Hopf algebraic structures on hypergraphs and multi-complexes

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Abstract

Using the formalism of species and twisted objects, we introduce two structures of cointeracting bialgebras on hypergraphs, induced by two notions of induced sub-hypergraphs. We study the associated unique morphisms of cointeracting bialgebras from hypergraphs to the polynomial algebra in one indeterminate: in the first case, this gives the chromatic polynomial of a graph attached to the considered hypergraph. In the second case, we obtained Helgason's notion of chromatic polynomial of a hypergraph. We obtain Hopf-algebraic proves of results about the values of this chromatic polynomial in -1 or about its coefficients, with the help of the action of a monoid of characters. This allows to give multiplicity-free formulas for the antipodes of these objects, using various notions of acyclic orientations of hypergraphs.

Mixing the two notions of induced sub-hypergraphs, we obtain a third Hopf algebra, firstly described by Aguiar and Ardila. We obtain negative results on the existence of a second coproduct making it a cointeracting bialgebra. Anyway, it is still possible to obtain a polynomial invariant from this structure, which is the chomatic polynomial described by Aval, Kharagbossian and Tanasa.

We finally study Iovanov and Jaiung's Hopf algebra of multi-complexes, making it a cointeracting bialgebra which has for quotient one of the preceding cointeracting bialgebras of hypergraphs.

Keywords. double bialgebra; hypergraphs; chromatic polynomial; acyclic orientations; multi-complexes

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Contents

Introduction

Hypergraphs (a name due to Claude Berge in the sixties) are generalisations of graphs, where edges can contain an arbitrary number of vertices. A lot of classical notions on graphs can be extended to hypergraphs: sub-hypergraphs, colourings, orientations, and so on, see for example [5, 7, 8, 18, 25, 26]. We are here interested in Hopf-algebraic aspects of hypergraphs, with applications to colouring and orientations in the spirit of the results obtained in [13] for graphs.

We firstly construct four graded and connected Hopf algebras of hypergraphs, all based on the space $\mathcal{F}[\mathbf{H}]$ generated by the isoclasses of hypergraphs. For convenience, we choose to work in the framework of species [20, 21], instead of "classical" Hopf algebras, which are obtained by application of the bosonic Fock functor defined by Aguiar and Mahajan [2]. Note that coloured versions of these Hopf algebras can also be obtained by the application of coloured Fock functors [16]. For all these Hopf algebras, the product is the disjoint union of hypergraphs. The coproducts are based on two different notions of induced sub-hypergraphs: if G is a hypergraph and I is a subset of the set of vertices of G, the edges of the induced sub-hypergraph $G_{|_\subset I}$ are the edges of G included in I, whereas the edges of the induced sub-hypergraph $G_{|_\cap I}$ are the intersections of edges of G with I. This allows to define four coproducts: for $(\leftthreetimes, \rightthreetimes) \in \{\subset, \cap\}^2$, the coproduct $\Delta^{(\leftthreetimes, \rightthreetimes)}$ is given on any hypergraph G by

$$\Delta^{(\leftthreetimes, \rightthreetimes)}(G) = \sum_{I \subset V(G)} G_{|_\leftthreetimes I} \otimes G_{|_\rightthreetimes V(G) \setminus I},$$

where $V(G)$ is the set of vertices of G. For example, denoting by T_n the hypergraph with n vertices and a unique edge containing all its vertices,

$$\begin{aligned}
\Delta^{(\subset,\subset)}(T_n) &= T_n \otimes 1 + 1 \otimes T_n + \sum_{k=1}^{n-1} \binom{n}{k} T_1^k \otimes T_1^{n-k},\\
\Delta^{(\cap,\cap)}(T_n) &= T_n \otimes 1 + 1 \otimes T_n + \sum_{k=1}^{n-1} \binom{n}{k} T_k \otimes T_{n-k},\\
\Delta^{(\cap,\subset)}(T_n) &= T_n \otimes 1 + 1 \otimes T_n + \sum_{k=1}^{n-1} \binom{n}{k} T_k \otimes T_1^{n-k},\\
\Delta^{(\subset,\cap)}(T_n) &= T_n \otimes 1 + 1 \otimes T_n + \sum_{k=1}^{n-1} \binom{n}{k} T_1^k \otimes T_{n-k}.
\end{aligned}$$

The two coproducts $\Delta^{(\subset,\subset)}$ and $\Delta^{(\cap,\cap)}$ are cocommutative, whereas $\Delta^{(\cap,\subset)}$ and $\Delta^{(\subset,\cap)}$ are opposite one from the other. All these Hopf algebras contain the Hopf algebra of graphs of [13]. Up to a quotient, $\Delta^{(\subset,\subset)}$ and $\Delta^{(\subset,\cap)}$ appear in the recent paper [10], under the notations Δ and Δ'. The coproduct $\Delta^{(\subset,\cap)}$ is introduced in [1] and studied in [4].

We then define two other coproducts $\delta^{(\subset)}$ and $\delta^{(\cap)}$ of contractions and extractions. This is done with the formalism of contraction-extraction coproducts exposed in [16]. If G is a hypergraph and $\sim$ is an equivalence on $V(G)$, we denote by $G/\sim$ the hypergraph which set of vertices is $V(G)/\sim$ and which edges are the nontrivial $\pi_\sim(e)$ for e edge of G, where $\pi_\sim : V(G) \longrightarrow V(G)/\sim$ is the canonical surjection. For $\leftthreetimes \in \{\subset, \cap\}$, we denote by $G \mid_\leftthreetimes \sim$ the disjoint union of induced subgraphs $G_{|_\leftthreetimes C}$ with $C \in V(G)/\sim$. We finally shall write that $\sim \in \mathcal{E}_\leftthreetimes[G]$ if each $G_{|_\leftthreetimes C}$ is a connected hypergraph. We then can define

$$\delta^{(\leftthreetimes)}(G) = \sum_{\sim \in \mathcal{E}_\leftthreetimes[G]} G/\sim \otimes G \mid_\leftthreetimes \sim .$$

For example, if $n \geqslant 2$,

$$\delta^{(\subset)}(T_n) = T_n \otimes T_1^n + T_1 \otimes T_n,$$
$$\delta^{(\cap)}(T_n) = \sum_{n=1k_1+\ldots+nk_n} \frac{n!}{1!^{k_1}\ldots n!^{k_n} k_1!\ldots k_n!} T_{k_1+\ldots+k_n} \otimes T_1^{k_1}\ldots T_n^{k_n}.$$

We then obtain a double bialgebra $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\curlywedge,\curlywedge)}, \delta^{(\curlywedge)})$, that is to say:

- $(\mathcal{F}[\mathbf{H}], m, \delta^{(\curlywedge)})$ is a bialgebra.
- $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\curlywedge,\curlywedge)})$ is a bialgebra in the category of right comodules over the bialgebra $(\mathcal{F}[\mathbf{H}], m, \delta^{(\curlywedge)})$, with the coaction $\delta^{(\curlywedge)}$.

In particular, this implies the compatibility

$$(\Delta^{(\curlywedge,\curlywedge)} \otimes \mathrm{Id}) \circ \delta^{(\curlywedge)} = m_{1,3,24} \circ (\delta^{(\curlywedge)} \otimes \delta^{(\curlywedge)}) \circ \Delta^{(\curlywedge,\curlywedge)},$$

where

$$m_{1,3,24} : \left\{ \begin{array}{rcl} \mathcal{F}[\mathbf{H}]^{\otimes 4} & \longrightarrow & \mathcal{F}[\mathbf{H}]^{\otimes 3} \\ a_1 \otimes a_2 \otimes a_3 \otimes a_4 & \longmapsto & a_1 \otimes a_3 \otimes a_2 a_4. \end{array} \right.$$

The coproduct $\delta^{(\subset)}$ is different from the coproduct δ of [10], the difference coming from a different gestion of the contractions, seen as equivalences on the set of vertices here, and seen as contractions of edges in [10]. We did not find a convenient second coproduct for $\Delta^{(\subset,\cap)}$, but we have negative results about it (Proposition 2.7 and Corollary 2.8).

These results have interesting consequences. Let us fix $\curlywedge \in \{\subset, \cap\}$. A polynomial invariant of $\mathcal{F}[\mathbf{H}]$ is any Hopf algebra morphism from $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\curlywedge,\curlywedge)})$ to $(\mathbb{K}[X], m, \Delta)$, where Δ is the coproduct defined by

$$\Delta(X) = X \otimes 1 + 1 \otimes X.$$

The following results have been proved in [13, 11, 14]:

1. There exists a unique polynomial invariant $P_\curlywedge$, that is to say a map from $\mathcal{F}[\mathbf{H}]$ to $\mathbb{K}[X]$, which is also compatible with both bialgebraic structures, the second coproduct of $\mathbb{K}[X]$ being defined by $\delta(X) = X \otimes X$.
2. Moreover, any polynomial invariant can be obtained from $P_\curlywedge$ by an action $\leftsquigarrow$ of the monoid $\mathrm{Char}(\mathcal{F}[\mathbf{H}])$ of characters of the bialgebra $(\mathcal{F}[\mathbf{H}], m, \delta^{(\curlywedge)})$. Denoting by $\mathcal{P}_\curlywedge$ the set of polynomial invariants of $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\curlywedge,\curlywedge)})$, the following maps are two bijections, inverse one from the other:
$$\left\{ \begin{array}{rcl} \mathrm{Char}(\mathcal{F}[\mathbf{H}]) & \longrightarrow & \mathcal{P}_\curlywedge \\ \lambda & \longmapsto & P_\curlywedge \leftsquigarrow_\curlywedge \lambda = (P_\curlywedge \otimes \lambda) \circ \delta^{(\curlywedge)}, \end{array} \right.$$
$$\left\{ \begin{array}{rcl} \mathcal{P}_\curlywedge & \longrightarrow & \mathrm{Char}(\mathcal{F}[\mathbf{H}]) \\ \phi & \longmapsto & \left\{ \begin{array}{rcl} \mathcal{F}[\mathbf{H}] & \longrightarrow & \mathbb{K} \\ G & \longmapsto & \phi(G)(1). \end{array} \right. \end{array} \right.$$
3. The antipode of $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\curlywedge,\curlywedge)})$, denoted by $S_\curlywedge$, is given by
$$S_\curlywedge = (P_{\curlywedge|X=-1} \otimes \mathrm{Id}) \circ \delta^{(\curlywedge)}.$$

We prove in Proposition 2.1 that, if $\curlywedge = \subset$, for any hypergraph G, $P_\subset(G)$ is a polynomial such that for any $N \geqslant 0$, $P_\subset(G)(N)$ is the number of N-colourings of G, that is to say maps $f : V(G) \longrightarrow \{1, \ldots, N\}$, such that f is not constant on any non trivial edge of G; if $\curlywedge = \cap$, for any hypergraph G, $P_\cap(G)$ is a polynomial such that for any $N \geqslant 0$, $P_\cap(G)(N)$ is the number of N-colourings of G such that f is injective on any edge of G. The polynomial $P_\cap(G)$ is in fact the chromatic polynomial of the graph $\Gamma(G)$ obtained by replacing any hyperdge of G by a complete graph with the same set of vertices. The polynomial $P_\subset(G)$ is generally not the chromatic polynomial of a graph. It seems that its first appearance can be found in [18], see also [6, 9, 25, 27]. It is denoted by $\chi_{E,V}$ in [10]. This method cannot be applied to $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\cap,\subset)})$ by lack of the second coproduct. Anyway, it is still possible to define a chromatic polynomial invariant $P_{\cap,\subset}$, which plays the role of the $P_\curlywedge$ in the sense that for any hypergraph G,

$$P_{\cap,\subset}(G)(1) = P_\subset(G)(1) = P_\cap(G)(1) = \begin{cases} 1 \text{ if } G \text{ has no non trivial edge,} \\ 0 \text{ otherwise.} \end{cases}$$

We prove in Proposition 2.1 that this invariant counts the number of colourings f such that on any edge of G, the maximum of f is obtained exactly one time: this is the chromatic polynomial of [3, 4].

In order to find the antipode, we need to consider values of $P_\curlywedge$ at -1. As for graphs (Stanley's theorem), this is related to acyclic orientations. Here, an acyclic orientation is a partial quasi-order on the vertices, satisfying certain conditions, see Definition 2.9. We then obtain interpretations of $P_\subset(G)(-1)$ and $P_\cap(G)(-1)$ in terms of these orientations (Theorem 2.11), and this is used to give explicit formulas for the antipodes of $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\cap,\cap)})$ and $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\subset,\subset)})$, see Corollary 2.14. A combinatorial interpretation of $P_{\cap,\subset}(G)(-1)$ is also given in Theorem 2.11 and the antipode of $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\cap,\subset)})$ is described, with the help of Takeuchi's formula, in proposition 2.16.

Using the inverse of a particular character, we give a new proof of a formula on the coefficients of the chromatic polynomial $P_\subset(G)$, which can be found in [25, 27], see Proposition 2.18. We also give some results on decorated versions of $\mathcal{F}[\mathbf{H}]$, where the space of decorations is taken into a commutative and cocommutative bialgebra. This allows to replace $\mathbb{K}[X]$ by a quasishuffle algebra, and the unique double bialgebra morphism replacing the chromatic polynomials are described in Propositions 2.20 and 2.21. They are also based on colourings of graphs.

The last section of this text is devoted to multi-complexes. These objects, introduced in [19], generalize graphs, multigraphs, hypergraphs, Δ-complexes, and simplicial complexes. We prove that the bialgebraic structure of [19] can be extended to a double bialgebra structure, and that $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\subset,\subset)}, \delta^{(\subset)})$ is a quotient of this structure. Consequently, the unique polynomial invariant of multi-complexes compatible with both coproducts factorizes through the chromatic polynomial $P_\subset$ of the underlying hypergraphs, which allows to give formulas for the antipode and the eulerian projector for mutli-complexes.

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Notations 0.1.

1. We denote by $\mathbb{K}$ a commutative field of characteristic zero. Any vector space in this field will be taken over $\mathbb{K}$.
2. For any $N \in \mathbb{N}$, we denote by $[N]$ the set $\{1, \ldots, N\}$. In particular, $[0] = \emptyset$.
3. If (C, Δ) is a (coassociative but not necessarily counitary) coalgebra, we denote by $\Delta^{(n)}$ the n-th iterated coproduct of C: $\Delta^{(1)} = \Delta$ and if $n \geqslant 2$,
$$\Delta^{(n)} = \left(\Delta \otimes \mathrm{Id}^{\otimes(n-1)}\right) \circ \Delta^{(n-1)} : C \longrightarrow C^{\otimes(n+1)}.$$
4. If (B, m, Δ) is a bialgebra of unit 1_B and of counit ε_B, let us denote by $B_+ = \mathrm{Ker}(\varepsilon_B)$ its augmentation ideal. We define a coproduct on B_+ by
$$\forall x \in B_+, \qquad \tilde{\Delta}(x) = \Delta(x) - x \otimes 1_B - 1_B \otimes x.$$

Then $(B_+, \tilde{\Delta})$ is a coassociative (not necessarily counitary) coalgebra.

5. Let $\mathbf{P}$ be a species. For any finite set X, the vector space associated to X by $\mathbf{P}$ is denoted by $\mathbf{P}[X]$. For any bijection $\sigma : X \longrightarrow Y$ between two finite sets, the linear map associated to σ by $\mathbf{P}$ is denoted by $\mathbf{P}[\sigma] : \mathbf{P}[X] \longrightarrow \mathbf{P}[Y]$. The Cauchy tensor product of species is denoted by $\otimes$: if $\mathbf{P}$ and $\mathbf{Q}$ are two species, for any finite set X,

$$\mathbf{P} \otimes \mathbf{Q}[X] = \bigoplus_{X=Y \sqcup Z} \mathbf{P}[Y] \otimes \mathbf{Q}[Z].$$

If $\sigma : X \longrightarrow Y$ is a bijection between two finite sets, then

$$\mathbf{P} \otimes \mathbf{Q}[\sigma] = \bigoplus_{X=Y \sqcup Z} \mathbf{P}[\sigma_{|Y}] \otimes \mathbf{Q}[\sigma_{|Z}].$$

A twisted algebra (resp. coalgebra, bialgebra) is an algebra (resp. coalgebra, bialgebra) in the symmetric monoidal category of species with the Cauchy tensor product. We refer to [12, 16] for details and notations on algebras, coalgebras and bialgebras in the category of species.

6. Let V be a vector space. The V-coloured Fock functor $\mathcal{F}_V$, defined in [16, Definition 3.2], sends any species $\mathbf{P}$ to

$$\begin{aligned}
\mathcal{F}_V[\mathbf{P}] &= \bigoplus_{n=0}^{\infty} \mathrm{coInv}(V^{\otimes n} \otimes \mathbf{P}[n]) \\
&= \bigoplus_{n=0}^{\infty} \frac{V^{\otimes n} \otimes \mathbf{P}[n]}{\mathrm{Vect}(v_1 \ldots v_n \otimes \mathbf{P}[\sigma](p) - v_{\sigma(1)} \ldots v_{\sigma(n)} \otimes p \mid \sigma \in \mathfrak{S}_n,\ p \in \mathbf{P}[n],\ v_1, \ldots, v_n \in V)} \\
&= V^{\otimes n} \otimes_{\mathfrak{S}_n} \mathbf{P}[n].
\end{aligned}$$

When $V = \mathbb{K}$, we obtain the bosonic Fock functor of [2]:

$$\mathcal{F}[\mathbf{P}] = \bigoplus_{n=0}^{\infty} \mathrm{coInv}(\mathbf{P}[n]) = \bigoplus_{n=0}^{\infty} \frac{\mathbf{P}[n]}{\mathrm{Vect}(\mathbf{P}[\sigma](p) - p \mid \sigma \in \mathfrak{S}_n,\ p \in \mathbf{P}[n])}.$$

1 Twisted bialgebras of hypergraphs

1.1 Definitions

Definition 1.1. *A hypergraph is a family $G = (V(G), E(G))$, where $V(G)$ is a finite set, called the set of vertices of G, and $E(G)$ is a subset of $\mathcal{P}(V(G))$, called the set of edges of G. For the sake of simplicity, for any hypergraph G we shall consider in this article, we shall assume that:*

- $\emptyset \in E(G)$.
- *For any $x \in V(G)$, $\{x\} \in E(G)$.*

If G is a hypergraph, we shall denote the set of its nontrivial edges by

$$E^+(G) = \{e \in E(G) \mid |e| \geqslant 2\}.$$

Under our assumption,

$$E(G) = E^+(G) \sqcup \{\emptyset\} \sqcup \{\{x\},\ x \in V(G)\}.$$

If I is a finite set, we shall denote by $\mathcal{H}[I]$ the set of hypergraphs G such that $V(G) = I$. This defines a set species $\mathcal{H}$. The linearization of this set species is denoted by $\mathbf{H}$: for any finite set I,

$$\mathbf{H}[I] = \mathrm{Vect}(\mathcal{H}[I]).$$

Remark 1.1. 1. A (simple) graph is a hypergraph G such that for any $e \in E^+(G)$, $|e| = 2$. This defines a set subspecies of $\mathcal{H}$ denoted by $\mathscr{G}_s$ and a subspecies of $\mathbf{H}$ denoted by $\mathbf{G}_s$. This species of graphs and its bialgebraic structures are studied in [14, 16, 17].

2. If I is a finite set of cardinality n, then

$$|\mathcal{H}[I]| = 2^{\sum_{k=2}^{n} \binom{n}{k}} = 2^{2^n-n-1}.$$

This is the de Bruijn's sequence, entry A016031 in the OEIS [22].

$\lvert I\rvert$	1	2	3	4	5	6
$\lvert\mathcal{H}[I]\rvert$	1	2	16	2048	67108864	144115188075855872

The following is the number h_n of isoclasses of hypergraphs according to the number of vertices n:

n	1	2	3	4	5	6
h_n	1	2	8	180	612032	200253854316544

This is sequence A317794 of the OEIS [22].

1.2 Twisted bialgebras of hypergraphs

Let us now define four twisted bialgebra structures on $\mathbf{H}$, with the help of two different notions of induced sub-hypergraphs.

Notations 1.1. 1. Let $G \in \mathcal{H}[X]$ and $I \subseteq X$.

(a) $G_{|\subset I}$ is the hypergraph such that

$$V(G_{|\subset I}) = I, \qquad E(G_{|\subset I}) = \{e \in E(G) \mid e \subset I\}.$$

(b) $G_{|\cap I}$ is the hypergraph such that

$$V(G_{|\cap I}) = I, \qquad E(G_{|\cap I}) = \{e \cap I \mid e \in E(G)\}.$$

Thanks to the conditions we imposed on hypergraphs, both $G_{|\subset I}$ and $G_{|\cap I}$ belong to $\mathcal{H}[I]$.

2. Let X and Y be two disjoint sets, $G \in \mathcal{H}[X]$ and $G' \in \mathcal{H}[Y]$. Then GG' is the hypergraph such that

$$V(GG') = X \sqcup Y, \qquad E(GG') = E(G) \sqcup E(G').$$

This defines an element of $\mathcal{H}[X \sqcup Y]$.

Lemma 1.2. *Let* $\curlywedge, \curlyvee \in \{\cap, \subset\}$.

1. *Let G be a hypergraph and $X \subseteq Y \subseteq G$. Then*

$$(G_{|\curlywedge Y})_{|\curlywedge X} = G_{|\curlywedge X}.$$

2. *Let G be a hypergraph and $I, J, K \subseteq V(G)$ such that $V(G) = I \sqcup J \sqcup K$. Then*

$$(G_{|\curlywedge J \sqcup K})_{|\curlyvee J} = (G_{|\curlyvee I \sqcup J})_{|\curlywedge J}.$$

Proof. 1. We first consider the case $\leftthreetimes = \cap$. Then

$$\begin{aligned}
E((G_{|\cap Y})_{|\cap X}) &= \{e \cap X \mid e \in E(G_{|\cap Y})\}\\
&= \{e \cap Y \cap X \mid e \in E(G)\}\\
&= \{e \cap X \mid e \in E(G)\}\\
&= E(G_{|\cap X}).
\end{aligned}$$

So $(G_{|\cap Y})_{|\cap X} = G_{|\cap X}$. Let us the consider the case $\leftthreetimes =\subset$.

$$\begin{aligned}
E((G_{|\subset Y})_{|\subset X}) &= \{e \in E(G_{|\subset Y}) \mid e \subset X\}\\
&= \{e \in E(G) \mid e \subset X,\ e \subset Y\}\\
&= \{e \in E(G) \mid e \subset X\}\\
&= E(G_{|\subset X}).
\end{aligned}$$

So $(G_{|\subset Y})_{|\subset X} = G_{|\subset X}$.

2. If $\leftthreetimes = \rightthreetimes$, both $(G_{|\leftthreetimes J\sqcup K})_{|\rightthreetimes J}$ and $(G_{|\rightthreetimes I\sqcup J})_{|\leftthreetimes J}$ are equal to $G_{|\leftthreetimes J}$ by the first point. Let us now consider the case $\leftthreetimes = \cap$ and $\rightthreetimes =\subset$.

$$\begin{aligned}
E((G_{|\cap J\sqcup K})_{|\subset J}) &= \{e \in E(G_{|\cap J\sqcup K}) \mid e \subseteq J\}\\
&= \{e \cap (J \sqcup K) \mid e \in E(G),\ e \cap (J \sqcup K) \subseteq J\}\\
&= \{e \cap J \mid e \in E(G),\ e \cap K = \emptyset\}.\\
E((G_{|\subset I\sqcup J})_{|\cap J}) &= \{e \cap J \mid e \in E(G_{|\subset I\sqcup J})\}\\
&= \{e \cap J \mid e \in E(G),\ e \subseteq I \sqcup J\}\\
&= \{e \cap J \mid e \in E(G),\ e \cap K = \emptyset\}.
\end{aligned}$$

Therefore, $(G_{|\cap J\sqcup K})_{|\subset J} = (G_{|\subset I\sqcup J})_{|\cap J}$. By symmetry of I and K, this also gives the proof for $\leftthreetimes =\subset$ and $\rightthreetimes = \cap$. □

Proposition 1.3. *Let $\leftthreetimes, \rightthreetimes \in \{\cap, \subset\}$. We define a twisted bialgebra structure $(\mathbf{H}, m, \Delta^{(\leftthreetimes,\rightthreetimes)})$ on $\mathbf{H}$ by the following:*

- *For any $G \in \mathcal{H}[X]$, for any $G' \in \mathcal{H}[Y]$, $m_{X,Y}(G \otimes G') = GG'$.*
- *For any $G \in \mathcal{H}[I \sqcup J]$, $\Delta^{(\leftthreetimes,\rightthreetimes)}_{I,J}(G) = G_{|\leftthreetimes I} \otimes G_{|\rightthreetimes J}$.*

The coopposite coproduct of $\Delta^{(\leftthreetimes,\rightthreetimes)}$ is $\Delta^{(\rightthreetimes,\leftthreetimes)}$. Consequently, $\Delta^{(\subset,\subset)}$ and $\Delta^{(\cap,\cap)}$ are cocommutative.

Proof. The product m is obviously associative and its unit is the empty hypergraph.

Let I, J, I', J' be finite sets such that $I' \sqcup J' = I \sqcup J$, and let $G \in \mathcal{H}[I']$ and $G' \in \mathcal{H}[J']$. As the edges of GG' are included in I or in J,

$$\begin{aligned}
\Delta^{(\leftthreetimes,\rightthreetimes)}_{I,J} \circ m_{I',J'}(G \otimes G') &= (GG')_{|\leftthreetimes I} \otimes (GG')_{|TJ}\\
&= G_{|\leftthreetimes I\cap I'} G'_{|\leftthreetimes I\cap J'} \otimes G_{|\rightthreetimes J\cap I'} G'_{|\rightthreetimes J\cap J'}\\
&= (m_{I\cap I',I'\cap J'} \otimes m_{J\cap I',J\cap J'}) \circ (\mathrm{Id}_{\mathbf{H}[I\cap I']} \otimes c_{\mathbf{H}[J\cap I'],\mathbf{H}[I\cap J']} \otimes \mathrm{Id}_{\mathbf{H}[J\cap J']})\\
&\circ (\Delta^{(\leftthreetimes,\rightthreetimes)}_{I\cap I',J\cap I'} \otimes \Delta^{(\leftthreetimes,\rightthreetimes)}_{I\cap J',J\cap J'})(G \otimes G'),
\end{aligned}$$

so $\Delta^{(\leftthreetimes,\rightthreetimes)}$ is an algebra morphism.

Let us now prove the coassociativity of $\Delta^{(\leftthreetimes,\rightthreetimes)}$. If $G\in\mathcal{H}[I\sqcup J\sqcup K]$, by Lemma 1.2, first item,

$$\begin{aligned}
(\Delta_{I,J}^{(\leftthreetimes,\rightthreetimes)}\otimes \mathrm{Id})\circ\Delta_{I\sqcup J,K}^{(\leftthreetimes,\rightthreetimes)}(G)&=(G_{|_\leftthreetimes I\sqcup J})_{|_\leftthreetimes I}\otimes (G_{|_\leftthreetimes I\sqcup J})_{|_\rightthreetimes J}\otimes G_{|_\rightthreetimes K}\\
&=G_{|_\leftthreetimes I}\otimes (G_{|_\leftthreetimes I\sqcup J})_{|_\rightthreetimes J}\otimes G_{|_\rightthreetimes K},\\
(\mathrm{Id}\otimes\Delta_{J,K}^{(\leftthreetimes,\rightthreetimes)})\circ\Delta_{I,J\sqcup K}^{(\leftthreetimes,\rightthreetimes)}(G)&=G_{|_\leftthreetimes I}\otimes (G_{|_\rightthreetimes J\sqcup K})_{|_\leftthreetimes J}\otimes (G_{|_\rightthreetimes J\sqcup K})_{|_\rightthreetimes K}\\
&=G_{|_\leftthreetimes I}\otimes (G_{|_\rightthreetimes J\sqcup K})_{|_\leftthreetimes J}\otimes G_{|_\rightthreetimes K}.
\end{aligned}$$

By Lemma 1.2, second item, $(G_{|_\leftthreetimes I\sqcup J})_{|_\rightthreetimes J}=(G_{|_\rightthreetimes J\sqcup K})_{|_\leftthreetimes J}$, so $\Delta^{(\leftthreetimes,\rightthreetimes)}$ is coassociative. These four coproducts share the same counit, defined by $\varepsilon(\emptyset)=1$. □

Example 1.1. For any finite set X with at least two elements, let us denote by T_X the hypergraph which vertices set is X, with X as a unique nontrivial edge. For any finite nonempty disjoint sets I and J,

$$\Delta_{I,J}^{(\subset,\subset)}(T_{I\sqcup J})=\prod_{x\in I}T_{\{x\}}\otimes\prod_{y\in J}T_{\{y\}},\qquad \Delta_{I,J}^{(\cap,\cap)}(T_{I\sqcup J})=T_I\otimes T_J,$$
$$\Delta_{I,J}^{(\cap,\subset)}(T_{I\sqcup J})=T_I\otimes\prod_{y\in J}T_{\{y\}},\qquad \Delta_{I,J}^{(\subset,\cap)}(T_{I\sqcup J})=\prod_{x\in I}T_{\{x\}}\otimes T_J.$$

Remark 1.2.

1. As seen in Remark 1.1, graphs are hypergraphs, so **H** contains a subspecies of graphs, which is a twisted subbialgebra. Moreover, if G is a graph,
$$\Delta^{(\subset,\subset)}(G)=\Delta^{(\cap,\subset)}(G)=\Delta^{(\subset,\cap)}(G)=\Delta^{(\cap,\cap)}(G).$$
We recover the twisted bialgebra of graphs of [14, 16, 17].

2. Let $I_1,\ldots,I_n$ be disjoint sets. For any hypergraph $G\in\mathcal{H}[I_1\sqcup\ldots\sqcup I_n]$,
$$\Delta_{I_1,\ldots,I_n}^{(\leftthreetimes,\leftthreetimes)}(G)=G_{|_\leftthreetimes I_1}\otimes\ldots\otimes G_{|_\leftthreetimes I_n},$$
whereas
$$\Delta_{I_1,\ldots,I_n}^{(\subset,\cap)}(G)=G_{|^{(1)}I_1}\otimes\ldots\otimes G_{|^{(n)}I_n},$$
where for any $p\in[n]$,
$$V(G_{|^{(p)}I_p})=I_p,\qquad E(G_{|^{(p)}I_p})=\{e\cap I_p\mid e\in E(G),\ e\subseteq I_1\sqcup\ldots\sqcup I_p\}.$$
In other words, the nonempty edges of $G_{|^{(p)}I_p}$ are the sets $e\cap I_p$, where e runs among the edges of G such that
$$\max\{i\in[n]\mid e\cap I_i\neq\emptyset\}=p.$$

Notations 1.2. In the following, we shall simply write $\Delta^{(\leftthreetimes)}$ for $\Delta^{(\leftthreetimes,\leftthreetimes)}$ for $\leftthreetimes\in\{\subset,\cap\}$.

1.3 Contraction-extraction coproducts

In order to define double bialgebras of graphs, we shall use here the formalism of contraction-extraction coproducts of [16]. We introduce for this contractions of hypergraphs, with connectedness constraints.

Definition 1.4. *Let G be a hypergraph. A path in G is a sequence $(x_0,\ldots,x_k)$ of vertices of G such that for any $i\in[k]$, there exists an edge $e\in E(G)$ containing both x_{i-1} and x_i. We shall say that G is connected if for any $x,y\in V(G)$, there exists a path in G from X to Y. Any hypergraph G can be uniquely written as the product of connected hypergraphs, called the connected components of G.*

Notations 1.3. We use the notations of [16, Notations 2.1] for the equivalence relations. For any finite set X, $\mathcal{E}[X]$ is the set of equivalence relations on X. It is partially ordered by the refinement order: if $\sim, \sim' \in \mathcal{E}[X]$, then

$$\sim \leqslant \sim' \Longleftrightarrow (\forall x, y \in X,\ x \sim' y \Longrightarrow x \sim y).$$

If $\sim' \in \mathcal{E}[X]$, then $\{\sim \in \mathcal{E}[X] \mid \sim \leqslant \sim'\}$ and $\mathcal{E}[X/\sim']$ are in bijection, via the map sending $\sim$ to $\overline{\sim}$ defined by

$$\overline{x}\overline{\sim}\overline{y} \Longleftrightarrow x \sim y.$$

We identify in this way $\{\sim \in \mathcal{E}[X] \mid \sim \leqslant \sim'\}$ and $\mathcal{E}[X/\sim']$.

Definition 1.5. *Let $G \in \mathcal{H}[X]$, $\sim \in \mathcal{E}[X]$ and let $\curlywedge \in \{\subset, \cap\}$.*

- *We define the hypergraph $G/\sim \in \mathcal{H}[X/\sim]$ by*

$$V(G/\sim) = X/\sim, \qquad E(G/\sim) = \{\pi_\sim(e) \mid e \in E(G)\},$$

where $\pi_\sim : X \longrightarrow X/\sim$ is the canonical surjection. By the conditions we imposed in Definition 1.1 on hypergraphs, this is indeed a hypergraph.

- *We define the hypergraph $G \mid_\curlywedge \sim \in \mathcal{H}[X]$ by*

$$G \mid_\curlywedge \sim = \prod_{C \in X/\sim} G_{|_\curlywedge C}.$$

- *We shall say that $\sim \in \mathcal{E}_\curlywedge[G]$ if for any class C of $\sim$, $G_{|_\curlywedge C}$ is connected.*

Remark 1.3. For any hypergraph G and $\sim \in \mathcal{E}[V(G)]$, $V(G \mid_\cap \sim) = V(G \mid_\subset \sim) = V(G)$, and

$$E(G \mid_\cap \sim) = \{e \cap C \mid e \in E(G),\ C \in V(G)/\sim\},$$
$$E(G \mid_\subset \sim) = \{e \in E(G) \mid |\pi_\sim(e)| \leqslant 1\}.$$

By definition, if $\sim \in \mathcal{E}_\curlywedge[G]$, the connected components of $G_{|_\curlywedge C}$ are the classes of $\sim$.

Theorem 1.6. *Let $\curlywedge \in \{\cap, \subset\}$. For any hypergraph $G \in \mathcal{H}[X]$ and for any $\sim \in \mathcal{E}[X]$, we put*

$$\delta_\sim^{(\curlywedge)}(G) = \begin{cases} G/\sim \otimes G \mid_\curlywedge \sim & \text{if } \sim \in \mathcal{E}_\curlywedge[G], \\ 0 & \text{otherwise.} \end{cases}$$

This defines a contraction-extraction coproduct on $\mathbf{H}$ in the sense of [16], compatible with m and $\Delta^{(\curlywedge)}$.

Let us start the proof of this theorem with a combinatorial lemma.

Lemma 1.7. *Let $G \in [\mathbf{X}]$ and $\curlywedge \in \{\subset, \cap\}$.*

1. *If $\sim \leqslant \sim' \in \mathcal{E}[X]$, then the hypergraphs $(G/\sim')/\overline{\sim}$ and $G/\sim$ are equal.*
2. *Let $\sim \in \mathcal{E}[X]$. Then*

$$\sim \in \mathcal{E}_\curlywedge[G] \Longleftrightarrow \text{the connected components of } G \mid_\curlywedge \sim \text{ are the classes of } \sim.$$

3. *Let $\sim \in \mathcal{E}_\curlywedge[G]$. The connected components of $G/\sim$ are the images by $\pi_\sim$ of the connected components of G.*
4. *Let $\sim \leqslant \sim' \in \mathcal{E}[X]$. Then*

$$\sim' \in \mathcal{E}_\curlywedge[G] \text{ and } \overline{\sim} \in \mathcal{E}_\curlywedge[G/\sim'] \Longleftrightarrow \sim \in \mathcal{E}_\curlywedge[G] \text{ and } \sim' \in \mathcal{E}_\curlywedge[G \mid_\curlywedge \sim].$$

If this holds, $(G/\sim') \mid_\curlywedge \overline{\sim} = (G \mid_\curlywedge \sim)/\sim'$.

Proof. 1. Firstly, $V((G/\sim')/\overline{\sim}) = V(G)/\sim = V(G/\sim)$ and secondly,

$$E((G/\sim')/\overline{\sim}) = \{\pi_{\overline{\sim}} \circ \pi_{\sim'}(e) \mid e \in E(G)\} = \{\pi_{\sim}(e) \mid e \in E(G)\} = E(G/\sim).$$

Hence, $(G/\sim')/\overline{\sim} = G/\sim$.

2. Immediate consequence of the definition of $\mathcal{E}_{\curlywedge}[G]$, as $\pi_{\sim} = \pi_{\overline{\sim}} \circ \pi_{\sim'}$.

3. By definition of the connectivity, if H is a connected hypergraph and $\sim\in \mathcal{E}[V(H)]$, then $H/\sim$ is connected. Consequently, if H is a connected component of G, $\pi_{\sim}(H)$ is connected, so is included in a connected component of $G/\sim$: we proved that the connected components of $G/\sim$ are union of images by $\pi_{\sim}$ of connected components of G.

Let us consider the equivalence $\sim_G$ defined on $V(G)$ by

$$x \sim_G y \text{ if there exists a path in } G \text{ from } x \text{ to } y.$$

By definition, the classes of G are the connected components of G, and $\sim_G\in \mathcal{E}_{\curlywedge}[G]$. As $\sim\in \mathcal{E}_{\curlywedge}[G]$, its classes are connected, so are included in a single connected component of G: $\sim_G\leqslant\sim$. Therefore, if x and y are in two different connected components of G, then $\pi_{\sim}(x) \neq \pi_{\sim}(y)$ and there is no edge containing these two elements in $G/\sim$: any connected component of $G/\sim$ is included in a single $\pi_{\sim}(H)$, where H is a connected component of G.

4. $\Longrightarrow$. Let C be a class of $\sim'$. As $\sim\leqslant\sim'$, $(G\,|_{\curlywedge}\sim)_{|_{\curlywedge}C} = G_{|_{\curlywedge}C}$, so is connected as $\sim'\in \mathcal{E}_{\curlywedge}[G]$. Therefore, $\sim'\in \mathcal{E}_{\curlywedge}[G\,|_{\curlywedge}\sim]$.

Let C be a class of $\sim$ and let $x, y \in C$. As $\overline{\sim} \in \mathcal{E}_{\curlywedge}[G/\sim']$, there exists a path from $\pi_{\sim'}(x)$ to $\pi_{\sim'}(y)$ in $(G/\sim')\,|_{\curlywedge}\,\overline{\sim}$. We denote this path by $(\pi_{\sim'}(x_0), \ldots, \pi_{\sim'}(x_k))$. Note that all the elements $\pi_{\sim'}(x_p)$ are $\overline{\sim}$-equivalent, so all the elements x_p are $\sim$-equivalent. By definition of $G/\sim'$, we can assume that for any p, there exists y_p such that $x_p \sim' y_p$, and with an edge in G containing both y_p and x_{p+1}. As $\sim'\in \mathcal{E}_{\curlywedge}[G]$, for any p there exists a path from x_p to y_p, with all the vertices being $\sim'$-equivalent, so also $\sim$-equivalent as $\sim\leqslant\sim'$. Hence, there exists in G a path from x to y with all vertices being $\sim$-equivalent: C is connected, which proves that $\sim\in \mathcal{E}_{\curlywedge}[G]$.

$\Longleftarrow$. Let $G \in [\mathbf{X}]$ and $\sim\leqslant\sim'\in \mathcal{E}[X]$. Let us prove that if $\sim\in \mathcal{E}_{\curlywedge}[G]$ and $\sim'\in \mathcal{E}_{\curlywedge}[G\,|_{\curlywedge}\sim]$, then $\sim'\in \mathcal{E}_{\curlywedge}[G]$ and $\overline{\sim} \in \mathcal{E}_{\curlywedge}[G/\sim']$.

Let C be a class of $\sim'$. Then it is connected in $G\,|_{\curlywedge}\sim$, so also in G: we proved that $\sim'\in \mathcal{E}_{\curlywedge}[G]$. Let $\pi_{\sim'}(C)$ be a class of $\overline{\sim}$: as $\sim\leqslant\sim'$, we can assume that C is a class of $\sim$. As $\sim\in \mathcal{E}_{\curlywedge}[G]$, C is connected. By the third item of Lemma 1.7, $\pi_{\sim'}(C)$ is connected in $G/\sim'$, so $\overline{\sim} \in \mathcal{E}_{\curlywedge}[G/\sim']$.

Let us now prove the equality $(G/\sim')\,|_{\subset}\,\overline{\sim} = (G\,|_{\subset}\sim)/\sim'$. As $\sim\leqslant\sim'$,

$$\begin{aligned}
E((G/\sim')\,|_{\subset}\,\overline{\sim}) &= \{\pi_{\sim'}(e) \mid e \in E(G), \text{ all the elements of } \pi_{\sim'}(e) \text{ are } \overline{\sim}\text{-equivalent}\}\\
&= \{\pi_{\sim'}(e) \mid e \in E(G), \text{ all the elements of } e \text{ are } \sim\text{-equivalent}\}\\
&= E((G\,|_{\subset}\sim)/\sim').
\end{aligned}$$

Let us finally prove the equality $(G/\sim')\,|_{\cap}\,\overline{\sim} = (G\,|_{\cap}\sim)/\sim'$.

$$\begin{aligned}
E((G/\sim')\,|_{\cap}\,\overline{\sim}) &= \{\pi_{\sim'}(e) \cap C \mid e \in E(G),\ C \in V(G/\sim')/\overline{\sim}\}\\
&= \{\pi_{\sim'}(e) \cap \pi_{\sim'}(C) \mid e \in E(G),\ C \in V(G)/\sim\},\\
E((G\,|_{\cap}\sim)/\sim') &= \{\pi_{\sim'}(e \cap C) \mid e \in E(G),\ C \in V(G)/\sim\}.
\end{aligned}$$

Let $e \in E(G)$ and $C \in V(G)/\sim$. Obviously, $\pi_{\sim'}(e \cap C) \subseteq \pi_{\sim'}(e) \cap \pi_{\sim'}(C)$. Let $\overline{y} \in \pi_{\sim'}(e) \cap \pi_{\sim'}(C)$. There exists $y' \in e$ and $y'' \in C$ such that $y \sim' y' \sim' y''$. As $\sim\leqslant\sim'$, $y \sim y' \sim y''$, so $y' \in C$ and $\overline{y} = \overline{y'} \in \pi_{\sim'}(e \cap C)$. We proved that $\pi_{\sim'}(e \cap C) = \pi_{\sim'}(e) \cap \pi_{\sim'}(C)$, which implies that $(G/\sim')\,|_{\cap}\,\overline{\sim} = (G\,|_{\cap}\sim)/\sim'$. ◻

Proof. (Theorem 1.6). Let us first prove the coassociativity of $\delta^{(\curlywedge)}$, see [16, Definition 2.2, third item]. Let $G \in [\mathbf{X}]$ and $\sim \leqslant \sim' \in \mathcal{E}[X]$. Then, by Lemma 1.7, first item,

$$(\delta^{(\curlywedge)}_{\sim} \otimes \mathrm{Id}) \circ \delta^{(\curlywedge)}_{\sim'}(G) = \begin{cases} (G/\sim')/\overline{\sim} \otimes (G/\sim') \mid_{\curlywedge} \overline{\sim} \otimes G \mid_{\curlywedge} \sim' & \text{if } \sim' \in \mathcal{E}_{\curlywedge}[G] \text{ and } \overline{\sim} \in \mathcal{E}_{\curlywedge}[G/\sim'] \\ 0 \text{ otherwise} \end{cases}$$

$$= \begin{cases} G/\sim \otimes (G/\sim') \mid_{\curlywedge} \overline{\sim} \otimes G \mid_{\curlywedge} \sim' & \text{if } \sim' \in \mathcal{E}_{\curlywedge}[G] \text{ and } \overline{\sim} \in \mathcal{E}_{\curlywedge}[G/\sim'] \\ 0 \text{ otherwise} \end{cases}$$

$$(\mathrm{Id} \otimes \delta^{(\curlywedge)}_{\sim'}) \circ \delta^{(\curlywedge)}_{\sim}(G) = \begin{cases} G/\sim \otimes (G \mid_{\curlywedge} \sim)/\sim' \otimes (G \mid_{\curlywedge} \sim) \mid_{\curlywedge} \sim' & \text{if } \sim \in \mathcal{E}_{\curlywedge}[G] \text{ and } \sim' \in \mathcal{E}_{\curlywedge}[G \mid_{\curlywedge} \sim], \\ 0 \text{ otherwise} \end{cases}$$

$$= \begin{cases} G/\sim \otimes (G \mid_{\curlywedge} \sim)/\sim' \otimes G \mid_{\curlywedge} \sim' & \text{if } \sim \in \mathcal{E}_{\curlywedge}[G] \text{ and } \sim' \in \mathcal{E}_{\curlywedge}[G \mid_{\curlywedge} \sim], \\ 0 \text{ otherwise.} \end{cases}$$

By Lemma 1.7, fourth item,

$$(\delta^{(\curlywedge)}_{\sim} \otimes \mathrm{Id}) \circ \delta^{(\curlywedge)}_{\sim'}(G) = (\mathrm{Id} \otimes \delta^{(\curlywedge)}_{\sim'}) \circ \delta^{(\curlywedge)}_{\sim}(G).$$

Let us now prove the multiplicativity of $\delta^{(\curlywedge)}$, see [16, Proposition 2.4]. Let $G \in \mathbf{H}[X]$ and $G' \in \mathbf{H}[Y]$, and $\sim \in \mathcal{E}[X \sqcup Y]$. If $\sim \neq \sim_X \sqcup \sim_Y$, because of the connectivity condition, $\sim \notin \mathcal{E}_{\curlywedge}[GG']$, so $\delta^{(\curlywedge)}_{\sim}(GG') = 0$. Otherwise, $\sim \in \mathcal{E}_{\curlywedge}[GG']$ if, and only if, $\sim_X \in \mathcal{E}_{\curlywedge}[G]$ and $\sim_Y \in \mathcal{E}_{\curlywedge}[G']$, and, if this holds:

$$(GG')/\sim = (G/\sim_X)(G'/\sim_Y), \qquad (GG') \mid_{\curlywedge} \sim = (G \mid_{\curlywedge} \sim_X)(G' \mid_{\curlywedge} \sim_Y).$$

This implies that $\delta^{(\curlywedge)}_{\sim}(GG') = \delta^{(\curlywedge)}_{\sim_X}(G)\delta^{(\curlywedge)}_{\sim_Y}(G')$.

Let us prove the compatibility of $\delta^{(\curlywedge)}$ with $\Delta^{(\curlywedge)}$, see [16, Proposition 2.5]. Let $G \in \mathcal{H}[X \sqcup Y]$, $\sim_X \in \mathcal{E}[X]$ and $\sim_Y \in \mathcal{E}[Y]$. We put $\sim = \sim_X \sqcup \sim_Y \in \mathcal{E}[X \sqcup Y]$.

$$\begin{aligned} &(\Delta^{(\curlywedge)}_{X/\sim_X, Y/\sim_Y} \otimes \mathrm{Id}) \circ \delta^{(\curlywedge)}_{\sim}(G) \\ &= \begin{cases} (G/\sim)_{\mid_{\curlywedge} X/\sim_X} \otimes (G/\sim)_{\mid_{\curlywedge} Y/\sim_Y} \otimes G \mid_{\curlywedge} \sim & \text{if } \sim \in \mathcal{E}_{\curlywedge}[G], \\ 0 \text{ otherwise} \end{cases} \\ &= \begin{cases} (G_{\mid_{\curlywedge} X})/\sim_X \otimes (G_{\mid_{\curlywedge} Y})/\sim_Y \otimes (G \mid_{\curlywedge} \sim)_{\mid_{\curlywedge} X} (G \mid_{\curlywedge} \sim)_{\mid_{\curlywedge} Y} & \text{if } \sim_X \in \mathcal{E}_{\curlywedge}[G_{\mid_{\curlywedge} X}] \text{ and } \sim_Y \in \mathcal{E}_{\curlywedge}[G_{\mid_{\curlywedge} Y}] \\ 0 \text{ otherwise} \end{cases} \\ &= m_{1,3,24} \circ (\delta^{(\curlywedge)}_{\sim_X} \otimes \delta^{(\curlywedge)}_{\sim_Y}) \circ \Delta^{(\curlywedge)}_{X,Y}(G). \end{aligned}$$

Let us finally prove that $\delta^{(\curlywedge)}$ has a counit, see [16, Definition 2.2, fourth item]. For any hypergraph $G \in \mathcal{H}[X]$, we put

$$\epsilon_\delta[X](G) = \begin{cases} 1 \text{ if } E^+(G) = \emptyset, \\ 0 \text{ otherwise.} \end{cases}$$

If $G \in \mathcal{H}[X]$, let us denote by $\sim_0$ the equivalence on X which classes are the connected components of G. By definition, $\sim_0 \in \mathcal{E}_{\curlywedge}[G]$, $G \mid \sim_0 = G$ and $G/\sim_0$ is a hypergraph with no nontrivial edge. Moreover, if $\sim \in \mathcal{E}_{\curlywedge}[G]$ is different from $\sim_0$, by the second item of Lemma 1.7, at least one of the connected component of $G/\sim$ is not reduced to a vertex, so has a nontrivial edge: $\epsilon_\delta[X/\sim](G/\sim) = 0$. Hence,

$$(\epsilon_\delta \otimes \mathrm{Id}) \circ \delta^{(\curlywedge)}(G) = G \mid \sim_0 + 0 = G.$$

Let $\sim_1$ be the equality of X. Then $\sim_1 \in \mathcal{E}_{\curlywedge}[G]$, $G \mid \sim_1$ is a hypergraph with no nontrivial edge and $G/\sim_1 = G$. Moreover, if $\sim \in \mathcal{E}_{\curlywedge}[G]$ is different from $\sim_1$, at least one of its class is not reduced to a vertex, so, as it is connected, has a non trivial edge: $\epsilon_\delta[X](G \mid \sim) = 0$. Hence,

$$(\mathrm{Id} \otimes \epsilon_\delta) \circ \delta^{(\curlywedge)}(G) = G/\sim_1 + 0 = G.$$

So ϵ_δ is the counit of δ. □

Example 1.2. With the notations of Example 1.1,

$$\begin{aligned}
\delta^{(\subset)}(T_{\{x\}}) &= T_{\{x\}} \otimes T_{\{x\}},\\
\delta^{(\cap)}(T_{\{x\}}) &= T_{\{x\}} \otimes T_{\{x\}},\\
\delta^{(\subset)}(T_{\{x,y\}}) &= T_{\{x,y\}} \otimes T_{\{x\}}T_{\{y\}} + T_{\{\{x,y\}\}} \otimes T_{\{x,y\}},\\
\delta^{(\cap)}(T_{\{x,y\}}) &= T_{\{x,y\}} \otimes T_{\{x\}}T_{\{y\}} + T_{\{\{x,y\}\}} \otimes T_{\{x,y\}},\\
\delta^{(\subset)}(T_{\{x,y,z\}}) &= T_{\{x,y,z\}} \otimes T_{\{x\}}T_{\{y\}}T_{\{z\}} + T_{\{\{x,y,z\}\}} \otimes T_{\{x,y,z\}},\\
\delta^{(\cap)}(T_{\{x,y,z\}}) &= T_{\{x,y,z\}} \otimes T_{\{x\}}T_{\{y\}}T_{\{z\}} + T_{\{\{x,y,z\}\}} \otimes T_{\{x,y,z\}}\\
&\quad + T_{\{\{x,y\},\{z\}\}} \otimes T_{\{x,y\}}T_{\{z\}} + T_{\{\{x,y\},\{z\}\}} \otimes T_{\{x,z\}}T_{\{y\}} + T_{\{\{x,y\},\{z\}\}} \otimes T_{\{y,z\}}T_{\{x\}}.
\end{aligned}$$

Consequently, if V is a vector space, we obtain four bialgebra structures on $\mathcal{F}_V[\mathbf{H}]$. As a vector space, they are generated by isomorphism classes of linearly V-decorated hypergraphs, that is to say pairs (H, d_H), where H is a hypergraph and $d_H : V(G) \longrightarrow V$ is a map, with relations such that these decorations are linear in any vertex. The product is given by disjoint union. The coproducts are given on any V-decorated hypergraph G by

$$\Delta^{(\curlywedge,\curlyvee)}(G) = \sum_{I \subseteq V(G)} G_{|_{\curlywedge} I} \otimes G_{|_{\curlyvee} V(G)\setminus I},$$

where $(\curlywedge, \curlyvee) \in \{\subset, \cap\}^2$. Moreover, if $(V, \cdot, \delta_V)$ is a not necessarily unitary, commutative and cocommutative bialgebra, we obtain two double bialgebras $(\mathcal{F}_V[\mathcal{H}], m, \Delta^{(\curlywedge)}, \delta^{(\curlywedge)})$, with $\curlywedge \in \{\subset, \cap\}$. The coproduct $\delta^{(\curlywedge)}$ is defined on any V-decorated hypergraph G by

$$\delta^{(\curlywedge)}(G) = \sum_{\sim \in \mathcal{E}_{\curlywedge}[G]} G/\sim \otimes G \mid_{\curlywedge} \sim,$$

where the vertices of $G/\sim \otimes G \mid_{\curlywedge} \sim$ are decorated in the following way: denoting by $d_G(x)$ the decoration of the vertex $x \in V(G)$, any vertex $\mathrm{cl}_\sim(x)$ of $G/\sim$ is decorated by the products of elements $d_G(y)'$, where $y \in \mathrm{cl}_\sim(x)$, whereas the vertex $x \in V(G \mid \sim) = V(G)$ is decorated by $d_G(x)''$, and everything being extended by multilinearity of each decoration. The counit ϵ_δ is given on any mixed graph G by

$$\epsilon_\delta(G) = \begin{cases} \displaystyle\prod_{x \in V(G)} \epsilon_V \circ d_G(x) \text{ if } E^+(G) = \emptyset,\\ 0 \text{ otherwise.} \end{cases}$$

This construction is functorial in V.

In the particular case where $V = \mathbb{K}$, we obtain the bosonic Fock functor $\mathcal{F}[\mathbf{H}]$. As a vector space, a basis is given by isomorphisms classes of hypergraphs. It is given four bialgebra structures $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\curlywedge,\curlyvee)})$ and two double bialgebra structures $(\mathcal{F}[\mathcal{H}], m, \Delta^{(\curlywedge)}, \delta^{(\curlywedge)})$, with $\curlywedge, \curlyvee \in \{\subset, \cap\}$.

Example 1.3. For example, if T_n is the hypergraph with n vertices and a unique nontrivial edge e containing all vertices, we obtain, for $n \geqslant 2$,

$$\begin{aligned}
\Delta^{(\subset,\subset)}(T_n) &= T_n \otimes 1 + 1 \otimes T_n + \sum_{k=1}^{n-1} \binom{n}{k} T_1^k \otimes T_1^{n-k},\\
\Delta^{(\cap,\cap)}(T_n) &= T_n \otimes 1 + 1 \otimes T_n + \sum_{k=1}^{n-1} \binom{n}{k} T_k \otimes T_{n-k},\\
\Delta^{(\cap,\subset)}(T_n) &= T_n \otimes 1 + 1 \otimes T_n + \sum_{k=1}^{n-1} \binom{n}{k} T_k \otimes T_1^{n-k},\\
\Delta^{(\subset,\cap)}(T_n) &= T_n \otimes 1 + 1 \otimes T_n + \sum_{k=1}^{n-1} \binom{n}{k} T_1^k \otimes T_{n-k};\\
\delta^{(\subset)}(T_n) &= T_n \otimes T_1^n + T_1 \otimes T_n,\\
\delta^{(\cap)}(T_n) &= \sum_{n=1k_1+\ldots+nk_n} \frac{n!}{1!^{k_1}\ldots n!^{k_n} k_1!\ldots k_n!} T_{k_1+\ldots+k_n} \otimes T_1^{k_1}\ldots T_n^{k_n}.
\end{aligned}$$

Remark 1.4. In [10], twelve coproducts on hypergraphs are introduced. The hypergraphs considered there are more general than ours, as the conditions we impose on edges of cardinality $\leqslant 1$ is not required. Let us denote by $\mathbf{H}'$ the set of hypergraphs of [10] and by $\mathcal{H}'$ the space generated by the isoclasses. We define a map θ from $\mathbf{H}'$ to $\mathcal{F}[\mathbf{H}]$ by sending any $G \in \mathcal{H}'$ to:

- 0 if G has an empty edge or an edge of cardinality 1.
- The unique hypergraph $\theta(G) \in \mathcal{H}$ such that $E^+(\theta(G)) = E^+(G)$ (that is to say we add the empty set and all the singletons as edges).

It is then not difficult to prove that θ is a bialgebra morphism from $(\mathcal{H}', m, \Delta)$ to $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\subset,\subset)})$, and from $(\mathcal{H}', m, \Delta')$ to $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\subset,\cap)})$. The other coproducts Δ^d, Δ^c dans Δ^{cd} of [10], using duality and complementation, do not fit well with our context, because of the restrictions we impose on hypergraphs. The coproduct δ of [10] is not $\delta^{(\subset)}$, as shown by [10, Example 4.3].

Remark 1.5. We assume that for any hypergraph G, $\varnothing$ and the singletons $\{v\}$, with $v \in V(G)$, belong to $E(G)$. We can relax this hypothesis by only assuming that $\varnothing \in E(G)$. The objects we obtain in this way will be called general hypergraphs. General hypergraphs are identified with hypergraphs decorated by the set $\{0,1\}$: for any general hypergraph G, decorate its vertex $v \in V(G)$ by 1 if $\{v\} \in E(G)$ and by 0 otherwise. Therefore, choosing any two-dimensional commutative and cocommutative bialgebra with a basis (e_0, e_1) gives rise to two double bialgebra structures on generalized hypergraph. For example, choosing the product and coproducts defined by

$$\begin{aligned}
&e_0 \cdot e_0 = e_0, && e_0 \cdot e_1 = e_1, && \delta_V(e_0) = e_0 \otimes e_0,\\
&e_1 \cdot e_0 = e_1 && e_1 \cdot e_1 = e_1, && \delta_V(e_1) = e_1 \otimes e_1,
\end{aligned}$$

we obtain coproducts $\Delta^{(\subset)}$ and $\Delta^{(\cap)}$ given by induction of sub-hypergraphs, cointeracting with coproducts $\delta^{(\subset)}$ and $\Delta^{(\cap)}$ of contractions and extractions. For the contraction part, the vertex obtained by the identification of a subset X of $V(G)$ is part of an edge of cardinality 1 if, and only if, at least one of the element of X is part of an edge of G of cardinality 1.

Proposition 1.8. *Let V be a (non necessarily unitary) commutative and cocommutative bialgebra. For any linearly V-decorated hypergraph (G, d_G). The following map is a double bialgebra morphism:*

$$\Theta_V : \left\{ \begin{array}{rcl} \mathcal{F}_V[\mathbf{H}] & \longrightarrow & \mathcal{F}[\mathbf{H}] \\ (G, d_G) & \longmapsto & \left(\displaystyle\prod_{x \in V(G)} \epsilon_V \circ d_G(x) \right) G. \end{array} \right.$$

Proof. The counit $\epsilon_V : V \longrightarrow \mathbb{K}$ is a bialgebra morphism. By functoriality, Θ_V is a double bialgebra morphism. □

2 Polynomial invariants

2.1 Chromatic polynomials

From [14, Theorem 3.9], if $\curlywedge \in \{\cap, \subset\}$, there exists a unique morphism $P_\curlywedge$ of double bialgebras from $(\mathcal{F}_V[\mathbf{H}], m, \Delta^{(\curlywedge)}, \delta^{(\curlywedge)})$ to the double bialgebra $(\mathbb{K}[X], m, \Delta, \delta)$, with

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad \delta(X) = X \otimes X.$$

Let us determine $P_\curlywedge$. Let $G \in \mathcal{H}[X]$ be a nonempty hypergraph. Then, still from [14, Theorem 3.9],

$$P_\curlywedge(G) = \sum_{k=0}^{\infty} \left(\epsilon_\delta^{\otimes(k-1)} \circ \left(\tilde{\Delta}^{(\curlywedge)}\right)^{(k-1)}(G)\right) H_k(X),$$

where H_k is the k-th Hilbert polynomial:

$$H_k(X) = \frac{X(X-1)\ldots(X-k+1)}{k!}.$$

Proposition 2.1. *Let $\curlywedge \in \{\cap, \subset\}$. The unique double bialgebra morphism $P_\curlywedge$ from $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\curlywedge)}, \delta^{(\curlywedge)})$ to $(\mathbb{K}[X], m, \Delta, \delta)$ sends any hypergraph G to a polynomial $P_\curlywedge(G)$ such that, for any $N \in \mathbb{N}_{>0}$:*

- *$P_\cap(G)(N)$ is the number of maps $f : V(G) \longrightarrow [N]$ such that if x and y are two distinct elements of an edge $e \in E(G)$, then $f(x) \neq f(y)$.*
- *$P_\subset(G)(N)$ is the number of maps $f : V(G) \longrightarrow [N]$ such that for any nontrivial edge $e \in E(G)$, f takes at least two different values on e.*

Proof. We obtain that

$$\begin{aligned}
P_\curlywedge(G) &= \sum_{k=1}^{\infty} \sum_{\substack{V(G)=I_1\sqcup\ldots\sqcup I_k\\ I_1,\ldots,I_k\neq\emptyset}} \epsilon'_\delta(G_{|\curlywedge I_1}) \ldots \epsilon'_\delta(G_{|\curlywedge I_k}) H_k(X)\\
&= \sum_{k=1}^{\infty} \sum_{\substack{f:V(G)\longrightarrow[k]\\ f \text{ surjective}\\ \forall i\in[k],\, E^+(G_{|\curlywedge f^{-1}(i)})=\emptyset}} H_k(X).
\end{aligned}$$

Hence, for any $N \in \mathbb{N}_{>0}$, $P_\curlywedge(G)(N)$ is the number of maps $f : V(G) \longrightarrow [N]$ such that for any $i \in [N]$, $E^+(G_{|\curlywedge f^{-1}(i)}) = \emptyset$.

If $S = \cap$, this is equivalent to the fact that $f^{-1}(i)$ contains at most one vertex of any $e \in E^+(G)$, which gives the interpretation of the proposition. If $S =\subset$, this is equivalent to the fact that any $f^{-1}(i)$ does not contain any $e \in E^+(G)$, which gives the interpretation of the proposition. □

Remark 2.1. If G is a graph, both $P_\cap(G)$ and $P_\subset(G)$ are equal to the chromatic polynomial of G.

Even without a coproduct δ making $(\mathcal{F}_V[\mathbf{H}], m, \Delta^{(\subset,\cap)})$ a double bialgebra (see Proposition 2.7), we can define a Hopf algebra morphism, recovering the chromatic polynomial of [3, 4]:

Proposition 2.2. *For any hypergraph G, there exists a polynomial $P_{\subset,\cap}(G)$ such that for any $N \in \mathbb{N}_{>0}$, $P_{\subset,\cap}(G)(N)$ is the number of maps $f : V(G) \longrightarrow [N]$ such that for any $e \in E^+(G)$, $\max\{f(x) \mid x \in e\}$ is obtained in exactly one element of e. Then $P_{\subset,\cap} : (\mathcal{F}[\mathbf{H}], m, \Delta^{(\subset,\cap)}) \longrightarrow (\mathbb{K}[X], m, \Delta)$ is a Hopf algebra morphism.*

Proof. The map ϵ_δ is a character of $\mathcal{F}_V[\mathbf{H}]$. Hence, we obtain a bialgebra map

$$P_{\subset,\cap}(G) = \sum_{k=1}^{\infty} \sum_{\substack{V(G)=I_1\sqcup\ldots\sqcup I_k\\ I_1,\ldots,I_k\neq\emptyset}} \epsilon_\delta(G_{|^{(1)}I_1})\ldots\epsilon_\delta(G_{|^{(k)}I_k})H_k(X).$$

In other words,

$$P_{\subset,\cap}(G) = \sum_{k=1}^{\infty} \sum_{\substack{f:V(G)\longrightarrow[k]\\ f\text{ surjective}\\ \forall i\in[k],\, E^+(G_{|^{(i)}f^{-1}(i)})=\emptyset}} H_k(X).$$

By construction, for any $N \in \mathbb{N}_{>0}$, $P_{\subset,\cap}(G)(N)$ is the number of maps $f : V(G) \longrightarrow [N]$ such that for any $i \in [N]$, $E^+(G_{|^{(i)}f^{-1}(i)}) = \emptyset$. this is equivalent to the fact that for any edge e, $f^{-1}(\max(f_{|e})) \cap e$ does not contain any nontrivial edge of G, which means that it is reduced to a single vertex. This gives the interpretation of the proposition. □

Example 2.1. Let us use the notations of Example 1.3. If $n \geqslant 2$,

$$P_\cap(T_n) = X(X-1)\ldots(X-n+1), \qquad P_\subset(T_n) = X^n - X.$$

Here are examples of $P_{\subset,\cap}(T_n)$:

$$\begin{aligned}
P_{\subset,\cap}(T_1) &= X\\
&= H_1(X),\\
P_{\subset,\cap}(T_2) &= X(X-1)\\
&= 2H_2(X),\\
P_{\subset,\cap}(T_3) &= \frac{X(X-1)(2X-1)}{2}\\
&= 3H_2(X) + 6H_3(X),\\
P_{\subset,\cap}(T_4) &= X^2(X-1)^2\\
&= 4H_2(X) + 24H_3(X) + 24H_4(X),\\
P_{\subset,\cap}(T_5) &= \frac{X(X-1)(2X-1)(3X^2-3X-1)}{6}\\
&= 5H_2(X) + 70H_3(X) + 180H_4(X) + 120H_5(X),\\
P_{\subset,\cap}(T_6) &= \frac{X^2(X-1)^2(2X^2-2X-1)}{2}\\
&= 6H_2(X) + 180H_3(X) + 900H_4(X) + 1440H_5(X) + 720H_6(X),\\
P_{\subset,\cap}(T_7) &= \frac{X(X-1)(2X-1)(3X^4-6X^3+3X+1)}{6}\\
&= 7H_2(X) + 434H_3(X) + 3780H_4(X) + 10920H_5(X) + 12600H_6(X) + 5040H_7(X).
\end{aligned}$$

The coefficients of $H_k(X)$ in $P_{\subset,\cap}(T_n)$ are given by Entry A282507 of the OEIS [22].

2.2 Homogeneous polynomial invariants

In all this paragraph, we fix $\curlywedge \in \{\subset, \cap\}$.

Proposition 2.3. *The following map is a bialgebra map from* $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\curlywedge)})$ *to* $(\mathbb{K}[X], m, \Delta)$*:*

$$P_0 : \begin{cases} \mathcal{F}[\mathbf{H}] & \longrightarrow & \mathbb{K}[X] \\ G & \longmapsto & X^{|V(G)|}. \end{cases}$$

Proof. With the help of [14, Propositions 3.10 and 5.2], let us define a homogeneous morphism $P_0 : \mathcal{F}[\mathbf{H}] \longrightarrow \mathbb{K}[X]$ with the help of the element $\mu \in \mathcal{F}[\mathbf{H}]_1^*$ defined by

$$\mu(\centerdot) = 1,$$

where $\centerdot$ is the unique hypergraph with one vertex. For any nonempty hypergraph G with n vertices,

$$\begin{aligned} P_0(G) &= \sum_{k=1}^{\infty} \mu^k \circ \left(\Delta^{(\curlywedge)}\right)^{(k-1)}(G)\frac{X^k}{k!} \\ &= \sum_{k=1}^{\infty} \sum_{V(G)=I_1 \sqcup \ldots \sqcup I_k} \mu(G_{|_\curlywedge I_1}) \ldots \mu(G_{|_\curlywedge I_k})\frac{X^k}{k!} \\ &= \sum_{k=1}^{\infty} \sum_{\substack{V(G)=I_1 \sqcup \ldots \sqcup I_k \\ |I_1|=\ldots=|I_k|=1}} \frac{X^k}{k!} \\ &= n!\frac{X^n}{n!} + 0 \\ &= X^n. \end{aligned}$$

□

Remark 2.2. The map P_0 is also a Hopf algebra morphism from $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\subset,\cap)})$ and from $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\cap,\subset)})$ to $(\mathbb{K}[X], m, \Delta)$.

We denote by $\leftsquigarrow_\curlywedge$ the action of the monoid $\mathrm{Char}(\mathcal{F}[\mathbf{H}])$ of characters of $(\mathcal{F}[\mathbf{H}], m, \delta^{(\curlywedge)})$ on the set of Hopf algebra morphisms from $(\mathcal{F}[\mathbf{H}], m\Delta^{(\curlywedge)})$ to $(\mathbb{K}[X], m, \Delta)$ induced by $\delta^{(\curlywedge)}$, as defined in [14]: for any $\lambda \in \mathrm{Char}(\mathcal{F}[\mathbf{H}])$, for any Hopf algebra morphism $\phi : (\mathcal{F}[\mathbf{H}], m, \Delta^{(\curlywedge)}) \longrightarrow (\mathbb{K}[X], m, \Delta)$,

$$\phi \leftsquigarrow_\curlywedge \lambda = (\phi \otimes \lambda) \circ \delta^{(\curlywedge)}.$$

Let λ_0 be the character $\epsilon_\delta \circ P_0$ of $\mathcal{F}[\mathbf{H}]$: for any hypergraph H,

$$\lambda_0(H) = P_0(H)(1) = 1.$$

By [14, Corollary 3.11],

$$P_0 = P_\curlywedge \leftsquigarrow_\curlywedge \lambda_0.$$

In order to "reverse" this formula, let us study the inverses of characters of $\mathcal{F}[\mathbf{H}]$.

Proposition 2.4. *We denote by* $\star_\curlywedge$ *the convolution induced by* $\delta^{(\curlywedge)}$ *on the set of characters of* $\mathcal{F}[\mathbf{H}]$*. Let* ζ *be a character of* $\mathcal{F}[\mathbf{H}]$*.*

1. *Then* ζ *is invertible for* $\star$ *if, and only if,* $\zeta(\centerdot) \neq 0$*.*
2. *If* $\zeta(\centerdot) = \pm 1$ *and for any hypergraph* G*,* $\zeta(G) \in \mathbb{Z}$*, then for any hypergraph* G*,* $\zeta^{\star_\curlywedge -1}(G) \in \mathbb{Z}$*.*

Proof. 1. For any hypergraph G, let us denote by $\mathrm{cc}(G)$ the number of its connected components. We put $\deg(G) = |V(G)| - \mathrm{cc}(G)$. For any hypergraph G, for any $\sim \in \mathcal{E}_c[G]$,

$$\begin{aligned} \mathrm{cc}(G/\sim) &= \mathrm{cc}(G), & |V(G/\sim)| &= \mathrm{cl}(\sim), \\ \mathrm{cc}(G\mid\sim) &= \mathrm{cl}(\sim), & |V(G\mid\sim)| &= |V(G)|, \end{aligned}$$

where $\mathrm{cl}(\sim)$ is the number of equivalence classes of $\sim$. Therefore, deg induces a graduation of the bialgebra $(\mathcal{F}[\mathbf{H}], m, \delta^{(\leftthreetimes)})$. The result is then a direct consequence of [17, Lemma 3.9], where the family of group-like elements is reduced to $\bullet$.

2. We proceed by induction on $\deg(G)$. If $\deg(G) = 0$, then $\delta^{(\leftthreetimes)}(G) = G \otimes G$ and we deduce that

$$\zeta^{\star_{\leftthreetimes}-1}(G) = \frac{1}{\zeta(G)} = \pm 1.$$

Let us assume that the result is satisfied for any graph H of degree $< \deg(G)$. Then

$$\delta^{(\leftthreetimes)}(G) = G \otimes \bullet^{|V(G)|} + \bullet^{|\mathrm{cc}(G)|} \otimes G + \sum G' \otimes G'',$$

with G' and G'' are hypergraphs with $\deg(G)', \deg(G'') < n$. As $\deg(G) > 0$, G has at least one edge, and $\epsilon_\delta(G) = 0$. Therefore, we put

$$\zeta^{\star_{\leftthreetimes}-1}(G) = -\frac{1}{\zeta(\bullet)^{|V(G)|}} \left(\frac{1}{\zeta(\bullet)^{\mathrm{cc}(G)}} \zeta(G) + \sum \zeta^{\star_{\leftthreetimes}-1}(G')\zeta(G'') \right) \in \mathbb{Z},$$

as $\zeta(\bullet) = \pm 1$ and $\zeta(G'')$, $\zeta^{\star_{\leftthreetimes}-1}(G') \in \mathbb{Z}$. □

This can be applied to λ_0:

Definition 2.5. *We denote by $\lambda_{\leftthreetimes}$ the inverse of λ_0 for the convolution product $\star_{\leftthreetimes}$ associated to $\delta^{(\leftthreetimes)}$. It exists, and for any hypergraph G, $\lambda_{\leftthreetimes}(G) \in \mathbb{Z}$.*

Proposition 2.6. *Let $\leftthreetimes \in \{\subset, \cap\}$. For any hypergraph G, $P_{\leftthreetimes}(G) \in \mathbb{Z}[X]$, and is a unitary polynomial of degree $|V(G)|$. Moreover, the opposite of the coefficient of $X^{|V(G)|-1}$ in $P_{\leftthreetimes}(G)$ is:*

- *the number of edges of G of cardinality 2 if $\leftthreetimes = \subset$.*
- $\displaystyle\sum_{e \in E^+(G)} \binom{|e|}{2}$ *if $\leftthreetimes = \cap$.*

Moreover, for any hypergraph G,

$$P_{\leftthreetimes}(G) = \sum_{\sim \in \mathcal{E}_{\leftthreetimes}[G]} \lambda_{\leftthreetimes}(G\mid\sim) X^{\mathrm{cl}(\sim)}. \tag{1}$$

Proof. By Proposition 2.4,

$$P_{\leftthreetimes} = P_0 \leftsquigarrow_{\leftthreetimes} \lambda_{\leftthreetimes}.$$

This gives (1). For any hypergraph H, $\lambda_{\leftthreetimes}(H) \in \mathbb{Z}$, which leads to the conclusion that the coefficients of $P_{\leftthreetimes}(G)$ are integers. Moreover, the degree of $\phi_{\leftthreetimes}(G)$ is smaller that $|V(G)|$. The unique $\sim$ contributing with a term of degree $|V(G)|$ is the equality of $V(G)$, for which $G(|\sim) = \bullet^{|V(G)|}$, so $\lambda_{\leftthreetimes}(G\mid_{\leftthreetimes}\sim) = 1$: P is unitary of degree $|V(G)|$.

The equivalences $\sim \in \mathcal{E}_{\leftthreetimes}[G]$ contributing with a term $X^{|V(G)|-1}$ have exactly one class $\{x, y\}$ of cardinality 2, the other ones being singleton. The connectedness condition implies that $G_{|_{\leftthreetimes}\{x,y\}}$ should be the graph $\mathbf{I}$. For such an equivalence $\sim$,

$$\lambda_{\leftthreetimes}(G\mid_{\leftthreetimes}\sim) = \lambda_{\leftthreetimes}(\mathbf{I}\bullet^{|V(G)|-2}) = \lambda_{\leftthreetimes}(\mathbf{I}) = -1.$$

Consequently, the coefficient of $X^{|V(G)|-1}$ is the opposite of the number of such equivalences $\sim$, that is to say the number of pairs $\{x, y\}$ of G such that $G_{|_{\leftthreetimes}\{x,y\}} = \mathbf{I}$. This leads directly to the conclusion. □

See Proposition 2.18 for more results on the coefficients of $P_{\curlywedge}(G)$.

Remark 2.3. In general, $P_{\subset,\cap}(H) \notin \mathbb{Z}[X]$, For example,

$$P_{\subset,\cap}(T_3) = X^3 - \frac{3}{2}X^2 + \frac{1}{2}X.$$

Proposition 2.7. *There is no coproduct $\delta^{(\subset,\cap)}$ on $\mathcal{F}[\mathbf{H}]$ such that:*

1. *$(\mathcal{F}[\mathbf{H}], m, \Delta^{(\subset,\cap)}, \delta^{(\subset,\cap)})$ is a double bialgebra.*

2. *The counit of $\delta^{(\subset,\cap)}$ is ϵ_δ.*

3. *The character λ_0 is invertible for the convolution $\star$ associated to $\delta^{(\subset,\cap)}$ and for any hypergraph G, $\lambda_0^{\star-1}(G) \in \mathbb{Z}$.*

4. *For any hypergraph G, we can write*

$$\delta^{(\subset,\cap)}(G) = \sum_{G_1,G_2 \text{ hypergraphs}} a_{G_1,G_2}(G) G_1 \otimes G_2,$$

with $a_{G_1,G_2}(G) \in \mathbb{Z}$ for any G_1, G_2.

Proof. Let us assume that such a $\delta^{(\subset,\cap)}$ exists. The unique double bialgebra morphism ϕ from $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\subset,\cap)}, \delta^{(\subset,\cap)})$ to $(\mathbb{K}[X], m, \Delta, \delta)$ is the unique bialgebra morphism ϕ from $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\subset,\cap)})$ to $(\mathbb{K}[X], m, \Delta)$ such that $\epsilon_\delta \circ \phi = \epsilon_\delta$: it is $P_{(\subset,\cap)}$. The morphism P_0 is also a bialgebra morphism from $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\subset,\cap)})$ to $(\mathbb{K}[X], m, \Delta)$. Denoting by $\leftsquigarrow$ the action induced by $\delta^{(\subset,\cap)}$,

$$P_0 = P_{(\subset,\cap)} \leftsquigarrow \lambda_0.$$

As λ_0 is invertible, $P_{(\subset,\cap)} = P_0 \leftsquigarrow \lambda_0^{\star-1}$. Therefore, for any hypergraph G,

$$P_{(\subset,\cap)}(G) = \sum_{G_1,G_2 \text{ hypergraphs}} a_{G_1,G_2}(G)\lambda_0^{\star-1}(G_2)X^{|V(G_1)|} \in \mathbb{Z}[X],$$

which is not the case for $G = T_3$. □

Corollary 2.8. *There is no coproduct δ making $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\subset,\cap)}, \delta)$ a double bialgebra, of the form*

$$\delta(G) = \sum_{\sim \in \mathcal{E}'[G]} G/\sim \otimes G_{|_\curlywedge \sim},$$

where $\mathcal{E}'[G]$ is a set of equivalences on $V(G)$ and $\curlywedge \in \{\cap, \subset\}$.

Proof. Indeed, for such a coproduct:

- The compatibility with the product implies that if $\sim \in \mathcal{E}'[G]$, then any class of $\sim$ is included into a single connected component of G.

- The existence of the counit implies then that the equality of $V(G)$ belongs to $\mathcal{E}'[G]$, as well as the one which classes are the connected components of G. Consequently, the counit is ϵ_δ.

- Adapting the proof of Proposition 2.4, we obtain the condition on $\lambda_0^{\star-1}$. □

2.3 Acyclic orientations

If G is a graph, Stanley's theorem [23] gives that

$$P_{chr}(G)(-1) = (-1)^{|V(G)|}\sharp\{\text{acyclic orientations of } G\}.$$

We here extend this result to $P_\subset(G)$ and $P_\cap(G)$ when G are hypergraphs.

Notations 2.1. Let X be a set. Recall that a quasi-order on X is a transitive and reflexive relation $\leqslant$ on X. It is called total if for any $x, y \in X$, $x \leqslant y$ or $y \leqslant x$ (note that $x \leqslant y$, $y \leqslant x$ and $x \neq y$ may happen). If $\leqslant$ is a quasi-order on X, we define an equivalence on X by

$$\forall x, y \in X, \qquad x \sim y \Longleftrightarrow x \leqslant y \text{ and } y \leqslant x.$$

The number of classes of $\sim$ is denoted by $\mathrm{cl}(\leqslant)$. The set $X/\sim$ is given an order by

$$\forall \overline{x}, \overline{y} \in X/\sim, \qquad \overline{x} \preceq \overline{y} \Longleftrightarrow x \leqslant y.$$

Definition 2.9. *Let G be a hypergraph.*

1. *An acyclic orientation of G is a quasi-order $\leqslant$ on $V(G)$ such that:*
   - *For any $e \in E^+(G)$, $\leqslant_{|e}$ is a total nontrivial quasi-order on e.*
   - *For any $x, y \in V(G)$ such that $x < y$, there exists a path $(x_0, \ldots, x_k)$ in G with $x = x_0$, $y = x_k$ and $x_0 < \ldots < x_k$.*
   - *For any $x, y \in V(G)$, if $x \leqslant y$ and $y \leqslant x$, then x, y belong to a same edge of G.*
2. *Let $\leqslant$ be an acyclic orientation of G.*
   - *We shall say that $\leqslant$ is total if for any edge e, $\leqslant_{|e}$ is an order (hence, a total order).*
   - *We shall say that $\leqslant$ is 1-max if for any edge e, the maximal class of $\leqslant_{|e}$ is a singleton.*

Remark 2.4. Let G be a graph, considered as a hypergraph and let $\leqslant$ be an acyclic orientation of G. By the first point, for any edge $\{x, y\}$ of G, $x < y$ or $y < x$: we obtain an orientation of G by orienting any edge e according to $<$. As the vertices in an oriented path of G are strictly increasing according to $<$, there is no cycle in this orientation: we recover an acyclic orientation of G in its usual sense. Conversely, if G' is an acyclic orientation of G in the usual sense, we define a partial order on $V(G)$ by $x \leqslant y$ if there exists an oriented path from x to y in G'. It is not difficult to see that this is an acyclic orientation of the hypergraph G. Hence, acyclic orientations of graphs G (seen as hypergraphs) are acyclic orientations in the usual sense.

Lemma 2.10. *For any hypergraph G, for any $\curlywedge \in \{\subset, \cap\}$,*

$$P_\curlywedge(G)(-1) = \sum_{n=1}^{|V(G)|} \sum_{\substack{V(G)=I_1\sqcup\ldots\sqcup I_n\\ I_1,\ldots,I_n\neq\varnothing,\\ \forall p\in[n],\, E^+(G_{|\curlywedge I_p})=\varnothing}} (-1)^n,$$

and

$$P_{(\subset,\cap)}(G)(-1) = \sum_{n=1}^{|V(G)|} \sum_{\substack{V(G)=I_1\sqcup\ldots\sqcup I_n\\ I_1,\ldots,I_n\neq\varnothing,\\ \forall p\in[n],\, E^+(G_{|(p)I_p})=\varnothing}} (-1)^n.$$

Proof. Recall that $P_\curlywedge : (\mathcal{F}[\mathbf{H}], m, \Delta^{(\curlywedge)}) \longrightarrow (\mathbb{K}[X], m, \Delta)$ is a bialgebra morphism, so is a Hopf algebra morphism. Let G be a hypergraph.

$$P_\curlywedge(G)(-X) = S \circ P_\curlywedge(G) = P_\curlywedge \circ S_\curlywedge(G),$$

where $S_\curlywedge(G)$ is the antipode of the Hopf algebra $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\curlywedge)})$ and S the antipode of $(\mathbb{K}[X], m, \Delta)$. Moreover, by Takeuchi's formula [24],

$$S_\curlywedge(G) = \sum_{n=1}^{|V(G)|} \sum_{\substack{V(G)=I_1\sqcup\ldots\sqcup I_n\\ I_1,\ldots,I_n\neq\emptyset}} (-1)^n G_{|_\curlywedge I_1} \ldots G_{|_\curlywedge I_n}.$$

Hence,

$$\begin{aligned}
P_\curlywedge(G)(-1) &= \sum_{n=1}^{|V(G)|} \sum_{\substack{V(G)=I_1\sqcup\ldots\sqcup I_n\\ I_1,\ldots,I_n\neq\emptyset}} (-1)^n P_\curlywedge(G_{|_\curlywedge I_1})(1) \ldots P_\curlywedge(G_{|_\curlywedge I_n})(1)\\
&= \sum_{n=1}^{|V(G)|} \sum_{\substack{V(G)=I_1\sqcup\ldots\sqcup I_n\\ I_1,\ldots,I_n\neq\emptyset}} (-1)^n \epsilon_\delta(G_{|_\curlywedge I_1}) \ldots \epsilon_\delta(G_{|_\curlywedge I_n})\\
&= \sum_{n=1}^{|V(G)|} \sum_{\substack{V(G)=I_1\sqcup\ldots\sqcup I_n\\ I_1,\ldots,I_n\neq\emptyset,\\ \forall p\in[n],\, E^+(G_{|_\curlywedge I_p})=\emptyset}} (-1)^n.
\end{aligned}$$

The proof is similar for $P_{(\subset,\cap)}$. □

Theorem 2.11. *Let G be a hypergraph.*

$$\begin{aligned}
P_\subset(G)(-1) &= \sum_{\leqslant \text{ acyclic orientation of } G} (-1)^{\mathrm{cl}(\leqslant)},\\
P_\cap(G)(-1) &= (-1)^{|V(G)|}|\{\text{total acyclic orientations of } G\}|,\\
P_{\subset,\cap}(G)(-1) &= \sum_{\leqslant \text{ 1-max acyclic orientation of } G} (-1)^{\mathrm{cl}(\leqslant)}.
\end{aligned}$$

Proof. Let $\leqslant$ be an acyclic orientation of the hypergraph G and let $\leqslant'$ be a linear extension of $\leqslant$: $\leqslant'$ is a total quasi-order on $V(G)$ such that

$$\begin{aligned}
\forall x, y \in V(G), \qquad && x \leqslant y \Longrightarrow x \leqslant' y,\\
&& x \leqslant y \text{ and } y \leqslant x \Longleftrightarrow x \leqslant' y \text{ and } y \leqslant' x.
\end{aligned}$$

Let $I_1, \ldots, I_k$ be the classes of $\sim'$, indexed in such a way that for any $(x_1, \ldots, x_k) \in I_1 \times \ldots \times I_k$, $x_1 \leqslant' \ldots \leqslant' x_k$. For any nontrivial edge $e \in E(G)$, $\leqslant_{|e}$ is a nontrivial total quasiorder, so is equal to $\leqslant'_{|e}$ which in turn is nontrivial. As a consequence, no nontrivial edge is included in a single class of $\sim'$: for any $p \in [k]$, $E^+(G_{|_\subset I_k}) = \emptyset$.

If $\leqslant$ is a quasi-order on a set X, a linear extension of $\leqslant$ is a total quasi-order $\leqslant'$ on the same set X, such that

$$\begin{aligned}
\forall x, y \in X, \qquad && x \leqslant y \text{ and } y \leqslant x \Longleftrightarrow x \leqslant' y \text{ and } y \leqslant' x,\\
&& x \leqslant y \Longrightarrow x \leqslant' y.
\end{aligned}$$

We put

$$\begin{aligned}
A &= \{(\leqslant, \leqslant') \mid \leqslant \text{ acyclic orientation of } G,\ \leqslant' \text{ linear extension of } \leqslant\},\\
B &= \{(I_1, \ldots, I_k) \mid V(G) = I_1 \sqcup \ldots \sqcup I_n,\ I_1, \ldots, I_n \neq \emptyset,\ \forall p \in [n],\ E^+(G_{|_\subset I_p}) = \emptyset\},
\end{aligned}$$

and, with the preceding notations, we obtain a map

$$\iota : \begin{cases} A & \longrightarrow & B \\ (\leqslant, \leqslant') & \longrightarrow & (I_1, \ldots, I_k). \end{cases}$$

Let us prove that ι is injective. If $\iota(\leqslant, \leqslant') = \iota(\preceq, \preceq')$, then the classes of $\leqslant'$ and $\preceq'$ are the same, and in the same order: $\leqslant' = \preceq'$. Let us assume that $x < y$. As $\leqslant$ is an acyclic orientation of G, there exists a path $(x = x_0, \ldots, x_k = y)$ in G, with $x_0 < \ldots < x_k$. Then $x_0 <' \ldots <' x_k$, so $x_0 \prec' \ldots \prec' x_k$. Let $p \in [k]$. x_{p-1} and x_p are in the same edge $e \in E(G)$. As $\preceq_{|e}$ is a total quasi-order, $\preceq_{|e} = \preceq'_{|e}$, so $x_{p-1} \prec x_p$. By transitivity, $x \prec y$. By symmetry, $\prec = <$, so $\preceq = \leqslant$.

Let us prove that ι is surjective. Let $(I_1, \ldots, I_n) \in B$. We define a total quasi-order $\leqslant'$ on $V(G)$ by $x \leqslant' y$ if $x \in I_p$ and $y \in I_q$, with $p \leqslant q$. We then define a partial quasi-order $\leqslant$ on $V(G)$ by $x \leqslant y$ if there exists a path $(x = x_0, \ldots, x_k = y)$ in G with for any $p \in [k]$, $x_{p-1} <' x_p$. Then $\leqslant'$ is a linear extension of $\leqslant$, and it is not difficult to prove that $\leqslant$ is an acyclic orientation of G. Moreover, $\iota(\leqslant, \leqslant') = (I_1, \ldots, I_k)$.

Therefore,

$$\begin{aligned} P_{\subset}(G)(-1) &= \sum_{(I_1,\ldots,I_n) \in B} (-1)^k = \sum_{(\leqslant,\leqslant') \in A} (-1)^{\mathrm{cl}(\leqslant)} \\ &= \sum_{\preceq \text{ acyclic orientation of } G} \left(\sum_{\preceq' \text{ linear extension of } \preceq} (-1)^{|V(G)/\sim|} \right). \end{aligned}$$

Let $\preceq$ be the partial order on $V(G)/\sim$ induced by $\leqslant$ and Hasse$(\preceq)$ its Hasse graph. Then, by the duality principle [17, Corollary 4.7], for any acyclic orientation $\leqslant$ of G,

$$\sum_{\preceq' \text{ linear extension of } \preceq} (-1)^{|V(G)/\sim|} = \mathrm{Ehr}_{Str}(\mathrm{Hasse}(\preceq))(-1) = (-1)^{\mathrm{cl}(\leqslant)},$$

where Ehr_{Str} is the strict Ehrhart polynomial [17, Proposition 4.4]. Hence,

$$P_{\subset}(G)(-1) = \sum_{\leqslant \text{ acyclic orientation of } G} (-1)^{\mathrm{cl}(\leqslant)}.$$

Let us now consider $P_\cap$. We put

$$B_\cap = \{(I_1, \ldots, I_n) \mid V(G) = I_1 \sqcup \ldots \sqcup I_n,\ I_1, \ldots, I_n \neq \emptyset,\ \forall p \in [n],\ E^+(G_{|\cap I_p}) = \emptyset\}.$$

If $(I_1, \ldots, I_k) \in B_\cap$, then for any I, $E^+(G_{|\cap I_p}) \subset E^+(G_{|\subset I_p}) = \emptyset$, so $(I_1, \ldots, I_k) \in B$. We proved that $B_\cap \subseteq B$. We put $A_\cap = \iota^{-1}(B_\cap)$. If $(\preceq, \preceq') \in A_\cap$ then for any edge e of G, for any class C of $\preceq'$, $C \cap e$ is $\emptyset$ or is a singleton. Therefore, $\preceq_{|e} = \preceq'_{|e}$ (as $\preceq_{|e}$ is total), so $\preceq_{|e}$ is a total order: $\preceq$ is a total acyclic orientation of G. Conversely, let $\preceq$ is a total acyclic orientation of G and $\preceq'$ a linear extension of $\preceq$. If $x \sim'_y$, then x and y belong to a same edge of G, and then $x \sim'_{|e} y$ and, as $\sim_{|e} = \sim'_{|e}$ is a total order, $x = y$ and finally $\preceq'$ is a total order. Hence, $I_1, \ldots, I_n$ are singletons. Therefore, obviously $(I_1, \ldots, I_n) \in B_\cap$. We obtain

$$P_\cap(G) = \sum_{\leqslant \text{ total acyclic orientation of } G} (-1)^{\mathrm{cl}(\leqslant)}.$$

Let $\leqslant$ be a total acyclic orientation of G. If $x \sim y$, then x and y belong to a common edge e of G. As $\leqslant_{|e}$ is a total order, $x = y$, so the classes of $\leqslant$ are singleton and $\mathrm{cl}(\leqslant) = |V(G)|$, which gives the result.

Let us finally consider $P_{\cap,\subset}$.

$$B_{\cap,\subset}=\{(I_1,\ldots,I_n)\mid V(G)=I_1\sqcup\ldots\sqcup I_n,\ I_1,\ldots,I_n\neq\emptyset,\ \forall p\in[n],\ E^+(G_{|\overset{(p)}{\cap}I_p})=\emptyset\}.$$

If $(I_1,\ldots,I_k)\in B_{\cap,\subset}$, then for any I, $E^+(G_{|I_p})\subset E^+(G_{|^{(i)}I_p})=\emptyset$, so $(I_1,\ldots,I_k)\in B$. We proved that $B_{\cap,\subset}\subseteq B$. We put $A_{\cap,\subset}=\iota^{-1}(B_{\cap,\subset})$. If $(\preceq,\preceq')\in A_{\cap,\subset}$, then for any p, for any edge e included in $I_1\sqcup\ldots\sqcup I_p$, $e\cap I_p$ is empty or is a singleton. Hence, for any edge e, the maximal class of e (for $\preceq$ or for $\preceq'$, as they coincide on e), is a singleton, so $\preceq$ is 1-max. Conversely, Let $\preceq$ be a 1-max acyclic orientation of G and $\preceq'$ be a linear extension of $\preceq$. Let $1\leqslant p\leqslant n$ and e be a nonempty edge of $G_{|\overset{(p)}{\cap}I_p}$. There exists an edge f such that $f\subseteq I_1\sqcup\ldots\sqcup I_p$ and $e=f\cap I_p$. As $\preceq$ is 1-max, the maximal class of f is a singleton, so $f\cap I_p$ is a singleton: we obtain that e is trivial, so $G_{|^{(p)}I_p}$ has no non trivial edge. Therefore, $(\preceq,\preceq')\in A_{\subset,\cap}$. We finally get

$$P_{\subset,\cap}(G)(-1)=\sum_{\leqslant\text{ acyclic 1-max orientation of }G}(-1)^{\mathrm{cl}(\leqslant)}.\qquad\square$$

Let us give another interpretation for $P_\cap$.

Notations 2.2. Let G be a hypergraph. We associate to G a graph $\Gamma(G)$, with $V(\Gamma(G))=V(G)$ and $E(\Gamma(G))$ is the set of pairs $\{x,y\}$ such that there exists $e\in E(G)$, $\{x,y\}\subseteq e$. In particular, if G is a graph, $\Gamma(G)=G$. This defines a species morphism from $\mathbf{H}$ to the species of simple graphs $\mathbf{G}_s$.

Proposition 2.12. *The map* $\Gamma:(\mathbf{H},m,\Delta^{(\cap,\cap)},\delta^{(\cap)})\longrightarrow(\mathbf{G}_s,m,\Delta,\delta)$ *is a morphism of twisted bialgebra with a contraction-extraction coproduct. Moreover,* $P_\cap=P_{chr}\circ\mathcal{F}[\Gamma]$.

Proof. Obviously, Γ is an algebra morphism. Let $G\in\mathcal{H}[X]$ be a hypergraph and $I\subset X$. Then $\Gamma(G_{|\cap I})=\Gamma(G)_{|I}$. If $\sim\in\mathcal{E}[X]$, then $\sim\in\mathcal{E}_\cap[G]$ if, and only if, $\sim\in\mathcal{E}_c[\Gamma(G)]$. Moreover, if this holds,

$$\Gamma(G/\sim)=\Gamma(G)/\sim,\qquad\qquad\Gamma(G\mid_\cap\sim)=\Gamma(G)\mid_\cap\sim.$$

This implies that Γ is a coalgebra morphism. As a consequence, for any nonunitary commutative bialgebra V, the map $\mathcal{F}_V[\Gamma]:\mathcal{F}_V[\mathbf{H}]\longrightarrow\mathcal{F}_V[\mathbf{G}]$ is a double bialgebra morphism. In the particular case $V=\mathbb{K}$, by unicity of the unique double bialgebra morphism from $\mathcal{F}[\mathbf{H}]$ to $\mathbb{K}[X]$,

$$P_\cap=P_{chr}\circ\mathcal{F}[\Gamma].\qquad\square$$

Therefore, by Stanley's theorem:

Proposition 2.13. *For any hypergraph G,*

$$P_\cap(G)(-1)=(-1)^{|V(G)|}\sharp\{\textit{acyclic orientations of }\Gamma(G)\}.$$

Remark 2.5. For any hypergraph G, $P_\cap(G)$ is the chromatic polynomial of a graph. This is generally not the case for $P_\subset(G)$. By Example 2.1, $P_\subset(T_n)=X^n-X$: if $n\geqslant 3$, this is not the chromatic polynomial of a graph, as for such a polynomial, the non-zero coefficients form a connected sequence. Similarly, $P_{\subset,\cap}(G)$ is generally not the chromatic polynomial of a graph, as they are generally not with integral coefficients.

2.4 Antipodes

From [14, Corollary 2.3]:

Corollary 2.14. *For $\curlywedge \in \{\cap, \subset\}$, let us denote by $S_\curlywedge$ the antipode of $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\curlywedge,\curlywedge)})$. For any hypergraph G,*

$$S_\subset(G) = \sum_{\sim \in \mathcal{E}_\subset[G]} \left(\sum_{\leqslant \text{ acyclic orientation of } G/\sim} (-1)^{\mathrm{cl}(\leqslant)} \right) G \mid_\subset \sim,$$

$$S_\subset(G) = \sum_{\sim \in \mathcal{E}_\cap[G]} (-1)^{\mathrm{cl}(\sim)} \sharp\{\leqslant \text{ acyclic orientation of } G/\sim\} G \mid_\cap \sim .$$

We cannot use the formalism of double bialgebras for the antipode of $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\subset,\cap)})$, which we simply denote by S. We shall use Takeuchi's formula [24]: for any nonempty hypergraph G,

$$S(G) = \sum_{k=1}^{|V(G)|} (-1)^k \sum_{\substack{V(G)=I_1\sqcup\ldots\sqcup I_k,\\ I_1,\ldots,I_k\neq\varnothing}} G_{|^{(1)} I_1} \ldots G_{|^{(k)} I_k}. \tag{2}$$

Let us consider the hypergraphs appearing in this sum. For such a hypergraph H, $V(H) = V(G)$, and the nonempty edges of H are sets of the form $e \cap I_{\theta(e)}$, where e is a nonempty edge of G and $\theta(e) \in [k]$. This leads to the following definition:

Definition 2.15. *Let G be a nonempty hypergraph, $\sim \in \mathcal{E}[V(G)]$ and $\theta : E(G)\backslash\{\varnothing\} \longrightarrow V(G)/\sim$ be a map such that for any nonempty edge e of G, $e \cap \theta(e) \neq \varnothing$.*

1. *We denote by $G \mid_\theta \sim$ the graph such that*

$$V(G \mid_\theta \sim) = V(G), \qquad E(G \mid_\theta \sim) = \{e \cap \theta(e) \mid e \in E(G)\backslash\{\varnothing\}\} \cup \{\varnothing\}.$$

2. *We denote by $G/_\theta \sim$ the oriented graph such that $V(G/_\theta \sim) = V(G)/\sim$ and with set of arcs defined by the following: for any edge $e \in E(G)$, for any $\pi \in V(G)/\sim$ such that $\pi \cap e \in \varnothing$ and $\pi \neq \theta(e)$, there is an arc from π to $\theta(e)$ in $G/_\theta \sim$.*

3. *We shall write that $(\sim, \theta) \in \mathcal{E}_{\subset,\cap}[G]$ if the connected components of $G \mid_\theta \sim$ are the classes of $\sim$ and if the oriented graph $G/_\theta \sim$ is acyclic.*

Proposition 2.16. *For any nonempty hypergraph G,*

$$S(G) = \sum_{(\sim,\theta)\in\mathcal{E}_{\subset,\cap}[G]} (-1)^{\mathrm{cl}(\sim)} G \mid_\theta \sim .$$

Proof. Let us denote by $\mathcal{E}'_{\subset,\cap}[G]$ the set of pairs $(\sim, \theta)$ such that the connected components of $G \mid_\theta \sim$ are the classes of $\sim$. Rewriting (2), we obtain that

$$S(G) = \sum_{(\sim,\theta)\in\mathcal{E}'_{\subset,\cap}[G]} \left(\sum_{\substack{V(G)=I_1\sqcup\ldots\sqcup I_k,\\ I_1,\ldots,I_k\neq\varnothing,\\ G_{|^{(1)}_{I_1}}\ldots G_{|^{(k)}_{I_k}}=G|_\theta\sim}} (-1)^k \right) G \mid_\theta \sim .$$

For any $(\sim, \theta) \in \mathcal{E}'_{\subset,\cap}[G]$, we put

$$P(\sim, \theta)(X) = \sum_{\substack{V(G)=I_1\sqcup\ldots\sqcup I_k,\\ I_1,\ldots,I_k\neq\varnothing,\\ G_{|^{(1)}_{I_1}}\ldots G_{|^{(k)}_{I_k}}=G|_\theta\sim}} H_k(X) \in \mathbb{K}[X],$$

in such a way that (2) is rewritten as

$$S(G) = \sum_{(\sim,\theta)\in\mathcal{E}'_{\subset,\cap}[G]} P(\sim,\theta)(-1) G\mid_\theta\sim .$$

By definition, for any $N \in \mathbb{N}$, $P(G,\sim)(N)$ is the number of maps $f : V(G) \longrightarrow [N]$ such that

$$G^{(1)}_{|f^{-1}(1)} \dots G^{(N)}_{|f^{-1}(N)} = G\mid_\theta\sim .$$

We denote by $\mathcal{A}_N$ the set of such maps f and by $\mathcal{B}_N$ the set of maps $g : V(G)/\sim\longrightarrow [N]$ such that if (π_1,π_2) is an arc of $G/_\theta\sim$, then $g(\pi_1) < g(\pi_2)$.

Let us now define a bijection between $\mathcal{A}_N$ and $\mathcal{B}_N$. Let $f \in \mathcal{A}_N$. If $x \sim y$, then by definition of $\mathcal{E}_{\subset,\cap}[G]$, x and y are in the same connected component of $G\mid_\theta\sim$, so they necessarily belong to the same $f^{-1}(i)$ and finally $f(x) = f(y)$. Therefore, f induces a map $\overline{f} : V(G)/\sim\longrightarrow [N]$ such that for any $x \in V(G)$, $\overline{f}(\overline{x}) = f(x)$. Let us prove that $\overline{f} \in \mathcal{B}_N$. If (π_1,π_2) is an arc of $G/_\theta\sim$, there exists an edge e of G, such that $e\cap\pi_1 \neq \emptyset$, $e\cap\pi_2 \neq \emptyset$, $\pi_1 \neq \pi_2 = \theta(e)$. As $e\cap\pi_2$ is an edge of $G\mid_\theta\sim = G^{(1)}_{|f^{-1}(1)} \dots G^{(N)}_{|f^{-1}(N)}$, necessarily

$$\overline{f}(\pi_2) = \max f_{|e},$$

and $\overline{f}(\pi_1) < \overline{f}(\pi_2)$, so $\overline{f} \in \mathcal{B}_N$. We have defined a map

$$\begin{cases} \mathcal{A}_N & \longrightarrow & \mathcal{B}_N \\ f & \longmapsto & \overline{f}. \end{cases}$$

It is obviously injective. Let $\overline{f} \in \mathcal{B}_N$ and let $f : V(G) \longrightarrow [N]$ be the unique map such that for any $x \in V(G)$, $f(x) = \overline{f}(\overline{x})$. Let e be a nonempty edge of G. By construction of $G/_\theta\sim$, the maximum of f over e is obtained on $e\cap\theta(e)$, so the contribution of e to the edges of $G^{(1)}_{|f^{-1}(1)} \dots G^{(N)}_{|f^{-1}(N)}$ is $e\cap\theta(e)$: we obtain that $G^{(1)}_{|f^{-1}(1)} \dots G^{(N)}_{|f^{-1}(N)} = G\mid_\theta\sim$. As a conclusion, $f \in \mathcal{A}_N$ and $\mathcal{A}_N$ and $\mathcal{B}_N$ are in bijection.

As a conclusion, $P(\sim,\theta)(X)$ is the strict Ehrhart polynomial $\mathrm{Ehr}_{str}(G/_\theta\sim)$ of the oriented graph $G\mid_\theta\sim$. If this oriented graph is acyclic, by the duality principle [17, Corollary 4.7], then

$$P(\sim,\theta)(-X) = (-1)^{|V(G)/\sim|}\mathrm{Ehr}(G/_\theta\sim)(X),$$

where $\mathrm{Ehr}(G/_\theta\sim)$ is the Ehrhart polynomial of $G/_\theta\sim$. In particular, $\mathrm{Ehr}(G/_\theta\sim)(1)$ is the number of maps $f : V(G)/\sim\longrightarrow [1]$ such that for any arc (π_1,π_2) of $G/_\theta\sim$, $f(\pi_1) \leqslant f(\pi_2)$: this is obviously 1. As a consequence, if $(\sim,\theta) \in \mathcal{E}'_{\subset,\cap}[G]$, then $P(\sim,\theta)(-1) = (-1)^{\mathrm{cl}(\sim)}$. Otherwise, $G/_\theta\sim$ is not acyclic, so $\mathrm{Ehr}_{str}(\sim,\theta)(X) = 0$, which implies that $P(G/_\theta\sim)(-1) = 0$. The results immediately follows. □

Remark 2.6. As $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\cap,\subset)})$ is the coopposite of $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\subset,\cap)})$, its antipode is S^{-1}. Moreover, as $(\mathcal{F}[\mathbf{H}], m)$ is commutative, S is involutive, so $S^{-1} = S$ and the antipode of $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\cap,\subset)})$ and $(\mathcal{F}[\mathbf{H}], m, \Delta^{(\subset,\cap)})$ are the same.

2.5 Coefficients of the chromatic polynomials

Notations 2.3. Let G be a hypergraph. For any $i, j \geqslant 1$, we denote by $\mathcal{N}_G(i,j)$ the set of hypergraphs G' of G such that

$$V(G') = V(G), \qquad E^+(G') \subset E^+(G), \qquad \mathrm{cc}(G') = i, \qquad |E^+(G')| = j.$$

We denote by $N_G(i,j)$ the cardinality of $\mathcal{N}_G(i,j)$.

Lemma 2.17. *Recall that $\lambda_\subset$ is the inverse of λ_0 for the convolution product $\star_\subset$ induced by $\delta^{(\subset)}$. For any hypergraph G,*

$$\lambda_\subset(G) = \sum_{j\geqslant 0}(-1)^j N_G(\mathrm{cc}(G), j).$$

Proof. We define $\mu \in \mathcal{F}[\mathbf{H}]^*$ by

$$\mu(G) = \sum_{j\geqslant 0}(-1)^j N_G(\mathrm{cc}(G), j),$$

for any hypergraph G. Let us prove that for any hypergraph G, $\lambda_0 \star_\subset \mu(G) = \epsilon_\delta(G)$.

$$\begin{aligned}\lambda_0 \star_\subset \mu(G) &= \sum_{\sim\in\mathcal{E}_\subset[G]} \mu(G\mid_\subset\sim)\\ &= \sum_{\sim\in\mathcal{E}_\subset[G]}\sum_{j\geqslant 0}(-1)^j N_G(\mathrm{cl}(\sim), j).\end{aligned}$$

There is an obvious bijection

$$\{F \subset E^+(G) \mid [F| = j\} \longrightarrow \bigsqcup_{\sim\in\mathcal{E}_\subset[G]} \mathcal{N}_{G|_\subset\sim}(\mathrm{cl}(\sim), j)$$

which sends $F \subset E^+(G)$ to the hypergraph $(V(G), F)$, belonging to $\mathcal{N}_{G|_\subset\sim}(\mathrm{cl}'\sim), j)$ where $\sim$ is the equivalence on $V(G)$ which classes are the connected components of the hypergraph $(V(G), F)$. Hence,

$$\begin{aligned}\lambda_0 \star_\subset \mu(G) &= \sum_{F\subseteq E^+(G)} (-1)^{|F|}\\ &= \begin{cases}1 \text{ if } |E^+(G)| = \varnothing,\\ 0 \text{ otherwise}\end{cases}\\ &= \epsilon_\delta(G).\end{aligned}$$

□

Let us deduce the following description of the coefficients of $P_\subset(G)$, which can be found in [6, 25]:

Proposition 2.18. *For any $i \geqslant 1$, the coefficient a_i of X^i in $P_\subset(G)$ is*

$$a_i = \sum_{j\geqslant 0}(-1)^j N_G(i, j).$$

Proof. From Proposition 2.6,

$$P_\subset(G) = \sum_{\sim\in\mathcal{E}_\subset[G]} \lambda_\subset(G\mid\sim)X^{\mathrm{cl}(\sim)}. \tag{3}$$

Consequently, combining with Lemma 2.17, for any $i \geqslant 1$,

$$a_i = \sum_{\substack{\sim\in\mathcal{E}_\subset[G],\\ \mathrm{cl}(\sim)=i}}\sum_{j\geqslant 0}(-1)^j N_{G|\sim}(i, j).$$

Moreover, there is an obvious bijection

$$\mathcal{N}_G(i, j) \longrightarrow \bigsqcup_{\substack{\sim\in\mathcal{E}_\subset[G],\\ \mathrm{cl}(\sim)=i}} \mathcal{N}_{G|_\subset\sim}(i, j),$$

sending G' to itself, seen as an an element of $\mathcal{N}_{G|_\subset\sim}(i, j)$, where $\sim$ is the equivalence which classes are the connected components of G'. Consequently,

$$a_i = \sum_{j\geqslant 0} N_G(i, j).$$

□

Remark 2.7. If G is a hypergraph with n vertices, then $N_G(n-1,j)=0$ if $j\neq 1$ and $N_G(n-1,1)$ is the number of edges of G of cardinality 2. We recover the result of Proposition 2.6.

Proposition 2.19. *We define a map ϖ on $\mathcal{F}[\mathbf{H}]$ by the following: for any hypergraph G,*

$$\varpi(G)=\sum_{\sim\in\mathcal{E}_\subset[G]}\left(\sum_{j\geqslant 0}(-1)^j N_{G/\sim}(1,j)\right) G\mid_\subset\sim .$$

Then ϖ is the projector on the space $\mathrm{Prim}(\mathcal{F}[\mathbf{H}])$ of primitive elements of $\mathcal{F}[\mathbf{H}]$ which vanishes on $(1)\oplus\mathrm{Ker}(\varepsilon)^2$ (eulerian idempotent). Consequently, a basis of $\mathrm{Prim}(\mathcal{F}[\mathbf{H}])$ is given by $(\varpi(G))_{G\ \text{connected hypergraph}}$.

Proof. By [14, Proposition 4.1], the infinitesimal character $\ln(\epsilon_\delta)$ is given on any hypergraph G by

$$\ln(\epsilon_\delta)(G)=\sum_{j\geqslant 0}(-1)^j N_G(1,j).$$

We conclude with [14, Corollary 4.5]. □

2.6 Morphisms to quasishuffle algebras

We assume in this paragraph that $(V,\cdot,\delta_V)$ is a nonunitary, commutative and cocommutative bialgebra. By [16, Proposition 3.9], $\mathcal{F}_V[\mathbf{H}]$ is a bialgebra over V, with the coaction ρ described as follows: if G is a V-decorated hypergraph with n vertices, we arbitrarily index these vertices and we denote by $G(v_1,\ldots,v_n)$ the hypergraph with for any i, the i-th vertex of G decorated by v_i. Then

$$\rho(G(v_1,\ldots,v_n))=G(v_1',\ldots,v_n')\otimes v_1''\cdot\ldots\cdot v_n''.$$

Notations 2.4. The map $\pi_V:T(V)\longrightarrow\mathbb{K}[X]$ is defined by

$$\forall v_1,\ldots,v_n\in V,\qquad \pi_V(v_1\ldots v_n)=\epsilon_V(v_1)\ldots\epsilon_V(v_n)\frac{X(X-1)\ldots(X-n+1)}{n!}.$$

It is a double bialgebra morphism.

By [15, Theorem 2.7]:

Proposition 2.20. *Let $(V,\cdot,\delta_V)$ be a commutative, not necessarily unitary bialgebra.*

1. *For any hypergraph G, we denote by $\mathrm{VC}_\cap(G)$ the set of surjective maps $f:V(G)\longrightarrow[k]$ such that if x and y are two distinct elements of an edge $e\in E(G)$, then $f(x)\neq f(y)$. The unique double bialgebra morphism over V from $(\mathcal{F}_V[\mathbf{H}],m,\Delta^{(\cap,\cap)},\delta^{(\cap)})$ to $(T(V),\uplus,\Delta,\delta)$ sends any V decorated hypergraph G to*

$$\Phi_\cap(G)=\sum_{f\in\mathrm{VC}_\cap[G]}\left(\prod_{f(i)=1}^{\cdot} v_i\right)\ldots\left(\prod_{f(i)=\max(f)}^{\cdot} v_i\right).$$

   Moreover, $P_\cap\circ\Theta_V=\pi_V\circ\Phi_\cap$.

2. *For any hypergraph G, we denote by $\mathrm{VC}_\subset(G)$ the set of surjective maps $f:V(G)\longrightarrow[k]$ such that if e is a nontrivial edge of G, then f takes at least two different values on e. The unique double bialgebra morphism over V from $(\mathcal{F}_V[\mathbf{H}],m,\Delta^{(\subset,\subset)},\delta^{(\subset)})$ to $(T(V),\uplus,\Delta,\delta)$ sends any V decorated hypergraph G to*

$$\Phi_\subset(G)=\sum_{f\in\mathrm{VC}_\subset[G]}\left(\prod_{f(i)=1}^{\cdot} v_i\right)\ldots\left(\prod_{f(i)=\max(f)}^{\cdot} v_i\right).$$

   Moreover, $P_\subset\circ\Theta_V=\pi_V\circ\Phi_\subset$.

Proof. Let $\curlywedge \in \{\cap, \subset\}$. Both maps $P_{\curlywedge} \circ \Theta_V$ and $\pi_V \circ \Phi_{\curlywedge}$ are double bialgebra morphisms from $(\mathcal{F}_V[\mathbf{H}], m, \Delta^{(\curlywedge)}, \delta^{(\curlywedge)})$ to $(\mathbb{K}[X], m, \Delta, \delta)$. By unicity of such a morphism, they are equal. □

Even without the double bialgebra structure, we can define a Hopf algebra morphism for $\Delta^{(\cap,\subset)}$, with [15, Theorem 2.3]:

Proposition 2.21. *Let $(V, \cdot)$ be a commutative, not necessarily unitary algebra. For any hypergraph G, we denote by $\mathrm{VC}_{\cap,\subset}(G)$ the set of surjective maps $f : V(G) \longrightarrow [k]$ such that if e is a nontrivial edge of G, then $\max\{f(x) \mid x \in e\}$ is obtained in exactly one element of e. The following defines a Hopf algebra morphism from $(\mathcal{F}_V[\mathbf{H}], m, \Delta^{(\cap,\subset)})$ to $(T(V), ⧢, \Delta)$: for any V-decorated hypergraph G,*

$$\Phi_{\cap,\subset}(G) = \sum_{f \in \mathrm{VC}_{\cap,\subset}[G]} \left(\prod_{f(i)=1}^{\cdot} v_i \right) \ldots \left(\prod_{f(i)=\max(f)}^{\cdot} v_i \right).$$

Moreover, $P_{\cap,\subset} \circ \Theta_V = \pi_V \circ \Phi_{\cap,\subset}$.

3 Multi-complexes

3.1 Definition

Recall that a multiset is a map $X : S \longrightarrow \mathbb{N}\backslash\{0\}$, where S is a set, called the support of X and denoted by $\mathrm{supp}(X)$. For any $x \in \mathrm{supp}(X)$, $X(x)$ is the multiplicity of x in X. Multisets are usually seen as "sets with repetitions of elements": for example, the multiset

$$X : \begin{cases} \{a, b, c\} & \longrightarrow & \mathbb{N}\backslash\{0\} \\ a & \longmapsto & 1 \\ b & \longmapsto & 3 \\ c & \longmapsto & 2 \end{cases}$$

is represented by $\{a, b, b, b, c, c\}$. If X and Y are two multisets, $X \subseteq Y$ if $\mathrm{supp}(X) \subseteq \mathrm{supp}(Y)$ and for any $x \in \mathrm{supp}(X)$, $X(x) \leqslant Y(x)$. For example, $\{a, a, b, b, b, c\} \subseteq \{a, a, a, b, b, b, c, c, c, d\}$.

The notion of multi-complexes is introduced in [19]. Let us give a slightly modified definition, adapted to our setting:

Definition 3.1. *A multi-complex is a triple $C = (V(C), E(C), \leqslant_C)$, where:*

- *$V(C)$ is a finite set, called the set of vertices of C.*
- *$E(C)$ is a multiset of multisets, such that:*
  - *For any $e \in E(C)$, the support $\mathrm{supp}(e)$ of e is a subset of $V(C)$.*
  - *For any $x \in V(C)$, $\{x\}$ is an element of the multiset $E(C)$ of multiplicity 1.*
  - *$\varnothing$ is an element of the multiset $E(C)$ of multiplicity 1.*

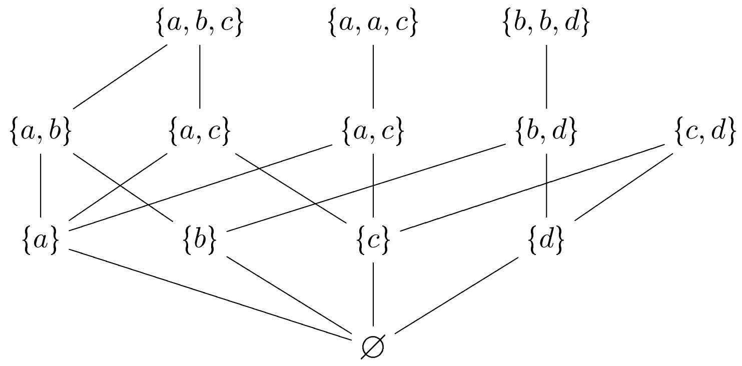
  The elements of $E(C)$ are called the edges of C.
- *$\leqslant_C$ is a partial order on the multiset $E(C)$ such that:*
  - *For any $x \in E(C)$, for any $e \in E(C)$, $\{x\} \leqslant_C e$ if, and only if, $x \in \mathrm{supp}(e)$.*
  - *For any $e \in E(C)$, $\varnothing \leqslant_C e$.*
  - *For any $e, f \in E(C)$, if $e \leqslant_C f$, then $e \subset f$.*

For any finite set X, the set of multi-complexes C with $V(C) = X$ is denoted by $\mathcal{MC}[X]$, and the vector space generated by $\mathcal{MC}[X]$ is denoted by $\mathbf{MC}[X]$. Then $\mathcal{MC}$ is a set species and $\mathbf{MC}$ is a species.

Example 3.1. Here is a multicomplex C. We put $V(C) = \{a, b, c, d\}$, and

$$E(C) = \left\{ \begin{array}{c} \varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \\ \{a,b\}, \{a,c\}, \{a,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,a,c\}, \{b,b,d\} \end{array} \right\},$$

with the partial order given by its Hasse graph:



Hypergraphs are multi-complexes, with $\leqslant$ given by the inclusion; note that in this case, the edges of C are sets, and the multiset of edges is also a set. Simplicial complexes and Δ-complexes are also multi-complexes, see [19].

3.2 Hopf algebraic structures multi-complexes

A Hopf algebra of multi-complexes is introduced in [19]. Let us lift this to the twisted level. Let X and Y be two disjoint sets. If $C \in \mathcal{MC}[X]$ and $D \in \mathcal{MC}[Y]$, the multi-complex $CD \in \mathcal{MC}[X \sqcup Y]$ is defined by

$$V(CD) = X \sqcup Y, \qquad E(CD) = E(C) \sqcup E(Y),$$

and for any $e, f \in E(CD)$,

$$e \leqslant_{CD} f \text{ if } (e, f \in E(C) \text{ and } e \leqslant_C f) \text{ or } (e, f \in E(D) \text{ and } e \leqslant_D f).$$

This defines a associative, commutative product m on $\mathbf{MC}$, which unit is the empty multi-complex.

For any finite sets $X \subset Y$ and for any multi-complex $C \in \mathcal{MC}[Y]$, we define $C_{|X}$ by

$$V(C_{|X}) = X, \qquad E(C_{|X}) = \{e \in E(C) \mid \mathrm{supp}(C) \subset X\}, \qquad \leqslant_{C_{|X}} = (\leqslant_C)_{|E(C_{|X})}.$$

This is indeed a multi-complex. We then define a coproduct Δ on $\mathbf{MC}$ by the following: for any finite sets X and Y, for any $C \in \mathcal{MC}[X \sqcup Y]$,

$$\Delta_{X,Y}(X) = C_{|X} \otimes C_{|Y}.$$

Proposition 3.2. $(\mathbf{MC}, m, \Delta)$ *is a twisted bialgebra. Moreover,* $\mathcal{F}[\mathbf{MC}]$ *is the Hopf algebra of multi-complexes of [19].*

Proof. Similar to the proof of Proposition 1.3. □

Let us now define an extraction-contraction coproduct on **MC**.

Let C be a multi-complex, and let $x, y \in V(C)$. A path from x to y is a sequence $(x_0, \ldots, x_k)$ of vertices of C such that:

- $x_0 = x$ and $x_k = y$.
- For any $i \in [k]$, there exists $e \in E(C)$ such that $x_{i-1}, x_i \in e$.

We shall say that C is connected if for any $x, y \in V(C)$, there exists a path from x to y in C.

Let X be a finite set, $\sim \in \mathcal{E}[X]$ and $C \in \mathcal{MC}[X]$.

1. We shall say that $\sim \in \mathcal{E}_c[C]$ if for any $\varpi \in X/\sim$, $C_{|\varpi}$ is connected.
2. We denote by $X \mid\sim$ the multi-complex defined by

$$\begin{aligned} V(C \mid\sim) &= V(C), \\ E(C \mid\sim) &= \{e \in E(C) \mid \forall x, y \in \mathrm{supp}(e),\ x \sim y\}, \\ \leqslant_{C\mid\sim} &= (\leqslant_C)_{|E(C\mid\sim)}. \end{aligned}$$

   In other words,

$$C \mid\sim = \prod_{\varpi \in X/\sim} C_{|\varpi}.$$

3. We denote by $X/\sim$ the multi-complex defined by

$$\begin{aligned} V(C/\sim) &= V(V)/\sim, \\ E(C/\sim) &= \{\pi(e) \mid e \in E(C)\}, \end{aligned}$$

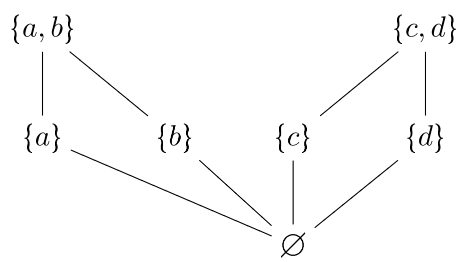
   where $\pi_\sim : V(C) \longrightarrow V(C)/\sim$ is the canonical surjection. It is noticeable that $E(C/\sim)$ is a multiset, that is to say we distinguish all the $\pi(e)$, $e \in E(C)$, in $E(C/\sim)$, except for the trivial edges (which are and the singletons, which remains of multiplicity 1). In other terms, if $\overline{e}$ is a multiset of support included in $V(C)/\sim$, its multiplicity in $E(C/\sim)$ is the sum of the multiplicities of the edges $e \in E(C/\sim)$ such that $\pi(e) = \overline{e}$. The partial order on $E(C/\sim)$ is defined by

$$\pi_\sim(e) \leqslant_{C/\sim} \pi_\sim(f) \Longleftrightarrow e \leqslant_C f.$$

Example 3.2. Let us consider the multi-complex C of Example 3.1 again. Let $\sim$ be the equivalence which classes are $\{a, b\}$ and $\{c, d\}$. Because its classes are edges of C, $\sim \in \mathcal{E}_c[C]$. Moreover, $V(C \mid\sim) = \{a, b, c, d\}$, and

$$E(C \mid\sim) = \left\{ \begin{array}{c} \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \\ \{a, b\}, \{b, d\} \end{array} \right\},$$

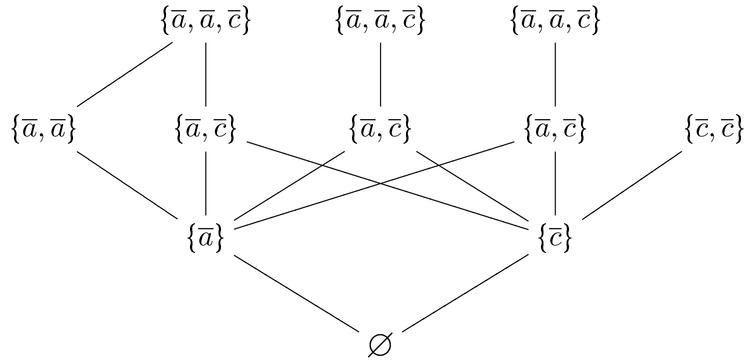
with the partial order given by its Hasse graph:



Moreover, $V(C/\sim) = \{\overline{a}, \overline{c}\}$, and

$$E(C) = \left\{ \begin{array}{c} \varnothing, \{\overline{a}\}, \{\overline{c}\}, \\ \{\overline{a},\overline{a}\}, \{\overline{a},\overline{c}\}, \{\overline{a},\overline{c}\}, \{\overline{a},\overline{c}\}, \{\overline{c},\overline{c}\}, \{\overline{a},\overline{a},\overline{c}\}, \{\overline{a},\overline{a},\overline{c}\}, \{\overline{a},\overline{a},\overline{c}\} \end{array} \right\},$$

with the partial order given by its Hasse graph:



Theorem 3.3. *For any multi-complex $C \in \mathcal{MC}[X]$ and for any $\sim \in \mathcal{E}[X]$, we put*

$$\delta_\sim(C) = \begin{cases} C/\sim \otimes C\mid\sim & \text{if } \sim \in \mathcal{E}_c[C], \\ 0 & \text{otherwise.} \end{cases}$$

This defines a contraction-extraction coproduct on **MC** *in the sense of [16], compatible with m and Δ.*

Proof. Similar to the proof of Theorem 1.6. □

4 Link with hypergraphs

Definition 4.1. *Let C be a multi-complex. We define the hypergraph $\kappa(C)$ by*

$$V(\kappa(C)) = V(C), \qquad E(\kappa(C)) = \mathrm{supp}\{\mathrm{supp}(e) \mid e \in E(C)\}.$$

In other words, $\kappa(C)$ is obtained from C by forgetting the partial order $\leqslant_C$ and the multiplicities in the edges and in $E(C)$. This defines a species morphism $\kappa : \mathbf{MC} \longrightarrow \mathbf{H}$.

The following is obtained by direct verifications:

Proposition 4.2. *$\kappa : (\mathbf{MC}, m, \Delta) \longrightarrow (\mathbf{H}, m, \Delta^{(\subset)})$ is a twisted bialgebra morphism. Moreover, it is compatible with the contraction-extraction coproducts δ and $\delta^{(\subset)}$.*

As a consequence, the unique double bialgebra morphism from $\mathcal{F}[\mathbf{MC}]$ to $\mathbb{K}[X]$ is $P_\subset \circ \mathcal{F}[\kappa]$.

From [14, Corollary 2.3]:

Corollary 4.3. *Let us denote by S the antipode of $(\mathcal{F}[\mathbf{MC}], m, \Delta)$. For any mutli-complex C,*

$$S(C) = \sum_{\sim \in \mathcal{E}_c[C]} \left(\sum_{\leqslant \text{ acyclic orientation of } \kappa(C/\sim)} (-1)^{\mathrm{cl}(\leqslant)} \right) C\mid\sim .$$

By [14, Corollary 4.5]:

Proposition 4.4. *We define a map ϖ on $\mathcal{F}[\mathbf{MC}]$ by the following: for any multi-complex C,*

$$\varpi(C) = \sum_{\sim \in \mathcal{E}_c[C]} \left(\sum_{j \geqslant 0} (-1)^j N_{\kappa(C/\sim)}(1, j) \right) C\mid\sim .$$

Then ϖ is the projector on the space $\mathrm{Prim}(\mathcal{F}[\mathbf{MC}])$ of primitive elements of $\mathcal{F}[\mathbf{MC}]$ which vanishes on $(1) \oplus \mathrm{Ker}(\varepsilon)^2$ (eulerian idempotent). Consequently, a basis of $\mathrm{Prim}(\mathcal{F}[\mathbf{MC}])$ is given by $(\varpi(C))_{C \text{ connected multi-complex}}$.

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