

# REDUCED TYPED ANGULARLY DECORATED PLANAR ROOTED TREES AND GENERALIZED TRIDENDRIFORM ALGEBRAS

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**ABSTRACT.** We introduce a generalization of tridendriform algebras, where each of the three products are replaced by a family of products indexed by a set  $\Omega$ . We study the needed structure on  $\Omega$  for free  $\Omega$ -tridendriform algebras to be built on Schröder trees (as it is the case in the classical case), with convenient decorations on their leaves. We obtain in this way extended triassociative semigroups. We describe commutative  $\Omega$ -tridendriform algebras in terms of typed words. We also study links with generalizations of Rota-Baxter algebras and describe the Koszul duals of the corresponding operads.

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## 1. INTRODUCTION

Shuffle algebras have been studied for a long time, starting from combinatorial problems of card shufflings, to operadic aspects. They were formalized in the fifties by Eilenberger and MacLane [2] and, independently, by Schützenberger [11]: a shuffle algebra is a vector space with a bilinear product  $<$  following the axiom

$$(x < y) < z = x < (y < z).$$

Free shuffle algebras are based on words, with the half-shuffle product: for example, if  $a, b, c, d$  are letters,

$$\begin{aligned} a < bcd &= abcd, \\ ab < cd &= abcd + acbd + acdb, \\ abc < d &= abcd + abdc + adbc. \end{aligned}$$

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It follows that the product  $*$  defined by  $x * y = x < y + y < x$  is commutative and associative. The noncommutative version, known as noncommutative shuffle algebras or dendriform algebras, is introduced by Loday and Ronco [9]: a dendriform algebra is a vector space with two products  $<$  and  $>$ , with the following axioms:

$$\begin{aligned}(x < y) < z &= x < (y < z + y > z), \\ (x > y) < z &= x > (y < z), \\ x > (y > z) &= (x < y + x > y) > z.\end{aligned}$$

It follows that the product  $*$  defined by  $x * y = x < y + x > y$  is associative. Loday and Ronco described the free dendriform algebra on one generator in terms of planar binary trees [8, 1]. Recently, various generalizations of dendriform algebras appeared: the two products  $<$  and  $>$  are replaced by families  $(<_{\alpha})_{\alpha \in \Omega}$  and  $(>_{\alpha})_{\alpha \in \Omega}$  of products parametrized by elements of a given set  $\Omega$ , which can have extra structures: for matching dendriform algebras [12],  $\Omega$  is just a set, whereas for family dendriform algebra it is a semigroup. More generally,  $\Omega$ -dendriform algebras over an extended diassociative semigroup have been introduced and studied in [3].

In the spirit of [3], we work here with parametrized versions of tridendriform algebras. Tridendriform algebras are introduced in [7]: they are vector spaces with three products  $<$ ,  $>$  and  $\circ$ , with the following axioms:

$$\begin{aligned}(a < b) < c &= a < (b > c) + a < (b < c) + a < (b \circ c) \\ (a > b) < c &= a > (b < c) \\ a > (b > c) &= (a > b) > c + (a < b) > c + (a \circ b) > c \\ (a > b) \circ c &= a > (b \circ c) \\ (a < b) \circ c &= a \circ (b > c) \\ (a \circ b) < c &= a \circ (b < c) \\ (a \circ b) \circ c &= a \circ (b \circ c).\end{aligned}$$

Summing, we obtain that the product  $* = < + > + \circ$  is associative; moreover,  $< + \circ$  and  $>$  define a dendriform algebra structure, as well as  $<$  and  $> + \circ$ . A classical example of tridendriform algebra is given by quasi-shuffle (or stuffle) algebras [6, 4]: if  $(V, \cdot)$  is an associative algebra, then the half-shuffle algebra  $T(V)$  is tridendriform. For example, if  $a, b, c, d \in V$ ,

$$\begin{aligned}ab < cd &= abcd + acbd + acdb + a(b \cdot c)d + ac(b \cdot d), \\ ab > cd &= cabd + cadb + cdab + c(a \cdot d)b + ca(b \cdot d), \\ ab > cd &= (a \cdot c)bd + (a \cdot c)db + (a \cdot c)(b \cdot d).\end{aligned}$$

Other examples of tridendriform algebras are based on packed words or on parking functions [10]. The free tridendriform algebra on one generator is described by Loday and Ronco in terms of planar reduced trees, which we call here Schröder trees.

We here give a parametrized version of tridendriform algebras. We start with a set  $\Omega$  with six products  $\leftarrow, \rightarrow, \triangleleft, \triangleright, * \text{ and } \cdot$ . An  $\Omega$ -tridendriform algebra is given three families  $(<_{\alpha})_{\alpha \in \Omega}$ ,  $(>_{\alpha})_{\alpha \in \Omega}$  and  $(\circ_{\alpha})_{\alpha \in \Omega}$  of bilinear products, with the following axioms:

$$\begin{aligned}(a <_{\alpha} b) <_{\beta} c &= a <_{\alpha \rightarrow \beta} (b >_{\alpha \triangleright \beta} c) + a <_{\alpha \leftarrow \beta} (b <_{\alpha \triangleleft \beta} c) + a <_{\alpha \cdot \beta} (b \circ_{\alpha * \beta} c) \\ (a >_{\alpha} b) <_{\beta} c &= a >_{\alpha} (b <_{\beta} c)\end{aligned}$$

$$\begin{aligned}
 a \succ_{\alpha} (b \succ_{\beta} c) &= (a \succ_{\alpha \triangleright \beta} b) \succ_{\alpha \rightarrow \beta} c + (a \prec_{\alpha \triangleleft \beta} b) \succ_{\alpha \leftarrow \beta} c + (a \circ_{\alpha * \beta} b) \succ_{\alpha \cdot \beta} c \\
 (a \succ_{\alpha} b) \circ_{\beta} c &= a \succ_{\alpha} (b \circ_{\beta} c) \\
 (a \prec_{\alpha} b) \circ_{\beta} c &= a \circ_{\beta} (b \succ_{\alpha} c) \\
 (a \circ_{\alpha} b) \prec_{\beta} c &= a \circ_{\alpha} (b \prec_{\beta} c) \\
 (a \circ_{\alpha} b) \circ_{\beta} c &= a \circ_{\alpha} (b \circ_{\beta} c).
 \end{aligned}$$

As in the classical case, particular examples of  $\Omega$ -tridendriform algebras are given by  $\Omega$ -Rota-Baxter algebras, as defined in [5], see Proposition 2.8. As a condition, we impose that free  $\Omega$ -tridendriform algebras are based on Schröder trees decorated in some sense by elements of  $\Omega$ . In the dendriform case, the elements of  $\Omega$  become types (that is to say decorations) of the internal edges of the planar binary trees which were the basis of free dendriform algebras: this is not possible for Schröder trees, as the number of internal edges does not uniquely depend on the number of internal vertices, which is a major difference with planar binary trees. Instead, we choose that the elements of  $\Omega$  become decorations of the leaves of the Schröder trees, at the exception of the leftmost and rightmost ones. We then define inductively three families of products  $\prec$ ,  $\succ$  and  $\circ$  on these trees and ask for them to define an  $\Omega$ -tridendriform algebra. This impose a strong constraint on the products taken on  $\Omega$ : they have to make  $\Omega$  an extended triassociative semigroup (briefly, ETS): see Definition 2.3 below for the list of 18 axioms defining these object below. If this holds, these Schröder trees indeed give free  $\Omega$ -tridendriform algebras (Theorem 3.2).

Moreover, if  $\Omega$  is a ETS and if  $A$  is an  $\Omega$ -tridendriform algebra, then  $\mathbf{k}\Omega \otimes A$  inherits a structure of (classical) tridendriform algebra (Proposition 3.6), which generalizes a similar result for  $\Omega$ -dendriform algebra proved in [3]. We consider the particular case where  $A$  is the  $\Omega$ -tridendriform algebra of Schröder trees in Proposition 3.7, where we give necessary and sufficient condition for  $\mathbf{k}\Omega \otimes \mathbf{k}\mathfrak{T}(X, \Omega)$  to be free.

We also study  $\Omega$ -tridendriform algebras of typed words. If  $A$  is a matching associative algebra (Definition 3.3), we prove in Theorem 3.4 that the algebra  $\text{Sh}_{\Omega}^{+}(A)$  of  $\Omega$ -typed words on  $A$  is an  $\Omega$ -tridendriform algebra. When  $A$  and  $\Omega$  are commutative, then we prove in Theorem 3.5 that  $\text{Sh}_{\Omega}^{+}(A)$  is the free commutative  $\Omega$ -tridendriform over  $A$ . Consequently, if  $A$  is the free matching algebra, generated by a set  $X$ , then  $\text{Sh}_{\Omega}^{+}(A)$  is the free commutative  $\Omega$ -tridendriform algebra generated by  $X$ .

The last section of the paper is devoted to the operad of  $\Omega$ -tridendriform algebras. In particular, we study operadic morphisms from tridendriform algebras to  $\Omega$ -tridendriform algebras and we compute its Koszul dual in Proposition 4.3, finding in this way a parametrised version of triassociative algebras [7].

**Notation.** Throughout this paper, let  $\mathbf{k}$  be a unitary commutative ring which will be the base ring of all modules, algebras, as well as linear maps, unless otherwise specified.

## 2. GENERALIZED TRIDENDRIFORM ALGEBRAS

**2.1. EDS and ETS.** First, we recall the definition of diassociative semigroups and extended diassociative semigroups in [3].

**Definition 2.1.** [3, Definition 1] A **diassociative semigroup** is a family  $(\Omega, \leftarrow, \rightarrow)$ , where  $\Omega$  is a set and  $\leftarrow, \rightarrow: \Omega \times \Omega \rightarrow \Omega$  are maps such that

$$\begin{aligned}
 (\alpha \leftarrow \beta) \leftarrow \gamma &= \alpha \leftarrow (\beta \leftarrow \gamma) = \alpha \leftarrow (\beta \rightarrow \gamma), \\
 (\alpha \rightarrow \beta) \leftarrow \gamma &= \alpha \rightarrow (\beta \leftarrow \gamma),
 \end{aligned}$$

$$(\alpha \rightarrow \beta) \rightarrow \gamma = (\alpha \leftarrow \beta) \rightarrow \gamma = \alpha \rightarrow (\beta \rightarrow \gamma),$$

for all  $\alpha, \beta, \gamma \in \Omega$ .

**Definition 2.2.** [3, Definition 2] An **extended diassociative semigroup** (abbr. EDS) is a family  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$ , where  $\Omega$  is a set and  $\leftarrow, \rightarrow, \triangleleft, \triangleright : \Omega \times \Omega \rightarrow \Omega$  such that  $(\Omega, \leftarrow, \rightarrow)$  is a diassociative semigroup and

$$\begin{aligned} (1) \quad & \alpha \triangleright (\beta \leftarrow \gamma) = \alpha \triangleright \beta, \\ (2) \quad & (\alpha \rightarrow \beta) \triangleleft \gamma = \beta \triangleleft \gamma, \\ (3) \quad & (\alpha \triangleleft \beta) \leftarrow ((\alpha \leftarrow \beta) \triangleleft \gamma) = \alpha \triangleleft (\beta \leftarrow \gamma), \\ (4) \quad & (\alpha \triangleleft \beta) \triangleleft ((\alpha \leftarrow \beta) \triangleleft \gamma) = \beta \triangleleft \gamma, \\ (5) \quad & (\alpha \triangleleft \beta) \rightarrow ((\alpha \leftarrow \beta) \triangleleft \gamma) = \alpha \triangleleft (\beta \rightarrow \gamma), \\ (6) \quad & (\alpha \triangleleft \beta) \triangleright ((\alpha \leftarrow \beta) \triangleleft \gamma) = \beta \triangleright \gamma, \\ (7) \quad & (\alpha \triangleright (\beta \rightarrow \gamma)) \leftarrow (\beta \triangleright \gamma) = (\alpha \leftarrow \beta) \triangleright \gamma, \\ (8) \quad & (\alpha \triangleright (\beta \rightarrow \gamma)) \triangleleft (\beta \triangleright \gamma) = \alpha \triangleleft \beta, \\ (9) \quad & (\alpha \triangleright (\beta \rightarrow \gamma)) \rightarrow (\beta \triangleright \gamma) = (\alpha \rightarrow \beta) \triangleright \gamma, \\ (10) \quad & (\alpha \triangleright (\beta \rightarrow \gamma)) \triangleright (\beta \triangleright \gamma) = \alpha \triangleright \beta, \end{aligned}$$

for all  $\alpha, \beta, \gamma \in \Omega$ .

**Definition 2.3.** An **extended triassociative semigroup** (abbr. ETS) is a family  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *)$ , where  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$  is an EDS and

$$\begin{aligned} (11) \quad & (\alpha \rightarrow \beta) * \gamma = \beta * \gamma, \\ (12) \quad & (\alpha \rightarrow \beta) \cdot \gamma = \alpha \rightarrow (\beta \cdot \gamma), \\ (13) \quad & \alpha \triangleright \beta = \alpha \triangleright (\beta \cdot \gamma), \\ (14) \quad & (\alpha \triangleleft \beta) * ((\alpha \leftarrow \beta) \triangleleft \gamma) = \beta * \gamma, \\ (15) \quad & (\alpha \triangleleft \beta) \cdot ((\alpha \leftarrow \beta) \triangleleft \gamma) = \alpha \triangleleft (\beta \cdot \gamma), \\ (16) \quad & (\alpha \leftarrow \beta) \leftarrow \gamma = \alpha \leftarrow (\beta \cdot \gamma), \\ (17) \quad & (\alpha \triangleright (\beta \rightarrow \gamma)) * (\beta \triangleright \gamma) = \alpha * \beta, \\ (18) \quad & \alpha \rightarrow (\beta \rightarrow \gamma) = (\alpha \cdot \beta) \rightarrow \gamma, \\ (19) \quad & (\alpha \triangleright (\beta \rightarrow \gamma)) \cdot (\beta \triangleright \gamma) = (\alpha \cdot \beta) \triangleright \gamma, \\ (20) \quad & (\alpha \leftarrow \beta) * \gamma = \alpha * (\beta \rightarrow \gamma), \\ (21) \quad & (\alpha \leftarrow \beta) \cdot \gamma = \alpha \cdot (\beta \rightarrow \gamma), \\ (22) \quad & \alpha \triangleleft \beta = \beta \triangleright \gamma, \\ (23) \quad & \alpha * \beta = \alpha * (\beta \leftarrow \gamma), \\ (24) \quad & (\alpha \cdot \beta) \triangleleft \gamma = \beta \triangleleft \gamma, \\ (25) \quad & (\alpha \cdot \beta) \leftarrow \gamma = \alpha \cdot (\beta \leftarrow \gamma), \\ (26) \quad & \alpha * \beta = \alpha * (\beta \cdot \gamma), \\ (27) \quad & (\alpha \cdot \beta) * \gamma = \beta * \gamma, \\ (28) \quad & (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma). \end{aligned}$$

**Example 2.4.** (a) Let  $(\Omega, *, \cdot)$  be a set with two products satisfying (26)–(28). This holds for example if for any  $\alpha, \beta \in \Omega$ ,

$$\alpha * \beta = \alpha, \quad \alpha \cdot \beta = \beta,$$

or if for any  $\alpha, \beta \in \Omega$ ,

$$\alpha * \beta = \beta, \quad \alpha \cdot \beta = \alpha.$$

We put, for any  $\alpha, \beta \in \Omega$ :

$$\begin{aligned} \alpha \leftarrow \beta &= \alpha, & \alpha \triangleleft \beta &= \beta, \\ \alpha \rightarrow \beta &= \alpha, & \alpha \triangleright \beta &= \alpha. \end{aligned}$$

Then  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *)$  is an ETS.

(b) Let  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *)$  be an ETS. For any  $\alpha, \beta \in \Omega$ , we put

$$\begin{aligned} \alpha \leftarrow^{op} \beta &= \beta \rightarrow \alpha, & \alpha \triangleleft^{op} \beta &= \beta \triangleright \alpha, \\ \alpha \rightarrow^{op} \beta &= \beta \leftarrow \alpha, & \alpha \triangleright^{op} \beta &= \beta \triangleleft \alpha, \\ \alpha *^{op} \beta &= \beta * \alpha, & \alpha \cdot^{op} \beta &= \beta \cdot \alpha. \end{aligned}$$

Then  $(\Omega, \leftarrow^{op}, \rightarrow^{op}, \triangleleft^{op}, \triangleright^{op}, *^{op}, \cdot^{op})$  is also an ETS, called the opposite of  $\Omega$ . We shall say that  $\Omega$  is commutative if it is equal to its opposite.

**2.2.  $\Omega$ -tridendriform algebras.** Let us now give the concept of  $\Omega$ -tridendriform algebras as follows.

**Definition 2.5.** Let  $\Omega$  be a set with six products  $\leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *$ . An  $\Omega$ -tridendriform algebra is a family  $(A, (<_{\omega})_{\omega \in \Omega}, (>_{\omega})_{\omega \in \Omega}, (\circ_{\omega})_{\omega \in \Omega})$ , where  $A$  is a  $\mathbf{k}$ -module and  $<_{\omega}, >_{\omega}, \circ_{\omega} : A \otimes A \rightarrow A$  are linear maps such that

$$(29) \quad (a <_{\alpha} b) <_{\beta} c = a <_{\alpha \rightarrow \beta} (b >_{\alpha \triangleright \beta} c) + a <_{\alpha \leftarrow \beta} (b <_{\alpha \triangleleft \beta} c) + a <_{\alpha \cdot \beta} (b \circ_{\alpha * \beta} c)$$

$$(30) \quad (a >_{\alpha} b) <_{\beta} c = a >_{\alpha} (b <_{\beta} c)$$

$$(31) \quad a >_{\alpha} (b >_{\beta} c) = (a >_{\alpha \triangleright \beta} b) >_{\alpha \rightarrow \beta} c + (a <_{\alpha \triangleleft \beta} b) >_{\alpha \leftarrow \beta} c + (a \circ_{\alpha * \beta} b) >_{\alpha \cdot \beta} c$$

$$(32) \quad (a >_{\alpha} b) \circ_{\beta} c = a >_{\alpha} (b \circ_{\beta} c)$$

$$(33) \quad (a <_{\alpha} b) \circ_{\beta} c = a \circ_{\beta} (b >_{\alpha} c)$$

$$(34) \quad (a \circ_{\alpha} b) <_{\beta} c = a \circ_{\alpha} (b <_{\beta} c)$$

$$(35) \quad (a \circ_{\alpha} b) \circ_{\beta} c = a \circ_{\alpha} (b \circ_{\beta} c)$$

for all  $a, b, c \in A$  and  $\alpha, \beta \in \Omega$ . If moreover, for all  $a, b \in A$  and  $\alpha \in \Omega$ ,

$$a <_{\alpha} b = b >_{\alpha} a \quad \text{and} \quad a \circ_{\alpha} b = b \circ_{\alpha} a,$$

then  $A$  is called a **commutative  $\Omega$ -tridendriform algebra**.

**Example 2.6.** (a) If  $\Omega$  is a semigroup, we take all maps  $\circ_{\omega}$  with  $\omega \in \Omega$  to be equal to an associative product  $\star$  and

$$\alpha \rightarrow \beta = \alpha \leftarrow \beta = \alpha \cdot \beta = \alpha \star \beta, \quad \alpha \triangleright \beta = \alpha, \quad \alpha \triangleleft \beta = \beta,$$

then  $\Omega$ -tridendriform algebra are tridendriform family algebras [13].

(b) For a set  $\Omega$ , define, as in Example 2.4,

$$\alpha \rightarrow \beta = \alpha \triangleleft \beta = \alpha * \beta = \beta, \quad \alpha \triangleright \beta = \alpha \leftarrow \beta = \alpha \cdot \beta = \alpha.$$

Then  $\Omega$ -tridendriform algebra are matching tridendriform algebras [12].

**Remark 2.7.**  $\Omega$ -dendriform algebras, as defined in [3], are  $\Omega$ -tridendriform algebras such that, for any  $\alpha \in \Omega$ ,  $\circ_\alpha = 0$ .

Next we show the relationship between  $\Omega$ -Rota-Baxter algebras [5] and  $\Omega$ -tridendriform algebras.

**Proposition 2.8.** *Let  $\Omega$  be a set with six products  $\leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *$  and, for any  $\alpha \in \Omega$ , let  $\mu_\alpha \in \Omega$ . We put, for any  $\alpha, \beta \in \Omega$ ,  $\lambda_{\alpha, \beta} = \mu_{\alpha * \beta}$ . Let  $(A, (P_\omega)_{\omega \in \Omega})$  be an  $\Omega$ -Rota-Baxter algebra of weight  $\lambda$ . Then  $(A, (\leftarrow_\omega)_{\omega \in \Omega}, (\triangleright_\omega)_{\omega \in \Omega}, (\circ_\omega)_{\omega \in \Omega})$  is an  $\Omega$ -tridendriform algebra, where*

$$a \leftarrow_\omega b := aP_\omega(b), \quad a \triangleright_\omega b := P_\omega(a)b, \quad a \circ_\omega b := \mu_\omega ab$$

for all  $a, b \in A$  and  $\omega \in \Omega$ .

*Proof.* For  $a, b, c \in A$  and  $\alpha, \beta \in \Omega$ ,

$$\begin{aligned} (a \leftarrow_\alpha b) \leftarrow_\beta c &= (aP_\alpha(b))P_\beta(c) = a(P_\alpha(b)P_\beta(c)) \\ &= aP_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(b)c) + aP_{\alpha \leftarrow \beta}(bP_{\alpha \triangleleft \beta}(c)) + aP_{\alpha \cdot \beta}(\lambda_{\alpha, \beta}bc) \\ &= a \leftarrow_{\alpha \rightarrow \beta} (b \triangleright_{\alpha \triangleright \beta} c) + a \leftarrow_{\alpha \leftarrow \beta} (b \triangleleft_{\alpha \triangleleft \beta} c) + a \leftarrow_{\alpha \cdot \beta} (b \circ_{\alpha * \beta} c) \\ (a \triangleright_\alpha b) \leftarrow_\beta c &= (P_\alpha(a)b)P_\beta(c) = P_\alpha(a)(bP_\beta(c)) = a \triangleright_\alpha (b \leftarrow_\beta c) \\ a \triangleright_\alpha (b \triangleright_\beta c) &= P_\alpha(a)(P_\beta(b)c) = (P_\alpha(a)P_\beta(b))c \\ &= P_{\alpha \rightarrow \beta}(P_{\alpha \triangleright \beta}(a)b)c + P_{\alpha \leftarrow \beta}(aP_{\alpha \triangleleft \beta}(b))c + P_{\alpha \cdot \beta}(\lambda_{\alpha, \beta}ab)c \\ &= (a \triangleright_{\alpha \triangleright \beta} b) \triangleright_{\alpha \rightarrow \beta} c + (a \triangleleft_{\alpha \triangleleft \beta} b) \triangleright_{\alpha \leftarrow \beta} c + (a \circ_{\alpha * \beta} b) \triangleright_{\alpha \cdot \beta} c \\ (a \triangleright_\alpha b) \circ_\beta c &= \mu_\beta(P_\alpha(a)b)c = P_\alpha(a)(\mu_\beta bc) = a \triangleright_\alpha (b \circ_\beta c) \\ (a \leftarrow_\alpha b) \circ_\beta c &= \mu_\beta(aP_\alpha(b))c = \mu_\beta a(P_\alpha(b)c) = a \circ_\beta (b \triangleright_\alpha c) \\ (a \circ_\alpha b) \leftarrow_\beta c &= \mu_\alpha(ab)P_\beta(c) = \mu_\alpha a(bP_\beta(c)) = a \circ_\alpha (b \leftarrow_\beta c) \\ (a \circ_\alpha b) \circ_\beta c &= \mu_\beta \mu_\alpha(ab)c = \mu_\alpha \mu_\beta a(bc) = a \circ_\alpha (b \circ_\beta c). \quad \square \end{aligned}$$

### 3. FREE $\Omega$ -TRIDENDRIFORM ALGEBRAS

**3.1.  $\Omega$ -tridendriform algebras on leaf-typed angularly decorated Schröder trees.** Recall from [14] that Schröder trees are planar rooted trees such that there are at least two incoming edges for each vertex. For a Schröder tree  $T$ , we still view the root and the leaves of  $T$  as edges rather than vertices. Denote by  $L(T)$  the set of leaf edges of  $T$ , i.e. edges which represent the leaves of  $T$  and denote by  $IL(T)$  the subset of  $L(T)$  consisting of leaf edges which are neither the leftmost one nor the rightmost one.

**Definition 3.1.** Let  $X$  and  $\Omega$  be two sets. An  $X$ -angularly decorated  $\Omega$ -leaf typed (abbr. **leaf-typed angularly decorated**) Schröder tree is a triple  $T = (T, \text{dec}, \text{type})$ , where  $T$  is a Schröder tree,  $\text{dec} : A(T) \rightarrow X$  and  $\text{type} : IL(T) \rightarrow \Omega$  are maps.

For  $n \geq 1$ , let  $\mathfrak{T}_n(X, \Omega)$  be the set of  $X$ -angularly decorated  $\Omega$ -leaf typed Schröder trees with  $n + 1$  leaves and at least one internal vertex. Denote by

$$\mathfrak{T}(X, \Omega) := \bigsqcup_{n \geq 1} \mathfrak{T}_n(X, \Omega) \quad \text{and} \quad \mathbf{k}\mathfrak{T}(X, \Omega) := \bigoplus_{n \geq 1} \mathbf{k}\mathfrak{T}_n(X, \Omega).$$

Here are some examples.

$$\mathfrak{T}_1(X, \Omega) = \left\{ \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad \quad x \\ \diagdown \quad \diagup \\ \quad \quad \quad | \\ \quad \quad \quad x \in X \end{array} \right\},$$

$$\mathfrak{T}_2(X, \Omega) = \left\{ \begin{array}{c} \begin{array}{c} y \quad \alpha \\ \diagdown \quad / \\ x \end{array}, \begin{array}{c} \alpha \quad y \\ / \quad \diagdown \\ x \end{array}, \begin{array}{c} \alpha \\ / \quad \diagdown \\ x \quad y \end{array} \mid x, y \in X, \alpha \in \Omega \right\},$$

$$\mathfrak{T}_3(X, \Omega) = \left\{ \begin{array}{c} \begin{array}{c} z \quad \beta \quad \alpha \\ \diagdown \quad / \quad / \\ x \end{array}, \begin{array}{c} \alpha \quad z \quad \beta \\ / \quad \diagdown \quad / \\ x \end{array}, \begin{array}{c} \alpha \quad \beta \\ / \quad \diagdown \\ x \quad y \quad z \end{array}, \begin{array}{c} \alpha \quad \beta \\ / \quad \diagdown \\ x \quad y \quad z \end{array}, \begin{array}{c} \alpha \quad z \quad \beta \\ / \quad \diagdown \\ x \quad y \end{array}, \dots \mid x, y, z \in X, \alpha, \beta \in \Omega \right\}.$$

For  $T \in \mathfrak{T}(X, \Omega)$  and  $\omega \in \Omega$ , let  ${}^\omega T$  be the tree  $T$  whose leftmost leaf edge is typed by  $\omega$  and let  $T^\omega$  be the tree  $T$  whose rightmost leaf edge is typed by  $\omega$ . We also define  ${}^\omega | := |^\omega := |$  and say  $l({}^\omega T) = r(T^\omega) = \omega$  if  $T \neq |$ . Graphically, an element  $T \in \mathfrak{T}(X, \Omega)$  is of the form

$$T = \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \diagdown \quad \dots \quad / \\ x_1 \quad \dots \quad x_m \\ \circ \quad \dots \quad \circ \\ T_1 \quad \dots \quad T_{m+1} \end{array},$$

with  $n \geq 1$ ,  $x_1, \dots, x_n \in X$  and  $T_1 = |$  or  $U_1^{\alpha_{1,r}}$ ,  $T_i = |$  or  $\alpha_{i,l} U_i^{\alpha_{i,r}}$  for  $2 \leq i \leq m$  and  $T_{m+1} = |$  or  $\alpha_{m+1,l} U_{m+1}$  for some  $U_j \in \mathfrak{T}(X, \Omega)$  and  $\alpha_{j,l}, \alpha_{j,r} \in \Omega$ . For each  $T \in \mathfrak{T}(X, \Omega)$ , denote by  $\text{leaf}(T)$  the number of leaf edges of  $T$ .

Now, let  $\Omega$  be a set with six products  $\leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *$ . For  $\omega \in \Omega$ , we define products  $\prec_\omega, \succ_\omega, \circ_\omega$  on  $\mathbf{k}\mathfrak{T}(X, \Omega)$  recursively as follows.

For

$$T = \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \diagdown \quad \dots \quad / \\ x_1 \quad \dots \quad x_m \\ \circ \quad \dots \quad \circ \\ T_1 \quad \dots \quad T_{m+1} \end{array} \quad \text{and} \quad U = \begin{array}{c} U_2 \quad U_n \\ \circ \quad \circ \\ \diagdown \quad \dots \quad / \\ y_1 \quad \dots \quad y_n \\ \circ \quad \dots \quad \circ \\ U_1 \quad \dots \quad U_{n+1} \end{array} \in \mathfrak{T}(X, \Omega),$$

we define  $T \prec_\omega U, T \succ_\omega U, T \circ_\omega U$  by induction on  $\text{leaf}(T) + \text{leaf}(U)$ . If  $\text{leaf}(T) + \text{leaf}(U) = 4$ , we have

$$T = \begin{array}{c} \diagdown \quad / \\ x \end{array} \quad \text{and} \quad U = \begin{array}{c} \diagdown \quad / \\ y \end{array}.$$

Then

$$T \prec_\omega U := \begin{array}{c} \omega \quad y \\ \diagdown \quad / \\ x \end{array}, \quad T \succ_\omega U := \begin{array}{c} x \quad \omega \\ / \quad \diagdown \\ y \end{array} \quad \text{and} \quad T \circ_\omega U := \begin{array}{c} \omega \\ / \quad \diagdown \\ x \quad y \end{array}.$$

For the induction step of  $\text{leaf}(T) + \text{leaf}(U) \geq 5$ , to define  $T \prec_\omega U$ , we consider the following two cases.

**Case 1:**  $T_{m+1} = |$ . Then

$$T \prec_\omega U := \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \diagdown \quad \dots \quad / \\ x_1 \quad \dots \quad x_m \\ \circ \quad \dots \quad \circ \\ T_1 \quad \dots \quad \omega U \end{array}.$$

**Case 2:**  $T_{m+1} \neq |$  and  $l(T_{m+1}) = \alpha_{m+1}$ . Then

$$T \prec_\omega U := \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \diagdown \quad \dots \quad / \\ x_1 \quad \dots \quad x_m \\ \circ \quad \dots \quad \circ \\ T_1 \quad \dots \quad T_{m+1} \end{array} \left( \begin{array}{l} \alpha_{m+1} \rightarrow \omega T_{m+1} \succ_{\alpha_{m+1} \triangleright \omega} U \\ + \alpha_{m+1} \leftarrow \omega T_{m+1} \prec_{\alpha_{m+1} \triangleleft \omega} U \\ + \alpha_{m+1} \cdot \omega T_{m+1} \circ_{\alpha_{m+1} * \omega} U \end{array} \right).$$

To define  $T \succ_{\omega} U$ , we consider the following two cases.

**Case 3:**  $U_1 = |$ . Then

$$T \succ_{\omega} U := T^{\omega} \circ \begin{array}{c} U_2 \quad U_n \\ \circ \quad \circ \\ \vdots \\ y_1 \quad \cdots \quad y_n \\ \circ \quad \circ \\ \vdots \\ \circ \quad \circ \\ U_{n+1} \end{array}.$$

**Case 4:**  $U_1 \neq |$  and  $r(U_1) = \beta_1$ . Then

$$T \succ_{\omega} U := \left( \begin{array}{l} T \succ_{\omega \triangleright \beta_1} U_1^{\omega \rightarrow \beta_1} \\ + T \prec_{\omega \triangleleft \beta_1} U_1^{\omega \leftarrow \beta_1} \\ + T \circ_{\omega * \beta_1} U_1^{\omega \beta_1} \end{array} \right) \begin{array}{c} U_2 \quad U_n \\ \circ \quad \circ \\ \vdots \\ y_1 \quad \cdots \quad y_n \\ \circ \quad \circ \\ \vdots \\ \circ \quad \circ \\ U_{n+1} \end{array}.$$

To define  $T \circ_{\omega} U$ , we consider the following four cases.

**Case 5:**  $T = \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ x \end{array}$ . Then

$$T \circ_{\omega} U := \begin{array}{c} \omega U_1 \quad U_n \\ \circ \quad \circ \\ \vdots \\ x \quad \cdots \quad y_n \\ \circ \quad \circ \\ \vdots \\ \circ \quad \circ \\ U_{n+1} \end{array}.$$

**Case 6:**  $T = \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ x \quad \circ \\ \vdots \\ \circ \end{array}$ . Then

$$T \circ_{\omega} U := \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ x \end{array} \circ_{\omega} (T_2 \succ_{\alpha} U).$$

**Case 7:**  $T = \begin{array}{c} \circ \quad \circ \\ \diagup \quad \vdots \quad \diagdown \\ x_1 \quad \cdots \quad x_m \\ \circ \quad \circ \\ \vdots \\ \circ \quad \circ \\ T_{m+1} \end{array}$  with  $m \geq 2$ . Then

$$T \circ_{\omega} U := \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ x_1 \end{array} \circ_{\alpha} \left( \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \vdots \\ x_1 \quad \cdots \quad x_m \\ \circ \quad \circ \\ \vdots \\ \circ \quad \circ \\ T_{m+1} \end{array} \circ_{\omega} U \right).$$

**Case 8:**  $T = \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \vdots \\ x_1 \quad \cdots \quad x_m \\ \circ \quad \circ \\ \vdots \\ \circ \quad \circ \\ T_{m+1} \end{array}$  with  $T_1 \neq |$ . Then

$$T \circ_{\omega} U := T_1 \succ_{\alpha} \left( \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \vdots \\ x_1 \quad \cdots \quad x_m \\ \circ \quad \circ \\ \vdots \\ \circ \quad \circ \\ T_{m+1} \end{array} \circ_{\omega} U \right).$$

Note that for  $T, U \in \mathfrak{T}(X, \Omega)$  with  $\text{leaf}(T) = m, \text{leaf}(U) = n$ , then the trees in  $T \prec_{\omega} U, T \succ_{\omega} U, T \circ_{\omega} U$  which are defined in Case 1-7 have  $m + n - 1$  leaf edges. Hence Case 8 is in the induction step.



Let  $j : X \rightarrow \mathbf{k}\mathfrak{T}(X, \Omega)$ ,  $x \mapsto \begin{array}{c} x \\ \diagdown \quad \diagup \\ \quad \quad \quad \end{array}$  be the natural inclusion. Then with these products defined as above, we obtain the following result:

**Theorem 3.2.** *Let  $\Omega$  be a set with six products  $\leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *$ . Then the following conditions are equivalent:*

- (a) *With these products,  $\mathbf{k}\mathfrak{T}(X, \Omega)$  and the map  $j$  is the free  $\Omega$ -tridendriform algebra generated by  $X$ .*
- (b) *With these products,  $\mathbf{k}\mathfrak{T}(X, \Omega)$  is an  $\Omega$ -tridendriform algebra.*
- (c)  *$(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *)$  is an ETS.*

*Proof.* (a)  $\implies$  (b) It is obvious.

(b)  $\implies$  (c) For  $\alpha, \beta, \gamma \in \Omega$  and  $\begin{array}{c} a \\ \diagdown \quad \diagup \\ \quad \quad \quad \end{array}, \begin{array}{c} b \\ \diagdown \quad \diagup \\ \quad \quad \quad \end{array}, \begin{array}{c} c \\ \diagdown \quad \diagup \\ \quad \quad \quad \end{array} \in \mathbf{k}\mathfrak{T}(X, \Omega)$ , by calculation there are both eleven trees in the expression of

$$\begin{array}{c} a \\ \diagdown \quad \diagup \\ \quad \quad \quad \end{array} \triangleright_{\alpha} \left( \begin{array}{c} b \\ \diagdown \quad \diagup \\ \quad \quad \quad \end{array} \triangleright_{\beta} \begin{array}{c} c \\ \diagdown \quad \diagup \\ \quad \quad \quad \end{array} \right)$$

and of

$$\begin{aligned} & \left( \begin{array}{c} a \\ \diagdown \quad \diagup \\ \quad \quad \quad \end{array} \triangleright_{\alpha \triangleright \beta} \begin{array}{c} b \\ \diagdown \quad \diagup \\ \quad \quad \quad \end{array} \right) \triangleright_{\alpha \rightarrow \beta} \begin{array}{c} c \\ \diagdown \quad \diagup \\ \quad \quad \quad \end{array} + \left( \begin{array}{c} a \\ \diagdown \quad \diagup \\ \quad \quad \quad \end{array} \triangleleft_{\alpha \triangleleft \beta} \begin{array}{c} b \\ \diagdown \quad \diagup \\ \quad \quad \quad \end{array} \right) \triangleright_{\alpha \leftarrow \beta} \begin{array}{c} c \\ \diagdown \quad \diagup \\ \quad \quad \quad \end{array} \\ & + \left( \begin{array}{c} a \\ \diagdown \quad \diagup \\ \quad \quad \quad \end{array} \circ_{\alpha * \beta} \begin{array}{c} b \\ \diagdown \quad \diagup \\ \quad \quad \quad \end{array} \right) \triangleright_{\alpha \cdot \beta} \begin{array}{c} c \\ \diagdown \quad \diagup \\ \quad \quad \quad \end{array}. \end{aligned}$$

Identifying the types of the trees in these expressions, we get that  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *)$  is an ETS.

(c)  $\implies$  (b) Extending the products  $\leftarrow_{\omega}, \triangleright_{\omega}, \circ_{\omega}$  to the space

$$\left( \mathbf{k}\mathfrak{T}(X, \Omega) \otimes \mathbf{k}\mathfrak{T}(X, \Omega) \right) \oplus \left( \mathbf{k} \mid \mathbf{k}\mathfrak{T}(X, \Omega) \right) \oplus \left( \mathbf{k}\mathfrak{T}(X, \Omega) \otimes \mathbf{k} \right)$$

by

$$\mid \triangleright_{\omega} T := T \leftarrow_{\omega} \mid := T, \quad \mid \triangleleft_{\omega} T := T \triangleright_{\omega} \mid := 0 \quad \text{and} \quad \mid \circ_{\omega} T := T \circ_{\omega} \mid := 0,$$

for  $\omega \in \Omega$  and  $T \in \mathbf{k}\mathfrak{T}(X, \Omega)$ . By convention, we consider the added element  $\emptyset$  as a unit for the six products of  $\Omega$ . Then the products  $\leftarrow_{\omega}, \triangleright_{\omega}$  can be rewritten in the following way: for

$$T = \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \diagdown \quad \dots \quad \diagup \\ x_1 \quad \dots \quad x_m \\ \quad \quad \quad \end{array} \circ_{\alpha_{m+1}} T_{m+1} \quad \text{and} \quad U = \begin{array}{c} U_2 \quad U_n \\ \circ \quad \circ \\ \diagdown \quad \dots \quad \diagup \\ y_1 \quad \dots \quad y_n \\ \quad \quad \quad \end{array} \in \mathfrak{T}(X, \Omega),$$

then

$$T \leftarrow_{\omega} \mid = \mid \triangleright_{\omega} T = T,$$

$$\mid \triangleleft_{\omega} T = T \triangleright_{\omega} \mid = 0,$$

$$\mid \circ_{\omega} T = T \circ_{\omega} \mid = 0,$$

$$T \leftarrow_{\omega} U := \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \diagdown \quad \dots \quad \diagup \\ x_1 \quad \dots \quad x_m \\ \quad \quad \quad \end{array} \left( \begin{array}{l} \alpha_{m+1} \rightarrow \omega T_{m+1} \triangleright_{\alpha_{m+1} \triangleright \omega} U \\ + \alpha_{m+1} \leftarrow \omega T_{m+1} \triangleleft_{\alpha_{m+1} \triangleleft \omega} U \\ + \alpha_{m+1} \cdot \omega T_{m+1} \circ_{\alpha_{m+1} * \omega} U \end{array} \right),$$

$$T \succ_{\omega} U := \left( \begin{array}{l} T \succ_{\omega \triangleright \beta_1} U_1^{\omega \rightarrow \beta_1} \\ + T \prec_{\omega \triangleleft \beta_1} U_1^{\omega \leftarrow \beta_1} \\ + T \circ_{\omega * \beta_1} U_1^{\omega \beta_1} \end{array} \right) \begin{array}{c} U_2 \quad U_n \\ \vdots \\ y_1 \quad y_n \\ \circ U_{n+1} \end{array}$$

We first show that  $\mathbf{k}\mathfrak{T}(X, \Omega)$  is an  $\Omega$ -tridendriform algebra. We prove Eqs. (29)-(35) hold for

$$T = \begin{array}{c} T_2 \quad T_l \\ \vdots \\ x_1 \quad x_m \\ \circ T_{l+1} \end{array}, \quad U = \begin{array}{c} U_2 \quad U_m \\ \vdots \\ y_1 \quad y_n \\ \circ U_{m+1} \end{array} \quad \text{and} \quad V = \begin{array}{c} V_2 \quad V_n \\ \vdots \\ z_1 \quad z_n \\ \circ V_{n+1} \end{array} \in \mathfrak{T}(X, \Omega)$$

by induction on the sum  $p := \text{leaf}(T) + \text{leaf}(U) + \text{leaf}(V)$ . If  $p = 6$ , then  $\text{leaf}(T) = \text{leaf}(U) = \text{leaf}(V) = 2$  and  $T, U, V$  are of the form

$$T = \begin{array}{c} x \\ \swarrow \quad \searrow \\ \circ \end{array}, \quad U = \begin{array}{c} y \\ \swarrow \quad \searrow \\ \circ \end{array} \quad \text{and} \quad V = \begin{array}{c} z \\ \swarrow \quad \searrow \\ \circ \end{array}.$$

Eqs. (29)-(35) hold by direct calculation.

Suppose that Eqs. (29)-(35) hold for  $p \leq q$ , where  $q \geq 6$  is a fixed positive integer. Consider the case of  $p = q + 1$ . First, we prove Eq. (29) and we assume  $l(T_{l+1}) = \alpha_{l+1}$  if  $T_{l+1} \neq \circ$ . Then

$$\begin{aligned} & (T \prec_{\alpha} U) \prec_{\beta} V \\ &= \left( \begin{array}{c} T_2 \quad T_l \\ \vdots \\ x_1 \quad x_l \\ \circ \end{array} \left( \begin{array}{l} \alpha_{m+1} \rightarrow \alpha T_{m+1} \succ_{\alpha_{m+1} \triangleright \alpha} U \\ + \alpha_{m+1} \leftarrow \alpha T_{m+1} \prec_{\alpha_{m+1} \triangleleft \alpha} U \\ + \alpha_{m+1} \cdot \alpha T_{m+1} \circ_{\alpha_{m+1} * \alpha} U \end{array} \right) \right) \prec_{\beta} V \\ &= \begin{array}{c} T_2 \quad T_l \\ \vdots \\ x_1 \quad x_l \\ \circ \end{array} \left( \begin{array}{l} (\alpha_{l+1} \rightarrow \alpha) \rightarrow \beta T_{l+1} \succ_{\alpha_{l+1} \triangleright \alpha} U \succ_{(\alpha_{l+1} \rightarrow \alpha) \triangleright \beta} V \\ + (\alpha_{l+1} \rightarrow \alpha) \leftarrow \beta T_{l+1} \succ_{\alpha_{l+1} \triangleright \alpha} U \prec_{(\alpha_{l+1} \rightarrow \alpha) \triangleleft \beta} V \\ + (\alpha_{l+1} \rightarrow \alpha) \cdot \beta T_{l+1} \succ_{\alpha_{l+1} \triangleright \alpha} U \circ_{(\alpha_{l+1} \rightarrow \alpha) * \beta} V \\ + (\alpha_{l+1} \leftarrow \alpha) \rightarrow \beta T_{l+1} \prec_{\alpha_{l+1} \triangleleft \alpha} U \succ_{(\alpha_{l+1} \leftarrow \alpha) \triangleright \beta} V \\ + (\alpha_{l+1} \leftarrow \alpha) \leftarrow \beta T_{l+1} \prec_{\alpha_{l+1} \triangleleft \alpha} U \prec_{(\alpha_{l+1} \leftarrow \alpha) \triangleleft \beta} V \\ + (\alpha_{l+1} \leftarrow \alpha) \cdot \beta T_{l+1} \prec_{\alpha_{l+1} \triangleleft \alpha} U \circ_{(\alpha_{l+1} \leftarrow \alpha) * \beta} V \\ + (\alpha_{l+1} \cdot \alpha) \rightarrow \beta T_{l+1} \circ_{\alpha_{l+1} * \alpha} U \succ_{(\alpha_{l+1} \cdot \alpha) \triangleright \beta} V \\ + (\alpha_{l+1} \cdot \alpha) \leftarrow \beta T_{l+1} \circ_{\alpha_{l+1} * \alpha} U \prec_{(\alpha_{l+1} \cdot \alpha) \triangleleft \beta} V \\ + (\alpha_{l+1} \cdot \alpha) \cdot \beta T_{l+1} \circ_{\alpha_{l+1} * \alpha} U \circ_{(\alpha_{l+1} \cdot \alpha) * \beta} V \end{array} \right) \\ &= \begin{array}{c} T_2 \quad T_l \\ \vdots \\ x_1 \quad x_l \\ \circ \end{array} \left( \begin{array}{l} \alpha_{l+1} \rightarrow (\alpha \rightarrow \beta) T_{l+1} \succ_{\alpha_{l+1} \triangleright (\alpha \rightarrow \beta)} (U \succ_{\alpha \triangleright \beta} V) \\ + \alpha_{l+1} \leftarrow (\alpha \rightarrow \beta) T_{l+1} \prec_{\alpha_{l+1} \triangleleft (\alpha \rightarrow \beta)} (U \succ_{\alpha \triangleright \beta} V) \\ + \alpha_{l+1} \cdot (\alpha \rightarrow \beta) T_{l+1} \circ_{\alpha_{l+1} * (\alpha \rightarrow \beta)} (U \succ_{\alpha \triangleright \beta} V) \\ + \alpha_{l+1} \rightarrow (\alpha \leftarrow \beta) T_{l+1} \succ_{\alpha_{l+1} \triangleright (\alpha \leftarrow \beta)} (U \prec_{\alpha \triangleleft \beta} V) \\ + \alpha_{l+1} \leftarrow (\alpha \leftarrow \beta) T_{l+1} \prec_{\alpha_{l+1} \triangleleft (\alpha \leftarrow \beta)} (U \prec_{\alpha \triangleleft \beta} V) \\ + \alpha_{l+1} \cdot (\alpha \leftarrow \beta) T_{l+1} \circ_{\alpha_{l+1} * (\alpha \leftarrow \beta)} (U \prec_{\alpha \triangleleft \beta} V) \\ + \alpha_{l+1} \rightarrow (\alpha \beta) T_{l+1} \succ_{\alpha_{l+1} \triangleright (\alpha \beta)} (U \circ_{\alpha * \beta} V) \\ + \alpha_{l+1} \leftarrow (\alpha \beta) T_{l+1} \prec_{\alpha_{l+1} \triangleleft (\alpha \beta)} (U \circ_{\alpha * \beta} V) \\ + \alpha_{l+1} \cdot (\alpha \beta) T_{l+1} \circ_{\alpha_{l+1} * (\alpha \beta)} (U \circ_{\alpha * \beta} V) \end{array} \right) \end{aligned}$$

(by induction hypothesis and  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *)$  being an ETS)

$$= T \prec_{\alpha \rightarrow \beta} (U \succ_{\alpha \triangleright \beta} V) + T \prec_{\alpha \leftarrow \beta} (U \prec_{\alpha \triangleleft \beta} V) + T \prec_{\alpha \cdot \beta} (U \circ_{\alpha * \beta} V).$$

Hence Eq. (29) holds. Eq.(31) can be proved similarly. Eq. (30) holds directly as  $T \succ_\alpha U$  changes the leftmost branch of  $U$  and  $U \prec_\beta V$  changes the rightmost branch of  $U$  and  $U$  has at least two branches. Now we show Eq. (32) holds. If  $U_1 = |$ , then  $(T \succ_\alpha U) \circ_\beta V = T \succ_\alpha (U \circ_\beta V)$  by the definition of  $\circ_\omega$  in Case 8. If  $U_1 \neq |$ , we assume  $r(U_1) = \beta_1$ , then

$$\begin{aligned}
 & (T \succ_\alpha U) \circ_\beta V \\
 &= \left( (T \succ_{\alpha \triangleright \beta_1} U_1^{\alpha \rightarrow \beta_1} + T \prec_{\alpha \triangleleft \beta_1} U_1^{\alpha \leftarrow \beta_1} + T \circ_{\alpha * \beta_1} U_1^{\alpha \beta_1}) \begin{array}{c} U_2 \quad U_n \\ \circ \quad \circ \\ \vdots \\ y_1 \cdots y_n \\ \circ \quad \circ \\ U_{n+1} \end{array} \right) \circ_\beta V \\
 &= \left( (T \succ_{\alpha \triangleright \beta_1} U_1) \succ_{\alpha \rightarrow \beta_1} \begin{array}{c} U_2 \quad U_n \\ \circ \quad \circ \\ \vdots \\ y_1 \cdots y_n \\ \circ \quad \circ \\ U_{n+1} \end{array} + (T \prec_{\alpha \triangleleft \beta_1} U_1) \succ_{\alpha \leftarrow \beta_1} \begin{array}{c} U_2 \quad U_n \\ \circ \quad \circ \\ \vdots \\ y_1 \cdots y_n \\ \circ \quad \circ \\ U_{n+1} \end{array} \right. \\
 &\quad \left. + (T \circ_{\alpha * \beta_1} U_1) \succ_{\alpha \beta_1} \begin{array}{c} U_2 \quad U_n \\ \circ \quad \circ \\ \vdots \\ y_1 \cdots y_n \\ \circ \quad \circ \\ U_{n+1} \end{array} \right) \circ_\beta V \\
 &= (T \succ_{\alpha \triangleright \beta_1} U_1) \succ_{\alpha \rightarrow \beta_1} \left( \begin{array}{c} U_2 \quad U_n \\ \circ \quad \circ \\ \vdots \\ y_1 \cdots y_n \\ \circ \quad \circ \\ U_{n+1} \end{array} \circ_\beta V \right) + (T \prec_{\alpha \triangleleft \beta_1} U_1) \succ_{\alpha \leftarrow \beta_1} \left( \begin{array}{c} U_2 \quad U_n \\ \circ \quad \circ \\ \vdots \\ y_1 \cdots y_n \\ \circ \quad \circ \\ U_{n+1} \end{array} \circ_\beta V \right) \\
 &\quad + (T \circ_{\alpha * \beta_1} U_1) \succ_{\alpha \beta_1} \left( \begin{array}{c} U_2 \quad U_n \\ \circ \quad \circ \\ \vdots \\ y_1 \cdots y_n \\ \circ \quad \circ \\ U_{n+1} \end{array} \circ_\beta V \right) \quad \text{(by Case 8)} \\
 &= T \succ_\alpha \left( U_1 \succ_{\beta_1} \left( \begin{array}{c} U_2 \quad U_n \\ \circ \quad \circ \\ \vdots \\ y_1 \cdots y_n \\ \circ \quad \circ \\ U_{n+1} \end{array} \circ_\beta V \right) \right) \quad \text{(by Eq. (31))} \\
 &= T \succ_\alpha \left( \left( U_1 \succ_{\beta_1} \begin{array}{c} U_2 \quad U_n \\ \circ \quad \circ \\ \vdots \\ y_1 \cdots y_n \\ \circ \quad \circ \\ U_{n+1} \end{array} \right) \circ_\beta V \right) \quad \text{(by the induction hypothesis)} \\
 &= T \succ_\alpha (U \circ_\beta V).
 \end{aligned}$$

Hence Eq. (32) holds. Now we show Eq. (33) holds. If  $T = \begin{array}{c} x \\ \swarrow \quad \searrow \end{array}$ , then  $(T \prec_\alpha U) \circ_\beta V = T \circ_\beta (U \succ_\alpha V)$  by the definition of  $\circ_\omega$  in Case 6. For general case, we assume  $l(T_{l+1}) = \alpha_{l+1}$  if  $T_{l+1} \neq |$ , then

$$\begin{aligned}
 & (T \prec_\alpha U) \circ_\beta V \\
 &= \left( T_1 \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \vdots \\ x_1 \cdots x_m \\ \circ \quad \circ \end{array} \left( \begin{array}{c} \alpha_{m+1} \rightarrow \alpha T_{m+1} \succ_{\alpha_{m+1} \triangleright \alpha} U + \alpha_{m+1} \leftarrow \alpha T_{m+1} \prec_{\alpha_{m+1} \triangleleft \alpha} U \\ + \alpha_{m+1} \cdot \alpha T_{m+1} \circ_{\alpha_{m+1} * \alpha} U \end{array} \right) \right) \circ_\beta V
 \end{aligned}$$

$$= \left( \begin{array}{c} \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \vdots \quad \vdots \\ x_1 \quad \dots \quad x_m \\ \diagdown \quad \diagup \\ T_1 \circ \quad \circ \end{array} \langle_{\alpha_{l+1} \rightarrow \alpha} (T_{m+1} \triangleright_{\alpha_{m+1} \triangleright \alpha} U) + \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \vdots \quad \vdots \\ x_1 \quad \dots \quad x_m \\ \diagdown \quad \diagup \\ T_1 \circ \quad \circ \end{array} \langle_{\alpha_{l+1} \leftarrow \alpha} (T_{m+1} \triangleleft_{\alpha_{m+1} \triangleleft \alpha} U) \\ + \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \vdots \quad \vdots \\ x_1 \quad \dots \quad x_m \\ \diagdown \quad \diagup \\ T_1 \circ \quad \circ \end{array} \langle_{\alpha_{l+1} \cdot \alpha} (T_{m+1} \circ_{\alpha_{m+1} * \alpha} U) \end{array} \right) \circ_{\beta} V.$$

Next we consider the form of  $T_1 \circ \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \vdots \quad \vdots \\ x_1 \quad \dots \quad x_m \\ \diagdown \quad \diagup \\ T_1 \circ \quad \circ \end{array}$  as following:

(a) If  $T_1 \circ \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \vdots \quad \vdots \\ x_1 \quad \dots \quad x_m \\ \diagdown \quad \diagup \\ T_1 \circ \quad \circ \end{array} = \begin{array}{c} x \\ \diagdown \quad \diagup \\ \circ \end{array}$  for some  $x \in X$ , then

$$\begin{aligned} & \left( \begin{array}{c} \begin{array}{c} x \\ \diagdown \quad \diagup \\ \circ \end{array} \langle_{\alpha_{l+1} \rightarrow \alpha} (T_{m+1} \triangleright_{\alpha_{m+1} \triangleright \alpha} U) + \begin{array}{c} x \\ \diagdown \quad \diagup \\ \circ \end{array} \langle_{\alpha_{l+1} \leftarrow \alpha} (T_{m+1} \triangleleft_{\alpha_{m+1} \triangleleft \alpha} U) \\ + \begin{array}{c} x \\ \diagdown \quad \diagup \\ \circ \end{array} \langle_{\alpha_{l+1} \cdot \alpha} (T_{m+1} \circ_{\alpha_{m+1} * \alpha} U) \end{array} \right) \circ_{\beta} V \\ &= \begin{array}{c} x \\ \diagdown \quad \diagup \\ \circ \end{array} \circ_{\beta} \left( \begin{array}{c} (T_{l+1} \triangleright_{\alpha_{l+1} \triangleright \alpha} U) \triangleright_{\alpha_{l+1} \rightarrow \alpha} V + (T_{l+1} \triangleleft_{\alpha_{l+1} \triangleleft \alpha} U) \triangleright_{\alpha_{l+1} \leftarrow \alpha} V \\ + (T_{l+1} \circ_{\alpha_{l+1} * \alpha} U) \triangleright_{\alpha_{l+1} \cdot \alpha} V \end{array} \right) \quad (\text{by Case 6}) \\ &= \begin{array}{c} x \\ \diagdown \quad \diagup \\ \circ \end{array} \circ_{\beta} (T_{l+1} \triangleright_{\alpha_{l+1}} (U \triangleright_{\alpha} V)) \quad (\text{by Eq. (31)}) \\ &= \left( \begin{array}{c} x \\ \diagdown \quad \diagup \\ \circ \end{array} \langle_{\alpha_{l+1}} T_{l+1} \right) \circ_{\beta} (U \triangleright_{\alpha} V) \quad (\text{by Case 6}) \\ &= T \circ_{\beta} (U \triangleright_{\alpha} V). \end{aligned}$$

(b) If there are  $T', T'' \in \mathfrak{T}(X, \Omega)$  and  $\omega \in \Omega$  such that  $T_1 \circ \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \vdots \quad \vdots \\ x_1 \quad \dots \quad x_m \\ \diagdown \quad \diagup \\ T_1 \circ \quad \circ \end{array} = T' \triangleright_{\omega} T''$ , then

$$\begin{aligned} & \left( \begin{array}{c} (T' \triangleright_{\omega} T'') \langle_{\alpha_{l+1} \rightarrow \alpha} (T_{m+1} \triangleright_{\alpha_{m+1} \triangleright \alpha} U) \\ + (T' \triangleright_{\omega} T'') \langle_{\alpha_{l+1} \leftarrow \alpha} (T_{m+1} \triangleleft_{\alpha_{m+1} \triangleleft \alpha} U) \\ + (T' \triangleright_{\omega} T'') \langle_{\alpha_{l+1} \cdot \alpha} (T_{m+1} \circ_{\alpha_{m+1} * \alpha} U) \end{array} \right) \circ_{\beta} V \\ &= \left( \begin{array}{c} T' \triangleright_{\omega} (T'' \triangleleft_{\alpha_{l+1} \rightarrow \alpha} (T_{m+1} \triangleright_{\alpha_{m+1} \triangleright \alpha} U) \\ + T'' \triangleleft_{\alpha_{l+1} \leftarrow \alpha} (T_{m+1} \triangleleft_{\alpha_{m+1} \triangleleft \alpha} U) \\ + T'' \triangleleft_{\alpha_{l+1} \cdot \alpha} (T_{m+1} \circ_{\alpha_{m+1} * \alpha} U) \end{array} \right) \circ_{\beta} V \quad (\text{by Eq. (30)}) \\ &= (T' \triangleright_{\omega} ((T'' \triangleleft_{\alpha_{l+1}} T_{l+1}) \triangleleft_{\alpha} U)) \circ_{\beta} V \quad (\text{by Eq. (29)}) \\ &= T' \triangleright_{\omega} (((T'' \triangleleft_{\alpha_{l+1}} T_{l+1}) \triangleleft_{\alpha} U) \circ_{\beta} V) \quad (\text{by Case 8}) \\ &= T' \triangleright_{\omega} ((T'' \triangleleft_{\alpha_{l+1}} T_{l+1}) \circ_{\beta} (U \triangleright_{\alpha} V)) \quad (\text{by induction hypothesis}) \\ &= (T' \triangleright_{\omega} (T'' \triangleleft_{\alpha_{l+1}} T_{l+1})) \circ_{\beta} (U \triangleright_{\alpha} V) \quad (\text{by Case 8}) \\ &= ((T' \triangleright_{\omega} T'') \triangleleft_{\alpha_{l+1}} T_{l+1}) \circ_{\beta} (U \triangleright_{\alpha} V) \quad (\text{by Eq. (30)}) \\ &= T \circ_{\beta} (U \triangleright_{\alpha} V). \end{aligned}$$

(c) If there are  $\begin{array}{c} x \\ \swarrow \quad \searrow \end{array}$ ,  $T'' \in \mathfrak{T}(X, \Omega)$  and  $\omega \in \Omega$  such that  $T_1 \circ_{x_1} \dots \circ_{x_m} T_m = \begin{array}{c} x \\ \swarrow \quad \searrow \end{array} \circ_{\omega} T''$ , then

$$\begin{aligned}
 & \left( \begin{array}{l} \left( \begin{array}{c} x \\ \swarrow \quad \searrow \end{array} \circ_{\omega} T'' \right) \prec_{\alpha_{l+1} \rightarrow \alpha} (T_{m+1} \succ_{\alpha_{m+1} \triangleright \alpha} U) \\ + \left( \begin{array}{c} x \\ \swarrow \quad \searrow \end{array} \circ_{\omega} T'' \right) \prec_{\alpha_{l+1} \leftarrow \alpha} (T_{m+1} \prec_{\alpha_{m+1} \triangleleft \alpha} U) \\ + \left( \begin{array}{c} x \\ \swarrow \quad \searrow \end{array} \circ_{\omega} T'' \right) \prec_{\alpha_{l+1} \cdot \alpha} (T_{m+1} \circ_{\alpha_{m+1} * \alpha} U) \end{array} \right) \circ_{\beta} V \\
 &= \left( \begin{array}{c} \begin{array}{c} x \\ \swarrow \quad \searrow \end{array} \circ_{\omega} \left( \begin{array}{l} T'' \prec_{\alpha_{l+1} \rightarrow \alpha} (T_{m+1} \succ_{\alpha_{m+1} \triangleright \alpha} U) \\ + T'' \prec_{\alpha_{l+1} \leftarrow \alpha} (T_{m+1} \prec_{\alpha_{m+1} \triangleleft \alpha} U) \\ + T'' \prec_{\alpha_{l+1} \cdot \alpha} (T_{m+1} \circ_{\alpha_{m+1} * \alpha} U) \end{array} \right) \end{array} \right) \circ_{\beta} V \quad (\text{by Case 7}) \\
 &= \left( \begin{array}{c} x \\ \swarrow \quad \searrow \end{array} \circ_{\omega} ((T'' \prec_{\alpha_{l+1}} T_{l+1}) \prec_{\alpha} U) \right) \circ_{\beta} V \quad (\text{by Eq. (29)}) \\
 &= \begin{array}{c} x \\ \swarrow \quad \searrow \end{array} \circ_{\omega} (((T'' \prec_{\alpha_{l+1}} T_{l+1}) \prec_{\alpha} U) \circ_{\beta} V) \quad (\text{by Case 7}) \\
 &= \begin{array}{c} x \\ \swarrow \quad \searrow \end{array} \circ_{\omega} ((T'' \prec_{\alpha_{l+1}} T_{l+1}) \circ_{\beta} (U \succ_{\alpha} V)) \quad (\text{by induction hypothesis}) \\
 &= \left( \begin{array}{c} x \\ \swarrow \quad \searrow \end{array} \circ_{\omega} (T'' \prec_{\alpha_{l+1}} T_{l+1}) \right) \circ_{\beta} (U \succ_{\alpha} V) \quad (\text{by Case 7}) \\
 &= \left( \left( \begin{array}{c} x \\ \swarrow \quad \searrow \end{array} \circ_{\omega} T'' \right) \prec_{\alpha_{l+1}} T_{l+1} \right) \circ_{\beta} (U \succ_{\alpha} V) \quad (\text{by induction hypothesis and Eq. (34)}) \\
 &= T \circ_{\beta} (U \succ_{\alpha} V).
 \end{aligned}$$

Hence Eq. (33) holds. Eq. (34) holds directly as  $T \circ_{\alpha} U$  does not change the rightmost branch of  $U$  and  $U \prec_{\beta} V$  only changes the rightmost branch of  $U$  and  $U$  has at least two branches.

Finally, we show Eq. (35) holds by induction on  $\text{leaf}(T)$ . If  $T = \begin{array}{c} x \\ \swarrow \quad \searrow \end{array}$  for some  $x \in X$ , then  $(T \circ_{\alpha} U) \circ_{\beta} V = T \circ_{\alpha} (U \circ_{\beta} V)$  by the definition of  $\circ_{\omega}$  in Case 7. Suppose Eq. (35) holds for all  $T$ , where  $\text{leaf}(T) \leq q$  with  $q$  a fixed integer. Assume  $\text{leaf}(T) = p + 1$ , we consider the form of  $T$  as follows.

(a) If there are  $T', T'' \in \mathfrak{T}(X, \Omega)$  and  $\omega \in \Omega$  such that  $T = T' \prec_{\omega} T''$ , then

$$\begin{aligned}
 & (T \circ_{\alpha} U) \circ_{\beta} V = ((T' \prec_{\omega} T'') \circ_{\alpha} U) \circ_{\beta} V \\
 &= (T' \circ_{\alpha} (T'' \succ_{\omega} U)) \circ_{\beta} V \quad (\text{by Eq. 33}) \\
 &= T' \circ_{\alpha} ((T'' \succ_{\omega} U) \circ_{\beta} V) \quad (\text{by induction hypothesis}) \\
 &= T' \circ_{\alpha} (T'' \succ_{\omega} (U \circ_{\beta} V)) \quad (\text{by Eq. (32)}) \\
 &= (T' \prec_{\omega} T'') \circ_{\alpha} (U \circ_{\beta} V) \quad (\text{by Case 6}) \\
 &= T \circ_{\alpha} (U \circ_{\beta} V).
 \end{aligned}$$

(b) If there are  $T', T'' \in \mathfrak{T}(X, \Omega)$  and  $\omega \in \Omega$  such that  $T = T' \succ_{\omega} T''$ , then

$$\begin{aligned}
 & (T \circ_{\alpha} U) \circ_{\beta} V = ((T' \succ_{\omega} T'') \circ_{\alpha} U) \circ_{\beta} V \\
 &= (T' \succ_{\omega} (T'' \circ_{\alpha} U)) \circ_{\beta} V \quad (\text{by Case 8}) \\
 &= T' \succ_{\omega} ((T'' \circ_{\alpha} U) \circ_{\beta} V) \quad (\text{by Case 8}) \\
 &= T' \succ_{\omega} (T'' \circ_{\alpha} (U \circ_{\beta} V)) \quad (\text{by induction hypothesis})
 \end{aligned}$$

$$\begin{aligned}
&= (T' \succ_{\omega} T'') \circ_{\alpha} (U \circ_{\beta} V) && \text{(by Case 8)} \\
&= T \circ_{\alpha} (U \circ_{\beta} V).
\end{aligned}$$

(c) If there are  $\begin{array}{c} x \\ \diagup \quad \diagdown \\ \text{---} \end{array}$ ,  $T'' \in \mathfrak{T}(X, \Omega)$  and  $\omega \in \Omega$  such that  $T = \begin{array}{c} x \\ \diagup \quad \diagdown \\ \text{---} \end{array} \circ_{\omega} T''$ , then

$$\begin{aligned}
(T \circ_{\alpha} U) \circ_{\beta} V &= \left( \left( \begin{array}{c} x \\ \diagup \quad \diagdown \\ \text{---} \end{array} \circ_{\omega} T'' \right) \circ_{\alpha} U \right) \circ_{\beta} V \\
&= \left( \begin{array}{c} x \\ \diagup \quad \diagdown \\ \text{---} \end{array} \circ_{\omega} (T'' \circ_{\alpha} U) \right) \circ_{\beta} V && \text{(by Case 7)} \\
&= \begin{array}{c} x \\ \diagup \quad \diagdown \\ \text{---} \end{array} \circ_{\omega} ((T'' \circ_{\alpha} U) \circ_{\beta} V) && \text{(by Case 7)} \\
&= \begin{array}{c} x \\ \diagup \quad \diagdown \\ \text{---} \end{array} \circ_{\omega} (T'' \circ_{\alpha} (U \circ_{\beta} V)) && \text{(by induction hypothesis)} \\
&= \left( \begin{array}{c} x \\ \diagup \quad \diagdown \\ \text{---} \end{array} \circ_{\omega} T'' \right) \circ_{\alpha} (U \circ_{\beta} V) && \text{(by Case 7)} \\
&= T \circ_{\alpha} (U \circ_{\beta} V).
\end{aligned}$$

Hence Eq. (35) holds. So  $\mathbf{k}\mathfrak{T}(X, \Omega)$  is an  $\Omega$ -tridendriform algebra.

Let  $(A, (\prec_{\omega}, \succ_{\omega}, \circ_{\omega})_{\omega \in \Omega})$  be an  $\Omega$ -tridendriform algebra and  $f : X \rightarrow A$  a set map. We extend  $f$  to be an  $\Omega$ -tridendriform algebra  $\bar{f} : \mathbf{k}\mathfrak{T}(X, \Omega) \rightarrow A$  such that  $\bar{f} \circ j = f$ . For  $T \in \mathfrak{T}(X, \Omega)$  with  $\text{leaf}(T) = 2$ , i.e.  $T = \begin{array}{c} x \\ \diagup \quad \diagdown \\ \text{---} \end{array}$  for some  $x \in X$ , define  $\bar{f}(T) = f(x)$ . Suppose  $\bar{f}(T)$  has been defined for all  $T$  with  $\text{leaf}(T) \leq q$ , where  $q \geq 2$  is a fixed integer. Consider the case of  $\text{leaf}(T) = q + 1$ . We consider the form of  $T$  as follows.

(a) If  $T = \begin{array}{c} \circledast T_2 \\ \diagup \quad \diagdown \\ x \\ \text{---} \end{array}$ , then define

$$\bar{f}(T) := f(x) \prec_{\alpha} \bar{f}(T_2).$$

(b) If  $T = \begin{array}{c} \circledast T_2 \quad T_m \\ \diagup \quad \cdots \quad \diagdown \\ x_1 \quad \cdots \quad x_m \\ \text{---} \end{array} \circ T_{m+1}$  with  $m \geq 2$ , then define

$$\bar{f}(T) := f(x) \circ_{\alpha} \bar{f} \left( \begin{array}{c} T_2 \quad T_m \\ \diagup \quad \cdots \quad \diagdown \\ \circ \quad \cdots \quad \circ \\ \text{---} \end{array} \right).$$

(c) If  $T = T_1^{\alpha} \begin{array}{c} T_2 \quad T_m \\ \diagup \quad \cdots \quad \diagdown \\ x_1 \quad \cdots \quad x_m \\ \text{---} \end{array} \circ T_{m+1}$  with  $T_1 \neq |$ , then define

$$\bar{f}(T) := \bar{f}(T_1) \succ_{\alpha} \bar{f} \left( \begin{array}{c} T_2 \quad T_m \\ \diagup \quad \cdots \quad \diagdown \\ x_1 \quad \cdots \quad x_m \\ \text{---} \end{array} \right).$$

We can get that it is the unique way to extend  $f$  as an  $\Omega$ -tridendriform algebra morphism. Hence,  $\mathbf{k}\mathfrak{T}(X, \Omega)$  and the map  $j$  is the free  $\Omega$ -tridendriform algebra generated by  $X$ .  $\square$

**3.2. Commutative  $\Omega$ -tridendriform algebras on typed words.** Let us first recall the concept of associative matching algebras [12].

**Definition 3.3.** An **associative matching algebra** is a tuple  $(A, (\star_\omega)_{\omega \in \Omega})$ , where  $A$  is a vector space and for each  $\omega \in \Omega$ ,  $\star_\omega : A \otimes A \rightarrow A$  is a linear map such that

$$(a \star_\alpha b) \star_\beta c = a \star_\alpha (b \star_\beta c)$$

for all  $a, b, c \in A$  and  $\alpha, \beta \in \Omega$ .

As in [5], the space of  $\Omega$ -typed words in  $A$  is

$$\text{Sh}_\Omega^+(A) = \bigoplus_{n \geq 1} \underbrace{A \otimes (\mathbf{k}\Omega) \otimes \cdots \otimes (\mathbf{k}\Omega) \otimes A}_{n\text{'s } A \text{ and } (n-1)\text{'s } (\mathbf{k}\Omega)}.$$

For ease of statement, we shall write each pure tensor  $\mathbf{v} = v_0 \otimes \omega_1 \otimes \cdots \otimes \omega_n \otimes v_n \in \Omega$  under the form

$$\mathbf{v} = v_0 \otimes_{\omega_1} v_1 \otimes_{\omega_2} \cdots \otimes_{\omega_n} v_n,$$

where  $n \geq 0$ ,  $\omega_1, \dots, \omega_n \in \Omega$  and  $v_0, \dots, v_n \in V$  with the convention  $\mathbf{v} = v_0$  if  $n = 0$ . We call  $\mathbf{v}$  an  **$\Omega$ -typed word** in  $V$  and define its **length**  $\ell(\mathbf{v}) := n + 1$ .

Let  $\Omega$  be a set with six products  $\leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *$ . For  $\omega \in \Omega$ , we define products  $\prec_\omega, \succ_\omega, \circ_\omega$  on  $\text{Sh}_\Omega^+(A)$  recursively in the following way:

For  $\mathbf{a}, \mathbf{b} \in \text{Sh}_\Omega^+(A)$ , if  $\ell(\mathbf{a}) + \ell(\mathbf{b}) = 2$ , then

$$\ell(\mathbf{a}) = \ell(\mathbf{b}) = 1, \mathbf{a} = a \text{ and } \mathbf{b} = b \text{ where } a, b \in A.$$

Define

$$\mathbf{a} \prec_\omega \mathbf{b} := a \otimes_\omega b, \mathbf{a} \circ_\omega \mathbf{b} = a \star_\omega b \text{ and } \mathbf{a} \succ_\omega \mathbf{b} = b \otimes_\omega a.$$

For the induction step of  $\ell(\mathbf{a}) + \ell(\mathbf{b}) > 1$ , to define  $\mathbf{a} \prec_\omega \mathbf{b}$ , we consider the following two cases. If  $\ell(\mathbf{a}) = 1$  and  $\mathbf{a} = a_1 \in A$ , then define

$$\mathbf{a} \prec_\omega \mathbf{b} := a_1 \otimes_\omega \mathbf{b}.$$

Otherwise  $\ell(\mathbf{a}) \geq 2$  and write  $\mathbf{a} = a_1 \otimes_{\alpha_1} \mathbf{a}'$  where  $a_1 \in A$ , then define

$$\mathbf{a} \prec_\omega \mathbf{b} := a_1 \otimes_{\alpha_1 \rightarrow \omega} (\mathbf{a}' \succ_{\alpha_1 \triangleright \omega} \mathbf{b}) + a_1 \otimes_{\alpha_1 \leftarrow \omega} (\mathbf{a}' \prec_{\alpha_1 \triangleleft \omega} \mathbf{b}) + a_1 \otimes_{\alpha_1 \cdot \omega} (\mathbf{a}' \circ_{\alpha_1 * \omega} \mathbf{b}).$$

To define  $\mathbf{a} \succ_\omega \mathbf{b}$ , we consider the following two cases.

If  $\ell(\mathbf{b}) = 1$  and  $\mathbf{b} = b_1 \in A$ , then define

$$\mathbf{a} \succ_\omega \mathbf{b} := b_1 \otimes_\omega \mathbf{a}.$$

Otherwise  $\ell(\mathbf{b}) \geq 2$  and write  $\mathbf{b} = b_1 \otimes_{\beta_1} \mathbf{b}'$  where  $b_1 \in A$ , then define

$$\mathbf{a} \succ_\omega \mathbf{b} := b_1 \otimes_{\omega \rightarrow \beta_1} (\mathbf{a} \succ_{\omega \triangleright \beta_1} \mathbf{b}') + b_1 \otimes_{\omega \leftarrow \beta_1} (\mathbf{a} \prec_{\omega \triangleleft \beta_1} \mathbf{b}') + b_1 \otimes_{\omega \cdot \beta_1} (\mathbf{a} \circ_{\omega * \beta_1} \mathbf{b}').$$

To define  $\mathbf{a} \circ_\omega \mathbf{b}$ , we consider the following three cases.

If  $\ell(\mathbf{a}) = 1$  and  $\ell(\mathbf{b}) \geq 2$ , write  $\mathbf{a} = a_1$  and  $\mathbf{b} = b_1 \otimes_{\beta_1} \mathbf{b}'$  where  $a_1, b_1 \in A$ , then define

$$\mathbf{a} \circ_\omega \mathbf{b} := (a_1 \star_\omega b_1) \otimes_{\beta_1} \mathbf{b}'.$$

Otherwise, if  $\ell(\mathbf{a}) \geq 2$  and  $\ell(\mathbf{b}) = 1$ , write  $\mathbf{a} = a_1 \otimes_{\alpha_1} \mathbf{a}'$  and  $\mathbf{b} = b_1$  where  $a_1, b_1 \in A$ , then define

$$\mathbf{a} \circ_{\omega} \mathbf{b} := (a_1 \star_{\omega} b_1) \otimes_{\alpha_1} \mathbf{a}'.$$

Otherwise,  $\mathbf{a} \geq 2$  and  $\mathbf{b} \geq 2$ , write  $\mathbf{a} = a_1 \otimes_{\alpha_1} \mathbf{a}'$  and  $\mathbf{b} = b_1 \otimes_{\beta_1} \mathbf{b}'$  where  $a_1, b_1 \in A$ , then define

$$\begin{aligned} \mathbf{a} \circ_{\omega} \mathbf{b} := & (a_1 \star_{\omega} b_1) \otimes_{\alpha_1 \rightarrow \beta_1} (\mathbf{a}' \succ_{\alpha_1 \triangleright \beta_1} \mathbf{b}') + (a_1 \star_{\omega} b_1) \otimes_{\alpha_1 \leftarrow \beta_1} (\mathbf{a}' \prec_{\alpha_1 \triangleleft \beta_1} \mathbf{b}') \\ & + (a_1 \star_{\omega} b_1) \otimes_{\alpha_1 * \beta_1} (\mathbf{a}' \circ_{\alpha_1 * \beta_1} \mathbf{b}'). \end{aligned}$$

We obtain the following result:

**Theorem 3.4.** *With these products defined as above, if  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *)$  is an ETS, then  $\text{Sh}_{\Omega}^+(A)$  is an  $\Omega$ -tridendriform algebra. Moreover, if  $\Omega$  is commutative and if  $(A, (\star_{\omega})_{\omega \in \Omega})$  is commutative, that is to say: for any  $a, b \in A$ , for any  $\omega \in \Omega$ ,  $a \star_{\omega} b = b \star_{\omega} a$ , then  $\text{Sh}_{\Omega}^+(A)$  is a commutative  $\Omega$ -tridendriform algebra.*

*Proof.* Denote by  $1$  the empty  $A$ -typed word and  $\text{Sh}_{\Omega}(A) := \mathbf{k}1 \oplus \text{Sh}_{\Omega}^+(A)$ . Extending the products  $\prec_{\omega}, \succ_{\omega}, \circ_{\omega}$  to the space  $\text{Sh}_{\Omega}(A) \otimes \text{Sh}_{\Omega}^+(A) \oplus \text{Sh}_{\Omega}^+(A) \otimes \text{Sh}_{\Omega}(A)$  by

$$1 \succ_{\omega} \mathbf{a} := \mathbf{a} \prec_{\omega} 1 := \mathbf{a}, \quad 1 \prec_{\omega} \mathbf{a} := \mathbf{a} \succ_{\omega} 1 := 0 \quad \text{and} \quad 1 \circ_{\omega} \mathbf{a} := \mathbf{a} \circ_{\omega} 1 := 0,$$

for all  $\omega \in \Omega$  and  $\mathbf{a} \in \text{Sh}_{\Omega}^+(A)$ . The products  $\prec_{\omega}, \succ_{\omega}, \circ_{\omega}$  can now be rewritten in the following way: for  $\mathbf{a} = a_1 \otimes_{\alpha_1} \mathbf{a}'$ ,  $\mathbf{b} = b_1 \otimes_{\beta_1} \mathbf{b}' \in \text{Sh}_{\Omega}^+(A)$ . By convention, we add an element  $\emptyset$  to  $\Omega$ , as a unit for the six products of  $\Omega$ . Note that  $\mathbf{a} = a_1$  and  $\alpha_1 = \emptyset$  if  $\ell(\mathbf{a}) = 1$ ;  $\mathbf{b} = b_1$  and  $\beta_1 = \emptyset$  if  $\ell(\mathbf{b}) = 1$ .

$$\begin{aligned} \mathbf{a} \prec_{\omega} \mathbf{b} := & a_1 \otimes_{\alpha_1 \rightarrow \omega} (\mathbf{a}' \succ_{\alpha_1 \triangleright \omega} \mathbf{b}') + a_1 \otimes_{\alpha_1 \leftarrow \omega} (\mathbf{a}' \prec_{\alpha_1 \triangleleft \omega} \mathbf{b}') + a_1 \otimes_{\alpha_1 \cdot \omega} (\mathbf{a}' \circ_{\alpha_1 * \omega} \mathbf{b}'), \\ \mathbf{a} \succ_{\omega} \mathbf{b} := & b_1 \otimes_{\omega \rightarrow \beta_1} (\mathbf{a}' \succ_{\omega \triangleright \beta_1} \mathbf{b}') + b_1 \otimes_{\omega \leftarrow \beta_1} (\mathbf{a}' \prec_{\omega \triangleleft \beta_1} \mathbf{b}') + b_1 \otimes_{\omega * \beta_1} (\mathbf{a}' \circ_{\omega * \beta_1} \mathbf{b}'), \\ \mathbf{a} \circ_{\omega} \mathbf{b} := & (a_1 \star_{\omega} b_1) \otimes_{\alpha_1 \rightarrow \beta_1} (\mathbf{a}' \succ_{\alpha_1 \triangleright \beta_1} \mathbf{b}') + (a_1 \star_{\omega} b_1) \otimes_{\alpha_1 \leftarrow \beta_1} (\mathbf{a}' \prec_{\alpha_1 \triangleleft \beta_1} \mathbf{b}') \\ & + (a_1 \star_{\omega} b_1) \otimes_{\alpha_1 * \beta_1} (\mathbf{a}' \circ_{\alpha_1 * \beta_1} \mathbf{b}'). \end{aligned}$$

Now we show that  $\text{Sh}_{\Omega}^+(A)$  is an  $\Omega$ -tridendriform algebra. For  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \text{Sh}_{\Omega}^+(A)$ , we prove Eqs. (29)-(35) hold by induction on the sum  $\ell(\mathbf{a}) + \ell(\mathbf{b}) + \ell(\mathbf{c})$ . If  $\ell(\mathbf{a}) + \ell(\mathbf{b}) + \ell(\mathbf{c}) = 3$ , then  $\ell(\mathbf{a}) = \ell(\mathbf{b}) = \ell(\mathbf{c}) = 1$  and  $\mathbf{a} = a_1, \mathbf{b} = b_1, \mathbf{c} = c_1 \in A$ , Eqs. (29)-(35) hold by direct calculation.

For the induction step of  $\ell(\mathbf{a}) + \ell(\mathbf{b}) + \ell(\mathbf{c}) \geq 4$ , assume  $\mathbf{a} = a_1 \otimes_{\alpha_1} \mathbf{a}'$ , then

$$\begin{aligned} (\mathbf{a} \prec_{\alpha} \mathbf{b}) \prec_{\beta} \mathbf{c} &= ((a_1 \otimes_{\alpha_1} \mathbf{a}') \prec_{\alpha} \mathbf{b}) \prec_{\beta} \mathbf{c} \\ &= (a_1 \otimes_{\alpha_1 \rightarrow \alpha} (\mathbf{a}' \succ_{\alpha_1 \triangleright \alpha} \mathbf{b}') + a_1 \otimes_{\alpha_1 \leftarrow \alpha} (\mathbf{a}' \prec_{\alpha_1 \triangleleft \alpha} \mathbf{b}') + a_1 \otimes_{\alpha_1 \cdot \alpha} (\mathbf{a}' \circ_{\alpha_1 * \alpha} \mathbf{b}')) \prec_{\beta} \mathbf{c} \\ &= a_1 \otimes_{(\alpha_1 \rightarrow \alpha) \rightarrow \beta} ((\mathbf{a}' \succ_{\alpha_1 \triangleright \alpha} \mathbf{b}') \succ_{(\alpha_1 \rightarrow \alpha) \triangleright \beta} \mathbf{c}') + a_1 \otimes_{(\alpha_1 \rightarrow \alpha) \leftarrow \beta} ((\mathbf{a}' \succ_{\alpha_1 \triangleright \alpha} \mathbf{b}') \prec_{(\alpha_1 \rightarrow \alpha) \triangleleft \beta} \mathbf{c}') \\ &\quad + a_1 \otimes_{(\alpha_1 \rightarrow \alpha) * \beta} ((\mathbf{a}' \succ_{\alpha_1 \triangleright \alpha} \mathbf{b}') \circ_{(\alpha_1 \rightarrow \alpha) * \beta} \mathbf{c}') + a_1 \otimes_{(\alpha_1 \leftarrow \alpha) \rightarrow \beta} ((\mathbf{a}' \prec_{\alpha_1 \triangleleft \alpha} \mathbf{b}') \succ_{(\alpha_1 \leftarrow \alpha) \triangleright \beta} \mathbf{c}') \\ &\quad + a_1 \otimes_{(\alpha_1 \leftarrow \alpha) \leftarrow \beta} ((\mathbf{a}' \prec_{\alpha_1 \triangleleft \alpha} \mathbf{b}') \prec_{(\alpha_1 \leftarrow \alpha) \triangleleft \beta} \mathbf{c}') + a_1 \otimes_{(\alpha_1 \leftarrow \alpha) * \beta} ((\mathbf{a}' \prec_{\alpha_1 \triangleleft \alpha} \mathbf{b}') \circ_{(\alpha_1 \leftarrow \alpha) * \beta} \mathbf{c}') \\ &\quad + a_1 \otimes_{(\alpha_1 \cdot \alpha) \rightarrow \beta} ((\mathbf{a}' \circ_{\alpha_1 * \alpha} \mathbf{b}') \succ_{(\alpha_1 \cdot \alpha) \triangleright \beta} \mathbf{c}') + a_1 \otimes_{(\alpha_1 \cdot \alpha) \leftarrow \beta} ((\mathbf{a}' \circ_{\alpha_1 * \alpha} \mathbf{b}') \prec_{(\alpha_1 \cdot \alpha) \triangleleft \beta} \mathbf{c}') \\ &\quad + a_1 \otimes_{(\alpha_1 \cdot \alpha) * \beta} ((\mathbf{a}' \circ_{\alpha_1 * \alpha} \mathbf{b}') \circ_{(\alpha_1 \cdot \alpha) * \beta} \mathbf{c}') \\ &= a_1 \otimes_{\alpha_1 \rightarrow (\alpha \rightarrow \beta)} (\mathbf{a}' \succ_{\alpha_1 \triangleright (\alpha \rightarrow \beta)} (\mathbf{b}' \succ_{\alpha \triangleright \beta} \mathbf{c}')) + a_1 \otimes_{\alpha_1 \leftarrow (\alpha \rightarrow \beta)} (\mathbf{a}' \prec_{\alpha_1 \triangleleft (\alpha \rightarrow \beta)} (\mathbf{b}' \succ_{\alpha \triangleright \beta} \mathbf{c}')) \\ &\quad + a_1 \otimes_{\alpha_1 \cdot (\alpha \rightarrow \beta)} (\mathbf{a}' \circ_{\alpha_1 * (\alpha \rightarrow \beta)} (\mathbf{b}' \succ_{\alpha \triangleright \beta} \mathbf{c}')) + a_1 \otimes_{\alpha_1 \rightarrow (\alpha \leftarrow \beta)} (\mathbf{a}' \succ_{\alpha_1 \triangleright (\alpha \leftarrow \beta)} (\mathbf{b}' \prec_{\alpha \triangleleft \beta} \mathbf{c}')) \\ &\quad + a_1 \otimes_{\alpha_1 \leftarrow (\alpha \leftarrow \beta)} (\mathbf{a}' \prec_{\alpha_1 \triangleleft (\alpha \leftarrow \beta)} (\mathbf{b}' \prec_{\alpha \triangleleft \beta} \mathbf{c}')) + a_1 \otimes_{\alpha_1 \cdot (\alpha \leftarrow \beta)} (\mathbf{a}' \circ_{\alpha_1 * (\alpha \leftarrow \beta)} (\mathbf{b}' \prec_{\alpha \triangleleft \beta} \mathbf{c}')) \\ &\quad + a_1 \otimes_{\alpha_1 \rightarrow (\alpha * \beta)} (\mathbf{a}' \succ_{\alpha_1 \triangleright (\alpha * \beta)} (\mathbf{b}' \circ_{\alpha * \beta} \mathbf{c}')) + a_1 \otimes_{\alpha_1 \leftarrow (\alpha * \beta)} (\mathbf{a}' \prec_{\alpha_1 \triangleleft (\alpha * \beta)} (\mathbf{b}' \circ_{\alpha * \beta} \mathbf{c}')) \\ &\quad + a_1 \otimes_{\alpha_1 \cdot (\alpha * \beta)} (\mathbf{a}' \circ_{\alpha_1 * (\alpha * \beta)} (\mathbf{b}' \circ_{\alpha * \beta} \mathbf{c}')) \quad (\text{by induction hypothesis and } \Omega \text{ being an ETS}) \\ &= \mathbf{a} \prec_{\alpha \rightarrow \beta} (\mathbf{b}' \succ_{\alpha \rightarrow \beta} \mathbf{c}') + \mathbf{a} \prec_{\alpha \leftarrow \beta} (\mathbf{b}' \prec_{\alpha \triangleleft \beta} \mathbf{c}') + \mathbf{a} \prec_{\alpha * \beta} (\mathbf{b}' \circ_{\alpha * \beta} \mathbf{c}'). \end{aligned}$$



Hence Eq. (29) holds. Similarly, it can be proved that Eqs. (30)-(35) hold. So  $\text{Sh}_\Omega^+(A)$  is an  $\Omega$ -tridendriform algebra.

Next, assume  $\Omega$  is commutative and  $(A, (\star_\omega)_{\omega \in \Omega})$  is commutative, we prove that

$$\mathbf{a} <_\alpha \mathbf{b} = \mathbf{b} >_\alpha \mathbf{a} \quad \text{and} \quad \mathbf{a} \circ_\alpha \mathbf{b} = \mathbf{b} \circ_\alpha \mathbf{a}$$

for all  $\mathbf{a}, \mathbf{b} \in \text{Sh}_\Omega^+(A)$  and  $\alpha \in \Omega$  by induction on  $\ell(\mathbf{a}) + \ell(\mathbf{b})$ . If  $\ell(\mathbf{a}) + \ell(\mathbf{b}) = 2$ , then  $\ell(\mathbf{a}) = \ell(\mathbf{b}) = 1$  and  $\mathbf{a} = a, \mathbf{b} = b \in A$ . So

$$\mathbf{a} <_\alpha \mathbf{b} = a \otimes_\alpha b = \mathbf{b} >_\alpha \mathbf{a} \quad \text{and} \quad \mathbf{a} \circ_\alpha \mathbf{b} = a \star_\alpha b = b \star_\alpha a = \mathbf{b} \circ_\alpha \mathbf{a}.$$

For the inductive step of  $\ell(\mathbf{a}) + \ell(\mathbf{b}) \geq 3$ , assume  $\mathbf{a} = a_1 \otimes_{\alpha_1} \mathbf{a}'$ , then

$$\begin{aligned} \mathbf{a} <_\alpha \mathbf{b} &= a_1 \otimes_{\alpha_1 \rightarrow \alpha} (\mathbf{a}' >_{\alpha_1 \triangleright \alpha} \mathbf{b}) + a_1 \otimes_{\alpha_1 \leftarrow \alpha} (\mathbf{a}' <_{\alpha_1 \triangleleft \alpha} \mathbf{b}) + a_1 \otimes_{\alpha_1 \cdot \alpha} (\mathbf{a}' \circ_{\alpha_1 * \alpha} \mathbf{b}) \\ &= a_1 \otimes_{\alpha \leftarrow \alpha_1} (\mathbf{b} <_{\alpha \triangleleft \alpha_1} \mathbf{a}') + a_1 \otimes_{\alpha \rightarrow \alpha_1} (\mathbf{b} >_{\alpha \triangleright \alpha_1} \mathbf{a}') + a_1 \otimes_{\alpha \cdot \alpha_1} (\mathbf{b} \circ_{\alpha * \alpha_1} \mathbf{a}') \\ &\quad \text{(by induction hypothesis and } \Omega \text{ being commutative)} \\ &= \mathbf{b} >_\alpha \mathbf{a}. \end{aligned}$$

Similarly, it can be proved that  $\mathbf{a} \circ_\alpha \mathbf{b} = \mathbf{b} \circ_\alpha \mathbf{a}$ . Hence  $\text{Sh}_\Omega^+(A)$  is a commutative  $\Omega$ -tridendriform algebra.  $\square$

Let  $i : A \rightarrow \text{Sh}_\Omega^+(A)$  be the natural inclusion. We obtain the following result:

**Theorem 3.5.** *Let  $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright, \cdot, *)$  be a commutative ETS and let  $(A, (\star_\omega)_{\omega \in \Omega})$  be a commutative matching associative algebra. If  $(B, (<_\omega)_{\omega \in \Omega}, (>_\omega)_{\omega \in \Omega}, (\circ_\omega)_{\omega \in \Omega})$  is a commutative  $\Omega$ -tridendriform algebra and  $\phi : (A, (\star_\omega)_{\omega \in \Omega}) \rightarrow (B, (\circ_\omega)_{\omega \in \Omega})$  is a morphism of matching associative algebras, then there exists a unique morphism  $\Phi : \text{Sh}_\Omega^+(A) \rightarrow B$  of  $\Omega$ -tridendriform algebras such that  $\phi = \Phi \circ i$ .*

In other terms,  $\text{Sh}_\Omega^+$  is the left adjoint functor of the forgetful functor from commutative  $\Omega$ -tridendriform algebras to commutative  $\Omega$ -associative algebras (which consists to forget  $<$  and  $>$ ). As a consequence, the free commutative  $\Omega$ -tridendriform algebra generated by  $A$  is  $\text{Sh}_\Omega^+(A')$  where  $A'$  is the free matching commutative algebra generated by  $A$ .

*Proof.* For  $\mathbf{a} \in \text{Sh}_\Omega^+(A)$ , we define  $\Phi(\mathbf{a})$  by induction on  $\ell(\mathbf{a})$ . If  $\ell(\mathbf{a}) = 1$ , then  $\mathbf{a} = a_1 \in A$  and define  $\Phi(\mathbf{a}) = \phi(a_1)$ . For the induction step of  $\ell(\mathbf{a}) \geq 2$ , suppose  $\mathbf{a} = a_1 \otimes_{\alpha_1} \mathbf{a}'$ , then define

$$\Phi(\mathbf{a}) = \Phi(a_1 \otimes_{\alpha_1} \mathbf{a}') = \Phi(a_1 <_{\alpha_1} \mathbf{a}') := \Phi(a_1) <_{\alpha_1} \Phi(\mathbf{a}').$$

We can get that it is the unique way to extend  $\phi$  to an  $\Omega$ -tridendriform algebra morphism  $\Phi$  such that  $\phi = \Phi \circ i$ .  $\square$

**3.3. From  $\Omega$ -tridendriform algebras to tridendriform algebras.** If  $\Omega$  is an ETS, we consider the three maps

$$\begin{aligned} \varphi_{\leftarrow} &: \begin{cases} \mathbf{k}\Omega^{\otimes 2} & \longrightarrow & \mathbf{k}\Omega^{\otimes 2} \\ \alpha \otimes \beta & \longmapsto & \alpha \leftarrow \beta \otimes \alpha \triangleleft \beta, \end{cases} & \varphi_{\rightarrow} &: \begin{cases} \mathbf{k}\Omega^{\otimes 2} & \longrightarrow & \mathbf{k}\Omega^{\otimes 2} \\ \alpha \otimes \beta & \longmapsto & \alpha \rightarrow \beta \otimes \alpha \triangleright \beta, \end{cases} \\ \varphi_{*} &: \begin{cases} \mathbf{k}\Omega^{\otimes 2} & \longrightarrow & \mathbf{k}\Omega^{\otimes 2} \\ \alpha \otimes \beta & \longmapsto & \alpha \cdot \beta \otimes \alpha * \beta. \end{cases} \end{aligned}$$

**Proposition 3.6.** *Let  $\Omega$  be an ETS and let  $A$  be a vector space equipped with bilinear products  $<_\omega, >_\omega, \circ_\omega$ . We equip  $\mathbf{k}\Omega \otimes A$  with three bilinear products  $<, >, \circ$  defined by*

$$\alpha \otimes x < \beta \otimes y := \alpha \leftarrow \beta \otimes x <_{\alpha \triangleleft \beta} y,$$

$$\begin{aligned}\alpha \otimes x > \beta \otimes y &:= \alpha \rightarrow \beta \otimes x >_{\alpha \triangleright \beta} y, \\ \alpha \otimes x \circ \beta \otimes y &:= \alpha \cdot \beta \otimes x \circ_{\alpha * \beta} y,\end{aligned}$$

for all  $\alpha, \beta \in \Omega$  and  $x, y \in A$ .

- (a) If  $(A, (<_{\omega})_{\omega \in \Omega}, (>_{\omega})_{\omega \in \Omega}, (\circ_{\omega})_{\omega \in \Omega})$  is an  $\Omega$ -tridendriform algebra, then  $(\mathbf{k}\Omega \otimes A, <, >, \circ)$  is a tridendriform algebra.  
(b) If  $\varphi_{\leftarrow}, \varphi_{\rightarrow}$  and  $\varphi_*$  are surjective, then the converse implication is true.

*Proof.* (a) For  $\alpha, \beta, \gamma \in \Omega$  and  $a, b, c \in A$ ,

$$\begin{aligned}(\alpha \otimes a < \beta \otimes b) < \gamma \otimes c &= (\alpha \leftarrow \beta \otimes a <_{\alpha \triangleleft \beta} b) < \gamma \otimes c \\ &= (\alpha \leftarrow \beta) \leftarrow \gamma \otimes (a <_{\alpha \triangleleft \beta} b) <_{(\alpha \leftarrow \beta) \triangleleft \gamma} c \\ &= (\alpha \leftarrow \beta) \leftarrow \gamma \otimes \left( \begin{array}{l} a <_{(\alpha \triangleleft \beta) \rightarrow ((\alpha \leftarrow \beta) \triangleleft \gamma)} (b >_{(\alpha \triangleleft \beta) \triangleright ((\alpha \leftarrow \beta) \triangleleft \gamma)} c) \\ + a <_{(\alpha \triangleleft \beta) \leftarrow ((\alpha \leftarrow \beta) \triangleleft \gamma)} (b <_{(\alpha \triangleleft \beta) \triangleleft ((\alpha \leftarrow \beta) \triangleleft \gamma)} c) \\ + a <_{(\alpha \triangleleft \beta) \cdot ((\alpha \leftarrow \beta) \triangleleft \gamma)} (b \circ_{(\alpha \triangleleft \beta) * ((\alpha \leftarrow \beta) \triangleleft \gamma)} c) \end{array} \right) \\ &\quad \text{(by } A \text{ being an } \Omega\text{-tridendriform algebra)} \\ &= \alpha \leftarrow (\beta \leftarrow \gamma) \otimes a <_{\alpha \triangleleft (\beta \leftarrow \gamma)} (b \leftarrow_{\beta \triangleleft \gamma} c) + \alpha \leftarrow (\beta \rightarrow \gamma) \otimes a <_{\alpha \triangleleft (\beta \rightarrow \gamma)} (b >_{\beta \triangleright \gamma} c) \\ &\quad + \alpha \leftarrow (\beta \cdot \gamma) \otimes a <_{\alpha \triangleleft (\beta \cdot \gamma)} (b \circ_{\beta * \gamma} c) \quad \text{(by } \Omega \text{ being an ETS)} \\ &= \alpha \otimes a < (\beta \leftarrow \gamma \otimes b <_{\beta \triangleleft \gamma} c + \beta \rightarrow \gamma \otimes b >_{\beta \triangleright \gamma} c + \beta \cdot \gamma \otimes b \circ_{\beta * \gamma} c) \\ &= \alpha \otimes a < (\beta \otimes b < \gamma \otimes c + \beta \otimes b > \gamma \otimes c + \beta \otimes b \circ \gamma \otimes c).\end{aligned}$$

The other equations can be proved in the same way. Hence  $(\mathbf{k}\Omega \otimes A, <, >, \circ)$  is a tridendriform algebra.

- (b) For  $\alpha, \beta, \gamma \in \Omega$  and  $a, b, c \in A$ , by  $\Omega$  being an ETS and

$$(\alpha \otimes a < \beta \otimes b) < \gamma \otimes c = \alpha \otimes a < (\beta \otimes b < \gamma \otimes c + \beta \otimes b > \gamma \otimes c + \beta \otimes b \circ \gamma \otimes c),$$

we get

$$(a <_{\alpha \triangleleft \beta} b) <_{(\alpha \leftarrow \beta) \triangleleft \gamma} c = a <_{\alpha \triangleleft (\beta \leftarrow \gamma)} (b \leftarrow_{\beta \triangleleft \gamma} c) + a <_{\alpha \triangleleft (\beta \rightarrow \gamma)} (b >_{\beta \triangleright \gamma} c) + a <_{\alpha \triangleleft (\beta \cdot \gamma)} (b \circ_{\beta * \gamma} c)$$

By hypothesis, the following map is surjective:

$$\phi'_{\leftarrow} : \begin{cases} \Omega^2 & \longrightarrow \Omega^2 \\ (\alpha, \beta) & \longrightarrow (\alpha \triangleleft \beta, \alpha \leftarrow \beta). \end{cases}$$

Hence, the following composition is surjective:

$$(\text{Id} \otimes \phi'_{\leftarrow}) \circ (\phi'_{\leftarrow} \otimes \text{Id}) : \begin{cases} \Omega^3 & \longrightarrow \Omega^3 \\ (\alpha, \beta, \gamma) & \longrightarrow (\alpha \triangleleft \beta, (\alpha \leftarrow \beta) \triangleleft \gamma, (\alpha \leftarrow \beta) \leftarrow \gamma) \end{cases}$$

Let  $(\alpha, \beta, \gamma), (\alpha', \beta', \gamma') \in \Omega^3$  such that

$$\alpha' = \alpha \triangleleft \beta, \quad \beta' = (\alpha \leftarrow \beta) \triangleleft \gamma, \quad \gamma' = (\alpha \leftarrow \beta) \leftarrow \gamma.$$

Then

$$\begin{aligned}(a <_{\alpha'} b) <_{\beta'} c &= a <_{\alpha \triangleleft (\beta \leftarrow \gamma)} (b \leftarrow_{\beta \triangleleft \gamma} c) + a <_{\alpha \triangleleft (\beta \rightarrow \gamma)} (b >_{\beta \triangleright \gamma} c) + a <_{\alpha \triangleleft (\beta \cdot \gamma)} (b \circ_{\beta * \gamma} c) \\ &= a <_{\alpha' \leftarrow \beta'} (b <_{\alpha' \triangleleft \beta'} c) + a <_{\alpha' \rightarrow \beta'} (b >_{\alpha' \triangleright \beta'} c) + a <_{\alpha' \cdot \beta'} (b \circ_{\alpha' * \beta'} c).\end{aligned}$$

So Eq. (29) holds. Eqs. (30)-(35) can be proved similarly.  $\square$

**Proposition 3.7.** *Let  $\Omega$  be an ETS.*

(1) *The following assertions are equivalent:*

- (a) *The tridendriform algebra  $\mathbf{k}\Omega \otimes \mathbf{k}\mathfrak{T}(X, \Omega)$  is generated by the elements  $\omega \otimes \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad x \\ \diagdown \quad \diagup \end{array}$ , where  $\omega \in \Omega$  and  $x \in X$ .*  
 (b) *The maps  $\varphi_{\leftarrow}$ ,  $\varphi_{\rightarrow}$  and  $\varphi_*$  are surjective.*

(2) *The following assertions are equivalent:*

- (a) *The tridendriform subalgebra of  $\mathbf{k}\Omega \otimes \mathbf{k}\mathfrak{T}(X, \Omega)$  generated by the elements  $\omega \otimes \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad x \\ \diagdown \quad \diagup \end{array}$ , where  $\omega \in \Omega$  and  $x \in X$ , is free.*  
 (b) *The maps  $\varphi_{\leftarrow}$ ,  $\varphi_{\rightarrow}$  and  $\varphi_*$  are injective.*

*Proof.* Note that the tridendriform algebra  $\mathbf{k}\Omega \otimes \mathfrak{T}(X, \Omega)$  is graded, with for each  $n \geq 1$ ,

$$(\mathbf{k}\Omega \otimes \mathfrak{T}(X, \Omega))_n = \mathbf{k}\Omega \otimes \mathfrak{T}_n(X, \Omega).$$

(1) (a)  $\implies$  (b) As  $\mathbf{k}\Omega \otimes \mathfrak{T}(X, \Omega)$  is graded, by hypothesis, for any  $\alpha \otimes \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad x \\ \diagdown \quad \diagup \end{array} \in (\Omega \otimes \mathfrak{T}(X, \Omega))_2$ , where  $\alpha, \beta \in \Omega$  and  $x, y \in X$ , there are  $\alpha_1 \otimes \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad y' \\ \diagdown \quad \diagup \end{array}, \beta_1 \otimes \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad x' \\ \diagdown \quad \diagup \end{array} \in (\Omega \otimes \mathfrak{T}(X, \Omega))_2$  and  $p_{\alpha_1, \beta_1} \in \mathbf{k}$  such that

$$\alpha \otimes \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad x \\ \diagdown \quad \diagup \end{array} = \sum_{\alpha_1, \beta_1 \in \Omega} p_{\alpha_1, \beta_1} \alpha_1 \otimes \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad y' \\ \diagdown \quad \diagup \end{array} \succ \beta_1 \otimes \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad x' \\ \diagdown \quad \diagup \end{array} = \sum_{\alpha_1, \beta_1 \in \Omega} p_{\alpha_1, \beta_1} \alpha_1 \rightarrow \beta_1 \otimes \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad x' \\ \diagdown \quad \diagup \end{array}.$$

Hence, there exists  $(\alpha_1, \beta_1) \in \Omega^2$  such that  $\alpha_1 \rightarrow \beta_1 = \alpha$  and  $\alpha_1 \triangleright \beta_1 = \beta$ . So  $\varphi_{\rightarrow}$  is surjective. Similarly,  $\varphi_{\leftarrow}$  and  $\varphi_*$  are surjective.

(b)  $\implies$  (a) We prove that any  $\alpha \otimes T \in \mathbf{k}\Omega \otimes \mathbf{k}\mathfrak{T}(X, \Omega)$ , where  $\alpha \in \Omega$  and  $T \in \mathfrak{T}(X, \Omega)$ , is generated by  $\omega \otimes \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad x \\ \diagdown \quad \diagup \end{array}$  by induction on the number  $N$  of leaves of  $T$ . If  $N = 2$ , then  $T = \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad x \\ \diagdown \quad \diagup \end{array}$  for some  $x \in X$  and it is obvious. Suppose  $\alpha \otimes T$  is generated by  $\omega \otimes \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad x \\ \diagdown \quad \diagup \end{array}$  for  $N \leq p$ , where  $p \geq 2$  is a fixed integer. Consider the case of  $N = p + 1$ . We consider the form of  $T$  as follows.

(a) If  $T = \begin{array}{c} \beta T_2 \\ \diagup \quad \diagdown \\ \quad \quad x \\ \diagdown \quad \diagup \end{array}$ , let  $(\alpha_1, \beta_1) \in \Omega^2$  such that  $\varphi_{\leftarrow}(\alpha_1, \beta_1) = (\alpha_1 \leftarrow \beta_1, \alpha_1 \triangleleft \beta_1) = (\alpha, \beta)$ . Then

$$\alpha_1 \otimes \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad x \\ \diagdown \quad \diagup \end{array} \triangleleft \beta_1 \otimes T_2 = (\alpha_1 \leftarrow \beta_1) \otimes \begin{array}{c} \alpha_1 \triangleleft \beta_1 T_2 \\ \diagup \quad \diagdown \\ \quad \quad x \\ \diagdown \quad \diagup \end{array} = \alpha \otimes T.$$

Hence, by induction hypothesis,  $\alpha \otimes T$  is generated by  $\omega \otimes \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad x \\ \diagdown \quad \diagup \end{array}$ .

(b) If  $T = \begin{array}{c} \beta T_2 \quad T_m \\ \circ \quad \circ \\ \vdots \quad \vdots \\ x_1 \cdots x_m \\ \circ T_{m+1} \end{array}$  with  $m \geq 2$ , let  $(\alpha_1, \beta_1) \in \Omega^2$  such that  $\varphi_*(\alpha_1, \beta_1) = (\alpha_1 * \beta_1, \alpha_1 \cdot \beta_1) = (\beta, \alpha)$ . Then

$$\alpha_1 \otimes \begin{array}{c} \diagup \quad \diagdown \\ x \end{array} \circ \beta_1 \otimes \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \vdots \quad \vdots \\ x_1 \cdots x_m \\ \circ T_{m+1} \end{array} = \alpha_1 \cdot \beta_1 \otimes \begin{array}{c} \alpha_1 * \beta_1 T_2 \quad T_m \\ \circ \quad \circ \\ \vdots \quad \vdots \\ x_1 \cdots x_m \\ \circ T_{m+1} \end{array} = \alpha \otimes T.$$

Hence, by induction hypothesis,  $\alpha \otimes T$  is generated by  $\omega \otimes \begin{array}{c} \diagup \quad \diagdown \\ x \end{array}$ .

(c) If  $T = \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \vdots \quad \vdots \\ x_1 \cdots x_m \\ \circ T_{m+1} \end{array}$  with  $T_1 \neq |$ , let  $(\alpha_1, \beta_1) \in \Omega^2$  such that  $\varphi_{\rightarrow}(\alpha_1, \beta_1) = (\alpha_1 \rightarrow \beta_1, \alpha_1 \triangleright \beta_1) = (\alpha, \beta)$ . Then

$$\alpha_1 \otimes T_1 \triangleright \beta_1 \otimes \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \vdots \quad \vdots \\ x_1 \cdots x_m \\ \circ T_{m+1} \end{array} = (\alpha_1 \rightarrow \beta_1) \otimes T_1^{\alpha_1 \triangleright \beta_1} \begin{array}{c} T_2 \quad T_m \\ \circ \quad \circ \\ \vdots \quad \vdots \\ x_1 \cdots x_m \\ \circ T_{m+1} \end{array} = \alpha \otimes T.$$

Hence, by induction hypothesis,  $\alpha \otimes T$  is generated by  $\omega \otimes \begin{array}{c} \diagup \quad \diagdown \\ x \end{array}$ .

(2) (a)  $\implies$  (b) Denote by  $A$  the tridendriform subalgebra of  $\mathbf{k}\Omega \otimes \mathbf{k}\mathfrak{I}(X, \Omega)$  generated by elements  $\omega \otimes \begin{array}{c} \diagup \quad \diagdown \\ x \end{array}$ . Let  $(\alpha, \beta), (\alpha', \beta') \in \Omega^2$  such that  $\varphi_{\leftarrow}(\alpha, \beta) = \varphi_{\leftarrow}(\alpha', \beta')$ . Then

$$\alpha \otimes \begin{array}{c} \diagup \quad \diagdown \\ x \end{array} \triangleleft \beta \otimes \begin{array}{c} \diagup \quad \diagdown \\ y \end{array} = \alpha \leftarrow \beta \otimes \begin{array}{c} \alpha \triangleleft \beta \\ \diagup \quad \diagdown \\ x \end{array} = \alpha' \leftarrow \beta' \otimes \begin{array}{c} \alpha' \triangleleft \beta' \\ \diagup \quad \diagdown \\ x \end{array} = \alpha' \otimes \begin{array}{c} \diagup \quad \diagdown \\ x \end{array} \triangleleft \beta' \otimes \begin{array}{c} \diagup \quad \diagdown \\ y \end{array}.$$

By the freeness of  $A$ ,  $(\alpha, \beta) = (\alpha', \beta')$  and so  $\varphi_{\leftarrow}$  is injective. The maps  $\varphi_{\rightarrow}$  and  $\varphi_*$  can be proved to be injective similarly.

(b)  $\implies$  (a) Let  $\text{TDend}(\Omega)$  be the free tridendriform algebra generated by  $\Omega \otimes X$ . As a vector space, it is generated by Schröder trees which angles are decorated by  $\Omega \otimes X$ . Let  $\Phi : \text{TDend}(\Omega) \rightarrow \mathbf{k}\Omega \otimes \mathbf{k}\mathfrak{I}(X, \Omega)$  be the unique tridendriform algebra sending  $\begin{array}{c} \diagup \quad \diagdown \\ \alpha \otimes x \end{array}$  to  $\alpha \otimes \begin{array}{c} \diagup \quad \diagdown \\ x \end{array}$ . We prove that  $\Phi$  is injective, i.e.

$$\Phi(T) = \Phi(T') \implies T = T'$$

by induction on the number  $N$  of leaves of  $T$ . By the construction of  $\Phi$ , if  $\Phi(T) = \Phi(T')$ , then  $T, T'$  are of the same form. If  $N = 2$ , then  $T = \begin{array}{c} \diagup \quad \diagdown \\ \alpha \otimes x \end{array}$  for some  $\alpha \in \Omega$  and  $x \in X$  and obviously  $T' = T$ . Suppose  $\Phi$  is injective for all  $T$  with  $N \leq p$ , where  $p$  is a fixed integer. Consider the case of  $N = p + 1$ . If

$$T = \begin{array}{c} T_2 \\ \circ \\ \diagup \quad \diagdown \\ \alpha \otimes x \end{array} \quad T' = \begin{array}{c} T'_2 \\ \circ \\ \diagup \quad \diagdown \\ \alpha' \otimes x' \end{array}$$

and assume  $\Phi(T_2) = \beta \otimes U_2, \Phi(T'_2) = \beta' \otimes U'_2$ . Then

$$\begin{aligned} \Phi(T) &= \Phi\left(\begin{array}{c} \alpha \otimes \alpha' \\ \diagdown \quad \diagup \\ \text{Y} \\ \text{Y} \end{array} \prec T_2\right) = \Phi\left(\begin{array}{c} \alpha \otimes \alpha' \\ \diagdown \quad \diagup \\ \text{Y} \\ \text{Y} \end{array}\right) \prec \Phi(T_2) = \alpha \leftarrow \beta \otimes \begin{array}{c} \alpha \prec \beta T_2 \\ \diagdown \quad \diagup \\ \text{Y} \\ \text{Y} \end{array} \\ \Phi(T') &= \Phi\left(\begin{array}{c} \alpha' \otimes \alpha' \\ \diagdown \quad \diagup \\ \text{Y} \\ \text{Y} \end{array} \prec T'_2\right) = \Phi\left(\begin{array}{c} \alpha' \otimes \alpha' \\ \diagdown \quad \diagup \\ \text{Y} \\ \text{Y} \end{array}\right) \prec \Phi(T'_2) = \alpha' \leftarrow \beta' \otimes \begin{array}{c} \alpha' \prec \beta' T_2 \\ \diagdown \quad \diagup \\ \text{Y} \\ \text{Y} \end{array}. \end{aligned}$$

Since  $\Phi(T) = \Phi(T')$  and  $\varphi_{\leftarrow}$  is injective,  $\alpha = \alpha', x = x'$  and  $\Phi(T_2) = \Phi(T'_2)$ . Hence by induction hypothesis,  $T = T'$ . For other forms of  $T, T'$ , the injectivity of  $\Phi$  is proved similarly. Hence  $A$  is isomorphic to the free tridendriform algebra  $\text{TDend}(\Omega)$  and so  $A$  is free.  $\square$

#### 4. OPERAD OF $\Omega$ -TRIDENDRIFORM ALGEBRAS

Denote by  $\mathcal{P}_\Omega$  the (nonsymmetric) operad of  $\Omega$ -tridendriform algebras. It is generated by  $\prec_\alpha, \circ_\omega$  and  $\succ_\omega \in \mathcal{P}_\Omega(2)$  with  $\alpha \in \Omega$  and the relations:

$$\begin{aligned} \prec_\beta \circ (\prec_\alpha, I) &= \prec_{\alpha \rightarrow \beta} \circ (I, \succ_{\alpha \triangleright \beta}) + \prec_{\alpha \leftarrow \beta} \circ (I, \prec_{\alpha \triangleleft \beta}) + \prec_{\alpha \cdot \beta} \circ (I, \circ_{\alpha * \beta}), \\ \prec_\beta \circ (\succ_\alpha, I) &= \succ_\alpha \circ (I, \prec_\beta), \\ \succ_\alpha \circ (I, \succ_\beta) &= \succ_{\alpha \rightarrow \beta} \circ (\succ_{\alpha \triangleright \beta}, I) + \succ_{\alpha \leftarrow \beta} \circ (\prec_{\alpha \triangleleft \beta}, I) + \succ_{\alpha \cdot \beta} \circ (\circ_{\alpha * \beta}, I), \\ \circ_\beta \circ (\succ_\alpha, I) &= \succ_\alpha \circ (I, \circ_\beta), \\ \circ_\beta \circ (\prec_\alpha, I) &= \circ_\beta \circ (I, \succ_\alpha), \\ \prec_\beta \circ (\circ_\alpha, I) &= \circ_\alpha \circ (I, \prec_\beta), \\ \circ_\beta \circ (\circ_\alpha, I) &= \circ_\alpha \circ (I, \circ_\beta), \end{aligned}$$

for all  $\alpha, \beta \in \Omega$ .

As in [3, Proposition 21], we obtain the following result:

**Proposition 4.1.** *Suppose  $m \in \mathcal{P}_\Omega(2)$  is of the form*

$$m = \sum_{\alpha \in \Omega} a_\alpha \prec_\alpha + \sum_{\alpha \in \Omega} b_\alpha \circ_\alpha + \sum_{\alpha \in \Omega} c_\alpha \succ_\alpha,$$

where  $a_\alpha, b_\alpha, c_\alpha \in \mathbf{k}$ . Then  $m \circ (I, m) = m \circ (m, I)$  if and only if for any  $\alpha, \beta \in \Omega$ ,

$$\begin{aligned} a_\alpha a_\beta &= \sum_{\varphi_{\leftarrow}(\alpha', \beta') = (\alpha, \beta)} a_{\alpha'} a_{\beta'}, & a_\alpha b_\beta &= \sum_{\varphi_{*}(\alpha', \beta') = (\alpha, \beta)} a_{\alpha'} a_{\beta'}, \\ a_\alpha c_\beta &= \sum_{\varphi_{\rightarrow}(\alpha', \beta') = (\alpha, \beta)} a_{\alpha'} a_{\beta'}, & b_\alpha c_\beta &= b_\alpha a_\beta, \\ c_\alpha a_\beta &= \sum_{\varphi_{\leftarrow}(\alpha', \beta') = (\alpha, \beta)} c_{\alpha'} c_{\beta'}, & c_\alpha b_\beta &= \sum_{\varphi_{*}(\alpha', \beta') = (\alpha, \beta)} c_{\alpha'} c_{\beta'}, \\ c_\alpha c_\beta &= \sum_{\varphi_{\rightarrow}(\alpha', \beta') = (\alpha, \beta)} c_{\alpha'} c_{\beta'}. \end{aligned}$$

*Proof.* By the relations of the operad of  $\Omega$ -tridendriform algebras,

$$m \circ (I, m) = (a_\alpha \prec_\alpha + b_\alpha \circ_\alpha + c_\alpha \succ_\alpha) \circ (I, a_\beta \prec_\beta + b_\beta \circ_\beta + c_\beta \succ_\beta)$$

$$\begin{aligned}
&= \sum_{\alpha,\beta} a_\alpha a_\beta \prec_\alpha \circ(I, \prec_\beta) + \sum_{\alpha,\beta} a_\alpha b_\beta \prec_\alpha \circ(I, \circ_\beta) + \sum_{\alpha,\beta} \prec_\alpha \circ(I, \succ_\beta) + \sum_{\alpha,\beta} b_\alpha a_\beta \circ_\alpha \circ(I, \prec_\beta) \\
&+ \sum_{\alpha,\beta} b_\alpha b_\beta \circ_\alpha \circ(I, \circ_\beta) + \sum_{\alpha,\beta} b_\alpha c_\beta \circ_\alpha \circ(I, \succ_\beta) + \sum_{\alpha,\beta} c_\alpha a_\beta \succ_\alpha \circ(I, \prec_\beta) + \sum_{\alpha,\beta} c_\alpha b_\beta \succ_\alpha \circ(I, \circ_\beta) \\
&+ \sum_{\alpha,\beta} c_\alpha c_\beta \succ_{\alpha \rightarrow \beta} \circ(\succ_{\alpha \triangleright \beta}, I) + \sum_{\alpha,\beta} c_\alpha c_\beta \succ_{\alpha \leftarrow \beta} (\prec_{\alpha \triangleleft \beta}, I) + \sum_{\alpha,\beta} c_\alpha c_\beta \succ_{\alpha \cdot \beta} \circ(\circ_{\alpha * \beta}, I)
\end{aligned}$$

and

$$\begin{aligned}
m \circ (m, I) &= (a_\alpha \prec_\alpha + b_\alpha \circ_\alpha + c_\alpha \succ_\alpha) \circ (a_\alpha \prec_\alpha + b_\alpha \circ_\alpha + c_\alpha \succ_\alpha, I) \\
&= \sum_{\alpha,\beta} a_\alpha a_\beta \prec_{\alpha \rightarrow \beta} \circ(I, \succ_{\alpha \triangleright \beta}) + \sum_{\alpha,\beta} a_\alpha a_\beta \prec_{\alpha \leftarrow \beta} (I, \prec_{\alpha \triangleleft \beta}) + \sum_{\alpha,\beta} a_\alpha a_\beta \prec_{\alpha \cdot \beta} \circ(I, \circ_{\alpha * \beta}) \\
&+ \sum_{\alpha,\beta} a_\alpha b_\beta \prec_\alpha \circ(\circ_\beta, I) + \sum_{\alpha,\beta} a_\alpha c_\beta \prec_\alpha \circ(\succ_\beta, I) + \sum_{\alpha,\beta} b_\alpha a_\beta \circ_\alpha \circ(\prec_\beta, I) + \sum_{\alpha,\beta} b_\alpha b_\beta \circ_\alpha \circ(\circ_\beta, I) \\
&+ \sum_{\alpha,\beta} b_\alpha c_\beta \circ_\alpha \circ(\succ_\beta, I) + \sum_{\alpha,\beta} c_\alpha a_\beta \succ_\alpha \circ(\prec_\beta, I) + \sum_{\alpha,\beta} c_\alpha b_\beta \succ_\alpha \circ(\circ_\beta, I) + \sum_{\alpha,\beta} c_\alpha c_\beta \succ_\alpha \circ(\succ_\beta, I).
\end{aligned}$$

Hence  $m \circ (I, m) = m \circ (m, I)$  if and only if the above equations hold.  $\square$

**Remark 4.2.** These conditions can be reformulated as follows. We extend  $\varphi_\leftarrow$ ,  $\varphi_\rightarrow$  and  $\varphi_*$  as linear endomorphisms  $\mathbf{k}\Omega^\otimes$ . We then consider the three elements of  $\mathbf{k}\Omega$ :

$$a = \sum_{\alpha \in \Omega} a_\alpha \alpha, \quad b = \sum_{\alpha \in \Omega} b_\alpha \alpha, \quad c = \sum_{\alpha \in \Omega} c_\alpha \alpha.$$

Then  $m$  is associative if, and only if

$$\begin{aligned}
\varphi_\leftarrow(a \otimes a) &= a \otimes a, & \varphi_*(a \otimes a) &= a \otimes b, \\
\varphi_\rightarrow(a \otimes a) &= a \otimes c, & b &= 0 \text{ or } a = c, \\
\varphi_\leftarrow(c \otimes c) &= c \otimes a, & \varphi_*(c \otimes c) &= c \otimes b, \\
\varphi_\rightarrow(c \otimes c) &= c \otimes c, & &
\end{aligned}$$

which is equivalent to

$$\left\{ \begin{array}{l} b = 0, \\ \varphi_\leftarrow(a \otimes a) = a \otimes a, \\ \varphi_\rightarrow(a \otimes a) = a \otimes c, \\ \varphi_*(a \otimes a) = 0, \\ \varphi_\leftarrow(c \otimes c) = c \otimes a, \\ \varphi_\rightarrow(c \otimes c) = c \otimes c, \\ \varphi_*(c \otimes c) = 0, \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} c = a, \\ \varphi_\leftarrow(a \otimes a) = a \otimes a, \\ \varphi_\rightarrow(a \otimes a) = a \otimes a, \\ \varphi_*(a \otimes a) = a \otimes b. \end{array} \right.$$

If  $\Omega$  is finite, the operad  $\mathcal{P}_\Omega$  is a finite generated quadratic operad. By computation, we obtain the Koszul dual of  $\mathcal{P}_\Omega$  as follows.

**Proposition 4.3.** *Let  $\Omega$  be a finite ETS. The Koszul dual  $\mathcal{P}_\Omega^!$  of  $\mathcal{P}_\Omega$  is generated by  $\dashv_\alpha, \perp_\alpha, \vdash_\alpha$  with  $\alpha \in \Omega$  and the relations*

$$\begin{aligned}
 \dashv_\alpha \circ (I, \dashv_\beta) &= \sum_{\substack{(\gamma, \delta) \in \Omega^2, \\ \gamma \leftarrow \delta = \alpha, \\ \gamma < \delta = \beta}} \dashv_\delta \circ (\dashv_\gamma, I), & \dashv_\alpha \circ (I, \vdash_\beta) &= \sum_{\substack{(\gamma, \delta) \in \Omega^2, \\ \gamma \rightarrow \delta = \alpha, \\ \gamma > \delta = \beta}} \dashv_\delta \circ (\dashv_\gamma, I), \\
 \dashv_\alpha \circ (I, \perp_\beta) &= \sum_{\substack{(\gamma, \delta) \in \Omega^2, \\ \gamma \cdot \delta = \alpha, \\ \gamma * \delta = \beta}} \dashv_\delta \circ (\dashv_\gamma, I), \\
 \vdash_\beta \circ (\vdash_\alpha, I) &= \sum_{\substack{(\gamma, \delta) \in \Omega^2, \\ \gamma \rightarrow \delta = \alpha, \\ \gamma > \delta = \beta}} \vdash_\gamma \circ (I, \vdash_\delta), & \vdash_\beta \circ (\dashv_\alpha, I) &= \sum_{\substack{(\gamma, \delta) \in \Omega^2, \\ \gamma \leftarrow \delta = \alpha, \\ \gamma < \delta = \beta}} \vdash_\gamma \circ (I, \vdash_\delta), \\
 \vdash_\beta \circ (\perp_\alpha, I) &= \sum_{\substack{(\gamma, \delta) \in \Omega^2, \\ \gamma \cdot \delta = \alpha, \\ \gamma * \delta = \beta}} \vdash_\gamma \circ (I, \vdash_\delta), \\
 \dashv_\beta \circ (\vdash_\alpha, I) &= \vdash_\alpha \circ (I, \dashv_\beta), & \perp_\beta \circ (\vdash_\alpha, I) &= \vdash_\alpha \circ (I, \perp_\beta), \\
 \perp_\beta \circ (\dashv_\alpha, I) &= \perp_\beta \circ (I, \vdash_\alpha), & \dashv_\beta \circ (\perp_\alpha, I) &= \perp_\alpha \circ (I, \dashv_\beta), \\
 \perp_\beta \circ (\perp_\alpha, I) &= \perp_\alpha \circ (I, \perp_\beta),
 \end{aligned}$$

for all  $\alpha, \beta \in \Omega$ .

In particular, if  $|\Omega| = 1$ , we recover the definition of triassociative algebras, which operad is the Koszul dual of the operad of tridendriform algebras [7].

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