Bialgebras overs another bialgebras and quasishuffle double bialgebras

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Abstract

Quasishuffle Hopf algebras, usually defined on a commutative monoid, can be more generally defined on any associative algebra V. If V is a commutative and cocommutative bialgebra, the associated quasishuffle bialgebra $\mathrm{QSh}(V)$ inherits a second coproduct δ of contraction and extraction of words, cointeracting with the deconcatenation coproduct Δ, making $\mathrm{QSh}(V)$ a double bialgebra. In order to generalize the universal property of the Hopf algebra of quasisymmetric functions **QSym** (a particular case of quasishuffle Hopf algebra) as exposed by Aguiar, Bergeron and Sottile, we introduce the notion of double bialgebra over V. A bialgebra over V is a bialgebra in the category of right V-comodules and an extra condition is required on the second coproduct for double bialgebras over V.

We prove that the quasishuffle bialgebra $\mathrm{QSh}(V)$ is a double bialgebra over V, and that it satisfies a universal property: for any bialgebra B over V and for any character λ of B, under a connectedness condition, there exists a unique morphism ϕ of bialgebras over V from B to $\mathrm{QSh}(V)$ such that $\varepsilon_\delta \circ \phi = \lambda$. When V is a double bialgebra over V, we obtain a unique morphism of double bialgebras over V from B to $\mathrm{QSh}(V)$, and show that this morphism ϕ_1 allows to obtain any morphism of bialgebra over V from B to $\mathrm{QSh}(V)$ thanks to an action of a monoid of characters. This formalism is applied to a double bialgebra of V-decorated graphs.

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Contents

Introduction

Quasishuffle bialgebras are Hopf algebras based on words, used in particular for the study of relations between multizêtas [10, 11]. They also appear in Ecalle's mould calculus, as a symmetrel mould can be interpreted as a character on a quasishuffle bialgebras [3]. Hoffman's construction is based on commutative countable semigroups, but it can be extended to any associative algebra $(V,\cdot)$, not necessarily unitary [6]. The associated quasishuffle bialgebra $\mathrm{QSh}(V)$ is, as a vector space, the tensor algebra $T(V)$. Its product is the quasishuffle product $\uplus$, inductively defined as follows: if $x,y\in V$ and $v,w\in T(V)$,

$$\begin{aligned}1\uplus w&=w,\\ v\uplus 1&=v,\\ xv\uplus yw&=x(v\uplus yw)+y(xv\uplus w)+(x\cdot y)(v\uplus w).\end{aligned}$$

For example, if $x,y,z,t\in V$,

$$\begin{aligned}x\uplus y&=xy+yx+x\cdot y,\\ xy\uplus z&=xyz+xzy+zxy+(x\cdot z)y+x(y\cdot z),\\ xy\uplus zt&=xyzt+xzyt+zxyt+xzty+zxty+ztxy\\ &\quad+(x\cdot z)ty+(x\cdot z)yt+xz(y\cdot t)+zx(y\cdot t)+(x\cdot z)(y\cdot t).\end{aligned}$$

The coproduct Δ is the deconcatenation: if $x_1,\ldots,x_n\in V$,

$$\Delta(x_1\ldots x_n)=\sum_{i=0}^{n}x_1\ldots x_i\otimes x_{i+1}\ldots x_n.$$

When $(V,\cdot,\delta_V)$ is a commutative bialgebra, not necessarily unitary, then $\mathrm{QSh}(V)$ inherits a second, less known coproduct δ: if $x_1,\ldots,x_n\in V$,

$$\delta(v_1\ldots v_n)=\sum_{1\leqslant i_1<\ldots<i_p<k}\left(\dot{\prod_{1\leqslant i\leqslant i_1}}v'_i\right)\ldots\left(\dot{\prod_{i_p+1\leqslant i\leqslant k}}v'_i\right)\otimes(v''_1\ldots v''_{i_1})\uplus\ldots\uplus(v''_{i_p+1}\ldots v''_k),$$

with Sweedler's notation for δ_V and where the symbols $\dot{\prod}$ mean that the products are taken in $(V,\cdot)$. The counit ϵ_δ is given as follows: for any word w of length $n\geqslant 1$,

$$\epsilon_\delta(w)=\begin{cases}\epsilon_V(w)\text{ if }n=1,\\ 0\text{ otherwise.}\end{cases}$$

Then $(T(V),\uplus,\delta)$ is a bialgebra, and $(T(V),\uplus,\Delta)$ is a bialgebra in the category of right $(T(V),\uplus,\delta)$-comodules, which in particular implies that

$$(\Delta\otimes\mathrm{Id})\circ\delta=\uplus_{1,3,24}\circ(\delta\otimes\delta)\circ\Delta,$$

where $\uplus_{1,3,24}:T(V)^{\otimes 4}\longrightarrow T(V)^{\otimes 3}$ send $w_1\otimes w_2\otimes w_3\otimes w_4$ to $w_1\otimes w_3\otimes w_2\uplus w_4$. Two particular cases will be considered all along this paper:

- $V=\mathbb{K}$, with its usual bialgebraic structure. The quasishuffle algebra $\mathrm{QSh}(\mathbb{K})$ is isomorphic to the polynomial algebra $\mathbb{K}[X]$, with its two coproducts defined by

$$\Delta(X)=X\otimes 1+1\otimes X,\qquad\qquad \delta(X)=X\otimes X.$$

- V is the algebra of the semigroup $(\mathbb{N}_{>0},+)$. We recover the double Hopf algebra of quasisymmetric functions $\mathbf{QSym}$ [8, 9, 12, 14]. This Hopf algebra is studied in [2], where it is proved to be the terminal object in a category of combinatorial Hopf algebras: If B is a graded and connected Hopf algebra and λ is a character of B, then there exists a unique homogeneous Hopf algebra morphism $\phi_\lambda:B\longrightarrow\mathbf{QSym}$ such that $\epsilon_\delta\circ\phi_\lambda=\lambda$. We proved in [4, 5] that when (B,m,Δ,δ) is a double bialgebra, such that:

– (B, m, Δ) is a graded and connected Hopf algebra,
– for any $n \in \mathbb{N}$, $\delta(B_n) \subseteq B_n \otimes B$,

then $\phi_{\varepsilon_\delta}$ is the unique homogeneous double bialgebra morphism from B to **QSym**. A similar result exists for $\mathbb{K}[X]$, where the hypothesis "graded and connected" on B is replaced by the weaker hypothesis "connected".

In this paper, we generalize these results to any quasishuffle $\mathrm{QSh}(V)$ associated to a commutative and cocommutative bialgebra $(V, \cdot, \delta_V)$, not necessarily unitary. We firstly show that $(T(V), \cdot, \Delta)$ is a bialgebra in the category of right V-comodules, with the coaction ρ defined by

$$\forall v_1, \ldots, v_n \in V, \qquad \rho(v_1 \ldots v_n) = v_1' \ldots v_n' \otimes v_1'' \cdot \ldots \cdot v_n''.$$

Moreover, the second coproduct δ satisfies this compatibility with ρ:

$$(\mathrm{Id} \otimes c) \circ (\rho \otimes \mathrm{Id}) \circ \delta = (\delta \otimes \mathrm{Id}) \circ \rho,$$

where $c : V \otimes T(V) \longrightarrow T(V) \otimes V$ is the usual flip. Equivalently, $(T(V), \uplus, \Delta)$ is a comodule over the coalgebra $(V, \delta_V^{op}) \otimes (T(V), \delta)$. This observation leads us to study bialgebras over V, that is to say bialgebras in the category of right $(V, \cdot, \delta_V)$-comodules (Definition 1.1 when V is unitary). Technical difficulties occur when V is not unitary, a case that cannot be neglected as it includes **QSym**: this is the object of Definition 1.3, where we use the unitary extension uV of V, which is also a bialgebra. We define double bialgebras over V in Definition 1.4 in the unitary case and Definition 1.3 in the nonunitary case. When $V = \mathbb{K}$, bialgebras over V are bialgebras B with a decomposition $B = B_1 \oplus B_{\bar{1}}$, where B_1 is a subbialgebra and $B_{\bar{1}}$ is a biideal. This includes any bialgebra B, taking $B_1 = \mathbb{K}1_B$ and $B_{\bar{1}}$ the kernel of the counit. When $V = \mathbb{K}(\mathbb{N}_{>0}, +)$, bialgebras over V are $\mathbb{N}$-graded and connected bialgebras, in other words $\mathbb{N}$-graded bialgebras B with $B_0 = \mathbb{K}1_B$.

We prove that the antipode of a bialgebra (B, m, Δ, ρ) over V, such that (B, m, Δ) is a Hopf algebra, is automatically a comodule morphism (Proposition 1.2), that is to say

$$\rho \circ S = (S \otimes \mathrm{Id}_V) \circ \rho.$$

In the case of $\mathbb{N}$-graded bialgebras, this means that S is automatically homogeneous; more generally, if Ω is a commutative semigroup and B is an Ω-graded bialgebra and a Hopf algebra, then its antipode is automatically Ω-homogeneous.

Let us now consider the double quasishuffle algebra $\mathrm{QSh}(V) = (T(V), \uplus, m, \Delta, \delta)$, which is over V with the coaction ρ. We obtain a generalization of Aguiar, Bergeron and Sottile's result: Theorem 2.3 states that for any connected bialgebra B over V and for any character λ of B, there exists a unique morphism ϕ_λ from B to $\mathrm{QSh}(V)$ of bialgebras over V such that $\epsilon_\delta \circ \phi_\lambda = \lambda$, given by an explicit formula implying the iterations of the reduced coproduct $\tilde{\Delta}$ associated to the coproduct Δ of B.

When B is moreover a double bialgebra over V, we prove that the unique morphism of double bialgebras over V from B to $\mathrm{QSh}(V)$ is Φ_{ϵ_δ} (Theorem 2.7). Moreover, for any bialgebra B' over V, the second coproduct δ induces an action $\leftsquigarrow$ of the monoid of characters $\mathrm{Char}(B)$ (with the product induced by δ) onto the set of morphisms of bialgebras over V from B to B' (Proposition 2.11. When $B' = \mathrm{QSh}(V)$, we obtain that this action is simply transitive (Corollary 2.13), which gives a bijection between the set of characters of B and the set of morphisms of double bialgebras over V from B to $\mathrm{QSh}(V)$. This is finally applied to the twisted bialgebra of graphs $\mathbf{G}$: for any V, we obtain a double bialgebra $\mathcal{H}_V$ of V-decorated graphs, and the unique morphism of double bialgebras over V from $\mathcal{H}_V$ to $\mathrm{QSh}(V)$ is a generalization of the chromatic polynomial and of the chromatic (quasi)symmetric series. Taking $V = \mathbb{K}$ or $\mathbb{K}(\mathbb{N}_{>0}, +)$, we recover the terminal property ok $\mathbb{K}[X]$ and **QSym**.

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Notations 0.1. 1. We denote by $\mathbb{K}$ a commutative field of characteristic zero. Any vector space in this field will be taken over $\mathbb{K}$.

2. For any $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, \ldots, n\}$. In particular, $[0] = \emptyset$.

1 Bialgebras over another bialgebra

1.1 Définitions and notations

Let $(V, \cdot, \delta_V)$ be a commutative bialgebra, which we firstly assume to be unitary and counitary. Its counit is denoted by ϵ_V and its unit by 1_V.

Definition 1.1. *A bialgebra over V is a bialgebra in the category of right V-comodules, that is to say a family (B, m, Δ, ρ) where (B, m, Δ) is a bialgebra and $\rho : B \longrightarrow B \otimes V$ such that:*

- *ρ is a right coaction of V over B:*

$$(\rho \otimes \mathrm{Id}_V) \circ \rho = (\mathrm{Id}_B \otimes \delta_V) \circ \rho, \qquad (\mathrm{Id}_B \otimes \epsilon_V) \circ \rho = \mathrm{Id}_B.$$

- *The unit of B is a V-comodule morphism:*

$$\rho(1_B) = 1_B \otimes 1_V.$$

- *The product m of B is a V-comodule morphism:*

$$\rho \circ m = (m \otimes \cdot) \circ (\mathrm{Id} \otimes c \otimes \mathrm{Id}) \circ (\rho \otimes \rho),$$

where $c : B \otimes B \longrightarrow B \otimes B$ is the usual flip, sending $a \otimes b$ to $b \otimes a$.

- *The counit ε_Δ of B is a V-comodule morphism:*

$$\forall x \in B, \qquad (\varepsilon_\Delta \otimes \mathrm{Id}) \circ \rho(x) = \varepsilon_\Delta(x) 1_V.$$

- *The coproduct Δ of B is a V-comodule morphism:*

$$(\Delta \otimes \mathrm{Id}) \circ \rho = m_{1,3,24} \circ (\rho \otimes \rho) \circ \Delta,$$

where

$$m_{1,3,24} : \begin{cases} B \otimes V \otimes B \otimes V & \longrightarrow & B \otimes B \otimes V \\ b_1 \otimes v_2 \otimes b_3 \otimes v_4 & \longrightarrow & b_1 \otimes b_3 \otimes v_2 \cdot v_4. \end{cases}$$

Notice that the second and third items are equivalent to the fact that ρ is an algebra morphism.

Example 1.1. • Let $(\Omega, \star)$ be a monoid and let $V = \mathbb{K}\Omega$ be the associated bialgebra. Let B be a bialgebra over V. For any $\alpha \in \Omega$, we put

$$B_\alpha = \{x \in B \mid \rho(x) = x \otimes \alpha\}.$$

Then $B = \bigoplus_{\alpha \in \Omega} B_\alpha$. Indeed, if $x \in B$, we can write

$$\rho(x) = \sum_{\alpha \in \Omega} x_\alpha \otimes \alpha.$$

Then

$$(\rho\otimes \mathrm{Id})\circ\rho(x)=\sum_{\alpha\in\Omega}\rho(x_\alpha)\otimes\alpha=(\mathrm{Id}\otimes\delta_V)\circ\rho(x)=\sum_{\alpha\in\Omega}x_\alpha\otimes\alpha\otimes\alpha.$$

Therefore, for any $\alpha\in\Omega$, $\rho(x_\alpha)=x_\alpha\otimes\alpha$, that is to say $x_\alpha\in B_\alpha$. Moreover,

$$x=(\mathrm{Id}\otimes\epsilon_V)\circ\rho(x)=\sum_{\alpha\in\Omega}x_\alpha.$$

The second item of Definition 1.1 is equivalent to $1_B\in B_{1_\Omega}$. The third item is equivalent to

$$\forall\alpha,\beta\in\Omega,\qquad B_\alpha B_\beta\subseteq B_{\alpha\star\beta}.$$

The fourth item is equivalent to $\bigoplus_{\alpha\neq 1_\Omega}B_\alpha\subseteq\mathrm{Ker}(\varepsilon_\Delta)$. The last item is equivalent to

$$\forall\alpha\in\Omega,\qquad \Delta(B_\alpha)\subseteq\bigoplus_{\alpha'\star\alpha''=\alpha}B_{\alpha'}\otimes B_{\alpha''}.$$

In other words, a bialgebra over $\mathbb{K}\Omega$ is an Ω-graded bialgebra.

- Let $V=\mathbb{K}(\mathbb{Z}/2\mathbb{Z},\times)$. A bialgebra over V admits a decomposition $B=B_{\overline{0}}\oplus B_{\overline{1}}$, with $1_B\in B_{\overline{0}}$, $\varepsilon_\Delta(B_{\overline{1}})=(0)$, and

$$\begin{aligned}
&B_{\overline{0}}B_{\overline{1}}+B_{\overline{1}}B_{\overline{0}}+B_{\overline{1}}B_{\overline{1}}\subseteq B_{\overline{1}}, && B_{\overline{0}}B_{\overline{0}}\subseteq B_{\overline{0}},\\
&\Delta(B_{\overline{1}})\subseteq B_{\overline{1}}\otimes B_{\overline{0}}+B_{\overline{0}}\otimes B_{\overline{1}}+B_{\overline{1}}\otimes B_{\overline{1}}, && \Delta(B_{\overline{0}})\subseteq B_{\overline{0}}\otimes B_{\overline{0}}.
\end{aligned}$$

  In other words, a bialgebra over V is a bialgebra with a decomposition $B=B_{\overline{0}}\oplus B_{\overline{1}}$, such that $B_{\overline{0}}$ is a subbialgebra and $B_{\overline{1}}$ is a biideal. In particular, any bialgebra (B,m,Δ) is trivially a bialgebra over V, with $B_{\overline{0}}=\mathbb{K}1_B$ and $B_{\overline{1}}=\mathrm{Ker}(\varepsilon_\Delta)$, or equivalently, for any $x\in B$,

$$\rho(x)=\varepsilon(x)1_B\otimes 1+(x-\varepsilon(x)1_B)\otimes X.$$

- Let Ω be a finite monoid and let $\mathbb{K}[\Omega]$ be the bialgebra of functions over G, dual of the bialgebra $\mathbb{K}\Omega$. A bialgebra over $\mathbb{K}[\Omega]$ is a family $(B,m,\Delta,\lhd)$ where (B,m,Δ) is a bialgebra and $\lhd$ is a right action of Ω on B such that:

$$\begin{aligned}
&\forall x,y\in B, && \forall\omega\in\Omega, && (xy)\lhd\omega=(x\lhd\omega)(y\lhd\omega),\\
&\forall x\in B, && \forall\omega\in\Omega, && \Delta(x\lhd\omega)=\Delta(x)\lhd(\omega\otimes\omega),\\
& && \forall\omega\in\Omega, && 1_B\lhd\omega=1_B,\\
&\forall x\in B, && \forall\omega\in\Omega, && \varepsilon_\Delta(x\lhd\omega)=\varepsilon_\Delta(x).
\end{aligned}$$

Notations 1.1. We shall use the Sweedler's notation $\rho(x)=x_0\otimes x_1$. The five items of Definition 1.1 become

$$\begin{aligned}
(x_0)_0\otimes(x_0)_1\otimes x_1&=x_0\otimes x_1'\otimes x_1'',\\
x_0\varepsilon(x_1)&=x,\\
(1_B)_0\otimes(1_B)_1&=1_B\otimes 1_V,\\
(xy)_0\otimes(xy)_1&=x_0y_0\otimes x_1y_1,\\
\varepsilon_\Delta(x_0)x_1&=\varepsilon_\Delta(x)1_V,\\
(x_0)^{(1)}\otimes(x_0)^{(2)}\otimes x_1&=(x^{(1)})_0\otimes(x^{(2)})_0\otimes(x^{(1)})_1(x^{(2)})_1.
\end{aligned}$$

1.2 Antipode

Proposition 1.2. *Let (V, m_V, δ_V) be a bialgebra and let (B, m, Δ, ρ) be a bialgebra over V. If (B, m, Δ) is a Hopf algebra of antipode S, then S is a comodule morphism:*

$$\rho \circ S = (S \otimes \mathrm{Id}_V) \circ \rho.$$

Proof. Let us give $\mathrm{Hom}(B, B \otimes V)$ its convolution product $*$: for any linear maps f, g from B to $B \otimes V$,

$$f * g = m_{B \otimes V} \circ (f \otimes g) \circ \Delta.$$

In this convolution algebra,

$$\begin{aligned}
((S \otimes \mathrm{Id}_V) \circ \rho) * \rho &= m_{B\otimes V} \circ (S \otimes \mathrm{Id}_V \otimes \mathrm{Id}_B \otimes \mathrm{Id}_V) \circ (\rho \otimes \rho) \circ \Delta \\
&= (m \circ (S \otimes \mathrm{Id}_B) \circ \Delta \otimes \mathrm{Id}_V) \circ m_{1,3,24} \circ (\rho \otimes \rho) \circ \Delta \\
&= (m \circ (S \otimes \mathrm{Id}_B) \circ \Delta \otimes \mathrm{Id}_V) \circ (\Delta \otimes \mathrm{Id}) \circ \rho \\
&= (m \circ (S \otimes \mathrm{Id}_B) \circ \Delta \otimes \mathrm{Id}_V) \circ \rho \\
&= (\iota_B \circ \varepsilon_\Delta \otimes \mathrm{Id}_V) \circ \rho \\
&= \iota_{B \otimes V} \circ \varepsilon_\Delta.
\end{aligned}$$

So $(S \otimes \mathrm{Id}_V) \circ \rho$ is a right inverse of ρ in $(\mathrm{Hom}(B, B \otimes V), *)$.

$$\begin{aligned}
\rho * (\rho \circ S) &= m_{B \otimes V} \circ (\rho \otimes \rho) \circ (\mathrm{Id} \otimes S) \circ \Delta \\
&= \rho \circ m \circ (\mathrm{Id} \otimes S) \circ \Delta \\
&= \rho \circ \iota_B \circ \varepsilon_\Delta \\
&= \iota_{B \otimes V} \circ \varepsilon_\Delta.
\end{aligned}$$

So $\rho \circ S$ is a left inverse of ρ in $(\mathrm{Hom}(B, B \otimes V), *)$. As $*$ is associative, $(S \otimes \mathrm{Id}_V) \circ \rho = \rho \circ S$. □

Example 1.2.

1. Let $(\Omega, \star)$ be a semigroup. If V is the bialgebra of $(\Omega, \star)$, we recover that if B is an Ω-graded bialgebra and a Hopf algebra, then, S is Ω-homogeneous, that is to say, for any $\alpha \in \Omega$,
$$S(B_\alpha) \subseteq B_\alpha.$$
2. Let Ω be a finite monoid. If $(B, m, \Delta, \triangleleft)$ is a bialgebra over $\mathbb{K}[\Omega]$ and a Hopf algebra, then for any $x \in B$, for any $\alpha \in \Omega$,
$$S(x \triangleleft \alpha) = S(x) \triangleleft \alpha.$$

1.3 Nonunitary cases

We shall work with not necessarily unitary bialgebras $(V, \cdot, \delta_V)$. If so, we put $uV = \mathbb{K} \oplus V$ and we give it a product and a coproduct defined as follows:

$$\begin{aligned}
&\forall \lambda, \mu \in \mathbb{K}, && \forall v, w \in V, && (\lambda + v) \cdot (\mu + w) = \lambda\mu + \lambda w + \mu v + v \cdot w, \\
&\forall \lambda \in \mathbb{K}, && \forall v \in V, && \delta_{uV}(\lambda + v) = \lambda 1 \otimes 1 + \delta_V(v).
\end{aligned}$$

Then $(uV, \cdot, \delta_{uV})$ is a counitary and unitary bialgebra, and V is a nonunitary subbialgebra of uV.

Definition 1.3. *Let $(V, \cdot, \delta_V)$ be a not necessarily unitary bialgebra and $(uV, \cdot, \delta_{uV})$ be its unitary extension. A bialgebra over V is a bialgebra (B, m, Δ, ρ) over uV such that*

$$\rho(\mathrm{Ker}(\varepsilon_\Delta)) \subseteq B \otimes V.$$

Remark 1.1. If (B, m, Δ, ρ) is a bialgebra over the nonunitary bialgebra $(V, \cdot, \delta_V)$, then

$$\{b \in B \mid \rho(b) = b \otimes 1\} = \mathbb{K}1_B.$$

Indeed, if $\rho(b) = b \otimes 1$, putting $b' = b - \varepsilon_\Delta(b)1_B$, then $b' \in \mathrm{Ker}(\varepsilon_\Delta)$. Hence,

$$\rho(b') = \rho(b) - \varepsilon_\Delta(b)1_B \otimes 1 = (b - \varepsilon(b)1_B) \otimes 1 \in B \otimes V,$$

so $b = \varepsilon_\Delta(b)1_B$.

In the sequel, we will mention that we work with a nonunitary bialgebra $(V, \cdot, \delta_V)$ if we want to use Definition 1.3 instead of Definition 1.1, even if $(V, \cdot)$ has a unit – that will happen when we will work with $\mathbb{K}$.

Example 1.3.

1. If Ω is a semigroup, then a bialgebra (B, m, Δ) over $\mathbb{K}\Omega$ is a connected $u\Omega$-graded bialgebra, where $u\Omega = \{e\} \sqcup \Omega$ with the extension of the product of Ω such that e is a unit:

$$\begin{aligned} & & B &= \bigoplus_{\alpha \in u\Omega} B_\alpha, \\ \forall \alpha, \beta \in \Omega, & & \Delta(B_\alpha) &\subseteq \sum_{\substack{\alpha', \alpha'' \in \Omega, \\ \alpha' \times \alpha'' = \alpha}} B_{\alpha'} \otimes B_{\alpha''} + B_\alpha \otimes B_e + B_e \otimes B_\alpha, \\ & & B_e &= \mathbb{K}1_B, \\ \forall \alpha \in \Omega, & & \varepsilon_\Delta(B_\alpha) &= (0). \end{aligned}$$

2. If $V = \mathbb{K}$[1], as $u\mathbb{K}$ is isomorphic to $\mathbb{K}(\mathbb{Z}/2\mathbb{Z}, \times)$, any bialgebra (B, m, Δ) is a bialgebra over V with $B_{\overline{0}} = \mathbb{K}1_B$ and $B_{\overline{1}} = \mathrm{Ker}(\varepsilon_\Delta)$.

1.4 Double bialgebras over V

Definition 1.4. *Let (B, m, Δ, δ) be a double bialgebra, $(V, \cdot, \delta_V)$ be a bialgebra and $\rho : B \longrightarrow B \otimes V$ be a right coaction of V over B. We shall say that $(B, m, \Delta, \delta, \rho)$ is a double bialgebra over V if (B, m, Δ, ρ) is a bialgebra over V and*

$$(\mathrm{Id} \otimes c) \circ (\rho \otimes \mathrm{Id}) \circ \delta = (\delta \otimes \mathrm{Id}) \circ \rho : B \longrightarrow B \otimes B \otimes V,$$

where $c : V \otimes B \longrightarrow B \otimes V$ is the usual flip. In other words, with Sweedler's notation $\delta(x) = x' \otimes x''$ for any $x \in B$,

$$(x')_0 \otimes x'' \otimes (x')_1 = (x_0)' \otimes (x_0)'' \otimes x_1.$$

Remark 1.2. In other words, in a double bialgebra B over V, considering the left coaction ρ^{op} of $V^{cop} = (V, \delta_V^{op})$ on B,

$$(\rho^{op} \otimes \mathrm{Id}) \circ \delta = (\mathrm{Id} \otimes \delta) \circ \rho^{op},$$

which means that B is a (V, δ_V^{op})–(B, δ)-bicomodule.

Example 1.4. Let Ω be a finite monoid. A double bialgebra $(B, m, \Delta, \triangleleft)$ over $\mathbb{K}[\Omega]$ is a bialgebra over $\mathbb{K}[\Omega]$ and a double bialgebra such that for any $x \in B$, for any $\alpha \in \Omega$,

$$\delta(x \triangleleft \alpha) = \delta(x) \triangleleft (\alpha \otimes e_\Omega),$$

where e_Ω is the unit of Ω.

In the nonunitary case:

Definition 1.5. *Let $(V, \cdot, \delta_V)$ be a not necessarily unitary bialgebra. A double bialgebra over V is a double bialgebra $(B, m, \Delta, \delta, \rho)$ over uV such that (B, m, Δ, ρ) is a bialgebra over V.*

[1] which is of course unitary, but which we treat as a nonunitary bialgebra, as mentioned before.

Example 1.5. 1. Let Ω be a semigroup. A double bialgebra (B, m, Δ, δ) over $\mathbb{K}\Omega$ is a bialgebra over $\mathbb{K}\Omega$ such that for any $\alpha \in \Omega$,

$$\delta(B_\alpha) \subseteq B_\alpha \otimes B.$$

2. If $V = \mathbb{K}$, as $u\mathbb{K}$ is isomorphic to $\mathbb{K}(\mathbb{Z}/2\mathbb{Z}, \times)$, any double bialgebra (B, m, Δ, δ) is a double bialgebra over V with $B_{\overline{0}} = \mathbb{K}1_B$ and $B_{\overline{1}} = \mathrm{Ker}(\varepsilon_\Delta)$.

2 Quasishuffle bialgebras

2.1 Definition

[3, 6, 10, 11] Let $(V, \cdot)$ be a nonunitary bialgebra. The tensor algebra $T(V)$ is given the quasishuffle product associated to V: For any $v_1, \ldots, v_{k+l} \in V$,

$$v_1 \ldots v_k \uplus v_{k+1} \ldots v_{k+l} = \sum_{\sigma \in \mathrm{QSh}(k,l)} \left(\dot{\prod_{i \in \sigma^{-1}(1)}} v_i \right) \ldots \left(\dot{\prod_{i \in \sigma^{-1}(\max(\sigma))}} v_i \right),$$

where $\mathrm{QSh}(k, l)$ is the set of (k, l)-quasishuffles, that is to say surjections $\sigma : [k+l] \longrightarrow [\max(\sigma)]$ such that $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(k+1) < \ldots < \sigma(k+l)$. The symbol $\dot{\prod}$ means that the corresponding products are taken in $(V, \cdot)$. The coproduct Δ is given by deconcatenation: for any $v_1, \ldots, v_n \in V$,

$$\Delta(v_1 \ldots v_n) = \sum_{k=0}^{n} v_1 \ldots v_k \otimes v_{k+1} \ldots v_n.$$

A special case is given when $\cdot$ is the zero product of V. In this case, we obtain the shuffle product $\sqcup\!\sqcup$ of $T(V)$. The bialgebra $(T(V), \sqcup\!\sqcup, \Delta)$ is denoted by $\mathrm{Sh}(V)$.

If $(V, \cdot, \delta_V)$ is a not necessarily unitary commutative bialgebra, then $\mathrm{QSh}(V)$ inherits a second coproduct δ making it a double bialgebra. For any $v_1, \ldots, v_k \in V$, with Sweeder's notation $\delta_V(v) = v' \otimes v''$,

$$\delta(v_1 \ldots v_n) = \sum_{1 \leqslant i_1 < \ldots < i_p < k} \left(\dot{\prod_{1 \leqslant i \leqslant i_1}} v'_i \right) \ldots \left(\dot{\prod_{i_p+1 \leqslant i \leqslant k}} v'_i \right) \otimes (v''_1 \ldots v''_{i_1}) \uplus \ldots \uplus (v''_{i_p+1} \ldots v''_k).$$

Proposition 2.1. *Let $(V, \cdot, \delta_V)$ be a nonunitary bialgebra. We define a coaction of V on $\mathrm{QSh}(V)$ by*

$$\forall v_1, \ldots, v_n \in V, \qquad \rho(v_1 \ldots v_n) = v'_1 \ldots v'_n \otimes v''_1 \cdot \ldots \cdot v''_n.$$

1. *The quasishuffle bialgebra $\mathrm{QSh}(V) = (T(V), \uplus, \Delta, \rho)$ is a bialgebra over V if and only if $(V, \cdot)$ is commutative.*

2. *The quasishuffle double bialgebra $\mathrm{QSh}(V) = (T(V), \uplus, \Delta, \delta, \rho)$ is a bialgebra over V if and only if $(V, \cdot)$ is commutative and cocommutative.*

Proof. 1. Let us assume that $\mathrm{QSh}(V)$ is a double bialgebra over V with this coaction ρ. For any $v, w \in V$,

$$\begin{aligned}
\rho(v \uplus w) &= \rho(vw + wv + v \cdot w) \\
&= v'w' \otimes v'' \cdot w'' + w'v' \otimes w'' \cdot v'' + v' \cdot w' \otimes v'' \cdot w'', \\
(\uplus \otimes m) \circ (\rho \otimes \rho)(v \otimes w) &= v' \uplus w' \otimes v'' \cdot w'' \\
&= (v'w' + w'v' + v' \otimes w') \otimes v'' \cdot w''.
\end{aligned}$$

As $\uplus$ is comodule morphism, we obtain that for any $v, w \in V$,

$$w' \otimes v' \otimes w'' \cdot v'' = w' \otimes v' \otimes v'' \cdot w''.$$

Applying $\epsilon_V \otimes \epsilon_V \otimes \mathrm{Id}_V$, this gives $v \cdot w = w \cdot v$, so V is commutative.

Let us now assume that V is commutative. The compatibilities of the unit and of the counit with the coaction ρ are obvious. Let $v_1, \ldots, v_{k+l} \in V$ and let $\sigma \in \mathrm{QSh}(k,l)$.

$$\begin{aligned}
&\rho\left(\left(\prod_{i\in\sigma^{-1}(1)}^{\cdot} v_i\right)\cdots\left(\prod_{i\in\sigma^{-1}(\max(\sigma))}^{\cdot} v_i\right)\right)\\
&=\left(\prod_{i\in\sigma^{-1}(1)}^{\cdot} v_i\right)'\cdots\left(\prod_{i\in\sigma^{-1}(\max(\sigma))}^{\cdot} v_i\right)'\otimes\left(\prod_{i\in\sigma^{-1}(1)}^{\cdot} v_i\right)''\cdot\ldots\cdot\left(\prod_{i\in\sigma^{-1}(\max(\sigma))}^{\cdot} v_i\right)''\\
&=\left(\prod_{i\in\sigma^{-1}(1)}^{\cdot} v_i'\right)\cdots\left(\prod_{i\in\sigma^{-1}(\max(\sigma))}^{\cdot} v_i'\right)\otimes\left(\prod_{i\in\sigma^{-1}(1)}^{\cdot} v_i''\right)\cdot\ldots\cdot\left(\prod_{i\in\sigma^{-1}(\max(\sigma))}^{\cdot} v_i''\right)\\
&=\left(\prod_{i\in\sigma^{-1}(1)}^{\cdot} v_i'\right)\cdots\left(\prod_{i\in\sigma^{-1}(\max(\sigma))}^{\cdot} v_i'\right)\otimes v_1''\cdot\ldots\cdot v_n'',
\end{aligned}$$

as $(V,\cdot)$ is commutative. Summing over all possible σ, we obtain

$$\begin{aligned}
\rho(v_1\ldots v_k \uplus v_{k+1}\ldots v_{k+l}) &= \left(\sum_{\sigma\in\mathrm{QSh}(k,l)}\left(\prod_{i\in\sigma^{-1}(1)}^{\cdot} v_i'\right)\cdots\left(\prod_{i\in\sigma^{-1}(\max(\sigma))}^{\cdot} v_i'\right)\right)\otimes v_1''\cdot\ldots\cdot v_n''\\
&= (v_1'\ldots v_k' \uplus v_{k+1}'\ldots v_{k+l}')\otimes(v_1''\cdot\ldots\cdot v_k'')\cdot(v_{k+1}''\cdot\ldots\cdot v_{k+l}'')\\
&= \rho(v_1\ldots v_k)\rho(v_{k+1}\ldots v_{k+l}).
\end{aligned}$$

Let $v_1, \ldots, v_k \in V$. If $0 \leqslant i \leqslant k$,

$$m_{1,3,24}\circ(\rho\otimes\rho)(v_1\ldots v_i\otimes v_{i+1}\ldots v_k) = v_1'\ldots v_i'\otimes v_{i+1}'\ldots v_n'\otimes v_1''\cdot\ldots\cdot v_k''.$$

Summing over all possible i, we obtain

$$\begin{aligned}
m_{1,3,24}\circ(\rho\otimes\rho)\circ\Delta(v_1\ldots v_k) &= \left(\sum_{i=0}^{k} v_1'\ldots v_i'\otimes v_{i+1}'\ldots v_k'\right)\otimes v_1''\cdot\ldots\cdot v_k''\\
&= (\Delta\otimes\mathrm{Id})\circ\rho(v_1\ldots v_k).
\end{aligned}$$

2. Let us assume that $\mathrm{QSh}(V)$ is a double bialgebra over V. By the first part of this proof, V is commutative. For any $v \in V$,

$$\begin{aligned}
(\mathrm{Id}\otimes\delta_V)\circ\delta_V(v) &= (\delta_V\otimes\mathrm{Id})\circ\delta_V(v)\\
&= (\delta\otimes\mathrm{Id})\circ\rho(v)\\
&= (\mathrm{Id}\otimes c)\circ(\rho\otimes\mathrm{Id})\circ\delta(v)\\
&= (\mathrm{Id}\otimes c)\circ(\delta\otimes\mathrm{Id})\circ\delta(v)\\
&= (\mathrm{Id}\otimes\delta_V^{op})\circ\delta_V(v).
\end{aligned}$$

Applying $\epsilon_V\otimes\mathrm{Id}\otimes\mathrm{Id}$, we obtain that $\delta_V^{op} = \delta_V$, so V is cocommutative.

Let us assume that V is commutative and cocommutative. It is proved in [6] that $\mathrm{QSh}(V)$ is a double bialgebra. By the first item, $\mathrm{QSh}(V)$ is a bialgebra over V. For any $v_1, \ldots, v_n \in V$,

$$\begin{aligned}
&(\delta \otimes \mathrm{Id}) \circ \rho(v_1 \ldots v_k)\\
&= \sum_{1 \leqslant i_1 < \ldots < i_p < k} \left(\dot{\prod_{1 \leqslant i \leqslant i_1}} v'_i \right) \ldots \left(\dot{\prod_{i_p+1 \leqslant i \leqslant k}} v'_i \right) \otimes (v''_1 \ldots v''_{i_1}) \uplus \ldots \uplus (v''_{i_p+1} \ldots v''_k) \otimes v'''_1 \cdot \ldots \cdot v'''_k,
\end{aligned}$$

whereas

$$\begin{aligned}
&(\mathrm{Id} \otimes c) \circ (\rho \otimes \mathrm{Id}) \circ \delta(v_1 \ldots v_k)\\
&= \sum_{1 \leqslant i_1 < \ldots < i_p < k} \left(\dot{\prod_{1 \leqslant i \leqslant i_1}} v'_i \right)' \ldots \left(\dot{\prod_{i_p+1 \leqslant i \leqslant k}} v'_i \right)' \otimes (v''_1 \ldots v''_{i_1}) \uplus \ldots \uplus (v''_{i_p+1} \ldots v''_k)\\
&\otimes \left(\dot{\prod_{1 \leqslant i \leqslant i_1}} v'_i \right)'' \cdot \ldots \cdot \left(\dot{\prod_{i_p+1 \leqslant i \leqslant k}} v'_i \right)''\\
&= \sum_{1 \leqslant i_1 < \ldots < i_p < k} \left(\dot{\prod_{1 \leqslant i \leqslant i_1}} v'_i \right) \ldots \left(\dot{\prod_{i_p+1 \leqslant i \leqslant k}} v'_i \right) \otimes (v''_1 \ldots v''_{i_1}) \uplus \ldots \uplus (v'''_{i_p+1} \ldots v'''_k)\\
&\otimes \left(\dot{\prod_{1 \leqslant i \leqslant i_1}} v''_i \right) \cdot \ldots \cdot \left(\dot{\prod_{i_p+1 \leqslant i \leqslant k}} v''_i \right)\\
&= \sum_{1 \leqslant i_1 < \ldots < i_p < k} \left(\dot{\prod_{1 \leqslant i \leqslant i_1}} v'_i \right) \ldots \left(\dot{\prod_{i_p+1 \leqslant i \leqslant k}} v'_i \right) \otimes (v'''_1 \ldots v'''_{i_1}) \uplus \ldots \uplus (v'''_{i_p+1} \ldots v'''_k) \otimes v''_1 \cdot \ldots \cdot v''_k,
\end{aligned}$$

as V is commutative. By the cocommutativity of δ_V,

$$(\delta \otimes \mathrm{Id}) \circ \rho(v_1 \ldots v_k) = (\mathrm{Id} \otimes c) \circ (\rho \otimes \mathrm{Id}) \circ \delta(v_1 \ldots v_k),$$

so $(T(V), \uplus, \Delta, \delta, \rho)$ is a double bialgebra over V. □

2.2 Universal property of quasishuffle bialgebras

Let us recall the definition of connectivity for bialgebras:

Notations 2.1.

1. Let (B, m, Δ) be a bialgebra, of unit 1_B and of counit ε_Δ. For any $x \in \mathrm{Ker}(\varepsilon_\Delta)$, we put
$$\tilde{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x.$$
Then $\tilde{\Delta}$ is a coassociative coproduct on $\mathrm{Ker}(\varepsilon_\Delta)$. Its iterations will be denoted by $\tilde{\Delta}^{(n)} : \mathrm{Ker}(\varepsilon_\Delta) \longrightarrow \mathrm{Ker}(\varepsilon_\Delta)^{\otimes(n+1)}$, inductively defined by
$$\tilde{\Delta}^{(n)} = \begin{cases} \mathrm{Id}_{\mathrm{Ker}(\varepsilon_\Delta)} \text{ if } n = 0,\\ (\tilde{\Delta}^{(n-1)} \otimes \mathrm{Id}) \circ \tilde{\Delta} \text{ otherwise.} \end{cases}$$

2. The bialgebra (B, m, Δ) is connected if
$$\mathrm{Ker}(\varepsilon_\Delta) = \bigcup_{n=0}^{\infty} \mathrm{Ker}\left(\tilde{\Delta}^{(n)}\right).$$

3. If (B, m, Δ) is a connected bialgebra, we put, for $n \geqslant 0$,

$$B_{\leqslant n} = \mathbb{K}1_B \oplus \mathrm{Ker}\left(\tilde{\Delta}^{(n)}\right).$$

As B is a connected, this is a filtration of B, known as the coradical filtration [1, 15]. Moreover, for any $n \geqslant 1$, because of the coassociativity of $\tilde{\Delta}$,

$$\tilde{\Delta}(B_{\leqslant n}) \subseteq B_{\leqslant n-1}^{\otimes 2}.$$

In the case of bialgebras over a bialgebra $(V, \cdot, \delta_V)$, the connectedness is sometimes automatic:

Proposition 2.2. *Let $(V, \cdot, \Delta)$ be a nonunitary bialgebra. For any $n \geqslant 1$, we put*

$$V^{\cdot n} = \mathrm{Vect}(v_1 \cdot \ldots \cdot v_n,\ v_1, \ldots, v_n \in V).$$

If $\bigcap_{n \geqslant 1} V^{\cdot n} = (0)$, then any bialgebra over V is a connected bialgebra.

Proof. Let (B, m, Δ, ρ) be a bialgebra over V and let $x \in \mathrm{Ker}(\varepsilon_\Delta)$. We put

$$\rho(x) = \sum_{i=1}^{p} x_i \otimes v_i.$$

Let us denote by W the vector space generated by the elements v_i. By definition, this is a finite-dimensional vector space and $\rho(x) \in B \otimes W$. As W is finite-dimensional, the decreasing sequence of vector spaces $(W \cap V^{\cdot n})_{n \geqslant 1}$ is stationary, so there exists $N \geqslant 1$ such that if $n \geqslant N$, $W \cap V^{\cdot n} = W \cap V^{\cdot N}$. Therefore

$$W \cap V^{\cdot N} = W \cap \bigcap_{n \geqslant 1} V^{\cdot n} = (0).$$

Moreover,

$$\underbrace{m_{1,3,\ldots,2N-1,24\ldots2N} \circ \rho^{\otimes N} \circ \tilde{\Delta}^{(N-1)}(x)}_{\in B^{\otimes N} \otimes V^{\cdot N}} = \underbrace{(\tilde{\Delta}^{(N-1)} \otimes \mathrm{Id}) \circ \rho(x)}_{\in B^{\otimes N} \otimes W}.$$

As $V^{\cdot N} \cap W = (0)$, $(\tilde{\Delta}^{(N-1)} \otimes \mathrm{Id}) \circ \rho(x) = 0$. Then

$$(\mathrm{Id}^{\otimes N} \otimes \epsilon_V) \circ (\tilde{\Delta}^{(N-1)} \otimes \mathrm{Id}) \circ \rho(x) = \tilde{\Delta}^{(N-1)}(x) = 0.$$

So (B, m, Δ) is connected. □

Example 2.1. 1. If $(V, \cdot, \delta_V)$ is the bialgebra of the semigroup $(\mathbb{N}_{>0}, +)$, then $\bigcap_{n \geqslant 1} V^{\cdot n} = (0)$. We recover the classical result that any $\mathbb{N}$-graded bialgebra B such that $B_0 = \mathbb{K}1_B$ is connected. This also works for algebras of semigroups $\mathbb{N}^n \setminus \{0\}$, for example.

2. This does not hold if V is unitary, as then $V^{\cdot n} = V$ for any $n \in \mathbb{N}$.

Theorem 2.3. *Let V be a nonunitary, commutative bialgebra and let (B, m, Δ, ρ) be a connected bialgebra over V. For any character λ of B, there exists a unique morphism ϕ from (B, m, Δ, ρ) to $(T(V), \uplus, \Delta, \rho)$ of bialgebras over V such that $\epsilon_\delta \circ \phi = \lambda$. Moreover, for any $x \in \mathrm{Ker}(\varepsilon_\Delta)$,*

$$\phi(x) = \sum_{n=1}^{\infty} \underbrace{((\lambda \otimes \mathrm{Id}) \circ \rho)^{\otimes n} \circ \tilde{\Delta}^{(n-1)}(x)}_{\in V^{\otimes n}}. \tag{1}$$

Proof. Let us first prove that for any $\lambda \in V^*$ such that $\lambda(1_B) = 1$, there exists a unique coalgebra morphism $\phi : (B, \Delta, \rho) \longrightarrow (T(V), \Delta, \rho)$ of coalgebras over V such that $\epsilon_\delta \circ \phi = \lambda$.

Existence. Let $\phi : B \longrightarrow \mathrm{QSh}(V)$ defined by (1) and by $\phi(1_B) = 1$. By connectivity of B, (1) makes perfectly sense. Let us prove that ϕ is a coalgebra morphism. As $\phi(1_B) = 1$, it is enough to prove that for any $x \in \mathrm{Ker}(\varepsilon_\Delta)$, $\tilde{\Delta} \circ \phi(x) = (\phi \otimes \phi) \circ \tilde{\Delta}(x)$. We shall use Sweedler's notation $\tilde{\Delta}^{(n-1)}(x) = x^{(1)} \otimes \ldots \otimes x^{(n)}$.

$$\begin{aligned}
&\tilde{\Delta} \circ \phi(x)\\
&= \sum_{n=1}^{\infty} \lambda\left(x_0^{(1)}\right) \ldots \lambda\left(x_0^{(n)}\right) \tilde{\Delta}\left(x_1^{(1)} \ldots x_1^{(n)}\right)\\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} \lambda\left(x_0^{(1)}\right) \ldots \lambda\left(x_0^{(n)}\right) x_1^{(1)} \ldots x_1^{(i)} \otimes x_1^{(i+1)} \ldots x_1^{(n)}\\
&= \sum_{i,j \geqslant 1} \lambda\left(x_0^{(1)(1)}\right) \ldots \lambda\left(x_0^{(1)(i)}\right) \lambda\left(x_0^{(2)(1)}\right) \ldots \lambda\left(x_0^{(2)(j)}\right) x_1^{(1)(1)} \ldots x_1^{(1)(i)} \otimes x_1^{(2)(1)} \ldots x_1^{(2)(j)}\\
&= (\phi \otimes \phi)\left(x^{(1)} \otimes x^{(2)}\right)\\
&= (\phi \otimes \phi) \circ \tilde{\Delta}(x).
\end{aligned}$$

Let us prove that $\epsilon_\delta \circ \phi = \lambda$. If $x = 1_B$, then $\epsilon_\delta \circ \phi(1_B) = \epsilon_\delta(1) = 1 = \lambda(1_B)$. If $x \in \mathrm{Ker}(\varepsilon_\Delta)$, as $\epsilon_\delta(V^{\otimes n}) = (0)$ for any $n \geqslant 2$,

$$\epsilon_\delta \circ \phi(x) = \epsilon_\delta \circ (\lambda \otimes \mathrm{Id}) \circ \rho \circ \tilde{\Delta}^{(0)}(x) + 0 = \lambda\left((\mathrm{Id} \otimes \epsilon_\delta) \circ \rho(x)\right) = \lambda(x).$$

Let us prove that ϕ is a comodule morphism. If $x = 1_B$, then

$$\rho \circ \phi(1_B) = 1 \otimes 1 = (\phi \otimes \mathrm{Id})(1_B \otimes 1) = (\phi \otimes \mathrm{Id}) \circ \rho(1_B).$$

Let us assume that $x \in \mathrm{Ker}(\varepsilon_\Delta)$.

$$\begin{aligned}
(\phi \otimes \mathrm{Id}) \circ \rho(x) &= \phi(x_0) \otimes x_1\\
&= \sum_{n=1}^{\infty} \lambda\left((x_0)_0^{(1)}\right) \ldots \lambda\left((x_0)_0^{(n)}\right) (x_0)_1^{(1)} \ldots (x_0)_1^{(n)} \otimes x_1\\
&= \sum_{n=1}^{\infty} \lambda\left(x_{00}^{(1)}\right) \ldots \lambda\left(x_{00}^{(n)}\right) x_{01}^{(1)} \ldots x_{01}^{(n)} \otimes x_1^{(1)} \cdot \ldots x_1^{(n)}\\
&= \sum_{n=1}^{\infty} \lambda\left(x_0^{(1)}\right) \ldots \lambda\left(x_0^{(n)}\right) x_1^{(1)} \ldots x_1^{(n)} \otimes x_2^{(1)} \cdot \ldots \cdot x_2^{(n)}\\
&= \sum_{n=1}^{\infty} \lambda\left(x_0^{(1)}\right) \ldots \lambda\left(x_0^{(n)}\right) \rho\left(x_1^{(1)} \ldots x_1^{(n)}\right)\\
&= \rho \circ \phi(x).
\end{aligned}$$

Uniqueness. Let $\psi : (B, \Delta, \rho) \longrightarrow (T(V), \Delta, \rho)$ such that $\epsilon_\delta \circ \psi = \lambda$. As 1 is the unique group-like element of $\mathrm{QSh}(V)$, necessarily $\psi(1_B) = 1 = \phi(1_B)$. It is now enough to prove that $\psi(x) = \phi(x)$ for any $x \in \mathrm{Ker}(\varepsilon_\Delta)$. We assume that $x \in B_{\leqslant n}$ and we proceed by induction on n. If $n = 0$, there is nothing to prove. Let us assume that $n \geqslant 1$. As $\tilde{\Delta}(x) \in B_{\leqslant n-1}^{\otimes 2}$, by the induction hypothesis,

$$\tilde{\Delta} \circ \psi(x) = (\psi \otimes \psi) \circ \tilde{\Delta}(x) = (\phi \otimes \phi) \circ \tilde{\Delta}(x) = \tilde{\Delta} \circ \phi(x),$$

so $\psi(x)-\phi(x)\in \mathrm{Ker}(\tilde{\Delta})=V$. We put $\psi(x)-\phi(x)=v\in V$. Then

$$\begin{aligned} v &= (\epsilon_V\otimes \mathrm{Id})\circ \delta_V(v)\\ &= (\epsilon_\delta\otimes \mathrm{Id})\circ \rho(v)\\ &= (\epsilon_\delta\otimes \mathrm{Id})\circ \rho\circ\phi(x)-(\epsilon_\delta\otimes \mathrm{Id})\circ \rho\circ\psi(x)\\ &= (\epsilon_\delta\otimes \mathrm{Id})\circ(\phi\otimes \mathrm{Id})(x)-(\epsilon_\delta\otimes \mathrm{Id})\circ(\psi\otimes \mathrm{Id})(x)\\ &= (\lambda\otimes \mathrm{Id})(x)-(\lambda\otimes \mathrm{Id})(x)\\ &= 0. \end{aligned}$$

So $\psi(x)=\phi(x)$.

Let us now consider a character λ. As $\lambda(1_B)=1$, we already proved that there exists a unique coalgebra morphism $\phi:(B,\Delta,\rho)\longrightarrow(T(V),\Delta,\rho)$ such that $\epsilon_\delta\circ\phi=\lambda$. Let us prove that it is an algebra morphism. We consider the two morphisms $\phi_1=\uplus\circ(\phi\otimes\phi)$ and $\phi_2:\phi\circ m$, both from $B\otimes B$ to $\mathrm{QSh}(V)$. As ϕ, $\uplus$ and m are both comodule and coalgebra morphisms, ϕ_1 and ϕ_2 are comodule and coalgebra morphisms. Moreover, $B\otimes B$ is connected and, as ϵ_δ is a character of $(T(V),\uplus)$ and λ is a character of (B,m),

$$\epsilon_\delta\circ\uplus\circ(\phi\otimes\phi)=(\epsilon_\delta\otimes\epsilon_\delta)\circ(\phi\otimes\phi)=\lambda\otimes\lambda=\lambda\otimes m=\epsilon_\delta\circ\phi\circ m.$$

So $\epsilon_\delta\circ\phi_1=\epsilon_\delta\circ\phi_2$. By the *uniqueness* part, $\phi_1=\phi_2$. □

Lemma 2.4. *1. The double bialgebras* $\mathrm{QSh}(\mathbb{K})=(T(\mathbb{K}),\uplus,\Delta,\delta)$ *and* $(\mathbb{K}[X],m,\Delta,\delta)$ *are isomorphic, through the map*

$$\Upsilon:\left\{\begin{array}{rcl}\mathrm{QSh}(\mathbb{K})&\longrightarrow&\mathbb{K}[X]\\ \lambda_1\ldots\lambda_n&\longrightarrow&\lambda_1\ldots\lambda_nH_n(X),\end{array}\right.$$

where H_n is the n-th Hilbert polynomial

$$H_n(X)=\frac{X(X-1)\ldots(X-n+1)}{n!}.$$

2. Let V be a nonunitary, commutative and cocommutative bialgebra. The following map is a morphism of double bialgebras:

$$\Upsilon_V:\left\{\begin{array}{rcl}\mathrm{QSh}(V)&\longrightarrow&\mathbb{K}[X]\\ v_1\ldots v_n&\longrightarrow&\epsilon_V(v_1)\ldots\epsilon(v_n)H_n(X).\end{array}\right.$$

Proof. 1. In order to simplify the reading of the proof, the element $1\in\mathbb{K}\subseteq\mathrm{QSh}(\mathbb{K})$ is denoted by x. We apply Theorem 2.3 with $B=\mathbb{K}[X]$, with its usual product m and coproducts Δ and δ. with the character ϵ_δ of $\mathbb{K}[X]$, which sends any polynomial P on $P(1)$. Let us denote by ϕ the following morphism. Then $\phi(X)=\epsilon_\delta(X)x=x$. By multiplicativity, for any $n\geqslant 1$,

$$\phi(X^n)=x^{\uplus n}=n!x^n+\text{a linear span of } x^k \text{ with } k<n.$$

By triangularity, ϕ is an isomorphism. Let us denote by Υ the inverse isomorphism, and let us prove that $\Upsilon(x^n)=H_n(X)$ for any n by induction on n. This obvious if $n=0$ or 1. Let us assume that $n\geqslant 2$. Let us prove that for any $0\leqslant k\leqslant n-1$, $\Upsilon(x^n)(k)=0$ by induction on k. As $\varepsilon_\Delta\circ\Upsilon=\varepsilon_\Delta$,

$$\Upsilon(x^n)(0)=\varepsilon_\Delta\circ\Upsilon(x^n)=\varepsilon_\Delta(x^n)=0.$$

If $k \geqslant 1$, as Υ is a coalgebra morphism,

$$\begin{aligned}\Upsilon(x^n)(k) &= \Upsilon(x^n)(k-1+1)\\ &= \Delta \circ \Upsilon(x^n)(k-1,1)\\ &= (\Upsilon \otimes \Upsilon) \circ \Delta(x^n)(k-1,k)\\ &= \sum_{l=0}^{n} \Upsilon(x^l)(k-1)\Upsilon(x^{n-l})(1)\\ &= \Upsilon(x^n)(k-1) + \sum_{l=1}^{n-1} \Upsilon_l(k-1)\Upsilon_{n-l}(1) + \Upsilon(x^n)(1)\\ &= \Upsilon(x^n)(1),\end{aligned}$$

by the induction hypotheses on k and n. As $\epsilon_\delta \circ \phi = \epsilon_\delta$, we obtain that $\epsilon_\delta \circ \Upsilon = \epsilon_\delta$,

$$\Upsilon(x^n)(1) = \epsilon_\delta \circ \Upsilon(x^n) = \epsilon_\delta(x^n) = 0.$$

Therefore, $\Upsilon(x^n)$ is a multiple of $X(X-1)\dots(X-n+1)$. By triangularity of ϕ, we obtain that

$$\Upsilon(x^n) = \frac{X^n}{n!} + \text{terms of degree} < n.$$

Consequently, $\Upsilon(x^n) = H_n(X)$.

2. The counit $\epsilon_V : V \longrightarrow \mathbb{K}$ is a bialgebra morphism. By functoriality, we obtain a double bialgebra morphism from $\mathrm{QSh}(V)$ to $\mathrm{QSh}(\mathbb{K})$, which sends $v_1 \dots v_n \in V^{\otimes n}$ to $\epsilon_V(v_1)\dots\epsilon_V(v_n)x^n$. Composing with the isomorphism of the preceding item, we obtain Υ_V. □

As any bialgebra is trivially a bialgebra over $\mathbb{K}$, we immediately obtain:

Corollary 2.5. *Let (B, m, Δ) be a connected bialgebra and let λ be a character of B. There exists a unique bialgebra morphism $\phi : (B, m, \Delta) \longrightarrow (\mathbb{K}[X], m, \Delta)$ such that for any $x \in B$, $\phi(x)(1) = \lambda(x)$. For any $x \in \mathrm{Ker}(\varepsilon_\Delta)$,*

$$\phi(x) = \sum_{n=1}^{\infty} \lambda^{\otimes n} \circ \tilde{\Delta}^{(n-1)}(x) H_n(X).$$

When V is the bialgebra of the semigroup $(\mathbb{N}_{>0}, +)$, we recover Aguiar, Bergeron and Sottile's result [2], with Proposition 2.2:

Corollary 2.6. *Let (B, m, Δ) be a graded bialgebra with $B_0 = \mathbb{K}1_B$ and let λ be a character of B. There exists a unique bialgebra morphism $\phi : (B, m, \Delta) \longrightarrow (\mathbf{QSym}, \uplus, \Delta)$ such that $\epsilon_\delta \circ \phi = \lambda$.*

2.3 Double bialgebra morphisms

Theorem 2.7. *Let V be a nonunitary, commutative and cocommutative bialgebra, and let $(B, m, \Delta, \delta, \rho)$ be a connected double bialgebra over V. There exists a unique morphism ϕ from $(B, m, \Delta, \delta, \rho)$ to $(T(V), \uplus, \Delta, \delta, \rho)$ of double bialgebras over V. For any $x \in \mathrm{Ker}(\varepsilon_\Delta)$,*

$$\phi(x) = \sum_{n=1}^{\infty} \underbrace{((\epsilon_\delta \otimes \mathrm{Id}) \circ \rho)^{\otimes n} \circ \tilde{\Delta}^{(n-1)}(x)}_{\in V^{\otimes n}}.$$

Proof. Uniqueness: such a morphism is a morphism ϕ from (B, m, Δ, ρ) to (B, m, Δ, ρ) with $\epsilon_\delta \circ \phi = \epsilon_\delta$. By Theorem 2.3, it is unique.

Existence: let $\phi : (B, m, \Delta, \rho) \longrightarrow (B, m, \Delta, \rho)$ be the (unique) morphism such that $\epsilon_\delta \circ \phi = \epsilon_\delta$. Let us prove that for any $x \in B_{\leqslant n}$, $\delta \circ \phi(x) = (\phi \otimes \phi) \circ \delta(x)$ by induction on n. If $n = 0$, we can assume that $x = 1_B$. Then

$$\delta \circ \phi(1_B) = (\phi \otimes \phi) \circ \delta(1_B) = 1 \otimes 1.$$

Let us assume the result at all ranks $< n$, with $n \geqslant 2$. Let $x \in \mathrm{Ker}(\varepsilon_\Delta)$. As $(\varepsilon_\Delta \otimes \mathrm{Id}) \circ \delta(x) = \varepsilon_\Delta(x)1$, $\delta(x) \in \mathrm{Ker}(\varepsilon_\Delta) \otimes B$.

$$\begin{aligned}
(\tilde{\Delta} \otimes \mathrm{Id}) \circ \delta \circ \phi(x) &= m_{1,3,24} \circ (\delta \otimes \delta) \circ \tilde{\Delta} \circ \phi(x) \\
&= m_{1,3,24} \circ (\delta \otimes \delta) \circ (\phi \otimes \phi) \circ \tilde{\Delta}(x) \\
&= m_{1,3,24} \circ (\phi \otimes \phi \otimes \phi \otimes \phi) \circ (\delta \otimes \delta) \circ \tilde{\Delta}(x) \\
&= (\phi \otimes \phi \otimes \phi) \circ m_{1,3,24} \circ (\delta \otimes \delta) \circ \tilde{\Delta}(x) \\
&= (\phi \otimes \phi \otimes \phi) \circ (\tilde{\Delta} \otimes \mathrm{Id}) \circ \delta(x) \\
&= (\tilde{\Delta} \otimes \mathrm{Id}) \circ (\phi \otimes \phi) \circ \tilde{\Delta}(x).
\end{aligned}$$

We used the induction hypothesis on the both sides of the tensors appearing in $\tilde{\Delta}(x)$ for the third equality. We deduce that $(\delta \circ \phi - \phi \otimes \phi) \circ \delta(x) \in \mathrm{Ker}(\tilde{\Delta} \otimes \mathrm{Id}) = V \otimes T(V)$. Moreover,

$$\begin{aligned}
(\mathrm{Id} \otimes c) \circ (\rho \otimes \mathrm{Id}) \circ \delta \circ \phi(x) &= (\delta \otimes \mathrm{Id}) \circ \rho \circ \phi(x) \\
&= (\delta \otimes \mathrm{Id}) \circ (\phi \otimes \mathrm{Id}) \circ \rho(x), \\
(\mathrm{Id} \otimes c) \circ (\rho \otimes \mathrm{Id}) \circ (\phi \otimes \phi) \circ \delta(x) &= (\mathrm{Id} \otimes c) \circ (\phi \otimes \mathrm{Id} \otimes \phi) \circ (\rho \otimes \mathrm{Id}) \circ \delta(x) \\
&= (\phi \otimes \phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes c) \circ (\rho \otimes \mathrm{Id}) \circ \delta(x) \\
&= (\phi \otimes \phi \otimes \mathrm{Id}) \circ (\delta \otimes \mathrm{Id}) \circ \rho(x).
\end{aligned}$$

Putting $y = (\delta \circ \phi - \phi \otimes \phi) \circ \delta(x) \in V \otimes T(V)$, we proved that

$$(\mathrm{Id} \otimes c) \circ (\rho \otimes \mathrm{Id})(y) = ((\delta \circ \phi - (\phi \otimes \phi) \circ \delta) \otimes \mathrm{Id}) \circ \rho(x).$$

As $y \in V \otimes T(V)$,

$$\rho \otimes \mathrm{Id}(y) = \delta_V \otimes \mathrm{Id}(y).$$

Consequently,

$$(\epsilon_\delta \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ (\rho \otimes \mathrm{Id})(y) = (\epsilon_V \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ (\delta_V \otimes \mathrm{Id})(y) = y.$$

Moreover,

$$\begin{aligned}
(\epsilon_\delta \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ (\rho \otimes \mathrm{Id})(y) &= (\epsilon_\delta \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ (\delta \circ \phi \otimes \mathrm{Id}) \circ \rho(x) \\
&\quad - (\epsilon_\delta \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ (((\phi \otimes \phi) \circ \delta) \otimes \mathrm{Id}) \circ \rho(x) \\
&= (\phi \otimes \mathrm{Id}) \circ \rho(x) - ((((\epsilon_\delta \circ \phi) \otimes \phi) \circ \delta) \otimes \mathrm{Id}) \circ \rho(x) \\
&= (\phi \otimes \mathrm{Id}) \circ \rho(x) - (((\epsilon_\delta \otimes \phi) \circ \delta) \otimes \mathrm{Id}) \circ \rho(x) \\
&= (\phi \otimes \mathrm{Id}) \circ \rho(x) - (\phi \otimes \mathrm{Id}) \circ \rho(x) \\
&= 0.
\end{aligned}$$

Hence, $y = 0$, so $\delta \circ \phi(x) = (\phi \otimes \phi) \circ \delta(x)$. □

Applying to $V = \mathbb{K}$ or $V = \mathbb{K}(> 0, +)$:

Corollary 2.8. *1. Let (B, m, Δ) be a connected double bialgebra. There exists a unique double bialgebra morphism ϕ from (B, m, Δ, δ) to $(\mathbb{K}[X], m, \Delta, \delta)$. For any $x \in \mathrm{Ker}(\varepsilon_\Delta)$,*

$$\phi(x) = \sum_{n=1}^{\infty} \epsilon_\delta^{\otimes n} \circ \tilde{\Delta}^{(n-1)}(x) H_n(X).$$

2. Let (B, m, Δ) be a graded, connected double bialgebra, such that for any $n \in \mathbb{N}$,

$$\delta(B_n) \subseteq B_n \otimes B.$$

There exists a unique homogeneous double bialgebra morphism ϕ from (B, m, Δ, δ) to $(\mathbf{QSym}, ⧢, \Delta, \delta)$. For any $x \in \mathrm{Ker}(\varepsilon_\Delta)$,

$$\phi(x) = \sum_{n=1}^{\infty} \sum_{k_1, \ldots, k_n \geqslant 1} \epsilon_\delta^{\otimes n} \circ (\pi_{k_1} \otimes \ldots \otimes \pi_{k_n}) \circ \tilde{\Delta}^{(n-1)}(x)(k_1, \ldots, k_n).$$

3. Let Ω be a commutative monoid and let (B, m, Δ) be a connected Ω-graded double bialgebra, connected as a coalgebra, such that for any $\alpha \in \Omega$,

$$\delta(B_\alpha) \subseteq B_\alpha \otimes B.$$

There exists a unique homogeneous double bialgebra morphism ϕ from (B, m, Δ, δ) to $\mathrm{QSh}(\mathbb{K}\Omega)$. For any $x \in \mathrm{Ker}(\varepsilon_\Delta)$,

$$\phi(x) = \sum_{n=1}^{\infty} \sum_{\alpha_1, \ldots, \alpha_n \in \Omega} \epsilon_\delta^{\otimes n} \circ (\pi_{\alpha_1} \otimes \ldots \otimes \pi_{\alpha_n}) \circ \tilde{\Delta}^{(n-1)}(x)(\alpha_1, \ldots, \alpha_n).$$

As an application, let us give a generalization of Hoffman's isomorphism between shuffle and quasishuffle algebras [10, 11]:

Theorem 2.9. *Let $(V, \cdot)$ be a nonunitary, commutative algebra. The following map is a Hopf algebra isomorphism:*

$$\Theta_V : \left\{ \begin{array}{rcl} \mathrm{Sh}(V) = (T(V), ⧢, \Delta) & \longrightarrow & \mathrm{QSh}(V) = (T(V), ⧢, \Delta) \\ w & \longrightarrow & \displaystyle\sum_{\substack{w = w_1 \ldots w_k, \\ w_1, \ldots, w_k \neq \emptyset}} \frac{1}{\ell(w_1)! \ldots \ell(w_k)!} |w_1| \ldots |w_k|, \end{array} \right.$$

where for any word w, $|w|$ is the product in V of its letters, and $\ell(w)$ its length.

Proof. We first prove this result when $(V, \cdot, \delta_V)$ is a commutative, cocommutative, counitary bialgebra, of counit ϵ_V. First, observe that $(T(V), ⧢, \Delta, \rho)$ is a bialgebra over $(V, \cdot, \delta_V)$ and that the following map is a character of $(T(V), ⧢)$: for any word $w = x_1 \ldots x_k$,

$$\lambda(w) = \frac{1}{k!} \epsilon_V(x_1) \ldots \epsilon_V(x_k).$$

By the universal property of the quasishuffle algebra, there exists a unique Hopf algebra morphism $\Theta_V : (T(V), ⧢, \Delta) \longrightarrow (T(V), ⧢, \Delta)$ such that $\epsilon \circ \Theta_V = \lambda$. For any word $w = v_1 \ldots v_k$,

$$\begin{aligned} (\lambda \otimes \mathrm{Id}) \circ \rho(w) &= \lambda(v_1' \ldots v_k') \; v_1'' \cdot \ldots \cdot v_k'' \\ &= \frac{1}{k!} \epsilon_V(v_1') \ldots \epsilon_V(v_k') \; v_1'' \cdot \ldots \cdot v_k'' \\ &= \frac{1}{k!} v_1 \cdot \ldots \cdot v_k \\ &= \frac{1}{\ell(w)!} |w|. \end{aligned}$$

Hence,

$$\begin{aligned}
\Theta_V(w) &= \sum_{k=1}^{\infty} ((\lambda \otimes \mathrm{Id}) \circ \rho)^{\otimes k} \circ \tilde{\Delta}^{(k-1)}(w) \\
&= \sum_{\substack{w=w_1\ldots w_k,\\ w_1,\ldots,w_k \neq \emptyset}} ((\lambda \otimes \mathrm{Id}) \circ \rho)^{\otimes k}(w_1 \otimes \ldots \otimes w_k) \\
&= \sum_{\substack{w=w_1\ldots w_k,\\ w_1,\ldots,w_k \neq \emptyset}} \frac{1}{\ell(w_1)!\ldots \ell(w_k)!}|w_1|\ldots|w_k|.
\end{aligned}$$

Let us now consider an commutative algebra $(V,\cdot)$. Let $(S(V),m,\Delta)$ be the symmetric algebra generated by V, with its usual product and coproduct. Applying the first item to $S(V)$, we obtain a Hopf algebra morphism $\Theta_{S(V)} : (T(S(V)), \sqcup\!\sqcup, \Delta) \longrightarrow (T(S(V)), \uplus, \Delta)$. By restriction, we obtain a Hopf algebra morphism $\Theta_{S_+(V)} : (T(S_+(V)), \sqcup\!\sqcup, \Delta) \longrightarrow (T(S_+(V)), \uplus, \Delta)$. The canonical algebra morphism $\pi : S_+(V) \longrightarrow V$, sending $v_1 \ldots v_k$ to $v_1 \cdot \ldots \cdot v_k$ (which exists as V is commutative), induces a surjective morphism $\varpi : T(S_+(V)) \longrightarrow T(V)$, which is obviously a Hopf algebra morphism from $(T(S_+(V)), \sqcup\!\sqcup, \Delta)$ to $(T(V), \sqcup\!\sqcup, \Delta)$ and from $(T(S_+(V)), \uplus, \Delta)$ to $(T(V), \uplus, \Delta)$. Moreover, the following diagram is commutative:

$$\begin{array}{ccc}
(T(S_+(V)), \sqcup\!\sqcup, \Delta) & \xrightarrow{\Theta_{S+(V)}} & (T(S_+(V)), \uplus, \Delta) \\
\downarrow \varpi & & \downarrow \varpi \\
(T(V), \sqcup\!\sqcup, \Delta) & \xrightarrow[\Theta_V]{} & (T(V), \uplus, \Delta)
\end{array}$$

As the vertical arrows are surjective Hopf algebra morphisms and the top horizontal arrow is also a Hopf algebra morphism, the bottom horizontal arrow is also a Hopf algebra morphism. For any word w, $\Theta_V(w) - w$ is a linear span of words of length $< \ell(w)$. By a triangularity argument, Θ_V is bijective. □

Remark 2.1. Using the same argument as in [10], it is not difficult to prove that for any nonempty word $w \in T(V)$,

$$\Theta_V^{-1}(w) = \sum_{\substack{w=w_1\ldots w_k,\\ w_1,\ldots,w_k \neq \emptyset}} \frac{(-1)^{\ell(w)+k}}{\ell(w_1)\ldots \ell(w_k)}|w_1|\ldots|w_k|.$$

It is immediate to show that Θ is a natural transformation from the functor Sh to the functor QSh, that is to say, if $\alpha : V \longrightarrow W$ is a morphism between two commutative non unitary algebras, then $T(\alpha) \circ \Theta_V = \Theta_W \circ T(\beta)$, as Hopf algebra morphisms from $\mathrm{Sh}(V)$ to $\mathrm{QSh}(W)$. Let us prove a unicity result:

Proposition 2.10. *Let Υ be a natural transformation from the functor* Sh *to the functor* QSh *(functors from the category of commutative nonunitary algebras to the category of Hopf algebras). There exists $\mu \in \mathbb{K}$ such that $\Upsilon = \Theta \circ \Phi^{(\mu)}$, where $\Phi^{(\mu)}$ is the natural transformation from* Sh *to* Sh *defined for any commutative nonunitary algebra V by*

$$\forall v_1, \ldots, v_n \in V, \qquad \Phi_V^{(\mu)}(v_1 \ldots v_n) = \mu^n v_1 \ldots v_n.$$

Proof. Let Υ be a natural transformation from Sh to QSh. For any commutative nonunitary algebra V, let us denote by $\pi_V : T(V) \longrightarrow V$ the canonical projection on V and let us put $\varpi_V = \pi_V \circ \Upsilon_V$. As Υ_V is an endomorphism of the cofree coalgebra $(T(V),\Delta)$, for any nonempty word $w \in T(V)$,

$$\Upsilon_V(w) = \sum_{\substack{w=w_1\ldots w_k,\\ w_1,\ldots,w_k \neq \emptyset}} \varpi_V(w_1)\ldots \varpi_V(w_k). \tag{2}$$

Let V be the augmentation ideal of $\mathbb{K}[X_1,\ldots,X_n]$. We consider $(\lambda_1,\ldots,\lambda_n)\in\mathbb{K}^n$ and the endomorphism α of V defined by $\alpha(x_i)=\lambda_i X_i$. By naturality of Υ,

$$\lambda_1\ldots\lambda_n\Upsilon_V(X_1\ldots X_n)=\Upsilon_V\circ T(\alpha)(X_1\ldots X_n)=T(\alpha)\circ\Upsilon_V(X_1\ldots X_n).$$

Applying π_V, we obtain

$$\lambda_1\ldots\lambda_n\varpi_V(X_1\ldots X_n)=\alpha\circ\varpi_V(X_1\ldots X_n).$$

Therefore, there exists $\mu_n\in\mathbb{K}$ such that

$$\varpi_V(X_1\ldots X_n)=\mu_n X_1\cdot\ldots\cdot X_n.$$

Let W be any nonunitary commutative algebra, $v_1,\ldots,v_n\in V$ and let $\beta:V\longrightarrow W$ be the morphism defined by $\phi(X_i)=v_i$. By naturality of Υ,

$$T(\beta)\circ\Upsilon_V(X_1\ldots X_n)=\Upsilon_W\circ T(\beta)(X_1\ldots X_n).$$

Applying π_W, we obtain

$$\beta\circ\varpi_V(X_1\ldots X_n)=\beta(\mu_n X_1\cdot\ldots\cdot X_n)=\mu_n v_1\cdot\ldots\cdot v_n=\varpi_W(v_1\ldots v_n).$$

We proved the existence of a family of scalars $(\mu_n)_{n\geqslant 0}$ such that for any commutative nonunitary algebra V, for any $v_1,\ldots,v_n\in V$, $\varpi_V(v_1\ldots v_n)=\mu_n v_1\cdot\ldots\cdot v_n$.

Let us study this sequence $(\mu_n)_{n\geqslant 0}$. Let V be the augmentation ideal of $\mathbb{K}[X]$. For any $k,l\geqslant 1$, as Υ_V is an algebra morphism from $\mathrm{Sh}(V)$ to $\mathrm{QSh}(V)$,

$$\begin{aligned}\varpi_V(X^{\otimes k}\sqcup\!\sqcup X^{\otimes l})&=\frac{(k+l)!}{k!l!}\varpi\left(X^{\otimes(k+l)}\right)\\&=\frac{(k+l)!}{k!l!}\mu_{k+l}X^{k+l},\\&=\pi_V\left(\Upsilon_V\left(X^{\otimes k}\sqcup\!\sqcup X^{\otimes l}\right)\right)\\&=\pi_V\left(\Upsilon_V\left(X^{\otimes k}\right)\uplus\Upsilon_V\left(X^{\otimes k}\right)\right)\\&=\varpi_V\left(X^{\otimes k}\right)\cdot\varpi_V\left(X^{\otimes l}\right)\\&=\mu_k\mu_l X^{k+l}.\end{aligned}$$

Hence, $(k+l)!\mu_{k+l}=k!\mu_k l!\mu_l$. This implies that for any $k\in\mathbb{N}$, $\mu_k=\dfrac{\mu^k}{k!}$, with $\mu=\mu_1$. Therefore, by (2), for any nonunitary commutative algebra, for any nonempty word $w\in T(V)$,

$$\begin{aligned}\Upsilon_V(w)&=\sum_{\substack{w=w_1\ldots w_k,\\ w_1,\ldots,w_k\neq\emptyset}}\frac{\mu^{\ell(w_1)+\ldots+\ell(w_k)}}{\ell(w_1)!\ldots\ell(w_k)!}|w_1|\ldots|w_k|\\&=\mu^{\ell(w)}\sum_{\substack{w=w_1\ldots w_k,\\ w_1,\ldots,w_k\neq\emptyset}}\frac{1}{\ell(w_1)!\ldots\ell(w_k)!}|w_1|\ldots|w_k|\\&=\Theta_V\circ\Phi_V^{(\mu)}(w).\end{aligned}$$

In other words, $\Upsilon=\Theta\circ\Phi^{(\mu)}$. □

Remark 2.2. For any $\mu\in\mathbb{K}$, for any commutative nonunitary algebra V $\Phi_V^{(\mu)}$ is indeed a Hopf algebra endomorphism of $\mathrm{Sh}(V)$, as $\mathrm{Sh}(V)$ is graded by the length of words.

2.4 Action on bialgebra morphisms

We here fix a bialgebra $(V, \cdot, \delta_V)$, nonunitary, commutative and cocommutative.

Notations 2.2. 1. Let (B, m, Δ) and (B', m', Δ') be bialgebras. We denote by $M_{B\to B'}$ the set of bialgebra morphisms from (B, m, Δ) to (B', m', Δ').

2. Let (B, m, Δ, ρ) and (B', m', Δ', ρ') be bialgebras over V. We denote by $M^\rho_{B\to B'}$ the set of morphisms of bialgebra over V from B to B', that is to say morphisms both of bialgebras and of comodules over V.

Proposition 2.11. *Let $(B, m, \Delta, \delta, \rho)$ be a double bialgebra over V and (B', m', Δ', ρ') be a bialgebra over V. The following map is a right action of the monoid of characters* $(\mathrm{Char}(B), \star)$ *attached to (B, m, δ) on $M^\rho_{B\to B'}$,*

$$\leftsquigarrow : \begin{cases} M^\rho_{B\to B'} \times \mathrm{Char}(B) & \longrightarrow & M^\rho_{B\to B'} \\ (\phi, \lambda) & \longrightarrow & \phi \leftsquigarrow \lambda = (\phi \otimes \lambda) \circ \delta. \end{cases}$$

Proof. Let $(\phi, \lambda) \in M^\rho_{B\to B'} \times \mathrm{Char}(B)$. Let us prove that $\psi = (\phi \otimes \lambda) \circ \delta$ is a bialgebra morphism. As ϕ, λ and δ are algebra morphisms, by composition ψ is an algebra morphism.

$$\begin{aligned}
\Delta' \circ \psi &= \Delta' \circ (\phi \otimes \lambda) \circ \delta \\
&= (\phi \otimes \phi) \circ \Delta \circ (\mathrm{Id} \otimes \lambda) \circ \delta \\
&= (\phi \otimes \phi \otimes \lambda) \circ (\Delta \otimes \mathrm{Id}) \circ \delta \\
&= (\phi \otimes \phi \otimes \lambda) \circ m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta \\
&= (\phi \otimes \lambda \otimes \phi \otimes \lambda) \circ (\delta \otimes \delta) \circ \Delta \\
&= (\psi \otimes \psi) \circ \Delta.
\end{aligned}$$

We used that λ is a character for the fifth equality. Moreover,

$$\varepsilon'_\Delta \circ \Psi = (\varepsilon'_\Delta \otimes \lambda) \circ \delta = \lambda \circ \eta \circ \varepsilon_\Delta = \varepsilon_\Delta,$$

as $\lambda(1_B) = 1$ so $\lambda \circ \eta = \mathrm{Id}_\mathbb{K}$. So $\psi \in M_{B\to B'}$. Let us now prove that ψ is a comodule morphism. As $\rho' \circ \phi = (\phi \otimes \mathrm{Id}) \circ \rho$,

$$\begin{aligned}
\rho' \circ \psi &= \rho' \circ (\phi \otimes \lambda) \circ \delta \\
&= (\phi \otimes \mathrm{Id} \otimes \lambda) \circ (\rho \otimes \mathrm{Id}) \circ \delta \\
&= (\phi \otimes \mathrm{Id} \otimes \lambda) \circ (\mathrm{Id} \otimes c) \circ (\delta \otimes \mathrm{Id}) \circ \rho \\
&= (\phi \otimes \lambda \otimes \mathrm{Id}) \circ (\delta \otimes \mathrm{Id}) \circ \rho \\
&= (\psi \otimes \mathrm{Id}) \circ \rho.
\end{aligned}$$

So $\psi \in M^\rho_{B\to B'}$.

Let $\phi \in M^\rho_{B\to B'}$, $\lambda, \mu \in \mathrm{Char}(B)$.

$$\begin{aligned}
(\phi \leftsquigarrow \lambda) \leftsquigarrow \mu &= (\phi \otimes \lambda \otimes \mu) \circ (\delta \otimes \mathrm{Id}) \circ \delta \\
&= (\phi \otimes \lambda \otimes \mu) \circ (\mathrm{Id} \otimes \delta) \circ \delta \\
&= (\phi \otimes \lambda \star \mu) \circ \delta \\
&= \phi \leftsquigarrow (\lambda \star \mu).
\end{aligned}$$

Moreover,

$$\phi \leftsquigarrow \epsilon_\delta = (\phi \otimes \epsilon_\delta) \circ \delta = \phi.$$

Therefore, $\leftsquigarrow$ is an action. □

Moreover, any bialgebra morphism is compatible with these actions:

Proposition 2.12. *Let $(B, m, \Delta, \delta, \rho)$ be a double bialgebra over V and B' and B'' be bialgebras over V. For any morphisms $\phi : B \longrightarrow B'$ and $\psi : B' \longrightarrow B''$ of bialgebras over V, for any character λ of B,*

$$(\psi \circ \phi) \leftsquigarrow \lambda = \psi \circ (\phi \leftsquigarrow \lambda).$$

Proof. Indeed,

$$(\psi \circ \phi) \leftsquigarrow \lambda = ((\psi \circ \phi) \otimes \lambda) \circ \delta = \psi \circ (\phi \otimes \lambda) \circ \delta = \psi \circ (\phi \leftsquigarrow \lambda). \qquad \square$$

Corollary 2.13. *Let $(B, m, \Delta, \delta, \rho)$ be a connected double bialgebra over V. Let us denote by $\phi_1 : B \longrightarrow \mathrm{QSh}(V)$ the unique morphism of double bialgebras of Theorem 2.7. The following maps are bijections, inverse one from the other:*

$$\theta : \begin{cases} \mathrm{Char}(B) & \longrightarrow & M^{\rho}_{B \to \mathrm{QSh}(V)} \\ \lambda & \longrightarrow & \phi_1 \leftsquigarrow \lambda, \end{cases} \qquad \theta' : \begin{cases} M^{\rho}_{B \to \mathrm{QSh}(V)} & \longrightarrow & \mathrm{Char}(B) \\ \phi & \longrightarrow & \epsilon_\delta \circ \phi. \end{cases}$$

Proof. Let $\phi \in M^{\rho}_{B \to \mathrm{QSh}(V)}$. We put $\phi' = \theta \circ \theta'$ and $\lambda = \epsilon_\delta \circ \phi$. Then

$$\epsilon_\delta \circ \phi' = \epsilon_\delta \circ (\phi_1 \leftsquigarrow \lambda) = (\epsilon_\delta \circ \phi_1) \star \lambda = \epsilon_\delta \star \lambda = \lambda = \epsilon_\delta \circ \phi.$$

By the uniqueness in Theorem 2.3, $\phi = \phi'$.

Let $\lambda \in \mathrm{Char}(B)$ and let $\lambda' = \theta' \circ \theta(\lambda)$. Then

$$\lambda' = \epsilon_\delta \circ (\phi_1 \leftsquigarrow \lambda) = (\epsilon_\delta \circ \phi_1 \otimes \lambda) \circ \delta = (\epsilon_\delta \otimes \lambda) \circ \delta = \epsilon_\delta \star \lambda = \lambda.$$

So θ and θ' are bijections, inverse one from the other. $\square$

Corollary 2.14.

1. *Let (B, m, Δ, δ) be a connected double bialgebra. Let us denote by ϕ_1 the unique morphism of double bialgebras from B to $\mathbb{K}[X]$ of Theorem 2.7. The following maps are bijections, inverse one from the other:*

$$\theta : \begin{cases} \mathrm{Char}(B) & \longrightarrow & M_{B \to \mathbb{K}[X]} \\ \lambda & \longrightarrow & \phi_1 \leftsquigarrow \lambda, \end{cases} \qquad \theta' : \begin{cases} M_{B \to \mathbb{K}[X]} & \longrightarrow & \mathrm{Char}(B) \\ \phi & \longrightarrow & \epsilon_\delta \circ \phi. \end{cases}$$

2. *Let (B, m, Δ, δ) be a connected, graded double bialgebra such that for any $n \in \mathbb{N}$,*

$$\delta(B_n) \subseteq B_n \otimes B.$$

Let us denote by ϕ_1 the unique homogeneous morphism of double bialgebras from B to $\mathbf{QSym}$ of Theorem 2.7. We denote by $M^0_{B \to \mathbf{QSym}}$ the set of bialgebra morphisms from (B, m, Δ) to $(\mathbf{QSym}, \uplus, \Delta)$ which are homogeneous of degree 0. The following maps are bijections, inverse one from the other:

$$\theta : \begin{cases} \mathrm{Char}(B) & \longrightarrow & M^0_{B \to \mathbf{QSym}} \\ \lambda & \longrightarrow & \phi_1 \leftsquigarrow \lambda, \end{cases} \qquad \theta' : \begin{cases} M^0_{B \to \mathbf{QSym}} & \longrightarrow & \mathrm{Char}(B) \\ \phi & \longrightarrow & \epsilon_\delta \circ \phi. \end{cases}$$

3. *Let Ω be a commutative monoid and let (B, m, Δ, δ) be a connected, Ω-graded double bialgebra, connected as a coalgebra, such that for any $\alpha \in \Omega$,*

$$\delta(B_\alpha) \subseteq B_\alpha \otimes B.$$

Let us denote by ϕ_1 the unique homogeneous morphism of double bialgebras from B to $\mathrm{QSh}(\mathbb{K}\Omega)$ of Theorem 2.7. We denote by $M^0_{B \to \mathrm{QSh}(\mathbb{K}\Omega)}$ the set of bialgebra morphisms from (B, m, Δ) to $\mathrm{QSh}(\mathbb{K}\Omega)$ which are homogeneous of degree the unit of Ω. The following maps are bijections, inverse one from the other:

$$\theta : \begin{cases} \mathrm{Char}(B) & \longrightarrow & M^0_{B \to \mathrm{QSh}(\mathbb{K}\Omega)} \\ \lambda & \longrightarrow & \phi_1 \leftsquigarrow \lambda, \end{cases} \qquad \theta' : \begin{cases} M^0_{B \to \mathrm{QSh}(\mathbb{K}\Omega)} & \longrightarrow & \mathrm{Char}(B) \\ \phi & \longrightarrow & \epsilon_\delta \circ \phi. \end{cases}$$

2.5 Applications to graphs

We postpone the detailed construction of the double bialgebras of V-decorated graphs to a forthcoming paper [7]. For any nonunitary commutative bialgebra $(V, \cdot, \delta_V)$, we obtain a double bialgebra over V of V-decorated graphs $\mathcal{H}_V[\mathbf{G}]$, generated by graphs G which any vertex v is decorated by an element $d_G(v)$, with conditions of linearity in each vertex. For example, if $v_1, v_2, v_3, v_4 \in V$ and $\lambda_2, \lambda_4 \in \mathbb{K}$, if $w_1 = v_1 + \lambda_2 v_2$ and $w_2 = v_3 + \lambda_4 v_4$,

$$\mathbf{I}\begin{smallmatrix}w_2\\w_1\end{smallmatrix} = \mathbf{I}\begin{smallmatrix}v_3\\v_1\end{smallmatrix} + \lambda_4 \mathbf{I}\begin{smallmatrix}v_4\\v_1\end{smallmatrix} + \lambda_2 \mathbf{I}\begin{smallmatrix}v_3\\v_2\end{smallmatrix} + \lambda_2\lambda_4 \mathbf{I}\begin{smallmatrix}v_4\\v_2\end{smallmatrix}.$$

The product is given by the disjoint union of graphs, the decorations being untouched. For any graph G, for any $X \subseteq V(G)$, we denote by $G_{|X}$ the graph defined by

$$G_{|X} = X, \qquad E(G_{|X}) = \{\{x, y\} \in E(G) \mid x, y \in X\}.$$

Then

$$\Delta(G) = \sum_{V(G) = A \sqcup B} G_{|A} \otimes G_{|B},$$

the decorations being untouched. For any equivalence relation $\sim$ on $V(G)$:

- $G/\sim$ is the graph defined by

  $$V(G/\sim) = V(G)/\sim, \qquad E(G/\sim) = \{\{\overline{x}, \overline{y}\} \mid \{x, y\} \in E(G),\ \overline{x} \neq \overline{y}\},$$

  where for any $z \in V(G)$, $\overline{z}$ is its class in $V(G)/\sim$.
- $G \mid\sim$ is the graph defined by

  $$V(G \mid\sim) = V(G), \qquad E(G \mid\sim) = \{\{x, y\} \in E(G) \mid x \sim y\}.$$

- We shall say that $\sim \in \mathcal{E}_c[G]$ if for any equivalence class X of $\sim$, $G_{|X}$ is connected.

With these notations, the second coproduct δ is given by

$$\delta(G) = \sum_{\sim \in \mathcal{E}_c[G]} G/\sim \otimes G \mid\sim.$$

Any vertex $w \in V(G/\sim) = V(G)/\sim$ is decorated by

$$\dot{\prod_{v \in w}} d_G(v)',$$

where the symbol $\dot{\prod}$ means that the product is taken in V (recall that any vertex of $V(G/\sim)$ is a subset of $V(G)$). Any vertex $v \in V(G \mid\sim) = V(G)$ is decorated by $d_G(v)''$. We use Sweedler's notation $\delta_V(v) = v' \otimes v''$, and it is implicit that in the expression of $\delta(G)$, everything is developed by multilinearity in the vertices. For example, if $v_1, v_2, v_3 \in V$,

$$\begin{aligned}\Delta(\mathbf{I}\begin{smallmatrix}v_3\\v_2\\v_1\end{smallmatrix}) &= \mathbf{I}\begin{smallmatrix}v_3\\v_2\\v_1\end{smallmatrix} \otimes 1 + 1 \otimes \mathbf{I}\begin{smallmatrix}v_3\\v_2\\v_1\end{smallmatrix} + \mathbf{I}\begin{smallmatrix}v_2\\v_1\end{smallmatrix} \otimes \bullet_{v_3} + \mathbf{I}\begin{smallmatrix}v_3\\v_2\end{smallmatrix} \otimes \bullet_{v_1} + \bullet_{v_1}\bullet_{v_3} \otimes \bullet_{v_2}\\ &\quad + \bullet_{v_3} \otimes \mathbf{I}\begin{smallmatrix}v_2\\v_1\end{smallmatrix} + \bullet_{v_1} \otimes \mathbf{I}\begin{smallmatrix}v_3\\v_2\end{smallmatrix} + \bullet_{v_2} \otimes \bullet_{v_1}\bullet_{v_3},\end{aligned}$$

$$\delta(\mathbf{I}\begin{smallmatrix}v_3\\v_2\\v_1\end{smallmatrix}) = \mathbf{I}\begin{smallmatrix}v_3'\\v_2'\\v_1'\end{smallmatrix} \otimes \bullet_{v_1''}\bullet_{v_2''}\bullet_{v_3''} + \bullet_{v_1' \cdot v_2' \cdot v_3'} \otimes \mathbf{I}\begin{smallmatrix}v_3''\\v_2''\\v_1''\end{smallmatrix} + \mathbf{I}\begin{smallmatrix}v_3'\\v_1' \cdot v_2'\end{smallmatrix} \otimes \mathbf{I}\begin{smallmatrix}v_2''\\v_1''\end{smallmatrix}\bullet_{v_3''} + \mathbf{I}\begin{smallmatrix}v_2' \cdot v_3'\\v_1'\end{smallmatrix} \otimes \mathbf{I}\begin{smallmatrix}v_3''\\v_2''\end{smallmatrix}\bullet_{v_1''}.$$

For any V-decorated graph,

$$\epsilon_\delta(G) = \begin{cases} \displaystyle\prod_{v \in V(G)} \epsilon_V(d_G(v)) \text{ if } E(G) = \emptyset,\\ 0 \text{ otherwise.}\end{cases}$$

Proposition 2.15. *For any graph G, we denote by $\mathcal{C}(G)$ the set of packed valid colourations of G, that is to say surjective maps $c : V[G] \longrightarrow [\max(f)]$ such that for any $\{x,y\} \in E(g)$, $c(x) \neq c(y)$. We denote by Φ_1 the unique morphism of double bialgebras over V from $\mathcal{H}_V[\mathbf{G}]$ to* $\mathrm{QSh}(V)$*. For any V-decorated graph G,*

$$\Phi_1(G) = \sum_{c\in\mathcal{C}(G)} \left(\dot{\prod_{c(x)=1}} d_V(x), \ldots, \dot{\prod_{c(x)=\max(c))}} d_V(x) \right),$$

where for any vertex $x \in V(G)$, $d_V(x) \in V$ is its decoration.

Proof. Let G be a V-decorated graph. For any vertex i of G, we denote by $v_i \in V$ the decoration of i. The number of vertices of G is denoted by n.

$$\begin{aligned}
\Phi_1(G) &= \sum_{k=1}^{n} \sum_{\substack{V(G)=I_1\sqcup\ldots\sqcup I_k,\\ I_1,\ldots,I_k\neq\emptyset}} \epsilon_\delta(G_{|I_1})\ldots\epsilon_\delta(G_{|I_k}) \left(\dot{\prod_{i\in I_1}} v_i, \ldots, \dot{\prod_{i\in I_k}} v_i \right)\\
&= \sum_{k=1}^{n} \sum_{c:V[G]\longrightarrow[k],\ \text{surjective}} \epsilon_\delta(G_{|c^{-1}(1)})\ldots\epsilon_\delta(G_{|c^{-1}(k)}) \left(\dot{\prod_{c(x)=1}} d_V(x), \ldots, \dot{\prod_{c(x)=k}} d_V(x) \right)\\
&= \sum_{c\in\mathcal{C}(G)} \left(\dot{\prod_{c(x)=1)}} d_V(x), \ldots, \dot{\prod_{c(x)=\max(c))}} d_V(x) \right),
\end{aligned}$$

as for any surjective map $c : V[G] \longrightarrow [\max(f)]$,

$$\epsilon_\delta(G_{|c^{-1}(1)})\ldots\epsilon_\delta(G_{|c^{-1}(k)}) = \begin{cases} 1 \text{ if } c \in \mathcal{C}(G),\\ 0 \text{ otherwise.}\end{cases}$$

□

Example 2.2. For any $v_1, v_2, v_3 \in V$,

$$\begin{aligned}
\Phi_1(\mathbf{1}^{v_2}_{v_1}) &= v_1v_2 + v_2v_1,\\
\Phi_1(\mathbf{1}^{v_3}_{v_2\,v_1}) &= v_1v_2v_3 + v_1v_3v_2 + v_2v_1v_3 + v_2v_3v_1 + v_3v_1v_2 + v_3v_2v_1 + (v_1\cdot v_3)v_2 + v_2(v_1\cdot v_3),\\
\Phi_1({}^{v_2}_{v_1}\nabla^{v_3}) &= v_1v_2v_3 + v_1v_3v_2 + v_2v_1v_3 + v_2v_3v_1 + v_3v_1v_2 + v_3v_2v_1.
\end{aligned}$$

If $V = \mathbb{K}$, we obtain the double bialgebra morphism $\phi_{chr} : \mathcal{H}[\mathbf{G}] \longrightarrow \mathbb{K}[X]$, sending any graph on its chromatic polynomial. If V is the algebra of the semigroup $(>0,+)$, we obtain the morphism $\Phi_{chr} : \mathcal{H}_V[\mathbf{G}] \longrightarrow \mathbf{QSym}$, sending any graphs which vertices are decorated by positive integers to its chromatic (quasi)symmetric function [13].

References

[1] Eiichi Abe, *Hopf algebras*, Cambridge Tracts in Mathematics, vol. 74, Cambridge University Press, Cambridge-New York, 1980, Translated from the Japanese by Hisae Kinoshita and Hiroko Tanaka.

[2] Marcelo Aguiar, Nantel Bergeron, and Frank Sottile, *Combinatorial Hopf algebras and generalized Dehn-Sommerville relations*, Compos. Math. **142** (2006), no. 1, 1–30.

[3] Kurusch Ebrahimi-Fard, Frédéric Fauvet, and Dominique Manchon, *A comodule-bialgebra structure for word-series substitution and mould composition*, J. Algebra **489** (2017), 552–581.

[4] Loïc Foissy, *Commutative and non-commutative bialgebras of quasi-posets and applications to Ehrhart polynomials*, Adv. Pure Appl. Math. **10** (2019), no. 1, 27–63.

[5] ______, *Chromatic polynomials and bialgebras of graphs*, Int. Electron. J. Algebra **30** (2021), 116–167.

[6] ______, *Bialgebras in cointeraction, the antipode and the eulerian idempotent*, arXiv:2201.11974, 2022.

[7] ______, *Contractions and extractions on twisted bialgebras and coloured Fock functors*, in preparation, 2022.

[8] Israel M. Gelfand, Daniel Krob, Alain Lascoux, Bernard Leclerc, Vladimir S. Retakh, and Jean-Yves Thibon, *Noncommutative symmetric functions*, Adv. Math. **112** (1995), no. 2, 218–348.

[9] Michiel Hazewinkel, *Symmetric functions, noncommutative symmetric functions and quasisymmetric functions. II*, Acta Appl. Math. **85** (2005), no. 1-3, 319–340.

[10] Michael E. Hoffman, *Quasi-shuffle products*, J. Algebr. Comb. **11** (2000), no. 1, 49–68.

[11] ______, *Quasi-shuffle algebras and applications*, Algebraic combinatorics, resurgence, moulds and applications (CARMA). Volume 2, Berlin: European Mathematical Society (EMS), 2020, pp. 327–348.

[12] Claudia Malvenuto and Christophe Reutenauer, *A self paired Hopf algebra on double posets and a Littlewood-Richardson rule*, J. Combin. Theory Ser. A **118** (2011), no. 4, 1322–1333.

[13] Mercedes H. Rosas, *MacMahon symmetric functions, the partition lattice, and Young subgroups*, J. Combin. Theory Ser. A **96** (2001), no. 2, 326–340.

[14] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[15] Moss E. Sweedler, *Hopf algebras*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969.