Hopf algebraic structures on mixed graphs

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Abstract

We introduce two coproducts on mixed graphs (that is to say graphs with both edges and arcs), the first one by separation of the vertices into two parts, and the second one given by contraction and extractions of subgraphs. We show that, with the disjoint union product, this gives a double bialgebra, that is to say that the first coproduct makes it a Hopf algebra in the category of righ comodules over the second coproduct.

This structures implies the existence of a unique polynomial invariants on mixed graphs compatible with the product and both coproducts: we prove that it is the (strong) chromatic polynomial of Beck, Bogart and Pham. Using the action of the monoid of characters, we relate it to the weak chromatic polynomial, as well to Ehrhart polynomials and to a polynomial invariants related to linear extensions. As applications, we give an algebraic proof of the link between the values of the strong chromatic polynomial at negative values and acyclic orientations (a result due to Beck, Blado, Crawford, Jean-Louis and Young) and obtain a combinatorial description of the antipode of the Hopf algebra of mixed graphs.

Keywords. double bialgebra; mixed graph; chromatic polynomial; Ehrhart polynomial

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Contents

Introduction

Mixed graphs are graphs with both unoriented and oriented edges. They are used for example to study scheduling problems with disjunctive (represented by unoriented edges) and precedence (represented by oriented edges) constraints [17, 17, 13]. In this context, a notion of admissible colouring, similar to the notion used for classical graphs, give a solution of the scheduling problem represented by the mixed graph. These admissible colourings can be counted according to the number of colours, which gives a chromatic polynomial [5, 4]. The aim of this text is the study of this chromatic polynomial for mixed graphs in the context of double bialgebras, as this has be done for graphs in [9] and for posets and finite topologies in [7]. A double bialgebra is a family (A, m, Δ, δ) such that:

- (A, m, δ) is a bialgebra.
- (A, m, Δ) is a bialgebra in the category of right comodules over (A, m, δ), with the coaction given by δ itself.

A simple example of such an object is $\mathbb{K}[X]$, with its usual algebra structures and its coproducts defined by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad \delta(X) = X \otimes X.$$

We proved in [7, 9] the following results: if (A, m, Δ, δ) is a double bialgebra, under a condition of connectedness of (A, m, Δ),

- There exists a unique double bialgebra morphism $\phi_0 : (A, m, \Delta, \delta) \longrightarrow (\mathbb{K}[X], m, \Delta, \delta)$, which can be explicitly described with iterations of the coproduct Δ of A and with the counit ϵ_δ of its coproduct δ.
- Let us denote by $\mathrm{Char}(A)$ the set of characters of A. It inherits a convolution product $\star$, dual to δ. We denote by $\mathrm{End}_B(A, \mathbb{K}[X])$ the set of bialgebra morphisms from (A, m, Δ) to $(\mathbb{K}[X], m, \Delta)$. Then the following defines an action of the monoid $(\mathrm{Char}(A), \star)$ on $\mathrm{End}_B(A, \mathbb{K}[X])$:

  $$\forall \lambda \in \mathrm{Char}(A),\ \forall \phi \in \mathrm{End}_B(A, \mathbb{K}[X]), \qquad \phi \leftsquigarrow \lambda = (\phi \otimes \lambda) \circ \delta.$$

  Moreover, the two following maps are bijection, inverse one from the other:

  $$\begin{cases} \mathrm{End}_B(A, \mathbb{K}[X]) & \longrightarrow & \mathrm{Char}(A) \\ \phi & \longmapsto & \epsilon_\delta \circ \phi, \end{cases} \qquad \begin{cases} \mathrm{Char}(A) & \longrightarrow & \mathrm{End}_B(A, \mathbb{K}[X]) \\ \lambda & \longmapsto & \phi_0 \leftsquigarrow \lambda, \end{cases}$$

  where here ϵ_δ is the counit of $(\mathbb{K}[X], m, \delta)$, which send any $P \in \mathbb{K}[X]$ onto $P(1)$.
- When (A, m, Δ) is a graded and connected bialgebra, then under a technical condition on δ, the set of homogeneous bialgebra morphisms from (A, m, Δ) to $(\mathbb{K}[X], m, \Delta)$ is in bijection with the dual of the homogeneous component A_1 of A of degree 1.
- If (A, m, Δ) is a Hopf algebra, then its antipode is given by

  $$S = (\epsilon_\delta^{*-1} \otimes \mathrm{Id}) \circ \delta,$$

  where ϵ_δ^{*-1} is the inverse of the counit ϵ_δ of δ for the convolution product $*$ associated to Δ. Moreover, for any $a \in A$,

  $$\epsilon_\delta^{-1}(a) = \phi_0(a)(-1).$$

We apply here these results on mixed graphs. In the first section, we define a structure of double bialgebras on mixed graphs. We work in the frame of species and use the formalism built in [12] of contraction-extraction coproduct. We first give in Proposition 1.4 the species of mixed graphs $\mathbf{G}$ a bialgebra structure in the category of species (what is commonly called a twisted bialgebra structure), and then a contraction-extraction coproduct in Proposition 1.6. Consequently, applying the bosonic Fock functor of [1], we obtain a double bialgebra of mixed graphs $\mathcal{F}[\mathbf{G}]$, and more generally, for any commutative and cocommutative bialgebra V, a double bialgebra $\mathcal{F}_V[\mathbf{G}]$ of mixed graphs which vertices are decorated by elements of V. Using Loday and Ronco's rigidity theorem, we prove that $(\mathcal{F}_V[\mathbf{G}], \Delta)$ is a cofree coalgebra (Corollary 1.9).

We study certain subobjects and quotients of $\mathbf{G}$ in the second section. Obviously, simple (i.e., unoriented) graphs and oriented graphs define twisted double subbialgebras of $\mathbf{G}$, denoted respectively by $\mathbf{G}_s$ and $\mathbf{G}_o$ (Proposition 2.1). Applying the bosonic Fock functor to $\mathbf{G}_s$, we obtain again the double bialgebra of graphs of [9, 12]. We also consider the subspecies of acyclic mixed graphs $\mathbf{G}_{aco}$, which turns out to be stable under the product and the first coproduct Δ, but not on the second one. However, quotienting by non acyclic mixed graphs, $\mathbf{G}_{ac}$ can be seen as a twisted double bialgebra, quotient of $\mathbf{G}$ (Proposition 2.5). It contains a twisted double subbialgebra of oriented acyclic mixed graphs $\mathbf{G}_{aco}$, which has itself for quotient the twisted double bialgebra of topologies of [7] (Proposition 2.6).

The third part is devoted to polynomial invariants of mixed graphs, that is to say bialgebra morphisms from the bialgebra of mixed graphs $(\mathcal{F}[\mathbf{G}], m, \Delta)$ to $(\mathbb{K}[X], m, \Delta)$. We first describe the unique polynomial invariant compatible with the second coalgebraic structure of mixed graphs: it turns out to be the strong chromatic polynomial P_{chr_S} of [5], see Proposition 3.2. In other words, for any mixed graph G, for any $n \geqslant 1$, $P_{chr_S}(G)(n)$ is the number of n-valid colourations of G, that is to say maps c from the set of vertices $V(G)$ of G to $\{1, \ldots, n\}$ such that, for any pair of vertices x, y of G,

- If x and y are related by an edge of G, then $c(x) \neq c(y)$.
- If x and y are related by an arc of G, then $c(c) < c(y)$.

A notion of weak valid colouration is also defined in [5]. A weak n-valid colouration is a map c from the set of vertices $V(G)$ of G to $\{1, \ldots, n\}$ such that, for any pair of vertices x, y of G,

- If x and y are related by an edge of G, then $c(x) \neq c(y)$.
- If x and y are related by an arc of G, then $c(c) \leqslant c(y)$.

The polynomial counting the number of weak n-valid colourations of G is denoted by $P_{chr_W}(G)$. Using the the action of the monoid of characters described earlier and the character of $\mathcal{F}[\mathbf{G}]$ defined by

$$\lambda_W(G) = \begin{cases} 1 \text{ if } G \text{ is an oriented graph,} \\ 0 \text{ otherwise,} \end{cases}$$

we obtain that $\phi_{chr_W} = \phi_{chr_S} \leftsquigarrow \lambda_W$, which implies that P_{chr_w} is a bialgebra morphism from $(\mathcal{F}[\mathbf{G}], m, \Delta)$ to $(\mathbb{K}[X], m, \Delta)$, see Corollary 3.4. Finally, using the correspondence between homogeneous polynomial invariants and elements of $\mathcal{F}[\mathbf{G}]_1$, we construct a homogenous bialgebra morphism ϕ_0 from $(\mathcal{F}[\mathbf{G}], m, \Delta)$ to $(\mathbb{K}[X], m, \Delta)$, related to the number of linear extensions (Corollary 3.5) and to a character λ_0. When V is a commutative and cocommutative bialgebra, similar results are given for $\mathcal{F}_V[\mathbf{G}]$, where $\mathbb{K}[X]$ is replaced by a double bialgebra of quasishuffles, see Theorem 3.7 and Corollary 3.8. After the determination of invertible characters of $\mathcal{F}[\mathbf{G}]$ for the convolution product $\star$ dual to δ, we prove that both characters λ_0 and λ_W are invertible, which allows to express P_{chr_S} in terms of P_{chr_W} or ϕ_0, with the help of certain characters ν_W and μ_W (Proposition 3.13). This allows to give a formula for the leading monomial of $P_{chr_W}(G)$ and $P_{chr_S}(G)$ in Corollary 3.14, with coefficients (in the case of $P_{chr_S}(G)$) related to Murua's coefficients [15].

In section fourth, we give an algebraic proof of the result [5], which gives a combinatorial interpretation of $P_{chr_S}(G)(-1)$ in terms of acyclic orientations. We firstly introduce a surjective double bialgebra morphism Θ from $\mathcal{F}[\mathbf{G}]$ to the double bialgebra of acyclic oriented mixed graphs $\mathcal{F}[\mathbf{G}_{aco}]$ in Theorem 4.2. The unicity of the double bialgebra morphism to $\mathbb{K}[X]$ immediately implies for example that the strong chromatic polynomial of a graph G is the sum of the strong chromatic polynomial of all its acyclic orientations (Corollary 4.3). Introducing two one-parameter families of characters of $\mathcal{F}[\mathbf{G}]$ in Proposition 4.4, we introduce two new polynomial invariants, which turn out to be Ehrhart polynomials and satisfy a duality principle (Corollary 4.7) on acyclic mixed graphs. Mixing this duality principle with the morphism Θ, we obtain a new proof that for any mixed graph, $P_{chr_S}(G)(-1)$ is the number of acyclic orientations of G, up to a sign (Corollary 4.9). This allows to give a formula for the antipode of $(\mathcal{F}[\mathbf{G}], m, \Delta)$ involving the number of acyclic orientations of G, see Corollary 4.10.

The last section is devoted to combinatorial interpretations of special characters of mixed graphs. We give an algebraic proof of a result of [5] in an algebraic way (Proposition 5.1 and Corollary 5.2) about values of the weak chromatic polynomial on negative integers (for totally mixed graphs only), and we give a combinatorial interpretation of $\nu_W(G)$ when G is a simple graph or an oriented graph (Proposition 5.3), where ν_W is the inverse of λ_W for the convolution product $\star$.

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Notations 0.1.

1. We denote by $\mathbb{K}$ a commutative field of characteristic zero. Any vector space in this field will be taken over $\mathbb{K}$.

2. For any $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, \ldots, n\}$. In particular, $[0] = \emptyset$.

3. If (C, Δ) is a (coassociative but not necessarily counitary) coalgebra, we denote by $\Delta^{(n)}$ the n-th iterated coproduct of C: $\Delta^{(1)} = \Delta$ and if $n \geqslant 2$,
$$\Delta^{(n)} = \left(\Delta \otimes \mathrm{Id}^{\otimes(n-1)}\right) \circ \Delta^{(n-1)} : C \longrightarrow C^{\otimes(n+1)}.$$

4. If (B, m, Δ) is a bialgebra of unit 1_B and of counit ε_B, let us denote by $B_+ = \mathrm{Ker}(\varepsilon_B)$ its augmentation ideal. We define a coproduct on B_+ by
$$\forall x \in B_+, \qquad \tilde{\Delta}(x) = \Delta(x) - x \otimes 1_B - 1_B \otimes x.$$
Then $(B_+, \tilde{\Delta})$ is a coassociative (not necessarily counitary) coalgebra.

5. Let $\mathbf{P}$ be a species. For any finite set X, the vector space associated to X by $\mathbf{P}$ is denoted by $\mathbf{P}[X]$. For any bijection $\sigma : X \longrightarrow Y$ between two finite sets, the linear map associated to σ by $\mathbf{P}$ is denoted by $\mathbf{P}[\sigma] : \mathbf{P}[X] \longrightarrow \mathbf{P}[Y]$. The Cauchy tensor product of species is denoted by $\otimes$: if $\mathbf{P}$ and $\mathbf{Q}$ are two species, for any finite set X,
$$\mathbf{P} \otimes \mathbf{Q}[X] = \bigoplus_{X = Y \sqcup Z} \mathbf{P}[Y] \otimes \mathbf{Q}[Z].$$
If $\sigma : X \longrightarrow Y$ is a bijection between two finite sets, then
$$\mathbf{P} \otimes \mathbf{Q}[\sigma] = \bigoplus_{X = Y \sqcup Z} \mathbf{P}[\sigma_{|Y}] \otimes \mathbf{Q}[\sigma_{|Z}].$$

A twisted algebra (resp. coalgebra, bialgebra) is an algebra (resp. coalgebra, bialgebra) in the symmetric monoidal category of species with the Cauchy tensor product. We refer to [8, 12] for details and notations on algebras, coalgebras and bialgebras in the category of species.

6. Let V be a vector space. The V-coloured Fock functor $\mathcal{F}_V$, defined in [12, Definition 3.2], sends any species $\mathbf{P}$ to

$$\begin{aligned}
\mathcal{F}_V[\mathbf{P}] &= \bigoplus_{n=0}^{\infty} \mathrm{coInv}(V^{\otimes n} \otimes \mathbf{P}[n]) \\
&= \bigoplus_{n=0}^{\infty} \frac{V^{\otimes n} \otimes \mathbf{P}[n]}{\mathrm{Vect}(v_1 \ldots v_n \otimes \mathbf{P}[\sigma](p) - v_{\sigma(1)} \ldots v_{\sigma(n)} \otimes p \mid \sigma \in \mathfrak{S}_n,\ p \in \mathbf{P}[n],\ v_1, \ldots, v_n \in V)} \\
&= V^{\otimes n} \otimes_{\mathfrak{S}_n} \mathbf{P}[n].
\end{aligned}$$

When $V = \mathbb{K}$, we obtain the bosonic Fock functor of [1]:

$$\mathcal{F}[\mathbf{P}] = \bigoplus_{n=0}^{\infty} \mathrm{coInv}(\mathbf{P}[n]) = \bigoplus_{n=0}^{\infty} \frac{\mathbf{P}[n]}{\mathrm{Vect}(\mathbf{P}[\sigma](p) - p \mid \sigma \in \mathfrak{S}_n,\ p \in \mathbf{P}[n])}.$$

1 The species of mixed graphs

1.1 Mixed graphs

Definition 1.1. *1. A* mixed graph *is a triple $G = (V(G), E(G), A(G))$ where:*

(a) $V(G)$ is a finite set, called the set of vertices of G,

(b) $E(G)$ is a subset of $\{\{x,y\} \subseteq V(G),\ x \neq y\}$, called the set of edges of G,

(c) $A(G)$ is a subset of $\{(x,y) \in X^2,\ x \neq y\}$, called the set of arcs of G,

such that, for any $x, y \in V(G)$, with $x \neq y$,

$$\begin{aligned}
\{x,y\} \in E(G) &\Longrightarrow (x,y) \notin A(G) \text{ and } (y,x) \notin A(G), \\
(x,y) \in A(G) &\Longrightarrow \{x,y\} \notin E(G).
\end{aligned}$$

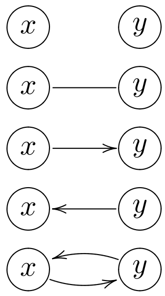
2. For any finite set X, we denote by $\mathscr{G}[X]$ the set of mixed graphs G such that $V(G) = X$. This defines a set species of mixed graphs. The vector space generated by $\mathscr{G}[X]$ is denoted by $\mathbf{G}[X]$. This defines a species $\mathbf{G}$.

3. A mixed graph is an oriented graph *if $E(G) = \emptyset$: this defines a subspecies of $\mathscr{G}$ denoted by $\mathscr{G}_o$, and a subspecies of $\mathbf{G}$ denoted by $\mathbf{G}_o$.*

4. A mixed graph G is a simple graph *if $A(G) = \emptyset$: this defines a set subspecies of $\mathscr{G}$ denoted by $\mathscr{G}_s$, and a subspecies of $\mathbf{G}$ denoted by $\mathbf{G}_s$.*

Example 1.1. Let $X = \{x, y\}$ be a set with two elements. There are five elements in $\mathscr{G}[X]$, which we graphically represent on the right:

$$\begin{aligned}
G_1 &= (X, \emptyset, \emptyset), \\
G_2 &= (X, \{\{x,y\}\}, \emptyset), \\
G_3 &= (X, \emptyset, \{(x,y)\}), \\
G_4 &= (X, \emptyset, \{(y,x)\}), \\
G_5 &= (X, \emptyset, \{(x,y), (y,x)\}),
\end{aligned}$$



Moreover,

$$\mathscr{G}_o[X] = \{G_1, G_3, G_4, G_5\}, \qquad \mathscr{G}_s[X] = \{G_1, G_2\}.$$

Remark 1.1. In a mixed graph G, for any pair of vertices $\{x, y\}$ of G, there are five possibilities to define edges or arcs between x and y. Hence, if X is of cardinality n, then

$$|\mathscr{G}[X]| = 5^{\frac{n(n-1)}{2}}.$$

Similarly,

$$|\mathscr{G}_o[X]| = 4^{\frac{n(n-1)}{2}}, \qquad |\mathscr{G}_s[X]| = 2^{\frac{n(n-1)}{2}}.$$

These gives respectively sequences A109345, A053763 and A006125 of the OEIS [16].

Notations 1.1. Let G be a mixed graph and $x, y \in V(G)$. We shall write $x \xrightarrow{G} y$ if $(x, y) \in A(G)$ and $x \overset{G}{-} y$ if $\{x, y\} \in E(G)$.

Definition 1.2. *Let G be a mixed graph.*

1. *An oriented path in G is a finite sequence $P = (x_0, \ldots, x_n)$ of vertices of G such that for any $i \in \{0, \ldots, n-1\}$, $x_i \xrightarrow{G} x_{i+1}$. The vertices x_0 and x_n are respectively the beginning and the end of P.*

2. *A mixed path in G is a finite sequence $P = (x_0, \ldots, x_n)$ of vertices of G such that for any $i \in \{0, \ldots, n-1\}$, $x_i \xrightarrow{G} x_{i+1}$ or $x_i \overset{G}{-} x_{i+1}$. The vertices x_0 and x_n are the extremities of P.*

3. *A path in G is a finite sequence $P = (x_0, \ldots, x_n)$ of vertices of G such that for any $i \in \{0, \ldots, n-1\}$, $x_i \xrightarrow{G} x_{i+1}$ or $x_{i+1} \xrightarrow{G} x_i$ or $x_i \overset{G}{-} x_{i+1}$. The vertices x_0 and x_n are the extremities of P. The mixed graph G is connected if for any vertices $x, y \in V(G)$, there exists a path of extremities x and y.*

Example 1.2. Let us consider the mixed graphs of Example 1.1. The connected ones are G_2, G_3, G_4 and G_5.

1.2 The twisted bialgebra of mixed graphs

We now define a product and a coproduct on the species $\mathbf{G}$. Let X, Y be two finite sets, $G \in \mathscr{G}[X]$ and $H \in \mathscr{G}[Y]$. The mixed graph $GH \in \mathscr{G}[X \sqcup Y]$ is defined by

$$V(GH) = V(G) \sqcup V(H), \qquad E(GH) = E(G) \sqcup E(H), \qquad A(GH) = A(G) \sqcup A(H).$$

In other words, GH is the disjoint union of G and H. This product is bilinearly extended to $\mathbf{G}$. it is clearly associative, commutative and has a unit, which is the empty graph $1 \in \mathscr{G}[\emptyset]$. With this product, $\mathbf{G}$ is a commutative twisted algebra. Note that $\mathbf{G}_s$ and $\mathbf{G}_o$ are subalgebras of $\mathbf{G}$.

Definition 1.3. *Let G be a mixed graph and let $I \subseteq V(G)$.*

1. *The mixed graph $G_{|I}$ is defined by*

$$\begin{aligned} V(G_{|I}) &= I, \\ E(G_{|I}) &= \{\{x, y\} \in E(G),\ x, y \in I\}, \\ A(G_{|I}) &= \{(x, y) \in A(G),\ x, y \in I\}. \end{aligned}$$

2. *We shall say that I is an ideal of G if*

$$\forall x, y \in V(G), \qquad (x \in I \text{ and } x \xrightarrow{G} y) \Longrightarrow (y \in I).$$

Proposition 1.4. *We define a coproduct Δ on the species $\mathbf{G}[X]$ in the following way: for any finite sets X and Y, for any mixed graph $G \in \mathscr{G}[X \sqcup Y]$,*

$$\Delta_{X,Y}(G) = \begin{cases} G_{|X} \otimes G_{|Y} \text{ if } Y \text{ is an ideal of } G, \\ 0 \text{ otherwise.} \end{cases}$$

Together with the product defined earlier, this coproduct makes $\mathbf{G}$ *a twisted bialgebra.*

Proof. Let us first prove the coassociativity of Δ. Let X, Y and Z be finite sets and $G \in \mathscr{G}[X \sqcup Y \sqcup Z]$. Then

$$(\Delta_{X,Y} \otimes \mathrm{Id}) \circ \Delta_{X \sqcup Y, Z}(G) = \begin{cases} G_{|X} \otimes G_{|Y} \otimes G_{|Z} \\ \qquad \text{if } Z \text{ is an ideal of } G \text{ and } Y \text{ is an ideal of } G_{|X \sqcup Y}, \\ 0 \text{ otherwise.} \end{cases}$$

$$(\mathrm{Id} \otimes \Delta_{Y,Z}) \circ \Delta_{X, Y \sqcup Z}(G) = \begin{cases} G_{|X} \otimes G_{|Y} \otimes G_{|Z} \\ \qquad \text{if } Y \sqcup Z \text{ is an ideal of } G \text{ and } Z \text{ is an ideal of } G_{|Y \sqcup Z}, \\ 0 \text{ otherwise.} \end{cases}$$

Moreover,

$$\begin{aligned} & Z \text{ is an ideal of } G \text{ and } Y \text{ is an ideal of } G_{|X \sqcup Y} \\ \iff & \left(\forall (x,y) \in V(G)^2,\ x \xrightarrow{G} y \Longrightarrow (x,y) \notin (Y \times X) \sqcup (Z \times X) \sqcup (Z \times Y)\right) \\ \iff & Y \sqcup Z \text{ is an ideal of } G \text{ and } Z \text{ is an ideal of } G_{|Y \sqcup Z}, \end{aligned}$$

so $(\Delta_{X,Y} \otimes \mathrm{Id}) \circ \Delta_{X \sqcup Y, Z} = (\mathrm{Id} \otimes \Delta_{Y,Z}) \circ \Delta_{X, Y \sqcup Z}$. As for any graph G, $\varnothing$ and $V(G)$ are ideals of G,

$$\Delta_{V(G),\varnothing}(G) = G \otimes 1, \qquad \Delta_{\varnothing, V(G)}(G) = 1 \otimes G.$$

So Δ is counitary, and the counit $\varepsilon_\Delta : \mathbf{G}[\varnothing] \longrightarrow \mathbb{K}$ sends the empty graph 1 to 1.

Let $G \in \mathscr{G}[X]$, $H \in \mathscr{G}[Y]$ and X'', Y' be sets such that $X \sqcup Y = X' \sqcup Y'$. In GH, there is no arc between any element of X and any element of Y, nor between any element of Y and any element of X. Hence, the ideals of GH are of the form $I \sqcup J$, where I is an ideal of G and J is an ideal of H. Therefore,

$$\begin{aligned} \Delta_{X',Y'}(GH) &= \begin{cases} G_{|X \cap X'} H_{|Y \cap X'} \otimes G_{|X \cap Y'} H_{|Y \cap Y'} \\ \qquad \text{if } Y' \cap X \text{ is an ideal of } G \text{ and } Y' \cap Y \text{ is an ideal of } H, \\ 0 \text{ otherwise} \end{cases} \\ &= \Delta_{X \cap X', X \cap Y'}(G) \Delta_{Y \cap X', Y \cap Y'}(H). \end{aligned}$$

Moreover, $\Delta_{\varnothing,\varnothing}(1) = 1 \otimes 1$. So Δ is a morphism of twisted algebras. $\square$

Example 1.3. With the notations of Example 1.1 (or Example 1.4 below):

	$\Delta_{\{x,y\},\varnothing}$	$\Delta_{\{x\},\{y\}}$	$\Delta_{\{y\},\{x\}}$	$\Delta_{\varnothing,\{x,y\}}$
G_1	$G_1 \otimes 1$	$x \otimes y$	$y \otimes x$	$1 \otimes G_1$
G_2	$G_2 \otimes 1$	$x \otimes y$	$y \otimes x$	$1 \otimes G_2$
G_3	$G_3 \otimes 1$	$x \otimes y$	0	$1 \otimes G_3$
G_4	$G_4 \otimes 1$	0	$y \otimes x$	$1 \otimes G_4$
G_5	$G_5 \otimes 1$	0	0	$1 \otimes G_5$

We can now consider the twisted bialgebra $\mathbf{G}' = \mathbf{G} \circ \mathbf{Com}$, as defined in [12, Proposition 2.1]:

- For any finite set X, $\mathbf{G}'[X]$ is the vector space generated by the set $\mathscr{G}'[X]$ of mixed graphs G such that $V(G)$ is a partition of X.

- For any finite sets X and Y and for any $(G, H) \in \mathscr{G}'[X] \times \mathscr{G}'[Y]$, $m_{X,Y}(GH)$ is the disjoint union of G and H.

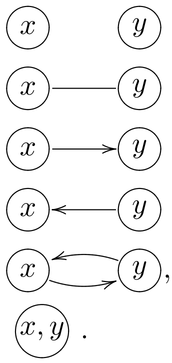
- For any finite sets X and Y and for any $G \in \mathscr{G}'[X]$,

$$\Delta_{X,Y}(G) = \begin{cases} G_{|X} \otimes G_{|Y} \text{ if } Y \text{ is the union of vertices of } X \text{ forming an ideal of } G, \\ 0 \text{ otherwise.} \end{cases}$$

We define similarly $\mathbf{G}'_o = \mathbf{G} \circ \mathbf{Com}$ and $\mathbf{G}'_s = \mathbf{G} \circ \mathbf{Com}$, and the set species $\mathscr{G}_o$ and $\mathscr{G}_s$.

Example 1.4. If $X = \{x, y\}$, there are six elements in $\mathscr{G}'[X]$, which we graphically represent on the right:

$$\begin{aligned}
G_1 &= (X, \varnothing, \varnothing), \\
G_2 &= (X, \{\{x, y\}\}, \varnothing), \\
G_3 &= (X, \varnothing, \{(x, y)\}), \\
G_4 &= (X, \varnothing, \{(y, x)\}), \\
G_5 &= (X, \varnothing, \{(x, y), (y, x)\}), \\
G_6 &= (\{\{x, y\}\}, \varnothing, \varnothing),
\end{aligned}$$



The coproduct of G_6 is given by the following:

	$\Delta_{\{x,y\},\varnothing}$	$\Delta_{\{x\},\{y\}}$	$\Delta_{\{y\},\{x\}}$	$\Delta_{\varnothing,\{x,y\}}$
G_6	$G_6 \otimes 1$	0	0	$1 \otimes G_6$

The cardinality of $\mathscr{G}'[X]$ can be computed with the help of Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$, which count the number of partitions of a set with n elements in k parts:

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0} (-1)^{k-j} \binom{k}{j} j^n.$$

If $|X| = n$,

$$|\mathscr{G}'[X]| = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} 5^{\frac{k(k-1)}{2}}.$$

Similarly,

$$|\mathscr{G}'_o[X]| = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} 4^{\frac{k(k-1)}{2}}, \qquad |\mathscr{G}'_s[X]| = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} 2^{\frac{k(k-1)}{2}}.$$

This gives the following array:

$\|X\|$	$\|\mathscr{G}'[X]\|$	$\|\mathscr{G}'_o[X]\|$	$\|\mathscr{G}'_s[X]\|$
1	1	1	1
2	6	5	3
3	141	77	15
4	16411	4509	127
5	9925076	1091197	1895
6	30665089531	1089742589	53071
7	477479400037941	4420743343165	2953575
8	37266262553005215136	72181026063598461	337064047

The sequence of cardinalities of $\mathscr{G}'_s[X]$ is entry A335390 of the OEIS [16].

1.3 Contraction-extraction on mixed graphs

We here shall the formalism of contraction-extraction coproducts of [12]. Recall that for any finite set X, $\mathcal{E}[X]$ is the set of equivalence relations on X. If δ is a contraction-extraction coproduct on a species $\mathbf{P}$, then for any finite set X and any $\sim\in\mathcal{E}[X]$, then $\delta_\sim$ sends $\mathbf{P}[X]$ onto $\mathbf{P}[X/\sim]\otimes\mathbf{P}[X]$.

Definition 1.5. *Let $G\in\mathscr{G}[X]$ and $\sim\in\mathcal{E}[X]$.*

1. *We define a mixed graph $G\mid\sim\in\mathscr{G}[X]$ by*

$$\begin{aligned}
V(G\mid\sim)&=V(G),\\
E(G\mid\sim)&=\{\{x,y\}\in E(G)\mid x\sim y\},\\
A(G\mid\sim)&=\{(x,y)\in A(G)\mid x\sim y\}.
\end{aligned}$$

   In other words, $G\mid\sim$ is obtained from G by deleting all the edges or arcs which extremities are not equivalent; or equivalently, $G\mid\sim$ is the disjoint union of the restrictions of G to the equivalence classes of $\sim$.

2. *We define a mixed graph $G/\sim\in\mathscr{G}[X/\sim]$ by*

$$\begin{aligned}
V(G/\sim)&=X/\sim,\\
E(G/\sim)&=\{\{\mathrm{cl}_\sim(x),\mathrm{cl}_\sim(y)\}\mid\{x,y\}\in E(G),(x,y)\notin A(G),\ (y,x)\notin A(G),\ \mathrm{cl}_\sim(x)\neq\mathrm{cl}_\sim(y)\},\\
A(G/\sim)&=\{(\mathrm{cl}_\sim(x),\mathrm{cl}_\sim(y))\mid (x,y)\in A(G),\ \mathrm{cl}_\sim(x)\neq\mathrm{cl}_\sim(y)\}.
\end{aligned}$$

   In other words, $G/\sim$ is obtained from G by identifying the vertices according to $\sim$, then deleting the loops created in the process and the redundant edges, giving priority to the oriented ones.

3. *We shall say that $\sim\in\mathcal{E}^c[G]$ if for any equivalence class C of $\sim$, $G_{|C}$ is connected.*

Proposition 1.6. *We define a contraction-extraction coproduct δ on $\mathbf{G}$ by the following: for any finite set X, for any $\sim\in\mathcal{E}[X]$, for any $G\in\mathscr{G}[X]$,*

$$\delta_\sim(G)=\begin{cases}G/\sim\otimes G\mid\sim & \text{if } \sim\in\mathcal{E}^c[G],\\ 0 & \text{otherwise.}\end{cases}$$

It is compatible with the product and the coproduct in the sense of [12, Proposition 2.5]

Proof. The compatibility of δ with the species structure [12, Definition 2.2, second item] is clear. Let us prove the coassociativity of δ [12, Definition 2.2, third item]. Let X be a finite set, $\sim, \sim' \in \mathcal{E}[X]$ and $G \in \mathscr{G}[X]$.

If $\sim \leqslant \sim'$, let us prove that $\sim \in \mathcal{E}^c[G/\sim']$ and $\sim' \in \mathcal{E}^c[G]$ if, and only if, $\sim' \in \mathcal{E}^c[G\mid\sim]$ and $\sim \in \mathcal{E}^c[G]$.

$\Longrightarrow$. Let C' be a class of $\sim'$. As $\sim' \in \mathcal{E}^c[G]$, it is a connected subgraph of G. Moreover, as $\sim \leqslant \sim'$, all its elements are in the same class of $\sim$, so $G_{|C'} = (G|\sim)_{|C'}$: as a consequence, $(G|\sim)_{|C'}$ is connected, so $\sim' \in \mathcal{E}^c[G\mid\sim]$. Let C be a class of $\sim$, and $x, y \in C$. As $\sim \in \mathcal{E}^c[G/\sim']$, it is connected in $G/\sim'$: there exists a path in $G/\sim'$ from $\mathrm{cl}_{\sim'}(x)$ to $\mathrm{cl}_{\sim'}(y)$. Moreover, as $\sim' \in \mathcal{E}^c[G]$, each $\mathrm{cl}_{\sim'}(z)$ is a connected subgraph of G, so there is a path from x to y in G: $\sim \in \mathcal{E}^c[G]$.

$\Longleftarrow$. Let C be a class of $\sim$. As $\sim \in \mathcal{E}^c[G]$, any of its class is a connected subgraph of G, so by contraction is a connected subgraph of $G/\sim'$: $\sim \in \mathcal{E}^c[G/\sim']$. Let C' be a class of $\sim'$. As $\sim' \in \mathcal{E}^c[G\mid\sim]$, it is a connected subgraph of $G\mid\sim$, so also of G: $\sim \in \mathcal{E}^c[G/\sim']$.

As a conclusion,

$$\begin{aligned}
(\delta_\sim \otimes \mathrm{Id}) \circ \delta_{\sim'}(G) &= \begin{cases} (G/\sim')/\sim \otimes (G/\sim')\mid\sim \otimes G\mid\sim' & \text{if } \sim \in \mathcal{E}^c[G/\sim'] \text{ and } \sim' \in \mathcal{E}^c[G], \\ 0 \text{ otherwise} \end{cases} \\
&= \begin{cases} G/\sim \otimes (G\mid\sim)/\sim' \otimes (G\mid\sim)\mid\sim' & \text{if } \sim' \in \mathcal{E}^c[G\mid\sim] \text{ and } \sim \in \mathcal{E}^c[G], \\ 0 \text{ otherwise} \end{cases} \\
&= (\mathrm{Id} \otimes \delta_{\sim'}) \circ \delta_\sim(G).
\end{aligned}$$

If we do not have $\sim \leqslant \sim'$, then at least one class C of $\sim$ intersects two classes of $\sim'$, so intersects two connected components of $|\sim'$: we obtain that $\sim \notin \mathcal{E}^c[G\mid\sim']$. So $\delta_\sim(G\mid\sim') = 0$ and finally $(\mathrm{Id} \otimes \delta_\sim) \circ \delta_{\sim'}(G) = 0$.

Let us now study the counity [12, Definition 2.2, fourth item]. We define a species morphism $\epsilon_\delta : \mathbf{G} \longrightarrow \mathbf{Com}$ by the following: if $G \in \mathscr{G}[X]$,

$$\epsilon_\delta[X](G) = \begin{cases} 1 \text{ if } E(G) = A(G) = \emptyset, \\ 0 \text{ otherwise.} \end{cases}$$

Let $G \in \mathscr{G}[X]$ and $\sim \in \mathcal{E}[X]$. If $\sim$ is the equality of X, then $\sim \in \mathcal{E}^c[G]$, $G/\sim = G$ and $G\mid\sim$ as no edge, so $(\mathrm{Id} \otimes \epsilon_\delta[X])(G) = G$. Otherwise, either $G \notin \mathcal{E}^c[G]$ or at least one class of $\sim$ contains an edge or an arc, so $\epsilon_\delta[X](G\mid\sim) = 0$. In both cases, $(\mathrm{Id} \otimes \epsilon_\delta[X])(G) = 0$.

Let $\sim \in \mathcal{E}^c[G]$, such that $E(G/\sim) = E(G\mid\sim) = \emptyset$. If two vertices of G are related by an edge of an arc, there are necessarily equivalent, so any connected component of G is included in a single class of $\sim$. As the classes of $\sim$ are connected, $\sim$ is the relation $\sim_c$ which classes are the connected components of G. Moreover, $G/\sim_c$ has no edge nor arc, and $G\mid\sim_c = G$. Therefore,

$$\begin{aligned}
\sum_{\sim \in \mathcal{E}[X]} (\epsilon_\delta[X/\sim] \otimes \mathrm{Id}) \circ \delta_\sim(G) &= \sum_{\sim \in \mathcal{E}^c[G]} (\epsilon_\delta[X/\sim] \otimes \mathrm{Id}) \circ \delta_\sim(G) \\
&= (\epsilon_\delta[X/\sim] \otimes \mathrm{Id}) \circ \delta_{\sim_c}(G) \\
&= G\mid\sim_c \\
&= G.
\end{aligned}$$

Let us prove the compatibility of δ with the product [12, Proposition 2.4]. Let X and Y be two finite sets, $\sim \in \mathcal{E}[X \sqcup Y]$, $G \in \mathscr{G}[X]$ and $H \in \mathscr{G}[Y]$. If $\sim \neq \sim_X \sqcup \sim_Y$, at least one class C of

$\sim$ intersects both X and Y, so is not connected in $GH = m_{X,Y}(G \otimes H)$. Therefore, $\sim \notin \mathcal{E}^c[GH]$ and

$$\delta_\sim \circ m_{X,Y}(G \otimes H) = 0.$$

Let us assume that $\sim = \sim_X \sqcup \sim_Y$. Then $\sim \in \mathcal{E}^c[GH]$ if, and only if, $\sim_X \in \mathcal{E}^c[G]$ and $\sim_Y \in \mathcal{E}^c[H]$, as the connected components of GH are the connected components of G and of H. If so, $(GH)/\sim = (G/\sim_X)(H/\sim_Y)$ and $(GH)\mid\sim = (G\mid\sim_X)(H\mid\sim_Y)$. Therefore,

$$\begin{aligned}
\delta_\sim \circ m_{X,Y}(G \otimes H) &= \begin{cases} (GH)/\sim \otimes (GH)\mid\sim \text{ if } \sim \in \mathcal{E}^c[GH], \\ 0 \text{ otherwise} \end{cases} \\
&= \begin{cases} (G/\sim_X)(H/\sim_Y) \otimes (G\mid\sim_X)(H\mid\sim_Y) \text{ if } \sim_X \in \mathcal{E}^c[G] \text{ and } \sim_Y \in \mathcal{E}^c[H], \\ 0 \text{ otherwise} \end{cases} \\
&= (m_{X/\sim_X, Y/\sim_Y} \otimes m_{X,Y}) \circ (\mathrm{Id} \otimes c \otimes \mathrm{Id}) \circ (\delta_{\sim_X} \otimes \delta_{\sim_Y})(G \otimes H).
\end{aligned}$$

Let us finally prove the compatibility of δ with the coproduct Δ [12, Proposition 2.5]. Let X and Y be two finite sets, $\sim_X \in \mathcal{E}[X]$, $\sim_Y \in \mathcal{E}[Y]$ and $G \in \mathscr{G}[X]$. We put $\sim = \sim_X \sqcup \sim_Y$.

$$(\Delta_{X/\sim_X, Y/\sim_Y} \otimes \mathrm{Id}) \circ \delta_\sim(G) = \begin{cases} (G/\sim)_{|X/\sim_X} \otimes (G/\sim)_{|Y/\sim_Y} \\ \qquad \text{if } \sim \in \mathcal{E}^c[G] \text{ and } Y/\sim_Y \text{ is an ideal of } G/\sim, \\ 0 \text{ otherwise,} \end{cases}$$

$$m_{1,3,24} \circ (\delta_{\sim_X} \otimes \delta_{\sim_Y}) \circ \Delta_{X,Y}(G) = \begin{cases} (G_{|X})/\sim_X \otimes (G_{|Y})/\sim_Y \\ \qquad \text{if } Y \text{ is an ideal of } G, \sim_X \in \mathcal{E}^c[G_{|X}] \text{ and } \sim_Y \in \mathcal{E}^c[G_{|Y}], \\ 0 \text{ otherwise,} \end{cases}$$

Let us prove that $\sim \in \mathcal{E}^c[G]$ and $Y/\sim_Y$ is an ideal of $G/\sim$ if, and only if, Y is an ideal of G, $\sim_X \in \mathcal{E}^c[G_{|X}]$ and $\sim_Y \in \mathcal{E}^c[G_{|Y}]$.

$\Longrightarrow$. Let $y \in Y$ and $z \in X \sqcup Y$ such that $x \overset{G}{\to} y$. Then either $\mathrm{cl}_\sim(y) = \mathrm{cl}_\sim(z)$ or $\mathrm{cl}_\sim(y) \overset{G/\sim}{\to} \mathrm{cl}_\sim(z)$. As $Y/\sim_Y$ is an ideal of $G/\sim$, in both cases $z \in Y$. As $\sim = \sim_X \sqcup \sim_Y$, its classes are the classes of $\sim_X$ and $\sim_Y$, and are connected by hypothesis. So $\sim_X \in \mathcal{E}^c[G_{|X}]$ and $\sim_Y \in \mathcal{E}^c[G_{|Y}]$.

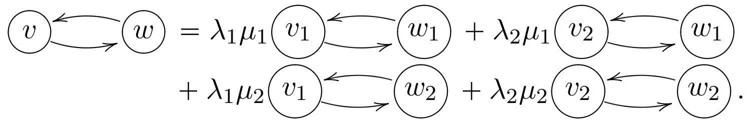
$\Longleftarrow$. As $\sim = \sim_X \sqcup \sim_Y$, its classes are the classes of $\sim_X$ and $\sim_Y$, which are connected by hypothesis. Hence, $\sim \in \mathcal{E}^c[G]$. Let $\mathrm{cl}_\sim(y) \in Y/\sim_Y$ and $\mathrm{cl}_\sim(z) \in [X \sqcup Y]/\sim$, such that $\mathrm{cl}_\sim(y) \overset{G/\sim}{\to} \mathrm{cl}_\sim(z)$. There exist $y', z' \in X \sqcup Y$ such that $y \sim y'$, $z \sim z'$ and $y' \overset{G}{\to} z'$. As $\sim = \sim_X \sqcup \sim_Y$, $y' \in Y$. As Y is an ideal of G, necessarily $z' \in Y$. As $\sim = \sim_X \sqcup \sim_Y$, $z' \in Y$ and finally $\mathrm{cl}_\sim(z') \in Y/\sim_Y$.

Moreover,

$$(G/\sim)_{|X/\sim_X} = (G_{|X})/\sim_X, \qquad (G/\sim)_{|Y/\sim_Y} = (G_{|Y})/\sim_Y,$$

which finally proves the compatibility between δ and Δ. □

As a consequence, for any vector space V, we obtain a graded bialgebra $\mathcal{F}_V[\mathbf{G}]$. This is the vector space of mixed graphs which vertices are decorated by elements of V, any graph being linear in any of its decorations: these objects will be called V-linearly decorated graphs. For example, if $v_1, v_2, w_1, w_2 \in V$ and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{K}$, in $\mathcal{F}_V[\mathbf{G}]$, if $v = \lambda_1 v_1 + \lambda_2 v_2$ and $w = \mu_1 w_1 + \mu_2 w_2$,



If $\mathcal{B}$ is a basis of V, a basis of $\mathcal{F}_V[\mathbf{G}]$ is the set of mixed graphs which vertices are decorated by elements of $\mathcal{B}$. The product is the disjoint union. For any V-linearly decorated graph G,

$$\Delta(G) = \sum_{I \text{ ideal of } G} G_{|V(G)\setminus I} \otimes G_{|I}.$$

For example, if $v, w \in V$,

$$
\begin{aligned}
\Delta(v \ w) &= v \ w \otimes 1 + 1 \otimes v \ w + v \otimes w + w \otimes v,\\
\Delta(v \text{---} w) &= v \text{---} w \otimes 1 + 1 \otimes v \text{---} w + v \otimes w + w \otimes v,\\
\Delta(v \longrightarrow w) &= v \longrightarrow w \otimes 1 + 1 \otimes v \longrightarrow w + v \otimes w,\\
\Delta(v \rightleftarrows w) &= v \rightleftarrows w \otimes 1 + 1 \otimes v \rightleftarrows w.
\end{aligned}
$$

The counit ε_Δ sends any mixed graph $G \neq 1$ to 0. If $(V, \cdot, \Delta)$ is a nonunitary, commutative and cocommutative bialgebra, then $\mathcal{F}_V[\mathbf{G}]$ inherits a second coproduct δ: if G is a V-linearly decorated graph,

$$\delta(G) = \sum_{\sim \in \mathcal{E}^c[G]} G/\sim \otimes G\mid\sim,$$

where the vertices of $G/\sim \otimes G\mid\sim$ are decorated in the following way: denoting by $d_G(x)$ the decoration of the vertex $x \in V(G)$, any vertex $\mathrm{cl}_\sim(x)$ of $G/\sim$ is decorated by the products of elements $d_G(y)'$, where $y \in \mathrm{cl}_\sim(x)$, whereas the vertex $x \in V(G\mid\sim) = V(G)$ is decorated by $d_G(x)''$, and everything being extended by multilinearity of each decoration. The counit ϵ_δ is given on any mixed graph G by

$$\epsilon_\delta(G) = \begin{cases} \displaystyle\prod_{x \in V(G)} \epsilon_V \circ d_G(x) \text{ if } A(G) = E(G) = \emptyset,\\ 0 \text{ otherwise.}\end{cases}$$

This construction is functorial in V.

Example 1.5. If $v, w \in V$,

$$
\begin{aligned}
\delta(v \ w) &= v' \ w' \otimes v'' \ w'',\\
\delta(v \text{---} w) &= v' \text{---} w'' \otimes v'' \ w'' + v' \cdot w' \otimes v'' \text{---} w'',\\
\delta(v \longrightarrow w) &= v' \longrightarrow w' \otimes v'' \ w'' + v' \cdot w' \otimes v'' \longrightarrow w'',\\
\delta(v \rightleftarrows w) &= v' \rightleftarrows w' \otimes v'' \ w'' + v' \cdot w' \otimes v'' \rightleftarrows w''.
\end{aligned}
$$

Remark 1.2. We shall often work with $V = \mathbb{K}$, with its usual bialgebraic structure defined by $\delta_{\mathbb{K}}(1) = 1 \otimes 1$. We shall then identify any V-decorated mixed graph which any vertex is decorated by 1 with the underlying mixed graph. The double bialgebra $\mathcal{F}_V[\mathbf{G}]$ is identified with $\mathcal{F}[\mathbf{G}]$ and has for basis the set of mixed graphs.

Proposition 1.7. *Let V be a (non necessarily unitary) commutative and cocommutative bialgebra. For any linearly V-decorated mixed graph G, we denote by $d_G : V(G) \longrightarrow V$ the decoration map of G and by $\overline{G}$ the underlying mixed graph. Then the following map is a double bialgebra morphism:*

$$\Theta_V : \begin{cases} \mathcal{F}_V[\mathbf{G}] & \longrightarrow & \mathcal{F}[\mathbf{G}]\\ G & \longmapsto & \left(\displaystyle\prod_{x \in V(G)} \epsilon_V \circ d_G(x)\right)\overline{G}.\end{cases}$$

Proof. The counit $\epsilon_V : V \longrightarrow \mathbb{K}$ is a bialgebra map. By functoriality, Θ_V is a double bialgebra morphism. □

1.4 Cofreeness of the coalgebra $\mathcal{F}[\mathbf{G}]$

Let us define a second product on $\mathcal{F}[\mathbf{G}]$.

Proposition 1.8. *Let G and H be two mixed graphs. The mixed graph $G \rightharpoondown H$ is defined by*

$$\begin{aligned}V(G \rightharpoondown H) &= V(G) \sqcup V(H),\\ E(G \rightharpoondown H) &= E(G) \sqcup E(H),\\ A(G \rightharpoondown H) &= A(G) \sqcup A(H) \sqcup (V(G) \times V(H)).\end{aligned}$$

This product is bilinearly extended to $\mathcal{F}[\mathbf{G}]$. Then $(\mathcal{F}[\mathbf{G}], \rightharpoondown, \Delta)$ is a unital infinitesimal bialgebra in the sense of [14, Definition 2.1].

Proof. As we already know that Δ is coassociative and unitary, it remains to prove that:

1. $\rightharpoondown$ is associative and unitary.
2. For any $x, y \in \mathcal{F}[\mathbf{G}]$, $\Delta(x \rightharpoondown y) = (x \otimes 1) \rightharpoondown \Delta(y) + \Delta(x) \rightharpoondown (1 \otimes y) - x \otimes y$.

1. Let G, H, K be three mixed graphs. Then

$$\begin{aligned}V((G \rightharpoondown H) \rightharpoondown K) &= V(G \rightharpoondown (H \rightharpoondown K)) = V(G) \sqcup V(H) \sqcup V(K),\\ E((G \rightharpoondown H) \rightharpoondown K) &= E(G \rightharpoondown (H \rightharpoondown K)) = E(G) \sqcup E(H) \sqcup E(K),\\ A((G \rightharpoondown H) \rightharpoondown K) &= A(G \rightharpoondown (H \rightharpoondown K)) = A(G) \sqcup A(H) \sqcup A(K)\\ (\sqcup V(G) \times V(H)) \sqcup (V(G) \times V(K)) \sqcup (V(H) \sqcup V(K)),\end{aligned}$$

so $(G \rightharpoondown H) \rightharpoondown K) = G \rightharpoondown (H \rightharpoondown K)$. Therefore, $\rightharpoondown$ is associative. The unit is the empty mixed graph 1.

2. Let G, H be two mixed graph. As there is an arc from any vertex of G to any vertex of H in $G \rightharpoondown H$, the ideals of $G \rightharpoondown H$ are:

- $I \sqcup V(H)$ where I is an ideal of G. For such an ideal,

$$(G \rightharpoondown H)_{|I \sqcup V(H)} = G_{|I} \rightharpoondown H, \qquad (G \rightharpoondown H)_{|V(G \rightharpoondown H) \setminus (I \sqcup V(H))} = G_{|V(G) \setminus I}.$$

- Ideals J of H. For such an ideal,

$$(G \rightharpoondown H)_{|J} = H_{|J}, \qquad (G \rightharpoondown H)_{|V(G \rightharpoondown H) \setminus J} = G \rightharpoondown H_{|V(H) \setminus J}.$$

Note that the ideal $V(H)$ appears twice in this list, for $I = \emptyset$ and $J = V(H)$. Therefore,

$$\begin{aligned}\Delta(G \rightharpoondown H) &= \sum_{I \text{ ideal of } G} G_{|V(G) \setminus I} \otimes G_{|I} \rightharpoondown H + \sum_{J \text{ ideal of } G} G \rightharpoondown H_{|V(H) \setminus J} \otimes H_{|J} - G \otimes H\\ &= \Delta(G) \rightharpoondown (1 \otimes H) + (G \otimes 1) \rightharpoondown \Delta(H) - G \otimes H.\end{aligned}$$

□

From [14, Theorem 2.6]:

Corollary 1.9. *The coalgebra $(\mathcal{F}[\mathbf{G}], \Delta)$ is isomorphic to the coalgebra $T(\mathrm{Prim}(\mathcal{F}[\mathbf{G}])$ with the deconcatenation coproduct.*

The same proof can be adapted to any $\mathcal{F}_V[\mathbf{G}]$.

2 Subobjects and quotients of mixed graphs

2.1 Simple and oriented graphs

Proposition 2.1. *$\mathbf{G}_s$ and $\mathbf{G}_o$ are twisted subbialgebras of $(\mathbf{G}, m, \Delta)$ and are stable under the contraction-extraction coproduct δ.*

Proof. If G and H are simple graphs, then GH is a simple graph. If G is a simple graph, then all its subgraphs are also simple graphs. Moreover, if $\sim \in \mathcal{E}^c[G]$, then $G/\sim$ and $G\mid\sim$ are also simple graphs. The proof is similar for oriented graphs. $\square$

Corollary 2.2. *For any vector space V, $\mathcal{F}_V[\mathbf{G}_s]$ is a subbialgebra of $\mathcal{F}_V[\mathbf{G}]$ and $\mathcal{F}_V[\mathbf{G}_o]$ is a subbialgebra of $\mathcal{F}_V[\mathbf{G}]$. For any (non necessarily unitary) commutative and cocommutative bialgebra V, $\mathcal{F}_V[\mathbf{G}_s]$ is a double subbialgebra of $\mathcal{F}_V[\mathbf{G}]$ and $\mathcal{F}_V[\mathbf{G}_o]$ is a double subbialgebra of $\mathcal{F}_V[\mathbf{G}]$.*

In particular, $\mathcal{F}_{\mathbb{K}}[\mathbf{G}_s] = \mathcal{F}[\mathbf{G}_s]$ is the double bialgebra of graphs of [8, 9] and $\mathcal{F}_{\mathbb{K}}[\mathbf{G}_o] = \mathcal{F}[\mathbf{G}_o]$ is the double bialgebra of [6].

2.2 Acyclic mixed graphs and finite topologies

Let $G \in \mathscr{G}[X]$ be a mixed graph. We denote by $\mathcal{O}[G]$ the set of ideals of G. if $I, J \in \mathcal{O}[G]$, then $I \cap J$ and $I \cup J$ belong to $\mathcal{O}[G]$: $\mathcal{O}[G]$ is a topology on the finite set X. We obtain:

Proposition 2.3. *For any finite set X, let us denote by $\mathscr{T}opo[X]$ the set of topologies on X and by $\mathbf{Topo}[X]$ the space generated by $\mathscr{T}opo[X]$. This defines a species. it is equipped with a twisted bialgebra structure by the following:*

- *For any finite sets X, Y, for any $(\mathcal{O}_X, \mathcal{O}_Y) \in \mathscr{T}opo[X] \times \mathscr{T}opo[Y]$,*

$$m_{X,Y}(O_X \times O_Y) = \{I \sqcup J,\ I \in \mathcal{O}_X,\ J \in \mathcal{O}_Y\}.$$

- *For any finite sets X, Y, for any $\mathcal{O} \in \mathscr{T}opo[X \sqcup Y]$,*

$$\Delta_{X,Y}(\mathcal{O}) = \begin{cases} \mathcal{O}_{|X} \otimes \mathcal{O}_{|Y} \text{ if } Y \in \mathcal{O}, \\ 0 \text{ otherwise.} \end{cases}$$

Moreover, the following map is a sujective morphism of twisted bialgebras:

$$\Upsilon : \left\{ \begin{array}{rcl} \mathbf{G} & \longrightarrow & \mathbf{Topo} \\ G \in \mathscr{G}[X] & \longrightarrow & \mathcal{O}[G] \in \mathscr{T}opo[X]. \end{array} \right.$$

Proof. Let $\mathcal{O}$ be a finite topology on a finite set X. We define a quasi-order on X (that is, a transitive and reflexive relation on X) by the following: for any $x, y \in G$, $x \leqslant y$ if any $O \in \mathcal{O}$ containing x also contains y. By Alexandroff's theorem [2], $\mathcal{O}$ is the set of ideals of the quasi-order $\leqslant$. Let us consider the arrow graph G of $\leqslant$: $V(G) = X$ and for any $x \neq y$ in X, there is an arc between x and y if, and only if $x \leqslant y$. Then $\mathcal{O} = \mathcal{O}[G]$, so Υ is surjective. Let G, G', H, H' be graphs such that $\mathcal{O}[G] = \mathcal{O}[G']$ and $\mathcal{O}[H] = \mathcal{O}[H']$. Then

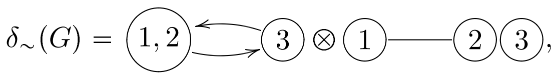
$$\mathcal{O}[GH] = \mathcal{O}[G]\mathcal{O}[H] = \mathcal{O}[G']\mathcal{O}[H'] = \mathcal{O}[G'H'].$$

Therefore, the product of $\mathbf{G}$ is compatible with the products of $\mathbf{G}$ and $\mathbf{Topo}$.

For any graph $G \in \mathscr{G}[X]$ and for any $Y \subset X$, $\mathcal{O}[G_{|Y}] = \mathcal{O}[G]_{|Y}$. This implies that υ is compatible with the coproducts of $\mathbf{G}$ and $\mathbf{Topo}$. As Υ is surjective and $\mathbf{G}$ is a twisted bialgebra, $\mathbf{Topo}$ is also a twisted bialgebra. $\square$

This map Υ is not compatible with the contraction-extraction coproduct: for example, if

then $\mathcal{O}[G_1] = \mathcal{O}[G_2]$. Let us denote by $\sim$ the equivalence with classes $\{1,2\}$ and $\{3\}$. Then

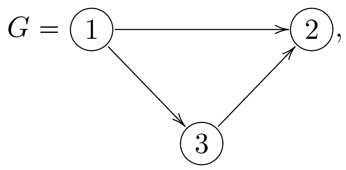


$$\delta_\sim(G') = 0.$$

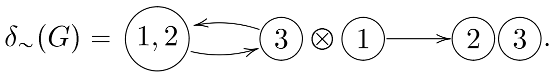
In order to obtain a second coproduct, we restrict ourselves to acyclic mixed graphs:

Definition 2.4. *Let G be a mixed graph. We shall say that G is* acyclic *if does not contain any oriented path $x_0 \overset{G}{\to} \ldots \overset{G}{\to} x_n$ with $x_0 = x_n$ and $n \geqslant 2$. Acyclic mixed graphs form a subspecies $\mathbf{G}_{ac}$ of $\mathbf{G}$, and acyclic oriented graphs form a subspecies $\mathbf{G}_{aco}$ of $\mathbf{G}_o$.*

Obviously, if G and H are acyclic mixed graphs, then GH is acyclic; if G is acyclic and $I \subseteq V(G)$, then $G_{|I}$ is also acyclic. Therefore, $\mathbf{G}_{ac}$ is a twisted subbialgebra of $\mathbf{G}$ and $\mathbf{G}_{aco}$ is a twisted subbialgebra of $\mathbf{G}_o$. But $\mathbf{G}_{ac}$ is not stable under δ. For example, let us consider the following acylic oriented graph:



Let us denote by $\sim$ the equivalence with classes $\{1,2\}$ and $\{3\}$. Then



Proposition 2.5. *A contraction-extraction coproduct on $\mathbf{G}_{ac}$ is defined by the following: for any acyclic graph $G \in \mathscr{G}_{ac}[X]$, for any $\sim \in \mathcal{E}[X]$,*

$$\delta_\sim(G) = \begin{cases} G/\sim \otimes G \mid \sim & \text{if } \sim \in \mathcal{E}^{c,ac}[G], \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathcal{E}^{c,ac}[G]$ the set of equivalences on $V(G)$ such that the classes of G are connected and $G/\sim$ is acyclic. Moreover, the following map is a surjective morphims of twisted bialgebras, compatible with the contraction-extraction coproducts:

$$\varpi_0 : \left\{ \begin{array}{rcl} \mathbf{G} & \longrightarrow & \mathbf{G}_{ac} \\ G \in \mathscr{G}[X] & \longmapsto & \begin{cases} G \text{ if } G \text{ is acyclic,} \\ 0 \text{ otherwise.} \end{cases} \end{array} \right.$$

Proof. Let $\mathbf{I}$ be the subspecies of $\mathbf{G}$ of non acyclic mixed graphs. If G is a non acyclic mixed graph, then for any mixed graph H, GH is not acyclic: $\mathbf{I}$ is an ideal. If I is an ideal of G, if it contains a vertex on a cycle of G, then it contains all the vertices of the cycle: therefore, $G_{|I}$ or $G_{V(G)\setminus I}$ is not acyclic, which proves that $\mathbf{I}$ is an ideal for Δ. Let $\sim \in \mathcal{E}^c[G]$. Let us consider a cycle C of G. If all the vertices of C are equivalent for $\sim$, then $G \mid\sim$ contains a cycle, so is not acyclic. Otherwise, $G/\sim$ contains a cycle: $\mathbf{I}$ is a coideal for δ. Identifying the species $\mathbf{G}/\mathbf{I}$ and $\mathbf{G}_{ac}$ via ϖ_0, $\mathbf{G}_{ac}$ inherits a contraction-extraction coproduct δ, which is precisely the one defined in this proposition. $\square$

Similarly, restricting ϖ_0 to $\mathbf{G}_o$, its image $\mathbf{G}_{aco}$ inherits a contraction-extraction coproduct δ, as a subquotient of $\mathbf{G}$.

If G is an acyclic oriented graph, we define a relation on $V(G)$ by $x \leqslant_G y$ if there exists an oriented path from x to y. As G is acyclic, we obtain a partial order on $V(G)$. This defines a morphism from $\mathbf{G}_{aco}$ to the species $\mathbf{Pos}$ of posets. Considering a finite poset as a finite topology trough Alexandroff's theorem, this map is the restriction of Υ. If P is a poset, considering its Hasse graph G, we obtain an acyclic oriented graph such that $\Upsilon(G) = P$. Hence, Υ is surjective.

Proposition 2.6. *There exists a unique product, a unique coproduct and a unique contraction-extraction coproduct on* $\mathbf{Pos}$ *making the map* $\Upsilon_{|\mathbf{G}_{aco}} : \mathbf{G}_{aco} \longrightarrow \mathbf{Pos}$ *a morphism of twisted bialgebras, compatible with the contraction-extraction coproduct.*

Proof. The unicity comes from the surjectivity of Υ. The product and the coproduct are the restriction of the product and of the coproduct on finite topologies.

Let G be an oriented graph and $\sim \in \mathcal{E}[V(G)]$. The partial order on $G/\sim$ is the transitive closure of the relation defined by

$$\overline{x}\mathcal{R}\overline{y} \text{ if } x \overset{G}{\to} y.$$

Equivalently, it is the transitive closure of the relation defined by

$$\overline{x}\mathcal{R}\overline{y} \text{ if } x \leqslant_G y.$$

Therefore, if $\leqslant_G = \leqslant_{G'}$, for any equivalence $\sim \in \mathcal{E}[V(G)]$, $\Upsilon(G/\sim) = \Upsilon(G'/\sim)$. Obviously, $\Upsilon(G\mid\sim) = \Upsilon(G'\mid\sim)$.

Let $G, G' \in \mathscr{G}_o[X]$ such that $\leqslant_G = \leqslant_{G'}$: let us prove that $\mathcal{E}^{c,ac}[G] = \mathcal{E}^{c,ac}[G']$. Let $\sim \in \mathcal{E}^{c,ac}[G]$. We consider a class C of $\sim$. Then $\Upsilon(G/\sim)$ is an order, as $G/\sim$ is acyclic, so $\Upsilon(G/\sim')$ is also an order: $G/\sim'$ is acyclic. let $x, y \in G$. As C is connected, there exists a (non oriented) path from x to y in C, which we denote by $x = x_0 \overset{G}{-} \ldots \overset{G}{-} x_n = y$. For any i, x_i and x_{i+1} are comparable for $\leqslant_G$, so are comparable for $\leqslant_{G'}$: there exists in G' an oriented path from x_i to x_{i+1}. If all the vertices on this oriented path are not equivalent to x_i, then $G'/\sim$ is not acyclic, so $G/\sim = G'/\sim$ is not acyclic, which contradicts $\sim \in \mathcal{E}^{c,ac}[G]$. Therefore, there exists a path in $G'_{|C}$ from x_i to x_{i+1} for any i, so there exists a path in $G'_{|C}$ from x to y: we proved that $\sim \in \mathcal{E}^{c,ac}[G']$. By symmetry, we obtain $\mathcal{E}^{c,ac}[G] = \mathcal{E}^{c,ac}[G']$.

Let G, G' be two acyclic graphs such that $\Upsilon(G) = \Upsilon(G')$, and let $\sim \in \mathcal{E}[V(G)]$. Then

$$\begin{aligned}(\Upsilon \otimes \Upsilon) \circ \delta_\sim(G) &= \begin{cases} \Upsilon(G/\sim) \otimes \Upsilon(G\mid\sim) \text{ if } \sim \in \mathcal{E}^{c,ac}[G], \\ 0 \text{ otherwise} \end{cases} \\ &= \begin{cases} \Upsilon(G'/\sim) \otimes \Upsilon(G'\mid\sim) \text{ if } \sim \in \mathcal{E}^{c,ac}[G'], \\ 0 \text{ otherwise} \end{cases} \\ &= (\Upsilon \otimes \Upsilon) \circ \delta_\sim(G).\end{aligned}$$

Consequently, $\mathbf{Pos}$ inherits a contraction-extraction coproduct as a quotient of $\mathbf{G}_{aco}$. □

If $\leqslant$ is a quasi-order on a finite set X, we can define an equivalence on X by

$$x \sim y \text{ if } x \leqslant y \text{ and } y \leqslant x.$$

Then $X/\sim$ is partially ordered, with

$$\overline{x}\overline{\leqslant}\overline{y} \text{ if } x \leqslant y.$$

In other words, for any finite set X,

$$\mathbf{Topo}[X] = \bigoplus_{\sim \in \mathcal{E}[X]} \mathbf{Pos}[X/\sim],$$

that is to say $\mathbf{Pos} \circ \mathbf{Com} = \mathbf{Topo}$. The double twisted bialgebra structure which we obtain in this way is described in [8]. Applying Aguiar and Mahajan's bosonic Fock functor [1], we obtain the double algebra of finite topologies of [7].

2.3 Totally acyclic graphs

Definition 2.7. *Let G be a mixed graph. We shall say that it is totally acyclic if does not contain any mixed path $(x_0, \ldots, x_n)$, with $x_0 = x_n$ and $n \geqslant 2$. Totally acyclic graphs form a subspecies $\mathbf{G}_{tac}$ of $\mathbf{G}$.*

Note that totally acyclic mixed graphs are simply called acyclic in [5].

Proposition 2.8. *$\mathbf{G}_{tac}$ is a twisted subbialgebra of $\mathbf{G}$.*

Proof. If G and H are totally acyclic graphs, then GH is totally acyclic. So $\mathbf{G}_{tac}$ is a twisted subalgebra of $\mathbf{G}$. Let G be a totally acyclic mixed graph and $I \subseteq V(G)$. As G does not contain any mixed cycle, so does $G_{|I}$: $G_{|I}$ is totally acyclic. As a conclusion, $\mathbf{G}_{tac}$ is a twisted subcoalgebra of $\mathbf{G}$. □

Consequently, for any vector space V, $\mathcal{F}_V[\mathbf{G}_{tac}]$ is a subbialgebra of $(\mathcal{F}_V[\mathbf{G}], m, \Delta)$. The subspecies $\mathbf{G}_{tac}$ is not stable under δ. For example, considering the mixed graph

$$G = \begin{tikzcd} x \arrow[r] \arrow[d, dash] & y, \\ z \arrow[r, dash] & t \arrow[u] \end{tikzcd}$$

which is totally acyclic, the equivalence relation $\sim$ which classes are $\{x, y\}$, $\{z\}$ and $\{t\}$ belongs to $\mathcal{E}^c[G]$ (in fact, even to $\mathcal{E}^{c,ac}[G]$), and $G/\sim$ is not totally acyclic.

3 Applications

3.1 Three polynomial invariants

Let $(V, \cdot, \delta_V)$ be a non necessarily unitary, commutative bialgebra. From [10, Theorem 3.9], there exists a unique morphism ϕ_1 of double bialgebras from $(\mathcal{F}_V[\mathbf{G}], m, \Delta, \delta)$ onto $(\mathbb{K}[X], m, \Delta, \delta)$ where the two coproducts of $\mathbb{K}[X]$ are defined by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad \delta(X) = X \otimes X.$$

Let us determine ϕ_1, firstly when $V = \mathbb{K}$. Let $G \in \mathscr{G}[X]$, nonempty. Then, still by [10, Theorem 3.9],

$$\phi_1(G) = \sum_{k=0}^{\infty} \epsilon_\delta^{\otimes(k-1)} \circ \tilde{\Delta}^{(k-1)}(G) H_k(X),$$

where H_k is the k-th Hilbert polynomial:

$$H_k(X) = \frac{X(X-1)\ldots(X-k+1)}{k!}.$$

Notations 3.1. For any $k \geqslant 1$, for any mixed graph G, we denote by $L_k(G)$ the set of surjections $c : V(G) \longrightarrow [k]$ such that

$$\forall x, y \in V(G), \qquad x \overset{G}{\to} y \Longrightarrow c(x) \leqslant c(y).$$

By definition of the coproduct Δ, for any mixed graph G with $n \geqslant 1$ vertices,

$$\tilde{\Delta}^{(k-1)}(G) = \sum_{c \in L_k(G)} G_{|c^{-1}(1)} \otimes \ldots \otimes G_{|c^{-1}(k)}.$$

This leads to the following definition:

Definition 3.1. *Let G be a mixed graph.*

1. *A valid coloration of G is a map $c : V(G) \longrightarrow \mathbb{N}_{>0}$ such that*

$$\forall x, y \in V(G), \qquad \begin{aligned} x \overset{G}{\to} y &\Longrightarrow c(x) < c(y), \\ x \overset{G}{-} y &\Longrightarrow c(x) \neq c(y). \end{aligned}$$

2. *A valid coloration c of G is packed if $c(V(G)) = [\max(c)]$. The set of valid packed colorations of G is denoted by* $\mathrm{VPC}(G)$.

Thanks to this definition, we obtain, For any V-linearly decorated mixed graph G,

$$\epsilon_\delta^{\otimes(k-1)} \circ \tilde{\Delta}^{(k-1)}(G) = |\{c \in \mathrm{VPC}(G), | \max(c) = k\}|.$$

And finally:

Proposition 3.2. *The unique morphism of double bialgebras from $\mathcal{F}[\mathbf{G}]$ to $\mathbb{K}[X]$ is given on any mixed graph G by*

$$P_{chr_S}(G) = \sum_{c \in \mathrm{VPC}(G)} H_{\max(c)}.$$

Consequently, if $N \in \mathbb{N}$, $P_{chr_S}(G)(N)$ is the number of valid colorations c such that $\max(c) \leqslant N$*: we recover the (strong) chromatic polynomial $P_{chr_S}(G)$ of [5].*

Remark 3.1. If V is a non necessarily unitary, commutative bialgebra, the unique double bialgebra morphism from $(\mathcal{F}_V[\mathbf{G}], m, \Delta, \delta)$ to $(\mathbb{K}[X], m, \Delta, \delta)$ is $P_{chr_S} \circ \Theta_V$ (which is indeed a double bialgebra morphism by composition). It sends any V-linearly decorated mixed graph G to

$$P_{chr_S} \circ \Theta_V(G) = \left(\prod_{x \in V(G)} \epsilon_V \circ d_G(x) \right) P_{chr_S}(\overline{G}),$$

where d_G is the decoration map of G and $\overline{G}$ the underlying mixed graph.

Let us now recover the weak chromatic polynomial of [5].

Definition 3.3. *Let G be a mixed graph.*

1. *A weak valid coloration of G is a map $c : V(G) \longrightarrow \mathbb{N}_{>0}$ such that*

$$\forall x, y \in V(G), \qquad \begin{aligned} x \overset{G}{\to} y &\Longrightarrow c(x) \leqslant c(y), \\ x \overset{G}{-} y &\Longrightarrow c(x) \neq c(y). \end{aligned}$$

2. *A weak valid coloration c of G is packed if $c(V(G)) = [\max(c)]$. The set of weak valid packed colorations of G is denoted by* $\mathrm{WVPC}(G)$.

We are going to use the action $\leftsquigarrow$ of the monoid of characters on the set of morphisms [10, Proposition 2.5]. Let $\lambda_W : \mathcal{F}[G] \longrightarrow \mathbb{K}$ defined on any mixed graph G by

$$\lambda_W(G) = \begin{cases} 1 \text{ if } E(G) = \emptyset, \\ 0 \text{ otherwise.} \end{cases}$$

This is obviously a character. We consider $\theta(\lambda_W) = P_{chr_S} \leftsquigarrow \lambda_W = P_{chr_W}$, with the notations of [10, Corollary 3.12]. This is a bialgebra morphism from $(\mathcal{F}[\mathbf{G}], m, \Delta)$ to $(\mathbb{K}[X], m, \Delta)$. For any mixed graph G,

$$P_{chr_W}(G) = \sum_{c \in \mathrm{WVPC}(G)} H_{\max(c)}.$$

Consequently, if $N \in \mathbb{N}$, $P_{chr_W}(G)(N)$ is the number of weak valid colorations c such that $\max(c) \leqslant N$: we recover the weak chromatic polynomial of [5]. Therefore:

Corollary 3.4. $P_{chr_W} : (\mathcal{F}[\mathbf{G}], m, \Delta) \longrightarrow (\mathbb{K}[X], m, \Delta)$ *is a Hopf algebra morphism.*

Remark 3.2. Let $G \in \mathscr{G}[X]$. We denote by $\mathcal{E}^c_W[G]$ the set of equivalences $\sim \in \mathcal{E}^c[G]$ such that

$$\forall x, y \in V(G), \qquad x \xrightarrow{G} y \Longrightarrow x \nsim y.$$

Then, for any mixed graph G,

$$P_{chr_W}(G) = \sum_{\sim \in \mathcal{E}^c[G]} \lambda_W(G \mid \sim) P_{chr_S}(G/\sim) = \sum_{\sim \in \mathcal{E}^c_W[G]} P_{chr_S}(G/\sim).$$

Example 3.1. For $n \geqslant 3$. Let G_n be the following mixed graph:

$$V(G_n) = [n], \qquad E(G_n) = \{\{1, n\}\}, \qquad A(G_n) = \{(i, i+1) \mid i \in [n-1]\}.$$

In other terms,

$$G_n = \begin{tikzcd} 2 \arrow[r] & \ldots \arrow[r] & n-1 \arrow[d] \\ 1 \arrow[u] \arrow[rr, no head] & & n \end{tikzcd}.$$

Weak valid colorations of G_n are nondecreasing maps $c : [n] \longrightarrow \mathbb{N}_{>0}$, such that $c(1) \neq c(n)$. Therefore,

$$P_{chr_W}(G_n) = \frac{X(X+1)\ldots(X+n-1)}{n!} - X.$$

Valid colorations of G_n are strictly increasing maps $c : [n] \longrightarrow \mathbb{N}_{>0}$. Therefore,

$$P_{chr_S}(G_n) = \frac{X(X-1)\ldots(X-n+1)}{n!}.$$

With the help of [10, Propositions 3.10 and 5.2], we now define a homogeneous morphism $\phi_0 : \mathcal{F}[\mathbf{G}] \longrightarrow \mathbb{K}[X]$ with the help of the element $\mu \in \mathcal{F}[\mathbf{G}]_1^*$ defined by

$$\mu(\bullet) = 1.$$

Then, if G is a mixed graph,

$$\mu^{\otimes k} \circ \tilde{\Delta}^{(k-1)}(G) = \begin{cases} |L_k(G)| \text{ if } k = |V(G)|, \\ 0 \text{ otherwise.} \end{cases}$$

We denote by $\ell(G)$ the cardinality of $L_n(G)$, that is to say the number of bijections $c : V(G) \longrightarrow [n]$ such that

$$\forall x, y \in V(G), \qquad x \xrightarrow{G} y \Longrightarrow c(x) < c(y),$$

and finally:

Corollary 3.5. *For any mixed graph G, we put*

$$\lambda_0(G)=\frac{\ell(G)}{|V(G)|!}, \qquad \phi_0(G)=\lambda_0(G)X^{|V(G)|}.$$

Then λ_0 is a character of $\mathcal{F}[\mathbf{G}]$ and $\phi_0:(\mathcal{F}[\mathbf{G}],m,\Delta)\longrightarrow(\mathbb{K}[X],m,\Delta)$ is a bialgebra morphism.

For any graph G, $\phi_0(G)(1)=\lambda_0(G)$. From [10, Corollary 3.11], $\phi_0=\phi_S\leftsquigarrow\lambda_0$. Therefore:

Corollary 3.6. *For any mixed graph G with n vertices,*

$$\ell(G)X^n=\sum_{\sim\in\mathcal{E}^c[G]}\ell(G\mid\sim)P_{chr_S}(G/\sim).$$

3.2 Morphisms to quasishuffle algebras

We assume in this paragraph that $(V,\cdot,\delta_V)$ is a nonunitary, commutative and cocommutative bialgebra. By [12, Proposition 3.9], $\mathcal{F}_V[\mathbf{G}]$ is a bialgebra over V, with the coaction ρ described as follows: if G is a V-decorated mixed graph with n vertices, we arbitrarily index these vertices and we denote by $G(v_1,\ldots,v_n)$ the mixed graph with for any i, the i-th vertex of G decorated by v_i. Then

$$\rho(G(v_1,\ldots,v_n))=G(v_1',\ldots,v_n')\otimes v_1''\cdot\ldots\cdot v_n''.$$

Notations 3.2. The map $\pi_V:T(V)\longrightarrow\mathbb{K}[X]$ is defined by

$$\forall v_1,\ldots,v_n\in V,\qquad \pi_V(v_1\ldots v_n)=\epsilon_V(v_1)\ldots\epsilon_V(v_n)\frac{X(X-1)\ldots(X-n+1)}{n!}.$$

It is a double bialgebra morphism.

By [11, Theorem 2.7]:

Theorem 3.7. *The unique morphism of double bialgebras over V from $\mathcal{F}_V[\mathbf{G}]$ to $(T(V),\uplus,\Delta,\delta,\rho)$ is*

$$\Phi_S:\left\{\begin{array}{rcl}\mathcal{F}_V[\mathbf{G}]&\longrightarrow&T(V)\\ G(v_1,\ldots,v_n)&\longrightarrow&\displaystyle\sum_{c\in\mathrm{VPC}(G)}\left(\prod_{c(i)=1}^{\cdot}v_i\right)\ldots\left(\prod_{c(i)=\max(l)}^{\cdot}v_i\right).\end{array}\right.$$

Moreover, $\pi_V\circ\Phi_S=P_{chr_S}\circ\Theta_V$.

Proof. The description of Φ_S comes directly from [11, Theorem 2.7]. By composition, $\pi_V\circ\Phi_S$ is a double algebra morphism from $\mathcal{F}_V[\mathbf{G}]$ to $\mathbb{K}[X]$, so is equal to $P_{chr_S}\circ\Theta_V$. □

By [11, Proposition 2.10 and Corollary 2.12], we can consider the morphisms

$$\Phi_W=\Phi_S\leftsquigarrow(\lambda_S\circ\Theta_V), \qquad \Phi_0=\Phi_S\leftsquigarrow(\lambda_0\circ\Theta_V).$$

We obtain:

Corollary 3.8. *The maps $\Phi_W,\Phi_0:(\mathcal{F}_V[\mathbf{G}],m,\Delta,\rho)\longrightarrow(T(V),\uplus,\Delta,\rho)$ are morphisms of bialgebras over V. For any V-decorated mixed graph $G(v_1,\ldots,v_n)$,*

$$\Phi_W(G(v_1,\ldots,v_n))=\sum_{c\in\mathrm{WVPC}(G)}\left(\prod_{c(i)=1}^{\cdot}v_i\right)\ldots\left(\prod_{c(i)=\max(l)}^{\cdot}v_i\right),$$

$$\Phi_0(G(v_1,\ldots,v_n))=\sum_f\prod_{i=1}^{\max(f)}\frac{\ell(G_{|f^{-1}(i)})}{|f^{-1}(i)|!}\left(\prod_{f(i)=1}^{\cdot}v_i\right)\ldots\left(\prod_{f(i)=\max(f)}^{\cdot}v_i\right),$$

where the sum is over all surjective maps $f : V(G) \longrightarrow [\max(f)]$ *such that*

$$\forall i, j \in V(G), \qquad i \xrightarrow{G} j \Longrightarrow f(i) \leqslant f(j).$$

Moreover,

$$\pi_V \circ \Phi_W = P_{chr_W} \circ \Theta_V, \qquad \pi_V \circ \Phi_0 = \phi_0 \circ \Theta_V.$$

Proof. The description of Φ_W and Φ_0 comes directly from [11, Proposition 2.10]. Moreover, by [11, Proposition 2.11], as Θ_V is a double bialgebra morphism,

$$\begin{aligned}
\pi_V \circ \Phi_W &= \pi_V \circ (\Phi_S \leftsquigarrow (\lambda_W \circ \Theta_V)) \\
&= (\pi_V \circ \Phi_S) \leftsquigarrow (\lambda_W \circ \Theta_V) \\
&= (P_{chr_S} \circ \Theta_V) \leftsquigarrow (\lambda_W \circ \Theta_V) \\
&= (P_{chr_S} \leftsquigarrow \lambda_W) \circ \Theta_V \\
&= P_{chr_W} \circ \Theta_V.
\end{aligned}$$

The proof is similar for Φ_0. $\square$

3.3 Invertible characters

Let us fix a non unitary, commutative and cocommutative bialgebra $(V, \cdot, \delta_V)$. The product of the dual algebra is denoted by $\star_V$; its unit is the counit ϵ_V. Let us now study the monoid of characters of $(\mathcal{F}_V[\mathbf{G}], m, \delta)$, which product is denoted by $\star$, and in particular let us look for its invertible. We shall use the following lemma:

Lemma 3.9. *Let* (B, m, δ) *be a graded bialgebra. In particular, its homogeneous component of degree* 0 *is a subbialgebra. Let* λ *be a character of* B. *Then* λ *is an invertible character of* B *if, and only if, its restriction* B_0 *to* B_0 *is invertible in the algebra* B_0^*. *In the particular case where* B_0 *is generated by a family* $(x_i)_{i\in I}$ *if group-like elements,* λ *is an invertible character if, and only if,* $\lambda(x_i) \neq 0$ *for any* $i \in I$.

Proof. We shall denote by π_k the canonical projection on B_k for any $k \in \mathbb{N}$. We put

$$\rho_L = (\pi_0 \otimes \mathrm{Id}) \circ \delta, \qquad \rho_R = (\pi \otimes \pi_0) \circ \delta.$$

As $\pi_0 : B \longrightarrow B_0$ is a bialgebra map, (B, ρ_L, ρ_R) is a B_0-bicomodule. For any $x \in B_n$ with $n \geqslant 1$, we put

$$\delta'(x) = \delta(x) - \rho_L(x) - \rho_R(x).$$

By homogeneity of δ,

$$\delta(x) = \sum_{i=0}^{n} (\pi_i \otimes \pi_{n-i}) \circ \delta(x) = \rho_L(x) + \underbrace{\sum_{i=1}^{n-1} (\pi_i \otimes \pi_{n-i}) \circ \delta(x)}_{\delta'(x)} + \rho_R(x).$$

$\Longrightarrow$. Let us denote by μ the inverse of λ in the monoid of characters of B. We put $\mu_0 = \mu_{|B_0}$. For any $x \in B_0$,

$$\lambda_0 * \mu_0(x) = (\lambda \otimes \mu) \circ \delta_0(x) = (\lambda \otimes \mu) \circ \delta(x) = \lambda * \mu(x) = \epsilon(x).$$

Similarly, $\mu_0 * \lambda_0(x) = \epsilon(x)$, so λ_0 is invertible in B_0^*.

$\Longleftarrow$. Let us define $\mu_n : B_n \longrightarrow \mathbb{K}$ for any n, such that if $x \in B_n$,

$$\left(\lambda \otimes \sum_{i=0}^{n} \mu_i\right) \circ \delta(x) = \epsilon(x).$$

We proceed by induction on n. If $n=0$, we us take μ_0 the inverse of λ_0 in B_0^*. Let us assume $n \geqslant 1$ and $\mu_0, \ldots, \mu_{n-1}$ defined. We first define $\nu : B_n \longrightarrow \mathbb{K}$ by

$$\forall x \in B_n, \qquad \nu(x) = \epsilon(x) - \left(\lambda \otimes \sum_{i=0}^{n-1} \mu_i\right)(\delta'(x) + \rho_R(x)).$$

This is well-defined, as $\delta'(x) + \rho(x) \in B \otimes \bigoplus_{i=0}^{n-1} B_i$.. We then put $\mu_n = (\mu_0 \otimes \nu) \circ \rho_L : B_n \longrightarrow \mathbb{K}$. As $\rho_L = (\pi_0 \otimes \mathrm{Id}) \circ \delta$, by homogeneity of δ, $\rho_L(B_n) \subseteq B_0 \otimes B_n$ and μ_n is well-defined. For any $x \in B_n$,

$$\begin{aligned}
\left(\lambda \otimes \sum_{i=0}^{n} \mu_n\right) \circ \delta(x) &= (\lambda \otimes \mu_n) \circ \rho_L(x) + \epsilon(x) - \nu(x) \\
&= (\lambda \otimes \mu \circ \nu) \circ (\mathrm{Id} \otimes \rho_L) \circ \rho_L(x) + \epsilon(x) - \nu(x) \\
&= (\lambda \otimes \mu \circ \nu) \circ (\delta_0 \otimes \mathrm{Id}) \circ \rho_L(x) + \epsilon(x) - \nu(x) \\
&= ((\lambda_0 * \mu_0) \circ \nu) \circ \rho_L(x) + \epsilon(x) - \nu(x) \\
&= (\epsilon_0 \circ \nu) \circ \rho_L(x) + \epsilon(x) - \mu(x) \\
&= \nu(x) + \epsilon(x) - \mu(x) \\
&= \epsilon(x).
\end{aligned}$$

Considering $\mu = \sum_{i=0}^{\infty} \mu_i \in B^*$, by construction $\lambda * \mu = \epsilon$. Similarly, we can define $\mu' \in B^*$ such that $\mu' * \lambda = \epsilon$. Then, as the convolution product $*$ is associative, $\mu' = \mu$ and λ is invertible in B^*. Let us now prove that μ is a character. We work in the algebra $(B \otimes B)^*$, which convolution product is also denoted by $*$. For any $x, y \in B$, with Sweedler's notation $\delta(z) = \sum_z z^{(1)} \otimes z^{(2)}$ for any $z \in B$,

$$\begin{aligned}
(\mu \circ m) * (\lambda \circ m)(x \otimes y) &= \sum_x \sum_y \mu(x^{(1)} y^{(1)}) \lambda(x^{(1)} y^{(1)}) \\
&= \sum_{xy} \mu((xy)^{(1)}) \lambda((xy)^{(2)}) \\
&= \epsilon(xy) \\
&= \epsilon(x)\epsilon(y) \\
&= \epsilon_{B \otimes B}(x \otimes y).
\end{aligned}$$

Similarly, $(\lambda \circ m) * (\mu \circ m) = \epsilon_{B \otimes B}$, so, as λ is a character,

$$\mu \circ m = (\lambda \circ m)^{*-1} = (\lambda \otimes \lambda)^{*-1} = \mu \otimes \mu.$$

So μ is indeed a character of B.

Let us now consider the particular case where B_0 is generated by a family $(x_i)_{i \in I}$ of group-like elements.

$\Longrightarrow$. If λ is an invertible character, denoting its inverse by μ, for any $i \in I$,

$$\lambda * \mu(x_i) = \epsilon(x_i) = 1 = \lambda(x_i)\mu(x_i),$$

so $\lambda(x_i) \neq 0$.

$\Longleftarrow$. Let us assume that λ is a character of B such that $\lambda(x_i) \neq 0$ for any $i \in I$. In order to prove that λ is an invertible character, it is enough to prove that λ_0 is invertible in B_0^*. By hypothesis, B_0 has a basis $(y_j)_{j\in J}$ of monomials in $(x_i)_{i\in I}$. By multiplicativity, for any $j \in J$, y_j is group-like and $\lambda(y_j) \neq 0$. We then define $\mu_0 \in B_0^*$ by

$$\forall j \in J, \qquad \mu(y_j) = \frac{1}{\lambda(y_j)}.$$

Then for any $j \in J$,

$$\lambda_0 * \mu_0(y_j) = \mu_0 * \lambda_0(y_j) = \lambda_0(y_j)\mu_0(y_j) = 1 = \epsilon(y_j),$$

so λ_0 is invertible in B_0^*. □

In order to use this lemma, let us introduce a gradation of $(\mathcal{F}_V[\mathbf{G}], m, \delta)$.

Proposition 3.10. *For any V-linearly decorated mixed graph G, we denote by $\mathrm{cc}(G)$ the number of connected components of G and put*

$$\deg(G) = |V(G)| - \mathrm{cc}(G).$$

This defines a gradation of the bialgebra $(\mathcal{F}_V[\mathbf{G}], m, \delta)$.

Proof. Note that for any graph G, $\deg(G) \geqslant 0$. Let G and H be two V-linearly decorated mixed graphs. Then

$$|V(GH)| = |V(G)| + |V(H)|, \qquad \mathrm{cc}(GH) = \mathrm{cc}(G) + \mathrm{cc}(H),$$

so $\deg(GH) = \deg(G) + \deg(H)$. Let G be a V-linearly decorated mixed graph and $\sim \in \mathcal{E}^c[G]$. We denote by k the number of equivalence classes of k. As $\sim \in \mathcal{E}^c[G]$,

$$|V(G \mid \sim)| = |V(G)|, \qquad \mathrm{cc}(G \mid \sim) = k.$$

Moreover, the connected components of $G/\sim$ are the contractions of the connected components of G, so

$$|V(G/\sim)| = k, \qquad \mathrm{cc}(G/\sim) = \mathrm{cc}(G).$$

We obtain that $\deg(G/\sim) + \deg(G \mid \sim) = |V(G)| - k + k - \mathrm{cc}(G) = \deg(G)$. So $(\mathcal{F}_V[\mathbf{G}], m, \delta)$ is graded. □

For any graph G, $\deg(G) = 0$ if, and only if, $E(G) = A(G) = \emptyset$. The subbialgebra $\mathcal{F}_V[\mathbf{G}]_{\deg=0}$ of elements of degree 0 is the symmetric algebra generated by elements v, with $v \in V$. The coproduct of such an element is given by the coproduct of V,

$$\delta(v) = v' \otimes v''.$$

Proposition 3.11. *Let $\lambda \in \mathrm{Char}(\mathcal{F}_V[\mathbf{G}])$. We define a map $\lambda_V \in V^*$ by*

$$\forall v \in V, \qquad \lambda_V(v) = \lambda(v).$$

Then λ is invertible in $(\mathrm{Char}(\mathcal{F}_V[\mathbf{G}]), \star)$ if, and only if, λ_V is invertible in $(V^*, \star_V)$.

Proof. $\Longrightarrow$. Let us assume that λ is an invertible character. Denoting by μ its inverse, $\mu_{|V}$ provides an inverse of λ_V in V^*.

$\Longleftarrow$. Let us assume that λ_V is invertible in V^*. By Lemma 3.9, it is enough to prove that λ_0 is invertible in the algebra $\mathcal{F}_V[\mathbf{G}]_0$. By construction of the graduation, $\mathcal{F}_V[\mathbf{G}]_0$ is the symmetric algebra generated by V. Extending multiplicatively the inverse of λ_V to $\mathcal{F}_V[\mathbf{G}]_0$, we obtain an inverse of λ_0. □

In the particular case where $V = \mathbb{K}$:

Corollary 3.12. *Let λ be a character of $\mathcal{F}[\mathbf{G}]$. It is invertible*$\mathrm{Char}(\mathcal{F}[\mathbf{G}]), \star)$ *if, and only if,*

$$\lambda(\bullet) \neq 0.$$

Proof. This is implied by Lemma 3.9, with the family of group-like elements reduced to $\bullet$. □

In particular, λ_W and λ_0 are invertible. Their inverses are denoted respectively by ν_W and μ_S. We also put $\mu_W = \mu_S \star \lambda_W$. Then, as $P_{chr_W} = P_{chr_S} \leftarrow\!\!\!\!\leadsto \lambda_W$ and $\phi_0 = P_{chr_S} \leftarrow\!\!\!\!\leadsto \lambda_0$, we obtain

$$P_{chr_S} = P_{chr_W} \leftarrow\!\!\!\!\leadsto \nu_W, \qquad P_{chr_S} = \phi_0 \leftarrow\!\!\!\!\leadsto \mu_S, \qquad P_{chr_W} = \phi_0 \leftarrow\!\!\!\!\leadsto \mu_W.$$

and, similarly,

$$P_{chr_S} = \Phi_W \leftarrow\!\!\!\!\leadsto \nu_W, \qquad P_{chr_S} = \Phi_0 \leftarrow\!\!\!\!\leadsto \mu_S, \qquad \Phi_W = \Phi_0 \leftarrow\!\!\!\!\leadsto \mu_W.$$

Proposition 3.13. *For any mixed graph G,*

$$P_{chr_S}(G) = \sum_{\sim \in \mathcal{E}^c[G]} \lambda_0(G/\sim)\mu_S(G\mid\sim)X^{\mathrm{cl}(\sim)} = \sum_{\sim \in \mathcal{E}^c[G]} \nu_W(G\mid\sim)P_{chr_W}(G/\sim),$$
$$P_{chr_W}(G) = \sum_{\sim \in \mathcal{E}^c[G]} \lambda_0(G/\sim)\mu_W(G\mid\sim)X^{\mathrm{cl}(\sim)},$$

where $\mathrm{cl}(\sim)$ is the number of classes of $\sim$.

Corollary 3.14. *If G is a connected mixed graph, then $\mu_S(G)$ is the coefficient of X in $P_{chr_S}(G)$ whereas $\mu_W(G)$ is the coefficient of X in $P_{chr_W}(G)$. Moreover, $\lambda_0(G)$ is the coefficient of $X^{|V(G)|}$ in both $P_{chr_S}(G)$ and $P_{chr_W}(G)$.*

Proof. As G is connected, the unique element $\sim$ of $\mathcal{E}^c[G]$ with $\mathrm{cl}(\sim) = 1$ is $\sim_L$. So the coefficient of X in $P_{chr_S}(G)$ is, as λ_0 and ϵ coincide on $\mathcal{F}_V[\mathbf{G}]_{\deg=0}$, equal to

$$\lambda_0(G/\sim_L)\mu_S(G\mid\sim_L) = (\lambda_0 \otimes \mu_S) \circ \rho_L(G) = (\epsilon \otimes \mu_S) \circ \rho_L(G) = \mu_S(G).$$

Similarly, the unique equivalence $\sim \in \mathcal{E}^c[G]$ such that $\mathrm{cl}(\sim) = |V(G)|$ is $\sim_R$. So the coefficient of $X^{|V(G)|}$ in $P_{chr_S}(G)$ is, as μ_S and ϵ coincide on $\mathcal{F}_V[\mathbf{G}]_{\deg=0}$ equal to

$$\lambda_0(G/\sim_R)\mu_S(G\mid\sim_R) = (\lambda_0 \otimes \mu_S) \circ \rho_R(G) = (\lambda_0 \otimes \epsilon) \circ \rho_R(G) = \lambda_0(G).$$

The proof is similar for the weak chromatic polynomial. □

Remark 3.3.

1. One can define an infinitesimal character μ'_S by the following: for any mixed graph G,

$$\mu'_S(G) = \begin{cases} \mu_S(G) \text{ if } G \text{ is connected,} \\ 0 \text{ otherwise.} \end{cases}$$

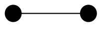
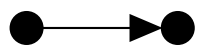
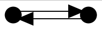
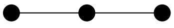
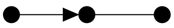
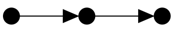
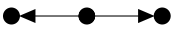
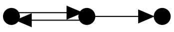
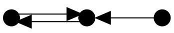
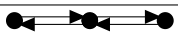
   This infinitesimal character is equal to $\ln(\epsilon_\delta)$ and is studied in [10, Proposition 4.1]. It is closely related to the eulerian idempotent.

2. Consequently, as the coefficient of X in $H_k(X)$ is $\dfrac{(-1)^{k+1}}{k}$ if $k \geqslant 1$, we obtain that for any connected mixed graph G,

$$\mu_S(G) = \sum_{c \in \mathrm{VPC}(G)} \frac{(1)^{\max(c)+1}}{\max(c)} = \sum_{k=1}^{\infty} |\{c \in \mathrm{VPC}(G) \mid \max(c) = k\} \frac{(-1)^{k+1}}{k}.$$

   When G is a rooted tree, we recover Murua's coefficients [3, 15], which appeared in the analysis of the continuous Baker–Campbell–Hausdorff problem.

Example 3.2.

G	$P_{chr_S}(G)$	$P_{chr_W}(G)$	$\lambda_0(G)$	$\mu_S(G)$	$\mu_W(G)$	$\nu_W(G)$
	$X(X-1)$	$X(X-1)$	1	-1	-1	0
	$\frac{X(X-1)}{2}$	$\frac{X(X+1)}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1
	0	X	0	0	1	-1
	$X(X-1)^2$	$X(X-1)^2$	1	1	1	0
	$\frac{X(X-1)^2}{2}$	$\frac{X(X+1)(X-1)}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0
	$\frac{X(X-1)^2}{2}$	$\frac{X(X+1)(X-1)}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0
	0	$X(X-1)$	0	0	-1	0
	$\frac{X(X-1)(X-2)}{6}$	$\frac{X(X+1)(X+2)}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	1
	$\frac{X(2X-1)(X-1)}{6}$	$\frac{X(2X+1)(X+1)}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	1
	0	$\frac{X(X+1)}{2}$	0	0	$\frac{1}{2}$	1
	0	$\frac{X(X+1)}{2}$	0	0	$\frac{1}{2}$	1
	0	X	0	0	1	1

Example 3.3. Let us consider again the graph G_n of Example 3.1. We obtain

$$\mu_S(G_n) = \frac{(-1)^n}{n!}, \qquad \mu_W(G_n) = \frac{(n-1)!}{n!} - 1 = -\frac{n-1}{n}, \qquad \lambda_0(G_n) = \frac{1}{n!}.$$

As an example of application:

Proposition 3.15. *1. $P_{chr_S} \circ \varpi_0 = P_{chr_S}$.*

2. Let G be an acyclic mixed graph. Then $P_{chr_S}(G)$ is of degree $|V(G)|$ and its leading term is $\ell(G)$.

Proof. 1. Note that $P_{chr_S} \circ \varpi_0$ and P_{chr_S} are bialgebra morphisms from $\mathcal{F}[\mathbf{G}]$ to $\mathbb{K}[X]$. Moreover,

$$\epsilon_\delta \circ P_{chr_S} \circ \varpi_0 = \epsilon_\delta \circ \varpi_0 = \epsilon_\delta = \epsilon_\delta \circ P_{chr_S}.$$

By unicity, $P_{chr_S} \circ \varpi_0 = P_{chr_S}$.

2. Let G be an acyclic graph. As $P_{chr_S} = \phi_0 \leftsquigarrow \mu_S$,

$$P_{chr_S}(G) = \sum_{\sim \in \mathcal{E}_c[G]} \mu_S(G\mid\sim)\ell(G/\sim)X^{\mathrm{cl}(\sim)},$$

which implies that $\deg(P_{chr_S}(G)) \leqslant |V(G)|$. If $\sim$ is the equality of $V(G)$, then $G\mid\sim$ is a graph with no edge, so $\mu_S(G\mid\sim) = 1$. We obtain that

$$P_{chr_S}(G) = \ell(G)X^{|V(G)|} + \text{terms of degree } < |V(G)|.$$

As G has no cycle, $\ell(G) \neq 0$, so $\deg(P_{chr_S}(G)) = |V(G)|$. $\square$

4 From mixed graphs to acyclic oriented graphs

4.1 A double bialgebra epimorphism

Definition 4.1. *Let G a mixed graph. An orientation of G is an oriented graph H with $V(G) = V(H)$ and $A(H) = A(G) \sqcup E'$, where E' is a set of arcs in bijection with $E(G)$, through a bijection respecting the extremities. Such an orientation of G is acyclic if H is an acyclic oriented graph. We denote by $O_{ac}(G)$ the set of acyclic orientations of G.*

Theorem 4.2. *Let G be a mixed graph. We put*

$$\Theta(G) = \sum_{G' \in \mathcal{O}_{ac}(G)} G',$$

with the usual convention that this sum is 0 if $\mathcal{O}_{ac}(G)$ is empty. Then Θ is a double twisted bialgebra morphism from $\mathbf{G}$ to $\mathbf{G}_{aco}$.

Proof. Firstly, Θ is indeed a species morphism from $\mathbf{G}$ to $\mathbf{G}_{aco}$. Let G and H be two mixed graphs, G' and H' be orientations of G and H. Then $G'H'$ is an acyclic orientation of GH if, and only if, G' and H' are acyclic orientations of G and H. This implies directly that $\Theta(GH) = \Theta(G)\Theta(H)$. So Θ is a twisted algebra morphism.

Let $G \in \mathbf{G}[I \sqcup J]$ be a mixed graph. If J is not an ideal of G, then $\Delta_{I,J}(G) = 0$. Moreover, for any orientation G' of G, I is not an ideal of G', so $\Delta_{I,J}(G') = 0$. Hence, in this case,

$$(\Theta \otimes \Theta) \circ \Delta_{I,J}(G) = \Delta_{I,J} \circ \Theta(G) = 0.$$

We now assume that J is an ideal of G. Then

$$\begin{aligned}(\Theta \otimes \Theta) \circ \Delta_{I,J}(G) &= \sum_{(G',G'') \in \mathcal{O}_{ac}(G_{|I}) \times \mathcal{O}_{ac}(G_{|J})} G' \otimes G'',\\ \Delta_{I,J} \circ \Theta(G) &= \sum_{\substack{H \in \mathcal{O}_{ac}(G),\\ J \text{ ideal of } H}} H_{|I} \otimes H_{|J}.\end{aligned}$$

We put

$$A = \mathcal{O}_{ac}(G_{|I}) \times \mathcal{O}_{ac}(G_{|J}), \qquad B = \{H \in \mathcal{O}_{ac}(G) \mid J \text{ ideal of } H\},$$

and we consider the map

$$\psi : \left\{ \begin{array}{rcl} B & \longrightarrow & A \\ H & \longrightarrow & (H_{|I}, H_{|J}). \end{array} \right.$$

This is obviously well-defined. We now consider the map $\psi' : A \longrightarrow B$, sending any pair (G', G'') to an orientation $\psi(G', G'') = H$ of G defined in this way: for any edge $\{x, y\}$ of G,

- If $x, y \in I$, then orient this edge as in G'.
- If $x, y \in J$, then orient this edge as in G''.
- If $x \in I$ and $y \in J$, then orient this edge from x to y.
- If $x \in J$ and $y \in I$, then orient this edge from y to x.

As there is no arc in H from J to I, J is an ideal of H. Moreover, as G' and G'' are acyclic, H is acyclic: ψ' is well-defined. We immediately obtain that $\psi' \circ \psi = \mathrm{Id}_B$ and $\psi \circ \psi' = \mathrm{Id}_A$, so ψ and ψ' are bijections. Therefore,

$$(\Theta \otimes \Theta) \circ \Delta_{I,J}(G) = \sum_{(G',G'))\in A} G' \otimes G'' = \sum_{H\in B} H_{|I} \otimes H_{|J} = \Delta_{I,J} \circ \Theta(G).$$

So $\Theta : (\mathbf{G}, m, \Delta) \longrightarrow (\mathbf{G}_{aco}, m, \Delta)$ is a twisted bialgebra morphism.

Let $G \in \mathbf{G}[I \sqcup J]$ be a mixed graph and $\sim \in \mathcal{E}(G)$. If H is an orientation of G, as the paths in G and H are the same, $\sim \in \mathcal{E}_\sim(G)$ if, and only if, $\sim \in \mathcal{E}_\sim(H)$. Hence, if $\sim \notin \mathcal{E}_c(G)$,

$$\delta_\sim \circ \Theta(G) = (\Theta \otimes \Theta) \circ \delta_\sim(G) = 0.$$

Let us now assume that $\sim \in \mathcal{E}_c(G)$. Then

$$\begin{aligned}
\delta_\sim \circ \Theta(G) &= \sum_{\substack{H\in\mathcal{O}_{ac}(G),\\ H/\sim \text{ acyclic}}} H/\sim \otimes H\mid\sim,\\
(\Theta \otimes \Theta) \circ \delta_\sim(G) &= \sum_{(G',G'')\in\mathcal{O}_{ac}(G/\sim)\times\mathcal{O}_{ac}(G\mid\sim)} G' \otimes G''.
\end{aligned}$$

We put

$$\begin{aligned}
C &= \mathcal{O}_{ac}(G/\sim) \times \mathcal{O}_{ac}(G\mid\sim),\\
D &= \{H \in \mathcal{O}_{ac}(G) \mid H/\sim \text{ acyclic}\}.
\end{aligned}$$

If $H \in D$, then $H/\sim$ is an acyclic orientation of $G/\sim$ by definition of D and $H\mid\sim$ is an acyclic orientation of $G\mid\sim$ by restriction. This defines a map

$$\phi : \left\{ \begin{array}{rcl} C & \longrightarrow & D \\ H & \longrightarrow & (H/\sim, H\mid\sim). \end{array} \right.$$

Let us now consider $(G', G'') \in C$. We define an orientation of G in the following way: if $\{x, y\}$ is an edge of G,

- If $x \sim y$, then $\{x, y\}$ is an edge of $G\mid\sim$: orient it as in G''.
- Otherwise, $\{\overline{x}, \overline{y}\}$ is an edge or an arc of $G/\sim$: orient $\{x, y\}$ as $\{\overline{x}, \overline{y}\}$ in G': as G' is acyclic, this is unambiguous.

Note that $H/\sim = G'$ and $H\mid\sim = G''$ by construction. Moreover, this is an acyclic orientation of G: if $x_1 \to \ldots \to x_k \to x_1$ is a cycle in H, as G' is acyclic, necessarily $x_1 \sim \ldots \sim x_k$, so this is a cycle in G'': as G'' is acyclic, this is not possible. Moreover, $H/\sim = G'$ is acyclic, so this defines a map $\phi' : D \longrightarrow C$ such that $\phi \circ \phi' = \mathrm{Id}_D$.

Let $H \in C$. We put $H' = \phi' \circ \phi(H)$. Let (x, y) be an arc of H. If $x \sim y$, then (x, y) is an arc of $H\mid\sim$, so is an arc of H'. Otherwise, $(\overline{x}, \overline{y})$ is an arc of $H/\sim = H'/\sim$. If (y, x) is an arc of H', then $\overline{x} \to \overline{y} \to \overline{x}$ is a cycle in $H'/\sim$, so $H/\sim$ is not acyclic: this is a contradiction. So (x, y) is an arc of H'. Therefore, H and H' have the same arcs, so are equal. We proved that $\phi' \circ \phi = \mathrm{Id}_C$, so ϕ is a bijection. We obtain:

$$\delta_\sim \circ \Theta(G) = \sum_{H\in D} H/\sim \otimes H\mid\sim = \sum_{(G',G'')\in C} G' \otimes G'' = (\Theta \otimes \Theta) \circ \delta_\sim(G).$$

So Θ is compatible with δ. It is obviously compatible with the unit and both counits. $\square$

Consequently, $\mathcal{F}[\Theta] : \mathcal{F}[\mathbf{G}] \longrightarrow \mathcal{F}[\mathbf{G}_{aco}]$ is a double bialgebra morphism. By composition, $P_{chr_S} \circ \mathcal{F}[\Theta] : \mathcal{F}[\mathbf{G}] \longrightarrow \mathbb{K}[X]$ is a double bialgebra morphism. By unicity of such a morphism,

$$P_{chr_S} \circ \mathcal{F}[\mathbf{G}] = P_{chr_S}.$$

Corollary 4.3. *For any mixed graph G,*

$$P_{chr_S}(G) = \sum_{H \in \mathcal{O}_{ac}(G)} P_{chr_S}(H).$$

4.2 Erhahrt polynomials for mixed graphs

Proposition 4.4. *Let $q \in \mathbb{K}$. The two following maps are characters of $\mathcal{F}[\mathbf{G}]$:*

$$\mathrm{ehr}_{str}^{(q)} : \begin{cases} \mathcal{F}[\mathbf{G}] & \longrightarrow \mathbb{K} \\ G & \longmapsto \begin{cases} q^{|V(G)|} \text{ if } A(G) = \emptyset, \\ 0 \text{ otherwise,} \end{cases} \end{cases} \qquad \mathrm{ehr}^{(q)} : \begin{cases} \mathcal{F}[\mathbf{G}] & \longrightarrow \mathbb{K} \\ G & \longmapsto q^{|V(G)|}, \end{cases}$$

with the convention $q^0 = 1$ even if $q = 0$. We denote by $\mathrm{Ehr}_{str}^{(q)}$ and by $\mathrm{Ehr}^{(q)}$ the unique Hopf algebra morphisms from $(\mathcal{F}[\mathbf{G}], m, \Delta)$ to $(\mathbb{K}[X], m, \Delta)$ such that

$$\epsilon_\delta \circ \mathrm{Ehr}_{str}^{(q)} = \mathrm{ehr}_{str}^{(q)}, \qquad \epsilon_\delta \circ \mathrm{Ehr}^{(q)} = \mathrm{ehr}^{(q)}.$$

Then, for any $n \in \mathbb{N}$, for any mixed graph G,

$$\begin{aligned} \mathrm{Ehr}_{str}^{(q)}(G)(n) &= q^{|V(G)|} \sharp\{f : V(G) \longrightarrow [n] \mid \forall x, y \in V(G),\ x \xrightarrow{G} y \Longrightarrow f(x) < f(y)\}, \\ \mathrm{Ehr}^{(q)}(G)(n) &= q^{|V(G)|} \sharp\{f : V(G) \longrightarrow [n] \mid \forall x, y \in V(G),\ x \xrightarrow{G} y \Longrightarrow f(x) \leqslant f(y)\}. \end{aligned}$$

Moreover, $\mathrm{Ehr}_{str}^{(q)} \circ \varpi_0 = \mathrm{Ehr}_{str}^{(q)}$.

From now, we shall write simply ehr_{str}, ehr, Ehr_{str} and Ehr for $\mathrm{ehr}_{str}^{(1)}$, $\mathrm{ehr}^{(1)}$, $\mathrm{Ehr}_{str}^{(1)}$ and $\mathrm{Ehr}^{(1)}$.

Proof. The maps $\mathrm{ehr}^{(q)}$ and $\mathrm{ehr}_{str}^{(q)}$ are obviously characters of $\mathcal{F}[\mathbf{G}]$. For any mixed graph G,

$$\begin{aligned} \mathrm{Ehr}_{str}^{(q)}(G) &= \sum_{k=1}^{\infty} \sum_f \mathrm{ehr}_{str}^{(q)}(G_{|f^{-1}(1)}) \ldots \mathrm{ehr}_{str}^{(q)}(G_{|f^{-1}(k)}) H_k(X) \\ &= \sum_{k=1}^{\infty} \sum_f q^{|f^{-1}(1)|} \mathrm{ehr}_{str}^{(1)}(G_{|f^{-1}(1)}) \ldots q^{|f^{-1}(k)|} \mathrm{ehr}_{str}^{(1)}(G_{|f^{-1}(k)}) H_k(X) \\ &= q^{|V(G)|} \sum_{k=1}^{\infty} \sum_f \mathrm{ehr}_{str}^{(1)}(G_{|f^{-1}(1)}) \ldots \mathrm{ehr}_{str}^{(1)}(G_{|f^{-1}(k)}) H_k(X) \\ &= q^{|V(G)|} \mathrm{Ehr}_{str}(G), \end{aligned}$$

where the second sum is over the surjective maps $f : V(G) \longrightarrow [k]$ such that for any $x, y \in V(G)$,

$$x \xrightarrow{G} y \Longrightarrow f(x) \leqslant f(y).$$

Similarly, $\mathrm{Ehr}^{(q)}(G) = q^{|V(G)|} \mathrm{Ehr}(G)$. We now study Ehr_{str} and Ehr.

$$\mathrm{Ehr}_{str}(G) = \sum_{k=1}^{\infty} \sum_f \mathrm{ehr}_{str}(G_{|f^{-1}(1)}) \ldots \mathrm{ehr}_{str}(G_{|f^{-1}(k)}) H_k(X),$$

where the second sum is over the surjective maps $f : V(G) \longrightarrow [k]$ such that for any $x, y \in V(G)$,

$$x \xrightarrow{G} y \Longrightarrow f(x) \leqslant f(y).$$

By definition of ehr_{str},

$$\mathrm{Ehr}_{str}(G) = \sum_{k=1}^{\infty} \sum_{f} H_k(X),$$

where the second sum is over the surjective maps $f : V(G) \longrightarrow [k]$ such that for any $x, y \in V(G)$,

$$x \xrightarrow{G} y \Longrightarrow f(x) < f(y),$$

which implies the announced result. Similarly,

$$\mathrm{Ehr}_{str}(G) = \sum_{k=1}^{\infty} \sum_{f} \mathrm{ehr}_{str}(G_{|f^{-1}(1)}) \ldots \mathrm{ehr}_{str}(G_{|f^{-1}(k)}) H_k(X) = \sum_{k=1}^{\infty} \sum_{f} H_k(X),$$

where the second sum is over the surjective maps $f : V(G) \longrightarrow [k]$ such that for any $x, y \in V(G)$,

$$x \xrightarrow{G} y \Longrightarrow f(x) \leqslant f(y).$$

This implies the announced result.

Let us prove that $\mathrm{Ehr}^{(q)}_{str} \circ \varpi_0 = \mathrm{Ehr}^{(q)}_{str}$. For this, it is enough to prove that $\epsilon_\delta \circ \mathrm{Ehr}^{(q)}_{str} \circ \varpi_0 = \epsilon_\delta \circ \mathrm{Ehr}^{(q)}_{str}$, that is to say $\mathrm{ehr}^{(q)}_{str} \circ \varpi_0 = \mathrm{ehr}^{(q)}_{str}$. Let G be a graph. If G is not acyclic, then $\mathrm{ehr}^{(q)}_{str} \circ \varpi_0(G) = 0$. Moreover, necessarily $A(G) \neq \emptyset$, so $\mathrm{ehr}^{(q)}_{str} = 0$. Otherwise, $\varpi_0(G) = G$ and $\mathrm{ehr}^{(q)}_{str} \circ \varpi_0(G) = \mathrm{ehr}^{(q)}_{str}(G)$. □

Remark 4.1. By [10, Corollary 3.12],

$$\mathrm{Ehr}_{str} = P_{chr_S} \leftsquigarrow \mathrm{ehr}_{str}, \qquad \mathrm{Ehr} = P_{chr_S} \leftsquigarrow \mathrm{ehr}.$$

Notations 4.1. Let G be a mixed graph. We denote by $S(G)$ the set of vertices $y \in V(G)$ such that there exists no $e \in A(G)$ such that y is the final vertex of e (set of *sources* of G) and by $W(G)$ the set of vertices $x \in V(G)$ such that there exists no $e \in A(G)$ such that x is the initial vertex of e (set of *wells* of G).

Proposition 4.5. *Let $q, q' \in \mathbb{K}$. For any mixed graph G, denoting by $*$ the convolution product associated to Δ,*

$$\begin{aligned} \mathrm{ehr}^{(q)}_{str} * \mathrm{ehr}^{(q')}(G) &= q'^{|V(G)\setminus S(G)|}(q+q')^{|S(G)|}, \\ \mathrm{ehr}^{(q)} * \mathrm{ehr}^{(q')}_{str}(G) &= q'^{|V(G)\setminus W(G)|}(q+q')^{|W(G)|}. \end{aligned}$$

Proof. Indeed,

$$\mathrm{ehr}^{(q)}_{str} * \mathrm{ehr}^{(q')}(G) = \sum q^{|I_1|} q'^{|I_2|},$$

where the sum is over all partitions $V(G) = I_1 \sqcup I_2$ such that if $x \xrightarrow{G} y$ in $V(G)$, then $(x, y) \in (I_1 \times I_2) \cup I_2^2$. Hence,

$$\begin{aligned} \mathrm{ehr}^{(q)}_{str} * \mathrm{ehr}^{(q')}(G) &= \sum_{I_1 \subseteq S(G)} q^{|I_1|} q'^{|V(G)\setminus I_1|} \\ &= q'^{|V(G)\setminus S(G)|} \sum_{I_1 \subseteq S(G)} q^{|I_1|} q'^{|S(G)\setminus I_1|} \\ &= q'^{|V(G)\setminus S(G)|}(q+q')^{|S(G)|}. \end{aligned}$$

Moreover,

$$\mathrm{ehr}^{(q)} * \mathrm{ehr}^{(q')}_{str}(G) = \sum q^{|I_1|}q'^{|I_2|},$$

where the sum is over all partitions $V(G) = I_1 \sqcup I_2$ such that if $x \xrightarrow{G} y$ in $V(G)$, then $(x,y) \in (I_1 \times I_2) \cup I_1^2$. Hence,

$$\begin{aligned}
\mathrm{ehr}^{(q)}_{str} * \mathrm{ehr}^{(q')}(G) &= \sum_{I_1 \subseteq W(G)} q^{|I_1|}q'^{|V(G)\setminus I_1|}\\
&= q'^{|V(G)\setminus W(G)|}\sum_{I_1 \subseteq W(G)} q^{|I_1|}q'^{|W(G)\setminus I_1|}\\
&= q'^{|V(G)\setminus W(G)|}(q+q')^{|W(G)|}.
\end{aligned}$$

□

Corollary 4.6. *The inverse (for the convolution product $*$) of the restriction of* $\mathrm{ehr}^{(q)}$ *to* $\mathcal{F}[\mathbf{G}_{ac}]$ *is the restriction of* $\mathrm{ehr}^{(-q)}_{str}$.

Proof. Let G be an acyclic mixed graph. Let us prove that

$$\mathrm{ehr}^{(-q)}_{str} * \mathrm{ehr}^{(q)}(G) = \varepsilon_\Delta(G).$$

This is obvious if $G = 1$. Otherwise, as G is acyclic, then $S(G) \neq \emptyset$ and $W(G) \neq \emptyset$. Hence,

$$\mathrm{ehr}^{(-q)}_{str} * \mathrm{ehr}^{(q)}(G) = q^{|V(G)\setminus S(G)|}(q-q)^{|S(G)|} = 0 = \varepsilon_\Delta(G).$$

□

Let us now prove the duality principle for Ehrhart polynomials:

Corollary 4.7. *Let G be an acyclic mixed graph. Then*

$$\mathrm{Ehr}_{str}(G)(-X) = (-1)^{|V(G)|}\mathrm{Ehr}(G)(X).$$

Proof. We denote by $S_{\mathbf{G}}$ the antipode of $(\mathcal{F}[\mathbf{G}_{ac}], m, \Delta)$ and by S the antipode of $(\mathbb{K}[X], m, \Delta)$: for any $P \in \mathbb{K}[X]$, $S(P(X)) = P(-X)$. As $\mathrm{Ehr} : (\mathcal{F}[\mathbf{G}_{ac}], m, \Delta) \longrightarrow (\mathbb{K}[X], m, \Delta)$ is a Hopf algebra morphism,

$$\mathrm{Ehr}_{str}(G)(-X) = S \circ \mathrm{Ehr}_{str}(G) = \mathrm{Ehr}_{str} \circ S_{\mathbf{G}}(G).$$

Therefore, by the duality principle,

$$\begin{aligned}
\mathrm{Ehr}_{str}(G)(-1) &= S \circ \mathrm{Ehr}_{str}(G)(1)\\
&= \mathrm{Ehr}_{str} \circ S_{\mathbf{G}}(G)(1)\\
&= \mathrm{ehr}_{str} \circ S_{\mathbf{G}}(G)\\
&= \mathrm{ehr}^{*-1}_{str}(G)\\
&= \mathrm{ehr}^{(-1)}(G).
\end{aligned}$$

This implies that $\mathrm{Ehr}_{str} \circ S$ is the Hopf algebra morphism $P_{chr_S} \leftsquigarrow \mathrm{ehr}^{(-1)}$:

$$\begin{aligned}
\mathrm{Ehr}_{str} \circ S(G) &= (P_{chr_S} \otimes \mathrm{ehr}^{(-1)}) \circ \delta(G)\\
&= (-1)^{|V(G)|}(P_{chr_S} \otimes \mathrm{ehr}) \circ \delta(G)\\
&= (-1)^{|V(G)|}\mathrm{Ehr}(G)(X).
\end{aligned}$$

□

Corollary 4.8. *Let G be an acyclic oriented graph. Then*

$$\begin{aligned}
\mathrm{Ehr}_{str}(G) &= P_{chr_S}(G), & P_{chr_S}(G)(-1) &= (-1)^{|V(G)|};\\
\mathrm{Ehr}(G) &= P_{chr_W}(G), & P_{chr_W}(G)(-1) &= (-1)^{|V(G)|}\epsilon_\delta(G).
\end{aligned}$$

Proof. We work in $\mathcal{F}_{\mathbb{K}}[\mathbf{G}] = \mathcal{F}[\mathbf{G}]$. For any oriented acyclic graph H,

$$\mathrm{ehr}_{str}(H) = \epsilon_\delta(H) = \begin{cases} 1 \text{ if } A(H) = \emptyset, \\ 0 \text{ otherwise.} \end{cases}$$

Hence, $\mathrm{Ehr}_{str}(G) = P_{chr_S}(G)$: the restriction of Ehr to $\mathcal{F}[\mathbf{G}_{aco}]$ is P_{chr_S}. Moreover,

$$\mathrm{Ehr}_{str}(G)(-1) = (-1)^{|V(G)|}\mathrm{Ehr}(G)(1) = (-1)^{|V(G)|}\mathrm{ehr}(G) = (-1)^{|V(G)|}.$$

For any oriented acyclic graph H, $\mathrm{ehr}(H) = 1 = \lambda_W(H)$, so $\mathrm{Ehr}(G) = P_{chr_W}(G)$. Moreover,

$$\mathrm{Ehr}(G)(-1) = (-1)^{|V(G)|}\mathrm{Ehr}_{str}(G)(1) = (-1)^{|V(G)|}\mathrm{ehr}_{str}(G) = (-1)^{|V(G)|}\epsilon_\delta(G). \qquad \square$$

From Corollary 4.3, we obtain another proof of the following result, proved in a different way in [5, Theorem 3]:

Corollary 4.9. *Let G be a mixed graph. Then*

$$P_{chr_S}(G)(-1) = (-1)^{|V(G)|}\sharp\mathcal{O}_{ac}(G).$$

Remark 4.2. We recover the classical result on the chromatic polynomial when this corollary is applied to graphs [18].

From [10, Corollary 2.3]:

Corollary 4.10. *Denoting by S the antipode of $(\mathcal{F}[\mathbf{G}], m, \Delta)$, for any mixed graph G,*

$$S(G) = \sum_{\sim\in\mathcal{E}^c[G]} (-1)^{\mathrm{cl}(\sim)}\sharp\mathcal{O}_{ac}(G/\sim)G\mid\sim.$$

5 Applications to characters on mixed graphs

5.1 Weak chromatic polynomial

Proposition 5.1. *Let G be a totally acyclic mixed graph. Then*

$$P_{chr_W}(G)(-1) = \begin{cases} 0 \text{ if } A(G) \neq \emptyset, \\ (-1)^{|V(G)|}|\mathcal{O}_{ac}(G)| \text{ otherwise.} \end{cases}$$

Proof. If $A(G) = \emptyset$, then $P_{chr_W}(G) = P_{chr_S}(G)$, and the result comes from Corollary 4.9. Let us assume that $A(G) \neq \emptyset$. We proceed by induction on $|E(G)|$. If $E(G) = \emptyset$, then by definition, $P_{chr_W}(G) = \mathrm{Ehr}(G)$. By the duality principle for Ehrhart polynomials (Corollary 4.7),

$$P_{chr_W}(G)(-1) = (-1)^{|V(G)|}\mathrm{Ehr}_{str}(G)(1) = 0,$$

as $A(G) \neq \emptyset$. Let us assume the result for all acyclic graph H such that $|E(G)| > |E(H)|$ and $A(H) \neq \emptyset$. Let e be an edge of G. We denote respectively by G/e and by $G\backslash e$ the mixed graph obtained from G by contraction of the edge e respectively by deleting the edge e. From [5, Proposition 6],

$$P_{chr_W}(G)(-1) = P_{chr_W}(G\backslash e)(-1) - P_{chr_W}(G/e)(-1).$$

Moreover, $G\backslash e$ and G/e are mixed graph with at least one arc and strictly less edges than G. Moreover, as G is totally acyclic, G/e and $G\backslash e$ are acyclic: we deduce that $P_{chr_W}(G\backslash e)(-1) = P_{chr_W}(G/e)(-1) = 0$. Hence, $P_{chr_W}(G)(-1) = 0$. $\square$

Remark 5.1. If G is not totally acyclic, no interpretation of $P_{chr_W}(G)(-1)$, and even of its sign, is known. For example, if G_n is the graph of Example 3.1, then $P_{chr_W}(G_n)(-1) = 1$ for any $n \geqslant 2$.

We recover the interpretation of [4] of the values of the weak chromatic polynomial at negative values:

Corollary 5.2. *Let G be a totally acyclic mixed graph and $k \in \mathbb{N}$. Then $(-1)^{|V(G)|}P_{chr_W}(G)(-k)$ is the number of pairs (H,f) such that:*

- *H is an acyclic orientation of G.*
- *f is a k-coloration of G compatible with H, that is, for any $x,y \in V(G)$,*

$$\begin{aligned} x \xrightarrow{G} y &\Longrightarrow f(x) < f(y),\\ x \xrightarrow{H} y &\Longrightarrow f(x) \leqslant f(y). \end{aligned}$$

Proof. By compatibility of P_{chr_W} with the coproduct Δ,

$$P_{chr_W}(G)(-k) = \sum_{\substack{f:V(G)\longrightarrow[k],\\ x\xrightarrow{G}y\Longrightarrow f(x)\leqslant f(y)}} P_{chr_W}\left(G_{|f^{-1}(1)}\right)(-1)\ldots P_{chr_W}\left(G_{|f^{-1}(k)}\right)(-1).$$

By the preceding proposition, if $G_{|f^{-1}(i)}$ has an arc, then $P_{chr_W}\left(G_{|f^{-1}(1)}\right)(-1) = 0$. Therefore,

$$\begin{aligned} P_{chr_W}(G)(-k) &= \sum_{\substack{f:V(G)\longrightarrow[k],\\ x\xrightarrow{G}y\Longrightarrow f(x)<f(y)}} P_{chr_W}\left(G_{|f^{-1}(1)}\right)(-1)\ldots P_{chr_W}\left(G_{|f^{-1}(k)}\right)(-1)\\ &= \sum_{\substack{f:V(G)\longrightarrow[k],\\ x\xrightarrow{G}y\Longrightarrow f(x)<f(y)}} (-1)^{|f^{-1}(1)|+\ldots+|f^{-1}(k)|}\prod_{i=1}^{k}\sharp\mathcal{O}_{ac}\left(G_{|f^{-1}(i)}\right)\\ &= (-1)^{|V(G)|}\sum_{\substack{f:V(G)\longrightarrow[k],\\ x\xrightarrow{G}y\Longrightarrow f(x)<f(y)}} \prod_{i=1}^{k}\sharp\mathcal{O}_{ac}\left(G_{|f^{-1}(i)}\right). \end{aligned}$$

We consider

$$\begin{aligned} A_k &= \{(f,H_1,\ldots,H_k) \mid f: V(G)\longrightarrow[k],\ H_i \in \mathcal{O}_{ac}\left(G_{|f^{-1}(i)}\right)\},\\ B_k &= \{(H,f) \mid H \in \mathcal{O}_{ac}(G),\ f: V(G)\longrightarrow[k] \text{ compatible with } H\}. \end{aligned}$$

The map $\phi : B_k \longrightarrow A_k$ which send (H,f) to $(f, H_{|f^{-1}(1)},\ldots,H_{|f^{-1}(k)})$ is well-defined. If $\phi(H,f) = \phi(H',f')$, then $f = f'$. Moreover, if $\{x,y\} \in E(G)$:

- If $f(x) = f(y) = i$, then $\{x,y\}$ is oriented in the same way in $H_{|f^{-1}(i)}$ and in $H'_{|f^{-1}(i)}$, as these oriented graphs are equal. So $\{x,y\}$ is oriented in the same way in H and in H'.
- If $f(x) < f(y)$, as f is compatible with H and in H', then $(x,y) \in E(H)$ and $(x,y) \in E(H')$.
- If $f(x) > f(y)$, as f is compatible with H and in H', then $(y,x) \in E(H)$ and $(y,x) \in E(H')$.

Therefore, $H = H'$: ϕ is injective. Let $(f,H_1,\ldots,H_k) \in A_k$. We define an orientation of G by the following: if $\{x,y\} \in E(G)$,

- if $f(x) = f(y) = i$, we keep the orientation of this edge in H_i.
- If $f(x) < f(y)$, we orient this edge in (x,y) in H.

- If $f(x) > f(y)$, we orient this edge in (y, x) in H.

By construction, for any $i \in [k]$, $H_{|f^{-1}(i)} = H_i$. Moreover, f is compatible with H, by construction. Consequently, f is constant on any cycle of H. As the oriented graphs H_i are acyclic, H is acyclic. We obtain that $(H, f) \in B_k$ and $\phi(H, f) = (f, H_1, \ldots, H_k)$. Finally,

$$P_{chr_W}(G)(-k) = (-1)^{|V(G)|}|A_k| = (-1)^{|V(G)|}|B_k|. \qquad \square$$

5.2 The character ν_W

Recall that ν_W is the inverse of λ_W for the convolution $\star$ associated to δ.

Proposition 5.3. *1. For any simple graph G with at least one edge, $\nu_W(G) = 0$.*

2. For any oriented graph G, $\nu_W(G) = (-1)^{|V(G)|+cc(G)}$, where $cc(G)$ is the number of connected components of G.

Proof. 1. We denote by λ_W^s the restriction of λ_W to $\mathcal{F}[\mathbf{G}_s]$. For any simple graph G, $\lambda_W(G) = \epsilon_\delta(G)$, so $\lambda_W^s = \epsilon_{\delta|\mathcal{F}_V[\mathbf{G}_s]}$. As $\mathcal{F}[\mathbf{G}_s]$ is a double subbialgebra of $\mathcal{F}[\mathbf{G}]$,

$$(\lambda_W^s)^{\star-1} = (\epsilon_{\delta|\mathcal{F}[\mathbf{G}_s]})^{\star-1} = (\epsilon_\delta{}^{\star-1})_{|\mathcal{F}[\mathbf{G}_s]} = \epsilon_{\delta|\mathcal{F}[\mathbf{G}_s]}.$$

2. Let λ be a character of $\mathcal{F}[\mathbf{G}_o]$, such that $\lambda(\bullet) \neq 0$. Then λ is invertible for the convolution product $\star$ associated to δ: its inverse is denoted by μ. Denoting by S the antipode of $(\mathcal{F}[\mathbf{G}_o], m, \Delta)$, let us prove that $\lambda \circ S$ is invertible for $\star$ and that its inverse is $\mu \star (\epsilon_\delta \circ S)$. Firstly, $\lambda \circ S(\bullet) = -\lambda(\bullet) \neq 0$, so $\lambda \circ S$ is invertible. Moreover,

$$\begin{aligned}
(\lambda \circ S) \star \mu \star (\epsilon_\delta \circ S) &= (\lambda \otimes \mu \circ \epsilon_\delta) \circ (S \otimes \mathrm{Id} \otimes S) \circ (\delta \otimes \mathrm{Id}) \circ \delta \\
&= (\lambda \otimes \mu \circ \epsilon_\delta) \circ (\mathrm{Id} \otimes \mathrm{Id} \otimes S) \circ (\delta \otimes \mathrm{Id}) \circ \delta \circ S \\
&= (\lambda \otimes \mu \circ \epsilon_\delta) \circ (\delta \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes S) \circ \delta \circ S \\
&= (\lambda \star \mu \otimes \epsilon_\delta) \circ (\mathrm{Id} \otimes S) \circ \delta \circ S \\
&= \epsilon_\delta \otimes \epsilon_\delta \circ (\mathrm{Id} \otimes S) \circ \delta \circ S \\
&= \epsilon_\delta \circ S^2 \\
&= \epsilon_\delta.
\end{aligned}$$

We used for the second equality that $(S \otimes \mathrm{Id}) \circ \delta = \delta \circ S$ (see [10, Proposition 2.1] and for the last equality that $S^2 = \mathrm{Id}$, as $\mathcal{F}[\mathbf{G}]$ is commutative. So $\mu \star (\epsilon_\delta \circ S) = (\lambda \circ S)^{\star-1}$.

In the particular case were $\lambda = \epsilon_\delta$, then $\mu = \epsilon_\delta$ and we obtain that $(\epsilon_\delta \circ S)^{\star-1} = \epsilon_\delta \circ S$.

Let G be an oriented graph. By definition of the weak Ehrhart polynomial and by the duality principle for Ehrhart polynomial (Corollary 4.7),

$$\begin{aligned}
\lambda_W(G) &= 1 \\
&= \mathrm{Ehr}_W(G)(1) \\
&= (-1)^{|V(G)|}\mathrm{Ehr}(G)(-1) \\
&= (-1)^{|V(G)|} S \circ \mathrm{Ehr}(G)(1) \\
&= (-1)^{|V(G)|}\mathrm{Ehr}(S(G))(1) \\
&= (-1)^{|V(G)|}\epsilon_\delta \circ S(G).
\end{aligned}$$

We now consider the three characters of $\mathcal{F}[\mathbf{G}_o]$ defined on any oriented graph G by

$$\lambda(G) = \epsilon_\delta \circ S(G), \qquad \mu(G) = (-1)^{|V(G)|}\lambda(G), \qquad \nu(G) = (-1)^{cc(G)}\lambda(G).$$

We already proved that $\lambda \star \lambda = \epsilon_\delta$. For any graph G,

$$\begin{aligned}
\mu \star \nu(G) &= \sum_{\sim \in \mathcal{E}^c[G]} (-1)^{|V(G/\sim)|+cc(G|\sim)} \lambda(G/\sim)\lambda(G\,|\!\sim) \\
&= \sum_{\sim \in \mathcal{E}^c[G]} (-1)^{2\mathrm{cl}(\sim)} \lambda(G/\sim)\lambda(G\,|\!\sim) \\
&= \sum_{\sim \in \mathcal{E}^c[G]} \lambda(G/\sim)\lambda(G\,|\!\sim) \\
&= \lambda \star \lambda(G) \\
&= \epsilon_\delta(G).
\end{aligned}$$

Therefore, $\nu = \mu^{\star -1}$. As $\mu = \lambda_W$, we obtain that $\nu = \nu_W$ and, for any oriented graph G,

$$\nu_W(G) = (-1)^{cc(G)}\lambda(G) = (-1)^{cc(G)+|V(G)|}\lambda_W(G) = (-1)^{cc(G)+|V(G)|}. \qquad \square$$

No interpretation of $\nu_W(G)$ is known in general. For example:

Proposition 5.4. *Let G_n be the mixed graph of Example 3.1, with the convention $G_2 = \bullet\!\longrightarrow\!\bullet$. For any $n \geqslant 2$, $\nu_W(G_n) = (-1)^{n-1}(n-1)$.*

Proof. For any $n \geqslant 2$,

$$\nu_W * \lambda_W(G_n) = \sum_{\sim \in \mathcal{E}^c[G_n]} \nu_W(G/\sim)\lambda_W(G\,|\!\sim) = \epsilon_\delta(G_n) = 0.$$

By definition of λ_W, for any $\sim \in \mathcal{E}^c[G_n]$, $\lambda_W(G\,|\!\sim) = 0$ if, and only if, $1 \sim n$. Therefore, the contributing terms corresponds to the equivalences which classes are intervals of $[n]$, at the exception of the one with only one class. For such an equivalence $\sim$, $G/\sim$ is isomorphic to $G_{\mathrm{cl}(\sim)}$. We obtain that if $n \geqslant 3$,

$$\sum_{k=2}^{n} \sum_{\substack{i_1+\ldots+i_k=n,\\ i_1,\ldots,i_k \geqslant 1}} \nu_W(G_k) = 0.$$

A direct computation shows that $\lambda_W(G_2) = -1$. Summing, we obtain in the ring of formal series $\mathbb{Q}[[X]]$ that

$$\sum_{k=2}^{\infty} \sum_{\substack{i_1+\ldots+i_k=n,\\ i_1,\ldots,i_k \geqslant 1}} \nu_W(G_k) X^{i_1+\ldots+i_n} = \sum_{k=2}^{\infty} \nu_W(G_k)\left(\frac{X}{1-X}\right)^k = -X^2.$$

Substituting $\dfrac{X}{1+X}$ to X, we obtain

$$\sum_{k=2}^{\infty} \nu_W(G_k) X^k = -\frac{X^2}{(1+X)^2} = \sum_{k=2}^{\infty} (-1)^{k+1}(k-1)X^k. \qquad \square$$

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