

Tensor products and the Milnor-Moore theorem in the locality setup

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Abstract

The present exploratory paper deals with tensor products in the locality framework developed in previous work, a natural setting for an algebraic formulation of the locality principle in quantum field theory. Locality tensor products of locality vector spaces raise challenging questions, such as whether the locality tensor product of two locality vector spaces is a locality vector space. A related question is whether the quotient of locality vector spaces is a locality vector space, which we first reinterpret in a group theoretic language and then in terms of short exact sequences. We prove a universal property for the locality tensor algebra and for the locality enveloping algebra, the analogs in the locality framework of the tensor algebra and of the enveloping algebra. These universal properties hold under compatibility assumptions between the locality and the multilinearity underlying the construction of tensor products which we formulate in the form of conjectural statements. Assuming they hold true, we generalise the Milnor-Moore theorem to the locality setup and discuss some of its consequences.

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Introduction

Locality: state of the art

The notion of locality is ubiquitous in the physics and the mathematics literature. The physical concept of locality states that an object or an event can only interact with objects or events in its vicinity. Depending on the approach one adopts, the concept of locality can be expressed in various ways including commutativity of observables, disjointedness of supports or (partial) multiplicativity. We focus on the latter which is relevant in the Hopf algebraic approach to renormalisation and led to an abstract mathematical formulation of the physical notion of locality proposed in [CGPZ1]. Roughly speaking, the "locality" framework consists in enhancing the mathematician workman's categories such as that of sets, vector spaces, monoids, algebras, coalgebras, Hopf algebras to locality categories by adjoining to a set, vector space, monoid, coalgebras algebras or Hopf algebra a symmetric binary relation \top that we call a locality. This locality should satisfy some compatibility properties with the underlying structure. Modulo some small adjustments, this "locality" framework can also host the related concept of causality in quantum field theory [Rej]. This locality setup tailored to ensure locality while renormalising was then implemented and discussed in various contexts [CGPZ2, CGPZ3, CGPZ4].

Tensor products are known to play an important role in renormalisation procedures e.g., in Hopf algebras used in Connes and Kreimer's approach to renormalisation [CK2]. This prompted the notion of locality tensor product and locality Hopf algebra introduced in [CGPZ1]. In the locality setup, tensor products require a special treatment which gives rise to challenging questions. Whether the locality tensor product of two locality vector spaces is a locality vector space, a property which we expect to hold, raises the question whether the quotient of locality vector spaces is a locality vector space. Both issues are discussed in this paper, leading to interesting open questions.

In the present exploratory paper, we formulate conjectural statements related to these questions and investigate their consequences. **The conjectures for which we could not find any counterexample and which we trust to hold, remain open in spite of much effort put into proving them, including an attempt by means of an a priori promising algorithmic method.** Our main results are generalisations to the locality setup, of the universality properties of tensor algebras and universal enveloping algebras as well as of the Milnor-Moore theorem. The locality Milnor-Moore theorem provides new information which is not accessible in the ordinary non-locality set up.

Objectives and main results

A first objective of this paper is to explore paths towards a full-fledged locality setup for tensor products. For this purpose we introduce **pre-locality vector spaces**, namely vector spaces equipped with a set locality structure, the locality not being required to be compatible with the linear structure. We show in Theorem 1.8 that the relation \top_E on pre-locality vector space $(E; \top_E)$ can be extended to a relation \top_E^ℓ such that (E, \top_E^ℓ) is a locality vector space. This defines a non essentially surjective functor from the category of pre-locality spaces to the category of locality spaces, whose morphisms are locality linear maps in both cases, namely linear maps that map the graph of a locality relation on the source space to the graph of the locality relation on the target space (Definition 1.1).

We then proceed to show that many of the important theorems which hold for usual (i.e. non pre-locality nor locality) tensor products still hold true in the pre-locality setup. We first show the universality of the pre-locality tensor product (Theorem 1.15). We then refine this result by formulating it in the pre-locality category (Theorem 2.13).

We further prove the universality of the locality tensor algebra over a pre-locality vector space (Theorem 4.15), and of the universal locality enveloping algebra in the category of pre-locality (Lie) algebras (Theorem 4.22). Whereas the universal property of the locality tensor product for pre-locality vector spaces turns out to be equivalent to the universal property of the usual tensor product of two vector spaces, the universality of the universal locality enveloping algebra only implies the universality of the universal enveloping algebra of a pre-locality algebra. Along the way, we prove various useful properties and in particular the associativity of the locality tensor product (Theorem 3.9).

The second objective of the paper is to formulate precise conditions implying the expected stability

property, namely that locality tensor products of locality vector spaces are locality vector spaces. Like ordinary tensor products, locality tensor products are defined as quotient spaces, which brings us to the study of quotients of locality vector spaces and raises the question (formulated in Question 35)) whether the quotient of a locality vector space by a linear subspace is a locality vector space. We first reformulate this question in group-theoretic terms (Theorem 5.3) and then (see Proposition 5.9) in terms of locality short exact sequences introduced in Definition 5.7. By means of some counterexamples, we further show that Question (35) does not always have a positive answer.

Whether the quotient of a locality vector space by a locality subspace is also a locality space, relates in turn to the existence of a **strong locality complement** (Definition-Proposition 6.1) of the locality subspace. Roughly speaking, a strong locality complement of a subspace of a locality vector space is an algebraic complement of the given vector space with some locality conditions on the canonical projections induced by the direct sum of the subspace and its complement. We show in Corollary 6.4 that the quotient of a locality space by a subspace with a strong locality complement is a locality space, which yields a sufficient condition under which Question 35 has a positive answer.

We then introduce a second sufficient condition – the **locality compatibility** condition (Definition 6.7) – on a linear subspace, which also ensures that the quotient of a locality space by this subspace is a locality space (Theorem 6.12). It is less stringent than the existence of a locality complement (see Proposition 6.9) and turns out to be easier to work with. Theorem 6.12 underlies most of the forthcoming constructions, in particular the locality Milnor-Moore theorem.

We are now ready to formulate two main conjectural statements in terms of the (weaker) locality compatibility condition:

1. Conjectural statement 6.16, which ensures that the tensor product of locality vector spaces is a locality vector space (Proposition 6.17)
2. Conjectural statement 6.19, which ensures that the locality tensor algebra of a locality vector space is indeed a locality algebra (Theorem 7.3).

Assuming these conjectural properties hold true, we can enhance the results of the first part from the pre-locality to the locality setup. Assuming they hold true, we prove the universal property of the tensor power of a locality vector space (Theorem 7.4), which is the locality version of Theorem 2.13. We also prove the universal property of the locality tensor algebra (Theorem 7.5), which is the locality version of Theorem 4.15. Assuming that Conjectural statement 6.19 holds, and under one further assumption (Conjectural statement 7.6), we moreover prove the locality version of Theorem 4.22, namely the universal property of the locality universal enveloping algebra in the category of locality (Lie) algebras. Its universal property is then proven in Theorem 7.8 under the assumption that the conjectural statements (7.6) and 6.19 both hold.

As a third and final objective of the paper, we prove a locality version of the celebrated Milnor-Moore Theorem (also called Cartier-Quillen-Milnor-Moore Theorem, [Car2, MM, Qui, Cha]), namely Theorem 9.5 which holds assuming the statements 6.19 and 7.6 hold true. We hope that this is a first step towards a classification of locality Hopf algebras, which was one of our original motivations.

The traditional proof of this theorem relies on a corollary of the Poincaré-Birkhoff-Witt theorem. Poincaré-Birkhoff-Witt theorem is not yet available in the locality setup mainly due to the fact that, unlike in the ordinary (non-locality) setup, one cannot reconstruct a locality vector space from a locality basis (whose generating vectors are mutually independent). Instead we prove the required intermediate step (Proposition 9.3) using Zorn’s lemma. To our knowledge, our proof is also new in the usual (non-locality) set up.

The article closes on some consequences of the Milnor-Moore Theorem inherent to the locality framework. Interestingly, it turns out that one cannot simply “switch on” locality, at least when the locality Milnor-Moore Theorem applies (Corollary 9.7).

The known correspondence between a class of sub-Hopf algebras and sub-Lie algebras of the set of their primitive elements extends to the locality setup, see Corollary 9.9 and the notations therein:

$$((H, \mathbb{T}) \supset (H', \mathbb{T}')) \iff ((\mathfrak{g}', \mathbb{T}') \subset (\mathfrak{g}, \mathbb{T})).$$

However it follows from Corollary 9.7 that $H = H'$ and $\mathfrak{g} = \mathfrak{g}'$ implies $\top = \top'$. We give an example (Proposition 9.13) for which $\mathfrak{g} = \mathfrak{g}'$ but $H \neq H'$, that is specific to the locality setup, since it cannot occur in the non-locality setup due to the Milnor-Moore Theorem.

Organisation of the paper

The paper is divided into three parts (and one appendix), each corresponding to one of the objectives described above. The properties proved in the pre-locality setup in this first part shed light on the challenges of the locality structures that we address in the second part of the paper. They also serve as a preparation to establish the corresponding properties in the full-fledged locality setup when assuming that the conjectural statements hold true. This preliminary study in the pre-locality framework which is interesting in its own right, serves to identify obstructions preventing tensor products of locality structures from being locality vector spaces.

The first part introduces and proves universality results in the pre-locality setup. In the second part, we question the locality property of the quotient of a locality vector space by a locality subspace, formulate conjectural statements that play a role when dealing with locality tensor products and further investigate useful consequences of these conjectural properties.

The various conjectural statements discussed in this second part are formulated in a way that suggests a strategy for proving them. Although such proofs are not within the scope of the current work, we expect to tackle them in future work. The fact that these conjectural statements allow to extend important results of the theory of Hopf algebras in the locality setup serves as a motivation. Hopf algebraic results are the object of the third part of this paper. This final third part is dedicated to more sophisticated consequences of these conjectural statements, namely the locality version of the Milnor-Moore Theorem. Each of these parts is divided into sections whose contents we now present in more detail.

The first section of the first part, namely Section 1 introduces the basic concepts we use throughout this paper: locality vector spaces, pre-locality vector spaces and locality tensor products of these structures. Section 2 introduces locality on quotients of locality vector spaces, applying them to locality tensor products of pre-locality spaces. Section 3 deals with products of more than two pre-locality vector spaces. Finally, Section 4 discusses properties of the locality tensor algebra and the locality universal enveloping algebra in the pre-locality setup.

The second part of the paper starts with Section 5 which discusses when the quotient of a locality vector space with a linear subspace, inherits a locality vector space structure. These considerations on quotients allow us to give in Section 6 two sufficient conditions, one stronger than the other, for quotients of locality vector spaces to be locality vector spaces. The remaining part of Section 6 is dedicated to the formulation and implications of two conjectural statements under which locality tensor products and locality tensor algebras are locality vector spaces and locality algebra respectively. In Section 7 we infer that assuming the conjectural properties to hold true, the universal properties shown in the first part of the paper in the pre-locality setup still hold in the locality one. One of these universal properties is proved under the assumption of a third conjectural property, also formulated in Section 7.

The third and final part of the paper focuses on the locality version of the Milnor-Moore Theorem. It starts in Section 8 by recalling (locality version of) the structures the Milnor-Moore Theorem implies, namely graded connected Hopf algebras, and establishing some useful preliminary results. In the final section 9 we state and prove, assuming the validity of the conjectural statements in the second part, the Milnor-Moore Theorem in the locality setup and state some of its consequences specific to the locality framework.

For the sake of completeness, we dedicate a short appendix to the description of an alternative locality tensor product which unlike the one we use throughout the paper, is not a subspace of the usual tensor product.

Openings

This paper provides first steps toward a full-fledge theory of locality tensor products. Thanks to the notion of pre-locality vector space, we were able to identify the challenges and formulate precise conditions under which we could merge locality and tensors products. We believe that discussing these issues paves the way to a complete theory of locality Hopf algebras. Beyond its relevance as a generalisation of the theory of

ordinary Hopf algebras, it is of importance in the context of renormalisation theory, which in turn was one of the original motivations for introducing locality structures.

Let us hint to what could be the next steps of our program. The three conjectural statements discussed in this paper raise challenging questions which we feel are interesting for their own sake. As shown in the paper, they are related to open questions in group theory. Yet other approaches might also be fruitful, such as rewriting systems and Gröbner bases which seem natural candidates to tackle these questions.

Another natural goal is to establish a locality version of the Poincaré-Birkhoff-Witt Theorem. Such a result would simplify our proof of the Milnor-Moore Theorem and (possibly) avoid the use of the axiom of choice. Indeed, while the original proof of Birkhoff and Witt requires the use of the axiom of choice, more recent generalisations of their theorem do not. It could also be of interest to see if the three conjectural statements of this paper (and in particular the third one) are enough to prove a locality version of the Poincaré-Birkhoff-Witt Theorem.

An interesting open question concerns the interplay of locality and duality. Indeed, there is not yet a natural notion of locality on the algebraic dual of a locality vector space, which turns the dual into a (non-trivial) locality vector space. Similarly, the algebraic dual of a locality algebra (resp. coalgebra) is not in general a locality coalgebra (resp. algebra).

The locality Milnor-Moore Theorem gives a classification of cocommutative graded connected locality Hopf algebras and we hope for other classification results for locality structures, and in particular for locality Lie algebras. Our motivation to search for a locality dual is that the dual version of the locality Milnor-Moore Theorem could potentially give rise to a classification of graded, connected commutative Hopf algebras derived from Theorem 9.5. Such a dual Milnor-Moore Theorem can also be proved in a more pedestrian way, dualising the proof of this paper, which for the sake of the length of the paper, we have omitted here.

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Notations

Throughout the document we use the notation $[n] := \{1, \dots, n\}$, for $n \in \mathbb{N}$ and with the convention $[0] = \emptyset$. \mathbb{K} denotes a commutative field of characteristic zero, and unless otherwise specified we take $\mathbb{K} = \mathbb{C}$ or \mathbb{R} to be the underlying field of every vector space. Given a set X , we denote the free span of X by $\mathbb{K}(X)$, $\mathbb{K}X$ or $\langle X \rangle$ indistinctly.

Part I

Universal properties in the pre-locality setup

This first part of the document is devoted to the construction of tensor products as quotient spaces in the locality and pre-locality setup and to derivation of their first properties.

1 (Pre-) locality spaces and the locality tensor product

Let us start by introducing the most important concepts of this paper, namely locality and pre-locality structures.

1.1 (Pre-) locality vector spaces

We recall and partially extend some basic notions introduced in [CGPZ1] relative to linear structures.

Definition 1.1. • A **locality set** is a pair (S, \top) where S is a set and $\top \subset S \times S$ is a symmetric relation on S . We sometimes denote $(x, y) \in \top$ as $x \top y$.

- A **locality map** is a map $f : (X, \top_X) \rightarrow (Y, \top_Y)$ between locality sets which preserves locality i.e., such that $(f \times f)(\top_X) \subset \top_Y$.
- Two maps $f, g : (X, \top_X) \rightarrow (Y, \top_Y)$ between locality sets are called locally independent (or simply independent if there is no risk of ambiguity) if

$$(f \times g)(\top_X) \subset \top_Y.$$

When $X = Y$ we sometimes denote f and g being locality independent as $f \top g$.

- A **pre-locality \mathbb{K} -vector space** is a locality set (V, \top) such that V has the structure of a \mathbb{K} -vector space, and $(0_V, 0_V) \in \top$.
- A **locality \mathbb{K} -vector space** or simply **locality vector space** is a pre-locality \mathbb{K} -vector space (V, \top) such that the polar set $U^\top = \{x \in V : (\forall u \in U), u \top x\}$ of any subset $U \subset V$ is a linear subspace of V . Equivalently, the following condition should be fulfilled

$$u \top v \text{ and } u' \top v \implies (\lambda u + \lambda' u') \top v \quad \forall (\lambda, \lambda') \in \mathbb{K}^2, \forall (u, u', v) \in V^3. \quad (1)$$

We sometimes refer to the previous property as linear locality.

- Let (V, \top_V) and (W, \top_W) be two pre-locality \mathbb{K} -vector spaces. We call a linear map $f : V \rightarrow W$ a **locality morphism** or **locality linear map** if it is also a locality map i.e., if it is locality independent of itself:

$$f \top f \iff (f \times f)(\top_V) \subset \top_W. \quad (2)$$

- Let (W, \top_W) be a (pre-)locality vector space and $V \subseteq W$. We call (V, \top_V) a **(pre-)locality subspace** of (W, \top_W) if the inclusion map $\iota : V \hookrightarrow W$ is a locality linear map.
- Let (V, \top_V) and (W, \top_W) be two pre-locality \mathbb{K} -vector spaces (resp. locality vector spaces). We say they are isomorphic as pre-locality (resp. locality) vector spaces if there is an invertible locality linear map $f : V \rightarrow W$ such that f^{-1} is also a locality linear map.

Remark 1.2. (Pre-)locality is a hereditary property. Indeed, any linear subspace W of a (resp. pre-) locality vector space (V, \top) can be endowed with a locality structure $\top_W := \top \cap (W \times W)$ induced by \top , in which case (W, \top_W) is a (resp. pre-) locality vector space. Our definition of (pre-)locality subspace (which is more general than the one in [CGPZ1]) amounts to require that $V \subseteq W$ and $\top_V \subseteq \top_W$ such that (V, \top_V) is a (pre-)locality vector space.

We recall that locality sets build a category \mathbf{S}_{loc} whose morphisms are locality maps.

Proposition 1.3. *Let (V, \top_V) and (W, \top_W) be two (resp. pre-)locality vector spaces and $f : V \rightarrow W$ be a locality linear map. Then the image of any (resp. pre-)locality subspace of (V, \top_V) by f is a (resp. pre-)locality subspace of (W, \top_W) and the collection \mathbf{V}_{loc} (resp. $\mathbf{V}_{\text{pre-loc}}$) of (resp. pre-)locality vector spaces, forms a category whose morphisms are locality morphisms.*

Proof. Let $f : V \rightarrow W$ be a morphism as in the statement.

- Since f is a locality map, it sends locality sets to locality sets, so it maps the (pre-)locality vector space (V, \top_V) to the vector space $f(V)$ equipped with the locality structure $\top_{f(V)} := \top_W \cap (f(V) \times f(V))$ inherited from \top_W , which makes $(f(V), \top_{f(V)})$ a pre-locality linear subspace of (W, \top_W) .
- If (V, \top_V) is a locality vector space, so is $(f(V), \top_{f(V)})$. Indeed, the polar set

$$f(X)^{\top_{f(V)}} = \{w \in f(V), w \top f(x), \forall x \in X\} = f(X)^{\top_W} \cap f(V)$$

of the range $f(X)$ of a subset $X \subset V$ is a vector space for by assumption $f(X)^{\top_W}$ is a vector space.

- The fact that the composition of two locality maps is a locality maps was already proven in [CGPZ1] (see Remark 2.6). Associativity of the composition of locality maps follows from the usual associativity of composition. Thus the fact that \mathbf{V}_{loc} and $\mathbf{V}_{\text{pre-loc}}$ are categories as claimed follows from the fact that the identity maps trivially are locality morphisms. \square

We have the following forgetful functors between the various categories:

$$\begin{array}{ccccc}
 & & & & \mathbf{V} \\
 & & & \nearrow \iota_V & \\
 & & & & \\
 \mathbf{V}_{\text{loc}} & \xleftarrow{\iota_{V_{pl}}} & \mathbf{V}_{\text{pre-loc}} & \searrow \iota_S & \\
 & & & & \mathbf{S}_{\text{loc}}
 \end{array}$$

where \mathbf{V} denotes the category of vector spaces. The map $\iota_{V_{pl}}$ takes a locality vector space to its underlying pre-locality vector space (thus ignoring the linearity condition on polar sets), ι_V ignores the pre-locality and takes it into (usual) vector spaces and ι_S ignores the linearity.

Definition 1.4. Let V and W be two linear subspaces of a (resp. pre-)locality vector space (E, \top) . The **locality Cartesian product** of V and W , denoted by $V \times_{\top} W$ is the restriction

$$V \times_{\top} W := \top \cap (V \times W)$$

Note that $(0_V \times 0_W) \in V \times_{\top} W$ since $0_V = 0_W = 0_E$.

Remark 1.5. Assume that (E, \top) is a locality vector space. For any subsets $X \subset V$ or $Y \subset W$, the relative polar sets $X^{\top} \cap W = \{w \in W : (x, w) \in \top \forall x \in X\}$ and $Y^{\top} \cap V := \{v \in V : (v, y) \in \top \forall y \in Y\}$ are linear subspaces of W and V since (E, \top) is a locality vector space. In the terminology of [CGPZ1, Sect. 4.1], the triple (V, W, \top) is a relative locality vector space.

Before generalising the concept of bilinear map to the locality set up, recall that the freely generated vector space $\mathbb{K}X$ of a set X obeys the following universal property.

Given a set X and a vector space G , any map $f : X \rightarrow G$ uniquely extends to a linear map $\bar{f} : \mathbb{K}X \rightarrow G$ as follows

$$\bar{f} \left(\sum_{x \in X} \alpha_x x \right) = \sum_{x \in X} \alpha_x f(x).$$

The following diagram, where ι stands for the canonical injection, therefore commutes:

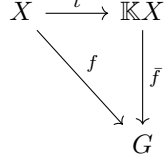


Figure 1: Universal property of freely generated vector spaces

In the following, V and W are two vector spaces over the same field \mathbb{K} . In the free linear span $\mathbb{K}(V \times W)$, we consider the linear subspace I_{bil} generated by all elements of the form

$$(a + b, x) - (a, x) - (b, x) \quad (3)$$

$$(a, x + y) - (a, x) - (a, y) \quad (4)$$

$$(ka, x) - k(a, x) \quad (5)$$

$$(a, kx) - k(a, x) \quad (6)$$

with $a, b \in V$, $x, y \in W$ and $k \in \mathbb{K}$ the underlying field. Clearly, given any vector space G , a map $f : X \times Y \rightarrow G$ is bilinear if and only if $\bar{f}(I_{\text{bil}}) = \{0_G\}$.

The notion of local bilinearity is inspired from [CGPZ1, Paragraph 3.3], adapted to our construction of the (pre-)locality tensor product.

Definition 1.6. Let V and W be subspaces of a pre-locality vector space (E, \top) and G any vector space. We call \top_{\times} -**bilinear** a map $f : V \times_{\top} W \rightarrow G$ which satisfies the \top_{\times} -bilinearity condition:

$$\bar{f}(I_{\text{bil}}^{\top_{\times}}) = \{0_G\}, \quad (7)$$

where we have set $I_{\text{bil}}^{\top_{\times}} := \mathbb{K}(V \times_{\top} W) \cap I_{\text{bil}}$.

Definition 1.7. Let V and W be subspaces of a pre-locality vector space (E, \top) and (G, \top_G) a pre-locality vector space. We call a map $f : V \times_{\top} W \rightarrow G$ **locality \top_{\times} -bilinear**, if it is \top_{\times} -bilinear (see (7)) and satisfies the locality condition

$$(f \times f)(\top_{V \times_{\top} W}) \subset \top_G, \quad \text{i.e., } (v, w) \top_{V \times_{\top} W} (v', w') \Rightarrow f(v, w) \top_G f(v', w'), \quad (8)$$

where $\top_{V \times_{\top} W}$ is a locality relation on $V \times_{\top} W$ given by

$$\top_{V \times_{\top} W} := \{(v, w), (v', w')\} \in (V \times_{\top} W)^2 : v \top_V v' \wedge w \top_W w'\}. \quad (9)$$

Our new concept of pre-locality vector space is related to the previous notion of locality vector space through the following result.

Theorem 1.8. Let (E, \top_E) be a pre-locality \mathbb{K} -vector space. There is a unique locality relation \top_E^{ℓ} on E containing \top_E such that for any locality space (F, \top_F) and any locality linear map $f : (E, \top_E) \rightarrow (F, \top_F)$, f is a locality linear map from (E, \top_E^{ℓ}) to (F, \top_F) .

The map

$$\begin{array}{ccc}
\mathcal{L} : \mathbf{V}_{\text{pre-loc}} & \longrightarrow & \mathbf{V}_{\text{loc}} \\
(E, \top_E) & \longmapsto & (E, \top_E^{\ell})
\end{array}$$

on the objects of the category combined with the identity map $\mathcal{L}(f) = f$ on the morphisms of the category given by locality linear maps f , defines a non essentially surjective functor between the two categories.

Proof. • Existence: With the notations of the theorem, let \mathcal{E} be the set of locality relations \top on E such that $\top_E \subseteq \top$. This is not empty, as it contains $E \times E$. We consider

$$\top_E^\ell = \bigcap_{\top \in \mathcal{E}} \top.$$

Then $\top_E \subseteq \top_E^\ell$. If $x \top_E^\ell y$, $x \top_E^\ell z$ and $\lambda \in \mathbb{K}$, then for any \top in \mathcal{E} , $x \top y$ and $x \top z$, so $x \top (y + \lambda z)$ since by definition of \mathcal{E} , \top is a locality relation. Hence, $x \top_E^\ell (y + \lambda z)$, and \top_E^ℓ is a locality relation on E .

Let $f : (E, \top_E) \longrightarrow (F, \top_F)$ be a locality linear map. We define a relation \top on E by

$$\top_f := (f \times f)^{-1}(\top_F).$$

Since f is a locality map, $(f \times f)(\top_E) \subseteq \top_F$ so $\top_E \subseteq \top_f$. Combining the locality of the map f , which implies

$$(x \top_f y \wedge x \top_f z) \implies (f(x) \top_F f(y) \wedge f(x) \top_F f(z)),$$

with the fact that (F, \top_F) is a locality space yields

$$f(x) \top_F (f(y) + \lambda f(z)) \implies f(x) \top_F f(y + \lambda z),$$

so $x \top_f (y + \lambda z)$, and hence \top_f lies in \mathcal{E} . Thus, $\top_E^\ell \subseteq \top_f$, which as before implies that $f : (E, \top_E^\ell) \longrightarrow (F, \top_F)$ is a locality linear map.

- Uniqueness: let \top' be another relation obeying the assumptions of the theorem. Since $\top_E \subseteq \top_E^\ell$, the identity map $\text{Id}_E : (E, \top_E) \longrightarrow (E, \top_E^\ell)$ is a pre-locality linear map. Thus, by the very definition of \top' , $\text{Id}_E : (E, \top') \longrightarrow (E, \top_E^\ell)$ is a locality linear map. Consequently, $\top' \subseteq \top_E^\ell$. Similarly, exchanging the roles of \top' and \top_E^ℓ , $\text{Id}_E : (E, \top_E^\ell) \longrightarrow (E, \top')$ is a locality linear map, so $\top_E^\ell = \top'$.
- \mathcal{L} defines a functor: Given two pre-locality spaces (E, \top_E) , (F, \top_F) , we want to show that a locality linear map $f : (E, \top_E) \longrightarrow (F, \top_F)$ extends to a locality linear map $f : (E, \top_E^\ell) \longrightarrow (F, \top_F^\ell)$. We first observe that since $\top_F \subseteq \top_F^\ell$, $f : (E, \top_E) \longrightarrow (F, \top_F^\ell)$ is also a locality linear map. It then follows from the definition of \top_E^ℓ , that the map $f : (E, \top_E^\ell) \longrightarrow (F, \top_F^\ell)$ is indeed a locality linear map.
- \mathcal{L} is not essentially surjective: the vector space (\mathbb{K}, \top) with $\top = \{(0, 0), (1, 1)\}$ is a pre-locality vector space. A linear map $f : \mathbb{K} \longrightarrow \mathbb{K}$ is a pre-locality map if and only if $(f(1), f(1)) \in \{(0, 0), (1, 1)\}$. Therefore,

$$\text{End}_{\text{pre-loc}}(\mathbb{K}, \top) = \{0, \text{Id}_{\mathbb{K}}\}.$$

For a locality relation \top' containing \top , we have $1 \top 1 \implies 1 \top' \lambda$ for any $\lambda \in \mathbb{K}$ and from there $\lambda \top' \mu$ for any $(\lambda, \mu) \in \mathbb{K}^2$ so that $\top' = \mathbb{K} \times \mathbb{K}$. It follows that

$$\text{End}_{\text{loc}}(\mathbb{K}, \top') = \{\lambda \text{Id}_{\mathbb{K}}, \lambda \in \mathbb{K}\}.$$

The map $f \mapsto \mathcal{L}(f)$ from $\text{End}_{\text{pre-loc}}(\mathbb{K}, \top)$ to $\text{End}_{\text{loc}}(\mathbb{K}, \top')$ is therefore not surjective. \square

1.2 Locality tensor product of two pre-locality spaces

We adapt to the pre-locality setup, concepts introduced in [CGPZ1] relative to locality tensor products. The assumptions are weakened, yet the definition stays the same so we keep the terminology "locality tensor product". As before, V and W are two vector spaces over the same field \mathbb{K} . Let us recall that the tensor product of two vector spaces V and W reads

$$V \otimes W := \mathbb{K}(V \times W) / I_{\text{bil}}.$$

It comes with a map

$$\otimes := \pi \circ \iota : V \times W \rightarrow V \otimes W$$

$$\begin{array}{ccc}
V \times W & \xrightarrow{\iota} & \mathbb{K}(V \times W) \\
& \searrow \otimes & \downarrow \pi \\
& & V \otimes W
\end{array}$$

Figure 2: The tensor product from the Cartesian product

built from the canonical inclusion $\iota : V \times W \rightarrow \mathbb{K}(V \times W)$ and the canonical quotient map $\pi : \mathbb{K}(V \times W) \rightarrow V \otimes W$, which makes the following diagram commute

Following [CGPZ1], we define the (pre-)locality counterpart.

Definition 1.9. Given V and W subspaces of a pre-locality vector space (E, \top) , the locality tensor product is the vector space

$$V \otimes_{\top} W := \mathbb{K}(V \times_{\top} W) / I_{\text{bil}}^{\top \times}. \quad (10)$$

Remark 1.10. Since $V \times_{\top} W \subset V \times W$ and $I_{\text{bil}}^{\top \times} := \mathbb{K}(V \times_{\top} W) \cap I_{\text{bil}}$, we have an inclusion of vector spaces $V \otimes_{\top} W \subset V \otimes W$. If $V \times_{\top} W = V \times W$, then $V \otimes_{\top} W = V \otimes W$.

$V \otimes_{\top} W$ comes with a map

$$\otimes_{\top} := \pi_{\top} \circ \iota_{\top} : V \times_{\top} W \rightarrow V \otimes_{\top} W \quad (11)$$

built from the canonical inclusion $\iota_{\top} : V \times_{\top} W \rightarrow \mathbb{K}(V \times_{\top} W)$ and the canonical quotient map $\pi_{\top} : \mathbb{K}(V \times_{\top} W) \rightarrow V \otimes_{\top} W$, which makes the following diagram commute:

$$\begin{array}{ccc}
V \times_{\top} W & \xrightarrow{\iota_{\top}} & \mathbb{K}(V \times_{\top} W) \\
& \searrow \otimes_{\top} & \downarrow \pi_{\top} \\
& & V \otimes_{\top} W
\end{array}$$

Figure 3: The locality tensor product from subspaces of a pre-locality vector space.

Proposition 1.11. Given V and W subspaces of a pre-locality vector space (E, \top) , the map

$$\otimes_{\top} : V \times_{\top} W \rightarrow V \otimes_{\top} W$$

is a \top_{\times} -bilinear map.

Proof. Let $\overline{\otimes}_{\top}$ be the linear extension of (11) to a map $\mathbb{K}(V \times_{\top} W) \rightarrow V \otimes_{\top} W$. By construction, $\overline{\otimes}_{\top}(I_{\text{bil}}^{\top \times}) = \pi_{\top}(I_{\text{bil}}^{\top \times})$ coincides with $\{0_{V \otimes_{\top} W}\}$. The map \otimes_{\top} therefore satisfies (7) and defines a \top_{\times} -bilinear map. \square

Properties of the non-locality tensor product commonly used are the distributivity with respect to direct sums

$$(V_1 \oplus V_2) \otimes W = (V_1 \otimes W) \oplus (V_2 \otimes W), \quad (12)$$

and with respect to intersections, namely $(V_1 \cap V_2) \otimes W = (V_1 \otimes W) \cap (V_2 \otimes W)$. These two properties rely on the existence of a direct complement to every subspace $V_1 \subset V$, i.e, the existence of a subspace $V_2 \subset V$ such that $V_1 \oplus V_2 = V$. However, for the locality tensor product, distributivity does not always hold as the following example illustrates.

Counter-example 1.12. Consider \mathbb{R}^2 with the orthogonality locality. Then

$$(\langle e_1 \rangle \otimes_{\top} \mathbb{R}^2) \oplus (\langle e_2 \rangle \otimes_{\top} \mathbb{R}^2) = \langle e_1 \otimes e_2 \rangle \oplus \langle e_2 \otimes e_1 \rangle, \quad (13)$$

is not $(\langle e_1 \rangle \oplus \langle e_2 \rangle) \otimes_{\top} \mathbb{R}^2$ since it does not contains $(e_1 + e_2) \otimes (e_1 - e_2) \in \mathbb{R}^2 \otimes_{\top} \mathbb{R}^2$.

In order to accomodate for distributivity properties of the tensor product in the locality set up, we need to ensure some compatibility of the splitting $V = V_1 \oplus V_2$ with the locality relation \top . The following proposition gives sufficient conditions to ensure the distributivity of the locality tensor product w.r. to the direct sum.

Proposition 1.13. *Let V and W be linear subspaces of a pre-locality vector space (E, \top) , and let V_1 and V_2 be subspaces of V such that $V_1 \oplus V_2 = V$.*

1. *If for $\{i, j\} = \{1, 2\}$ the projection maps $\pi_i : V \rightarrow V_i$ onto V_i along V_j are locally independent of the identity map Id_W on W (i.e., $(\pi_i \times \text{Id}_W)(\top|_{V \times W}) \subset \top|_{V_i \times W}$), then*

$$(V_1 \otimes_{\top} W) \oplus (V_2 \otimes_{\top} W) = V \otimes_{\top} W.$$

2. *If (E, \top) is a locality vector space, and one of the projections π_i is locally independent of Id_W , then the other projection is also locally independent of Id_W .*

Proof. 1. We first prove that $(V_1 \otimes_{\top} W) \oplus (V_2 \otimes_{\top} W) \subset V \otimes_{\top} W$. Since $V_i \subset V$ then $V_i \otimes_{\top} W \subset V \otimes_{\top} W$ and the expected inclusion follows. To prove the other direction, w.l.o.g. consider $a \otimes b$ in $V \otimes_{\top} W$ such that (a, b) lies in $\top|_{V \times W}$. Since π_i and Id_W are locally independent $(\pi_i(a), b) \in \top|_{V_i \times W}$ thus $\pi_i(a) \otimes b \in V_i \otimes_{\top} W$, which implies that

$$a \otimes b = (\pi_1(a) + \pi_2(a)) \otimes b = \pi_1(a) \otimes b + \pi_2(a) \otimes b \in (V_1 \otimes_{\top} W) \oplus (V_2 \otimes_{\top} W).$$

2. Moreover, if (E, \top) is a locality vector space and if π_1 is independent of Id_W , then for any pair $(a, b) \in V \times W$ we have $a \top b \implies \pi_1(a) \top b$. Using $\pi_2(a) = a - \pi_1(a)$, it follows that $\pi_2(a) \top b$ since $\{b\}^{\top}$ is a subvector space of E . Thus π_2 is independent of Id_W as claimed. \square

We prove a (weak) locality version of the distributivity property w.r. to the intersection $(V_1 \cap V_2) \otimes W = (V_1 \otimes W) \cap (V_2 \otimes W)$.

Corollary 1.14. *Let V and W be linear subspaces of a pre-locality vector space (E, \top) , and let V_1, V_2 be subspaces of V . Let V'_2 be a direct complement of the intersection $V_1 \cap V_2$ in V_2 i.e. $(V_1 \cap V_2) \oplus V'_2 = V_2$.*

1. *If the projection maps $\pi : V_2 \rightarrow V_1 \cap V_2$ onto $V_1 \cap V_2$ along V'_2 and $\text{Id}_{V_2} - \pi$ are locally independent of the identity map Id_W on W , then*

$$(V_1 \cap V_2) \otimes_{\top} W = (V_1 \otimes W) \cap (V_2 \otimes_{\top} W).$$

2. *In particular, if $V_1 \subset V$, and if the projection maps $\pi : V \rightarrow V_1$ onto V_1 along V'_2 and $\text{Id}_{V_2} - \pi$ are locally independent of the identity map Id_W on W , then*

$$V_1 \otimes_{\top} W = (V_1 \otimes W) \cap (V \otimes_{\top} W).$$

Proof. 1. Using the distributivity property of the locality tensor product $((V_1 \cap V_2) \otimes_{\top} W) \oplus (V'_2 \otimes_{\top} W) = V_2 \otimes_{\top} W$ which follows from Proposition 1.13, we have

$$(V_1 \otimes W) \cap (V_2 \otimes_{\top} W) = (V_1 \otimes W) \cap ((V_1 \cap V_2) \otimes_{\top} W) \oplus (V'_2 \otimes_{\top} W). \quad (14)$$

We now make use of a result of elementary linear algebra, namely $A \cap (B \oplus C) = B$ whenever A, B and C are linear subspaces of a linear space E with $B \subset A$ and $A \cap C = \{0\}$. Since $(V_1 \cap V_2) \otimes_{\top} W \subset V_1 \otimes W$ and $(V_1 \otimes W) \cap (V'_2 \otimes_{\top} W) = \{0\}$, the right hand side of (14) is equal to $(V_1 \cap V_2) \otimes_{\top} W$, which yields the result.

2. Setting $V_2 =: V$ in the previous item, yields the result since $V'_2 = \{0\}$. \square

1.3 A first universal property of the locality tensor product

We recall the universal property of the usual tensor product. Given three \mathbb{K} -linear spaces V, W and G , a \mathbb{K} -bilinear map $f : V \times W \rightarrow G$, there is a unique \mathbb{K} -linear map $\phi_f : V \otimes W \rightarrow G$ such that the following diagram commutes:

$$\begin{array}{ccc} V \times W & \xrightarrow{\otimes} & V \otimes W \\ & \searrow f & \downarrow \phi_f \\ & & G \end{array}$$

In a similar manner, the locality tensor product of a pair of subspaces of a pre-locality vector space also satisfy a universal property.

Theorem 1.15. Given V and W two subspaces of a pre-locality vector space (E, \top) over \mathbb{K} , G a \mathbb{K} -linear space and $f_{\top} : V \times_{\top} W \rightarrow G$ a \top_{\times} - \mathbb{K} -bilinear map, there is a unique \mathbb{K} -linear map $\phi_{f_{\top}} : V \otimes_{\top} W \rightarrow G$ such that the following diagram commutes:

$$\begin{array}{ccc} V \times_{\top} W & \xrightarrow{\otimes_{\top}} & V \otimes_{\top} W \\ & \searrow f_{\top} & \downarrow \phi_{f_{\top}} \\ & & G \end{array} \quad (15)$$

Proof. Existence of $\phi_{f_{\top}}$. There exists a unique linear map $\overline{f_{\top}}$ from $\mathbb{K}(V \times_{\top} W)$ to G sending $(v, w) \in V \times_{\top} W$ to $f_{\top}(v, w)$. As f_{\top} is \top_{\times} -bilinear, i.e., $\overline{f_{\top}}(I_{\text{bil}}^{\top_{\times}}) = \{0_G\}$, this map induces a linear map $\phi_{f_{\top}}$ from $V \otimes_{\top} W$ to G . For any $(v, w) \in V \times_{\top} W$,

$$\phi_{f_{\top}}(v \otimes w) = \phi_{f_{\top}}([(v, w)]) = \overline{f_{\top}}(v, w) = f_{\top}(v, w).$$

Therefore, $\phi_{f_{\top}} \circ \otimes = f_{\top}$.

Uniqueness of $\phi_{f_{\top}}$. As $V \times_{\top} W$ is a basis of $\mathbb{K}(V \times_{\top} W)$, the elements $(v \otimes w)_{(v, w) \in V \times_{\top} W}$ linearly generate $V \otimes_{\top} W$, which implies the uniqueness of $\phi_{f_{\top}}$. \square

We now prove the equivalence between Theorem 1.15 and the universal property of the usual tensor product. For that purpose, we recall that a subset B of a vector space V is a (Hamel or algebraic) basis if it satisfies the linear independence property i) for every finite subset $\{b_1, \dots, b_n\}$ of B and every $\alpha_1, \dots, \alpha_n$ in \mathbb{K} we have $\sum_{i=1}^n \alpha_i b_i = 0 \implies \alpha_1 = \dots = \alpha_n = 0$ and the spanning property ii) every vector v in V can be written as a finite linear combination $v = \sum_{k=1}^n \alpha_k b_k$ in which case there is an isomorphism of vector spaces $\mathbb{K}B \simeq V$.

We further recall a classical result of linear algebra which relies on Zorn's lemma. The latter reads: A partially ordered set \mathcal{P} whose chains all have an upper bound in (\mathcal{P}, \preceq) contains at least one maximal element. Consequently,

Lemma 1.16. Let G be a generating subset of a vector space V . A linearly independent set $A \subset G$ can be extended to a Hamel basis $B \subset G$ of V .

Proposition 1.17. Let V and W be subspaces of a pre-locality vector space (E, \top) and G a \mathbb{K} -linear space. Any \top_{\times} - (\mathbb{K}) -bilinear map $f : V \times_{\top} W \rightarrow G$ extends to a \mathbb{K} -bilinear map $g : V \times W \rightarrow G$ i.e., $g|_{V \times_{\top} W} = f$.

Proof. Consider the set $\mathcal{O} := \{B \subset V \times_{\top} W \mid \otimes B \text{ is a linearly independent subset of } V \otimes W\}$, where $\otimes B$ is the image of B under the canonical map $\otimes : V \times W \rightarrow V \otimes W$. We equip \mathcal{O} with the partial inclusion order $B_1 \subset B_2$ and consider a chain \mathcal{C} of \mathcal{O} . We observe that $\bigcup_{B \in \mathcal{C}} B \in \mathcal{O}$ since $\otimes(\bigcup_{B \in \mathcal{C}} B) = \bigcup_{B \in \mathcal{C}} \otimes(B)$, and the union of nested linearly independent sets is a linearly independent set.

Thus, \mathcal{O} satisfies the assumption of Zorn's Lemma which ensures the existence of a maximal element $B \in \mathcal{O}$ and correspondingly a linearly independent set $\otimes(B) \subset V \otimes W$. Since $\otimes(B) \subset \otimes(V \times W)$ and

$\otimes(V \times W)$ generates $V \otimes W$, by Lemma 1.16, we can complete $\otimes(\mathcal{B})$ to a basis $\overline{\otimes(\mathcal{B})} \subset \otimes(V \times W)$ of $V \otimes W$.

Since by construction, $\overline{\otimes(\mathcal{B})} \subset \otimes(V \times W)$, any element $y \in \overline{\otimes(\mathcal{B})} \setminus \otimes(\mathcal{B})$ can be written $y = \otimes(x_y)$ for some $x_y \in V \times W$. We claim that the set $\overline{\mathcal{B}} := \{x_y : y \in \overline{\otimes(\mathcal{B})} \setminus \otimes(\mathcal{B})\} \cup \mathcal{B}$ fulfills the relation

$$\overline{\otimes(\mathcal{B})} = \otimes(\overline{\mathcal{B}}).$$

Indeed, if $x \in \overline{\mathcal{B}}$, either $x \in \mathcal{B}$, in which case $y := \otimes(x) \in \otimes(\mathcal{B}) \subset \overline{\otimes(\mathcal{B})}$, or $y := \otimes(x) \in \overline{\otimes(\mathcal{B})} \setminus \otimes(\mathcal{B}) \subset \overline{\otimes(\mathcal{B})}$ and therefore $\otimes(\overline{\mathcal{B}}) \subset \overline{\otimes(\mathcal{B})}$. Conversely, for $y \in \overline{\otimes(\mathcal{B})}$ either $y \in \otimes(\mathcal{B}) \subset \otimes(\overline{\mathcal{B}})$ or $y \in \overline{\otimes(\mathcal{B})} \setminus \otimes(\mathcal{B})$ in which case $x_y \in \overline{\mathcal{B}}$ and therefore $\otimes(\overline{\mathcal{B}}) \subset \overline{\otimes(\mathcal{B})}$.

Let $g : V \times W \rightarrow G$ be the unique \mathbb{K} -bilinear map defined on $\overline{\mathcal{B}}$ by

$$g(x, y) := \begin{cases} f(x, y) & \text{if } (x, y) \in \mathcal{B} \\ 0 & \text{if } (x, y) \notin \mathcal{B} \end{cases}$$

It remains to show that $g|_{V \times_{\top} W} = f$. Given $(p, q) \in V \times_{\top} W$, the maximality of \mathcal{B} yields the existence of $(x_i, y_i) \in \mathcal{B}$ with $1 \leq i \leq n$ for some $n \in \mathbb{N}$ such that $\sum_{i=1}^n \alpha_i x_i \otimes y_i = p \otimes q$, and thus $\sum_{i=1}^n \alpha_i (x_i, y_i) = (p, q) + \omega$ which amounts to $\omega \in I_{\text{bil}}^{\top \times}$. Using the extension of f and g to \bar{f} and \bar{g} , and the fact that $\bar{f}(\omega) = \bar{g}(\omega) = 0$, this implies

$$f(p, q) = \sum_{i=1}^n \alpha_i f(x_i, y_i) = \sum_{i=1}^n \alpha_i g(x_i, y_i) = g(p, q).$$

It follows that $g|_{V \times_{\top} W} = f$ as expected. \square

Corollary 1.18. The universal property of the locality tensor product of two subspaces of a pre-locality vector space is equivalent to the universal property of the usual (by usual we mean non locality) tensor product of two subspaces of an ordinary vector space.

Proof. We know that the statement holds for $\top = E \times E$ as a result of the universal property of the usual tensor product. From there, we build the map $\phi_{f_{\top}}$ for any locality relation \top on E .

We now prove that the usual universal property implies the one in the locality set up. Thanks to Proposition 1.17, f_{\top} extends to a bilinear map $g : V \times W \rightarrow G$ such that $g|_{V \times_{\top} W} = f_{\top}$. The universal property of the usual tensor product yields the existence of a unique linear map $\phi_g : V \otimes W \rightarrow G$ such that

$$g = \phi_g \circ \otimes. \tag{16}$$

Since $V \otimes_{\top} W \subset V \otimes W$, we can restrict ϕ_g to $V \otimes_{\top} W$ and set $\phi_{f_{\top}} := \phi_g|_{V \otimes_{\top} W}$. Since $\otimes|_{V \times_{\top} W} = \otimes_{\top}$, we can further restrict (16) to $V \times_{\top} W$ and we have

$$f_{\top} = \phi_{f_{\top}} \circ \otimes_{\top}$$

as expected.

Now assuming that the universal property holds for locality tensor products, we want to show that it holds for ordinary tensor products. So, let V, W be two subspaces of a vector space E , G another vector space. We equip E with the trivial locality relation $\top = E \times E$, in which case $V \times_{\top} W = (V \times W) \cap \top = V \times W$ and the bilinear map $f : V \times W \rightarrow G$ can be interpreted as a \top_{\times} -bilinear map $f_{\top} := f : V \times_{\top} W \rightarrow G$. Applying Theorem 1.15 yields a linear map $\phi_f := \phi_{f_{\top}} : V \otimes_{\top} W = V \otimes W \rightarrow G$ such that $f = \phi_f \circ \otimes$ since $\otimes_{\top} = \pi_{\top} \circ \iota_{\top} = \pi \circ \iota = \otimes$. The uniqueness of ϕ_f then follows from that of $\phi_{f_{\top}}$. \square

2 A locality relation on locality tensor products

We want to define a locality relation \top_{\otimes} in such a way that \otimes_{\top} defined in (15) is a locality map. For this purpose, we define a notion of final locality inspired by that of final topology. Let us first recall some definitions.

2.1 Free locality vector spaces

We recall a rather straightforward construction from [CGPZ1, Lemma 2.3 (ii)]. A locality relation \top on a set X induces another one in the power set $\mathcal{P}(X)$, which with some abuse of notation, we denote by the same symbol \top :

$$\forall U, V \in \mathcal{P}(X), \quad U \top V \iff u \top v \quad \forall (u, v) \in U \times V.$$

Conversely a locality relation \top on a power set $\mathcal{P}(X)$ induces a locality relation, again written \top on the set X by restriction:

$$\forall (x, y) \in X^2, \quad x \top y \iff \{x\} \top \{y\}.$$

A locality relation \top on a set X further induces a locality relation (denoted with some abuse of notation by the same symbol \top) on the vector space $\mathbb{K}X$ generated by X given by the linear extension of the locality relation \top on X . Explicitly, two elements a and b in $\mathbb{K}X$ are independent if the basis elements from X appearing in a are independent of the basis elements arising in b . More precisely, the linear span $(\mathbb{K}X, \top)$ of a locality set (X, \top) is a pre-locality vector space when equipped with the symmetric binary relation

$$\left(a := \sum_{x \in X_a} \alpha_x x \right) \top \left(b := \sum_{x \in X_b} \beta_x x \right) \iff X_a \top_X X_b \quad (17)$$

where the coefficients α_x and β_x are all non zero.

Lemma 2.1. The linear span $(\mathbb{K}X, \top)$ of a locality set (X, \top) is a locality (and hence also a pre-locality) vector space.

Proof. By definition we have

$$\left(\sum_{i=1}^n \lambda_i u_i \right) \top \left(\sum_{j=1}^{n'} \lambda'_j u'_j \right) \iff u_i \top u'_j \quad \forall (i, j) \in \{1, \dots, n\} \times \{1, \dots, n'\}.$$

In order to check Condition (1), we take $v := \sum_{k=1}^m \mu_k v_k$, $u = \sum_{i=1}^n \lambda_i u_i$ in $\mathbb{K}X$, and $u' = \sum_{j=1}^{n'} \lambda'_j u'_j$ with $u \top v$ and $u' \top v$. For any $(\lambda, \lambda') \in \mathbb{K}^2$, the element $\lambda u + \lambda' u' = \sum_{i=1}^n \lambda \lambda_i u_i + \sum_{j=1}^{n'} \lambda' \lambda'_j u'_j$ is locality independent of v . Indeed, it follows from the definition of the linearly extended relation \top , and from the u_i 's and u'_j 's being locality independent of v for all $(i, j) \in \{1, \dots, n\} \times \{1, \dots, n'\}$. \square

2.2 Quotient locality as a final locality relation

We define a final locality in much the same way as a final topology. Recall that given two topologies τ_1, τ_2 on some set X , τ_1 is said to be **coarser (weaker or smaller)** than τ_2 , or equivalently τ_2 **finer (stronger or larger)** than τ_1 if, and only if $\tau_1 \subset \tau_2$. Also, given a set X and $(X_i, \tau_i)_{i \in I}$ a family of topological spaces together with a family of maps $f_i : X_i \rightarrow X$, the **final topology (or strong, colimit, coinduced, or inductive topology)** $\bar{\tau}$ is the finest topology on X such that all maps f_i are continuous. With a small abuse of language, one says that the topology $\bar{\tau}$ is final with respect to the maps f_i .

A typical example is the quotient topology on $X \times I$ where I is a subset of a locality set (X, \top) , defined as the final topology for the projection map $\pi : X \rightarrow X \setminus I$.

Let us now transpose this terminology to the locality setup.

Definition 2.2. Let \top_1 and \top_2 be two locality relations over a set A . We say \top_1 is **coarser** than \top_2 or equivalently, that \top_2 is **finer** than \top_1 if, and only if $\top_1 \subset \top_2$.

Remark 2.3. In Theorem 1.8, a pre-locality relation \top on a vector space is completed into a locality relation \top^ℓ on the same vector space. Then \top^ℓ can be defined as the coarsest locality relation that is finer than \top .

The following example provides a justification of the terminology in our transposition from a topological to a locality context.

Example 2.4. Let X be a set and $\mathcal{P}(X)$ its powerset. Disjointness of sets:

$$A \top B \iff A \cap B = \emptyset$$

defines a locality relation on any subset \mathcal{O} of $\mathcal{P}(X)$. If (X, \mathcal{O}) is a topological space with topology $\mathcal{O} \subset \mathcal{P}(X)$, this disjointness relation gives rise to another locality relation (which with some abuse of notation, we denote by the same notation) given by the separation of points:

$$x \top y \iff \exists U, V \in \mathcal{O}, \quad (U \top V) \wedge (x \in U \wedge y \in V).$$

The finer (coarser) the topology \mathcal{O} , the larger (smaller) the graph $\{(x, y), x \top y\}$ of the locality relation, hence the terminology we have chosen.

Definition 2.5. Let X be a set, $(X_i, \top_i)_{i \in I}$ a family of locality sets, and $f_i : X_i \rightarrow X$ a family of maps. The **final locality relation** $\overline{\top}$ on X is the coarsest locality relation among the locality relations \top on X for which

$$f_i : (X_i, \top_i) \longrightarrow (X, \top), \quad i \in I$$

are locality maps.

As before, with a slight abuse of language, we shall say that $\overline{\top}$ is a **final locality relation** on X for the maps f_i .

Proposition 2.6. Given a surjective map $\phi : A \rightarrow B$, the locality relation \top on A induces a locality relation $\overline{\top}$ on B defined by

$$b_1 \overline{\top} b_2 \iff (\exists (a_1, a_2) \in A \times A : \phi(a_i) = b_i \text{ and } a_1 \top a_2),$$

which is the final locality relation for the map ϕ .

Proof. It is clear from the definition of $\overline{\top}$, that $\phi : (A, \top) \longrightarrow (B, \overline{\top})$ is a locality map.

Let \top_B be a locality relation on B such that $\phi : (A, \top) \longrightarrow (B, \top_B)$ is a locality map. For any $(b_1, b_2) \in B^2$ we have

$$\begin{aligned} b_1 \overline{\top} b_2 &\implies (\exists (a_1, a_2) \in A^2 \mid \phi(a_i) = b_i \wedge a_1 \top a_2) \quad \text{for } i \in \{1, 2\} \\ &\implies (\exists (a_1, a_2) \in A^2 \mid \phi(a_i) = b_i \wedge \phi(a_1) \top_B \phi(a_2)) \quad \text{since } \phi \text{ is a locality map} \\ &\implies b_1 \top_B b_2. \end{aligned}$$

Therefore $\overline{\top} \subseteq \top_B$. □

Example 2.7. The map

$$\phi : \begin{cases} \phi : \mathbb{N} & \longrightarrow & 2\mathbb{N} \\ m & \longmapsto & 2m \end{cases}$$

is surjective. We equip $A := \mathbb{N}$ with the locality relation $m_1 \top m_2 \iff |m_1 - m_2| = 3$. Then $n_1 \overline{\top} n_2$ if and only if $|n_1 - n_2| = 6$.

Applying Proposition 2.6 to the canonical projection map $\pi : V \rightarrow V/W$ of a pre-locality vector space (V, \top) to its quotient V/W by a linear subspace W , we equip the quotient with the quotient locality relation.

Definition-Proposition 2.8. For a subspace W of a pre-locality vector space (V, \top) , we call **quotient locality** on the quotient V/W , the final locality relation

$$([u] \overline{\top} [v] \iff \exists (u', v') \in [u] \times [v] : u' \top v') \quad \forall ([u], [v]) \in (V/W)^2 \quad (18)$$

for the canonical projection map $\pi : V \rightarrow V/W$. This way, the pre-locality space (V, \top) gives rise to a pre-locality vector quotient space $(V/W, \overline{\top})$ and the projection map $\pi : (V, \top) \rightarrow (V/W, \overline{\top})$ is a morphism of pre-locality vector spaces.

Proof. The facts that $(V/W, \overline{\top})$ is a pre-locality space and that $\pi : (V, \top) \rightarrow (V/W, \overline{\top})$ is a morphism of pre-locality spaces hold by definition of $\overline{\top}$, since it is the coarsest locality relation such that π is a locality map. \square

The following simple examples illustrate this last concept.

Example 2.9. Consider the pre-locality vector space (\mathbb{R}^3, \top) where \top is the orthogonality relation, namely $v \top w \Leftrightarrow v \perp w$. Let $W = \mathbb{K}e_1 \subset \mathbb{R}^3$ be the span of e_1 where $\{e_i\}_{i=1}^3$ is the canonical basis of \mathbb{R}^3 . The quotient locality on \mathbb{R}^3/W is $\overline{\top} = (\mathbb{R}^3/W) \times (\mathbb{R}^3/W)$ since for any pair $([q_2e_2 + q_3e_3], [k_2e_2 + k_3e_3]) \in (\mathbb{R}^3/W)^2$ there are scalars q_1 and k_1 in \mathbb{K} such that $(q_1e_1 + q_2e_2 + q_3e_3) \perp (k_1e_1 + k_2e_2 + k_3e_3)$, so that $[q_2e_2 + q_3e_3] \overline{\top} [k_2e_2 + k_3e_3]$.

Example 2.10. Consider the pre-locality vector space (V, \top) where $V = \mathbb{R}^4$ and $\top = \mathbb{R}^4 \times \{0\} \cup \{0\} \times \mathbb{R}^4 \cup (\langle\{e_1, e_3\}\rangle \times \langle e_2 + e_4 \rangle) \cup (\langle e_2 + e_4 \rangle \times \langle\{e_1, e_3\}\rangle)$. For $W = \mathbb{K}(e_4)$, the quotient locality on V/W is given by $\overline{\top} = (V/W \times \{[0]\}) \cup (\{[0]\} \times V/W) \cup (\langle\{e_1 + e_3\}\rangle \times \langle[e_2]\rangle) \cup (\langle[e_2]\rangle \times \langle\{e_1 + e_3\}\rangle)$.

2.3 An enhanced universal property on tensor products of pre-locality spaces

Let (E, \top) be a pre-locality vector space, and let us consider the locality cartesian product $V \times_{\top} W = (V \times W) \cap \top$ of two linear subspaces V and W of E . Both subspaces inherit from \top a locality relation $\top_V = \top \cap (V \times V)$ and similarly for W , turning them into pre-locality subspaces. We want to equip their locality tensor product $V \otimes_{\top} W$ defined in (10) with a locality relation.

Definition 2.11. The locality relation \top_{\otimes} on $V \otimes_{\top} W$ is defined as the quotient relation (see Definition 2.8) for the quotient map $\pi : \mathbb{K}(V \times_{\top} W) \rightarrow V \otimes_{\top} W$, where the locality relation on $\mathbb{K}(V \times_{\top} W)$ is the relation $\top_{V \times_{\top} W}$ defined in Equation (9).

With this locality relation on the locality tensor product, the map \otimes_{\top} is a locality map:

Proposition 2.12. *The map $\otimes_{\top} : (V \times_{\top} W, \top_{V \times_{\top} W}) \rightarrow (V \otimes_{\top} W, \top_{\otimes})$ is a locality \top_{\times} -bilinear map.*

Proof. It was shown in Proposition 1.11 that \otimes_{\top} is \top_{\times} -bilinear. We therefore only need to show that it is a locality map. Recall that $\otimes_{\top} = \pi_{\top} \circ \iota_{\top}$, where $\iota_{\top} : V \times_{\top} W \rightarrow \mathbb{K}(V \times_{\top} W)$ is the canonical inclusion map. The latter is a locality map since the locality \top_{\times} on $\mathbb{K}(V \times_{\top} W)$ is a linear extension of the locality relation in $V \times_{\top} W$. The map $\pi_{\top} : \mathbb{K}(V \times_{\top} W) \rightarrow V \otimes_{\top} W$ is also a locality map by construction of the locality relation \top_{\otimes} . The statement then follows from the fact that the composition of locality maps is again a locality map. \square

Locality tensor products equipped with the locality relation \top_{\otimes} are pre-locality vector spaces and the universal property of the tensor product proved in Theorem 1.15 can be enhanced as follows.

Theorem 2.13 (Universal property of the locality tensor product on pre-locality vector spaces). Let (G, \top_G) be a pre-locality vector space, and $f : (V \times_{\top} W, \top_{V \times_{\top} W}) \rightarrow (G, \top_G)$ a locality \top_{\times} -bilinear map. There is a unique locality linear map $\phi : V \otimes_{\top} W \rightarrow G$ such that the following diagram commutes.

$$\begin{array}{ccc} (V \times_{\top} W, \top_{V \times_{\top} W}) & \xrightarrow{\otimes_{\top}} & (V \otimes_{\top} W, \top_{\otimes}) \\ & \searrow f & \downarrow \phi \\ & & (G, \top_G) \end{array}$$

Proof. Theorem 1.15 yields the existence and uniqueness of the linear map ϕ . We are only left to show that ϕ is a locality map. Recall that two equivalence classes $[a]$ and $[b]$ in $V \otimes_{\top} W$ verify $[a] \top_{\otimes} [b]$ if there are $\sum_{i=1}^n \alpha_i(x_i, y_i) \in [a]$ and $\sum_{j=1}^m \beta_j(u_j, v_j) \in [b]$ such that every possible pair taken from the set $\{x_i, y_i, u_j, v_j\}$ lies in $V \times_{\top} W$ for every $1 \leq i \leq n$ and every $1 \leq j \leq m$. Since f is locality bilinear, then $f(\sum_{i=1}^n \alpha_i(x_i, y_i)) \top_G f(\sum_{j=1}^m \beta_j(u_j, v_j))$ which amounts to $\phi([a]) \top_G \phi([b])$. Therefore ϕ is locality as expected. \square

Remark 2.14. The universal property of the locality tensor product on pre-local vector spaces clearly implies the universal property of the tensor product on ordinary vector spaces; take $\top_V = V \times V$, $\top_W = W \times W$, $\top_G = G \times G$ in the above theorem.

3 Higher locality tensor products

We generalise Definition 1.9 and build the locality tensor product of n pre-locality vector spaces, which we equip with a locality relation. In this paragraph, (E, \top) is a pre-locality vector space over \mathbb{K} , and V_1, \dots, V_n are linear subspaces of E . We build the quotient of $\mathbb{K}(V_1 \times \dots \times V_n)$ as the subspace $I_{\text{mult}}(V_1, \dots, V_n)$ generated by all elements of the form

$$(x_1, \dots, x_{i-1}, a_i + b_i, x_{i+1}, \dots, x_n) - (x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_n) - (x_1, \dots, x_{i-1}, b_i, x_{i+1}, \dots, x_n) \quad (19)$$

$$(x_1, \dots, kx_i, \dots, x_n) - k(x_1, \dots, x_i, \dots, x_n) \quad (20)$$

for every $i \in [n]$, $k \in \mathbb{K}$ and $a_i, b_i, x_i \in V_i$. If $V_1 = \dots = V_n = V$, we write $I_{\text{mult},n}(V)$.

Definition 3.1. We define

- [CGPZ1, §3.1] the **locality cartesian product**

$$V_1 \times_{\top} \dots \times_{\top} V_n := \{(x_1, \dots, x_n) \in V_1 \times \dots \times V_n \mid \forall (i, j) \in [n] : i \neq j \Rightarrow (x_i, x_j) \in V_i \times_{\top} V_j := (V_i \times V_j) \cap \top\}; \quad (21)$$

If $V_i = V$ for any $i \in [n]$, we set $V^{\times_{\top}^n} := V_1 \times_{\top} \dots \times_{\top} V_n = V \times_{\top} \dots \times_{\top} V$. In particular $V^{\times_{\top}^2} = \top$, $V^{\times_{\top}^1} = V$ and we set by convention $V^{\times_{\top}^0} = \mathbb{K}$ and $V^{\times_{\top}^{\infty}} := \bigcup_{n \geq 0} V^{\times_{\top}^n}$. Note that $V_1 \times_{\top} \dots \times_{\top} V_n \ni (0, 0, \dots, 0)$ where 0 is the zero element in E , since $\top \ni (0, 0)$ by definition of a pre-locality vector space.

- [CGPZ1, §4.1] the **locality tensor product**

$$V_1 \otimes_{\top} \dots \otimes_{\top} V_n := \mathbb{K}(V_1 \times_{\top} \dots \times_{\top} V_n) / (I_{\text{mult}}(V_1, \dots, V_n) \cap \mathbb{K}(V_1 \times_{\top} \dots \times_{\top} V_n)) \quad (22)$$

If $V_i = V$ for any $i \in [n]$, we set $V^{\otimes_{\top}^n} := V_1 \otimes_{\top} \dots \otimes_{\top} V_n$.

- We endow the locality tensor product $V_1 \otimes_{\top} \dots \otimes_{\top} V_n$ with the locality relation $\top_{\otimes n}$ defined as the quotient locality (see Definition 2.8) for the quotient map $\pi_n : \mathbb{K}(V_1 \times_{\top} \dots \times_{\top} V_n) \longrightarrow V_1 \otimes_{\top} \dots \otimes_{\top} V_n$.

Remark 3.2. For $n = 2$ we recover Definition 1.9.

The size of $V^{\otimes_{\top}^n} \subset V^{\otimes n}$ depends on the locality relation, namely on how many elements mutually independent it allows.

Example 3.3. Consider the pre-locality vector space (\mathbb{R}^2, \perp) , then $V^{\times_{\top}^n} = \{0\}$ for all $n \geq 3$ since there are no three pairwise orthogonal non zero elements in \mathbb{R}^2 .

This contrasts with the following example.

Example 3.4. We equip the vector space $V := \mathbb{R}^{\infty}$ with the canonical orthogonality relation \perp as locality relation: $u \perp v \iff \langle u, v \rangle = 0$ (see [CGPZ1, Subsection 2.2.1] for details). One easily checks that $(\mathbb{R}^{\infty}, \perp)$ is a locality vector space. In this case, there is no integer n in \mathbb{N} such that $V^{\otimes \perp n} = \{0\}$.

Definition 3.5. • We define the relation $\top_{\times m, n} \subset (V_1 \times_{\top} \dots \times_{\top} V_m) \times (V_{m+1} \times_{\top} \dots \times_{\top} V_{m+n})$ as follows

$$(x, y) \in \top_{\times m, n} \iff \forall (i, j) \in [m] \times [n], (x_i, y_{m+j}) \in \top, \quad (23)$$

extend it linearly to $\mathbb{K}(V_1 \times_{\top} \dots \times_{\top} V_m) \times \mathbb{K}(V_{m+1} \times_{\top} \dots \times_{\top} V_{m+n})$ as follows

$$\left(\sum_{k \in K} \alpha_k x_k, \sum_{l \in L} \beta_l y_l \right) \in \mathbb{K}(V_1 \times_{\top} \dots \times_{\top} V_m) \times_{\top_{\times m, n}} \mathbb{K}(V_{m+1} \times_{\top} \dots \times_{\top} V_{m+n}) \\ \iff \forall (k, l) \in K \times L, (x_k, y_l) \in \top,$$

for any K and L finite sets, for every $k \in K$ and every $l \in L$, $(\alpha_k, \beta_l) \in (\mathbb{K} \setminus \{0_{\mathbb{K}}\})^2$, $x_k \in V_k$ and $y_l \in V_l$.

- Finally, we define the relation $\top_{\otimes m,n} \subset (V_1 \otimes_{\top} \cdots \otimes_{\top} V_m) \times (V_{m+1} \otimes_{\top} \cdots \otimes_{\top} V_{m+n})$ induced from $\top_{\times m,n}$ as follows:

$$(x, y) \in \top_{\otimes m,n} \Leftrightarrow (\exists x' \in x) \wedge (\exists y' \in y) : (x', y') \in \top_{\times m,n}, \quad (24)$$

where $x \in V_1 \otimes_{\top} \cdots \otimes_{\top} V_m$ and $y \in V_{m+1} \otimes_{\top} \cdots \otimes_{\top} V_{m+n}$ (notice that according to Definition 3.1, x and y are equivalence classes so that the notation $x' \in x$ makes sense).

Remark 3.6. $\top_{\times m,n}$ and $\top_{\otimes m,n}$ are not locality relations since they are not in general subset of a set of the form $S \times S$. Instead, $\top_{\times m,n}$ can be seen as a relation between $\mathbb{K}(V_1 \times_{\top} \cdots \times_{\top} V_m)$ and $\mathbb{K}(V_{m+1} \times_{\top} \cdots \times_{\top} V_{m+n})$; and $\top_{\otimes m,n}$ as a relation between $V_1 \otimes_{\top} \cdots \otimes_{\top} V_m$ and $V_{m+1} \otimes_{\top} \cdots \otimes_{\top} V_{m+n}$. In contrast, $\top_{\otimes n}$ is a locality relation on $V_1 \otimes_{\top} \cdots \otimes_{\top} V_n$.

Lemma 3.7. The map

$$\Psi_{m,n} : \begin{cases} \mathbb{K}(V_1 \times_{\top} \cdots \times_{\top} V_m) \times_{\top} \mathbb{K}(V_{m+1} \times_{\top} \cdots \times_{\top} V_{m+n}) & \longrightarrow & \mathbb{K}(V_1 \times_{\top} \cdots \times_{\top} V_{m+n}) \\ ((x_1, \dots, x_m), (y_1, \dots, y_n)) & \longmapsto & (x_1, \dots, x_m, y_1, \dots, y_n) \end{cases} \quad (25)$$

linearly extends to a surjective morphism of pre-locality vector spaces:

$$\Psi_{m,n} : \mathbb{K}(\mathbb{K}(V_1 \times_{\top} \cdots \times_{\top} V_m) \times_{\top} \mathbb{K}(V_{m+1} \times_{\top} \cdots \times_{\top} V_{m+n})) \rightarrow \mathbb{K}(V_1 \times_{\top} \cdots \times_{\top} V_{m+n}). \quad (26)$$

Remark 3.8. Note that $\Psi_{m,n}$ is not expected to be an isomorphism. Let us take $m = n = 1$ to simplify. A basis of $\mathbb{K}(\mathbb{K}(V_1) \times_{\top} \mathbb{K}(V_2))$ is given by elements $(k_1 v_1 + \dots + k_p v_p, l_1 w_1 + \dots + l_q w_q)$, with p, q in \mathbb{N} , (v_i, w_j) in $V_1 \times_{\top} V_2$ and k_i, l_j in \mathbb{K} non zero for all indices i, j . A basis of $\mathbb{K}(V_1 \times_{\top} V_2)$ is given by pairs $(v_1, v_2) \in V_1 \times_{\top} V_2$. Since

$$\psi_{1,1}((k_1 v_1 + \dots + k_p v_p, l_1 w_1 + \dots + l_q w_q)) = \sum_{i,j} k_i l_j (v_i, w_j) = \psi_{1,1} \left(\sum_{i,j} k_i l_j (v_i, w_j) \right),$$

$\psi_{1,1}$ is not injective.

Proof. The fact that $\Psi_{m,n}$ extends to a surjective morphism of vector spaces is a classical result of linear algebra. To show that $\Psi_{m,n}$ is a locality morphism we just need to check that it is a locality map, which is an easy consequence of the following equivalence: for any $x := (x_1, \dots, x_m) \in V_1 \times_{\top} \cdots \times_{\top} V_m$ and $y := (y_1, \dots, y_n) \in V_{m+1} \times_{\top} \cdots \times_{\top} V_{m+n}$, we have

$$\begin{aligned} (x, y) \in \top_{\times m,n} &\iff (x_i, y_j) \in \top \quad \forall (i, j) \in [m] \times [n] \\ &\iff \Psi(x, y) = (x_1, \dots, x_m, y_1, \dots, y_n) \in \mathbb{K}(V_1 \times_{\top} \cdots \times_{\top} V_{m+n}) \end{aligned}$$

One simply needs to take two independent pairs (x, y) and (x', y') and work with these equivalences. We omit here the detailed proof which is straightforward but rather combersome to write. \square

The locality morphism (26) induces a locality morphism between locality tensor products.

Theorem 3.9. For any subspaces V_1, \dots, V_{m+n} of the pre-locality space (E, \top) , we set

$$\begin{aligned} &(V_1 \otimes_{\top} \cdots \otimes_{\top} V_m) \otimes_{\top} (V_{m+1} \otimes_{\top} \cdots \otimes_{\top} V_{m+n}) \\ &:= \mathbb{K}((V_1 \otimes_{\top} \cdots \otimes_{\top} V_m) \times_{\top_{\otimes m,n}} (V_{m+1} \otimes_{\top} \cdots \otimes_{\top} V_{m+n})) \\ &/ (I_{\text{bil}} \cap \mathbb{K}((V_1 \otimes_{\top} \cdots \otimes_{\top} V_m) \times_{\top_{\otimes m,n}} (V_{m+1} \otimes_{\top} \cdots \otimes_{\top} V_{m+n}))) \end{aligned}$$

and Definition 2.11 yields a pre-locality relation on this quotient.

There is an isomorphism of pre-locality vector spaces

$$\begin{aligned} \Phi_{m,n} : &((V_1 \otimes_{\top} \cdots \otimes_{\top} V_m) \otimes_{\top} (V_{m+1} \otimes_{\top} \cdots \otimes_{\top} V_{m+n}), \top_{\otimes}^{m,n}) \\ &\xrightarrow{\sim} (V_1 \otimes_{\top} \cdots \otimes_{\top} V_m \otimes_{\top} V_{m+1} \otimes_{\top} \cdots \otimes_{\top} V_{m+n}, \top_{\otimes(m+n)}). \end{aligned} \quad (27)$$

Remark 3.10. Notice that the locality relation $\top_{\otimes(m+n)}$ is the locality relation $\top_{\otimes N}$ of the third item of Definition 3.1 with $N = m + n$, not to be confused with the relation $\top_{\otimes m, n}$ of Definition 3.5, nor with the newly introduced locality $\top_{\otimes}^{m, n}$.

Proof. We build the isomorphism $\Phi_{m, n}$ from the morphism $\Psi_{m, n}$ defined in (25).

- An element $[y]$ of $(V_1 \otimes_{\top} \cdots \otimes_{\top} V_m) \otimes_{\top} (V_{m+1} \otimes_{\top} \cdots \otimes_{\top} V_{m+n})$ reads

$$[y] = \left[\sum_{i \in I} \alpha_i \left(\sum_{j \in J} \beta_j^i v_{j,1}^i \otimes \cdots \otimes v_{j,m}^i, \sum_{k \in K} \gamma_k^i v_{k,1}^i \otimes \cdots \otimes v_{k,n}^i \right) \right],$$

for some vectors $v_{j,r}^i \in V_r$ and $v_{k,s}^i \in V_s$ and scalars $\alpha_i, \beta_j^i, \gamma_k^i$ in \mathbb{K} with I, J, K three finite sets, such that $\sum_{j \in J} \sum_{k \in K} \beta_j^i \gamma_k^i (v_{j,1}^i, \dots, v_{j,m}^i, v_{k,1}^i, \dots, v_{k,n}^i) \in (V_1 \times_{\top} \cdots \times_{\top} V_m) \times_{\top n, m} (V_{m+1} \times_{\top} \cdots \times_{\top} V_{m+n}) \simeq (V_1 \times_{\top} \cdots \times_{\top} V_{m+n})$.

- The linear map $\Phi_{m, n} : (V_1 \otimes_{\top} \cdots \otimes_{\top} V_m) \otimes_{\top} (V_{m+1} \otimes_{\top} \cdots \otimes_{\top} V_{m+n}) \rightarrow (V_1 \otimes_{\top} \cdots \otimes_{\top} V_{m+n})$ is defined by the following action on an element $[y]$ of $(V_1 \otimes_{\top} \cdots \otimes_{\top} V_m) \otimes_{\top n, m} (V_1 \otimes_{\top} \cdots \otimes_{\top} V_n)$:

$$\Phi_{m, n}([y]) = \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \alpha_i \beta_j^i \gamma_k^i (v_{j,1}^i \otimes \cdots \otimes v_{j,n}^i \otimes v_{k,1}^i \otimes \cdots \otimes v_{k,m}^i).$$

Notice that $(v_{j,1}^i, \dots, v_{j,n}^i, v_{k,1}^i, \dots, v_{k,m}^i)$ is an element of $V_1 \times_{\top} \cdots \times_{\top} V_{m+n}$ by definition of $(V_1 \otimes_{\top} \cdots \otimes_{\top} V_m) \otimes_{\top} (V_{m+1} \otimes_{\top} \cdots \otimes_{\top} V_{m+n})$, and that the difference of two representatives of $[y]$ lies in I_{bil} whose image by Φ lies in $I_{\text{mult}}(V_1, \dots, V_{m+n})$. Thus $\Phi_{m, n}$ is well-defined.

- The injectivity of $\Phi_{m, n}$ follows from the commutativity of Diagram 4 in which the vertical arrows are quotient maps.

$$\begin{array}{ccc} \mathbb{K} \left(\mathbb{K} (V_1 \times_{\top} \cdots \times_{\top} V_m) \times_{\top \times m, n} \mathbb{K} (V_{m+1} \times_{\top} \cdots \times_{\top} V_{m+n}) \right) & \xrightarrow{\Psi_{m, n}} & \mathbb{K} (V_1 \times_{\top} \cdots \times_{\top} V_{m+n}) \\ \downarrow \pi_m \times \pi_n & & \downarrow \pi_{m+n} \\ \mathbb{K} \left((V_1 \otimes_{\top} \cdots \otimes_{\top} V_m) \times_{\top \otimes m, n} (V_{m+1} \otimes_{\top} \cdots \otimes_{\top} V_{m+n}) \right) & & \\ \downarrow & & \\ (V_1 \otimes_{\top} \cdots \otimes_{\top} V_m) \otimes_{\top} (V_{m+1} \otimes_{\top} \cdots \otimes_{\top} V_{m+n}) & \xrightarrow{\Phi_{m, n}} & V_1 \otimes_{\top} \cdots \otimes_{\top} V_{m+n} \end{array}$$

Figure 4: Tensor products of pre-locality spaces

Assume $\Phi_{m, n}([y]) = 0$ for some $[y] \in (V_1 \otimes_{\top} \cdots \otimes_{\top} V_m) \otimes_{\top} (V_{m+1} \otimes_{\top} \cdots \otimes_{\top} V_{m+n})$. Then the preimage of $\Phi_{m, n}([y])$ under the quotient map π_{m+n} in $V_1 \times_{\top} \cdots \times_{\top} V_{m+n}$ lies in $I_{\text{mult}}(V_1, \dots, V_{m+n})$. This implies that its preimage under $\pi_{m+n} \circ \Psi_{m, n}$ lies in $\mathbb{K}(I_{\text{mult}}(V_1, \dots, V_m) \times_{\top n, m} I_{\text{mult}}(V_{m+1}, \dots, V_{m+n}))$ or in $I_{\text{bil}}(\mathbb{K}(V_1 \times_{\top} \cdots \times_{\top} V_m), \mathbb{K}(V_{m+1} \times_{\top} \cdots \times_{\top} V_{m+n}))$ by construction of $\Psi_{m, n}$ in Lemma 3.7. This element of $\mathbb{K}(I_{\text{mult}}(V_1, \dots, V_m) \times_{\top n, m} I_{\text{mult}}(V_{m+1}, \dots, V_{m+n}))$ combined with $I_{\text{bil}}(\mathbb{K}(V_1 \times_{\top} \cdots \times_{\top} V_m), \mathbb{K}(V_{m+1} \times_{\top} \cdots \times_{\top} V_{m+n}))$ is the preimage of $[y]$ under the two projections on the left column of Figure 4, and therefore $[y] = 0$.

- The surjectivity of $\Phi_{m, n}$ follows from that of $\Psi_{m, n}$ combined with the commutativity of Diagram 4. Indeed, any element $y \in V_1 \otimes_{\top} \cdots \otimes_{\top} V_{m+n}$ has a preimage $\tilde{y} \in \mathbb{K}(\mathbb{K}(V_1 \times_{\top} \cdots \times_{\top} V_m) \times_{\top n, m} \mathbb{K}(V_{m+1} \times_{\top} \cdots \times_{\top} V_{m+n}))$ under $\pi_{m+n} \circ \Psi_{m, n}$ since $\Psi_{m, n}$ and $\pi_{m, n}$ are surjections. Taking the image of \tilde{y} under the two projections of the left column of Figure 4 we obtain a preimage of y in $(V_1 \otimes_{\top} \cdots \otimes_{\top} V_m) \otimes_{\top} (V_{m+1} \otimes_{\top} \cdots \otimes_{\top} V_{m+n})$. \square

4 Tensor and universal pre-locality algebras

We now turn to the pre-locality counterpart of the notions of tensor algebras and universal enveloping algebras, and discuss their universal properties. Both are pre-locality algebras, a notion we first introduce.

4.1 Pre-locality algebras

We slightly adapt various notions introduced in [CGPZ1] to the pre-locality framework.

Definition 4.1. • A **non-unital pre-locality algebra** is a triple (A, \top, m) , where (A, \top) is a pre-locality vector space, equipped with a partially defined product, namely a $\top \times$ -bilinear map $m : A \times_{\top} A \rightarrow A$, which is associative in the following sense

$$m(m(x, y), z) = m(x, m(y, z)) \quad \forall (x, y, z) \in A^{\times \top 3},$$

whenever $m(m(x, y), z)$ and $m(x, m(y, z))$ are defined.

- [CGPZ1, Definition 3.16 (ii)] We call **non-unital locality algebra** a non-unital pre-locality algebra (A, \top, m) , whose underlying vector space is a locality vector space (so that in particular U^{\top} is a vector space for any $U \subset A$), and whose partial product $m : A \times_{\top} A \rightarrow A$ is compatible with the locality relation in the following sense

$$m(U^{\top} \times_{\top} U^{\top}) \subset U^{\top} \quad \forall U \subset A, \quad (28)$$

where we have set $V \times_{\top} V := (V \times V) \cap \top$ for any subset $V \subset A$.

- A non-unital locality (resp. pre-locality) algebra is called **unital** (or simply (pre-)locality algebra) if there is a map $u : \mathbb{K} \rightarrow A$ such that $u(\mathbb{K}) \subseteq A^{\top}$, which makes the following diagram 4.1 commute

$$\begin{array}{ccccc} \mathbb{K} \otimes A & \xrightarrow{u \otimes Id} & A \otimes_{\top} A & \xleftarrow{Id \otimes u} & A \otimes \mathbb{K} \\ & \searrow \simeq & \downarrow \mu & \swarrow \simeq & \\ & & A & & \end{array}$$

Figure 5: The unit map

- A (resp. pre-)locality subspace I of a (resp. pre-)locality algebra (A, \top, m) is called a left, resp. right **(pre-)locality ideal** of A , if

$$m(I \times I^{\top}) \subset I; \text{ resp. } m(I^{\top} \times I) \subset I. \quad (29)$$

If it is both a left and a right ideal, we call it a (resp. pre-) locality ideal.

If (A, \top, m) is a locality algebra, then I^{\top} is a linear subspace of A , and we call I a locality ideal.

- [CGPZ1, Definition 3.16 (ii)] Given two (resp. pre-)locality algebras $(A_i, \top_i, m_i, u_i), i = 1, 2$, a locality linear map $f : A_1 \rightarrow A_2$ is called a **(resp. pre-) locality algebra morphism** if

$$f \circ m_1|_{\top_1} = m_2 \circ (f \times f)|_{\top_1}. \quad (30)$$

- We call $(A_1, \top_{A_1}, m_1, u_1)$ a (pre-)locality subalgebra of $(A_2, \top_{A_2}, m_2, u_2)$ if there is an inclusion map $\iota : A_1 \hookrightarrow A_2$ which is also a (pre-)locality algebra morphism.

Remark 4.2. Note that this definition is more general than the definition of sub-locality algebra given in [CGPZ1]. We will need this degree of generality for the locality version of the Milnor-Moore theorem. A case of particular importance is when $A_1 = A_2$ and $\top_2 \subseteq \top_1$.

Example 4.3. Let (A, \top, m) be a locality algebra, the polar set U^\top of any non-empty subset U of A gives rise to a non-unital locality subalgebra $(U^\top, \top_{U^\top}, m)$ of (A, \top, m) . Here $\top_{U^\top} = \top \cap (U^\top \times U^\top)$.

Remark 4.4. Notice that for a non-unital locality algebra (A, \top, m) , Condition (28) is equivalent to the product $m : (A \times_\top A, \top_{A \times_\top A}) \rightarrow (A, \top)$ being a locality map, and thus a locality \top_\times -bilinear map.

Lemma 4.5. Let $f : A_1 \rightarrow A_2$ be a locality linear map between two (resp. pre-)locality algebras $(A_i, m_i, \top_i), i \in \{1, 2\}$. Its kernel is a (resp. pre-)locality ideal of A_1 and its range is a (resp. pre-)locality subalgebra of A_2 .

Proof. We prove that the kernel $\text{Ker}(f)$ is a (resp. pre-)locality ideal. Take $a \in \text{Ker}(f)$ and $b \in \text{Ker}(f)^{\top_1}$, then $f(m_1(a \otimes b)) = m_2(f(a) \otimes f(b)) = m_2(0 \otimes f(b)) = 0$, hence $m_1(\text{Ker}(f) \times \text{Ker}(f)^{\top_1}) \subset \text{Ker}(f)$. Similarly we check that $m_1(\text{Ker}(f)^{\top_1} \times \text{Ker}(f)) \subset \text{Ker}(f)$.

If A_1 is a locality algebra, then $\text{Ker}(f)^\top$ is a linear subspace of A_1 and $\text{Ker}(f)$ a locality ideal in A_1 .

We prove that the range $\text{Im}(f)$ is a (resp. pre-)locality algebra. Given $(f(a), f(b)) \in (\text{Im}(f) \times \text{Im}(f)) \cap \top_2$, by (30) we have $m_2(f(a) \otimes f(b)) = f \circ m_1(a \otimes b) \in \text{Im}(f)$.

If A_2 is a locality algebra, then $\text{Im}(f)^\top$ is a linear subspace of A_2 . Moreover, setting $\top_{\text{Im}(f)} := \top_2 \cap (\text{Im}(f) \times \text{Im}(f))$, and given $U \subset \text{Im}(f)$,

$$\begin{aligned} m(U^{\top_{\text{Im}(f)}} \times_{\top_{\text{Im}(f)}} U^{\top_{\text{Im}(f)}}) &= m((U^{\top_2} \cap \text{Im}(f)) \times_{\top_2} (U^{\top_2} \cap \text{Im}(f))) \\ &= m((U^{\top_2} \times_{\top_2} U^{\top_2}) \cap \text{Im}(f)^{\times_{\top_2} 2}) \\ &\subset U^{\top_2} \cap \text{Im}(f) = U^{\top_{\text{Im}(f)}}. \end{aligned}$$

The last inclusion is a consequence of Condition (28) for A_2 and $\text{Im}(f)$ being closed under the product m . Therefore Condition (28) is satisfied for $\text{Im}(f)$, so that $\text{Im}(f)$ a locality subalgebra of A_2 . \square

4.2 The locality tensor algebra of a pre-locality vector space

The tensor algebra over a vector space V is defined as $\mathcal{T}(V) := \bigoplus_{n \geq 0} V^{\otimes n}$ with the convention that $V^{\otimes 0} = \mathbb{K}$ and $V^{\otimes 1} = V$. Following [CGPZ1], we define a locality tensor algebra in a similar manner, modulo the fact that we first build it on a pre-locality vector space.

Definition 4.6. • The n -th filtration of the locality tensor algebra over a pre-locality vector space (V, \top) is defined as

$$\mathcal{T}_\top^n(V) := \bigoplus_{k=0}^n V^{\otimes k}.$$

- For any integers $0 \leq m \leq n$, the canonical injections

$$\iota_{m,n} : \mathcal{T}_\top^m(V) \rightarrow \mathcal{T}_\top^n(V) \tag{31}$$

are linear maps.

- The **locality tensor algebra** over a pre-locality vector space (V, \top) is defined as

$$\mathcal{T}_\top(V) := \bigcup_{n \geq 0} \mathcal{T}_\top^n(V) = \bigoplus_{n \geq 0} V^{\otimes n}.$$

The following results set the basis to define a locality relation on $\mathcal{T}_\top(V)$ as a quotient relation.

Proposition 4.7. Given a pre-locality vector space (V, \top) , then

$$\mathcal{T}_\top^n(V) = \mathbb{K}(\bigcup_{k=0}^n V^{\times \top k}) / (I_{\text{mult}}^n(V) \cap \mathbb{K}(\bigcup_{k=0}^n V^{\times \top k}))$$

where we have set $I_{\text{mult}}^n(V) := \bigoplus_{k=1}^n I_{\text{mult},k}(V)$.

Proof. Given a direct sum of vector spaces $V = \bigoplus_{k=0}^n V_k$ and I_k a linear subspace of V_k for each $0 \leq k \leq n$, it is a standard result of linear algebra that the quotient space $\mathbb{V} = (\bigoplus_{k=0}^n V_k) / \bigoplus_{k=0}^n I_k$ inherits the structure of a direct sum of vector space $\mathbb{V} = \bigoplus_{k=0}^n \mathbb{V}_k$ with $\mathbb{V}_k = V_k / I_k$. Applying this to $V_k := \mathbb{K}(V^{\times \tau^k})$ and $I_k := I_{\text{mult},k}(V) \cap \mathbb{K}(V^{\times \tau^k})$ (we simplify the notation leaving out the dependence on V) and noticing that

$$\bigoplus_{k=0}^n I_k = \bigoplus_{k=0}^n (I_{\text{mult},k} \cap \mathbb{K}(V^{\times \tau^k})) = \bigoplus_{k=0}^1 I_{\text{mult},k} \cap \bigoplus_{k=0}^n \mathbb{K}(V^{\times \tau^k}) = I_{\text{mult}}^n \cap \mathbb{K}\left(\bigcup_{k=0}^n V^{\times \tau^k}\right).$$

then yields the result, where we have set $I_{\text{mult},0} := \{0\}$. \square

Definition 4.8. Let $N \in \mathbb{N}$ and (V, \top) be a pre-locality vector space.

- We define the locality relation \top_{\times}^N on $\bigcup_{k=0}^N V^{\times \tau^k}$ as

$$\top_{\times}^N := \bigcup_{k=1}^N \bigcup_{l=1}^N \top_{\times k,l} \quad (\text{see (23)}),$$

which we then linearly extend to $\mathbb{K}(\bigcup_{k=0}^N V^{\times \tau^k})$ as in (17).

- The locality relation \top_{\otimes}^N on $\mathcal{T}_{\top}^N(V)$ is defined as the quotient relation for the canonical map $\pi : \mathbb{K}(\bigcup_{k=0}^N V^{\times \tau^k}) \rightarrow \mathcal{T}_{\top}^N(V)$.

Remark 4.9. For pre-locality vector space (V, \top) and any $n \in \mathbb{N}$, the pair $(\mathcal{T}_{\top}^n(V), \top_{\otimes}^n)$ is a pre-locality vector space.

It follows from the definition of \top_{\times}^n and I_{mult}^n , that $\top_{\otimes}^n \subset \top_{\otimes}^{n+1}$ for any $n \in \mathbb{N}$. Consequently, for every pair $0 \leq m, n$ natural numbers, there is a canonical embedding $\iota_{m,m+n} : \top_{\otimes}^m \rightarrow \top_{\otimes}^{m+n}$. This leads to the following definition.

Definition 4.10. Given a pre-locality vector space (V, \top) , the locality relation on the locality tensor algebra $\mathcal{T}_{\top}(V)$ is defined as a direct limit $\top_{\otimes} := \varinjlim \top_{\otimes}^n$.

Remark 4.11. The pair $(\mathcal{T}_{\top}(V), \top_{\otimes})$ is a trivially a pre-locality vector space, since \top_{\otimes} is symmetric by construction.

Notations 4.2.1. From now on we use \top_{\otimes} instead of $\top_{\otimes m,n}$, $\top_{\otimes n}$, or \top_{\otimes}^n since they are all restrictions of the first one.

Remark 4.12. Since $V^{\times 0} := \mathbb{K}$, and $k\top_{\times}^N(v_1, \dots, v_n)$ for any $k \in \mathbb{K}$ and $(v_1, \dots, v_n) \in V^{\times \tau^n}$ with $N \geq n \geq 1$, then $k\top_{\otimes}(v_1 \otimes \dots \otimes v_n)$ for every $v_1 \otimes \dots \otimes v_n \in V^{\otimes \tau^n}$ for every $n \geq 1$ and hence $(\mathbb{K} = V^{\otimes 0})\top_{\otimes}\mathcal{T}_{\top}(V)$.

Proposition 4.13. Let (V, \top) be a pre-locality vector space. The usual product $\otimes : \mathcal{T}(V) \times \mathcal{T}(V) \rightarrow \mathcal{T}(V)$ on the tensor algebra restricts to $\top_{\otimes} = \mathcal{T}_{\top}(V) \times_{\top_{\otimes}} \mathcal{T}_{\top}(V)$ where it defines a \top_{\times} -bilinear map (see (7)) and

$$\left(\mathcal{T}_{\top}(V), \top_{\otimes}, \otimes, u\right)$$

defines a pre-locality algebra, where u is the canonical injection $u : \mathbb{K} \rightarrow V^{\otimes 0}$.

Proof. • Let us first check that the restriction is $\mathcal{T}_{\top}(V)$ -valued, namely that $\otimes(\top_{\otimes}) \subset \mathcal{T}_{\top}(V)$.

For $([a], [b]) \in \top_{\otimes}$, we may assume without loss of generality that $a = (a_1, \dots, a_m) \in V^{\times \tau^m}$, $b = (b_1, \dots, b_n) \in V^{\times \tau^n}$ and $(a, b) \in \top_{\times m,n}$. Therefore $(a_i, b_j) \in \top$ for every i and j implying that $ab := (a_1, \dots, a_m, b_1, \dots, b_n) \in V^{\times \tau(m+n)}$ so that $\otimes([a], [b]) = [ab] \in \mathcal{T}_{\top}(V)$.

- The fact that it is \top_{\times} -bilinear follows from the fact that \otimes is a restriction of the usual product on the tensor algebra.

- The associativity of the usual product \otimes is preserved when we restrict to \mathbb{T}_\otimes whenever it is well defined. Therefore $(\mathcal{T}_\mathbb{T}(V), \mathbb{T}_\otimes, \otimes, u)$ is indeed a pre-locality algebra. \square

The following Proposition will be of use in the sequel.

Proposition 4.14. *Let (W, \mathbb{T}') be a (pre-)locality subspace of (V, \mathbb{T}) a (pre-)locality vector space. Then $\mathcal{T}_{\mathbb{T}'}(W)$ is a (pre-)locality subalgebra of $\mathcal{T}_\mathbb{T}(V)$.*

Proof. We prove first that for every $n \in \mathbb{Z}_{\geq 0}$, $W^{\otimes_{\mathbb{T}'}}_n$ is a subspace of $V^{\otimes_{\mathbb{T}'}}_n$. This is trivially true for $n \in \{0, 1\}$. For $n \geq 2$. Notice that

$$I_{\text{mult}}(\underbrace{W, \dots, W}_{n\text{-times}}) = I_{\text{mult}}(\underbrace{V, \dots, V}_{n\text{-times}}) \cap \mathbb{K}(\underbrace{W \times \dots \times W}_{n\text{-times}}).$$

Intersecting both sides with $\mathbb{K}(\underbrace{W \times_{\mathbb{T}'} \dots \times_{\mathbb{T}'} W}_{n\text{-times}})$ it follows that

$$\left(I_{\text{mult}}(\underbrace{V, \dots, V}_{n\text{-times}}) \cap \mathbb{K}(\underbrace{W \times \dots \times W}_{n\text{-times}}) \right) \cap \mathbb{K}(\underbrace{W \times_{\mathbb{T}'} \dots \times_{\mathbb{T}'} W}_{n\text{-times}}) = I_{\text{mult}}(\underbrace{W, \dots, W}_{n\text{-times}}) \cap \mathbb{K}(\underbrace{W \times_{\mathbb{T}'} \dots \times_{\mathbb{T}'} W}_{n\text{-times}}).$$

Moreover, since $\mathbb{K}(W \times_{\mathbb{T}'} \dots \times_{\mathbb{T}'} W) \subset \mathbb{K}(W \times \dots \times W)$ and $\mathbb{K}(W \times_{\mathbb{T}'} \dots \times_{\mathbb{T}'} W) \subset \mathbb{K}(V \times_{\mathbb{T}} \dots \times_{\mathbb{T}} V)$ then

$$\left(I_{\text{mult}}(\underbrace{V, \dots, V}_{n\text{-times}}) \cap \mathbb{K}(\underbrace{V \times_{\mathbb{T}} \dots \times_{\mathbb{T}} V}_{n\text{-times}}) \right) \cap \mathbb{K}(\underbrace{W \times_{\mathbb{T}'} \dots \times_{\mathbb{T}'} W}_{n\text{-times}}) = I_{\text{mult}}(\underbrace{W, \dots, W}_{n\text{-times}}) \cap \mathbb{K}(\underbrace{W \times_{\mathbb{T}'} \dots \times_{\mathbb{T}'} W}_{n\text{-times}})$$

and hence, using the identity $\mathbb{K}(W \times_{\mathbb{T}'} \dots \times_{\mathbb{T}'} W) \cap \mathbb{K}(V \times_{\mathbb{T}} \dots \times_{\mathbb{T}} V) = \mathbb{K}(W \times_{\mathbb{T}'} \dots \times_{\mathbb{T}'} W)$ we have

$$\begin{aligned} W^{\otimes_{\mathbb{T}'}}_n &= \mathbb{K}(W \times_{\mathbb{T}'} \dots \times_{\mathbb{T}'} W) / I_{\text{mult}}(W, \dots, W) \cap \mathbb{K}(W \times_{\mathbb{T}'} \dots \times_{\mathbb{T}'} W) \\ &= \mathbb{K}(V \times_{\mathbb{T}} \dots \times_{\mathbb{T}} V) \cap \mathbb{K}(W \times_{\mathbb{T}'} \dots \times_{\mathbb{T}'} W) / I_{\text{mult}}(W, \dots, W) \cap \mathbb{K}(W \times_{\mathbb{T}'} \dots \times_{\mathbb{T}'} W) \\ &\subset \mathbb{K}(V \times_{\mathbb{T}} \dots \times_{\mathbb{T}} V) / I_{\text{mult}}(V, \dots, V) \cap \mathbb{K}(V \times_{\mathbb{T}} \dots \times_{\mathbb{T}} V) \\ &= V^{\otimes_{\mathbb{T}'}}_n, \end{aligned}$$

(since $(A \cap C)/(B \cap C) \subseteq A/B$ for $B \subseteq A$). In particular $\mathcal{T}_{\mathbb{T}'}(W)$ is a subspace of $\mathcal{T}_\mathbb{T}(V)$. We are only left to prove that the injection map $\iota : \mathcal{T}_{\mathbb{T}'}(W) \rightarrow \mathcal{T}_\mathbb{T}(V)$ is a locality map. The inclusion $\mathbb{T}' \subset \mathbb{T}$ implies that $\mathbb{T}'^n_x \subset \mathbb{T}^n_x$ for any n (See 23 and Definition 4.8), and therefore

$$\begin{aligned} ([w_1], [w_2]) \in \mathbb{T}'^n_\otimes &\Rightarrow (\exists w'_1 \in [w_1])(\exists w'_2 \in [w_2]) : (w'_1, w'_2) \in \mathbb{T}'^n_x \\ &\Rightarrow (\exists w'_1 \in [w_1])(\exists w'_2 \in [w_2]) : (w'_1, w'_2) \in \mathbb{T}^n_x \\ &\Rightarrow ([w_1], [w_2]) \in \mathbb{T}^n_\otimes. \end{aligned}$$

Thus ι is a morphism of (pre-)locality vector spaces. One easily checks that it is moreover a morphism of (pre-)locality algebras which proves the statement of the proposition. \square

As in the usual (i.e. non-locality) case, the pre-locality tensor algebra enjoys a universal property.

Theorem 4.15 (Universal property of locality tensor algebra over a pre-locality vector space). *Let (V, \mathbb{T}) be a pre-locality vector space, (A, \mathbb{T}_A) a pre-locality algebra whose product $m_A : (A \times_{\mathbb{T}} A, \mathbb{T}_{A \times_{\mathbb{T}} A}) \rightarrow (A, \mathbb{T}_A)$ is a locality map, and $f : V \rightarrow A$ a locality linear map. There is a unique pre-locality algebra morphism $\psi : \mathcal{T}_\mathbb{T}(V) \rightarrow A$ such that the following diagram commutes*

$$\begin{array}{ccc} (V, \mathbb{T}) & \xrightarrow{\otimes_{\mathbb{T}}} & (\mathcal{T}_\mathbb{T}(V), \mathbb{T}_\otimes) \\ & \searrow f & \downarrow \psi \\ & & (A, \mathbb{T}_A) \end{array}$$

where $\otimes_{\mathbb{T}} : V \rightarrow \mathcal{T}_\mathbb{T}(V)$ is the canonical (locality) injection map.

Proof. Let $f : V \rightarrow A$ be a locality linear map. We define for every $n \in \mathbb{N}$, a locality n -linear map $f_n : V^{\times \tau^n} \rightarrow A$ as $f_n(x_1, \dots, x_n) := m_A^{n-1}(f(x_1), \dots, f(x_n))$. Thanks to the universal property of the locality tensor product (Theorem 2.13), there are locality linear maps $\psi_n : V^{\otimes \tau^n} \rightarrow A$ such that $f_n = \psi_n \circ \otimes_{\tau^n}$ where \otimes_{τ^n} is the canonical map from $V^{\times \tau^n} \rightarrow V^{\otimes \tau^n}$. The map $\psi : \mathcal{T}_{\tau}(V) \rightarrow A$ defined as the sum of the ψ_n 's is a locality algebra morphism such that $f = \psi \circ \otimes_{\tau}$. It is unique due to the uniqueness of each of the maps ψ_n (Theorem 2.13). \square

Similarly to the universal property of the locality tensor product, this last statement is equivalent to the universal property of the usual (non-locality) tensor algebra.

Theorem 4.16. The universal property of the locality tensor algebra is equivalent to the universal property of the usual (non-locality) tensor algebra.

Proof. The fact that the universal property of the locality tensor algebra implies the usual one follows from choosing the trivial locality relation $\tau = V \times V$.

For the converse: Assuming the universal property of the usual tensor algebra and given a locality linear map $f : V \rightarrow A$, it is in particular a linear map. Applying the usual universal property, we get an algebra morphism $\phi : \mathcal{T}(V) \rightarrow A$ such that $f = \phi \circ \otimes$ where \otimes is the canonical injection of V into $\mathcal{T}(V)$. Recall that $\mathcal{T}_{\tau}(V)$ is a linear subspace of $\mathcal{T}(V)$ which contains V , so \otimes is also the canonical injection of V into $\mathcal{T}_{\tau}(V)$. Restricting $\phi|_{\mathcal{T}_{\tau}(V)}$ we therefore obtain a locality algebra morphism such that $f = \phi|_{\mathcal{T}_{\tau}(V)} \circ \otimes$. We conclude that the usual universal property implies the locality one. \square

4.3 The locality universal enveloping algebra

Let us first introduce some terminology inspired by [CGPZ1]. Here, instead of considering locality Lie algebras straightaway, we first introduce a notion of pre-locality Lie algebra.

Definition 4.17. • A **pre-locality Lie algebra** is a triple $(\mathfrak{g}, \tau_{\mathfrak{g}}, [\cdot, \cdot])$ where $(\mathfrak{g}, \tau_{\mathfrak{g}})$ is a pre-locality vector space, and $[\cdot, \cdot] : \tau_{\mathfrak{g}} \rightarrow \mathfrak{g}$ is a locality τ_{\times} -bilinear map antisymmetric which satisfies the following properties:

- For every $U \subset \mathfrak{g}$, the Lie bracket stabilises polar sets, i.e. it maps $(U^{\tau} \times U^{\tau}) \cap \tau_{\mathfrak{g}}$ into U^{τ} .
- For $(a, b, c) \in V^{\times \tau^3}$ then $[[a, b], c] + [[c, a], b] + [[b, c], a] = 0$.

- A **locality Lie algebra** is a pre-locality Lie algebra $(\mathfrak{g}, \tau_{\mathfrak{g}}, [\cdot, \cdot])$ such that $(\mathfrak{g}, \tau_{\mathfrak{g}})$ is also a locality vector space.
- Let $(\mathfrak{g}_1, \tau_{\mathfrak{g}_1}, [\cdot, \cdot]_1)$ and $(\mathfrak{g}_2, \tau_{\mathfrak{g}_2}, [\cdot, \cdot]_2)$ be two (resp. pre-)locality Lie algebras. A locality linear map $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is called a **(resp. pre-)locality Lie algebra morphism** if $f([x, y]_1) = [f(x), f(y)]_2$, for every independent pair $x \tau_1 y$.
- Let $(\mathfrak{g}_2, \tau_{\mathfrak{g}_2}, [\cdot, \cdot]_2)$ be a (pre-)locality Lie algebra and $\mathfrak{g}_1 \subseteq \mathfrak{g}_2$. We call $(\mathfrak{g}_1, \tau_{\mathfrak{g}_1}, [\cdot, \cdot]_1)$ a **(pre-)locality Lie subalgebra** of $(\mathfrak{g}_2, \tau_{\mathfrak{g}_2}, [\cdot, \cdot]_2)$ if the inclusion map $\iota : \mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2$ is a (pre-)locality Lie algebra morphism.

The following example provides a justification of the universal property of the enveloping algebra.

Example 4.18. If (A, τ_A, m_A) is a (pre-)locality associative algebra, then it is a (pre-)locality Lie algebra, with the bracket defined by

$$\forall (x, y) \in A \times_{\tau} A, [x, y] = xy - yx.$$

We now define the locality universal enveloping algebra similarly to the usual universal enveloping algebra

Definition 4.19. Let $(\mathfrak{g}, \tau_{\mathfrak{g}}, [\cdot, \cdot])$ be a pre-locality Lie algebra. Consider the pre-locality ideal $J_{\tau}(\mathfrak{g})$ (see Definition 4.1) of $\mathcal{T}_{\tau}(\mathfrak{g})$ generated by all terms of the form $a \otimes b - b \otimes a - [a, b]$ for $(a, b) \in \tau_{\mathfrak{g}}$. The **locality universal enveloping algebra** of \mathfrak{g} is defined as

$$U_{\tau}(\mathfrak{g}) := \mathcal{T}_{\tau}(\mathfrak{g})/J_{\tau}(\mathfrak{g}). \quad (32)$$

The locality relation \top_U on $U_{\top}(\mathfrak{g})$ is the final locality relation for the quotient map on $U_{\top}(\mathfrak{g})$ induced by \top_{\otimes} on $\mathcal{T}_{\top}(\mathfrak{g})$. This means that two equivalence classes x and y in $U_{\top}(\mathfrak{g})$ are locally independent i.e. $x \top_U y$ if, and only if there are elements a in x and $b \in y$ in $\mathcal{T}_{\top}(\mathfrak{g})$ such that $a \top_{\otimes} b$.

Notice that the locality relation \top_U is defined in a similar manner to \top_{\otimes} , namely it is induced by \top_{\times} on $\mathbb{K}(\mathfrak{g} \times_{\top} \mathfrak{g})$ (see Definition-Proposition 4.8).

Remark 4.20. The ideal $J_{\top}(\mathfrak{g})$ is not graded, so one does not expect $U_{\top}(\mathfrak{g})$ to inherit a grading from $\mathcal{T}_{\top}(\mathfrak{g})$. Yet it does inherit a filtration

$$(U_{\top}(\mathfrak{g}))^n = (\mathcal{T}_{\top}(\mathfrak{g}))^n / J_{\top}(\mathfrak{g}) \cap (\mathcal{T}_{\top}(\mathfrak{g}))^n,$$

where $(\mathcal{T}_{\top}(\mathfrak{g}))^n = \bigoplus_{i=0}^n \mathfrak{g}^{\otimes \top i}$. It is easy to check that this is indeed a filtered pre-locality algebra which induces the grading $(U_{\top}(\mathfrak{g}))_n = (U_{\top}(\mathfrak{g}))^n / (U_{\top}(\mathfrak{g}))^{n-1}$.

Similarly to Proposition 4.14, the following Proposition compares the universal enveloping algebras of two locality Lie algebras.

Proposition 4.21. *Let (\mathfrak{g}', \top') be a (pre-)locality Lie subalgebra of (\mathfrak{g}, \top) a (pre-)locality Lie algebra. Then $(U_{\top'}(\mathfrak{g}'), \top'_U)$ is a (pre-)locality subalgebra of $(U_{\top}(\mathfrak{g}), \top_U)$.*

Proof. By Proposition 4.14 $\mathcal{T}_{\top'}(\mathfrak{g}') \subset \mathcal{T}_{\top}(\mathfrak{g})$, and by construction $J_{\top'}(\mathfrak{g}') = J_{\top}(\mathfrak{g}) \cap \mathcal{T}_{\top'}(\mathfrak{g}')$. It follows that $U_{\top'}(\mathfrak{g}')$ is a subspace of $U_{\top}(\mathfrak{g})$. Proposition 4.14 also states that $\top'_{\otimes} \subset \top_{\otimes}$ and we can show the inclusion $\top'_U \subset \top_U$ in a similar manner. Therefore $U_{\top'}(\mathfrak{g}')$ is (pre-)locality subspace of $U_{\top}(\mathfrak{g})$. It is straightforward to see that it is moreover a (pre-)locality subalgebra as expected. \square

We are now ready to prove the universal property of the locality universal enveloping algebra $U_{\top}(\mathfrak{g})$.

Theorem 4.22. Let $(\mathfrak{g}, \top_{\mathfrak{g}}, [,])$ be a pre-locality Lie algebra, (A, \top_A) a pre-locality algebra with product $m_A : (A \times_{\top} A, \top_{A \times_{\top} A}) \rightarrow (A, \top_A)$, and $f : \mathfrak{g} \rightarrow A$ a pre-locality Lie algebra morphism for the Lie bracket on A defined as the commutator of the product. There is a unique pre-locality algebra morphism $\phi : U_{\top}(\mathfrak{g}) \rightarrow A$ such that $f = \phi \circ \iota_{\mathfrak{g}}$ where

$$\iota_{\mathfrak{g}} : \mathfrak{g} \longrightarrow U_{\top}(\mathfrak{g}) \tag{33}$$

is the canonical (locality) map from \mathfrak{g} to $U_{\top}(\mathfrak{g})$. In other words, the following diagram commutes

$$\begin{array}{ccc} (\mathfrak{g}, \top_{\mathfrak{g}}) & \xrightarrow{\iota_{\mathfrak{g}}} & (U_{\top}(\mathfrak{g}), \top_U) \\ & \searrow f & \downarrow \phi \\ & & (A, \top_A). \end{array}$$

Proof. • Existence: Since f is linear map, we may apply the universal property of the locality tensor algebra (Theorem 4.15) to obtain a pre-locality algebra morphism $\psi : \mathcal{T}_{\top}(\mathfrak{g}) \rightarrow A$ such that $f = \psi \circ \otimes_{\top}$ where \otimes_{\top} is the canonical map from \mathfrak{g} to $\mathcal{T}_{\top}(\mathfrak{g})$. We then define $\phi : U_{\top}(\mathfrak{g}) \rightarrow A$ by $\phi([x]) := \psi(x)$. This is clearly well defined since f is a pre-locality Lie algebra morphism for indeed we have:

$$\psi(x + a \otimes b - b \otimes a - [a, b]) = \psi(x) + \psi(a \otimes b - b \otimes a - [a, b]) = \psi(x) + f(a)f(b) - f(b)f(a) - f([a, b]) = \psi(x)$$

It also satisfies the equation $f = \phi \circ \iota_{\mathfrak{g}}$ since $\iota_{\mathfrak{g}}$ is the composition of \otimes_{\top} with the map which takes every element of $\mathcal{T}_{\top}(\mathfrak{g})$ to its equivalence class on $U_{\top}(\mathfrak{g})$. For the same reason, and because ψ is a locality map (Theorem 4.15), ϕ is a locality map. The fact that ϕ is a pre-algebra morphism is direct: with a small abuse of notation we have

$$\phi([a] \otimes [b]) = \phi([a \otimes b]) = \psi(a \otimes b) = f(ab) = f(a)f(b) = \phi([a])\phi([b]).$$

- Uniqueness: Let $\phi : U_{\top}(\mathfrak{g}) \rightarrow A$ be pre-locality algebra morphism such that $f = \phi \circ \iota_{\mathfrak{g}}$. By the universal property of the locality tensor algebra, the pre-locality Lie algebra morphism $f : \mathfrak{g} \rightarrow A$ uniquely extends to a pre-locality algebra morphism $\psi : \mathcal{T}_{\top}(\mathfrak{g}) \rightarrow A$. By the uniqueness of this extension, ψ has to coincide with $x \mapsto \phi([x])$, which proves that ϕ is determined by ψ and hence the uniqueness of ϕ . □

Proposition 4.23. *The universal property of the locality universal enveloping algebra $U_{\top}(\mathfrak{g})$ implies the universal property of the usual universal enveloping algebra $U(\mathfrak{g})$.*

Proof. Given a Lie algebra \mathfrak{g} , an algebra A and a Lie algebra morphism $f : \mathfrak{g} \rightarrow A$, it is enough to consider the trivial locality relation $\top = \mathfrak{g} \times \mathfrak{g}$ and this yields the existence and uniqueness of the algebra morphism $\phi : U(\mathfrak{g}) \rightarrow A$ required for the universal property of the universal enveloping algebra. □

Remark 4.24. For the equivalence of the universal properties of the universal enveloping algebras in the usual and locality set up, one would need to extend the Lie bracket of a pre-locality Lie algebra to the whole Lie algebra. However this is not always possible as can be seen from Counterexample 4.25.

Proposition 1.17 yields a bilinear map which is antisymmetric on $\mathfrak{g} \times_{\top} \mathfrak{g}$ by construction since \top is symmetric. Outside of the span of the image of the original (non-extended map), the extended map vanishes identically so that the extended map is antisymmetric. However, as the next counter-example shows, it does not in general satisfy Jacobi identity.

Counter-example 4.25. Three-dimensional Lie algebras are known, and there are finitely many (see Mubarakzyanov's Classification [Mub]). We build an infinite family of distinct locality Lie algebras, which as a consequence, does not correspond to an ordinary Lie algebra equipped with a locality algebra.

Take $\mathfrak{g} = \mathbb{R}^3$, (e_1, e_2, e_3) its canonical basis and the locality relation $\top_{\mathfrak{g}}$ defined by the subset of $\mathbb{R}^3 \times \mathbb{R}^3$ obtained by symmetrising the following set

$$\langle e_1 \rangle \times \langle e_1 \rangle \cup \langle e_2 \rangle \times \langle e_2 \rangle \cup \langle e_3 \rangle \times \langle e_3 \rangle \cup \langle e_1, e_2 \rangle \times \langle e_3 \rangle.$$

Let $[\cdot, \cdot] : \top_{\mathfrak{g}} \rightarrow \mathfrak{g}$ be the linear antisymmetric map defined by

$$[e_1, e_3] = \lambda e_1 + \mu e_3, \quad [e_2, e_3] = \mu' e_3 \tag{34}$$

with $(\lambda, \mu, \mu') \in (\mathbb{R}^*)^3$. Notice that since $(e_1, e_2) \notin \top_{\mathfrak{g}}$, $[e_1, e_2]$ does not need to be defined. We argue is that it *cannot* be defined such that $(\mathfrak{g}, [\cdot, \cdot])$ is a usual (non-locality) Lie algebra. Indeed, let us write $[e_1, e_2] = xe_1 + ye_2 + ze_3$. Then computing $[[e_1, e_2], e_3]$ and its permutations, we find that the Jacobi identity is satisfied if, and only if:

$$-\mu' \lambda e_1 - \lambda y e_2 + (x\mu + y\mu' - \lambda z) e_3 = 0.$$

This equation has no solutions since $\mu' \lambda \neq 0$.

However, it is easy to see that for any $(\lambda, \mu, \mu') \in (\mathbb{R}^*)^3$, $(\mathfrak{g}, \top_{\mathfrak{g}}, [\cdot, \cdot])$ is a locality Lie algebra. This follows from the fact that since $(e_1, e_2) \notin \top_{\mathfrak{g}}$, (e_i, e_j, e_k) cannot lie in $\mathfrak{g}^{\times 3}_{\top}$ if i, j and k are all different, $[\cdot, \cdot]$ trivially satisfies the locality Jacobi identity.

Remark 4.26. The previous counter-example indicates that the category of locality Lie algebras is much richer than the one of usual (non-locality) Lie algebras, even when they are finite-dimensional. This hints to the difficulty of classifying such structures. As pointed out in the introduction, this issue of classification is left for future studies.

Part II

Enhancement to the locality setup

We want to enhance to a locality setup the constructions of Part I carried out in a pre-locality framework. Since they involve tensor products, the question arises of the stability of locality vector spaces under locality tensor products. Like ordinary tensor products, locality tensor products are defined as quotient spaces, which brings us to the study of quotients of locality vector spaces.

5 Quotient of locality vector spaces

This section is dedicated to the following natural question:

When is the quotient of a locality vector space by a linear subspace, a locality vector space, if equipped with the quotient locality relation of Definition 2.8 ? (35)

Given a locality vector space (V, \top) and a subspace W , the question can be reformulated as follows: does the following implication

$$\forall (v_1, v_2, v_3) \in V^3, \quad ([v_1] \overline{\top} [v_2] \wedge [v_1] \overline{\top} [v_3]) \implies ([v_1] \overline{\top} [v_2 + v_3]),$$

hold, where $\overline{\top}$ stands for the quotient locality of Definition 2.8 ? This amounts to whether the subsequent implication holds:

$$\forall (v_1, v_2, v_3) \in V^3, \quad (\exists v'_i \in [v_i], \exists v''_1 \in [v_1], \quad v'_1 \top v'_2 \wedge v''_1 \top v'_3) \implies (\exists v'_{23} \in [v_2 + v_3], \exists \tilde{v}_1 \in [v_1], \quad \tilde{v}_1 \top v'_{23}). \quad (36)$$

Investigating this question in the context of locality tensor products leads to two main assumptions formulated in the conjectural statements 6.16 and 6.19.

We first reformulate Question (35) in order get a better grasp on it and to motivate the above statements.

5.1 A group theoretic interpretation

We can rephrase Question (35), namely whether or not a quotient of locality vector spaces is a locality vector space as whether or not a union of groups is a group. This question in full generality, is to our knowledge, still an open question. A good description of the state of the art on the question of when a group is the union of n proper subgroups is given in [Bha]. An answer was provided in [Coh] for $n \leq 6$ and in [Tom] for $n = 7$. To our knowledge, the question for any n or for infinite unions of groups remains open.

Notations 5.1.1. In the subsequent sections, given a locality set (S, \top) we will use the polar map

$$\begin{aligned} P^\top : \mathcal{P}(S) &\longrightarrow \mathcal{P}(S) \\ U &\longmapsto U^\top. \end{aligned}$$

By convention, we take $P^\top(\emptyset) := S$. Furthermore, in a small abuse of notation we write $P^\top(u)$ instead of $P^\top(\{u\})$ if $U = \{u\}$ has only one element.

Let us recall an elementary result.

Lemma 5.1. Given a vector space V equipped with a locality relation \top , the pair (V, \top) is a locality vector space if, and only if, $P^\top(u)$ is a vector subspace of V for any $u \in V$.

Proof. By definition, (V, \top) is a locality vector space if, and only if, $P^\top(U)$ is a vector subspace of V for any subset $U \subseteq V$. If this holds, then it is obvious that $P^\top(u)$ is a vector subspace of V for any $u \in V$.

Conversely, assume $P^\top(u)$ is a linear subspace of V for any $u \in V$. Then for any subset $U \subseteq V$, we have

$$P^\top(U) := \bigcap_{u \in U} P^\top(u)$$

which is an intersection of vector spaces and thus a vector space. \square

In order to make a precise statement, let us first recall some basic facts. Given a locality vector space (V, \top) , $W \subseteq V$ a linear subspace of V and $\pi : V \rightarrow V/W$ the linear quotient map, we write $[u] := \pi^{-1}(\pi(u)) \subseteq V$. We have $[u] = \{\bar{u} + w, w \in W\} = \bar{u} + W$ where \bar{u} is any representative in V of the class $\pi(u)$. Note that $0_V \in [u]$ implies $[u] = W$. In V/W , we write indifferently $[u]$ or $\pi(u)$ as it is more convenient.

We need one other intermediate result:

Lemma 5.2. (V, \top) a locality vector space, $W \subseteq V$ a linear subspace of V and $\pi : V \rightarrow V/W$ the quotient map. Then

$$P^\top([u]) = \bigcup_{u' \in [u]} \pi(P^\top(u'))$$

for any u in V .

Proof. Let (V, \top) , W and π be as in the statement and let u be any element of V .

- \subseteq : Let $[\alpha]$ be in V/W such that $[\alpha] \in P^\top([u])$ i.e., such that $[\alpha] \top [u]$. Then by definition of \top there is some $u' \in [u]$ and some $\alpha' \in [\alpha]$ such that $\alpha' \top u' \Leftrightarrow \alpha' \in P^\top(u')$. This implies

$$[\alpha] = \pi(\alpha) = \pi(\alpha') \in \pi(P^\top(u')) \subseteq \bigcup_{u' \in [u]} \pi(P^\top(u')).$$

Thus $P^\top([u]) \subseteq \bigcup_{u' \in [u]} \pi(P^\top(u'))$.

- \supseteq : Let $[\alpha]$ be in V/W such that $[\alpha] \in \bigcup_{u' \in [u]} \pi(P^\top(u'))$. Then

$$\exists u' \in [u] : [\alpha] \in \pi(P^\top(u')) \implies \exists (u', w) \in [u] \times W : (\alpha + w) \in P^\top(u').$$

In V , we have $\alpha' := (\alpha + w) \in [\alpha]$ and this in turn implies

$$\exists (u', \alpha') \in [u] \times [\alpha] : \alpha' \top u' \implies [\alpha] \top [u].$$

Thus $[\alpha] \in P^\top([u])$ and $\bigcup_{u' \in [u]} \pi(P^\top(u')) \subseteq P^\top([u])$ which proves the statement. \square

We now state the main result of this section.

Theorem 5.3. Let (V, \top) be a locality vector space, $W \subseteq V$ a linear subspace of V and $\pi : V \rightarrow V/W$ the quotient map. Then the following statements are equivalent:

1. $(V/W, \bar{\top})$ is a locality vector space,
2. The set

$$H_u := \bigcup_{u' \in [u]} \pi(P^\top(u'))$$

is a commutative group (for the internal operation $+$ induced on the quotient space by the internal operation $+$ on V) for any u in V .

3. The set H_u is a commutative semigroup (for the same product) for any u in V .

Remark 5.4. Notice that $P^\top(u')$ is a subset of V , thus $\pi(P^\top(u'))$ is a subset of V/W , thus the \bigcup notation in the Theorem.

Proof. • 2. \Leftrightarrow 3. If H_u is a group, it is in particular a semigroup. Let us show that the converse is also true. Assume that H_u is a semigroup (i.e. that it is closed under summation) for any u in V and observe that by Lemma 5.2 $H_u = P^{\overline{\top}}([u])$. Since (V, \top) is a locality vector space we have $[0] \ni 0 \top u \in [u]$ thus $[0] = 0_{V/W} \in P^{\overline{\top}}([u]) = H_u$ be definition of $\overline{\top}$. Hence H_u is a monoid for any u in V . We are left to show that if H_u is stable under addition, it is also stable under taking the inverse, that is multiplication by the scalar -1 .

For any $[\alpha] \in V/W$ we have:

$$[\alpha] \in P^{\overline{\top}}([u]) \implies \exists(\alpha', u') \in [\alpha] \times [u] : \alpha' \top u' \implies \exists(\alpha', u') \in [\alpha] \times [u] : \lambda \alpha' \top u' \forall \lambda \in \mathbb{K}$$

since $P^{\overline{\top}}(u)$ is a vector subspace of V . Then since $\alpha' \in [\alpha] \implies \lambda \alpha' \in \lambda[\alpha]$ we deduce that

$$[\alpha] \in P^{\overline{\top}}([u]) \implies \lambda[\alpha] \in P^{\overline{\top}}([u]) \forall \lambda \in \mathbb{K}$$

and in particular $H_u = P^{\overline{\top}}([u])$ is a group.

- 1. \Leftrightarrow 2. Let V, W and π be as in the statement of the Theorem. By Lemma 5.1, we know that $(V/W, \overline{\top})$ is a locality vector space if, and only if, $P^{\overline{\top}}([u]) = H_u$ is a vector subspace of V/W for any $[u] \in V/W$.

We have already shown that H_u is stable by scalar multiplication, thus it is a vector space if, and only if, it is a group for the addition. \square

5.2 Locality on quotients and locality exact sequences

In this paragraph, we rephrase Question (35) in terms of locality exact sequences. Let us first recall a known result of linear algebra.

Lemma 5.5. Let V_1, V_2, V be three linear spaces. The following properties are equivalent:

1. There is a short exact sequence $0 \rightarrow V_1 \xrightarrow{\iota_1} V \xrightarrow{\pi_2} V_2 \rightarrow 0$ i.e., ι_1 is an injective morphism, π_2 is a surjective morphism, and $\text{Im}(\iota_1) = \text{Ker}(\pi_2)$.
2. There is an injective morphism $\iota_1 : V_1 \rightarrow V$ such that $V/\iota_1(V_1) \simeq V_2$.

In the following, we denote by ϕ the resulting isomorphism:

$$\begin{aligned} \phi : V/\iota_1(V_1) &\longrightarrow V_2 \\ [v] &\longmapsto \phi([v]) := \pi_2(v), \end{aligned}$$

whose inverse map is given by $\phi^{-1}(v_2) = [w_2]$ for any $w_2 \in \pi_2^{-1}(v_2)$. It is well defined since $\phi^{-1}(v_2)$ is independent of the choice of w_2 .

Example 5.6. If $V_1 \subset V$ is a linear subspace of V , we can take ι_1 to be the identity map $\iota_1 : v_1 \mapsto v_1$ so that item 2. reads $V/V_1 \simeq V_2$.

To give a locality counterpart of this result, we define locality short exact sequences.

Definition 5.7. A **locality short exact sequence** (resp. a **pre-locality short exact sequence**) is a sequence $0 \rightarrow (V_1, \top_1) \xrightarrow{f_1} (V_2, \top_2) \xrightarrow{f_2} (V_3, \top_3) \rightarrow 0$ such that

1. (V_i, \top_i) are locality vector spaces (resp. pre-locality vector spaces) for $i \in \{1, 2, 3\}$,
2. f_1 and f_2 are locality maps,
3. $0 \rightarrow V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \rightarrow 0$ is a short exact sequence.

Note that items 2. and 3. imply that f_1 and f_2 are locality linear maps.

Lemma 5.8. Let $(V_1, \top_1), (V_2, \top_2), (V, \top)$ be three locality (resp. pre-locality) linear spaces. The following properties are equivalent:

1. There is a short locality (resp. pre-locality) exact sequence $0 \rightarrow (V_1, \top_1) \xrightarrow{\iota_1} (V, \top) \xrightarrow{\pi_2} (V_2, \top_2) \rightarrow 0$, where \top_2 is the final locality relation for the map π_2 (See Definition 2.5).
2. There is an injective locality morphism $\iota_1 : V_1 \rightarrow V$ such that the canonical isomorphism $(V/\iota_1(V_1)) \xrightarrow{\phi} V_2$ is a locality (resp. pre-locality) isomorphism for the independence relations $\overline{\top}$ and \top_2 .

Proof. • $1 \Rightarrow 2$: since the short exact sequence in item 1. implies the isomorphism $V/\iota_1(V_1) \xrightarrow{\phi} V_2$ as vector spaces, all we need to check is the locality of the maps ϕ and ϕ^{-1} . If $[v] \overline{\top} [v']$, there is some $w \in [v]$ and some $w' \in [v']$ such that $w \top w'$. The locality of π_2 then implies that $\pi_2(w) \top_2 \pi_2(w')$ so that $\phi([v]) \top_2 \phi([v'])$ which show the locality of ϕ .

The inverse map ϕ^{-1} is defined by $\phi^{-1}(v) = [w]$ for any $w \in \pi_2^{-1}(v)$. Since \top_2 is the final locality relation for π_2 , $v_1 \top_2 v_2$ implies that there are $w_1 \in \pi_2^{-1}(v_1)$ and $w_2 \in \pi_2^{-1}(v_2)$ such that $w_1 \top w_2$. By definition of the locality $\overline{\top}$ on the quotient space, $[w_1] \overline{\top} [w_2]$ or equivalently $\phi^{-1}(v_1) \overline{\top} \phi^{-1}(v_2)$.

- $2 \Rightarrow 1$: since the isomorphism $V/\iota_1(V_1) \simeq V_2$ holds as vector spaces, we have the short exact sequence of vector spaces in item 1. We want to show that the maps ι_1 and π_2 are locality maps, and that \top_2 is the final locality relation for π_2 . By assumption, ι_1 is a locality map. We show the locality of π_2 . Recall that the surjective map $\pi_2 : V \rightarrow V_2$ is given by $\pi_2(v) := \phi([v])$. Since $v \top v'$ it follows that $[v] \overline{\top} [v']$ so that from the locality of ϕ we infer that $\pi_2(v) = \phi([v])$ is independent of $\pi_2(v') = \phi([v'])$. We prove that \top_2 is the final locality for π_2 . Let \top'_2 be another locality relation on V_2 such that $\pi_2 : (V, \top) \rightarrow (V_2, \top'_2)$ is a locality map, and let $(v, v') \in V_2^2$ such that $v \top_2 v'$. Since ϕ^{-1} is a locality map $\phi^{-1}(v) \overline{\top} \phi^{-1}(v')$. By definition of $\overline{\top}$, there are elements $w \in \phi^{-1}(v)$ and $w' \in \phi^{-1}(v')$ such that $w \top w'$. Notice that in particular $\pi_2(w) = v$ and $\pi_2(w') = v'$. Since \top'_2 makes π_2 a locality map, then $v \top'_2 v'$ which implies $\top_2 \subset \top'_2$ as expected. \square

This has a simple consequence:

Proposition 5.9. Let (V, \top) and (V_1, \top_1) be two locality vector spaces and let $\iota_1 : V_1 \rightarrow V$ an injective locality linear map. Then $(V/\iota_1(V_1), \overline{\top})$ is a locality vector space if, and only if, there is a locality vector space (V_2, \top_2) and a map $\pi_2 : V \rightarrow V_2$ such that \top_2 is the final locality relation for π_2 , and $0 \rightarrow (V_1, \top_1) \xrightarrow{\iota_1} (V, \top) \xrightarrow{\pi_2} (V_2, \top_2) \rightarrow 0$ is a locality short exact sequence.

Proof. • \Leftarrow : if such a (V_2, \top_2) exists then by Lemma 5.8, $(V/\iota_1(V_1), \overline{\top}) \simeq (V_2, \top_2)$ as locality vector space. It follows that $(V/\iota_1(V_1), \overline{\top})$ is a locality vector space.

- \Rightarrow : if $(V/\iota_1(V_1), \overline{\top})$ is a locality vector space, take $(V_2, \top_2) := (V/\iota_1(V_1), \overline{\top})$ and $\pi_2(v) := [v]$. Then by construction $0 \rightarrow (V_1, \top_1) \xrightarrow{\iota_1} (V, \top) \xrightarrow{\pi_2} (V_2, \top_2) \rightarrow 0$ is a locality short exact sequence, and by definition of the quotient locality $\overline{\top}$, \top_2 is the final locality relation for π_2 . \square

We have now rephrased Question 35 in terms of locality short exact sequences as follows:

Given a locality vector space (V, \top) and V_1 a subspace of V , when is the pre-locality short exact sequence $0 \rightarrow (V_1, \top|_{V_1 \times V_1}) \xrightarrow{\iota_1} (V, \top) \xrightarrow{\pi_2} (V/V_1, \overline{\top}) \rightarrow 0$ a locality short exact sequence?

5.3 Locality quotient vector spaces: examples and counterexamples

The following statement provides a class of examples of locality quotient vector spaces for the locality relation $\overline{\top}$ (see Definition 2.8) induced by the orthogonality relation.

Proposition 5.10. Take V be any Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. We equip V with the locality relation \top given by the orthogonality relation: $v \top v' \iff \langle v, v' \rangle = 0$. Then for any closed linear subspace W of V not reduced to $\{0\}$, the quotient locality $\overline{\top}$ on V/W , induced by \top is the complete locality relation: $\overline{\top} = (V/W) \times (V/W)$.

In particular, for any closed linear subspace W of V , $(V/W, \overline{\top})$ is a locality vector space with $\overline{\top}$ induced from \top , which is given by the orthogonality on V .

Remark 5.11. When V is a finite dimensional euclidean space, then $(V/W, \overline{\top})$ is a locality vector space for any linear subspace $W \neq \{0\}$ since any linear subspace of a finite dimensional euclidean space is closed.

Proof. We want to show that if $W \neq \{0\}$ then for any $(v_1, v_2) \in V^2$, there exist $(w_1, w_2) \in W^2$ such that $\langle v_1 + w_1, v_2 + w_2 \rangle = 0$. Write $v_i = u_i + \tilde{w}_i$ for $i \in \{1, 2\}$ with $\tilde{w}_i \in W$ and $u_i \in W^\perp$. Then:

$$\langle v_1 + w_1, v_2 + w_2 \rangle = 0 \iff \langle v_1, v_2 \rangle + \langle \tilde{w}_1, w_2 \rangle + \langle w_1, \tilde{w}_2 \rangle + \langle w_1, w_2 \rangle = 0. \quad (37)$$

and we want to find w_1, w_2 that solve (37). Let us consider three different cases:

- If $\tilde{w}_1 \neq 0$, then

$$(w_1, w_2) = \left(0, -\frac{\langle v_1, v_2 \rangle}{\|\tilde{w}_1\|^2} \tilde{w}_1 \right) \in W^2$$

(with $\|w\| := \sqrt{\langle w, w \rangle}$) solves (37).

- If $\tilde{w}_1 = 0$ and $\tilde{w}_2 \neq 0$, then we find a solution to (37) as in the first item by exchanging the roles of w_1 and w_2 .
- If $\tilde{w}_1 = \tilde{w}_2 = 0$, we pick $w \in W \neq \{0\}$ and set

$$(w_1, w_2) = \left(\frac{w}{\|w\|}, -\frac{\langle v_1, v_2 \rangle}{\|w\|} w \right) \in W^2$$

which solve (37). □

The following counterexample gives a locality vector space with a linear subspace whose quotient is not a locality vector space for the locality relation $\overline{\top}$ (see Definition 2.8). It shows that the answer to question 35 cannot be always positive.

Counter-example 5.12. We equip the vector space V of real valued maps on \mathbb{R} with the locality relation \top given by disjoint supports: $f \top g \iff \text{supp}(f) \cap \text{supp}(g) = \emptyset$. Let W denote the linear subspace of constant functions. Take three functions $(u, v, w) \in V^3$ whose supports are respectively $\text{supp}(u) =]1, +\infty[$, $\text{supp}(v) =]0, +\infty[$ and $\text{supp}(w) =]2, +\infty[$ and $v(x) = u(x+1) = w(x+2) = 1$ for any $x > 0$. Then in V/W we have $[u] \overline{\top} [v]$ since $u \top (v-1)$ and $[u] \overline{\top} [w]$ since $(u-1) \top w$. However, $[u] \not\overline{\top} [v+w]$. Thus V/W is not a locality vector space for $\overline{\top}$.

The question (35) is particularly relevant if we want the tensor product of a pair of locality vector spaces to be again a locality (not only pre-locality) vector space. The following is an example of a quotient of locality spaces which is again a locality vector space.

The subsequent folklore result provides a justification for the identification $V^{\otimes 1} = V$ made in Section 4.2. We give a constructive proof for it since it will be of use in Proposition 5.15.

Lemma 5.13. Consider the subspace $I_{\text{lin}}(V)$ of $\mathbb{K}(V)$ generated by all elements of the form

$$(a + b) - (a) - (b), \quad (38)$$

and

$$k(a) - (ka) \quad (39)$$

for a and b in V and k in \mathbb{K} .

Then $V \simeq \mathbb{K}(V)/I_{\text{lin}}(V)$ are isomorphic as vector spaces.

Remark 5.14. We use the notation (a) with rounded brackets to denote elements of $\mathbb{K}V$ and to distinguish from elements of V . These brackets should not be confused with the squared one $[a]$ used to denote equivalence classes.

Notice that by construction of $I_{\text{lin}}(V)$, the only element a in V such that $(a) \in I_{\text{lin}}(V)$ is $a = 0$.

Proof. We give two proofs, the first of which does not use Zorn's lemma.

Let us consider the map

$$\theta : \begin{cases} V & \longrightarrow \mathbb{K}(V)/I_{\text{lin}}(V) \\ a & \longmapsto [a]. \end{cases}$$

By definition of $I_{\text{lin}}(V)$, for any $a, b \in V$, for any $k \in \mathbb{K}$,

$$[a + b] = [a] + [b], \quad [ka] = k[a],$$

which implies that θ is linear. Let us consider the linear map

$$\bar{\theta}' : \begin{cases} \mathbb{K}(V) & \longrightarrow V \\ (a) & \longmapsto a. \end{cases}$$

For any $a, b \in V$, for any $k \in \mathbb{K}$,

$$\begin{aligned} \bar{\theta}'((a + b) - (a) - (b)) &= a + b - a - b = 0, \\ \bar{\theta}'((ka) - k(a)) &= ka - ka = 0, \end{aligned}$$

so $I_{\text{lin}}(V) \subseteq \ker(\bar{\theta}')$. Consequently, we obtain a linear map

$$\theta' : \begin{cases} \mathbb{K}(V)/I_{\text{lin}}(V) & \longrightarrow V \\ [a] & \longmapsto a. \end{cases}$$

For any $a \in V$, $\theta' \circ \theta(a) = a$, so $\theta' \circ \theta = \text{id}_V$. For any $a \in V$, $\theta \circ \theta'([a]) = [a]$. As $\theta \circ \theta'$ is linear and the elements $[a]$ generate $\mathbb{K}(V)/I_{\text{lin}}(V)$, $\theta \circ \theta' = \text{id}_{\mathbb{K}(V)/I_{\text{lin}}(V)}$.

Here is an alternative proof which uses Zorn's lemma.

We define the equivalence relation \sim on $\mathbb{K}(V)$ as $x \sim y$ if, and only if, there is an element w in $I_{\text{lin}}(V)$ such that $x + w = y$. We show that the linear map $V \mapsto \mathbb{K}(V)/I_{\text{lin}}(V)$, which sends the basis $\{v_i\}_{i \in I}$ of V to the family $\{[v_i]\}_{i \in I}$ of vectors in $\mathbb{K}(V)/I_{\text{lin}}(V)$ is bijective. For this purpose we need to show that $\{[v_i]\}_{i \in I}$ is a basis of $\mathbb{K}(V)/I_{\text{lin}}(V)$. Since $\{v_i\}_{i \in I}$ generates V , for any $a \in V$, there are α_i 's in \mathbb{K} such that $a = \sum_{i \in I} \alpha_i v_i$, then $(a) \sim \sum_{i \in I} \alpha_i (v_i)$ (using equations (38) and (39)) which implies $[a] = \sum_{i \in I} \alpha_i [v_i]$. Since the elements (a) , with $a \in V$ generate $\mathbb{K}(V)$, it therefore follows that $\{[v_i]\}_{i \in I}$ generates $\mathbb{K}(V)/I_{\text{lin}}(V)$. We show now that $\{[v_i]\}_{i \in I}$ is free. Let $\sum_{i \in I} \alpha_i [v_i] = 0$, this is equivalent to $\sum_{i \in I} \alpha_i (v_i) \sim (\sum_{i \in I} \alpha_i v_i) \in I_{\text{lin}}(V)$, thus $\sum_{i \in I} \alpha_i v_i = 0$. Since $\{v_i\}_{i \in I}$ is a basis of V , this implies that for all $i \in I$, $\alpha_i = 0$ so that $\{[v_i]\}_{i \in I}$ is a basis of $V \simeq \mathbb{K}(V)/I_{\text{lin}}(V)$ as announced. \square

If (V, \top) is a locality vector space, $\mathbb{K}(V)$ can be endowed with the locality relation $\top_{\times 1, 1}$ (see 23), which leads to an isomorphism of locality vector spaces, which generalises the isomorphism of Lemma 5.13.

Proposition 5.15. *Let (V, \top) be a locality vector space, then (V, \top) and $(\mathbb{K}(V)/I_{\text{lin}}(V), \bar{\top})$ are isomorphic as locality vector spaces:*

$$(V, \top) \simeq (\mathbb{K}(V)/I_{\text{lin}}(V), \bar{\top}), \quad (40)$$

where $\bar{\top}$ is the quotient locality (See Definition 2.8).

Proof. We need to show that the isomorphism of Lemma 5.13 is a locality map as well as its inverse. Let $(x, y) \in \top$, then $((x), (y)) \in \top_{\times 1, 1}$ and thus $([x], [y]) \in \bar{\top}$. Conversely if $([x], [y]) \in \bar{\top}$, this implies that there exist $\sum_{i \in I} \alpha_i (x_i) \in [x]$ and $\sum_{j \in J} \beta_j (y_j) \in [y]$ such that $(\sum_{i \in I} \alpha_i (x_i), \sum_{j \in J} \beta_j (y_j)) \in \top_{\times 1, 1}$ or equivalently $(x_i, y_j) \in \top$ for every $(i, j) \in I \times J$. Since (V, \top) is a locality vector space the latter implies that $(\sum_{i \in I} \alpha_i x_i, \sum_{j \in J} \beta_j y_j) \in \top$. And finally, by means of Lemma 5.13, $[\sum_{i \in I} \alpha_i x_i] = [x]$ and $[\sum_{j \in J} \beta_j y_j] = [y]$ implies $\sum_{i \in I} \alpha_i x_i = x$ and $\sum_{j \in J} \beta_j y_j = y$, thus $(x, y) \in \top$ as announced. \square

We note that the proof of the above proposition uses the locality property of the vector space V . The following counterexample shows that the isomorphism is not necessarily a locality morphism when V is only a pre-locality vector space.

Counter-example 5.16. Let $V = \mathbb{R}^2$ and $\top = \{(e_1, e_2), (e_1, 3e_2), (e_2, e_1), (3e_2, e_1)\}$. Then (V, \top) and $(\mathbb{K}(V)/I_{\text{lin}}(V), \bar{\top})$ are not isomorphic as locality vector spaces, where $\bar{\top}$ is the quotient locality. Indeed $((3e_2) - (e_2), (e_1)) \in \top_{\times 1,1}$ and thus $([3e_2 - e_2], [e_1]) \in \bar{\top}$, but $(2e_2, e_1) \notin \top$.

6 Sufficient conditions for the locality of quotient vector spaces

In this section we discuss and compare sufficient conditions for the quotient of two vector spaces to be a locality vector space.

6.1 Split locality exact sequences and locality quotients

We start by giving a first sufficient (but not necessary) condition to answer Question 35.

Definition-Proposition 6.1. We call a **strong locality complement** of a subspace W of a locality vector space (V, \top) , a complement subspace \tilde{W} of W in (V, \top) i.e. such that $W \oplus \tilde{W} = V$, which satisfies one the following equivalent properties:

1. $\pi \top \pi$ and $\pi \top \tilde{\pi}$;
2. $\pi \top \text{Id}_V$;
3. $\tilde{\pi} \top \tilde{\pi}$ and $\pi \top \tilde{\pi}$;
4. $\tilde{\pi} \top \text{Id}_V$;

where π (resp. $\tilde{\pi}$) is the projection onto W (resp. onto \tilde{W}) parallel to \tilde{W} (resp. parallel to W).

We call W and \tilde{W} **strong locality complement subspaces** of the locality space (V, \top) .

Remark 6.2. Note that any subspace W of a strongly non-degenerate locality vector space (V, \top) discussed in [CGPZ3, Equation (30)] has an algebraic complement given by its polar set W^\top (or ortho-complement) so that $V = W \oplus W^\top$. However, unlike in the above definition, in [CGPZ3] the projections $\pi : V \rightarrow W$ and $\tilde{\pi} : V \rightarrow W^\top$ were not required to be locality maps.

Proof. We use the fact that

$$(\varphi \top \varphi_1 \text{ and } \varphi \top \varphi_2) \implies (\varphi \top (\varphi_1 \pm \varphi_2)) \quad (41)$$

and

$$\varphi \top \text{Id}_V \implies \varphi \top \varphi. \quad (42)$$

Indeed, by assumption $x \top y \implies \varphi(x) \top \varphi_i(y)$ for $i = 1, 2$ and the locality of the vector space (V, \top) implies that $\varphi(x) \top \varphi_1(y) \pm \varphi_2(y)$, which shows (41). Furthermore, if $\varphi \top \text{Id}_V$ then $x \top y \implies \varphi(x) \top y$ which, using the symmetry of \top implies $\varphi(y) \top \text{Id}_V(\varphi(y))$ and hence $\varphi(x) \top \varphi(y)$, which proves (42).

- 1) \implies 2) follows from (41) applied to $\varphi = \pi$ and $\varphi_1 = \tilde{\pi}, \varphi_2 = \pi$, which implies $\pi \top \text{Id}_V$ since $\text{Id}_V = \pi + \tilde{\pi}$.
- 2) \implies 1) : Follows from (42) applied to $\varphi = \pi$ followed by (41) applied to $\varphi = \pi$ and $\varphi_1 = \text{Id}_V, \varphi_2 = \pi$ using the fact that $\tilde{\pi} = \text{Id}_V - \pi$.
- 3) \Leftrightarrow 4) : Analogous to the last two points exchanging the roles of $\tilde{\pi}$ and π .
- 2) \implies 4) : Since $\text{Id}_V \top \text{Id}_V$, it follows from (41) applied to $\varphi = \text{Id}_V, \varphi_1 = \text{Id}_V, \text{ and } \varphi_2 = \pi$.
- 4) \implies 2) : Analogous to the last point exchanging the roles of π and $\tilde{\pi}$.

□

Remark 6.3. Notice that any two of the four points of Definition-Proposition 6.1 imply

$$\pi \top \pi \wedge \tilde{\pi} \top \tilde{\pi}$$

which we will freely use later on. However, $\pi \top \pi \wedge \tilde{\pi} \top \tilde{\pi}$ is not the fifth point of Definition-Proposition 6.1 since it does not imply any of the four points of the aforementioned Definition-Proposition.

The previous definition can be expressed in terms of short split exact sequences. Indeed, complement subspaces W and \tilde{W} in V , i.e. such that $W \oplus \tilde{W} = V$, give rise to a short exact sequence $0 \rightarrow W \xrightarrow{\iota} V \xrightarrow{\tilde{\pi}} \tilde{W} \rightarrow 0$, where ι is the inclusion map and $\tilde{\pi}$ is the projection onto \tilde{W} along W . The exact sequence is split since the projection π onto W along \tilde{W} yields a right inverse for ι . The equivalent conditions in Definition-Proposition 6.1 lead to the following generalisation.

Corollary 6.4. Two complement linear subspaces W and \tilde{W} of a locality space (V, \top) , i.e. such that $W \oplus \tilde{W} = V$, are strong locality complements if and only if the projection map $\tilde{\pi}$ and the right inverse π of ι in the split short exact sequence

$$0 \rightarrow W \xrightarrow[\pi]{\iota} V \xrightarrow{\tilde{\pi}} \tilde{W} \rightarrow 0 \quad (43)$$

satisfy one of the equivalent conditions of Definition 6.1, where Id_V corresponds to the sum $\pi + \tilde{\pi}$. We call such sequence a **locality split exact sequence**. Furthermore, in this case $(V/W, \bar{\top})$ is a locality vector space isomorphic to $(\tilde{W}, \top \cap (\tilde{W} \times \tilde{W}))$.

Remark 6.5. For W a subspace of (V, \top) , Proposition 5.9 shows the equivalence between

1. whether a quotient $(V/W, \bar{\top})$ of locality vector spaces is a locality vector space, and
2. the existence of a locality vector space $(\tilde{W}, \tilde{\top})$ which makes the short exact sequence (43) a locality short exact sequence, where $\tilde{\top}$ is the final locality relation for $\tilde{\pi}$.

Corollary 6.4 is a specialisation of Proposition 5.9 in so far as it requests the existence of a strong locality complement \tilde{W} of W in (V, \top) , and yields a locality split exact sequence.

Proof. By definition strong locality complements are equivalent to the existence of a split short exact sequence

$$0 \rightarrow W \xrightarrow[\pi]{\iota} V \xrightarrow{\tilde{\pi}} \tilde{W} \rightarrow 0$$

such that π and $\tilde{\pi}$ satisfy one of the four equivalent conditions of Proposition-Definition 6.1.

In this case we have $(V/W, \bar{\top}) \simeq (\tilde{W}, \top \cap (\tilde{W} \times \tilde{W}))$ by Proposition 5.9 □

Remark 6.6. Recall from [CGPZ4, Definition 6.2], that a locality relation \top on V is non degenerate if $v \top v \implies v = 0$ and called strongly non-degenerate if moreover for any subspace $U \subsetneq V$, the polar space U^\top is nonzero. A strongly non degenerate locality relation \top on V yields the splitting $V = W \oplus W^\top$ for any subspace W of V , so W has a complement space $\tilde{W} = W^\top$. The corresponding projections $\pi : V \rightarrow W$, resp. $\tilde{\pi} : V \rightarrow W^\top$ are locally independent of each other by construction. By [CGPZ4, Proposition 6.5], a strongly separating locality vector space of countable dimension admits a locality basis for \top , in which case the projections can be shown to be locality maps.

6.2 A weaker condition ensuring the locality of quotient spaces

This paragraph provides a weaker sufficient condition to answer (35) positively. This new condition is also more computational friendly than the a priori more natural condition (for a subspace to have a strong locality complement) given in the previous paragraph, which is why we use it in the sequel.

Definition 6.7. Let (V, \top) be a locality vector space and $W \subset V$ a linear subspace. We say that W is **locality compatible with \top** if $\forall(x, y, z) \in V^3, \forall w \in W,$

$$(x, y) \in \top \wedge (x + w, z) \in \top \implies (\exists w' \in W) : (x + w', y) \in \top \wedge (x + w', z) \in \top. \quad (44)$$

We shall see (Theorem 6.12) that if W is locality compatible with \top then the condition (36) is fulfilled so that the quotient V/W is a locality vector space.

Here are first rather trivial examples.

Example 6.8. Let (V, \top) be a locality vector space. Then V and $\{0\}$ are locality compatible with \top . Indeed, to see that V is locality compatible with \top it is enough to consider $w' = -x$ using the notations of Definition 6.7. To see that $\{0\}$ is locality compatible with \top is enough to notice that $w = w' = 0$ and thus $x \top y$ and $x \top z$.

Proposition 6.9. *If a subspace W of (V, \top) admits a strong locality complement, then it is locality compatible with \top .*

Proof. With the notations of (44), since $\tilde{\pi}$ is locality independent of $\text{Id}_V,$

$$(x, y) \in \top \wedge (x + w, z) \in \top \implies (\tilde{\pi}(x), y) \in \top \wedge (\tilde{\pi}(x), z) \in \top.$$

Setting $-w' := x - \tilde{\pi}(x)$ which lies in W , we have $(x + w' = \tilde{\pi}(x), y) \in \top$ and $(x + w' = \tilde{\pi}(x), z) \in \top$ as expected. □

The following counterexample shows that for a locality vector space (V, \top) and a subspace W , the condition of W having a strong locality complement is stronger than the condition of W being locality compatible with \top .

Counter-example 6.10. Consider the vector space \mathbb{R}^7 , its subspace $W = \langle \{e_1, e_2, e_3\} \rangle$ where $\{e_i\}_{i=1}^7$ are the elements of the canonical basis, and

$$\begin{aligned} \top = & \mathbb{R}^7 \times \{0\} \cup \langle e_1 + e_7 \rangle \times \langle \{e_4, e_5\} \rangle \cup \langle e_2 + e_7 \rangle \times \langle \{e_5, e_6\} \rangle \cup \langle e_3 + e_7 \rangle \times \langle \{e_4, e_6\} \rangle \\ & \cup \langle \{e_1 + e_7, e_3 + e_7\} \rangle \times \langle e_4 \rangle \cup \langle \{e_1 + e_7, e_2 + e_7\} \rangle \times \langle e_5 \rangle \cup \langle \{e_2 + e_7, e_3 + e_7\} \rangle \times \langle e_6 \rangle \cup \text{Sym. terms.} \end{aligned}$$

Notice that \top is invariant under the natural action of the subgroup $\Omega := \langle \sigma_1, \sigma_2 \rangle$ of the group of permutations S_7 generated by $\sigma_1 := (1, 2)(4, 6)$ and $\sigma_2 := (1, 3)(5, 6)$, where (i, j) stands for the transposition of i and j . Ω has six elements:

$$\Omega = \{\sigma_1 := (1, 2)(4, 6), \sigma_2 := (1, 3)(5, 6), \sigma_3 = (2, 3)(4, 5), \sigma_4 = (1, 2, 3)(4, 5, 6), \sigma_5 = (3, 2, 1)(6, 5, 4), \text{Id}_7\}.$$

This follows from the relations $\sigma_3 = \sigma_1 \circ \sigma_2 \circ \sigma_1, \sigma_4 = \sigma_1 \circ \sigma_2, \sigma_5 = \sigma_2 \circ \sigma_1, \sigma_5^2 = \sigma_4$ and $\sigma_4^2 = \sigma_5$, combined with the involutivity of the transpositions.

One can see that W is locality compatible with \top by checking all cases. For instance, if $k \in \mathbb{R}, k(e_1 + e_7) \top e_4$ and $k(e_1 + e_7) + k(e_2 - e_1) = k(e_2 + e_7) \top e_6$, there is $k(e_3 + e_7) = k(e_1 + e_7) + k(e_3 - e_1)$ such that $k(e_3 + e_7) \top e_4$ and $k(e_3 + e_7) \top e_6$. In terms of equation (44), $x = k(e_1 + e_7), w = k(e_2 - e_1), y = e_4, z = e_6,$ and $w' = k(e_3 - e_1)$. Another possible case is when $x = k(e_1 + e_7) + q(e_3 + e_7)$ for $k, q \in \mathbb{K}$, then $x \top e_4$. If $w = q(-e_3 + e_2)$, then $x + w = k(e_1 + e_7) + q(e_2 + e_7) \top e_5$. In this case $w' = q(-e_3 + e_1)$ makes $x + w' = (k + q)(e_1 + e_7)$ locality independent to both e_4 and e_5 . All other possible cases are analogous, in the sense that they are obtained from the previous two via the action of a permutation in Ω on the subindices of the e_i 's.

We show that W has no strong locality complement by proving there is no projection $\pi : \mathbb{R}^7 \rightarrow W$ such that $\pi \top \text{Id}_{\mathbb{R}^7}$. Indeed, if there were such projection, then $\pi(e_1 + e_7) \top e_4$, but $\pi(e_1 + e_7) = e_1 + \pi(e_7)$ where $\pi(e_7) \in W$. From the construction of \top , the only option is $\pi(e_7) = -e_1$. On the other hand, $(e_3 + e_7) \top e_4$ but $\pi(e_3 + e_7) = e_3 - e_1$ is not locality independent to e_4 which yields the contradiction.

The following Proposition yields an example of a locality compatible subspace for a vector space freely generated by another locality vector space.

Proposition 6.11. *Let (V, \top) be a locality vector space, then $I_{\text{lin}}(V) \subseteq \mathbb{K}(V)$ (see Lemma 5.13) is locality compatible with $\top_{\times 1,1}$.*

Proof. Let $(x, y, z) \in \mathbb{K}(V)^3$, with $x = \sum_{i \in I} \alpha_i(x_i)$, $y = \sum_{j \in J} \beta_j(y_j)$ and $z = \sum_{k \in K} \gamma_k(z_k)$ (where, as before, we use brackets to distinguish elements of V from elements of $\mathbb{K}V$), and $w \in I_{\text{lin}}(V)$ such that $(x, y) \in \top_{\times 1,1}$ and $(x+w, z) \in \top_{\times 1,1}$. We search for an element w' in $I_{\text{lin}}(V)$ such that $(x'+w', y) \in \top_{\times 1,1}$ and $(x+w, z) \in \top_{\times 1,1}$.

Since $(x, y) \in \top_{\times 1,1}$ and $(x+w, z) \in \top_{\times 1,1}$, we have $([x], [y]) \in \overline{\top}$ and $([x], [z]) \in \overline{\top}$. Take x_V to be the preimage in V of $[x]$ under the isomorphism of Proposition 5.15. Since $[x] = [x+w]$ and since the isomorphism of Proposition 5.15 is a locality isomorphism, we have that x_V is independent of the preimages of y and z . Thus, writing $x_V = \sum_{i \in I} \alpha_i x_i$, it follows that $x_V \top y_j$ and $x_V \top z_k$ for every $j \in J$ and every $k \in K$. In particular $(x_V, \sum_{j \in J} \beta_j(y_j)) \in \top$ and $(x_V, \sum_{k \in K} \gamma_k(z_k)) \in \top$. Hence, $((x_V), y) \in \top_{\times 1,1}$ and $((x_V), z) \in \top_{\times 1,1}$. We claim that the difference $(x_V) - x$ is a candidate for w' . Indeed, since x_V was the preimage of x by the isomorphism of Proposition 5.15, we have $w' = (x_V) - x \in I_{\text{lin}}(V)$ as required. \square

As a result of the above constructions we get that the quotient by a locality compatible subspace yields a locality quotient, a fact on which most of the forthcoming constructions will rely.

Theorem 6.12. *Let (V, \top) be a locality vector space and $W \subset V$ a subspace locality compatible with \top , then $(V/W, \overline{\top})$ is a locality vector space, where $\overline{\top}$ is the quotient locality (see Definition 2.8).*

Proof. Given any subset U of V/W , we want to prove that $U^{\overline{\top}}$ is a linear subspace. For any $[x] \in U^{\overline{\top}}$, for every $[u] \in U$, there is an element u' of $[u]$ and an element x' of $[x]$, such that $(u', x') \in \top$. The fact that V is a linear vector space implies that for $\lambda \in \mathbb{K}$, $(u', \lambda x') \in \top$, therefore $\lambda[x] \in U^{\overline{\top}}$.

On the other hand, for $[x]$ and $[y]$ equivalence classes in $U^{\overline{\top}}$ and for every $[u] \in U$, there are $u' \in [u]$ and $w \in W$ such that $(x', u') \in \top$ and $(y', u' + w) \in \top$ for some $x' \in [x]$ and some $y' \in [y]$. Since W is compatible with \top , there is a $w' \in W$ such that $(x', u' + w') \in \top$ and $(y', u' + w') \in \top$. Since (V, \top) is a locality vector space, it follows that $(x' + y', u' + w') \in \top$ and therefore $[x] + [y] \in U^{\overline{\top}}$. Hence V/W is a locality vector space. \square

The following counterexample shows that locality compatibility is not necessary to have a local quotient space (compare with Example 5.10 and Remark 5.11)).

Counter-example 6.13. Take V be any Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. We equip V with the locality relation \top given by the orthogonality relation: $v \top v' :\iff \langle v, v' \rangle = 0$. Then a closed linear subspace $\{0\} \subsetneq W \subsetneq V$ is not locality compatible with \top .

Proof. Since $W \neq V$ and $W \neq \{0\}$, we can choose vectors $w^\perp \in W^\perp \setminus \{0\}$, $w \in W \setminus \{0\}$ and we set

$$x := \frac{w^\perp}{\|w^\perp\|}, \quad y := \frac{w}{\|w\|} =: w_0 \in W, \quad z := \frac{w}{\|w\|} - \frac{w^\perp}{\|w^\perp\|}.$$

We have $\langle x, y \rangle = \langle x + w_0, z \rangle = 0$. However there is no $w_1 \in W$ such that $\langle x + w_1, y \rangle = \langle x + w_1, z \rangle = 0$. Indeed $\langle x + w_1, y \rangle = 0$ implies $\langle w_1, y \rangle = 0$. But since $z = y - x$, we have $\langle x + w_1, z \rangle = 0$ which implies $\langle w^\perp, w^\perp \rangle = 0$ leading to a contradiction. \square

In the case of a vector space freely generated by a locality set, the locality compatibility property actually implies a stronger version with $N \geq 2$ elements instead of two:

Proposition 6.14. *Let (S, \top) be a locality set, $(\mathbb{K}(S), \top)$ the locality vector space generated by extending linearly the relation \top , and $W \subset \mathbb{K}(S)$ a subspace locality compatible with \top . Let $x \in \mathbb{K}(S)$, $y_i \in \mathbb{K}(S)$ and $w_i \in W$ for $1 \leq i \leq N$, such that for every i $(x + w_i, y_i) \in \top$. Then there is an element w' of W such that $(x + w', y_i) \in \top$ for every $1 \leq i \leq N$.*

Proof. We prove the statement by induction on N . The case $N = 2$ is immediate since W is locality compatible with \top . Assume it is true for $N - 1$, and let $x \in \mathbb{K}(S)$, $y_i \in \mathbb{K}(S)$ and $w_i \in W$ for $1 \leq i \leq N$, such that for every i $(x + w_i, y_i) \in \top$. By induction there is a $w'_0 \in W$ such that $(x + w'_0, y_i) \in \top$ for every $1 \leq i \leq N - 1$. We can write every y_i in terms of the basis elements of S as $y_i = \sum_{j \in S} \alpha_j^i s_j$ where only finitely many $\alpha_j^i \neq 0$, and define $\bar{y} = \sum_{i=1}^N M_i y_i$ where $M_i = 1 + \max\{\{M_j : j < i\} \cup \{\alpha_j^i\}\}$. Notice that $(x + w'_0, \bar{y}) \in \top$ and moreover, if there is a $z \in \mathbb{K}(S)$ such that $(z, \bar{y}) \in \top$ then $(z, y_i) \in \top$ for every $i < N$.

Since W is locality compatible with \top , there is an element w' in W such that $(x + w', \bar{y}) \in \top$ and $(x + w', y_N) \in \top$ which implies the expected result. \square

6.3 Two conjectural statements

We now formulate two conjectural statements which will play an important role in the sequel. The first one is the conjectural statement 6.16 which gives a sufficient condition for the tensor product of n subspaces of a locality vector space to be again a locality vector space. The second one is the conjectural statement 6.19 which enhances the tensor algebra to a locality algebra by giving sufficient conditions for the filtered components from Definition 4.6 to become locality vector spaces. In the following V and W are subspaces of a pre-locality vector space (E, \top) . We equip $V \times_{\top} W$ with a locality relation $\top_{V \times_{\top} W}$ as in (9).

Remark 6.15. If $V = W$, we have $\top_{V \times_{\top} W} = V^{\times \top 4}$ with the notations of (21).

As described in (17), $\top_{V \times_{\top} W}$ linearly extends to a locality relation on $\mathbb{K}(V \times_{\top} W)$, which we denote by the same symbol $\top_{V \times_{\top} W}$. This induces a locality relation \top_{\otimes} on $V \otimes_{\top} W$, (see Definition 2.8). Recall that the locality tensor product was defined in Definition 1.9 as

$$V \otimes_{\top} W = \mathbb{K}(V \times_{\top} W) / I_{\text{bil}} \cap \mathbb{K}(V \times_{\top} W).$$

We defined the locality tensor product (see Definition 1.9) by means of the quotient of a locality vector space by a subspace of multilinear forms, and provided a sufficient condition called "locality compatibility" on the subspace for the quotient to be a locality space. Whether or not the various subspaces of multilinear forms involved are locality compatible, are challenging questions which we formulate as conjectural statements.

Conjectural statement 6.16. [Pair of locality vector spaces] *Given a locality vector space (E, \top) and V_1, V_2 two of its subspaces, the subspace $I_{\text{bil}}(V_1, V_2) \cap \mathbb{K}(V_1 \times_{\top} V_2) \subset \mathbb{K}(V_1 \times_{\top} V_2)$ is locality compatible with $\top_{V_1 \times_{\top} V_2}$.*

Proposition 6.17. *Statement 6.16 is equivalent to the following statement:*

Let $n \geq 2$ and V_1, \dots, V_n be linear subspaces of a locality vector space (E, \top) . The space $I_{\text{mult},n}(V_1, \dots, V_n) \cap \mathbb{K}(V_1 \times_{\top} \dots \times_{\top} V_n) \subset \mathbb{K}(V_1 \times_{\top} \dots \times_{\top} V_n)$ is locality compatible with the locality relation $\top_{\times n, n}$ (see 23) on the space $\mathbb{K}(V_1 \times_{\top} \dots \times_{\top} V_n)$.

Proof. We can recover Statement 6.16 from the statement of Proposition 6.17 in the case $n = 2$. Conversely, we now prove that Statement 6.16 implies that of Proposition 6.17.

For $n \geq 3$: let x, y, z and w be elements of $\mathbb{K}(V_1 \times_{\top} \dots \times_{\top} V_n)$ where $w \in I_{\text{mult},n}(V_1 \times_{\top} \dots \times_{\top} V_n)$ such that $(x, y) \in \top_{\times n, n}$ and $(x+w, z) \in \top_{\times n, n}$. Extending linearly the usual bijection between $(V_1 \times_{\top} \dots \times_{\top} V_{n-1}) \times_{\top} V_n$ and $V_1 \times_{\top} \dots \times_{\top} V_n$, one obtains an isomorphism of vector spaces $\mathbb{K}((V_1 \times_{\top} \dots \times_{\top} V_{n-1}) \times_{\top} V_n) \simeq \mathbb{K}(V_1 \times_{\top} \dots \times_{\top} V_n)$. It is straightforward to show that the restriction of such an isomorphism to $\mathbb{K}((V_1 \times_{\top} \dots \times_{\top} V_{n-1}) \times_{\top} V_n)$ (resp. $I_{\text{bil}}(V_1 \times_{\top} \dots \times_{\top} V_{n-1}, V_n)$) yields an isomorphism with $\mathbb{K}(V_1 \times_{\top} \dots \times_{\top} V_n)$ (resp. $I_{\text{mult},n}(V_1 \times_{\top} \dots \times_{\top} V_n)$). We may therefore see x, y, z and w as elements on $\mathbb{K}((V_1 \times_{\top} \dots \times_{\top} V_{n-1}) \times_{\top} V_n)$ as well as view w as an element of $I_{\text{bil}}(V_1 \times_{\top} \dots \times_{\top} V_{n-1}, V_n)$. Assuming Statement (6.16) holds, then yields the existence of $w' \in I_{\text{bil}}(V_1 \times_{\top} \dots \times_{\top} V_{n-1}, V_n) \cap \mathbb{K}((V_1 \times_{\top} \dots \times_{\top} V_{n-1}) \times_{\top} V_n)$ such that $(x + w', y) \in \top_{\times n, n}$ and $(x + w', z) \in \top_{\times n, n}$, which proves the statement. \square

The following Corollary is important to ensure the stability of locality vector spaces stable under tensor products and is the main reason for introducing the conjectural statement 6.16.

Corollary 6.18. Let $n \geq 1$ and V_1, \dots, V_n be linear subspaces of a locality vector space (V, \top) . Assuming the statement 6.16 holds true, the locality tensor product $(V_1 \otimes_{\top} \dots \otimes_{\top} V_n, \top_{\otimes})$ is a locality vector space.

Proof. This statement follows from Proposition 6.17 and Theorem 6.12. \square

Even though the last statement is essential for the rest of the paper, it is not enough to make the locality tensor algebra a locality vector space, since it fails to relate the different graded components. Therefore we formulate the following conjectural statement whose consequences will be used in Section 7.

Conjectural statement 6.19. [Locality tensor algebra] *Given a locality vector space (V, \top_V) and any $n \in \mathbb{N}$, the subspace $(I_{\text{mult}}^n(V) \cap \mathbb{K}(\bigcup_{k=0}^n V^{\times_k \top})) \subset \mathbb{K}(\bigcup_{k=0}^n V^{\times_k \top})$ is locality compatible with \top_{\times}^n . (See Definition-Proposition 4.8).*

Each of the statements 6.16 and 6.19 implies the following useful property:

Proposition 6.20. [Locality vector space] (the case $V = W$) *Given a locality vector space (V, \top_V) and assuming the statement 6.16 holds true, the subspace $I_{\text{bil}}(V) \subset \mathbb{K}(V \times_{\top} V)$ is locality compatible with $\top_{V \times_{\top} V}$.*

Even though they might seem rather natural, these conjectural statements turn out to be rather challenging. We devote the following paragraphs to getting a better grasp of these assumptions and their consequences.

7 First consequences: enhanced universal properties

We first show how assuming that statements 6.16 and 6.19 hold true, enables us to enhance the results of Part I to a full-fledged locality setup. We transpose the results of Sections 2.3, 4.2 and 4.3 (in particular Theorem 2.13, Proposition 4.13, Theorem 4.15 and Theorem 4.22) to the locality framework. We do not dwell on the proofs that are similar to the usual setup, and instead we focus on results whose proofs require a treatment that differs from the pre-locality context.

7.1 The locality tensor algebra

The following proposition is in the same spirit as Theorem 6.12.

Proposition 7.1. *Let (A, \top_A, m) be a non-unital locality algebra and $I \subset A$ a locality ideal of A which is locality compatible with \top_A . Then $(A/I, \overline{\top}, \overline{m})$ is a non-unital locality algebra where $\overline{\top}$ is the quotient locality (See Definition 2.8).*

Proof. By means of Theorem 6.12 $(A/I, \overline{\top})$ has the structure of a locality vector space. Analogous to the usual (non locality) set up, the induced product $\overline{m} : A/I \times_{\overline{\top}} A/I \rightarrow A/I$ is an associative \top_{\times} -bilinear map. We are left to prove that for any $U \subset A/I$, $\overline{m}(U^{\overline{\top}} \times_{\overline{\top}} U^{\overline{\top}}) \subset U^{\overline{\top}}$. Given $([x], [y]) \in \overline{\top}$ such that both $[x]$ and $[y]$ are in $U^{\overline{\top}}$, consider also a $[u] \in U$. Then there are $x' \in [x]$, $y' \in [y]$, $u' \in [u]$, and $(w, w_1, w_2) \in I^3$ such that

$$(x', y') \in \top_A, \tag{45}$$

$$(x' + w_1, u') \in \top_A, \tag{46}$$

$$(y' + w_2, u' + w) \in \top_A. \tag{47}$$

Since I is locality compatible with \top_A , from (45) and (46) we conclude that there is $w'_1 \in I$ such that

$$(x' + w'_1, y') \in \top_A, \text{ and} \tag{48}$$

$$(x' + w'_1, u') \in \top_A. \tag{49}$$

From (47) and (48), there is a $w'_2 \in I$ such that

$$(y' + w'_2, x' + w'_1) \in \top_A, \quad \text{and} \quad (50)$$

$$(y' + w'_2, u' + w) \in \top_A. \quad (51)$$

And finally from (49) and (51), there is a $w' \in I$ such that

$$(x' + w'_1, u' + w') \in \top_A, \quad \text{and} \quad (52)$$

$$(y' + w'_2, u' + w') \in \top_A. \quad (53)$$

By (50) $m(x' + w'_1, y' + w'_2)$ is well defined and the fact that A is a locality algebra together with (52) and (53) imply that $(m(x' + w'_1, y' + w'_2), u' + w') \in \top_A$. Hence $\bar{m}([x], [y]) \in U^{\top}$. \square

Definition 7.2. • A **graded locality algebra** is a locality algebra together with a sequence of vector spaces $\{A_n\}_{n \in \mathbb{N}}$ called the grading, such that

$$A = \bigoplus_{n \in \mathbb{N}} A_n, \quad m(A_p \otimes_{\top} A_q) \subset A_{p+q}, \quad u(\mathbb{K}) \subset A_0.$$

- A **filtered locality algebra** is a locality algebra together with a sequence of nested vector spaces $A^0 \subset A^1 \subset \dots \subset A^n \subset \dots$ called the filtration, such that

$$A = \bigcup_{n \in \mathbb{N}} A^n, \quad m(A^p \otimes_{\top} A^q) \subset A^{p+q}, \quad u(\mathbb{K}) \subset A^0.$$

A **graded (resp. filtered) locality Lie algebra** is a locality Lie algebra which is also a graded (resp. filtered) locality Lie algebra.

This is the locality counterpart of Proposition 4.13.

Theorem 7.3. Assuming the conjectural statement 6.19 holds true, then the locality tensor algebra over a locality vector space is a graded locality algebra.

Proof. Let (V, \top_V) be a locality vector space, it is straightforward that the locality vector space $(\mathbb{K}(V^{\times \top \infty}), \top_V)$, where $V^{\times \top \infty} := \bigcup_{k \geq 1} V^{\times \top k}$, together with the concatenation product m_c between locally independent elements is a locality algebra. The subspace I_{mult} is moreover a locality ideal. Therefore Proposition 7.1 implies $(\mathcal{T}_{\top}(V), \top_{\otimes}, \otimes)$ is a locality algebra. Since for any p and q , the concatenation product m_c preserves the grading of $\mathbb{K}(V^{\times \top \infty})$, namely $m_c(V^{\times \top p} \times_{\top} V^{\times \top q}) \subset V^{\times \top p+q}$, it follows that $\otimes(V^{\otimes \top p} \otimes_{\top} V^{\otimes \top q}) \subset V^{\otimes \top p+q}$. Then, the convention that $V^0 = \mathbb{K}$ yields the result. \square

7.2 The universal property of the locality tensor and universal enveloping algebras

Assuming the statements 6.16 and 6.19 hold true, we can enhance the previous universal properties we had on Part I and introduce a new conjectural statement to enhance the universal property of the universal enveloping algebra.

The following theorem generalises Theorem 2.13.

Theorem 7.4 (Universal property of the locality tensor product). Given V and W linear subspaces of a locality vector space (E, \top) over \mathbb{K} , (G, \top_G) a locality vector space and $f : (V \times_{\top} W, \top_{\times}) \rightarrow (G, \top_G)$ a locality \top_{\times} -bilinear map. Assuming that conjectural statement 6.16 holds true for the locality vector spaces V and W , there is a unique locality linear map $\phi : V \otimes_{\top} W \rightarrow G$ such that the following diagram commutes.

$$\begin{array}{ccc} (V \times_{\top} W, \top_{\times}) & \xrightarrow{\otimes_{\top}} & (V \otimes_{\top} W, \top_{\otimes}) \\ & \searrow f & \downarrow \phi \\ & & (G, \top_G) \end{array}$$

Proof. Theorem (1.15) yields the existence and uniqueness of the linear map ϕ . Assuming the statement 6.16 holds true, implies that $V \otimes_{\top} W$ is a locality vector space. We are only left to show that ϕ is a locality map. Recall that two equivalence classes $[a]$ and $[b]$ in $V \otimes_{\top} W$ verify $[a]_{\top_{\otimes}}[b]$ if there are $\sum_{i=1}^n \alpha_i(x_i, y_i) \in [a]$ and $\sum_{j=1}^m \beta_j(u_j, v_j) \in [b]$ such that every possible pair taken from the set $\{x_i, y_i, u_j, v_j\}$ lies in $V \times_{\top} W$ for every $1 \leq i \leq n$ and every $1 \leq j \leq m$. Since f is locality \top_{\times} bilinear, then $f(\sum_{i=1}^n \alpha_i(x_i, y_i))_{\top_A} f(\sum_{j=1}^m \beta_j(u_j, v_j))$ which amounts to $\phi([a])_{\top_A} \phi([b])$. Therefore ϕ is as expected. \square

Assuming the conjectural statement 6.19 holds true, as a consequence of the previous Theorem, we can state and prove an enhanced universal property Theorem 4.15 for the locality tensor algebra.

Theorem 7.5 (Universal property of locality tensor algebra). Let (V, \top) be a locality vector space, (A, \top_A) a locality algebra and $f : V \rightarrow A$ a locality linear map. Assuming the conjectural statement 6.19 holds for tensors powers of V , there is a unique locality algebra morphism $\phi : \mathcal{T}_{\top}(V) \rightarrow A$ such that the following diagram commutes.

$$\begin{array}{ccc} (V, \top) & \xrightarrow{\otimes_{\top}} & (\mathcal{T}_{\top}(V), \top_{\otimes}) \\ & \searrow f & \downarrow \phi \\ & & (A, \top_A) \end{array}$$

where $\otimes : V \rightarrow \mathcal{T}_{\top}(V)$ is the canonical (locality) injection map.

Proof. The locality tensor algebra is a pre-locality algebra by Proposition 4.13, and a locality algebra since we have assumed that the conjectural statement 6.19 holds true. Thus it is a locality algebra.

By means of Theorem 4.15 and Remark 4.4 the pre-locality algebra morphism ϕ exists and is unique. Given that $\mathcal{T}_{\top}(V)$ and A are locality algebras, then ϕ is also a locality algebra morphism as expected. \square

We study the universal property of the locality universal enveloping algebra $U_{\top}(\mathfrak{g})$ of a locality Lie algebra \mathfrak{g} . In the remaining part of the paper, assuming that the conjectural statement 6.19 holds true, we make the following further assumption.

Conjectural statement 7.6. for the universal enveloping algebra: given a locality Lie algebra $(\mathfrak{g}, \top_{\mathfrak{g}}, [,])$, the ideal $J_{\top}(\mathfrak{g})$ of $\mathcal{T}_{\top}(\mathfrak{g})$ introduced in Definition 4.19 is locality compatible with \top_{\otimes} .

Proposition 7.7. Assuming the conjectural statement 7.6 holds true for a locality Lie algebra $(\mathfrak{g}, \top_{\mathfrak{g}}, [,])$, then $U_{\top}(\mathfrak{g})$ defines a locality algebra.

Proof. This is a direct consequence of Proposition 7.1. \square

This theorem is the locality counterpart of Theorem 4.22.

Theorem 7.8. Let $(\mathfrak{g}, \top_{\mathfrak{g}}, [,])$ be a locality Lie algebra, (A, \top_A) a locality algebra and $f : \mathfrak{g} \rightarrow A$ a locality Lie algebra morphism where the Lie bracket on A is the commutator defined by the product. Assuming that the conjectural statements 6.19 and 7.6 hold true for \mathfrak{g} , there is a unique locality algebra morphism $\phi : U_{\top}(\mathfrak{g}) \rightarrow A$ such that the following diagram commutes, and where $\iota_{\mathfrak{g}} : \mathfrak{g} \rightarrow U_{\top}(\mathfrak{g})$ was defined in (33).

$$\begin{array}{ccc} (\mathfrak{g}, \top_{\mathfrak{g}}) & \xrightarrow{\iota_{\mathfrak{g}}} & (U_{\top}(\mathfrak{g}), \top_{\mathfrak{g}}) \\ & \searrow f & \downarrow \phi \\ & & (A, \top_A) \end{array}$$

Proof. The proof follows from Theorem 4.22 and the fact that $U_{\top}(\mathfrak{g})$ and A are locality algebras. \square

Part III

The Milnor-Moore theorem in the locality setup

8 Prerequisites on coalgebraic locality structures

We review here preliminary results we shall use for the actual proof of the locality version of the Milnor-Moore theorem.

8.1 Graded connected locality Hopf algebras

Let us recall some definitions from [CGPZ1]. Just as a locality algebra was defined as a locality vector space equipped with a partial product and a unit compatible with the locality relation (see Definition 4.1), a locality coalgebra is a locality vector space equipped with partial coproduct and a counit compatible with the locality relation.

Definition 8.1. [CGPZ1, Definition 4.3] A locality \mathbb{K} -coalgebra is a quadruple $(C, \Delta, \epsilon, \top)$ where (C, \top) is a locality \mathbb{K} -vector space, $\Delta : (C, \top) \rightarrow (C \otimes_{\top} C, \top_{\otimes})$ and $\epsilon : C \rightarrow \mathbb{K}$ are linear maps such that

- the coproduct Δ is coassociative, namely $(Id_C \otimes \Delta) \circ \Delta = (\Delta \otimes Id_C) \circ \Delta$ on C ;
- and compatible with the locality structure i.e., for any $U \subset C$, $\Delta(U^{\top}) \subset U^{\top} \otimes_{\top} U^{\top}$;
- the counit $\epsilon : C \rightarrow \mathbb{K}$ satisfies $(Id_C \otimes \epsilon)\Delta = (\epsilon \otimes Id_C)\Delta = Id_C$.

Remark 8.2. Note that the second condition tells us that Δ is a locality linear map in the sense of (2) in Definition 1.1.

Remark 8.3. The above definition only makes sense in the locality framework, where the polar set U^{\top} of a set is required to be a vector space. In the pre-locality setup, the fact that this condition is relaxed prevents us from building locality tensor products such as $U^{\top} \otimes_{\top} U^{\top}$. This suggests that the Milnor-Moore theorem we are about to prove does not hold in the more general pre-locality setup.

We now refine this definition.

Definition 8.4. • A **graded locality \mathbb{K} -coalgebra** is a locality \mathbb{K} -coalgebra together with a sequence of vector spaces $\{C_n\}_{n \in \mathbb{N}}$ called a **grading**, such that

$$C = \bigoplus_{n \in \mathbb{N}} C_n, \quad \Delta(C_n) \subset \bigoplus_{p+q=n} C_p \otimes_{\top} C_q, \quad \bigoplus_{n \geq 1} C_n \subset \ker(\epsilon).$$

We denote by $|x|$ the degree of x .

Moreover we call a graded locality \mathbb{K} -coalgebra **connected** if C_0 has dimension 1 and therefore $\bigoplus_{n \geq 1} C_n = \ker(\epsilon)$.

- A **filtered locality \mathbb{K} -coalgebra** is a locality \mathbb{K} -coalgebra together with a nested sequence of vector spaces $C^0 \subset C^1 \subset \dots \subset C^n \dots$, called a **filtration**, such that

$$C = \bigcup_{n \in \mathbb{N}} C^n, \quad \Delta(C^n) \subset \sum_{p+q=n} C^p \otimes_{\top} C^q, \quad \bigoplus_{n \geq 1} C^n \subset \ker(\epsilon).$$

The following definition provides the coalgebraic counterpart of a locality ideal see (29) and a locality morphism, see (30).

Definition 8.5. 1. A locality linear subspace J of a locality \mathbb{K} -coalgebra (C, \top, Δ) is called a left, (resp. right) **locality coideal** of C , if

$$\Delta(J) \subset J \otimes_{\top} C; \text{ (resp. } \Delta(J) \subset C \otimes_{\top} J) \quad \text{and} \quad \epsilon(J) = (0). \quad (54)$$

We call it a **locality coideal** if

$$\Delta(J) \subset J \otimes_{\top} C + C \otimes_{\top} J \quad \text{and} \quad \epsilon(J) = (0). \quad (55)$$

Note that the condition $\epsilon(J) = (0)$ does not involve the locality relation.

2. Given two locality coalgebras $(C_i, \top_i, \Delta_i, \epsilon_i), i = 1, 2$, a locality linear map $f : C_1 \rightarrow C_2$ is called a **locality \mathbb{K} -coalgebra morphism** if

$$(f \otimes f) \circ \Delta_1 = \Delta_2 \circ f, \quad \text{and} \quad \epsilon_1 = \epsilon_2 \circ f. \quad (56)$$

In other words, it is a \mathbb{K} -coalgebra morphism which is a locality map.

3. Let (C, \top, Δ) be a locality \mathbb{K} -coalgebra. A locality \mathbb{K} -coalgebra (C_1, \top_1, Δ_1) with $C_1 \subset C$ is a locality sub-coalgebra of (C, \top, Δ) if the inclusion map $\iota : (C_1, \top_1, \Delta_1) \hookrightarrow (C, \top, \Delta)$ is a \mathbb{K} -coalgebra morphism.

Remark 8.6. This definition of sub-locality coalgebra is more general than the one given in [CGPZ1]. This level of generalisation will be needed later.

We first prove an elementary result of linear algebra.

Lemma 8.7. For $i \in \{1, 2\}$, let $f_i : V_i \rightarrow W_i$ be linear maps from a vector space V_i to a vector space W_i . We have

$$\ker(f_1 \otimes f_2) = \ker f_1 \otimes V_2 + V_1 \otimes \ker f_2.$$

Proof. Let $K_i = \ker(f_i) \subset V_i$ and let $X_i \subset V_i$ be any direct complement space in V_i so that $V_i = K_i \oplus X_i$. It follows from the distributivity property (12) that

$$V_1 \otimes V_2 = (K_1 \otimes K_2) \oplus (V_1 \otimes K_2) \oplus (K_1 \otimes X_2) \oplus (X_1 \otimes X_2).$$

As a consequence of the first isomorphism theorem for vector spaces there are isomorphisms $\phi_i : X_i \rightarrow \text{Im}(f_i)$ from which we build a linear map $(\phi_1^{-1} \otimes \phi_2^{-1}) \circ (f_1 \otimes f_2)$ from $V_1 \otimes V_2$ onto $X_1 \otimes X_2$ which does not vanish on $X_1 \otimes X_2$ outside the null tensor.

Since $(K_1 \otimes K_2) \oplus (V_1 \otimes K_2) \oplus (K_1 \otimes X_2) \subset \ker((\phi_1^{-1} \otimes \phi_2^{-1}) \circ (f_1 \otimes f_2))$ and since by construction $(\phi_1^{-1} \otimes \phi_2^{-1}) \circ (f_1 \otimes f_2)$ does not vanish on $X_1 \otimes X_2 \setminus \{0\}$ (where 0 is the null tensor), we have $\ker((\phi_1^{-1} \otimes \phi_2^{-1}) \circ (f_1 \otimes f_2)) = (K_1 \otimes K_2) \oplus (V_1 \otimes K_2) \oplus (K_1 \otimes X_2) = \ker f_1 \otimes V_2 + V_1 \otimes \ker f_2$. The result follows from the fact that $\phi_1 \otimes \phi_2$ is an isomorphism of vector spaces. \square

The following is the locality version of Lemma 8.7. As in Proposition 1.13, we require a compatibility between the locality relation and direct sums.

Proposition 8.8. Let (E, \top) and (F, \top_F) be two locality vector spaces. For $i \in \{1, 2\}$, let $f_i : V_i \rightarrow W_i$ be locality linear maps from locality subspaces $V_i \subset E$ to vector subspaces W_i of F . We moreover assume that f_1 and f_2 are mutually locally independent and the existence of surjective projections $\pi_i : V_i \rightarrow \ker(f_i)$ such that π_i and Id_{V_j} are locally independent for $i \neq j$ (See Proposition 1.13). Then

$$\ker(f_1 \otimes f_2) \cap (V_1 \otimes_{\top} V_2) = \ker f_1 \otimes_{\top} V_2 + V_1 \otimes_{\top} \ker f_2. \quad (57)$$

Proof. We know from Lemma 8.7 that $\ker(f_1 \otimes f_2) = \ker f_1 \otimes V_2 + V_1 \otimes \ker f_2$. Taking the intersection with $V_1 \otimes_{\top} V_2$ yields

$$\ker(f_1 \otimes f_2) \cap (V_1 \otimes_{\top} V_2) = (\ker f_1 \otimes V_2 + V_1 \otimes \ker f_2) \cap (V_1 \otimes_{\top} V_2)$$

From elementary linear algebra, we obtain $V_1 = \ker(f_1) \oplus \ker(\pi_1)$. Since by hypothesis π_1 and Id_{V_2} are locally independent and since (E, \top) is a locality vector space, by the second point of Proposition 1.13,

the projection $\tilde{\pi}_1 : V_1 \mapsto \ker(\pi_1)$ onto $\ker(\pi_1)$ along $\ker(f_1)$ is also independent of Id_{V_2} . Thus we can use Corollary 1.14 with $\ker(f_1)$, V_1 and V_2 respectively playing the roles of V_1 , V and W in Corollary 1.14. This yields $(\ker f_1 \otimes V_2) \cap (V_1 \otimes_{\top} V_2) = \ker(f_1) \otimes_{\top} V_2$. Similarly, $(V_1 \otimes \ker f_2) \cap (V_1 \otimes_{\top} V_2) = V_1 \otimes_{\top} \ker(f_2)$ and hence

$$\ker(f_1 \otimes f_2) \cap (V_1 \otimes_{\top} V_2) = (\ker f_1 \otimes V_2 + V_1 \otimes \ker f_2) \cap (V_1 \otimes_{\top} V_2) = \ker f_1 \otimes_{\top} V_2 + V_1 \otimes_{\top} \ker f_2. \quad \square$$

The following lemma is the coalgebraic counterpart of Lemma 4.5.

Lemma 8.9. Let $(C_i, \Delta_i, \top_i), i \in \{1, 2\}$ be locality \mathbb{K} -coalgebras. The range of a locality K -coalgebra morphism $f : C_1 \rightarrow C_2$ is a locality \mathbb{K} -subcoalgebra of C_2 . Moreover, if there is a projection $\pi : C_1 \rightarrow \ker(f_1)$, which is locally independent of the identity map Id_{C_1} on C_1 , then $\ker(f_1)$ is a locality coideal of C_1 .

Proof. • We prove that the kernel $\ker(f)$ is a locality coideal. Let $c \in \ker(f) \subset C_1$. Since f is a locality coalgebra morphism, by (56), we have $(f \otimes f)(\Delta_1 c) = \Delta_2(f(c)) = 0$. Since Δ_1 is a locality coproduct, $\Delta_1(\ker(f)) \subset (C_1 \otimes_{\top_1} C_1) \cap \ker(f \otimes f)$. Thus $\Delta_1(\ker(f)) \subseteq \ker(f \otimes f)$, which shows that $\ker(f_1)$ is a locality coideal of C_1 .

We now apply Proposition 8.8 to $f_i = f$ and $V_i = C_1$, with $f \otimes f$ acting on $C_1 \otimes_{\top_1} C_1$. Since f is a locality morphism, it follows that $\ker(f \otimes f)|_{C_1 \otimes_{\top_1} C_1} = \ker(f) \otimes_{\top_1} C_1 + C_1 \otimes_{\top_1} \ker(f)$. Consequently, $\Delta_1(\ker(f)) \subset \ker(f) \otimes_{\top_1} C_1 + C_1 \otimes_{\top_1} \ker(f)$.

- To prove that the range $\text{Im}(f)$ is a locality coalgebra, for any $c \in C_1$ such that $\Delta_1 c = \sum_{(c)} c_1 \otimes c_2$ and $(c_1, c_2) \in \top_1$, using (56) we write $\Delta_2 f(c) = (f \otimes f) \circ \Delta_1 c = \sum_{(c)} f(c_1) \otimes f(c_2)$. Since f is a locality map, $(f(c_1), f(c_2)) \in \top_2$, which proves that $\Delta_2(\text{Im}(f)) \subset \text{Im}(f) \otimes_{\top_2} \text{Im}(f)$, showing that $\text{Im}(f)$ is a subcoalgebra of C_2 . \square

We are now ready to introduce further useful definitions.

Definition 8.10. • [CGPZ1, §5.1] A **locality \mathbb{K} -bialgebra** is a sextuple $(B, \top, m, u, \Delta, \epsilon)$ consisting of a locality \mathbb{K} -algebra (B, m, u, \top) and a locality \mathbb{K} -coalgebra $(B, \Delta, \epsilon, \top)$ that are locality compatible in the sense that Δ and ϵ are locality \mathbb{K} -algebra morphisms (30) and m and u are locality \mathbb{K} -coalgebra morphisms¹ i.e.,

$$\Delta \circ m|_{B^{\otimes \top 2}} = \underbrace{(m \otimes m)}_{\text{domain } B^{\otimes \top 4}} \circ (\text{Id}_B \otimes \tau_{23} \otimes \text{Id}_B) \circ \underbrace{(\Delta \otimes \Delta)}_{\text{range } B^{\otimes \top 4}}|_{B^{\otimes \top 2}}; \quad \epsilon \circ m = \epsilon \otimes \epsilon; \quad \Delta \circ u = u \otimes u; \quad \epsilon \circ u = \text{Id}_{\mathbb{K}},$$

where $B^{\otimes \top n}$ was defined in (22), and $\tau_{23} : B^{\otimes 4} \rightarrow B^{\otimes 4}$ is the map that switches the terms on the second and third position of the tensor.

- Let $(B_i, \top_i, m_i, u_i, \Delta_i, \epsilon_i)$ ($i \in \{1, 2\}$) be two locality \mathbb{K} -bialgebras. A **locality \mathbb{K} -bialgebra morphism** from B_1 to B_2 is a locality map $f : B_1 \rightarrow B_2$ that is a morphism of locality algebras and of locality \mathbb{K} -coalgebras.
- [CGPZ1, Proposition 4.9] Let $(B, \top, m, u, \Delta, \epsilon)$ be a locality bialgebra, and $\phi, \psi : B \rightarrow B$ two mutually independent locality linear maps. The locality convolution product of ϕ and ψ is a locality linear map $B \rightarrow B$ defined by

$$(\phi \star \psi) = m(\phi \otimes \psi)\Delta.$$

- [CGPZ1, Definition 5.3 and Remark 5.4] A **locality Hopf algebra** is a locality bialgebra $(H, \top, m, u, \Delta, \epsilon)$ together with a locality linear map $S : B \rightarrow B$ such that S and Id_H are mutually independent and

$$S \star \text{Id}_H = \text{Id}_H \star S = u \circ \epsilon.$$

¹This condition was missing in [CGPZ1].

- Let $(H_i, \top_i, m_i, u_i, \Delta_i, \epsilon_i, S_i)$ be $(i \in \{1, 2\})$ be two locality Hopf algebra. A **locality Hopf algebra morphism** between B_1 and B_2 is a morphism of locality bialgebras $f : B_1 \rightarrow B_2$ such that $f \circ S_1 = S_2 \circ f$.
- A **sub-locality Hopf algebra** of H of a locality Hopf algebra $(H, \top, m, u, \Delta, \epsilon, S)$ is a locality Hopf algebra $(H', \top', m', u', \Delta', \epsilon', S')$ contained in H such that the injection map $f : H \hookrightarrow H'$ is a locality Hopf algebra morphism.
- A **graded (resp. filtered) locality Hopf algebra** is a locality Hopf algebra together with a grading (resp. filtration) which makes it a graded (resp. filtered) algebra and coalgebra and such that

$$S(H_n) \subset H_n \text{ (resp. } S(H^n) \subset H^n).$$

A **connected locality Hopf algebra** is a graded locality algebra such that H_0 has dimension 1.

Remark 8.11. As in the algebra and coalgebra cases, our definition of sub-locality Hopf algebra is more general than the one used in [CGPZ1] and will be used later.

Example 8.12 (The locality tensor algebra as a locality Hopf algebra). $\mathcal{T}_{\top}(V)$ is a locality Hopf algebra in the sense of [CGPZ1, Definition 5.3] when equipped with the tensor product restricted to pairs in \top_{\otimes} and the deconcatenation coproduct defined on $x \in V$ by $\Delta_{\sqcup}(x) = 1 \otimes x + x \otimes 1$ and inductively on the degree by

$$\Delta_{\sqcup}(x_{i_1} \otimes \cdots \otimes x_{i_n}) = \sum_{J \subset \{i_1, \dots, i_n\}, w_J \top_{\otimes} w_{\bar{J}}} w_J \otimes w_{\bar{J}}, \quad (58)$$

where we have set $w_J := x_{j_1} \otimes \cdots \otimes x_{j_k}$ for $J = (j_1, \dots, j_k)$ and where \bar{J} stands for the complement of J in $\{i_1, \dots, i_n\}$. The counit is defined by $\epsilon(x) = 0$ for $x \in V$ and the antipode is given by $S(x_{j_1} \otimes \cdots \otimes x_{j_k}) = (-1)^k x_{j_k} \otimes \cdots \otimes x_{j_1}$.

It is a connected graded cocommutative locality Hopf algebra of finite type.

The usual result for the existence of an antipode in a graded connected bialgebra also holds in the locality setup.

Proposition 8.13. [CGPZ1, Proposition 5.5] *Let $(B, \top, m, u, \Delta, \epsilon)$ be a graded connected locality bialgebra. There exists an antipode $S : B \rightarrow B$ such that $(B, \top, m, u, \Delta, \epsilon, S)$ is a locality Hopf algebra.*

We end this paragraph with a transposition to the locality setup of the known Lie algebra structure of the space $\text{Prim}(B)$ of primitive elements of a bialgebra B (see for example [Abe, Theorem 2.1.3] for the usual result). Recall that a element $x \in B$ is called **primitive** if, and only if $\Delta x = x \otimes 1 + 1 \otimes x$ and that a **graded locality Lie algebra** is a locality Lie algebra which is also a graded algebra for the Lie product (see Definitions 4.17 and 7.2).

Proposition 8.14. *The space $(\text{Prim}(B), \top_{\text{Prim}(B)}, [; \cdot])$ of a (resp. graded) locality bialgebra (B, \top) equipped with $\top_{\text{Prim}(B)}$ the restriction of the locality relation \top to primitive elements and the usual commutator $[x, y] = xy - yx$ (resp. $[x, y] = xy - (-1)^{|x||y|}yx$), is a (resp. graded) locality Lie algebra.*

Proof. We carry out the proof for the graded case, since the ungraded case can be obtained by setting all degrees to zero.

Let m be the locality product of the locality bialgebra (B, \top) . Since for any $U \subseteq B$, $m(U^{\top} \otimes_{\top} U^{\top}) \subset U^{\top}$, we have that $[[x, y], z]$ (and its permutations) is well-defined for any triplet $(x, y, z) \in B^{\times \top 3}$. The rest of the proof goes exactly as for the non locality case. In particular, for any $(x, y) \in \text{Prim}(B) \times_{\top} \text{Prim}(B)$ since Δ is a locality algebra morphism, we have:

$$\begin{aligned} \Delta(xy) &= (m \otimes m) \circ \tau_{23} \circ (\Delta \otimes \Delta)(x, y) \\ &= (m \otimes m) \circ \tau_{23}((x \otimes 1 + 1 \otimes x) \otimes (y \otimes 1 + 1 \otimes y)) \quad \text{since } x \text{ and } y \text{ are primitive elements} \\ &= xy \otimes 1 + (-1)^{|x||y|} y \otimes x + x \otimes y + 1 \otimes xy. \end{aligned}$$

$\Delta(yx)$ is obtained by exchanging x and y . Putting everything together we obtain by linearity of Δ

$$\Delta([x, y]) = [x, y] \otimes 1 + 1 \otimes [x, y] \in B \otimes_{\top} B$$

as needed. This, together with the fact that Δ is linear, implies that $(\text{Prim}(B), \top, [; \cdot])$ is a locality algebra. \square

8.2 The reduced coproduct and primitive elements

This paragraph reviews preliminary well-known technical results, which we transpose to the locality setup. As before, \mathbb{K} is a commutative field of characteristic zero.

Lemma 8.15. Let H be a locality graded, connected \mathbb{K} -bialgebra. The projection onto $\ker(\epsilon)$ along $\mathbb{K}(1_H)$

$$\begin{aligned} \rho: H &\longrightarrow H \\ x &\longmapsto x - \epsilon(x)1_H \end{aligned} \quad (59)$$

is a locality linear map, which is independent of Δ in the following sense:

$$(x, y) \in \top \implies (\rho(x), \Delta(y)) \in \top_{\otimes} \quad (60)$$

with \top_{\otimes} the locality relation on $\mathcal{T}_{\otimes}(H) \supseteq H$ of Definition 4.10.

Proof. Let $x \top y$. Since $\mathbb{K} \subset H^{\top}$ we have $\epsilon(x)1_H \top y$ implying by linear locality that $(x - \epsilon(x)1_H) \top y$. Since $\epsilon(y)1_H \top (x - \epsilon(x)1_H)$, again by linear locality, we deduce that $x - \epsilon(x)1_H \top y - \epsilon(y)1_H$ and conclude that $\rho(x) \top \rho(y)$.

To check the mutual independence of ρ and Δ , we consider again $x \in H$ and $y \in H$ such that $x \top y$. We write $\Delta y = \sum y_i \otimes y'_i$, so that Equation (60) amounts to show that $\rho(x) \top y_i$ and $\rho(x) \top y'_i$. Since $\rho(x) = x - \epsilon(x)1_H$ and $\mathbb{K}1_H \top H$, by linearity, it suffices to show that $x \top y_i$ and $x \top y'_i$. But this follows from the fact that Δ maps $\{x\}^{\top}$ to $\{x\}^{\top} \otimes_{\top} \{x\}^{\top}$. \square

On a locality graded, connected \mathbb{K} -bialgebra H , equipped with a locality coproduct Δ , we consider the coassociative locality linear map $\tilde{\Delta}: H \rightarrow \ker(\epsilon) \otimes_{\top} \ker(\epsilon)$ defined as $\tilde{\Delta}(1) = 0$, and for $x \in \ker(\epsilon)$ as $\tilde{\Delta}(x) = \Delta(x) - 1 \otimes x - x \otimes 1$. For $n \geq 0$, we then define inductively $\tilde{\Delta}^{(n)}: H \rightarrow H^{\otimes_{\top}(n+1)}$ by:

- $\tilde{\Delta}^{(0)} = \rho$.
- $\tilde{\Delta}^{(1)} = \tilde{\Delta}$.
- $\tilde{\Delta}^{(n+1)} = (\tilde{\Delta} \otimes Id^{\otimes n}) \circ \tilde{\Delta}^{(n)}$.

Since Δ and ρ are locality linear maps, $\tilde{\Delta}$ and $\tilde{\Delta}^{(n)}$ are also locality maps. Moreover, ρ and Δ are mutually independent, and therefore ρ and $\tilde{\Delta}$ are also mutually independent in the sense of (60).

Proposition 8.16. Let (H, \top) with $H = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} H_k$, be a locality graded, connected \mathbb{K} -bialgebra. For every $n \geq 1$

$$\tilde{\Delta}^{(n)} = (\rho \otimes \dots \otimes \rho) \circ \Delta^{(n)}.$$

Proof. Recall that $H = u(\mathbb{K}) \oplus \ker(\epsilon) \simeq \mathbb{K} \oplus \ker(\epsilon)$. We proceed by induction over n . For $n = 1$ we have $(\rho \otimes \rho) \circ \Delta(1) = \rho(1) \otimes \rho(1) = 0 = \tilde{\Delta}(1)$. By linearity, this extends replacing 1 by any $x \in \mathbb{K}$. Let $x \in \ker(\epsilon)$. Since $\tilde{\Delta}(x) \in \ker(\epsilon) \otimes_{\top} \ker(\epsilon)$ and $(\rho \otimes \rho) \circ \tilde{\Delta} = \tilde{\Delta}$, it follows that $(\rho \otimes \rho) \circ \Delta(x) = \rho(1) \otimes \rho(x) + \rho(x) \otimes \rho(1) + (\rho \otimes \rho) \circ \tilde{\Delta}(x) = \Delta(x)$, so that the proposition holds for $n = 1$.

Suppose now that the proposition holds true for $n - 1$. We note that $\tilde{\Delta}(1) = 0$ implies that $\tilde{\Delta} \circ \rho = \tilde{\Delta}$. Hence,

$$\begin{aligned} (\rho \otimes \dots \otimes \rho) \circ \Delta^{(n)} &= ((\rho \otimes \rho) \circ \Delta \otimes \rho \otimes \dots \otimes \rho) \circ \Delta^{(n-1)} \\ &= (\tilde{\Delta} \otimes \rho \otimes \dots \otimes \rho) \circ \Delta^{(n-1)} \\ &= (\tilde{\Delta} \otimes Id \otimes \dots \otimes Id) \circ (\rho \otimes \dots \otimes \rho) \circ \Delta^{(n-1)} \quad \text{by the initial induction step} \\ &= (\tilde{\Delta} \otimes Id \otimes \dots \otimes Id) \circ \tilde{\Delta}^{(n-1)} \quad \text{by the induction step} \\ &= \tilde{\Delta}^{(n)}. \end{aligned} \quad \square$$

The subsequent proposition combines elementary known results, which we recall for the sake of completeness. If $H = \bigoplus_{n \in \mathbb{Z}} H_n$ is a graded algebra, we say $|x| = n$ if $x \in H$ is a homogeneous component of degree n , i.e. $x \in H_n$.

Proposition 8.17. *Let (H, \top) with $H = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} H_n$, be a locality graded, connected \mathbb{K} -bialgebra. Let $k \geq 1$.*

1. *For any $x \in H$ such that $\tilde{\Delta}^{(k-1)}(x) \neq 0$, we have $\tilde{\Delta}^{(k)}(x) = 0 \implies \tilde{\Delta}^{(k-1)}(x) \in \text{Prim}(H)^{\otimes_{\top} k}$.*
2. *For every $x \in H$, $\tilde{\Delta}^{(k)}(x) = 0$ if $k \geq |x|$ and $\tilde{\Delta}^{(k-1)}(x) \in \text{Prim}(H)^{\otimes_{\top} k}$ if $|x| = k$.*
3. *Let $n \geq 2$. For any $(v_1, \dots, v_n) \in \text{Prim}(H)^{\times_{\top} n}$, we have $\tilde{\Delta}^{(k)}(v_1 \cdots v_n) = 0$ for any $k \geq n$ and the following refinement of the first item holds:*

$$\tilde{\Delta}^{(n-1)}(v_1 \cdots v_n) = \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \in \text{Prim}(H)^{\otimes_{\top} n}, \quad (61)$$

where \mathfrak{S}_n is the n -th symmetric group.

Proof. 1. The coassociativity of $\tilde{\Delta}$ implies

$$(Id^{\otimes_{\top} (i-1)} \otimes \tilde{\Delta} \otimes Id^{\otimes_{\top} (k-i)}) \circ \tilde{\Delta}^{(k-1)}(x) = \tilde{\Delta}^{(k)}(x) = 0 \quad \forall 1 \leq i \leq k.$$

We infer that

$$\tilde{\Delta}^{(k-1)}(x) \in \ker(Id^{\otimes_{\top} (i-1)} \otimes \tilde{\Delta} \otimes Id^{\otimes_{\top} (k-i)}) = H^{\otimes_{\top} (i-1)} \otimes_{\top} (\mathbb{K} \oplus \text{Prim}(H)) \otimes_{\top} H^{\otimes_{\top} (k-i)}.$$

Assuming that $\tilde{\Delta}^{(k-1)}(x) \neq 0$, since $\tilde{\Delta}^{(k-1)}(x) \in \ker(\epsilon)^{\otimes_{\top} k}$ (which is a consequence of $\text{Im}(\rho) \subseteq \ker(\epsilon)$ and Proposition 8.16), it follows that

$$\tilde{\Delta}(x) \in H^{\otimes_{\top} (i-1)} \otimes_{\top} \text{Prim}(H) \otimes_{\top} H^{\otimes_{\top} (k-i)} \quad \forall 1 \leq i \leq k.$$

Thus, $\tilde{\Delta}^{(k-1)}(x)$ lies in the intersection of all such spaces, which yields the statement.

2. Using the coassociativity of $\tilde{\Delta}$, for any $x \in H$ with $|x| = n$, we have (by induction on k) $\tilde{\Delta}^{(k)}x = \sum_x x^{(1)} \otimes \cdots \otimes x^{(k+1)}$, with $(x^{(1)}, \dots, x^{(k+1)}) \in H^{\times_{\top} k+1}$, $\sum_{j=1}^{k+1} |x^{(j)}| = |x| = n$. If $k \geq n$, this imposes that $\tilde{\Delta}^{(k)}x = 0$. In particular, $\tilde{\Delta}^{(n)}x = 0$. It then follows from the previous item that $\tilde{\Delta}^{(n-1)}(x) \in \text{Prim}(H)^{\otimes_{\top} n}$.
3. Let $(v_1, \dots, v_n) \in \text{Prim}(H)^{\times_{\top} n}$ and $x := v_1 \cdots v_n$. To compute $\tilde{\Delta}^{(k)}(x)$ for any $k \leq n-1$, we proceed by induction on k . We prove first that $\Delta(v_1 \cdots v_n) = \sum_{I \subset [n]} v_I \otimes v_{I^c}$ by induction over n , setting $v_{\emptyset} = 1$. For $n = 1$,

$$\Delta(v_1) = v_1 \otimes 1 + 1 \otimes v_1 = \sum_{I \subset [1]} v_I \otimes v_{I^c} \in \text{Prim}(H)^{\otimes_{\top} 2}.$$

Now assume it is true for $n-1$. The compatibility of the product and the coproduct yields

$$\begin{aligned} \Delta(v_1 \cdots v_{n-1} v_n) &= \Delta(v_1 \cdots v_{n-1}) \Delta(v_n) \\ &= \left(\sum_{I \subset [n-1]} v_I \otimes v_{I^c} \right) (v_n \otimes 1 + 1 \otimes v_n) \\ &= \sum_{I \subset [n-1]} v_I \cdot v_n \otimes v_{I^c} + \sum_{I \subset [n-1]} v_I \otimes v_{I^c} \cdot v_n \\ &= \sum_{I \subset [n]} v_I \otimes v_{I^c} \in \text{Prim}(H)^{\otimes_{\top} n}. \end{aligned}$$

It then follows by induction on k that

$$\Delta^{(k)}(v_1 \cdots v_n) = \sum_{I_1 \sqcup \cdots \sqcup I_{k+1} = [n]} v_1 \otimes \cdots \otimes v_{I_{k+1}} \in \text{Prim}(H)^{\otimes_{\top} n}.$$

Using Proposition 8.16, we then easily derive the expression of $\tilde{\Delta}^k(v_1 \cdots v_n)$ composing with $\rho^{\otimes}(k+1)$. Note that for $I = \emptyset$, $\rho(v_I) = 0$, otherwise $\rho(v_I) = v_I$. We have

$$\Delta^{(k)}(v_1 \cdots v_n) = \sum_{\substack{I_1 \sqcup \cdots \sqcup I_{k+1} = [n] \\ I_1, \dots, I_{k+1} \neq \emptyset}} v_1 \otimes \cdots \otimes v_{I_{k+1}} \in \text{Prim}(H)^{\otimes_{\top}^{k+1}}.$$

For $k = n - 1$ each of the sets I_j only contains one element so that we get the expected formula. For $k \geq n$ this expression vanishes since some of the sets are empty. This ends the proof of the statement. \square

The following result will be useful in the sequel.

Proposition 8.18. *Let H be a filtered connected (with respect to the grading induced by the filtration) locality \mathbb{K} -bialgebra and $J \neq \{0\}$ a locality left, right or two-sided coideal of H such that $J \cap H^0 = \{0\}$. Then J contains non zero primitive elements of H .*

Proof. Consider the filtration $J^n = J \cap H^n$, $n \in \mathbb{Z}_{\geq 0}$ on J induced by the one on H . Since $J \neq \{0\}$, there is some element $0 \neq x \in J$ of minimum degree k among the elements of J . Explicitly, $x \in J^k$ and $J^n = \{0\}$ for every $n < k$. The existence of x is guaranteed. Indeed, let us write

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$$

with $\sum x' \otimes x'' \in \ker(\epsilon) \otimes_{\top} \ker(\epsilon)$. Since Δ respects the filtration, if x' and x'' are non zero, there are integers $0 < m, n < k$ such that $x' \in H^n$ and $x'' \in H^m$ with $n + m \leq k$. If J is a left, resp. right coideal, then x' , resp. x'' lies in J , which contradicts the minimality of k . Therefore at least one of the two elements x' or x'' vanishes, which implies $\sum x' \otimes x'' = 0$ so that x is primitive. \square

9 The Milnor-Moore theorem: locality version

The results of Subsection 8.2 hold independently of the validity of the statements 6.19 and 7.6. However, assuming these statements hold true, they will be used in an essential way in the proof of a locality version of a central theorem of Hopf algebra theory, namely the Milnor-Moore theorem.

In the sequel we therefore assume that the conjectural statements 6.19 and 7.6 hold true and transpose to the locality setup the well-known the Milnor-Moore theorem, also known as Cartier-Quillen-Milnor-Moore. The original theorem was proven by Milnor and Moore in 1965 [MM] using a rather categorical language. The idea of the proof we follow stems from Cartier [Car1] and Patras [Pat]. We found a nice explanation of this proof in lecture notes by Loïc Foissy [Foi1], on which most of the sequel is based.

9.1 The locality universal enveloping algebra as a Hopf algebra

The Milnor-Moore theorem relates a Hopf algebra H with a Hopf algebra built from the universal enveloping algebra of the Lie algebra of primitive elements of H . In order to show the Milnor-Moore theorem in the locality setup, we need to show that the locality universal enveloping algebra of a locality Lie algebra (\mathfrak{g}, \top) indeed admits a locality Hopf algebra structure.

Proposition 9.1. *Given a locality Lie algebra (\mathfrak{g}, \top) and assuming the statement 7.6 holds true for (\mathfrak{g}, \top) , the universal enveloping algebra $U_{\top}(\mathfrak{g})$ together with the locality relation \top_U (from Definition 4.19) can be equipped with a locality cocommutative Hopf algebra structure.*

Proof. So far, assuming that the conjectural statement 7.6 holds true for (\mathfrak{g}, \top) , we know from Proposition 7.7, that

$$(U_{\top}(\mathfrak{g}), \top_U, \bigotimes, u)$$

is an associative unital locality algebra. In order to equip it with a coproduct, we consider the map $\delta : \mathfrak{g} \rightarrow U_{\top}(\mathfrak{g}) \otimes_{\top} U_{\top}(\mathfrak{g})$ defined by $\delta(x) = \iota_{\mathfrak{g}}(x) \otimes 1 + 1 \otimes \iota_{\mathfrak{g}}(x)$, where $\iota_{\mathfrak{g}} : \mathfrak{g} \rightarrow U_{\top}(\mathfrak{g})$ is the canonical

map. One can check that it is locality linear and $\delta([x, y]) = \delta(x)\delta(y) - \delta(y)\delta(x)$. Hence, Theorem 7.8 (which applies since we have assumed that the conjectural statement 7.6 holds true for (\mathfrak{g}, \top)) gives the existence and uniqueness of a locality algebra morphism $\Delta : U_{\top}(\mathfrak{g}) \rightarrow U_{\top}(\mathfrak{g}) \otimes U_{\top}(\mathfrak{g})$ which extends δ . Note that by construction the elements in $\iota_{\mathfrak{g}}(\mathfrak{g})$ are primitive and Δ is cocommutative. For the counit we consider the zero map from \mathfrak{g} to \mathbb{K} . This is indeed a locality Lie algebra morphism and once again by Theorem 7.8, there is a unique locality algebra morphism $\epsilon : U_{\top}(\mathfrak{g}) \rightarrow \mathbb{K}$ which vanishes identically on $\iota_{\mathfrak{g}}(\mathfrak{g})$. Therefore $U_{\top}(\mathfrak{g})$ with this coproduct and counit is a filtered connected bialgebra over \mathbb{K} . Consider the locality Lie algebra morphism $\sigma : \mathfrak{g} \rightarrow U_{\top}(\mathfrak{g})$ defined by $\sigma(x) = -\iota_{\mathfrak{g}}(x)$. Once more, Theorem 7.8 gives the existence and uniqueness of a locality algebra morphism $S : U_{\top}(\mathfrak{g}) \rightarrow U_{\top}(\mathfrak{g})$ which extends σ . To prove that it is an antipode, for $\alpha \in \mathbb{K} \subset U_{\top}(\mathfrak{g})$, we see that $S \star I(\alpha) = \alpha(S(1)I(1)) = \alpha$ and for $x_i \in \iota_{\mathfrak{g}}(\mathfrak{g})$, we have $S \star I(x_1 \cdots x_k) = \sum_{J \subset [k]} (-1)^{|J|} x_1 \cdots x_k = 0$ which shows that $S \star I = u \circ \epsilon$. Similarly, one shows that $I \star S = u \circ \epsilon$, so that S is an antipode and $U_{\top}(\mathfrak{g})$ is a locality Hopf algebra. \square

Proposition 9.2. *Let $(\mathfrak{g}, \top, [,]_{\top})$ be locality Lie algebra and $(\mathfrak{g}', \top', [,]_{\top'})$ a sub-locality Lie algebra of \mathfrak{g} . Assuming that the conjectural statement 7.6 holds true for \mathfrak{g}' and \mathfrak{g} , then $U_{\top'}(\mathfrak{g}')$ is a sub-locality Hopf algebra of $U_{\top}(\mathfrak{g})$.*

Proof. By means of Proposition 4.21, $U_{\top'}(\mathfrak{g}')$ is a sub-locality algebra of $U_{\top}(\mathfrak{g})$. The inclusion map $\mathfrak{g}' \hookrightarrow \mathfrak{g}$ induces an injective map $\iota : U_{\top'}(\mathfrak{g}') \rightarrow U_{\top}(\mathfrak{g})$ it is easy to see that the counits relate by $\epsilon' = \epsilon \circ \iota$. Moreover, since the coproduct on the universal enveloping algebra are completely determined by the coproduct on $\iota_{\mathfrak{g}}(\mathfrak{g})$ (resp $\iota_{\mathfrak{g}'}(\mathfrak{g}')$) (see the proof of Proposition 9.1), it is easy to see that $U_{\top}(\mathfrak{g}')$ is a sub-locality bialgebra of $U_{\top}(\mathfrak{g})$.

Let $S_{U'}$ be the antipode on $U_{\top}(\mathfrak{g}')$, then

$$S_{U'} = S_U \circ \iota$$

follows from the fact the antipodes are completely determined by their action on $\iota_{\mathfrak{g}}(\mathfrak{g})$ (resp. $\iota_{\mathfrak{g}'}(\mathfrak{g}')$), and for $x \in \iota_{\mathfrak{g}'}(\mathfrak{g}')$,

$$S_{U'}(x) = -x = -\iota(x) = S_U(\iota(x)).$$

This proves that $U_{\top}(\mathfrak{g}')$ is indeed a sub-locality Hopf algebra of $U_{\top}(\mathfrak{g})$. \square

For the proof of locality version of the Milnor-Moore theorem, we use a description of the primitive elements of the universal enveloping algebra, which usually comes as a corollary of the Poincaré-Birkhoff-Witt theorem. Since there is no locality version of this theorem available yet, we provide an alternative proof using Zorn's lemma, of the following classical statement extended to a locality set up.

Proposition 9.3. *Given a locality Lie algebra (\mathfrak{g}, \top) over \mathbb{K} , the conjectural statement 7.6 holds true, and assuming the axiom of choice, the set of primitive elements of the locality universal enveloping algebra $U_{\top}(\mathfrak{g})$ coincides with $\iota_{\mathfrak{g}}(\mathfrak{g})$:*

$$\text{Prim}(U_{\top}(\mathfrak{g})) = \iota_{\mathfrak{g}}(\mathfrak{g}),$$

where $\iota_{\mathfrak{g}}$ is the canonical map from \mathfrak{g} to $U_{\top}(\mathfrak{g})$.

Proof. By the very construction of the coproduct on $U_{\top}(\mathfrak{g})$, we have $\iota_{\mathfrak{g}}(\mathfrak{g}) \subset \text{Prim}(U_{\top}(\mathfrak{g}))$. For the other inclusion, for any $n \in \mathbb{N}$ we consider the set

$$G_n := \{x_1^{k_1} \cdots x_m^{k_m} \in U_{\top}(\mathfrak{g}) \mid (x_1, \dots, x_m) \in \iota_{\mathfrak{g}}(\mathfrak{g})^{\times_m} \wedge k_i \in \mathbb{N} \wedge \sum_{i=1}^m k_i \leq n\}. \quad (62)$$

which by construction, generates the space $(U_{\top}(\mathfrak{g}))^n$ of filtration degree n in the natural filtration of $U_{\top}(\mathfrak{g})$. Lemma 1.16 applied to $E = \mathbb{K}1$ yields the existence of a vector space basis $\{1\} \subset B_1 \subset G_1$ of $(U_{\top}(\mathfrak{g}))^1$, so for filtration degree 1. Since $B_1 \subset G_2$, Lemma 1.16 yields a basis B_2 of $(U_{\top}(\mathfrak{g}))^2$ such that $B_1 \subset B_2 \subset G_2$. We proceed inductively to build $B := \bigcup_{n \in \mathbb{N}} B_n$ which is a Hammett (vector space) basis of $U_{\top}(\mathfrak{g})$. We use the simplified notation $\vec{x}^{\vec{k}} := x_1^{k_1} \cdots x_n^{k_n}$ and $|\vec{k}| := k_1 + \cdots + k_n$. Note that for $\vec{x}^{\vec{k}} \in B$ with $|\vec{k}| = n$, it is linearly independent of B_{n-1} .

A primitive element y of $U_{\top}(\mathfrak{g})$ can be expressed in terms of the basis B as

$$y = \sum_{\vec{x}^k \in B} \alpha_{\vec{x}^k} \vec{x}^k$$

where only finitely many $\alpha_{\vec{x}^k}$ are non zero. Let $N = \max\{|\vec{k}| : \alpha_{\vec{x}^k} \neq 0\}$. If $N = 1$ we have $y \in \iota_{\mathfrak{g}}(\mathfrak{g})$ as required. Let us now assume that $N > 1$. Then $\tilde{\Delta}^{(N-1)}(y) = 0$ since y is primitive. By (61), one can write

$$0 = \tilde{\Delta}^{(N-1)}(y) = \sum_{|\vec{k}|=N} \alpha_{\vec{x}^k} \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(N)}.$$

with $x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(N)} \in \text{Prim}(H)^{\otimes N}$.

Applying the product m yields

$$0 = m^{(N-1)}(\tilde{\Delta}^{(N-1)}(y)) = \sum_{|\vec{k}|=N} \alpha_{\vec{x}^k} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(N)} \in U_{\top}(\mathfrak{g}).$$

Since $\iota_{\mathfrak{g}}(\mathfrak{g}) = (U_{\top}(\mathfrak{g}))^1 \ni [x_i, x_j] = x_i x_j - x_j x_i$ for every i and j , we may reorder the x_i 's to get the original elements of B at the cost of adding some lower order terms (l.o.t.) (with respect to the natural filtration of $U_{\top}(\mathfrak{g})$ given by the sets (62)). The resulting products arising in the new linear combination are linearly independent of the leading term due to the very manner the basis B was constructed. Hence, we have

$$0 = \sum_{|\vec{k}|=N} \frac{\alpha_{\vec{x}^k}}{N!} \vec{x}^k + \text{l.o.t.}$$

Since the elements of the basis B are linearly independent, we may conclude that all $\alpha_{\vec{x}^k} = 0$ except if $N = 1$. Therefore $\text{Prim}(U_{\top}(\mathfrak{g})) \subset \iota_{\mathfrak{g}}(\mathfrak{g})$. Thus $\text{Prim}(U_{\top}(\mathfrak{g})) = \iota_{\mathfrak{g}}(\mathfrak{g})$. \square

Remark 9.4. In the non-locality case, the injectivity of the canonical map $\iota_{\mathfrak{g}} : \mathfrak{g} \rightarrow U(\mathfrak{g})$, see (33), follows from the Poincaré-Birkhoff-Witt theorem. However, this is not necessarily true in the locality setup. In general $\iota_{\mathfrak{g}}(\mathfrak{g})$ is a quotient of \mathfrak{g} , with the property that $U_{\top}(\mathfrak{g}) \simeq U_{\top}(\iota_{\mathfrak{g}}(\mathfrak{g}))$.

A case which will be of particular interest in the following section, is when \mathfrak{g} is the Lie algebra of primitive elements of a locality bialgebra (B, \top) . In that case, the universal property of the locality universal enveloping algebra (Theorem 4.22) yields the existence of a pre-locality algebra morphism $\phi : U_{\top}(\mathfrak{g}) \rightarrow B$ which extends the canonical injection $\iota : \mathfrak{g} \rightarrow B$. Therefore, the canonical map $\iota_{\mathfrak{g}} : \mathfrak{g} \rightarrow U_{\top}(B)$ is indeed injective in such cases.

9.2 A locality Cartier-Quillen-Milnor-Moore theorem

Assuming that the conjectural statement 6.19 holds true, we can now prove a locality version of the Cartier-Quillen-Milnor-Moore theorem.

Theorem 9.5 (Locality Cartier-Quillen-Milnor-Moore theorem). Let (H, \top) be a graded, connected, cocommutative locality Hopf algebra. H is generated by its primitive elements $\text{Prim}(H)$ as a locality algebra, which we assume satisfies the conjectural statements 7.6 and 6.19. In that case, we have the following isomorphism of locality Hopf algebras:

$$(H, \top) \underset{\text{loc}}{\simeq} (U_{\top}(\text{Prim}(H)), \top_U) \tag{63}$$

with \top_U the locality relation of Definition 4.19 in the case $\mathfrak{g} = \text{Prim}(H)$.

Proof. • We first prove that H is generated by its primitive elements $\text{Prim}(H)$ as a locality algebra. Let H' be the locality subalgebra of H generated by $\text{Prim}(H) \cup \{1\}$ and let $0 \neq x \in H$. By Proposition 8.17 2., for k big enough, $\tilde{\Delta}^{(k)}(x) = 0$. Set $\deg_p(x)$ to be the minimum of all such integers k . Using induction over $\deg_p(x)$, we show that $H \subset H'$. If $\deg_p(x) = 0$, then by definition

of $\tilde{\Delta}^{(0)}$, $\rho(x) = 0$. Hence $x \in \mathbb{K} \subset H'$. If $\deg_p(x) = 1$, then x is a primitive element so that $x \in H'$. If $n = \deg_p(x) > 0$, by means of Proposition 8.17 1. We have

$$\tilde{\Delta}^{(n-1)}(x) = \sum_i x_1^{(i)} \otimes \cdots \otimes x_n^{(i)}$$

where all the $x_j^{(i)}$ are primitive elements. The cocommutativity of H implies invariance under the natural action of the symmetric group so we have:

$$\tilde{\Delta}^{(n-1)}(x) = \frac{1}{n!} \sum_{\sigma \in S_n} \sum_i x_{\sigma(1)}^{(i)} \otimes \cdots \otimes x_{\sigma(n)}^{(i)}$$

Exchanging the two sums (both over finite sets), we then recognize in the innermost sum the expression of $\tilde{\Delta}^{(n-1)}(x_1^{(i)} \cdots x_n^{(i)})$ given by Proposition 8.17 3. The linearity of $\tilde{\Delta}^{(n-1)}$ then yields

$$\tilde{\Delta}^{(n-1)}(x) = \tilde{\Delta}^{(n-1)}\left(\frac{1}{n!} \sum_i x_1^{(i)} \cdots x_n^{(i)}\right) \iff \tilde{\Delta}^{(n-1)}\left(x - \frac{1}{n!} \sum_i x_1^{(i)} \cdots x_n^{(i)}\right) = 0.$$

Hence, by definition of the degree,

$$\deg_p\left(x - \frac{1}{n!} \sum_i x_1^{(i)} \cdots x_n^{(i)}\right) < n.$$

By the induction hypothesis, this element lies in H' , and so does $x \in H'$. We infer that $H' = H$ and H is generated by its primitive elements $\text{Prim}(H)$ as a locality algebra.

- To build the isomorphism (63) we assume that the conjectural statement 6.19 holds true. By Proposition 8.14, the set $\text{Prim}(H)$ of primitive elements of H has a graded locality Lie algebra structure. We can therefore build its enveloping algebra $U_{\top}(\text{Prim}(H))$. Assuming that the conjectural statement 6.19 holds true, we can apply the universal property (Theorem 7.8) to extend the locality Lie algebra morphism given by the injection $i : \text{Prim}(H) \rightarrow H$ to a locality algebra morphism

$$\phi : U_{\top}(\text{Prim}(H)) \rightarrow H,$$

which stabilizes the elements in $\text{Prim}(H)$. Moreover, since ϕ is a locality algebra morphism, again by Lemma 4.5 the range of ϕ is a locality subalgebra of H which contains H' . Hence, by the first part of this proof, ϕ is surjective.

- Let us show that ϕ is a coalgebra morphism. The coproducts Δ_U on $U_{\top}(\text{Prim}(H))$ and Δ on H are by definition algebra morphisms. Since they coincide on the set $\text{Prim}(H)$, $(\phi \otimes \phi) \circ \Delta_U = \Delta \circ \phi$ on $\text{Prim}(H)$. This identity extends everywhere since $\text{Prim}(H)$ generates both $U_{\top}(\text{Prim}(H))$ and H as locality algebras. We still need to show $\epsilon_U = \epsilon \circ \phi$, with ϵ_U the counit of $U_{\top}(\text{Prim}(H))$ and ϵ the counit of H . Again, since $\text{Prim}(H)$ generates both $U_{\top}(\text{Prim}(H))$ and H as locality algebras, it is enough to show that these maps coincide on $\text{Prim}(H)$. On the one hand, by definition of ϵ_U (see the proof of Proposition 9.1), ϵ_U vanishes on $\text{Prim}(H)$. On the other hand, for any $h \in \text{Prim}(H)$, using the property of ϵ and the canonical identification $\mathbb{K} \otimes H \simeq H$, we have:

$$h = (\epsilon \otimes \text{Id}_H) \circ \Delta(h) = \epsilon(h) \otimes 1_H + \epsilon(1) \otimes h = \epsilon(h)1_H + h.$$

Therefore ϵ vanishes on $\text{Prim}(H)$ as required and ϕ is a coalgebra morphism.

- We prove the injectivity of ϕ ad absurdum. The fact that ϕ is a locality coalgebra morphism, implies by Lemma 8.9, that its kernel $\text{Ker}(\phi)$ is a locality coideal of $U_{\top}(\text{Prim}(H))$. Since ϕ is an algebra morphism, $\phi(1) = 1$ so $\text{Ker}(\phi) \cap \mathbb{K}1 = \{0\}$. By Proposition 8.18 applied to $J = \text{Ker}(\phi)$, assuming the latter is non trivial, it must contain a primitive element of $U_{\top}(\text{Prim}(H))$. However this leads to a contradiction since, by Proposition 9.3, $\text{Prim}(U_{\top}(\text{Prim}(H))) = \text{Prim}(H)$ which is fixed by ϕ , therefore none of them lies in the kernel. Hence ϕ is injective.

- To show that ϕ is an locality isomorphism of Hopf algebras, we still need to prove that $\phi^{-1} : H \rightarrow U_{\top}(\text{Prim}(H))$ is a locality algebra morphism. The previous items give the existence of the inverse map ϕ^{-1} . By definition of ϕ , on any element h of H , ϕ^{-1} acts as:

$$\phi^{-1}(h) = x_1 \otimes \cdots \otimes x_n$$

for some $n \in \mathbb{N}^*$ and primitive elements x_i such that $h = x_1 \cdots x_n$. Here the right-hand-side actually stands for an equivalence class of tensor products, which we write as a tensor to simplify notations. We now distinguish two cases.

If $n = 1$, then h is a primitive element of H and ϕ^{-1} restricted to primitive elements is simply the projection from H to $U_{\top}(\text{Prim}(H))$. This map is a locality map by definition of the locality on the quotient space $U_{\top}(\text{Prim}(H))$.

If $n \geq 2$, using Equation (61) of Proposition 8.17, we have

$$\tilde{\Delta}^{(n-1)}(h) = \tilde{\Delta}^{(n-1)}(x_1 \cdots x_n) = \sum_{\sigma \in \mathcal{S}_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

The fact that ϕ^{-1} is a locality map then follows from combining together the fact that $\tilde{\Delta}^{(n-1)}$ is a locality map (which, as already pointed out in Section 8.2, follows from ρ and Δ being locality maps), that the identity permutation is one term of the right-hand-side of the above equation, and the definitions of the locality relations on $\mathcal{T}_{\top}(\text{Prim}(H))$ and $U_{\top}(\text{Prim}(H))$.

- Then by Proposition 9.1, $U_{\top}(\text{Prim}(H))$ is a Hopf algebra, and since ϕ is a morphism of graded locality bialgebras and an isomorphism, it is an isomorphism of locality Hopf algebras. \square

9.3 Consequences of the Milnor-Moore theorem

We end the paper with some useful consequences of the locality Milnor-Moore Theorem.

Throughout this paragraph, we assume that for any locality Hopf algebra H under consideration, the set $\text{Prim}(H)$ of its primitive elements, viewed as a locality Lie algebra, satisfies the conjectural statements 7.6 and 6.19.

It follows from the locality Milnor-Moore Theorem 9.5, that the locality relation on a graded, connected, cocommutative Hopf algebra is entirely determined by the locality relation on its primitive elements.

More precisely:

Corollary 9.6. Let $(H_1, m_1, \Delta_1, \top_1)$ and $(H_2, m_2, \Delta_2, \top_2)$ be two graded, connected, cocommutative locality Hopf algebras. Then if

$$(\text{Prim}(H_1), [\cdot, \cdot]_1, \tilde{\top}_1) \simeq (\text{Prim}(H_2), [\cdot, \cdot]_2, \tilde{\top}_2)$$

as locality Lie algebras (here $[\cdot, \cdot]_i$ is the antisymmetrisation of the product m_i) and we have set $\tilde{\top}_i := \top_i \cap (\text{Prim}(H_i) \times \text{Prim}(H_i))$, then

$$(H_1, m_1, \Delta_1, \top_1) \simeq (H_2, m_2, \Delta_2, \top_2)$$

as locality Hopf algebras.

Proof. By the universality property of universal locality algebras, the isomorphism $(\text{Prim}(H_1), [\cdot, \cdot]_1, \tilde{\top}_1) \simeq (\text{Prim}(H_2), [\cdot, \cdot]_2, \tilde{\top}_2)$ implies that the universal enveloping algebras of these Lie algebras are isomorphic as locality algebras. As a result of the construction of their respective coproducts (presented in Proposition 9.1), they are isomorphic as locality Hopf algebras. The result then follows from Theorem 9.5. \square

This further leads to the observation that locality Hopf algebras are not generally speaking ordinary locality Hopf algebras with an "added" locality relation. In other words, one cannot simply "turn on" locality, at least when the locality Milnor-Moore theorem applies.

Corollary 9.7. Let (H, m, Δ) be a graded, connected, cocommutative Hopf algebra. The trivial locality relation $\top = H \times H$ is the only locality relation \top on H such that $(H, \top, m|_{\top}, \Delta)$ is a locality Hopf algebra.

Proof. We proceed by contradiction, assuming such a non-trivial locality relation \top exists.

Since we saw that locality on H is determined by the locality on primitive elements, if \top is not the trivial locality relation, there are primitive elements a and b such that $(a, b) \notin \top$. Indeed, if this were not true, then $\top|_{\text{Prim}(H) \times \text{Prim}(H)} = \text{Prim}(H) \times \text{Prim}(H)$ would be the trivial locality on $\text{Prim}(H)$ and hence on H thanks to the locality Milnor-Moore theorem.

Let a and b be primitive elements such that $(a, b) \notin \top$. Setting $y = m(a, b)$, on the one hand $y \in (H, \top)$ implies that $\Delta(y) \subset H \otimes H$. On the other hand

$$\Delta(y) = \Delta(a)\Delta(b) = (a \otimes 1 + 1 \otimes a)(b \otimes 1 + 1 \otimes b) = y \otimes 1 + a \otimes b + b \otimes a + 1 \otimes y \notin H \otimes_{\top} H.$$

But this contradicts the inclusion $\Delta(H) \subset H \otimes_{\top} H$, which follows from the fact that the coproduct Δ of the Hopf algebra coincides with the locality coproduct of the locality Hopf algebra. \square

Theorem 9.5 also allows to describe examples of locality Hopf algebras.

Example 9.8. With the notations of Example 8.12, let P_V be the subspace of primitive elements of $\mathcal{T}_{\top}(V)$. Endowed with the locality Lie bracket induced by the commutator, namely $[a, b] = a \otimes b - b \otimes a$ whenever $(a, b) \in \top_{\otimes}$, P_V is a locality lie algebra. It then follows from Theorem 9.5 that

$$(\mathcal{T}_{\top}(V), \top_{\otimes}) \simeq (\mathcal{U}_{\top}(P_V), \top_U),$$

this last isomorphism is an isomorphism of locality Hopf algebras.

Theorem 9.5 allows to build sub-locality Hopf algebras from sub-locality Lie algebras.

Corollary 9.9. Let (H, \top) be a graded, connected, cocommutative locality Hopf algebra whose set $\mathfrak{g} := \text{Prim}(H)$ of primitive elements obeys the conjectural statements 6.19 and 7.6. There is a one-to-one correspondence

$$((H, \top) \supset (H', \top')) \longleftrightarrow ((\mathfrak{g}', \top') \subset (\mathfrak{g}, \top)) \quad (64)$$

between graded, connected, cocommutative sub-locality Hopf algebras of H and sub-locality Lie algebras of $(\mathfrak{g} = \text{Prim}(H), \top)$ where the localities \top and \top' on the r.h.s. are the restrictions of the ones of the l.h.s.

Proof. • Let \mathfrak{g}' be a subset of $\mathfrak{g} = \text{Prim}(H)$ with locality $\top' \subset \top$ so that $(\mathfrak{g}', \top', [.,.]|_{\top'})$ is sub-locality Lie algebra of $(\mathfrak{g}, \top, [.,.]|_{\top})$. Then by Proposition 9.2, $U_{\top'}(\mathfrak{g}')$ is a sub-locality algebra of $U_{\top}(\mathfrak{g})$. It is therefore isomorphic to a sub-locality Hopf algebra $(H', \top') \simeq U_{\top'}(\mathfrak{g}')$ of $(H, \top) \simeq U_{\top}(\mathfrak{g})$.

• Conversely, let H' be a sub-locality Hopf algebra of H with locality $\top' \subseteq \top$. Then $\mathfrak{g} := \text{Prim}(H') \subseteq \mathfrak{g} = \text{Prim}(H)$. By Proposition 8.14, setting $\top'' := \top' \cap (\mathfrak{g}' \times \mathfrak{g}')$, then $(\mathfrak{g}', \top'', [.,.]|_{\top''})$ is a sub-locality Lie algebra of \mathfrak{g} . Thus we have a map from sub-locality Hopf algebras of H and sub-locality Lie algebras of \mathfrak{g} .

• The two maps built above are inverse of each other by construction, which proves the corollary. \square

Remark 9.10. It follows from Corollary 9.7 that $H = H'$ and $\mathfrak{g} = \mathfrak{g}'$ implies $\top = \top'$. Note that this statement is trivially satisfied in the usual (non-locality) setup where $\top = \top'$ is the trivial locality. The subsequent Proposition 9.13 illustrates the case when $\mathfrak{g} = \mathfrak{g}'$ but $H \neq H'$, which is specific to the locality setup, since it cannot occur in the non-locality setup due to the Milnor-Moore theorem.

We apply Corollary 9.9 to the Hopf algebra of rooted forests. Let us start by recalling some classical combinatorial concepts [Foi2, Kre].

Definition 9.11. • An **admissible cut** of a rooted forest F is a subset c of edges of F such that any path from one the roots of F to a leaf of F meet c at most once. We write $\text{Adm}(F)$ the set of admissible cuts of F . For $c \in \text{Adm}(F)$, we write $R_c(F)$ the subforest of F below the c and $T_c(F)$ the subforest of F above the cut c . Notice that we have $|V(F)| = V(R_c(F)) + |V(T_c(F))|$.

- For F_1, F_2 and F three rooted forests, we set

$$n(F_1, F_2, F) := |\{c \in \text{Adm}(F) | R_c(F) = F_1 \wedge T_c(F) = F_2\}|.$$

Notice that for F_1 and F_2 two given rooted forests, $n(F_1, F_2, F) = 0$ except for a finite number of rooted forests F .

- We define the product of two rooted forests F_1 and F_2 by

$$F_1 * F_2 = \sum_{F \in \mathcal{F}} n(F_1, F_2, F) F.$$

It is well-defined since the sum contains finitely many non-zero terms.

- We further define the coproduct Δ_* by its action on rooted forests $F = T_1 \cdots T_n$ defined as

$$\Delta_*(F = T_1 \cdots T_n) = \sum_{I \subseteq [n]} T_I \otimes T_{[n] \setminus I}$$

with, for $I \subseteq [n]$, we have $T_I := \prod_{i \in I} T_i$.

$(\mathcal{F}, *, \Delta_*)$ is a Hopf algebra, which is the dual of the Connes-Kreimer Hopf algebra, sometimes called the Grossman-Larson Hopf algebra to which it is isomorphic, see [Pan] (with corrections in [Hof, Foi2]). The Grossman-Larson algebra, whose primitive elements are rooted trees, was introduced in [GL1, GL2, GL3].

In order to introduce locality, we first decorate the rooted forests. Recall that, for a set Ω , a **Ω -decorated forest** is a pair (F, d_F) with F a forest and $d_F : V(F) \rightarrow \Omega$ a map. We often omit the d_F to lighten the notation. We also write \mathcal{F}_Ω the set of Ω -decorated forests. The notions of Definition 9.11 easily generalise to decorated forests.

Definition 9.12. [CGPZ2, Definition 3.1] Let (Ω, \top) be a locality set. A **properly Ω -decorated forest** is a Ω -decorated forest (F, d_F) such that any disjoint pair of vertices of F are decorated by independent elements of Ω :

$$\forall (v_1, v_2) \in V(F) \times V(F), v_1 \neq v_2 \implies d_F(v_1) \top d_F(v_2).$$

We write $\mathcal{F}_\Omega^{\text{prop}}$ the set of properly Ω -decorated forests. We endow $\mathcal{F}_\Omega^{\text{prop}}$ with a locality relation $\top_{\mathcal{F}_\Omega}$ induced from the relation \top on Ω :

$$(F_1, d_1) \top_{\mathcal{F}_\Omega} (F_2, d_2) :\iff \forall (v_1, v_2) \in V(F_1) \times V(F_2), d_1(v_1) \top d_2(v_2).$$

The Hopf algebra $(\mathcal{F}, *|_{\top_{\mathcal{F}_\Omega}}, \Delta_*)$ induces a locality Hopf algebra structure on $\mathcal{F}_\Omega^{\text{prop}}$:

Proposition 9.13. *Given a locality set (Ω, \top) , the quadruple $(\mathcal{F}_\Omega^{\text{prop}}, \top_{\mathcal{F}_\Omega}, *|_{\top_{\mathcal{F}_\Omega}}, \Delta_*)$ is a graded, connected, cocommutative locality Hopf algebra equipped with the product $*|_{\top_{\mathcal{F}_\Omega}}$ given by the restriction of the $*$ product of Definition 9.11 to the graph $\top_{\mathcal{F}_\Omega}$ of the locality relation.*

Proof. Applying Corollary 9.9 to the locality Lie algebra $(\mathfrak{g} := \text{Prim}(\mathcal{F}_\Omega^{\text{prop}}), \top_{\text{triv}})$ equipped with the trivial locality relation $\top_{\text{triv}} := \mathfrak{g} \times \mathfrak{g}$ and $(\mathfrak{g}' := \text{Prim}(\mathcal{F}_\Omega^{\text{prop}}), \top_{\mathcal{F}_\Omega})$ shows that $\mathcal{F}_\Omega^{\text{prop}} := \mathcal{U}_{\top_{\mathcal{F}_\Omega}}(\mathfrak{g}')$ is a locality sub-Hopf algebra of $(\mathcal{F}_\Omega = \mathcal{U}_{\top_{\text{triv}}}(\mathfrak{g}), \top_{\text{triv}})$. \square

The locality Milnor-Moore theorem 9.5 also provides refinements of properties in the ordinary setup, here a decomposition involving mutually independent arguments.

Corollary 9.14. Given a locality set (Ω, \top) , any properly Ω -decorated rooted forest $F = T_1 \cdots T_n$ can be expressed as a linear combination of $*$ -products of finitely many pairwise independent properly Ω -decorated rooted trees $t_j^{(i)}$:

$$F = T_1 \cdots T_n = \sum_{n=1}^N \alpha_n t_1^{(n)} * \cdots * t_{p_n}^{(n)}$$

for $\alpha_1, \dots, \alpha_N \in \mathbb{R}$.

Proof. For the sake of simplicity, throughout the proof, we set $\top := \top_{\mathcal{F}_\Omega}$.

Let $\phi : U_\top(\text{Prim}(\mathcal{F}_\Omega^{\text{prop}})) \rightarrow \mathcal{F}_\Omega^{\text{prop}}$ be the isomorphism of locality Hopf algebra given by Theorem 9.5. Since $U_\top(\text{Prim}(\mathcal{F}_\Omega^{\text{prop}}))$ is the set Ω -decorated rooted trees, for any such tree t we have $\phi([t]) = t$.

The map ϕ being an isomorphism, for any properly Ω -decorated rooted forest $F = T_1 \cdots T_n$ there is an element $\sum_{n=1}^N \alpha_n t_1^{(n)} \cdots t_{p_n}^{(n)} \in U_\top(\text{Prim}(\mathcal{F}_\Omega^{\text{prop}}))$ (where we write \cdot for the product of $U_\top(\text{Prim}(\mathcal{F}_\Omega^{\text{prop}}))$) such that

$$F = T_1 \cdots T_n = \phi \left(\sum_{n=1}^N \alpha_n t_1^{(n)} \cdots t_{p_n}^{(n)} \right) = \sum_n \alpha_n \phi([t_1^{(n)}]) * \cdots * \phi([t_{p_n}^{(n)}]) = \sum_{n=1}^N \alpha_n t_1^{(n)} * \cdots * t_{p_n}^{(n)},$$

where we have used that ϕ is a morphism of algebras. □

Appendix: Alternative locality tensor product

For the sake of completeness we provide an alternative construction of the locality tensor product already suggested on [CGPZ1]. As in [CGPZ1], in practice, we choose to work with Definition 1.9 since the alternative tensor product is in general not a subspace of the usual (non-locality) tensor product. Notice that this alternative locality tensor product induces an alternative definition of \top_\times -bilinearity as mentioned in the sequel.

Alternatively to $I_{\text{bil}}(V)$ defined in Equations (3) to (6), we can consider the linear subspace $I_{\text{bil},\top} \subseteq \mathbb{K}(V \times_\top W)$ generated by all elements in $V \times_\top V$ of the forms (3) to (6) such that each argument in the linear combinations (3)–(6), lies in \top . In some cases $I_{\text{bil},\top}$ will coincide with $I_{\text{bil}} \cap \mathbb{K}(\top)$ but as we see in the following, this does not generally hold.

Example A.1. Consider the locality vector space $V = \mathbb{R}^2$,

$\top = \mathbb{R}^2 \times \{0\} \cup \{0\} \times \mathbb{R}^2 \cup \mathbb{K}(e_1 + e_2) \times \mathbb{K}(e_1) \cup \mathbb{K}(e_1 + 2e_2) \times \mathbb{K}(e_2) \cup \mathbb{K}(e_1) \times \mathbb{K}(e_1 + e_2) \cup \mathbb{K}(e_2) \times \mathbb{K}(e_1 + 2e_2)$
and the element of $\mathbb{K}(V \times_\top V)$

$$y = (-e_1 - e_2, e_1) + (-e_1 - 2e_2, e_2) + (e_1, e_1 + e_2) + (e_2, e_1 + 2e_2).$$

Now we can write $(-e_1 - e_2, e_1) = \left((-e_1 - e_2, e_1) + (e_1 + e_2, e_1) \right) - (e_1 + e_2, e_1) =: y_1 - (e_1 + e_2, e_1)$ with $y_1 \in I_{\text{bil}}$. Writing the same type of expansions for $(-e_1 - 2e_2, e_2)$, $(e_1, e_1 + e_2)$ and $y = y_1 + y_2 + y_3 + y_4 - (e_1 + e_2, e_1) - (e_1 + 2e_2, e_2) + (e_1, e_1) + (e_1, e_2) + (e_2, e_1) + (e_2, 2e_2)$ with $y_i \in I_{\text{bil}}$ for $i = 1, \dots, 4$. Writing further $-(e_1 + e_2, e_1) = \left(-(e_1 + e_2, e_1) + (e_2, e_1) + (e_1, e_1) \right) - (e_2, e_1) - (e_1, e_1) =: y_5 - (e_2, e_1) - (e_1, e_1)$ with $y_5 \in I_{\text{bil}}$. Writing the same type of expansions for $-(e_1 + 2e_2, e_2)$ and $(e_2 + 2e_2)$ we obtain

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4 + y_5 - (e_2, e_1) - (e_1, e_1) + y_6 - (e_1, e_2) - (2e_2, e_2) \\ &\quad + (e_1, e_1) + (e_1, e_2) + (e_2, e_1) + y_7 + (e_2, 2e_2) \\ &= y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 - (2e_2, e_2) + (e_2, 2e_2) \\ &= y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 \end{aligned}$$

with $y_i \in I_{\text{bil}}$ for $i = 1, \dots, 8$. Thus $y \in I_{\text{bil}}$. Note that we left $\mathbb{K}(V \times_\top V)$ in the last step to get to this conclusion: $y_i \notin I_{\text{bil},\top}$ for $i = 1, \dots, 8$, thus $y \notin I_{\text{bil},\top}$.

In general only the inclusion $I_{\text{bil},\top} \subset I_{\text{bil}} \cap \mathbb{K}(V \times_\top V)$ holds, which leaves us with another possible definition of the locality tensor product.

Definition A.2. Let (V, \top) be a locality vector space, the alternative locality tensor product is defined as

$$V \otimes^\top V := \mathbb{K}(V \times_\top V) / I_{\text{bil},\top} \tag{65}$$

Example A.3. Going back to Example (A.1), the alternative locality tensor product is

$$V \otimes^\top V = \mathbb{K}\{(e_1 + e_2) \otimes e_1, (e_1 + 2e_2) \otimes e_2, e_1 \otimes (e_1 + e_2), e_2 \otimes (e_1 + 2e_2)\} \not\subseteq V \otimes V$$

It has dimension 4 like the usual tensor product since we already saw that all those elements that span it are linearly independent as elements in $V \otimes^\top V$ but is not the same as the usual one, because in the usual tensor product those 4 elements mentioned above are not linearly independent. However, the locality tensor product in this case is

$$V \otimes_\top V = \mathbb{K}\{(e_1 + e_2) \otimes e_1, (e_1 + 2e_2) \otimes e_2, e_1 \otimes (e_1 + e_2)\} \subset V \otimes V$$

it has dimension 3 since some of the tensor products involve arguments that are not linearly independent as elements of $V \otimes_\top V$, and this is a vector subspace of the usual tensor product.

Using the same methods as in the paper, one can show that this alternative tensor product also enjoys universal properties (under similar conjectural assumptions as above) using an alternative definition of \top_\times -bilinearity, namely requiring that $f(I_{\text{bil},\top}) = \{0\}$.

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