$, so $1_G \in H$ and $\lambda \neq 0$; for $\beta = \alpha^{-1}$, we obtain $a_{\alpha^{-1}} = a_{1_G} = \lambda \neq 0$, so $\alpha^{-1} \in G$. If $\alpha, \beta \in H$, we obtain that $a_{\alpha\beta} = a_\beta \neq 0$, so $\alpha\beta \in H$. Hence, H is a subgroup and $a = \lambda \sum_{\alpha \in H} \alpha$.

Let $f \in \mathbb{K}^G$. We put $f(\alpha) = a_\alpha$ for any $\alpha \in G$.

$$\begin{aligned}
&f \text{ is a special vector of eigenvalue 1 of } \ell\mathbf{EAS}(\mathbb{K}^G)\\
&\Longleftrightarrow \forall \alpha, \beta \in G,\ a_\alpha a_\beta = a_{\alpha\beta}\\
&\Longleftrightarrow \forall \alpha, \beta \in G,\ a_\alpha(a_\beta - a_{\alpha\beta}) = 0;\\
&f \text{ is a special vector of eigenvalue 1 of } \ell\mathbf{EAS}'(\mathbb{K}^G)\\
&\Longleftrightarrow \forall \alpha, \beta \in G,\ a_\alpha a_\beta = a_\beta a_{\beta^{-1}\alpha}\\
&\Longleftrightarrow \forall \alpha, \beta \in G,\ a_\beta(a_\alpha - a_{\beta^{-1}\alpha}) = 0.
\end{aligned}$$

The conclusion is the same as for $\mathbb{K}G$. □

Remark 5.2.

1. From Proposition 5.4, the left units of $\ell\mathbf{EAS}(\mathbb{K}G)$ are the multiples of e_G, and its left counits are the multiples of its counit. If G is finite, the left units of $\ell\mathbf{EAS}(\mathbb{K}^G)$ are the multiple of $\sum_{g \in G} g$ are its left counits are the multiples of e_G.
2. From Proposition 5.11, it is not difficult to show that if G is finite, the left units of $\ell\mathbf{EAS}'(\mathbb{K}G)$ are the multiples of $\sum_{g \in G} g$; if G is not finite, $\ell\mathbf{EAS}'(\mathbb{K}G)$ has no nonzero left unit. The left counits of $\ell\mathbf{EAS}'(\mathbb{K}G)$ are the multiples of δ_{e_G}. By duality, if G is finite, the left units of $\ell\mathbf{EAS}'(\mathbb{K}^G)$ are the multiples of δ_{e_G} and its left counits are the multiples of $\sum_{g \in G} g$.

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